ON JUMPING LINES OF VECTOR BUNDLES ON $\mathbb{P}^n$.

PH. ELLIA

To the memory of Alexandru "Sacha" Lascu.

Introduction.

In the study of vector bundles on $\mathbb{P}^n$ (projective space over an algebraically closed field, k, with $ch(k) = 0$) a useful technique is to consider the restrictions, E_L, of E to lines $L \subset \mathbb{P}^n$. Thanks to a well known theorem E_L splits as a sum of line bundles: $E_L \simeq \bigoplus \mathcal{O}_L(a_i^{(L)})$. One says that $(a_1^{(L)}, ..., a_r^{(L)})$, $a_1^{(L)} \geq \cdots \geq a_r^{(L)}$, is the splitting type of E over L. By semi-continuity of cohomology there exists a dense open subset $\mathcal{U} \subset G(1, n)$ and a set of integers (a_i) such that the splitting type of E_L is $(a_1, ..., a_r)$ for every $L \in \mathcal{U}$. One says that $(a_1, ..., a_r)$ is the *generic splitting type* of E. If $L \notin \mathcal{U}$, the splitting type of E_L is different from the generic one and L is said to be a *jumping line* of E (although it is E which jumps and not L!).

The set, $\mathcal{J}(E)$, of jumping lines doesn't characterize E in general but it is very useful in understanding the structure of E.

The first case is when $\mathcal{J}(E) = \emptyset$, in this case E is said to be *uniform*. Observe that an homogeneous vector bundle is uniform. Uniform vector bundles have been studied for a while, in particular they are classified up to rank $n+1$ ([13], [11], [7] [8], [9], [1]). The classification shows that every uniform vector bundle of rank $\leq n+1$ on $\mathbb{P}^n$ is homogeneous. By the way there exist uniform, non homogeneous vector bundles of rank $2n$ on $\mathbb{P}^n$ ([5]).

In this paper we consider the next step that is to say when $\mathcal{J}(E)$ is non-empty but finite.

Definition 1. *A vector bundle, E, on $\mathbb{P}^n$ is said to be* almost-uniform *if it is not uniform but has only a* finite *number of jumping lines.*

Date: April 9, 2017.
2010 *Mathematics Subject Classification.* 14N05.
Key words and phrases. Vector bundles, uniform, jumping lines.

We set $\mathcal{A}(n) := min\{r \mid$ *there exists an almost-uniform vector bundle of rank* r *on* $\mathbb{P}^n\}$.

It is not clear a priori that such vector bundles even exist! Actually they do exist and our task will be to give bounds on the function $\mathcal{A}(n)$.

Our results may be summarized as follows:

Theorem 2.
With notations as above:
(1) $\mathcal{A}(2) = 2$.
(3) For every $n \geq 3$, $n + 1 \leq \mathcal{A}(n) \leq 2n - 1$.

In section 1 we treat the case of $\mathbb{P}^2$, which is easy. In Section 2 we show $\mathcal{A}(n) \geq n$ and $\mathcal{A}(n) \geq n+1$ if $n \geq 4$. For this we observe that if H is a "good" hyperplane for E (i.e. H doesn't contain any jumping line of E), then E_H is uniform and we use the classification of uniform bundles. In Section 3 we show $\mathcal{A}(3) \geq 4$. This case deserves a special attention because it may happen that $E_H \simeq S^2T_H$ for some "good" hyperplane. Finally in Section 4 we show $\mathcal{A}(n) \leq 2n - 1$ by constructing examples. The bound $\mathcal{A}(n) \leq 2n - 1$ could be not too far from being sharp (see Remark 17).

In conclusion "small" ($\leq n$) rank vector bundles on $\mathbb{P}^n, n \geq 3$, which are not a direct sum of line bundles or a twist of the tangent or cotangent bundle, have infinitely many jumping lines.

1. Almost uniform vector bundles on $\mathbb{P}^2$.

Let's first treat the case of $\mathbb{P}^2$ which is fairly easy.

Lemma 3. *There exist rank two vector bundles on* $\mathbb{P}^2$ *with a single jumping line (hence* $\mathcal{A}(2) = 2$*).*
More precisely if E *is a normalized, almost unifiorm rank two vector bundle on* $\mathbb{P}^2$*, then* $c_1(E) = -1$ *and* E *is stable.*

Proof. Let X be a set of d points on a line D. Consider the associated vector bundle:

$$0 \to \mathcal{O} \to E \to \mathcal{I}_X(1) \to 0$$

Then $c_1(E) = 1, c_2(E) = d$ and if $d > 1$, $h^0(E) = 2$ and E is stable. If $L \neq D$, then $E_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1)$, while $E_D \simeq \mathcal{O}_D(d) \oplus \mathcal{O}_D(-d+1)$. So D is the unique jumping line (if $d > 1$; if $d = 1$, $E \simeq T_{\mathbb{P}}(-1)$).

Assume E normalized ($c_1 = 0, -1$). If E is not stable ($h^0(E) \neq 0$), looking at the minimal twist having a section we have:

$$0 \to \mathcal{O} \to E(-t) \to \mathcal{I}_Z(-2t + c_1) \to 0$$

where Z has codimension two and degree $c_2(E(-t))$. Observe that $-2t+c_1 \leq 0$. If $L \cap Z = \emptyset$, then $E_L(-t) = \mathcal{O}_L \oplus \mathcal{O}_L(-2t + c_1)$. Let $p \in Z$ and let L be a line through p with $r = \#(Z \cap L)$, then $E_L(-t) = \mathcal{O}_L(r) \oplus \mathcal{O}_L(-2t + c_1 - r)$. Hence the pencil of lines through $p \in Z$ is made of jumping lines.

If E is stable with $c_1 = 0$, it is a result of Barth that there is a curve of jumping lines in the dual plane. □

Remark 4. *As proved in [10] a stable rank two vector bundle with c_1 odd has, in general, a finite number of jumping lines. More precisely Hulek defines jumping lines of the second order as lines L such $h^0(E|L^{(1)}) \neq 0$, where $c_1(E) = -1$. He shows that the locus of jumping lines of the second order is a curve, $C(E)$, and that $\mathcal{J}(E)$, the set of jumping lines, is the singular locus of $C(E)$. Hence if $C(E)$ is reduced, E has a finite number of jumping lines.*

2. A lower bound for $\mathcal{A}(n)$.

Goal of this section is to show that $\mathcal{A}(n) \geq n$, if $n \geq 2$.
Although not stated in this way, the following is proved in [6]

Proposition 5. (Glueing lemma)
Let $\mathcal{F}$ be a reflexive sheaf on $\mathbb{P}^n$. Suppose for a general hyperplane H and a general codimension two linear space K, that $h^0(\mathcal{F}_H) = h^0(\mathcal{F}_K) = s$. Furthermore assume that for any hyperplane, H, containing K, the restriction map $H^0(\mathcal{F}_H) \to H^0(\mathcal{F}_K)$ is an isomorphism. Then $h^0(\mathcal{F}) = s$.

Proof. The proof of Prop. 1.2 of [6] applies. The assumption $h^0(E_K(-1)) = 0$ in [6] is needed to show that the restriction map $H^0(E_H) \to H^0(E_K)$ is an isomorphism for every H containing K (first part of the proof). □

Lemma 6. *Let $n \geq 3$ be an integer. Let $R_1, ..., R_t \subset \mathbb{P}^n$ be lines. Let E be a rank r vector bundle such that:*

$$E_H \simeq T_H(-1) \oplus (r-n+1)\mathcal{O}_H \tag{1}$$

for every hyperplanes H containing no one of the lines R_i. Then $r \geq n$ and $E \simeq T(-1) \oplus (r-n).\mathcal{O}$.

Proof. Assume first $n = 3$. If L is a line different from the R_i's, then $E_L = \mathcal{O}_L(1) \oplus (r-1)\mathcal{O}_L$, hence $h^0(E_L) = r+1$. Let L be a line not meeting any of the R_i's. Then for every plane H containing L, $E_H = T_H(-1) \oplus (r-2).\mathcal{O}_H$. It follows that the restriction map $H^0(E_H) \to H^0(E_L)$ is an isomorphism. By Proposition 5, we conclude that $h^0(E) = r+1$. Let $x \in \mathbb{P}^3$ be a point and let H be a plane through x not containing any of the R_i's. We have:

$$\begin{array}{ccc} H^0(E) & \stackrel{ev}{\to} & E(x) \\ \downarrow r_H & & || \\ H^0(E_H) & \stackrel{ev_H}{\to} & E(x) \end{array}$$

It follows that the evaluation map has constant rank r, i.e. E is globally generated. So, considering Chern classes:

$$0 \to \mathcal{O}(-1) \to (r+1).\mathcal{O} \to E \to 0$$

It follows that E has rank $r \geq n$ because there is no injective vector bundle morphism $\mathcal{O}(-1) \to k.\mathcal{O}$ if $k \leq n$. Moreover E is uniform of splitting type $(1, 0, ..., 0)$ and the result follows by [9].

For $n > 3$ the argument is similar but easier. Since $H^1_*(E_H) = 0$, it follows that $H^1_*(E) = 0$ ($h^1(E(k-1)) \geq h^1(E(k)), \forall k$, but $h^1(E(-t)) = 0$ for $t >> 0$). So $H^0(E) \to H^0(E_H)$ is surjective for every H. Since $h^0(E_H(-1)) = 0$ if H doesn't contain any of the R_i's, for such an hyperplane the restriction map $H^0(E) \to H^0(E_H)$ is an isomorphism. We conclude as above. □

Corollary 7. *Let E be a rank r, almost uniform, vector bundle on $\mathbb{P}^n$, $n \geq 2$, then $r \geq n$ (i.e. $\mathcal{A}(n) \geq n$).*

Proof. We may assume $n \geq 3$. We can find an hyperplane H such that E_H has no jumping lines.

If $r < n-1$, by the classification of uniform vector bundles (cf [11], [8]), $E_H \simeq \bigoplus^r \mathcal{O}_H(a_i)$. This implies $E \simeq \bigoplus^r \mathcal{O}(a_i)$, which is absurd.

Assume $r = n-1$. If there exists one hyperplane such that E_H is a direct sum of line bundles, then as before we are done. By the classification of uniform bundles we may assume $E_H \simeq T_H(a), \Omega_H(b)$. The first Chern class will tell us if E_H is a twist of T_H or Ω_H (on $\mathbb{P}^2$, $\Omega = T(-3)$). So by dualizing and twisting, we may assume $E_H \simeq T_H(-1)$, for every hyperplane not containing any of the finitely many jumping lines of E. Lemma 6 says that no such bundle exists. □

According to [1] every rank $n+1$ uniform bundle on $\mathbb{P}^n$, $n \geq 3$, is a direct sum of bundles chosen among $T(a), \Omega(b), \mathcal{O}(c)$, $a, b, c \in \mathbb{Z}$. (This is no longer true on $\mathbb{P}^2$ since we have to add to the list $S^2T(m)$.) Using this fact we will show that $\mathcal{A}(n) \geq n+1$, if $n \geq 4$.

Lemma 8. *Let E be a rank $n+1$ almost uniform bundle on $\mathbb{P}^n$, $n \geq 4$. Then up to twisting or dualizing, there exists $a \in \mathbb{Z}$ such that: $E_H \simeq T_H(-1) \oplus \mathcal{O}_H(a)$, for every hyperplane not containing any of the finitely many jumping lines of E.*

Proof. If H is a good hyperplane (i.e. non containing any jumping line of E), then E_H is uniform, hence a direct sum of bundles chosen among $T(a), \Omega(b), \mathcal{O}(c)$, $a, b, c \in \mathbb{Z}$. If E_H is a direct sum of line bundles for one good hyperplane, then E is a direct sum of line bundles, but this is impossible. Assume $E_H \simeq T_H(b) \oplus \mathcal{O}_H(c)$ for *one* good hyperplane. Then $h^1(E_H(m)) = 0, \forall m$. By semi-continuity, this holds for H a general hyperplane. It follows that $E_H \simeq T_H(b) \oplus \mathcal{O}_H(c)$ for a general good hyperplane (looking at the splitting type, we see that b, c do not depend on H). Let H_0 be a good hyperplane such that $E_{H_0} \simeq \Omega_{H_0}(d) \oplus \mathcal{O}_{H_0}(e)$. Then $h^{n-2}(E_{H_0}(m)) = 0, \forall m$. By semi-continuity this should hold for a general hyperplane. But on a general hyperplane $E_H \simeq T_H(b) \oplus \mathcal{O}_H(c)$ and $h^{n-2}(E_H(-b-n+1)) \neq 0$. We conclude that $E_H(-b-1) \simeq T_H(-1) \oplus \mathcal{O}_H(a)$, $a = c-b-1$, for every good hyperplane. □

Proposition 9. *For $n \geq 4$, $\mathcal{A}(n) \geq n+1$.*

Proof. Let E be a rank n almost uniform bundle on $\mathbb{P}^n$, $n \geq 4$. By Lemma 8 we may assume, $E_H \simeq T_H(-1) \oplus \mathcal{O}_H(a)$, for every good hyperplane (i.e. H doesn't contain any jumping line of E).

• If $a = 0$, we conclude by Lemma 6 that $E \simeq T(-1)$, which is absurd.

• Assume $a > 0$. Since $H^1_*(E_H) = 0$ if H is a good hyperplane, we get $H^1_*(E) = 0$. Since $h^0(E_H(-a-1)) = 0$, we get $h^0(E(-a-1)) = 0$. Finally we see that $h^0(E(-a)) = h^0(E_H(-a)) = 1$. Let $s : \mathcal{O} \hookrightarrow E(-a)$, we claim that s doesn't vanish. Indeed if $s(x) = 0$, then $s|H = 0$ for every good hyperplane through x, which is absurd. So we have $0 \to \mathcal{O} \to E(-a) \to F(-a) \to 0$, where F is a rank $n-1$ vector bundle. If L is not a jumping line of E, then $E(-a)$ has splitting type $(1-a, -a, ..., -a, 0)$ on L and it follows that the splitting type of $F(-a)$ is $(1-a, -a, ..., -a)$. So F is (at least) almost uniform. Since there are no almost uniform bundles of rank $n-1$ (Corollary 7), we conclude that F is uniform, hence a direct sum of line bundles. It follows that the exact sequence splits and this yields a contradiction.

An alternative proof goes as follows: since $h^1(E(-1)) = 0$, $h^0(E) \to H^0(E_H)$ is surjective for every good hyperplane. This implies that E is globally generated. Hence F, also is globally generated. Since $c_1(F) = 1$, F is uniform and we conclude as above.

• Assume $a < 0$. This time $E_H^\vee \simeq \Omega_H(1) \oplus \mathcal{O}_H(b)$ $(-a = b > 0)$. Since $h^i(E_H^\vee(-b-m)) = 0$, for $0 \leq i \leq 1$ and $m > 0$, the same holds for $E^\vee(-b-m))$. It follows that $h^0(E^\vee(-b)) = h^0(E_H^\vee(-b)) = 1$. As before the section of $E^\vee(-b)$ doesn't vanish, so after dualizing and twisting we get: $0 \to F \to E \to \mathcal{O}(-b) \to 0$, where F is a rank $n-1$ vector bundle. If L is not a jumping line of E, the splitting type of F on L is $(1, 0, ..., 0)$. So F is (almost) uniform and, as above, we conclude that F is a direct sum of line bundles, which is a contradiction. □

3. Rank three bundles on $\mathbb{P}^3$.

Aim of this section is to prove that $\mathcal{A}(3) \geq 4$ (Theorem 15). Observe two differences with Proposition 9: we no longer have $H^1_*(T_H) = 0$, moreover it could be $E_H \simeq (S^2T_H)(m)$, for some good plane H.

Our major tool will be the classification of rank three uniform vector bundles on $\mathbb{P}^2$ due to Elencwajg ([7]

Theorem 10. *Let E be a rank three uniform vector bundles on $\mathbb{P}^2$, then E is isomorphic to one of the following:*

$$\bigoplus_{i=1}^{3}\mathcal{O}(a_i),\ T(a)\oplus\mathcal{O}(b),\ (S^2T)(m)$$

(we recall that on $\mathbb{P}^2$: $\Omega = T(-3)$)

Before to start let us recall some basic facts on $(S^2T)(-3)$ $(T := T_{\mathbb{P}^2})$. From $T\otimes T \simeq S^2T \oplus \wedge^2T$, we get $T\otimes T^* \simeq (S^2T)(-3)\oplus\mathcal{O}$. Since T is simple $End(T)\simeq k$ and $h^0((S^2T)(-3)) = 0$. To compute the Chern classes one uses the following:

Lemma 11. *Let E be a rank two vector bundles with Chern classes c_1, c_2. Then $c_2(E\otimes E^*) = -c_1^2 + 4c_2$.*

Proof. We use the splitting principle: $E = \mathcal{O}(a)\oplus\mathcal{O}(b)$. Then $E\otimes E^* = \mathcal{O}\oplus\mathcal{O}(a-b)\oplus\mathcal{O}(b-a)\oplus\mathcal{O}$. It follows that $c_2(E\otimes E^*) = -(a-b)^2$. We have $c_1 = a+b, c_2 = ab$, hence $c_1^2 = a^2+b^2+2c_2$. It follows that $(a-b)^2 = a^2+b^2-2ab = c_1^2-4c_2$. □

Let us set $\mathcal{E} = (S^2T_H)(-3)$, it is a rank 3 homogeneous vector bundle with splitting type $(1,0,-1)$. We have $h^0(\mathcal{E}) = 0$ and $h^1(\mathcal{E}(k)) = 0, k\geq 0$.

Lemma 12. *The vector bundle $\mathcal{E}(1) = (S^2T_H)(-2)$ is globally generated with $h^0(\mathcal{E}(1)) = 6$ and $c_1(\mathcal{E}(1)) = 3, c_2(\mathcal{E}(1)) = 6$.*

We also have $h^1(\mathcal{E}(-1)) = 3$.

Proof. Since $h^i(\mathcal{E}) = 0, 0\leq i\leq 1$, for every line $L\subset H$, the restriction map $H^0(\mathcal{E}(1))\to H^0(\mathcal{E}_L(1))$ is an isomorphism. Since $\mathcal{E}(1)$ has splitting type $(2,1,0)$, we see that $h^0(\mathcal{E}(1)) = 6$ and $\mathcal{E}(1)$ is globally generated.

We have $T\otimes T = S^2T\oplus\wedge^2T = S^2T\oplus\mathcal{O}(3)$. So $T\otimes T(-3) = (S^2T)(-3)\oplus\mathcal{O} = \mathcal{E}\oplus\mathcal{O}$. It follows that $c_i(\mathcal{E}) = c_i(T\otimes T^*)$. Since $c_i(T) = (3,3)$, by Lemma 11 $c_2(\mathcal{E}) = 3$ $(c_1(\mathcal{E}) = 0)$.

For a rank three coherent sheaf $c_2(\mathcal{F}(m)) = c_2 + 2mc_1 + 3m^2$.

From the exact sequence $0\to\mathcal{E}(-1)\to\mathcal{E}\to\mathcal{E}_L\to 0$, since $h^i(\mathcal{E}) = 0, i = 0,1$, we get $h^1(\mathcal{E}(-1)) = h^0(\mathcal{E}_L) = 3$. □

Going back to our problem, if E is an almost uniform bundle of rank three on $\mathbb{P}^3$, then for every good plane H, E_H will be one of the bundles $T_H(a)\oplus\mathcal{O}_H(b)$, $\mathcal{E}_H(m)$ ($\mathcal{E}_H := (S^2T_H)(-3)$). Indeed we can disregard the case where E_H is a direct sum of line bundles. A priori the restriction depends on the plane H, but we have:

Lemma 13. *Let E be an almost uniform rank three vector bundle on $\mathbb{P}^3$. Assume E normalized ($-2 \leq c_1(E) \leq 0$). Then one of the following occurs:*
(1) For every good plane H, $E_H \simeq (S^2T_H)(-3)$
(2) There exists $a \in \mathbb{Z}$ such that for every good plane H, $E_H \simeq T_H(a)\oplus\mathcal{O}_H(b)$, where $b = -3-2a+c_1(E)$.

Proof. First of all observe that since two good planes intersect along a line, the splitting type is the same for all good planes.

Assume $E_{H_0} = \mathcal{E}_{H_0}$ for *one* good plane ($\mathcal{E} = (S^2T)(-3)$). Then $h^0(E_{H_0}) = 0$ and $c_1(E) = 0$, also the splitting type is $(1,0,-1)$. If there exists a good plane H_1 with $E_{H_1} \neq \mathcal{E}_{H_1}$, then necessarily $E_{H_1} \simeq \begin{cases} T_{H_1}(-1)\oplus\mathcal{O}_{H_1}(-1) \\ \Omega_{H_1}(1)\oplus\mathcal{O}_{H_1}(1). \end{cases}$
So $h^1(E_{H_1}(-1)) \leq 1$. So if H is a general plane, then H is a good plane and $h^0(E_H) = 0, h^1(E_H(-1)) \leq 1$. By Lemma 12, this is impossible. So $E_H \simeq \mathcal{E}_H$ for every good plane.

We may now assume that E_H is of the form $T_H(a)\oplus\mathcal{O}_H(b)$ for every good plane H (with a,b depending on H). Since the splitting type is constant it is not hard to see that a,b are constant except if $c_1 = 0$ with splitting type $(1,0,-1)$, where the two cases (a) $T_H(-1)\oplus\mathcal{O}_H(-1)$, (b) $T_H(-2)\oplus\mathcal{O}_H(1)$ are possible. Looking at $h^0(E_H(-1))$, by semi-continuity, case (a) is the general one. But then looking at $h^0(E_H^\vee(-1))$, case (b) is the general one: contradiction. □

Lemma 14. *Let E be a normalized, almost uniform, vector bundle of rank three on $\mathbb{P}^3$. Then for every good plane, H, $E_H \simeq (S^2T_H)(-3)$.*

Proof. We have to show that case (2) of Lemma 13 cannot happen. So assume to the contrary that $E_H \simeq T_H(a)\oplus\mathcal{O}_H(-3-2a+c_1(E))$, for every good plane. Twisting by $-a-1$, we may assume $E_H \simeq T_H(-1)\oplus\mathcal{O}_H(c)$.

Assume $c > 1$. Since $h^0(E_H(-c-1)) = 0$ and $h^1(E_H(-c-k)) = 0$, if $k \geq 1$, we get $H^0(E(-c)) \simeq H^0(E_H(-c)) \simeq k$ and we conclude as in the proof of Proposition 9.

If $c = 0$, by Lemma 6 we should have $E \simeq T(-1)$, but this is impossible.

If $c < 0$ or $c = 1$, we consider $E^\vee$: we have $E_H^\vee \simeq \Omega_H(1) \oplus \mathcal{O}_H(-c)$. Since $\Omega_H(1) \simeq T_H(-2)$, we get: $E^\vee(1)_H \simeq T_H(-1) \oplus \mathcal{O}_H(-c+1)$ and we conclude by the previous cases. □

We may now prove the main result of this section:

Theorem 15. *Let E be a rank three vector bundle on $\mathbb{P}^3$. If E is not uniform, then E has infinitely many jumping lines.*

Proof. From Lemma 14 we may assume that $E_H \simeq (S^2T_H)(-3) =: \mathcal{E}_H$, for every good plane.

Let L be a line not meeting any of the jumping lines of E. If H is a plane through L then H doesn't contain any of the jumping line, so H is a good plane and $E_H = (S^2T_H)(-3)$. Now the restriction map $H^0(E_H(1)) \to H^0(E_L(1))$ is an isomorphism. By the Glueing Lemma 5, we get $h^0(E(1)) = h^0(E_H(1)) = 6$. Moreover from $h^0(E_H) = 0$ it follows that $h^0(E) = 0$ so the restriction map $H^0(E(1)) \to H^0(E_H(1))$ is an isomorphism. Now let $x \in \mathbb{P}^3$ be a point. Let H be a good plane through x. The evaluation map $H^0(E(1)) \overset{ev(x)}{\to} E(1)(x)$ factors through the restriction to H. Since $E_H(1)$ is globally generated (Lemma 12), we get that $E(1)$ is globally generated, with $h^0(E(1)) = 6$, $c_1 = 3, c_2 = 6$.

Now since $\mathcal{E}_H^\vee \simeq \mathcal{E}_H$ ($\mathcal{E}_H = (S^2T_H)(-3)$). We conclude, in exactly the same way, that $E^\vee(1)$ too is globally generated, with $h^0(E^\vee(1)) = 6$, $c_1 = 3, c_2 = 6$.

Now globally generated vector bundles with $c_1 = 3$ on $\mathbb{P}^n$ are classified in [12]. According to this classification if $\mathcal{F}$ is such a bundle on $\mathbb{P}^3$ with $c_2 = 6$, then one of the following occurs:

(a) $\mathcal{F} \simeq 3.T(-1)$

(b) there is an exact sequence: $0 \to \mathcal{O}(-2) \oplus \Omega(1) \to 7.\mathcal{O} \to \mathcal{F} \to 0$

(c) there is an exact sequence: $0 \to s.\mathcal{O} \to \mathcal{G} \oplus r.\mathcal{O} \to \mathcal{F} \to 0$, where $s = h^1(\mathcal{F}^\vee)$, $r = h^0(\mathcal{F}^\vee)$ and where $\mathcal{G}$ is as above, i.e.

(α) $\mathcal{G} = 3.T(-1)$ or

(β) there is an exact sequence: $0 \to \mathcal{O}(-2) \oplus \Omega(1) \to 7.\mathcal{O} \to \mathcal{G} \to 0$.

Now $E(1)$ and $E^\vee(1)$ have to fit with this classification. Clearly, under our assumptions, case (a) is impossible. Case (b) also is impossible. Indeed restricting to a good plane H we get: $0 \to \mathcal{O}_H(-3) \oplus \Omega_H \oplus \mathcal{O}_H(-1) \to 7.\mathcal{O}_H(-1) \to \mathcal{E}_H \to 0$. Since $h^0(\mathcal{E}_H) = 0, h^1(\Omega_H) = 1$, we get a contradiction. So both $E(1)$ and $E^\vee(1)$ come from (c). In any case we have $H^1_*(\mathcal{G}) = 0$. This implies $H^1_*(E) = H^1_*(E^\vee) = 0$. By Serre duality $H^1_*(E) = H^2_*(E) = 0$. By Horrocks' theorem E is a direct sum of line bundles, which is impossible. □

4. An upper-bound for $\mathcal{A}(n)$.

Let $\mathbb{P}^n = \mathbb{P}(V)$, *the projective space of lines* of the vector space V. A point $x \in \mathbb{P}^n$ corresponds to a line $d_x \subset V$, this line is the (vector bundle) fiber $\mathcal{O}(-1)(x)$ and Euler's sequence

$$0 \to \mathcal{O}(-1) \to V \otimes \mathcal{O} \to T(-1) \to 0$$

can be seen at the point x as: $0 \to d_x \to V \to V/d_x \to 0$. In particular $T(-1)(x) \simeq V/d_x$. We have $H^0(T(-1)) \simeq V$ and the section given by $u \in V$ vanishes exactly at the point x corresponding to the line $\langle u \rangle$.

Let $L \subset \mathbb{P}^n$ be a line. Then $T(-1)_L \simeq \mathcal{O}_L(1) \oplus (n-1).\mathcal{O}_L$. We can recover this isomorphism in the following way. The line L corresponds to a two-dimensional vector space $E \subset V$. Write $V = E \oplus W$. Then for any $x \in L$, $T(-1)_L(x) \simeq E/d_x \oplus W$. If $u \in V \simeq H^0(T(-1))$, then the value of u_L at the point $x \in L$ is $(\overline{u}_E, u_W) \in E/d_x \oplus W$, where $u = u_E + u_W \in E \oplus W$.

This being said we have:

Proposition 16. *Assume $n \geq 2$ and let $\sigma : \mathcal{O} \stackrel{(u,v)}{\to} T(-1) \oplus T(-1)$ be a section given by two linearly independent vectors. Then the quotient, E, is locally free of rank $2n-1$ and has a unique jumping line (the line corresponding to the plane $\langle u, v \rangle$). This shows $\mathcal{A}(n) \leq 2n-1$.*

Proof. Since u and v are linearly independent, σ has rank one at every point $x \in \mathbb{P}^n$. The exact sequence $0 \to \mathcal{O} \to 2.T(-1) \to E \to 0$ shows that E is globally generated with $c_1(E) = 2$. It follows that for a line L we have only two possibilities: (a) $E_L = 2.\mathcal{O}_L(1) \oplus (2n-3).\mathcal{O}_L$, (b) $E_L = \mathcal{O}_L(2) \oplus (2n-2).\mathcal{O}_L$. Of course case (a) is the generic case, so the jumping lines of E are precisely those of type (b).

With notations as above ($T(-1)_L \simeq \mathcal{O}_L(1) \oplus (W \otimes \mathcal{O}_L)$), consider the following diagram:

$$\begin{array}{ccc} \mathcal{O}_L & \stackrel{\sigma_L}{\to} & 2.\mathcal{O}_L(1) \oplus (W \oplus W) \otimes \mathcal{O}_L \\ & & \downarrow p_L \\ & & (W \oplus W) \otimes \mathcal{O}_L \end{array}$$

We see that we are in case (b) if and only if the composed map $\psi_L = p_L \circ \sigma_L$ is the zero map. Using our earlier description this means that $(u_W, v_W) \in W \oplus W$ is the zero vector. It follows that $u = u_E, v = v_E$, hence $E = \langle u, v\rangle$. In conclusion $\psi_L \equiv 0$ if and only if L is the line corresponding to the plane $\langle u, v\rangle$. □

Using a construction of Drezet ([5]) we can give another example (always of rank $2n-1$).

Let V be a k vector space of dimension $n+1$ and let $H \subset S^2V$ be a sub-vector space. Consider $\mathcal{O}(-2) \to (S^2V/H) \otimes \mathcal{O}$. At a point $x \in \mathbb{P}(V)$ corresponding to the line $\langle u\rangle$ the vector bundle map is given by $\overline{u}^2 \in S^2V/H$. It follows that the quotient $F(H)$ is a vector bundle if and only if H doesn't contain any non-zero square ($u^2 \in H \Leftrightarrow u = 0$). Assume this is the case. Then $F(H)$ is globally generated with $c_1(F(H)) = 2$. It follows that for a line $L \subset \mathbb{P}(V)$, the splitting type of $F(H)_L$ is (a) $(1, 1, 0, ..., 0)$ or (b) $(2, 0, ..., 0)$. The jumping lines of $F(H)$ are the lines of type (b). Clearly we are in case (a) if and only if $h^0(F(H)_L(-2)) = 0$. By Serre duality this is equivalent to $h^1(F(H)_L^\vee) = 0$. Dualizing and taking coomology on L this is equivalent to require that $f : H^0((S^2V/H)^\vee \otimes \mathcal{O}_L) \to \mathcal{O}_L(2)$ is surjective. If L corresponds to the plane $E \subset V$, then $H^0(\mathcal{O}_L(2)) \simeq S^2(E^\vee)$ and f is the transpose of the natural map $S^2E \to S^2V/H$.

In conclusion we are in case (a) if and only if $H \cap S^2E = \{0\}$, where $E \subset V$ is such that $\mathbb{P}(E) = L$.

It follows that $F(H)$ will be uniform if and only if we can find H such that $S^2E \cap H = \{0\}$, for every plane $E \subset V$.

Let $Y \subset \mathbb{P}(S^2V)$ be the union of the planes $\mathbb{P}(S^2E)$, $E \in Gr(2, V)$. We have the Veronese embedding $\nu : \mathbb{P}(V) \to \mathbb{P}(S^2V).\langle u\rangle \to \langle u^2\rangle$. The secant variety to $X := \nu(\mathbb{P}(V))$ can be described as follows. Given two points $x = \langle u\rangle$, $y = \langle v\rangle$ of $\mathbb{P}(V)$, the line $E = \langle u, v\rangle$ is mapped to a conic K_E $(= \{(\alpha u + \beta v)^2\})$. Every line of the plane $\langle K_E\rangle$ is a secant to X. It follows that $Sec(X)$ is the union of the planes $\langle K_E\rangle$. Since $G(1, n)$ has dimension $2n-2$, we get

$\dim(Sec(X)) = 2n$. Now $\langle K_E \rangle = \mathbb{P}(S^2E)$ (indeed $S^2E = \langle u^2, v^2, uv \rangle$, $u^2, v^2 \in K_E$ and $uv = ((u+v)/2)^2 - ((u-v)/2)^2$ is on the line spanned by two points of K_E). We conclude that $Y = Sec(X)$ has dimension $2n$.

So if $H \subset S^2V$ is a general subspace of codimension $2n+1$, the bundle $F(H)$ is uniform of rank $2n$ (and is not homogeneous, [5]).

Now if H is a general subspace of codimension $2n$, it will intersect Y at $\deg(Y)$ distinct points (not on X) and $F(H)$ will be a vector bundle of rank $2n-1$ with $\deg(Y)$ jumping lines. Since $\deg(Y) > 1$, $F(H)$ is not isomorphic to the vector bundle E of Proposition 16.

Remark 17. *Homogeneous vector bundles of rank $\leq 2n-1$ on $\mathbb{P}^n$ are classified ([2]) and are those one expects i.e. those obtained by algebraic operations $(\oplus, \otimes, \wedge)$ from $\mathcal{O}(1), T$. It is conjectured that every uniform bundle of rank $\leq 2n-1$ is homogeneous. This is true if $n = 2, 3$ ([7], [3]). Taking into account Drezet's example this should be sharp. For this reason I suspect the bound $\mathcal{A}(n) \leq 2n-1$ not too far from being sharp.*

References

[1] Ballico, E.: *Uniform vector bundles of rank $n+1$ on $\mathbb{P}^n$*, Tsukuba J. Math., **7**, 215-226 (1983)

[2] Ballico, E.-Ellia, Ph.: *Fibrés homogènes sur $\mathbb{P}^n$*, C. R. Acad. Sc. Paris, Série I, t.294, 403-406 (1982)

[3] Ballico, E.-Ellia, Ph.: *Fibrés uniformes de rang 5 sur $\mathbb{P}^3$*, Bull. Soc. Math. France, **111**, n.1, 59-87 (1983)

[4] Barth, W.: *Some properties of stable rank-2 vector bundles on $\mathbb{P}^n$*, Math. Ann., **226**, 125-150 (1977)

[5] Drezet, J.M.: *Exemples de fibrés uniformes non homogènes sur $\mathbb{P}^n$*, C.R. Acad. Sc. Paris, Série A, t.**291**, 125-128 (1980)

[6] Ein, L.-Hartshorne, R.-Vogelaar, H.: *Restriction theorems for stable rank 3 vector bundles on $\mathbb{P}^n$*, Math. Ann., **259**, 541-569 (1982)

[7] Elencwajg, G.: *Les fibrés uniformes de rang trois sur $\mathbb{P}^2(\mathbb{C})$ sont homogènes*, Math. Ann., **231**, 217-227 (1978)

[8] Elencwajg, G.-Hirschowitz, A.-Schneider, M.: *Les fibrés uniformes de rang au plus n sur $\mathbb{P}^n(\mathbb{C})$ sont ceux qu'on croit*, Progress in Math., **7**, 37-63 (1980)

[9] Ellia, Ph.: *Sur les fibrés uniformes de rang $n+1$ sur $\mathbb{P}^n$*, Mémoire Soc. Math. France, nouvelle série, **7**, 1-60 (1982)

[10] Hulek, K.: *Stable rank-2 vector bundles on $\mathbb{P}^2$ with c_1 odd*, Math. Ann., **242**, 241-266 (1979)

[11] Sato, E.: *Uniform vector bundles on a projective space*, J. Math. Soc. Japan, **28**, 123-132 (1976)

[12] Sierra, J.C.-Ugaglia, L.: *On globally generated vector bundles on projective spaces, II*, J. Pure and Applied Algebra, **218**, 174-180 (2014)

[13] Van de Ven, A.: *On uniform vector bundles*, Math. Ann., **195**, 245-248 (1972)

Dipartimento di Matematica, 35 via Machiavelli, 44100 Ferrara
E-mail address: phe@unife.it