Università degli Studi di Ferrara

DOTTORATO DI RICERCA IN
MATEMATICA

CICLO XXIX

COORDINATORE Prof. Massimiliano Mella

On the Cesàro average of some counting functions

Settore Scientifico Disciplinare MAT/05

Dottorando
Dott. Cantarini Marco

Tutore
Prof. Zaccagnini Alessandro

Anni 2014/2016

Sezioni

Università degli Studi di Ferrara - Ex labore fructus -

Dottorati di ricerca

Dichiarazione di conformità

AL MAGNIFICO RETTORE

UNIVERSITÀ DEGLI STUDI DI FERRARA
UFFICIO DOTTORATO
Via delle Scienze, n. 41/B – 44121
Ferrara Tel. 0532/455284 - Fax
0532/455262 e-mail dottorato@unife.it

ATTENZIONE: il presente modulo è da stampare, firmare e inserire come parte integrante della tesi in formato cartaceo; va inoltre inserito all'interno dei DVD o CD-ROM della tesi come file formato pdf.

Il tuo indirizzo e-mail Inserisci qui l'e-mail che ti ha fornito l'Università

Oggetto: Indicare qui: "Dichiarazione di conformità della tesi di Dottorato"

Io sottoscritto Dott. (Cognome e Nome)

Nato a: Indicare il Comune di nascita

Provincia: Scrivere per esteso la provincia di nascita oppure lo Stato (se nato all'estero)

Il giorno: Trascrivere la data di nascita

Avendo frequentato il Dottorato di Ricerca in: Specificare il nome del corso

Ciclo di Dottorato Scrivere in cifre (es. 26)

Titolo della tesi: Attenzione: deve essere lo stesso riportato sulla tesi cartacea

Titolo della tesi (traduzione): Se la stesura della tesi è autorizzata dal Collegio dei docenti in una lingua diversa dall'italiano, è obbligatorio compilare il campo secondo queste modalità: - Tesi in inglese: traduzione in lingua italiana - - Tesi in francese, spagnolo, tedesco etc.: traduzione del titolo sia in italiano che in inglese

Tutore: Prof. (Cognome e Nome)

Settore Scientifico Disciplinare (S.S.D.) Indicare il settore della tesi. Ad esempio, BIO/01

Parole chiave della tesi (max 10): In Italiano, Inglese ed eventuale terza lingua

Consapevole, dichiara Selezionare con un click

○ CONSAPEVOLE: (1) del fatto che in caso di dichiarazioni mendaci, oltre alle sanzioni previste dal codice penale e dalle Leggi speciali per l'ipotesi di falsità in atti ed uso di atti falsi, decade fin dall'inizio e senza necessità di alcuna formalità dai benefici conseguenti al provvedimento emanato sulla base di tali dichiarazioni; (2) dell'obbligo per l'Università di provvedere al deposito di legge delle tesi di dottorato al fine di assicurarne la conservazione e la consultabilità da parte di terzi; (3) della procedura adottata dall'Università di Ferrara ove si richiede che la tesi sia consegnata dal dottorando in 2 copie di cui una in formato cartaceo e una in formato pdf non modificabile su idonei supporti (CD-ROM, DVD) secondo le istruzioni pubblicate sul sito: http://www.unife.it/studenti/dottorato alla voce ESAME FINALE – disposizioni e modulistica; (4) del fatto che l'Università, sulla base dei dati forniti, archivierà e renderà consultabile in rete il testo completo della tesi di

dottorato di cui alla presente dichiarazione attraverso l'Archivio istituzionale ad accesso aperto "EPRINTS.unife.it" oltre che attraverso i Cataloghi delle Biblioteche Nazionali Centrali di Roma e Firenze; DICHIARO SOTTO LA MIA RESPONSABILITA': (1) che la copia della tesi depositata presso l'Università di Ferrara in formato cartaceo è del tutto identica a quella presentata in formato elettronico (CD-ROM, DVD), a quelle da inviare ai Commissari di esame finale e alla copia che produrrò in seduta d'esame finale. Di conseguenza va esclusa qualsiasi responsabilità dell'Ateneo stesso per quanto riguarda eventuali errori, imprecisioni o omissioni nei contenuti della tesi; (2) di prendere atto che la tesi in formato cartaceo è l'unica alla quale farà riferimento l'Università per rilasciare, a mia richiesta, la dichiarazione di conformità di eventuali copie; (3) che il contenuto e l'organizzazione della tesi è opera originale da me realizzata e non compromette in alcun modo i diritti di terzi, ivi compresi quelli relativi alla sicurezza dei dati personali; che pertanto l'Università è in ogni caso esente da responsabilità di qualsivoglia natura civile, amministrativa o penale e sarà da me tenuta indenne da qualsiasi richiesta o rivendicazione da parte di terzi; (4) che la tesi di dottorato non è il risultato di attività rientranti nella normativa sulla proprietà industriale, non è stata prodotta nell'ambito di progetti finanziati da soggetti pubblici o privati con vincoli alla divulgazione dei risultati, non è oggetto di eventuali registrazioni di tipo brevettale o di tutela. PER
ACCETTAZIONE DI QUANTO SOPRA RIPORTATO

Firma del dottorando

Ferrara, li
___________________________ (data)

Firma del Tutore

Visto: Il Tutore
Si approva
Firma del Tutore

Selezionando il tasto "INVIA" in fondo alla pagina, la dichiarazione di conformità viene spedita automaticamente alla posta elettronica dell'Ufficio Dottorato. Per ottenerne una copia in FORMATO PDF, fare click sull'icona della stampante del proprio browser (Internet Explorer, Firefox, Google Chrome etc.), quindi selezionare l'opzione Pdf Creator o simili (a seconda del programma in uso).

Contents

1 Ringraziamenti

Desidero ringraziare il prof. Alessandro Zaccagnini, relatore di questa tesi, per tutto il lavoro di supporto, revisione e incoraggiamento, sempre offerto con massima disponibilità. Lo ringrazio inoltre per avermi introdotto nell'affascinante mondo della Teoria Analitica dei Numeri e della fiducia che mi ha dimostrato in tutti questi anni. Desidero inoltre ringraziare il prof. Alessandro Languasco per le sue importanti osservazioni riguardanti questa tesi.

2 The Goldbach conjectures

In a letter to Euler dated 1742, the Prussian mathematician Christian Goldbach stated the following conjecture:
if N is an integer greater than 2, then $N = p_1 + p_2 + p_3$ where $p_i, i = 1, 2, 3$ are primes.
In his reply letter, Euler wrote another conjecture:
if N is a positive even integer, then $N = p_1 + p_2$ where $p_i, i = 1, 2$ are primes.
It is important to note that in Euler and Goldbach's time the number 1 was considered a prime and today these two conjectures are known to be equivalent (see [35]). In the modern version of the conjectures (i.e. not considering 1 as a prime number) the statement is:
if N is a positive odd integer greater than 5, then $N = p_1 + p_2 + p_3$ where $p_i, i = 1, 2, 3$ are primes (TGC)
and
if N is a positive even integer greater than 2, then $N = p_1 + p_2$ where $p_i, i = 1, 2$ are primes. (GC)
It is well know that the two conjectures are not equivalent, but

$$(\mathrm{GC}) \Rightarrow (\mathrm{TGC}).$$

Despite the very simple statement, the Goldbach conjectures are very hard to prove. Indeed GC is still an open problem and the TGC was completely proved only in 2013. For a more detailed survey see [35].

2.1 Results related to the Goldbach conjectures

The first partial result for the TGC was obtained by Hardy and Littlewood. In 1923 they proved, under the generalized Riemann hypothesis (GRH for brevity), that TGC holds for every sufficiently large odd integer (see [12] for details). In 1937 Vinogradov (see [40] for details) proved TGC unconditionally (i.e. without GRH) for every sufficiently large odd integer. In 2012 and 2013 Harald Helfgott published on ArXiv some papers (see [17], [18], [19] and [20], the latter in collaboration with D. Platt) which complete the proof of TGC for all odd integers larger than 5. The papers are still in preprint, but they are accepted by the mathematical community.
In 1923 Hardy and Littlewood (see [13]) attacked the GC studying the size of the exceptional set for the Goldbach's problem, that is

$$E(N) = \left|\{n < N : n \neq p_1 + p_2,\, p_1, p_2 \in \mathfrak{P}\}\right|$$

where $\mathfrak{P}$ is the set of the primes. They proved, assuming GRH, that almost all even numbers are sums of two primes, i.e.

$$E(N) = o(N)$$

as $N \to \infty$. In particular they proved that

$$E(N) \ll N^{1/2+\epsilon}$$

for every $\epsilon > 0$ and sufficiently large N. In 1937 Vinogradov (see [39]) was able to remove the dependence of GRH and proved the TGC for sufficiently large odd integers. In 1975 Montgomery and Vaughan (see [33]) proved unconditionally that there exists a positive and computable $\delta > 0$ such that

$$E(N) \ll N^{1-\delta}.$$

Pintz in 2006 (see [35]) announced that he proved the theorem with $\delta = 1/3$ but the proof has not been published yet. Assuming GRH Goldston [11] proved that

$$E(N) \ll N^{1/2} \log^4(N)$$

and then he improved it to

$$E(N) \ll N^{1/2} \log^3(N).$$

The exceptional set has been analyzed also in the so called *short interval* context, that is, an interval of the type $[N, N+H]$ when $N \to \infty$ and $H = o(N)$. The best unconditional results have been obtained in 1993 by Perelli and Pintz [34] where they proved that almost all even numbers in the interval $\left[N, N + N^{7/36+\epsilon}\right]$ are sum of two primes, where $0 < \epsilon < 2/3$. Assuming GRH and assuming that $H \log^{-10}(N) \to \infty$, Kaczorowski, Perelli and Pintz [22] proved in 1993 that all even numbers in any interval $[N, N+H]$ with at most $O\left(H^{1/2} \log^5(N)\right)$ exceptions are sum of two primes.

2.2 Structure of the thesis

Chapter 3 is of preliminary character and it collects some well-known results used in this work.

In Chapter 4 we will introduce our first theorem which is about the Cesàro mean of the numbers that can be written as sum of a prime and two squares of integer (that we call "Linnik numbers" for brevity). We will prove that the technique has a limitation and so we can not expect to get results that look real conjecturally. We now present a short introduction to our work; it will help us to a better comprehension of the proof.

We want to study the mean of some counting function with order k Cesàro weight, that is,

$$\sum_{n\leq N} f\left(n\right)\frac{\left(N-n\right)^{k}}{\Gamma\left(k+1\right)}.$$

So in Chapter 4 we will consider $f\left(n\right)=r_{Q}\left(n\right)$ as the weighted counting function of the representation of n as a sum of a prime and two squares of integers. We will prove that there are other two functions $\widetilde{S}\left(z\right)$ (see (7)) and $\omega_{2}\left(z\right)$ (see (8)) such that

$$\widetilde{S}\left(z\right)\omega_{2}^{2}\left(z\right)=\sum_{n\geq1}r_{Q}\left(n\right)e^{-nz}. \tag{1}$$

The presence of this two functions is linked to the fact that we are working with primes ans quares of integers. We will consider the integral

$$\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\widetilde{S}\left(z\right)\omega_{2}^{2}\left(z\right)dz$$

where $\int_{(a)}$ means $\int_{a-i\infty}^{a+i\infty}$ and using (1) and the fundamental identity

$$\frac{1}{2\pi i}\int_{(a)}v^{-s}e^{v}dv=\frac{1}{\Gamma\left(s\right)},\ \mathrm{Re}\left(s\right)>0,\ a>0$$

we will prove, after a convergence control, that

$$\sum_{n\leq N}r_{Q}\left(n\right)\frac{\left(N-n\right)^{k}}{\Gamma\left(k+1\right)}=\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\widetilde{S}\left(z\right)\omega_{2}^{2}\left(z\right)dz. \tag{2}$$

We can see now that we moved from an arithmetic problem to a problem that can be dealt with analytic tools. We would like to have lower bounds for k as small as possible since for $k=0$ the Cesàro weight is equal to 1.
We will prove that $\widetilde{S}\left(z\right)$ has a asymptotic formula, then we will substitute that formula in the RHS of (2) and we will obtain three integrals and an error term. In each integral we will study the absolute convergence that depends on k. After that we will exchange the integrals with the series in the integrand and, using some identities involving the Gamma function and the Bessel functions, we will able to find an asymptotic for the LHS of (2). As we said, the convergence depends on how large is k. We would like to get $k\geq0$ but unfortunately using this technique it is not possible. In fact we will also prove that for this problem the bound $k>3/2$ is optimal.
In Chapter 5 we will show our second theorem which is about the Cesàro mean of the numbers that can be written as sum of a prime and two squares of primes. The approach is the same of the earlier part.
In Chapter 6 we will describe the circle method of Hardy, Littlewood and Ramanujan and

we will see the classical applications to the Goldbach ternary problem. It is the “starting point” of our work, since our main results are based on a variant of this technique.

3 Preliminaries

3.1 Prime Number Theorem (PNT)

Given the prime counting function

$$\pi(x) = \sum_{p \leq x} 1$$

then exists some $C > 0$ such that

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp\left(-C\sqrt{\log(x)}\right)\right)$$

where

$$\mathrm{li}(x) = \lim_{\epsilon \to 0^+} \left(\int_0^{1-\epsilon} \frac{dt}{\log(t)} + \int_{1+\epsilon}^{x} \frac{dt}{\log(t)}\right), \ x > 1$$

is the logarithmic integral function. Sometimes it is defined as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)} = \mathrm{li}(x) - \mathrm{li}(2)$$

and this is called the "European" definition. It is easy to prove that

$$\mathrm{li}(x) \sim \frac{x}{\log(x)}$$

as $x \to \infty$ so

$$\pi(x) \sim \frac{x}{\log(x)}$$

as $x \to \infty$. For a proof see [1], chap. 13.

3.2 Poisson summation formula

We recall a very important summation technique due to Poisson. We need first of the following

Definition. Let $f \in C^{\infty}(\mathbb{R})$. Then f is a **Schwartz function** if $\forall c \in \mathbb{R}$, $\forall n \in \mathbb{N}_0$ we have

$$\left|f^{(n)}(x)\right| = o\left(|x|^c\right).$$

as $|x| \to \infty$.

Now we can prove the following

Theorem (Poisson summation formula). Let be f a Schwartz function and let

$$\widehat{f}(z)=\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i z x} dx$$

be its Fourier transform. Then

$$\sum_{n\in\mathbb{Z}} f(n)=\sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

For a proof see for example [45]. Actually it is possible to prove the theorem for more general functions, such as functions of bounded variation supported over a finite interval.

3.3 Riemann Hypothesis (RH)

Let

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},\ \forall s\in\mathbb{C} \text{ such that } \operatorname{Re}(s)>1$$

be the Riemann zeta function. It has an analytic continuation on $\mathbb{C}\setminus\{1\}$ and at $s=1$ it has a simple pole. The points $s=-2n$, $n\in\mathbb{N}_0$ are simple zeros of $\zeta(s)$ and they are called trivial zeros. It is known that every other zero (called nontrivial zero) of $\zeta(s)$ is a complex number such that $0<\operatorname{Re}(s)<1$. The Riemann hypothesis states that every nontrivial zero has real part equal to $1/2$. For more details see [38].

3.4 Generalized Riemann Hypothesis (GRH)

We recall that a primitive Dirichlet character χ to the modulus q is an arithmetic, periodic with period q and completely multiplicative function. For more details see Davenport's book [7], chapters 1,4 and 5. Let

$$L(s,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^s}=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1},\ \forall s\in\mathbb{C} \text{ such that } \operatorname{Re}(s)>1$$

be a Dirichlet L function, $q\geq 3$. If $\chi\neq\chi_0$, where χ_0 is the principal character, then $L(s,\chi)$ has an analytic continuation on $\mathbb{C}$. The points $s=-2n$, $n\in\mathbb{N}$ are simple zeros of $L(s,\chi)$ if $\chi(-1)=1$ and in $s=-2n+1$ with $n\in\mathbb{N}_0$ if $\chi(-1)=-1$. They are called trivial zeros. It is known that every other zero (called nontrivial zero) of $L(s,\chi)$ is a complex number such that $0<\operatorname{Re}(s)<1$. The Generalized Riemann hypothesis states that every nontrivial zero of $L(s,\chi)$, where χ is a primitive character, has real part equal to $1/2$. For more details see [7], chapters 4,5,9 and 14.

3.5 Laplace transform

Let $f(x)$ be a function locally integrable on $[0,\infty)$. Then the Laplace transform of f is

$$\mathfrak{L}(f)(s) := \int_0^\infty f(x)\, e^{-sx} dx = F(s)$$

and the inversion formula is given by

$$\mathfrak{L}^{-1}(F)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{sx} ds.$$

3.6 The Mellin transform and the Mellin-Barnes integrals

The Mellin transform of a function $f(x)$ is given by

$$\mathfrak{M}(f)(s) := \int_0^\infty f(x)\, x^{s-1} dx = F(s)$$

and the inversion formula is given by

$$\mathfrak{M}^{-1}(F)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s} ds,\ c > 0.$$

The Mellin transform is closely connected to the Laplace transform, since

$$\mathfrak{M}(f)(s) = \mathfrak{L}\left(f\left(e^{-x}\right)\right)(s).$$

Now let us consider the Gamma function

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx,\ \mathrm{Re}(s) > 0$$

which is the Mellin transform of e^{-x}. Then if we apply the inversion formula we get

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\, x^{-s} ds,\ \mathrm{Re}(x) > 0,\ c > 0. \tag{3}$$

Formula (3) (which is also known as Cahen–Mellin integral) will be useful later. This identity is a particular case of the Mellin-Barnes integrals:

Definition. The integrals of the form

$$f(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{n=1}^{N}\Gamma(a_n + A_n s)\prod_{l=1}^{L}\Gamma(c_l - C_l s)}{\prod_{m=1}^{M}\Gamma(b_m + B_m s)\prod_{h=1}^{H}\Gamma(d_h - D_h s)} z^s ds$$

where $c \in \mathbb{R}$ and $a_j,\ b_j,\ c_j,\ d_j \in \mathbb{C}$ are such that no poles of the integrand are on the

complex line $(c-i\infty, c+i\infty)$ and A_j, B_j, C_j and D_j are positive are known as *Mellin-Barnes integrals.*
Actually it is possible to give a more generally definition; see [3] for details.

3.7 Perron's formula

Let $a(n)$ be an arithmetic function and let

$$f(s)=\sum_{n\geq1}\frac{a(n)}{n^s}$$

be the corrisponding Dirichlet series. Assume that the series is convergent if $\mathrm{Re}(s)>\sigma$. Then

$$\sum_{n\leq x}{}' a(n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s)\frac{x^s}{s}ds$$

for $c>0$, $c>\sigma$. The symbol $\sum_{n\leq x}'$ indicates that the last term of the sum must be multiplied by $1/2$ when x is an integer. The Perron's formula describes the inverse Mellin transform applied to a Dirichlet series.

4 A Cesàro average of the "Linnik numbers"

In this part we will prove a result in the same spirit of the recent work of Languasco and Zaccagnini on additive problems with prime summands. In [24] and [25] they study the Cesàro weighted explicit formula for the Goldbach numbers (the integers that can be written as sum of two primes) and for the Hardy-Littlewood numbers (the integers that can be written as sum of a prime and a square). In a similar manner, we will study a Cesàro weighted explicit formula for the integers that can be written as sum of a prime and two squares (that we call "Linnik numbers"). We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function.

The study of these numbers is classical. For example Hardy and Littlewood in [12] studied the number of solutions of the equation

$$n = p + a^2 + b^2$$

and Linnik in [31] derived an asymptotic formula for the number of representations of these numbers. Similar averages of arithmetical functions are common in literature, see, e.g., Chandrasekharan - Narasimhan [6] and Berndt [2] who built on earlier classical work.

4.1 Bessel functions of the first kind and Laplace transform

For our work we will need the Bessel functions $J_v(u)$ of complex order v and real argument u. For their definition and main properties we refer to Watson [43], but we recall that they were introduced by Daniel Bernoulli and they are the canonical solution of the differential equation

$$u^2 \frac{d^2 J}{du^2} + u \frac{dJ}{du} + \left(u^2 - v^2\right) J = 0$$

for any complex number v. These solutions can be written as

$$J_v(u) = \left(\frac{u}{2}\right)^v \sum_{m \geq 0} \frac{(-1)^m u^{2m}}{4^m m! \Gamma(v + m + 1)}$$

and they are analytic functions of $u \in \mathbb{C}$, except for a branch point at $u = 0$ when v is not an integer. The principal branch of $J_v(u)$ corresponds to the principal value of $(u/2)^v$ and is analytic in the u-plane cut along the interval $(-\infty, 0]$. The Bessel functions with integer order are also known as cylinder functions or the cylindrical harmonics because they appear in the solution to Laplace's equation in cylindrical coordinates. Spherical

Bessel functions with half-integer order are obtained when the Helmholtz equation

$$\nabla^2 f + k^2 f = 0,$$

where ∇^2 is the Laplacian and k is the wave vector, is solved in spherical coordinates. In particular, equation (8) on page 177 of [43] gives the Sonine representation

$$J_\nu (u) = \frac{(u/2)^\nu}{2\pi i} \int_{(a)} e^s s^{-\nu-1} e^{-u^2/(4s)} ds \tag{4}$$

which is the basis of our future work. As noted by Languasco and Zaccagnini in [25] the estimates of such Bessel functions are harder to perform than the ones already present in the Number Theory literature (as far as we know, Bessel functions of complex order arise in a similar problem for the first time in [25]) since the real argument and the complex order are both unbounded while, in the previous papers, either the real order or the complex argument is bounded. As we said in the previous sections, the method we will use in this additive problem is based on a formula due to Laplace [27], namely

$$\frac{1}{2\pi i} \int_{(a)} v^{-s} e^v dv = \frac{1}{\Gamma (s)} \tag{5}$$

with $\mathrm{Re}(s) > 0$ and $a > 0$ (see, e.g., formula 5.4 (1) on page 238 of [9]). As in [25], we combine this approach with line integrals with the classical methods dealing with infinite sum over primes and integers. Similarly as [25] the problem naturally involves the modular relation for the complex Jacobi theta 3 function (see (10)); the presence of the Bessel functions in our statement strictly depends on such modularity relation.

4.2 Preliminary definitions and Lemmas

Let $z = a + iy$, $a > 0$, and

$$\theta_3 (z) = \sum_{m \in \mathbb{Z}} e^{-m^2 z} \tag{6}$$

$$\widetilde{S} (z) = \sum_{m \geq 1} \Lambda (m) e^{-mz} \tag{7}$$

$$\omega_2 (z) = \sum_{m \geq 1} e^{-m^2 z} \tag{8}$$

and we can see that

$$\theta_3 (z) = 1 + 2\omega_2 (z) . \tag{9}$$

Furthermore we have the functional equation (see, for example, the proposition VI.4.3 of Freitag-Busam [10] page 340)

$$\theta_3(z) = \left(\frac{\pi}{z}\right)^{1/2} \theta_3\left(\frac{\pi^2}{z}\right) \tag{10}$$

which follows from the Poisson summation formula described above. We will show briefly the proof. We have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2 z - 2\pi i x m} dx =& z^{-1/2} \int_{-\infty}^{\infty} e^{-u^2 - 2\pi i u m/\sqrt{z}} dx \\ =& z^{-1/2} e^{-\pi^2 m^2/z} \int_{-\infty}^{\infty} e^{-\left(u + \pi i m/\sqrt{z}\right)^2} du \\ =& z^{-1/2} e^{-\pi^2 m^2/z} \int_{-\infty}^{\infty} e^{-v^2} dv \\ =& z^{-1/2} e^{-\pi^2 m^2/z} 2 \int_{0}^{\infty} e^{-v^2} dv \\ =& \left(\frac{\pi}{z}\right)^{1/2} e^{-\pi^2 m^2/z}\end{aligned}$$

hence

$$\sum_{m\in\mathbb{Z}} e^{-m^2 z} = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-x^2 z - 2\pi i x m} dx = \left(\frac{\pi}{z}\right)^{1/2} \sum_{m\in\mathbb{Z}} e^{-\pi^2 m^2/z}$$

as wanted. So we have

$$\omega_2^2(z) = \left(\frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2}\right)^2 + \frac{\pi}{z}\omega_2^2\left(\frac{\pi^2}{z}\right) + \left(\left(\frac{\pi}{z}\right)^{1/2} - 1\right)\left(\left(\frac{\pi}{z}\right)^{1/2}\omega_2\left(\frac{\pi^2}{z}\right)\right). \tag{11}$$

A trivial but important estimation is

$$|\omega_2(z)| \leq \omega_2(a) \leq \int_0^{\infty} e^{-at^2} dt = \frac{\sqrt{\pi}}{2\sqrt{a}} \ll a^{-1/2}. \tag{12}$$

Let us introduce the following

Lemma 2. Let $z = a + iy$, $a > 0$ and $y \in \mathbb{R}$. Then

$$\widetilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho) + E(a, y) \tag{13}$$

where $\rho=\beta+i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(a,y)\ll|z|^{1/2}\begin{cases}1, & |y|\leq a\\ 1+\log^{2}(|y|/a), & |y|>a.\end{cases} \tag{14}$$

(For a proof see Lemma 1 of [24]. The bound for $E(a,y)$ has been corrected in [23]).
It is interesting to observe that the starting pointo of the proof of Lemma 2 is the identity

$$\widetilde{S}(z)=\frac{1}{2\pi i}\int_{(\alpha)}\frac{\zeta'}{\zeta}(w)\,\Gamma(w)\,z^{-w}dw,\ \alpha>1 \tag{15}$$

since

$$\frac{\zeta'}{\zeta}(s)=\sum_{n\geq1}\frac{\Lambda(n)}{n^{s}},\ \mathrm{Re}(s)>1.$$

So if we apply the Perron's formula or we apply (3), observing that we can exchange the series with the integral since the series converge absolutely, we can prove (15).
In particular, taking $z=\frac{1}{N}+iy$ we have

$$\left|\sum_{\rho}z^{-\rho}\Gamma(\rho)\right|=\left|\frac{1}{z}-\widetilde{S}(z)+E\left(\frac{1}{N},y\right)\right|\ll N+\frac{1}{|z|}+\left|E\left(\frac{1}{N},y\right)\right|$$

$$\ll\begin{cases}N, & |y|\leq1/N\\ N+|z|^{1/2}\log^{2}(2N|y|), & |y|>1/N.\end{cases} \tag{16}$$

Now we have to recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 2, to the statement

$$\widetilde{S}(a)\sim a^{-1},\ \text{when } a\rightarrow0^{+} \tag{17}$$

(see Lemma 9 of [12]). For our purposes it is important to introduce the Stirling approximation

$$|\Gamma(x+iy)|\sim\sqrt{2\pi}e^{-\pi|y|/2}\,|y|^{x-1/2} \tag{18}$$

(see for example §4.42 of [37]) uniformly for $x\in[x_1,x_2]$, x_1 and x_2 fixed, and the identity

$$\left|z^{-w}\right|=|z|^{-\mathrm{Re}(w)}\exp\left(\mathrm{Im}(w)\arctan(y/a)\right). \tag{19}$$

We now quote Lemmas 2 and 3 from [24]:

Lemma 3. Let $\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function and let $\alpha > 1$ be a parameter. The series

$$\sum_{\rho,\,\gamma>0} \gamma^{\beta-1/2} \int_{1}^{\infty} \exp\left(-\gamma \arctan\left(1/u\right)\right) \frac{dy}{u^{\alpha+\beta}}$$

converges provided that $\alpha > 3/2$. For $\alpha \leq 3/2$ the series does not converge. The result remains true if we insert in the integral a factor $\log^{c}(u)$, for any fixed $c \geq 0$.

Lemma 4. Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function, let $z = a + iy$, $a \in (0,1)$, $y \in \mathbb{R}$ and $\alpha > 1$. We have

$$\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{Y}_1 \cup \mathbb{Y}_2} \exp\left(\gamma \arctan\left(\frac{y}{a}\right) - \frac{\pi}{2}|\gamma|\right) \frac{dy}{|z|^{\alpha+\beta}} \ll_{\alpha} a^{-\alpha}$$

where $\mathbb{Y}_1 = \{y \in \mathbb{R} : \gamma y \leq 0\}$ and $\mathbb{Y}_2 = \{y \in [-a,a] : y\gamma > 0\}$. The result remains true if we insert in the integral a factor $\log^{c}(|y|/a)$, for any fixed $c \geq 0$.

We now establish an important Lemma. We will use it to prove that there is a limitation in our technique. Essentially the lower bound of k is linked to the number of squares in the problem. We have

Lemma 5. Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function, let N, d be positive integers and $k > 0$ be a real number. Then the series

$$\sum_{\bar{l} \in (0,\infty)^d} \sum_{\gamma>0} \gamma^{-k-3/2} \int_{0}^{\gamma} e^{-N\left\|\bar{l}\right\|^2 v^2/\gamma^2} e^{-v} v^{k+\beta} dv,$$

where

$$\sum_{\bar{l} \in (0,\infty)^d} = \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \cdots \sum_{l_d \geq 1},$$

converges if $k > d - 1/2$ and this result is optimal.

Proof. From (9) we have that

$$\omega_2^d(z) = \frac{1}{2^d} \sum_{m=0}^{d} \binom{d}{m} (-1)^{d-m} \theta_3^m(z)$$

then

$$\begin{aligned}I=&\sum_{\bar{l}\in(0,\infty)^d}\sum_{\gamma>0}\gamma^{-k-3/2}\int_0^\gamma e^{-N\left\|\bar{l}\right\|^2v^2/\gamma^2}e^{-v}v^{k+\beta}dv\\=&\sum_{\gamma>0}\gamma^{-k-3/2}\int_0^\gamma\omega_2^d\left(\frac{Nv^2}{\gamma^2}\right)e^{-v}v^{k+\beta}dv\\=&\frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\sum_{\gamma>0}\gamma^{-k-3/2}\int_0^\gamma\theta_3^m\left(\frac{Nv^2}{\gamma^2}\right)e^{-v}v^{k+\beta}dv.\end{aligned}$$

Now, using the functional equation (10) we have that

$$\begin{aligned}I=&\frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\frac{\pi^{m/2}}{N^{m/2}}\sum_{\gamma>0}\gamma^{m-k-3/2}\int_0^\gamma\theta_3^m\left(\frac{\pi^2\gamma^2}{Nv^2}\right)e^{-v}v^{k+\beta-m}dv\\=&\frac{1}{2^d}\sum_{m=0}^{d}\binom{d}{m}(-1)^{d-m}\frac{\pi^{m/2}}{N^{m/2}}\sum_{\gamma>0}\gamma^{m-k-3/2}I_{\gamma,m},\end{aligned}$$

say. Now we claim that

$$\theta_3\left(\frac{\pi^2\gamma^2}{Nv^2}\right)\asymp1$$

since $\theta_3(z)$ is a continuous function in the interval $\left[\frac{\pi^2}{N},\infty\right)$ (i.e. the range of $1/v^2$) and

$$\lim_{z\to\infty}\theta_3(z)=1.$$

So we have

$$I_{\gamma,m}\asymp\sum_{\gamma>0}\gamma^{m-k-3/2}\int_0^\gamma e^{-v}v^{k+\beta-m}dv$$

and now assuming $k+\beta-m+1>0$ we have

$$\int_0^\gamma e^{-v}v^{k+\beta-m}dv\asymp1,$$

hence

$$I_{\gamma,m}\asymp_k\sum_{\gamma>0}\gamma^{m-k-3/2}$$

and the last series converges if $k>m-1/2$. Since $m=0,\dots,d$ for a global convergence we must have $k>d-1/2$ and this result is optimal.$\square$
Let us introduce another lemma

Lemma 6. Let $\rho=\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function, let $z=\frac{1}{N}+iy$, $N>1$ be natural number, $y\in\mathbb{R}$ and $\alpha>3/2$. We have

$$\sum_{\rho}\left|\Gamma\left(\rho\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-\rho}\right|\left|z\right|^{-\alpha}\left|dz\right|\ll_{\alpha}N^{\alpha}.$$

Proof. Put $a=\frac{1}{N}$. Using the identity (19) and (18) we get that the left hand side in the statement above is

$$\sum_{\rho}\left|\gamma\right|^{\beta-1/2}\int_{\mathbb{R}}\exp\left(\gamma\arctan\left(\frac{y}{a}\right)-\frac{\pi}{2}\left|\gamma\right|\right)\frac{dy}{\left|z\right|^{\alpha+\beta}},\tag{20}$$

and so by Lemma 4 (20) is $\ll_{\alpha}a^{-\alpha}$ in $\mathbb{Y}_1\cup\mathbb{Y}_2$. For the other part we can see that

$$\sum_{\rho}\gamma^{\beta-1/2}\int_{a}^{\infty}\exp\left(-\gamma\arctan\left(\frac{a}{y}\right)\right)\frac{dy}{\left|z\right|^{\alpha+\beta}}$$

$$=a^{-\alpha-\beta+1}\sum_{\rho}\gamma^{\beta-1/2}\int_{1}^{\infty}\exp\left(-\gamma\arctan\left(\frac{1}{u}\right)\right)\frac{dy}{u^{\alpha+\beta}}$$

since

$$\left|z\right|^{-1}\asymp\begin{cases}a^{-1} & \left|y\right|\leq a,\\ \left|y\right|^{-1} & \left|y\right|\geq a,\end{cases}\tag{21}$$

and so by Lemma 3 we have the convergence if $\alpha>3/2$. $\square$

4.3 Settings

Using (6), (7) and (8) it is not hard to see that

$$\widetilde{S}\left(z\right)\omega_2^2\left(z\right)=\sum_{m_1\geq1}\sum_{m_2\geq1}\sum_{m_3\geq1}\Lambda\left(m_1\right)e^{-\left(m_1+m_2^2+m_3^2\right)z}=\sum_{n\geq1}r_Q\left(n\right)e^{-nz}$$

where

$$r_Q\left(n\right)=\sum_{m_1+m_2^2+m_3^2=n}\Lambda\left(n\right)$$

so let $z=a+iy$, $a>0$ and let us consider

$$\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\widetilde{S}\left(z\right)\omega_2^2\left(z\right)dz=\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\sum_{n\geq1}r_Q\left(n\right)e^{-nz}dz.$$

Now we prove that we can exchange the integral with the series. From (12) and the Prime Number Theorem in the form quoted above we have

$$\sum_{n\geq 1}\left|r_{Q}\left(n\right)e^{-nz}\right|=\widetilde{S}\left(a\right)\omega_{2}^{2}\left(a\right)\ll a^{-2}$$

hence

$$\begin{aligned}\int_{(a)}\left|e^{Nz}z^{-k-1}\right|\left|\widetilde{S}\left(z\right)\omega_{2}^{2}\left(z\right)\right|\left|dz\right| \ll & a^{-2}e^{Na}\left(\int_{-a}^{a}a^{-k-1}dy+2\int_{a}^{\infty}y^{-k-1}dy\right)\\ \ll_{k} & a^{-2-k}e^{Na}\end{aligned}$$

assuming $k>0$. So finally we have

$$\sum_{n\leq N}r_{Q}\left(n\right)\frac{\left(N-n\right)^{k}}{\Gamma\left(k+1\right)}=\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\widetilde{S}\left(z\right)\omega_{2}^{2}\left(z\right)dz. \tag{22}$$

Now, using (13), we can write (22) as

$$\begin{aligned}\sum_{n\leq N}r_{Q}\left(n\right)\frac{\left(N-n\right)^{k}}{\Gamma\left(k+1\right)}=\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho}z^{-\rho}\Gamma\left(\rho\right)\right)\omega_{2}^{2}\left(z\right)dz+\\ +O\left(\int_{(a)}\left|e^{Nz}\right|\left|z\right|^{-k-1}\left|\omega_{2}^{2}\left(z\right)\right|\left|E\left(a,y\right)\right|\left|dz\right|\right)\end{aligned} \tag{23}$$

and the error term can be estimated, using Lemma 2, (12) and (21) as

$$a^{-1}e^{Na}\left(\int_{-a}^{a}a^{-k-1}dy+\int_{a}^{\infty}y^{-k-1/2}\left(1+\log^{2}\left(y/a\right)\right)dy\right)\ll_{k}e^{Na}a^{-k-1}$$

assuming $k>1/2$. Hereafter we will consider $a=1/N$. We have

$$\sum_{n\leq N}r_{Q}\left(n\right)\frac{\left(N-n\right)^{k}}{\Gamma\left(k+1\right)}=\frac{1}{2\pi i}\int_{(1/N)}e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho}z^{-\rho}\Gamma\left(\rho\right)\right)\omega_{2}^{2}\left(z\right)dz+O\left(N^{k+1}\right)$$

and now, using the functional equation (11), we get

$$
\begin{aligned}
\sum_{n\leq N} r_Q(n)\frac{(N-n)^k}{\Gamma(k+1)} =& \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)^2 dz \\
&+\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\frac{\pi}{z}\omega_2^2\left(\frac{\pi^2}{z}\right) dz \\
&+\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\left(\frac{\pi}{z}\omega_2\left(\frac{\pi^2}{z}\right)\right) dz \\
&-\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma(\rho)\right)\left(\left(\frac{\pi}{z}\right)^{1/2}\omega_2\left(\frac{\pi^2}{z}\right)\right) dz \\
&+O\left(N^{k+1}\right) \\
=& I_1+I_2+I_3+O\left(N^{k+1}\right),
\end{aligned}
$$

say.

4.4 Evaluation of I_1

From I_1 we will find the main terms $M_1(N,k)$ and $M_2(N,k)$ of our asymptotic formula. We have

$$
\begin{aligned}
I_1 =& \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)^2 dz \\
&-\frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\sum_{\rho} z^{-\rho}\Gamma(\rho)\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)^2 dz \\
=& I_{1,1}-I_{1,2},
\end{aligned}
$$

say. From $I_{1,1}$ we have

$$
I_{1,1} = \frac{\pi}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-3}dz + \frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}dz - \frac{\pi^{1/2}}{4\pi i}\int_{(1/N)} e^{Nz}z^{-k-5/2}dz
$$

so, if we put $Nz=s$, $ds=Ndz$ and use (5) we have immediately

$$
\begin{aligned}
I_{1,1} =& \frac{\pi}{4}\frac{N^{k+2}}{2\pi i}\int_{(1)} e^{s}s^{-k-3}ds + \frac{N^{k+1}}{4}\frac{1}{2\pi i}\int_{(1)} e^{s}s^{-k-2}ds - \frac{\pi}{2}\frac{N^{k+3/2}}{2\pi i}\int_{(1)} e^{s}s^{-k-5/2}ds \\
=& M_1(N,k).
\end{aligned}
$$

From $I_{1,2}$ we have

$$\begin{aligned}I_{1,2} =&\frac{\pi}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz \\ &+\frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz \\ &-\frac{\pi^{1/2}}{4\pi i}\int_{(1/N)} e^{Nz}z^{-k-3/2}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz \\ &=\mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3,\end{aligned}$$

say. We have to study

$$\sum_{\rho}\left|\Gamma\left(\rho\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-2}\right|\left|z^{-\rho}\right|\left|dz\right|$$

$$\sum_{\rho}\left|\Gamma\left(\rho\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-1}\right|\left|z^{-\rho}\right|\left|dz\right|$$

$$\sum_{\rho}\left|\Gamma\left(\rho\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-3/2}\right|\left|z^{-\rho}\right|\left|dz\right|$$

and we observe that by Lemma 6 we have the absolute convergence of these integrals if, respectively, we have $k>-1/2$, $k>1/2$ and $k>0$. Hence for $k>1/2$ we have

$$\mathcal{I}_1=\frac{\pi}{4}\sum_{\rho}\Gamma\left(\rho\right)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-2-\rho}dz=\frac{\pi}{4}\sum_{\rho}\frac{\Gamma\left(\rho\right)}{\Gamma\left(k+2+\rho\right)}N^{k+1+\rho}$$

$$\mathcal{I}_2=\frac{1}{4}\sum_{\rho}\Gamma\left(\rho\right)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1-\rho}dz=\frac{1}{4}\sum_{\rho}\frac{\Gamma\left(\rho\right)}{\Gamma\left(k+1+\rho\right)}N^{k+\rho}$$

$$\mathcal{I}_3=\frac{\pi^{1/2}}{2}\sum_{\rho}\Gamma\left(\rho\right)\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-3/2-\rho}dz=\frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma\left(\rho\right)}{\Gamma\left(k+3/2+\rho\right)}N^{k+1/2+\rho}.$$

4.5 Evaluation of I_2

We have

$$
\begin{aligned}
I_2 =& \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2^2 \left(\frac{\pi^2}{z}\right) dz \\
-& \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma\left(\rho\right) \omega_2^2 \left(\frac{\pi^2}{z}\right) dz \\
=& I_{2,1} - I_{2,2},
\end{aligned}
$$

say.

4.5.1 Evaluation of $I_{2,1}$

We have that

$$
\begin{aligned}
I_{2,1} =& \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2^2 \left(\frac{\pi^2}{z}\right) dz \\
=& \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \left(\sum_{l_1 \geq 1} e^{-l_1^2 \pi^2 / z}\right) \left(\sum_{l_2 \geq 1} e^{-l_2^2 \pi^2 / z}\right) dz
\end{aligned}
$$

so let us prove that we can exchange the integral with the series. Let us consider

$$
A_1 = \sum_{l_1 \geq 1} \int_{(1/N)} \left|e^{Nz}\right| |z|^{-k-3} e^{-l_1^2 \pi^2 \mathrm{Re}(1/z)} \left|\omega_2 \left(\frac{\pi^2}{z}\right)\right| |dz| .
$$

From

$$
\mathrm{Re}\left(1/z\right) = \frac{N}{1 + N^2 y^2} \gg \begin{cases} N & |y| \leq 1/N \\ 1/\left(N y^2\right) & |y| > 1/N \end{cases} \tag{24}
$$

we have

$$
A_1 \ll \sum_{l_1 \geq 1} \int_0^{1/N} \frac{e^{-l_1^2 N}}{|z|^{k+3}} \omega_2\left(N\right) dy + N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{y e^{-l_1^2/\left(N y^2\right)}}{|z|^{k+3}} dy = U_1 + U_2,
$$

say. Hence, recalling (12) and (21),

$$
\begin{aligned}
U_1 \ll& \omega_2^2\left(N\right) N^{k+3} \int_0^{1/N} 1 dy \\
\ll& N^{k+1}
\end{aligned}
$$

and from (21) (with $a = 1/N$) we get

$$\begin{aligned}U_2 \ll & N^{1/2}\sum_{l_1\geq 1}\int_{1/N}^{\infty}\frac{e^{-l_1^2/\left(Ny^2\right)}}{y^{k+2}}dy \\ \ll & N^{k/2+1}\sum_{l_1\geq 1}\frac{1}{l_1^{k+1}}\int_{0}^{l_1^2 N}u^{k/2-1/2}e^{-u}du \\ \leq & \Gamma\left(\frac{k+1}{2}\right)N^{k/2+1}\sum_{l_1\geq 1}\frac{1}{l_1^{k+1}} \\ \ll_k & N^{k/2+1}\end{aligned}$$

assuming $k>0$. Now we have to study the convergence of

$$A_2=\sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{(1/N)}\left|e^{Nz}\right|\left|z\right|^{-k-3}e^{-l_1^2\pi^2\mathrm{Re}(1/z)}e^{-l_2^2\pi^2\mathrm{Re}(1/z)}\left|dz\right|$$

and again from (21) we have

$$\begin{aligned}A_2 \ll \sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{0}^{1/N}\frac{e^{-\left(l_1^2+l_2^2\right)N}}{\left|z\right|^{k+3}}dy+\sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{\left|z\right|^{k+3}}dy \\ =V_1+V_2,\end{aligned}$$

say. For V_1 we have

$$\begin{aligned}V_1 \ll & N^{k+3}\omega_2^2\left(N\right)\int_{0}^{1/N}1dy \\ \ll & N^{k+1}\end{aligned}$$

and for V_2, assuming $k>1$ and taking $u=\frac{l_1^2+l_2^2}{Ny^2}$, we have

$$\begin{aligned}V_2 \ll & \sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{y^{k+3}}dy \\ \ll & N^{k/2+1/2}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{\left(l_1^2+l_2^2\right)^{k/2+1/2}}\int_{0}^{\infty}u^{k/2}e^{-u}du \\ \ll_k & N^{k/2+1/2},\end{aligned}$$

recalling that

$$\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{\left(l_1^2+l_2^2\right)^{\alpha}}<\infty$$

holds if $\alpha > 1$. A simple way to prove it is using the AM-GM inequality

$$\sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{\left(l_1^2 + l_2^2\right)^{\alpha}} \leq 2^{\alpha} \sum_{l_1 \geq 1} \frac{1}{l_1^{\alpha}} \sum_{l_2 \geq 1} \frac{1}{l_2^{\alpha}}.$$

If $\alpha = 1$ the series diverges. We will prove this fact later.
Then finally we have

$$\begin{aligned} I_{2,1} =& \frac{\pi}{2\pi i} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{(1/N)} e^{Nz} z^{-k-3} e^{-\left(l_1^2 + l_2^2\right)\pi^2/z} dz \\ =& N^{k+2} \pi \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2\pi i} \int_{(1)} e^{s} s^{-k-3} e^{-\left(l_1^2 + l_2^2\right)\pi^2 N/s} ds \end{aligned}$$

from which, recalling the definition of the Bessel functions (4) we have, taking $u = 2\pi \left(l_1^2 + l_2^2\right)^{1/2} N^{1/2}$ and assuming $k > 1$,

$$J_{2,1} = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{J_{k+2}\left(2\pi \left(l_1^2 + l_2^2\right)^{1/2} N^{1/2}\right)}{\left(l_1^2 + l_2^2\right)^{k/2+1}}.$$

4.5.2 Evaluation of $I_{2,2}$

We have to calculate

$$I_{2,2} = \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma(\rho) \left(\sum_{l_1 \geq 1} e^{-l_1^2 \pi^2/z}\right) \left(\sum_{l_2 \geq 1} e^{-l_2^2 \pi^2/z}\right) dz$$

and again we have to prove that is possible to exchange the integral with the series. So let us consider

$$A_3 = \sum_{l_1 \geq 1} \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|\sum_{\rho} z^{-\rho} \Gamma(\rho)\right| e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} \left|\omega_2\left(\frac{\pi^2}{z}\right)\right| |dz|.$$

Now using (16) and (12) we have

$$\begin{aligned} A_3 \ll& N^{1/2} \sum_{l_1 \geq 1} \int_0^{1/N} \frac{e^{-l_1^2 N}}{|z|^{k+2}} dy + N^{3/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{y e^{-l_1^2/\left(Ny^2\right)}}{|z|^{k+2}} dy \\ &+ N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} y \log^2(2Ny) \frac{e^{-l_1^2/\left(Ny^2\right)}}{|z|^{k+3/2}} dy \\ =& W_1 + W_2 + W_3, \end{aligned}$$

say. For W_1 and W_2 we can easily see that

$$\begin{aligned} W_1 \ll & N^{k+5/2}\omega_2(N)\int_0^{1/N} 1dy \\ \ll & N^{k+1} \end{aligned}$$

and taking $u = l_1^2/(Ny^2)$

$$\begin{aligned} W_2 \ll & N^{3/2}\sum_{l_1\geq 1}\int_{1/N}^{\infty}\frac{e^{-l_1^2/(Ny^2)}}{y^{k+1}}dy \\ \ll & N^{k/2+3/2}\sum_{l_1\geq 1}\frac{1}{l_1^k}\int_0^{l_1^2N} e^{-u}u^{k/2-1}du \\ \ll_k & N^{k/2+3/2} \end{aligned}$$

assuming $k > 1$. We have now to check W_3. Taking again $u = l_1^2/(Ny^2)$ we have, assuming $k > 3/2$,

$$\begin{aligned} W_3 \ll & N^{k/2-1/4}\sum_{l_1\geq 1}\frac{1}{l_1^{k-1/2}}\int_0^{l_1^2N}\log^2\left(\frac{4Nl_1^2}{u}\right)e^{-u}u^{k/2-5/4}du \\ \ll & N^{k/2-1/4}\sum_{l_1\geq 1}\frac{\log^2(4Nl_1^2)}{l_1^{k-1/2}}\int_0^{\infty} e^{-u}u^{k/2-5/4}du \\ & -N^{k/2-1/4}\sum_{l_1\geq 1}\frac{\log\left(2\sqrt{N}l_1\right)}{l_1^{k-1/2}}\int_0^{\infty}\log(u)\, e^{-u}u^{k/2-5/4}du \\ & +N^{k/2-1/4}\sum_{l_1\geq 1}\frac{1}{l_1^{k-1/2}}\int_0^{\infty}\log^2(u)\, e^{-u}u^{k/2-5/4}du \\ \ll & N^{k/2-1/4}\sum_{l_1\geq 1}\frac{1}{l_1^{k-1/2}} \\ \ll_k & N^{k/2-1/4}. \end{aligned}$$

Let us consider

$$A_4 = \sum_{l_1\geq 1}\sum_{l_2\geq 2}\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-2}\right|\left|\sum_{\rho} z^{-\rho}\Gamma(\rho)\right| e^{-l_1^2\pi^2\mathrm{Re}(1/z)}e^{-l_2^2\pi^2\mathrm{Re}(1/z)}\,|dz|$$

and again for the estimation of $\left|\sum_{\rho} z^{-\rho}\Gamma(\rho)\right|$ we have, using (16),

$$\begin{aligned}
A_4 \ll & N\sum_{l_1\geq 1}\sum_{l_2\geq 2}\int_{0}^{1/N}\frac{e^{-\left(l_1^2+l_2^2\right)N}}{\left|z\right|^{k+2}}dy+\sum_{l_1\geq 1}\sum_{l_2\geq 2}\int_{1/N}^{\infty}\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{\left|z\right|^{k+2}}dy \\
& +\sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\log^2\left(2Ny\right)\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{\left|z\right|^{k+3/2}}dy \\
= & R_1+R_2+R_3,
\end{aligned}$$

say. So we have immediately

$$\begin{aligned}
R_1 \ll & N^{k+3}\omega^2\left(N\right)\int_{0}^{1/N}1dy \\
\ll & N^{k+1}
\end{aligned}$$

and if we take $u=\left(l_1^2+l_2^2\right)/\left(Ny^2\right)$ we have

$$\begin{aligned}
R_2 \ll & \sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{y^{k+2}}dy \\
\ll & N^{k/2+1/2}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{\left(l_1^2+l_2^2\right)^{k/2+1/2}}\int_{0}^{\infty}u^{k/2-1/2}e^{-u}dy \\
\ll_k & N^{k/2+1/2}
\end{aligned}$$

for $k>1$. So it remains to evaluate R_3. Again we take $u=\left(l_1^2+l_2^2\right)/\left(Ny^2\right)$ and we have

$$\begin{aligned}
R_3 \ll & \sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{\log^2\left(4N\left(l_1^2+l_2^2\right)\right)}{\left(l_1^2+l_2^2\right)^{k/2+1/4}}\int_{0}^{\infty}e^{-u}u^{k/2-3/4}du \\
& -\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{\log\left(2\sqrt{N\left(l_1^2+l_2^2\right)}\right)}{\left(l_1^2+l_2^2\right)^{k/2+1/4}}\int_{0}^{\infty}\log\left(u\right)e^{-u}u^{k/2-3/4}du \\
& +N^{k/2+1/4}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{1}{\left(l_1^2+l_2^2\right)^{k/2+1/4}}\int_{0}^{\infty}\log^2\left(u\right)e^{-u}u^{k/2-3/4}du
\end{aligned}$$

and the convergence follows if $k>3/2$. Note that the estimation of R_3 is optimal. For proving it, take $c=\left(l_1^2+l_2^2\right)/N$, assume $k\leq 3/2$ and $y>1$. We have

$$S=\sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1/N}^{\infty}\log^2\left(2Ny\right)\frac{e^{-c/y^2}}{y^{k+3/2}}dy\geq\sum_{l_1\geq 1}\sum_{l_2\geq 1}\int_{1}^{\infty}\log^2\left(2Ny\right)\frac{e^{-c/y^2}}{y^{k+3/2}}dy.$$

Now, since $y \geq 1$ we have $\log^2(2Ny) \geq \log^2(2N)$ and since $k \leq 3/2$ we have

$$\begin{aligned} S &\geq \log^2(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_1^{\infty} \frac{e^{-c/y^2}}{y^{k+3/2}} dy \\ &\geq \log^2(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_1^{\infty} \frac{e^{-c/y^2}}{y^3} dy \\ &= \log^2(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2c} \left(1 - e^{-c}\right) \\ &\geq \frac{N \log^2(2N) \left(1 - e^{-2/N}\right)}{2} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{l_1^2 + l_2^2} \end{aligned}$$

and the last double series is divergent, since

$$\sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{l_1^2 + l_2^2} \geq \sum_{l_1 \geq 1} \sum_{l_2 \leq l_1} \frac{1}{l_1^2 + l_2^2}$$

$$\geq \sum_{l_1 \geq 1} \sum_{l_2 \leq l_1} \frac{1}{2 l_1^2} \geq \frac{1}{2} \sum_{l_1 \geq 1} \frac{1}{l_1}.$$

Now we have to estimate

$$A_5 = \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|z^{-\rho}\right| e^{-\mathrm{Re}(1/z)\left(l_1^2 + l_2^2\right)} |dz|.$$

Using (18) and (19) we have

$$A_5 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} \gamma^{\beta - 1/2} \int_{(1/N)} |z|^{-k-2} |z|^{-\beta} \exp\left(\gamma \left(\arctan(Ny) - \frac{\pi}{2}\right)\right) e^{-\mathrm{Re}(1/z)\left(l_1^2 + l_2^2\right)} |dz|.$$

Let $Q_k = \sup_{\beta} \left\{\Gamma\left(\frac{k}{2} + \frac{\beta}{2} + \frac{1}{2}\right)\right\}$ and assume $y < 0$ (we will A_n^- the condition $y < 0$, with A_n^+ the contidion $y > 0$) Using the obvious bound $\gamma \arctan(Ny) - \gamma \frac{\pi}{2} \leq -\gamma \frac{\pi}{2}$ and taking

$u = \frac{l_1^2+l_2^2}{Ny^2}$ we get

$$
\begin{aligned}
A_5^- \ll N^{k+2} \sum_{l_1\geq 1} e^{-l_1^2 N} \sum_{l_2\geq 1} e^{-l_2^2 N} \sum_{\rho,\gamma>0} N^{\beta} \gamma^{\beta-1/2} e^{-\gamma\pi/2} \int_{-1/N}^{0} 1 dy \\
+ \sum_{l_1\geq 1} \sum_{l_2\geq 1} \sum_{\rho,\gamma>0} \gamma^{\beta-1/2} e^{-\gamma\pi/2} \int_{-\infty}^{-1/N} |y|^{-k-2-\beta} e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)} dy \\
\ll N^{k+1} \sum_{l_1\geq 1} e^{-l_1^2 N} \sum_{l_2\geq 1} e^{-l_2^2 N} \sum_{\rho,\gamma>0} N^{\beta} e^{-\pi\gamma/2} \gamma^{\beta-1/2} \\
+ \sum_{l_1\geq 1} \sum_{l_2\geq 1} \sum_{\rho,\gamma>0} \gamma^{\beta-1/2} e^{-\gamma\pi/2} \frac{N^{(k+1+\beta)/2}}{(l_1^2+l_2^2)^{(k+1+\beta)/2}} \int_0^{\infty} u^{(k+\beta-1)/2} e^{-u} du \\
\ll N^{k+1} \sum_{l_1\geq 1} e^{-l_1^2 N} \sum_{l_2\geq 1} e^{-l_2^2 N} \sum_{\rho,\gamma>0} N^{\beta} e^{-\pi\gamma/2} \gamma^{\beta-1/2} \\
+ N^{(k+1)/2} Q_k \sum_{l_1\geq 1} \sum_{l_2\geq 1} \frac{1}{(l_1^2+l_2^2)^{(k+1)/2}} \sum_{\rho,\gamma>0} N^{\beta/2} \frac{e^{-\pi\gamma/2}\gamma^{\beta-1/2}}{(l_1^2+l_2^2)^{\beta/2}} \\
\ll_k N^k \qquad (25)
\end{aligned}
$$

for $k > 1$, where (25) follows from the density estimate

$$\gamma_m \sim \frac{2\pi m}{\log(m)}$$

where γ_m is the imaginary part of the m-th non trivial zero of the Riemann Zeta function. If $y > 0$ we have

$$
\begin{aligned}
A_5^+ \ll \sum_{l_1\geq 1} \sum_{l_2\geq 1} \sum_{\rho:\gamma>0} \gamma^{\beta-1/2} \int_0^{1/N} |z|^{-k-2-\beta} \exp\left(\gamma\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) e^{-\left(l_1^2+l_2^2\right)\mathrm{Re}(1/z)} dy \\
+ \sum_{l_1\geq 1} \sum_{l_2\geq 1} \sum_{\rho:\gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\left(\gamma\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{y^{k+2+\beta}} dy.
\end{aligned}
$$

Obivously if $|y| < 1/N$ we have $\arctan(Ny) - \frac{\pi}{2} \leq -\frac{\pi}{4}$ hence

$$\sum_{l_1\geq 1} \sum_{l_2\geq 1} \sum_{\rho:\gamma>0} \gamma^{\beta-1/2} \int_0^{1/N} |z|^{-k-2-\beta} \exp\left(\gamma\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) e^{-\left(l_1^2+l_2^2\right)\mathrm{Re}(1/z)} dy$$

$$\ll N^{k+2} \sum_{l_1\geq 1} e^{-l_1^2 N} \sum_{l_2\geq 1} e^{-l_2^2 N} \sum_{\rho:\gamma>0} N^{\beta} e^{-\pi\gamma/4} \gamma^{\beta-1/2} \int_0^{1/N} 1 dy$$

$$\ll N^{k+1}$$

and from $\arctan(x) + \arctan(1/x) = \pi/2$ follows that

$$\begin{aligned}
A_5^+ \ll & N^{k+1} + \sum_{l_1\geq 1}\sum_{l_2\geq 1}\sum_{\rho:\gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\left(-\gamma \arctan\left(\frac{1}{Ny}\right)\right) \frac{e^{-\left(l_1^2+l_2^2\right)/\left(Ny^2\right)}}{y^{k+2+\beta}} dy \\
\ll & N^{k+1} + \sum_{l_1\geq 1}\sum_{l_2\geq 1}\sum_{\rho:\gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\left(-\frac{\gamma}{Ny} - \frac{l_1^2+l_2^2}{Ny^2}\right) y^{-k-2-\beta} dy
\end{aligned}$$

and if we put $\frac{\gamma}{Ny} = v$ we get

$$\begin{aligned}
A_5^+ \ll & N^{k+1} + \sum_{l_1\geq 1}\sum_{l_2\geq 1}\sum_{\rho:\gamma>0} \gamma^{\beta-1/2} \int_{0}^{\gamma} e^{-v} e^{-\left(Nv^2\left(l_1^2+l_2^2\right)/\gamma^2\right)} \left(\frac{\gamma}{Nv}\right)^{-k-2-\beta} \frac{\gamma}{Nv^2} dv \\
\ll & N^{k+1} + \sum_{l_1\geq 1}\sum_{l_2\geq 1}\sum_{\rho:\gamma>0} \gamma^{-k-3/2} \int_{0}^{\infty} e^{-v} e^{-\left(Nv^2\left(l_1^2+l_2^2\right)/\gamma^2\right)} v^{k+\beta} dv. \qquad (26)
\end{aligned}$$

Now we can observe that we are in the situation of Lemma 5 with $d = 2$ and so we can conclude immediately that we have the convergence for $k > 3/2$ and this result is optimal. We studied the convergence, so we finally have, using again the identity (4), that

$$I_{2,2} = \pi^{-k} N^{k/2+1/2} \sum_{\rho} \frac{\Gamma(\rho)}{\pi^{\rho}} N^{\rho/2} \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{J_{k+1+\rho}\left(2\pi\left(l_1^2+l_2^2\right)^{1/2} N^{1/2}\right)}{\left(l_1^2+l_2^2\right)^{(k+1+\rho)/2}}.$$

4.6 Evaluation of I_3

We have

$$\begin{aligned}
I_3 =& \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho)\right) \left(\frac{\pi}{z}\omega_2\left(\frac{\pi^2}{z}\right)\right) dz \\
&- \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho)\right) \left(\left(\frac{\pi}{z}\right)^{1/2} \omega_2\left(\frac{\pi^2}{z}\right)\right) dz \\
=& \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2\left(\frac{\pi^2}{z}\right) dz - \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho}\Gamma(\rho)\,\omega_2\left(\frac{\pi^2}{z}\right) dz \\
&- \frac{1}{2\pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-5/2} \omega_2\left(\frac{\pi^2}{z}\right) + \frac{1}{2\pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_{\rho} z^{-\rho}\Gamma(\rho)\,\omega_2\left(\frac{\pi^2}{z}\right) dz \\
=& I_{3,1} - I_{3,2} - I_{3,3} + I_{3,4}.
\end{aligned}$$

4.6.1 Evaluation of $I_{3,1}$

We have

$$I_{3,1} = \frac{1}{2i}\int_{(1/N)} e^{Nz}z^{-k-3}\omega_2\left(\frac{\pi^2}{z}\right)dz = \frac{1}{2i}\int_{(1/N)} e^{Nz}z^{-k-3}\sum_{m\geq 1} e^{-m^2\pi^2/z}dz$$

hence we have to establish the convergence of

$$A_6 = \sum_{m\geq 1}\int_{(1/N)} \left|e^{Nz}\right| |z|^{-k-3}\, e^{-m^2\mathrm{Re}(1/z)}dz.$$

Using (12), (21) and (24) we have

$$\begin{aligned} A_6 \ll& N^{k+3/2} + \sum_{m\geq 1}\int_0^\infty y^{-k-3}e^{-m^2/\left(Ny^2\right)}dy \\ =& N^{k+3/2} + N^{k/2+1}\sum_{m\geq 1}\frac{1}{m^{k+2}}\int_0^\infty u^{k/2}e^{-u}du \\ \ll_k& N^{k+3/2} \end{aligned}$$

for $k > -1$. So we obtain, recalling (4), that

$$J_{3,1} = \frac{N^{k/2+1}}{\pi^{k+1}}\sum_{m\geq 1}\frac{J_{k+2}\left(2m\pi N^{1/2}\right)}{m^{k+2}}.$$

4.6.2 Evaluation of $I_{3,3}$

We have

$$I_{3,3} = \frac{1}{2\pi^{1/2}i}\int_{(1/N)} e^{Nz}z^{-k-5/2}\sum_{m\geq 1} e^{-m^2\pi^2/z}dz$$

so we have to establish the convergence of

$$\begin{aligned} \sum_{m\geq 1}\int_{(1/N)} \left|e^{Nz}\right| |z|^{-k-5/2}\, e^{-m^2\mathrm{Re}(1/z)}dz \ll& N^{k+5/2}\sum_{m\geq 1} e^{-m^2N}\int_0^{1/N} 1dy \\ &+ \sum_{m\geq 1}\int_{1/N}^\infty y^{-k-5/2}e^{-m^2/\left(Ny^2\right)}dy \\ \ll& N^{k+5/2} + N^{k/2+3/4}\sum_{m\geq 1}\frac{1}{m^{k+3/2}}\int_0^\infty u^{k+1}e^{-u}du \\ \ll_k& N^{k+5/2} \end{aligned}$$

hence we have the convergence for $k > -1/2$. So

$$I_{3,3} = \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{m\geq 1} \frac{J_{k+3/2}\left(2m\pi N^{1/2}\right)}{m^{k+3/2}}.$$

4.6.3 Evaluation of $I_{3,2}$

We have to establish the convergence of

$$A_7 = \sum_{m\geq 1} \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|\sum_{\rho} z^{-\rho}\Gamma(\rho)\right| \left|e^{-m^2/z}\right| |dz|$$

so using (12), (21), (24) and (16) we get

$$\begin{aligned} A_7 \ll & N^{k+1/2} + N \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-2} e^{-m^2/\left(Ny^2\right)} dy \\ & + \log^2(2N) \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-3/2} e^{-m^2/\left(Ny^2\right)} dy \\ & + 2\log(2N) \sum_{m\geq 1} \int_{1/N}^{\infty} \log(y)\, y^{-k-3/2} e^{-m^2/\left(Ny^2\right)} dy \\ & + \sum_{m\geq 1} \int_{1/N}^{\infty} \log^2(y)\, y^{-k-3/2} e^{-m^2/\left(Ny^2\right)} dy. \end{aligned}$$

Now if we put $m^2/\left(Ny^2\right) = u$ we have

$$N \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-2} e^{-m^2/\left(Ny^2\right)} dy \ll N^{k/2+3/2} \Gamma\left(\frac{k+1}{2}\right) \sum_{m\geq 1} m^{-k-1}$$

which converges if $k > 0$. With the same substitution we get

$$\log^2(2N) \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-3/2} e^{-m^2/\left(Ny^2\right)} dy \ll \log^2(2N)\, N^{k/2+1/4} \Gamma\left(\frac{k}{2} + \frac{1}{4}\right) \sum_{m\geq 1} m^{-k-1/2}$$

and so the convergence for $k > 1/2$. For the estimation of the last integral in the bound of A_7 we observe that if we take $\epsilon > 0$ we have

$$\sum_{m\geq 1} \int_{1/N}^{\infty} \log^{\alpha}(y)\, y^{-k-3/2} e^{-m^2/\left(Ny^2\right)} dy \ll \sum_{m\geq 1} \int_{1/N}^{\infty} y^{-k-3/2+\epsilon} e^{-m^2/\left(Ny^2\right)} dy,\ \alpha \geq 0$$

so

$$A_7 \ll N^{k/2+1/4-\epsilon/2}\Gamma\left(\frac{k}{2}+\frac{1}{4}-\frac{\epsilon}{2}\right)\sum_{m\geq 1} m^{-k-1/2+\epsilon}$$

and for the arbitrariness of ϵ we have the convergence for $k > 1/2$. We have now to study

$$A_8 = \sum_{m\geq 1}\sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|z^{-\rho}\right| \left|e^{-m^2\pi^2/z}\right| |dz|.$$

By symmetry we may assume that $\gamma > 0$. If $y \leq 0$ we have $\gamma \arctan(y/a) - \frac{\pi}{2}\gamma \leq -\frac{\pi}{2}\gamma$ and so using (18) and (19) we get

$$\begin{aligned}
A_8^- \ll & \sum_{m\geq 1}\sum_{\gamma>0} \gamma^{\beta-1/2} e^{-\pi\gamma/2} \int_{-1/N}^{0} |z|^{-k-2-\beta}\, e^{-m^2 \mathrm{Re}(1/z)} dy \\
& + \sum_{m\geq 1}\sum_{\gamma>0} \gamma^{\beta-1/2} e^{-\pi\gamma/2} \int_{-\infty}^{-1/N} \frac{e^{-m^2/\left(Ny^2\right)}}{|y|^{k+2+\beta}} dy \\
\ll & N^{k+2} \sum_{m\geq 1} e^{-m^2 N} \sum_{\gamma>0} \gamma^{\beta-1/2} e^{-\pi\gamma/2} N^{\beta} \int_{-1/N}^{0} 1 dy \\
& + N^{(k+1)/2} \sum_{m\geq 1} \frac{1}{m^{k+1}} \sum_{\gamma>0} \frac{N^{\beta/2}\gamma^{\beta-1/2}e^{-\pi\gamma/2}}{m^{\beta}} \int_{0}^{\infty} u^{(k-1+\beta)/2} e^{-u} du \\
\ll & N^{k+2} + N^{k/2+1/2} Q_k \sum_{m\geq 1} \frac{1}{m^{k+1}} \sum_{\gamma>0} \frac{N^{\beta/2}\gamma^{\beta-1/2}e^{-\pi\gamma/2}}{m^{\beta}} \\
\ll_k & N^{k+2}
\end{aligned}$$

provided that $k > 0$ and $Q_k = \sup_{\beta}\left\{\Gamma\left(\frac{k}{2}+\frac{1}{2}+\frac{\beta}{2}\right)\right\}$. Let $y > 0$. We have

$$\begin{aligned}
A_8^+ \ll & \sum_{m\geq 1}\sum_{\gamma>0} \gamma^{\beta-1/2} \exp\left(-\frac{\pi}{4}\gamma\right) \int_{0}^{1/N} N^{k+2+\beta} e^{-m^2 N} dy \\
& + \sum_{m\geq 1}\sum_{\gamma>0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\left(\gamma \arctan(Ny) - \frac{\pi}{2}\gamma\right) \frac{e^{-m^2/\left(Ny^2\right)}}{y^{k+2+\beta}} dy \\
= & L_1 + L_2,
\end{aligned}$$

say. From (12) and (21) we have

$$L_1 \ll N^{k+1} \sum_{m\geq 1} e^{-m^2 N} \sum_{\gamma>0} N^{\beta}\gamma^{\beta-1/2} \exp\left(-\frac{\pi}{4}\gamma\right) \ll_k N^{k+3/2}$$

and recalling the identity $\arctan(x)+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}$ and taking $v=m/\left(N^{1/2}y\right)$ we have

$$\begin{aligned}L_2 \ll & \sum_{m\geq 1}\sum_{\gamma>0}\gamma^{\beta-1/2}\int_{1/N}^{\infty}\exp\left(-\frac{\gamma}{Ny}-\frac{m^2}{Ny^2}\right)\frac{dy}{y^{k+2+\beta}}\\ =&N^{(k+1)/2}\sum_{m\geq 1}\frac{1}{m^{k+1}}\sum_{\gamma>0}\frac{N^{\beta/2}}{m^{\beta}}\gamma^{\beta-1/2}\int_0^{m\sqrt{N}}\exp\left(-\frac{\gamma v}{N^{1/2}m}-v^2\right)v^{k+\beta}dv\end{aligned}$$

and now since $e^{-v^2}v^k=O_k(1)$ if $k>0$ we have, taking $s=\gamma v/\left(N^{1/2}m\right)$,

$$\begin{aligned}\ll & N^{k/2+1}\sum_{m\geq 1}\frac{1}{m^k}\sum_{\gamma>0}N^{\beta}\gamma^{-3/2}\int_0^{\infty}\exp(-s)\,s^{\beta}ds\\ \ll_k & N^{k/2+2}\end{aligned}$$

for $k>1$. Now we can exchange the series with the integral and so we have

$$I_{3,2}=\pi^{-k}N^{(k+1)/2}\sum_{\rho}\pi^{-\rho}N^{\rho/2}\Gamma(\rho)\sum_{m\geq 1}\frac{J_{k+1+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1+\rho}}.$$

4.6.4 Evaluation of $I_{3,4}$

We have to establish the convergence of

$$I_{3,4}=\frac{1}{2\pi^{1/2}i}\int_{(1/N)}e^{Nz}z^{-k-3/2}\sum_{\rho}z^{-\rho}\Gamma(\rho)\,\omega_2\left(\frac{\pi^2}{z}\right)dz$$

and so we have the similar situation of $I_{3,2}$.
So we have to study

$$A_9=\sum_{m\geq 1}\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-3/2}\right|\left|\sum_{\rho}z^{-\rho}\Gamma(\rho)\right|\left|e^{-m^2/z}\right||dz|$$

so using (12), (21), (24) and (16) we get

$$\begin{aligned}A_9 \ll & N^{k+5/2}\sum_{m\geq1}e^{-m^2N}\int_0^{1/N}1dy\\ & +N\sum_{m\geq1}\int_{1/N}^{\infty}y^{-k-3/2}e^{-m^2/\left(Ny^2\right)}dy\\ & +\log^2\left(2N\right)\sum_{m\geq1}\int_{1/N}^{\infty}y^{-k-1}e^{-m^2/\left(Ny^2\right)}dy\\ & +2\log\left(2N\right)\sum_{m\geq1}\int_{1/N}^{\infty}\log\left(y\right)y^{-k-1}e^{-m^2/\left(Ny^2\right)}dy\\ & +\sum_{m\geq1}\int_{1/N}^{\infty}\log^2\left(y\right)y^{-k-1}e^{-m^2/\left(Ny^2\right)}dy\end{aligned}$$

so taking $u=m/\left(Ny^2\right)$ we get

$$\begin{aligned}\sum_{m\geq1}\int_{1/N}^{\infty}y^{-k-3/2}e^{-m^2/\left(Ny^2\right)}dy \ll & N^{k/2+1/4}\sum_{m\geq1}\frac{1}{m^{k+1/2}}\int_0^{\infty}u^{k/2-3/4}e^{-u}du\\ \ll_k & N^{k/2+1/4}\end{aligned}$$

and

$$\begin{aligned}\sum_{m\geq1}\int_{1/N}^{\infty}y^{-k-1}e^{-m^2/\left(Ny^2\right)}dy \ll & N^{k/2}\sum_{m\geq1}\frac{1}{m^{k}}\int_0^{\infty}u^{k/2-1}e^{-u}du\\ \ll_k & N^{k/2}\end{aligned}$$

and again the presence of $\log^{\alpha}\left(y\right)$ does not alter the evalutation since for every fixed $\epsilon>0$ holds

$$\int_{1/N}^{\infty}\log^{\alpha}\left(y\right)y^{-k-1}e^{-m^2/\left(Ny^2\right)}\ll\int_{1/N}^{\infty}y^{-k-1+\epsilon}e^{-m^2/\left(Ny^2\right)},\ \alpha\geq0,$$

then we have the convergence if $k>1$.
Now we have to study

$$A_{10}=\sum_{m\geq1}\sum_{\rho}\left|\Gamma\left(\rho\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-3/2}\right|\left|z^{-\rho}\right|\left|e^{-m^2/z}\right|\left|dz\right|.$$

By symmetry we may assume that $\gamma>0$. If $y\leq0$ we have $\gamma\arctan\left(y/a\right)-\frac{\pi}{2}\gamma\leq-\frac{\pi}{2}\gamma$ and so using (18) and (19) we get

$$
\begin{aligned}
A_{10}^{-} \ll & \sum_{m\geq 1}\sum_{\gamma>0}\gamma^{\beta-1/2}e^{-\pi\gamma/2}\int_{-1/N}^{0}|z|^{-k-3/2-\beta}\, e^{-m^2\mathrm{Re}(1/z)}dy \\
&+\sum_{m\geq 1}\sum_{\gamma>0}\gamma^{\beta-1/2}e^{-\pi\gamma/2}\int_{-\infty}^{-1/N}\frac{e^{-m^2/\left(Ny^2\right)}}{|y|^{k+3/2+\beta}}dy \\
\ll & N^{k+3/2}\sum_{m\geq 1}e^{-m^2N}\sum_{\gamma>0}\gamma^{\beta-1/2}e^{-\pi\gamma/2}N^{\beta}\int_{-1/N}^{0}1dy \\
&+N^{k/2+1/4}\sum_{m\geq 1}\frac{1}{m^{k+1/2}}\sum_{\gamma>0}\frac{N^{\beta/2}\gamma^{\beta-1/2}e^{-\pi\gamma/2}}{m^{\beta}}\int_{0}^{\infty}u^{(k+\beta)/2-3/4}e^{-u}du \\
\ll & N^{k+3/2}+N^{k/2+1/4}G_k\sum_{m\geq 1}\frac{1}{m^{k+1/2}}\sum_{\gamma>0}\frac{N^{\beta/2}\gamma^{\beta-1/2}e^{-\pi\gamma/2}}{m^{\beta}} \\
\ll_k & N^{k+3/2}
\end{aligned}
$$

provided that $k>1/2$ and $G_k=\sup_{\beta}\left\{\Gamma\left(\frac{k}{2}+\frac{1}{4}+\frac{\beta}{2}\right)\right\}$. Let $y>0$. We have

$$
\begin{aligned}
A_{10}^{+} \ll & \sum_{m\geq 1}\sum_{\gamma>0}\gamma^{\beta-1/2}\exp\left(-\frac{\pi}{4}\gamma\right)\int_{0}^{1/N}N^{k+3/2+\beta}e^{-m^2N}dy \\
&+\sum_{m\geq 1}\sum_{\gamma>0}\gamma^{\beta-1/2}\int_{1/N}^{\infty}\exp\left(\gamma\arctan\left(Ny\right)-\frac{\pi}{2}\gamma\right)\frac{e^{-m^2/\left(Ny^2\right)}}{y^{k+3/2+\beta}}dy \\
= & F_1+F_2,
\end{aligned}
$$

say. From (12) and (21) we have

$$
F_1\ll N^{k+1/2}\sum_{m\geq 1}e^{-m^2N}\sum_{\gamma>0}N^{\beta}\gamma^{\beta-1/2}\exp\left(-\frac{\pi}{4}\gamma\right)\ll_k N^{k+1}
$$

and recalling the identity $\arctan(x)+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}$ and taking $v=m/\left(N^{1/2}y\right)$ we have

$$
\begin{aligned}
F_2\ll & \sum_{m\geq 1}\sum_{\gamma>0}\gamma^{\beta-1/2}\int_{1/N}^{\infty}\exp\left(-\frac{\gamma}{Ny}-\frac{m^2}{Ny^2}\right)\frac{dy}{y^{k+3/2+\beta}} \\
= & N^{k/2+1/4}\sum_{m\geq 1}\frac{1}{m^{k+1/2}}\sum_{\gamma>0}\frac{N^{\beta/2}}{m^{\beta}}\gamma^{\beta-1/2}\int_{0}^{m\sqrt{N}}\exp\left(-\frac{\gamma v}{N^{1/2}m}-v^2\right)v^{k-1/2+\beta}dv
\end{aligned}
$$

and now since $e^{-v^2}v^{k-1/2}=O_k(1)$ if $k>1/2$ we have, taking $s=\gamma v/\left(N^{1/2}m\right)$,

$$\begin{aligned}F_2 \ll& N^{k/2+3/4}\sum_{m\geq1}\frac{1}{m^{k-1/2}}\sum_{\gamma>0}N^{\beta}\gamma^{-3/2}\int_0^{\infty}\exp\left(-s\right)s^{\beta}ds\\ \ll_k& N^{k/2+3/4}\end{aligned}$$

and so the convergence if $k>3/2$. Now we can exchange the series with the integral and obtain

$$I_{3,4}=\pi^{-k}N^{k/2+1/4}\sum_{\rho}\pi^{-\rho}N^{\rho}\Gamma\left(\rho\right)\sum_{m\geq1}\frac{J_{k+1/2+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1/2+\rho}}.$$

Summing up, we have proved the following

Theorem 1. *Let N be a sufficient large integer. We have*

$$\sum_{n\leq N}r_Q\left(n\right)\frac{\left(N-n\right)^k}{\Gamma\left(k+1\right)}=M_1\left(N,k\right)+M_2\left(N,k\right)+M_3\left(N,k\right)+M_4\left(N,k\right)+O\left(N^{k+1}\right)$$

for $k>3/2$, where ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta\left(s\right)$, $J_v\left(u\right)$ is the Bessel function of complex order v and real argument u and

$$M_1(N,k) = \frac{\pi N^{k+2}}{4\Gamma(k+3)} + \frac{N^{k+1}}{4\Gamma(k+2)} - \frac{\pi^{1/2} N^{k+3/2}}{2\Gamma(k+5/2)} \tag{27}$$

$$\begin{aligned} M_2(N,k) = & -\frac{\pi}{4}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+2+\rho)}N^{k+1+\rho} - \frac{1}{4}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+1+\rho)}N^{k+\rho} \\ & + \frac{\pi^{1/2}}{2}\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)}N^{k+1/2+\rho} \end{aligned} \tag{28}$$

$$\begin{aligned} M_3(N,k) = & \frac{N^{k/2+1}}{\pi^{k+1}}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{J_{k+2}\left(2\pi\left(l_1^2+l_2^2\right)^{1/2}N^{1/2}\right)}{\left(l_1^2+l_2^2\right)^{k/2+1}} \\ & -\pi^{-k}N^{k/2+1/2}\sum_{\rho}\frac{\Gamma(\rho)}{\pi^{\rho}}N^{\rho/2}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{J_{k+1+\rho}\left(2\pi\left(l_1^2+l_2^2\right)^{1/2}N^{1/2}\right)}{\left(l_1^2+l_2^2\right)^{(k+1+\rho)/2}} \end{aligned} \tag{29}$$

$$\begin{aligned} M_4(N,k) = & \frac{N^{k/2+1}}{\pi^{k+1}}\sum_{m\geq 1}\frac{J_{k+2}\left(2m\pi N^{1/2}\right)}{m^{k+2}} - \frac{N^{k/2+3/4}}{\pi^{k+1}}\sum_{m\geq 1}\frac{J_{k+3/2}\left(2m\pi N^{1/2}\right)}{m^{k+3/2}} \\ & -\pi^{-k}N^{(k+1)/2}\sum_{\rho}\pi^{-\rho}N^{\rho/2}\Gamma(\rho)\sum_{m\geq 1}\frac{J_{k+1+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1+\rho}} \\ & +\pi^{-k}N^{k/2+1/4}\sum_{\rho}\pi^{-\rho}N^{\rho/2}\Gamma(\rho)\sum_{m\geq 1}\frac{J_{k+1/2+\rho}\left(2m\pi\sqrt{N}\right)}{m^{k+1/2+\rho}}. \end{aligned} \tag{30}$$

Furthermore the bound $k > 3/2$ is optimal using this technique.

5 On the Cesàro average of the numbers that can be written as a sum of a prime and two squares of primes

In this part we will study another Cesàro average for the numbers that can be written as sum of a prime and two squares of primes. We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function. The problem of representing an integer as sum of a prime and two prime squares is classical. Let

$$A = \{n \in \mathbb{N} : n \equiv 1 \mod 2; n \not\equiv 2 \mod 3\};$$

it is conjectured that every sufficiently large natural number $n \in A$ is a sum of a prime and two prime squares. Many authors studied the cardinality $E(N)$ of the set of integers $n \leq N$, $n \in A$ that are not representable as a sum of prime and two square of primes. We recall Hua [21], Schwarz [36], Leung-Liu [28], Wang [41], Wang-Meng [42], Li [29], Harman-Kumchev [16] and Zhao [44]. Languasco and Zaccagnini [26] proved that, assuming the RH, in every interval $[N, N+H]$ contains a number that can be written as a sum of a prime and two squares of primes, where $H \geq C\log^4(N)$ and $C > 0$ is a constant.
Let us define $z = a + iy$, $a > 0$ and $y \in \mathbb{R}$ and let us consider the functions

$$\widetilde{S}_1(z) = \widetilde{S}(z)$$

$$\widetilde{S}_2(z) = \sum_{m\geq 1} \Lambda(m)\, e^{-m^2 z} \tag{31}$$

and

$$r_{SP}(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3).$$

For our purpose we need a generalization of the Lemma 2. So we introduce

Lemma 7. *Let $z = a + iy$, $a > 0$, $y \in \mathbb{R}$ and $l \in \mathbb{N}_0$. Then*

$$\widetilde{S}_l(z) = \frac{\Gamma(1/l)}{lz^{1/l}} - \frac{1}{l}\sum_{\rho} z^{-\rho/l}\Gamma\left(\frac{\rho}{l}\right) + E_l(a,y) \tag{32}$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E_l(a,y) \ll_l E(a,y). \tag{33}$$

Proof. It is well know that (see for example formula 5 of [26]) that, for $l \in \mathbb{N}_0$,

$$\begin{aligned}\widetilde{S}_l(z) &= \sum_{m\geq 1}\Lambda(m)\, e^{-m^l z}\\ &= \frac{\Gamma(1/l)}{lz^{1/l}} - \frac{1}{l}\sum_{\rho} z^{-\rho/l}\Gamma\left(\frac{\rho}{l}\right) - \frac{\zeta'}{\zeta}(0) - \frac{1}{2\pi i}\int_{(-1/2)}\frac{\zeta'}{\zeta}(lw)\,\Gamma(w)\, z^{-w}dw \end{aligned}\tag{34}$$

so, taking $w = -\frac{1}{2} + it$, following the proof of the Lemma 2 and observing that

$$\left|\frac{\zeta'}{\zeta}(lw)\right| \ll_l \log(|t|+2)$$

we can conclude that we may estimate the integral in (34) exactly as in [24], so the claim follows. $\square$

From Lemma 7 it is quite simple to note that

$$\widetilde{S}_2(a) \sim \frac{\sqrt{\pi}}{2a^{1/2}},\ \text{when } a \to 0^+. \tag{35}$$

We now introduce another

Lemma 8. *Let $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function, let $z = \frac{1}{N} + iy$, $N > 1$ be natural number, $y \in \mathbb{R}$, $l \in \mathbb{N}_0$ and $\alpha > 3/2$. We have*

$$\sum_{\rho}\left|\Gamma\left(\frac{\rho}{l}\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-\rho/l}\right||z|^{-\alpha}|dz| \ll_{\alpha} N^{\alpha}.$$

Proof. Put $a = \frac{1}{N}$. Using the identity (19), (18) and (21) we get that the left hand side in the statement above is

$$\sum_{\rho}|\gamma|^{\beta/l-1/2}\int_{\mathbb{R}}\exp\left(\frac{\gamma}{l}\arctan\left(\frac{y}{a}\right) - \frac{\pi}{2}\frac{|\gamma|}{l}\right)\frac{dy}{|z|^{\alpha+\beta/l}}. \tag{36}$$

The case $l = 1$ has already been discussed in Lemma 6. For $l > 1$, observing Lemmas 2 and 3 of [24] and Lemma 6, we can conclude that the presence of l does not alter the proofs, so using the same argumentation of the Lemma 6 that we have the convergence for $\alpha > 3/2$. $\square$

5.1 Setting

From (7) and (31) it is not hard to see that

$$\widetilde{S}_1(z)\,\widetilde{S}_2^2(z)=\sum_{m_1\geq 1}\sum_{m_2\geq 1}\sum_{m_3\geq 1}\Lambda(m_1)\,\Lambda(m_2)\,\Lambda(m_3)\,e^{-\left(m_1+m_2^2+m_3^2\right)z}=\sum_{n\geq 1}r_{SP}(n)\,e^{-nz}$$

so let $z=a+iy$ and $a>0$ and let us consider

$$\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\widetilde{S}_1(z)\,\widetilde{S}_2^2(z)\,dz=\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\sum_{n\geq 1}r_{SP}(n)\,e^{-nz}dz.$$

Now we prove that we can exchange the integral with the series. From (17) and (35) we have

$$\sum_{n\geq 1}\left|r_{SP}(n)\,e^{-nz}\right|=\widetilde{S}_1(a)\,\widetilde{S}_2^2(a)\ll a^{-2};$$

hence

$$\begin{aligned}\int_{(a)}\left|e^{Nz}z^{-k-1}\right|\left|\widetilde{S}_1(z)\,\widetilde{S}_2^2(z)\right||dz|\ll&a^{-2}e^{Na}\left(\int_{-a}^{a}a^{-k-1}dy+2\int_{a}^{\infty}y^{-k-1}dy\right)\\ \ll_k&a^{-2-k}e^{Na}\end{aligned}$$

assuming $k>0$, so we have that

$$\sum_{n\leq N}r_{SP}(n)\,\frac{(N-n)^k}{\Gamma(k+1)}=\frac{1}{2\pi i}\int_{(a)}e^{Nz}z^{-k-1}\widetilde{S}_1(z)\,\widetilde{S}_2^2(z)\,dz. \qquad (37)$$

Now from (13), (32), (17) and (35) and observing that, for $l\geq 1$,

$$\frac{\Gamma(1/l)}{lz^{1/l}}-\frac{1}{l}\sum_{\rho}z^{-\rho/l}\Gamma\left(\frac{\rho}{l}\right)=\widetilde{S}_l(z)-E_l(a,y)\ll a^{-1/l}+|E_l(a,y)|$$

we have

$$\begin{aligned}\widetilde{S}_1(z)\,\widetilde{S}_2^2(z)=&\left(\frac{1}{z}-\sum_{\rho}z^{-\rho}\Gamma(\rho)+E_1(a,y)\right)\left(\frac{\sqrt{\pi}}{2z^{1/2}}-\frac{1}{2}\sum_{\rho}z^{-\rho/2}\Gamma\left(\frac{\rho}{2}\right)+E_2(a,y)\right)^2\\ =&\left(\frac{1}{z}-\sum_{\rho}z^{-\rho}\Gamma(\rho)\right)\left(\frac{\sqrt{\pi}}{2z^{1/2}}-\frac{1}{2}\sum_{\rho}z^{-\rho/2}\Gamma\left(\frac{\rho}{2}\right)\right)^2\\ &+O\left(|E_1(a,y)|\,a^{-1}+|E_1(a,y)|\,|E_2(a,y)|^2+|E_1(a,y)|\,|E_2(a,y)|\,a^{-1/2}\right. \qquad (38)\\ &\left.+\,|E_2(a,y)|^2\,a^{-1}+a^{-3/2}\,|E_2(a,y)|\right). \qquad (39)\end{aligned}$$

Not let us consider integers $l, m, r, s \geq 1$. From (14) and (33) we have that

$$\int_{(a)} \left|e^{Nz} z^{-k-1}\right| \left|E_l(a,y)\right|^r \left|E_m(a,y)\right|^s |dz|$$

$$\ll_{l,m,r,s} e^{Na} \left(a^{-k-1+\frac{r+s}{2}} \int_0^a dy + \int_a^\infty y^{-k-1+\frac{r+s}{2}} \log^{2r+2s}\left(\frac{y}{a}\right) dy \right)$$

$$\ll_{l,m,r,s} e^{Na} a^{-k+\frac{r+s}{2}}$$

assuming $k > \frac{r+s}{2}$. So taking $a = 1/N$ from (38) and (39) we can observe that

$$\int_{(1/N)} \left|e^{Nz} z^{-k-1}\right| \left|E_2(1/N,y)\right|^2 \left|E_1(1/N,y)\right| |dz| \ll_k N^{k-3/2},$$

$$N^{1/2} \int_{(1/N)} \left|e^{Nz} z^{-k-1}\right| \left|E_2(1/N,y)\right| \left|E_1(1/N,y)\right| |dz| \ll N^{k-1/2},$$

$$N \int_{(1/N)} \left|e^{Nz} z^{-k-1}\right| \left|E_2(1/N,y)\right|^2 |dz| \ll N^k$$

$$N^{3/2} \int_{(1/N)} \left|e^{Nz} z^{-k-1}\right| \left|E_2(1/N,y)\right| |dz| \ll N^{k+1}$$

and

$$N \int_{(1/N)} \left|e^{Nz} z^{-k-1}\right| \left|E_1(1/N,y)\right| |dz| \ll N^{k+1/2};$$

hence

$$\begin{aligned}\sum_{n\leq N} r_{SP}\left(n\right)\frac{\left(N-n\right)^{k}}{\Gamma\left(k+1\right)} =&\frac{1}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\frac{1}{z}-\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)\right)\\ &\cdot\left(\frac{\sqrt{\pi}}{2z^{1/2}}-\frac{1}{2}\sum_{\rho} z^{-\rho/2}\Gamma\left(\frac{\rho}{2}\right)\right)^{2} dz+O_{k}\left(N^{k+1}\right)\\ =&\frac{1}{8i}\int_{(1/N)} e^{Nz}z^{-k-3}dz+\frac{1}{8i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz\\ &-\frac{1}{4\sqrt{\pi}i}\int_{(1/N)} e^{Nz}z^{-k-5/2}\sum_{\rho} z^{-\rho/2}\Gamma\left(\frac{\rho}{2}\right)dz\\ &+\frac{1}{4\sqrt{\pi}i}\int_{(1/N)} e^{Nz}z^{-k-3/2}\sum_{\rho_1} z^{-\rho_1}\Gamma\left(\rho_1\right)\sum_{\rho_2} z^{-\rho_2/2}\Gamma\left(\frac{\rho_2}{2}\right)dz\\ &+\frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_{\rho_1} z^{-\rho_1/2}\Gamma\left(\frac{\rho_1}{2}\right)\sum_{\rho_2} z^{-\rho_2/2}\Gamma\left(\frac{\rho_2}{2}\right)dz\\ &-\frac{1}{8\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\sum_{\rho_1} z^{-\rho_1}\Gamma\left(\rho_1\right)\sum_{\rho_2} z^{-\rho_2/2}\Gamma\left(\frac{\rho_2}{2}\right)\\ &\cdot\sum_{\rho_3} z^{-\rho_3/2}\Gamma\left(\frac{\rho_3}{2}\right)dz+O_{k}\left(N^{k+1}\right)\\ =&I_1+I_2+I_3+I_4+I_5+I_6+O_{k}\left(N^{k+1}\right),\end{aligned}$$

say.

5.2 Evaluation of I_1

From I_1 we will find the main term. If we put $Nz=s$ we get

$$I_1=\frac{1}{8i}\int_{(1/N)} e^{Nz}z^{-k-3}dz=\frac{N^{k+2}}{8i}\int_{(1)} e^{s}s^{-k-3}ds=\frac{N^{k+2}\pi}{4\Gamma\left(k+3\right)}$$

using (5).

5.3 Evaluation of I_2

We have

$$I_2=\frac{1}{8i}\int_{(1/N)} e^{Nz}z^{-k-2}\sum_{\rho} z^{-\rho}\Gamma\left(\rho\right)dz$$

so we have to study

$$A_2=\left|\sum_{\rho}\Gamma\left(\rho\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-2}\right|\left|z^{-\rho}\right|\left|dz\right|$$

and from Lemma 8 we have the convergence for $k > -1/2$. So we can switch the integral and the series and get

$$I_2 = \frac{1}{8i}\sum_{\rho}\Gamma(\rho)\int_{(1/N)} e^{Nz}z^{-k-2-\rho}dz = \frac{N^{k+1}\pi}{4}\sum_{\rho}N^{\rho}\frac{\Gamma(\rho)}{\Gamma(k+2+\rho)}.$$

5.4 Evaluation of I_3

We have to estimate

$$I_3 = -\frac{1}{4\sqrt{\pi}i}\int_{(1/N)} e^{Nz}z^{-k-5/2}\sum_{\rho}z^{-\rho/2}\Gamma\left(\frac{\rho}{2}\right)dz$$

and, as before, we have to study

$$A_3 = \left|\sum_{\rho}\Gamma\left(\frac{\rho}{2}\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-5/2}\right|\left|z^{-\rho/2}\right||dz|$$

and using again Lemma 8 we have the convergence for $k > -1$. So we have

$$I_3 = -\frac{1}{4\sqrt{\pi}i}\sum_{\rho}\Gamma\left(\frac{\rho}{2}\right)\int_{(1/N)} e^{Nz}z^{-k-5/2-\rho/2}dz = -\frac{N^{k+3/2}\sqrt{\pi}}{2}\sum_{\rho}N^{\rho/2}\frac{\Gamma(\rho/2)}{\Gamma(k+5/2+\rho/2)}.$$

5.5 Evaluation of I_4

We have to evaluate

$$I_4 = \frac{1}{4\sqrt{\pi}i}\int_{(1/N)} e^{Nz}z^{-k-3/2}\sum_{\rho_1}z^{-\rho_1}\Gamma(\rho_1)\sum_{\rho_2}z^{-\rho_2/2}\Gamma\left(\frac{\rho_2}{2}\right)dz$$

so we consider

$$A_{4,1} := \sum_{\rho_1}|\Gamma(\rho_1)|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-3/2}\right|\left|z^{-\rho_1}\right|\left|\sum_{\rho_2}z^{-\rho_2/2}\Gamma\left(\frac{\rho_2}{2}\right)\right||dz|.$$

Assume that $A_{m,n} := \int_{(1/N)}\dots|dz| = \int_{1/N-i\infty}^{1/N+i\infty}\dots|dz|$. Hereafter we will indicate with the symbol $A_{m,n}^{+}$ the integral $\int_{0}^{1/N+i\infty}\dots|dz|$and with $A_{m,n}^{-}$ the integral $\int_{1/N-i\infty}^{0}\dots|dz|$. From (32) we can see that

$$\left|\sum_{\rho_2}z^{-\rho_2/2}\Gamma\left(\frac{\rho_2}{2}\right)\right| = \left|\widetilde{S}_2(z) - \frac{\sqrt{\pi}}{2z^{1/2}} - E_2(1/N,y)\right| \ll N^{1/2} + \frac{1}{|z|^{1/2}} + |E_2(1/N,y)|$$

$$\ll \begin{cases} N, & |y| \leq 1/N \\ N + |z|^{1/2} \log^2 (2N|y|), & |y| > 1/N. \end{cases}$$

Let us consider $y \leq 0$ and, recalling the notation $\rho_j = \beta_j + i\gamma_j$ and assuming $\gamma_1 > 0$ for symmetry, we have to study

$$\begin{aligned} A_{4,1}^{-} \ll & N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{-1/N}^{0} \frac{\exp\left(\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1\right)}{|z|^{k+3/2+\beta_1}} |dz| \\ & + N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{-\infty}^{-1/N} \frac{\exp\left(\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1\right)}{|y|^{k+3/2+\beta_1}} dy \\ & + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{-\infty}^{-1/N} \frac{\exp\left(\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1\right) \log^2 (2N|y|)}{|y|^{k+1+\beta_1}} dy. \end{aligned}$$

Now since $y \leq 0$ we have

$$\arctan(Ny) - \frac{\pi}{2} \leq -\frac{\pi}{2}$$

so from (21) we have

$$\begin{aligned} A_{4,1}^{-} \ll & N^{k+3/2} \sum_{\rho_1 \colon \gamma_1 > 0} N^{\beta} \gamma_1^{\beta_1 - 1/2} e^{-\pi\gamma_1/2} \\ & + N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} e^{-\pi\gamma_1/2} \int_{1/N}^{\infty} \frac{1}{y^{k+3/2+\beta_1}} dy \\ & + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} e^{-\pi\gamma_1/2} \int_{1/N}^{\infty} \frac{\log^2 (2Ny)}{y^{k+1+\beta_1}} dy \end{aligned}$$

and setting $\epsilon > 0$ we observe that $\log^2 (2Ny) \ll y^{\epsilon}$ as $y \to \infty$so

$$A_{4,1}^{-} \ll_k N^{k+5/2}$$

assuming that $k > 0$.

Now let us consider $y > 0$. We have to study

$$A_{4,1}^{+} \ll N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{0}^{1/N} \frac{\exp\left(\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1\right)}{|z|^{k+3/2+\beta_1}} |dz|$$

$$+ \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^{\infty} \exp\left(\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1\right) \frac{N + y^{1/2} \log^2 (2Ny)}{y^{k+3/2+\beta_1}} dy = \mathcal{A}_1 + \mathcal{A}_2$$

say, and we have that

$$\mathcal{A}_1 \ll N \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} e^{-\pi\gamma_1/4} \int_0^{1/N} \frac{1}{|z|^{k+3/2+\beta_1}} dy$$
$$\ll_k N^{k+5/2}$$

and for $\mathcal{A}_2$, taking $Ny = u$ and using the usual trigonometric identity we have

$$\mathcal{A}_2 \ll N^{k+5/2} \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \int_1^{\infty} \frac{\exp\left(-\gamma_1 \arctan\left(1/u\right)\right)}{u^{k+3/2+\beta_1}} du$$

$$+N^{k+1} \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \int_1^{\infty} \frac{\exp\left(-\gamma_1 \arctan\left(1/u\right)\right) \log^2\left(2u\right)}{u^{k+1/2+\beta_1}} du$$

$$\ll_k N^{k+5/2}$$

from Lemma 3, assuming $k > 1/2$.
Now let us consider

$$A_{4,2} = \sum_{\rho_1} \left|\Gamma\left(\rho_1\right)\right| \sum_{\rho_2} \left|\Gamma\left(\frac{\rho_2}{2}\right)\right| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-3/2}\right| \left|z^{-\rho_1}\right| \left|z^{-\rho_2/2}\right| |dz| .$$

We can consider only the cases $\gamma_1, \gamma_2 > 0$ or $\gamma_1 > 0$, $\gamma_2 < 0$, by symmetry. Hereafter we will use the symbol $\hat{A}_{m,n}$ when we consider $A_{m,n}$ with the assumption $\gamma_1, \gamma_2 > 0$ and the symbol $\check{A}_{m,n}$ when we consider $A_{m,n}$ with the assumption $\gamma_1 > 0$, $\gamma_2 < 0$.
Since

$$\arctan\left(Ny\right) - \frac{\pi}{2} \leq -\frac{\pi}{2} \tag{40}$$

we have, for $y \leq 0$,

$$\hat{A}_{4,2}^{-} \ll \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \left(\int_{-\infty}^{0} \frac{dy}{|z|^{k+3/2+\beta_1+\beta_2/2}}\right)$$

$$\ll_k N^{k+2} \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right)$$

$$\ll_k N^{k+2}.$$

Now let us consider $y > 0$. We have

$$\hat{A}_{4,2}^{+} \ll \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \int_0^{1/N} \frac{\exp\left(\left(\gamma_1 + \frac{\gamma_2}{2}\right)\left(\arctan\left(Ny\right) - \frac{\pi}{2}\right)\right)}{|z|^{k+3/2+\beta_1+\beta_2/2}} dy$$

$$+\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\sum_{\rho_2:\,\gamma_2>0}\gamma_2^{\beta_2/2-1/2}\int_{1/N}^{\infty}\frac{\exp\left(\left(\gamma_1+\frac{\gamma_2}{2}\right)\left(\arctan\left(Ny\right)-\frac{\pi}{2}\right)\right)}{y^{k+3/2+\beta_1+\beta_2/2}}dy$$
$$=\mathcal{A}_3+\mathcal{A}_4,$$

say. If $y\in(0,1/N]$ we obviously have $\arctan\left(Ny\right)-\frac{\pi}{2}\leq-\frac{\pi}{4}$ and so

$$\begin{aligned}\mathcal{A}_3\ll_k&\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{4}\gamma_1\right)\sum_{\rho_2:\,\gamma_2>0}\gamma_2^{\beta_2/2-1/2}\exp\left(-\frac{\pi}{8}\gamma_2\right)\int_0^{1/N}N^{k+3/2+\beta_1+\beta_2/2}dy\\ \ll_k&N^{k+2}\end{aligned}$$

For $\mathcal{A}_4$ we can observe that, taking $\arctan\left(1/u\right)=v$,

$$\begin{aligned}\mathcal{A}_4\ll&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}N^{\beta_1}\gamma_1^{\beta_1-1/2}\sum_{\rho_2:\,\gamma_2>0}N^{\beta_2/2}\gamma_2^{\beta_2/2-1/2}\int_1^{\infty}\frac{\exp\left(-\left(\gamma_1+\frac{\gamma_2}{2}\right)\arctan\left(\frac{1}{u}\right)\right)}{u^{k+3/2+\beta_1+\beta_2/2}}du\\
=&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}N^{\beta_1}\gamma_1^{\beta_1-1/2}\sum_{\rho_2:\,\gamma_2>0}N^{\beta_2/2}\gamma_2^{\beta_2/2-1/2}\int_0^{\pi/4}\frac{\exp\left(-\left(\gamma_1+\frac{\gamma_2}{2}\right)v\right)\left(\sin\left(v\right)\right)^{k-1/2+\beta_1+\beta_2/2}}{\left(\cos\left(v\right)\right)^{k+3/2+\beta_1+\beta_2/2}}dv\\
\ll&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}N^{\beta_1}\gamma_1^{\beta_1-1/2}\sum_{\rho_2:\,\gamma_2>0}N^{\beta_2/2}\gamma_2^{\beta_2/2-1/2}\int_0^{\pi/4}\exp\left(-\left(\gamma_1+\frac{\gamma_2}{2}\right)v\right)v^{k-1/2+\beta_1+\beta_2/2}dv\\
=&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}N^{\beta_1}\gamma_1^{\beta_1-1/2}\sum_{\rho_2:\,\gamma_2>0}\frac{N^{\beta_2/2}\gamma_2^{\beta_2/2-1/2}}{\left(\gamma_1+\frac{\gamma_2}{2}\right)^{1/2+k+\beta_1+\beta_2/2}}\int_0^{\pi\left(\gamma_1+\frac{\gamma_2}{2}\right)/4}\exp\left(-w\right)w^{k-1/2+\beta_1+\beta_2/2}dw\\
\asymp_k&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}\sum_{\rho_2:\,\gamma_2>0}N^{\beta_1+\beta_2/2}\frac{\gamma_1^{\beta_1-1/2}\gamma_2^{\beta_2/2-1/2}}{\left(\gamma_1+\frac{\gamma_2}{2}\right)^{k+1/2+\beta_1+\beta_2/2}}\end{aligned}$$

and observing that

$$\frac{\gamma_1^{\beta_1}\gamma_2^{\beta_2/2}}{2}\leq\left(\gamma_1+\frac{\gamma_2}{2}\right)^{\beta_1+\beta_2/2}$$

we get

$$\begin{aligned}\mathcal{A}_4\ll_k&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}\sum_{\rho_2:\,\gamma_2>0}N^{\beta_1+\beta_2/2}\frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}\left(\gamma_1+\frac{\gamma_2}{2}\right)^{k+1/2}}\\
\ll_k&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}\frac{1}{\gamma_1^{k+1}}\sum_{\rho_2:\,0<\gamma_2\leq\gamma_1}\frac{1}{\gamma_2^{1/2}}\\
\ll_k&N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}\frac{\log\left(\gamma_1\right)}{\gamma_1^{k+1/2}}\end{aligned}$$

and so we proved the convergence if $k>1/2$ using well-known density estimates.

Let us consider the case $\gamma_1 > 0$, $\gamma_2 < 0$ and let $y \le 0$. Using again (40) we have to study

$$\check{A}^-_{4,2} \ll \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \int_{-\infty}^{0} \frac{\exp\left(\frac{\gamma_2 \arctan(Ny)}{2} - \frac{\pi|\gamma_2|}{4}\right)}{|z|^{k+3/2+\beta_1+\beta_2/2}} |dz|$$

and using Lemma 3, Lemma 4 and the identity $\arctan(x) + \arctan(1/x) = -\pi/2$, $x < 0$ we have

$$\begin{aligned}
\check{A}^-_{4,2} \ll_k & N^{k+3} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{8}|\gamma_2|\right) \\
& + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \\
& \cdot \int_{-\infty}^{-1/N} \frac{\exp\left(-\frac{|\gamma_2|}{2}\left(\arctan(Ny) + \frac{\pi}{2}\right)\right)}{|y|^{k+3/2+\beta_1+\beta_2/2}} dy \\
\ll_k & N^{k+3} + N^{k+2} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \\
& \cdot \int_{1}^{\infty} \frac{\exp\left(-\frac{|\gamma_2|}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+3/2+\beta_1+\beta_2/2}} du \\
\ll_k & N^{k+3}
\end{aligned}$$

for $k > -1/2$.

If $y > 0$ we have essentially the same situation exchanging the role of γ_1 and γ_2. We have

$$\begin{aligned}
\check{A}^+_{4,2} \ll & \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_{0}^{\infty} \frac{\exp\left(\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1\right)}{|z|^{k+3/2+\beta_1+\beta_2/2}} |dz| \\
\ll_k & N^{k+3} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{4}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \\
& + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_{1/N}^{\infty} \frac{\exp\left(\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1\right)}{y^{k+3/2+\beta_1+\beta_2/2}} dy \\
\ll_k & N^{k+3} + N^{k+2} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_{1}^{\infty} \frac{\exp\left(-\gamma_1 \arctan\left(\frac{1}{u}\right)\right)}{u^{k+3/2+\beta_1+\beta_2/2}} du \\
\ll_k & N^{k+3}.
\end{aligned}$$

So we can switch the integral with the series and get

$$I_4 = \frac{1}{4\sqrt{\pi}i} \sum_{\rho_1} \Gamma(\rho_1) \sum_{\rho_2} \Gamma\left(\frac{\rho_2}{2}\right) \int_{(1/N)} e^{Nz} z^{-k-3/2-\rho_1-\rho_2/2} dz$$

$$= \frac{N^{k+1/2}\sqrt{\pi}}{2} \sum_{\rho_1} \sum_{\rho_2} N^{\rho_1+\rho_2/2} \frac{\Gamma(\rho_1)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma(k+3/2+\rho_1+\rho_2/2)}.$$

5.6 Evaluation of I_5

We have to evaluate

$$I_5 = \frac{1}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho_1} z^{-\rho_1/2} \Gamma\left(\frac{\rho_1}{2}\right) \sum_{\rho_2} z^{-\rho_2/2} \Gamma\left(\frac{\rho_2}{2}\right) dz.$$

We define

$$A_{5,1} := \sum_{\rho_1} \left|\Gamma\left(\frac{\rho_1}{2}\right)\right| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|z^{-\rho_1/2}\right| \left|\sum_{\rho_2} z^{-\rho_2/2} \Gamma\left(\frac{\rho_2}{2}\right)\right| |dz|.$$

By symmetry we can consider only the case $\gamma_1 > 0$. So taking $y \leq 0$ and using the same argument used in I_4 we get

$$\begin{aligned}
A_{5,1}^{-} \ll & N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} e^{-\pi\gamma_1/4} \int_{-1/N}^{0} |z|^{-k-2-\beta_1/2} |dz| \\
& + N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \int_{-\infty}^{-1/N} \frac{\exp\left(\frac{\gamma_1}{2}\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{|y|^{k+2+\beta_1/2}} dy \\
& + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \int_{-\infty}^{-1/N} \frac{\exp\left(\frac{\gamma_1}{2}\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \log^2(2N|y|)}{|y|^{k+3/2+\beta_1/2}} dy \\
\ll_k & N^{k+5/2} + N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} e^{-\pi\gamma_1/4} \int_{1/N}^{\infty} \frac{1}{y^{k+2+\beta_1/2}} \\
& + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} e^{-\pi\gamma_1/4} \int_{1/N}^{\infty} \frac{\log^2(2Ny)}{y^{k+3/2+\beta_1/2}} \\
\ll & N^{k+5/2}.
\end{aligned}$$

assuming $k > -1/2$.

Now let us consider $y > 0$.We have

$$\begin{aligned}
A_{5,1}^{+} \ll & N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} e^{-\pi\gamma_1/8} \int_0^{1/N} |z|^{-k-2-\beta_1/2} |dz| \\
& + N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \int_{1/N}^{\infty} \frac{\exp\left(\frac{\gamma_1}{2}\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{y^{k+2+\beta_1/2}} dy \\
& + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \int_{1/N}^{\infty} \frac{\exp\left(\frac{\gamma_1}{2}\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \log^2(2Ny)}{y^{k+3/2+\beta_1/2}} dy \\
\ll_k & N^{k+5/2} + N^{k+2} \sum_{\rho_1 \colon \gamma_1 > 0} N^{\beta_1/2} \gamma_1^{\beta_1/2-1/2} \int_1^{\infty} \frac{\exp\left(-\frac{\gamma_1}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+2+\beta_1/2}} du \\
& + N^{k+1/2} \sum_{\rho_1 \colon \gamma_1 > 0} N^{\beta_1/2} \gamma_1^{\beta_1/2-1/2} \int_1^{\infty} \frac{\exp\left(-\frac{\gamma_1}{2}\arctan\left(\frac{1}{u}\right)\right) \log^2(2u)}{u^{k+3/2+\beta_1/2}} du \\
\ll_k & N^{k+5/2}
\end{aligned}$$

assuming $k > 0$, from Lemma 3.
Now we consider

$$A_{5,2} := \sum_{\rho_1} \left|\Gamma\left(\frac{\rho_1}{2}\right)\right| \sum_{\rho_2} \left|\Gamma\left(\frac{\rho_2}{2}\right)\right| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-2}\right| \left|z^{-\rho_1/2}\right| \left|z^{-\rho_2/2}\right| |dz| .$$

We may consider only the cases $\gamma_1, \gamma_2 > 0$ or $\gamma_1 > 0$, $\gamma_2 < 0$, by symmetry. Using the same argument used in I_4 we get

$$\hat{A}_{5,2}^{-} \ll \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \exp\left(-\frac{\pi\gamma_1}{4}\right) \sum_{\rho_2 \colon \gamma_2 > 0} \gamma_2^{\beta_1/2-1/2} \exp\left(-\frac{\pi\gamma_1}{4}\right) \int_{-\infty}^{0} |z|^{-k-2-\beta_1/2-\beta_2/2} |dz|$$

$$\ll_k N^{k+3}$$

for $k > -1$. Taking $y > 0$ we get

$$\begin{aligned}
\hat{A}_{5,2}^{+} \ll & \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \exp\left(-\frac{\pi\gamma_1}{8}\right) \sum_{\rho_2 \colon \gamma_2 > 0} \gamma_2^{\beta_1/2-1/2} \exp\left(-\frac{\pi\gamma_1}{8}\right) \int_0^{1/N} |z|^{-k-2-\beta_1/2-\beta_2/2} |dz| \\
& + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \sum_{\rho_2 \colon \gamma_2 > 0} \gamma_2^{\beta_1/2-1/2} \int_{1/N}^{\infty} \frac{\exp\left(\frac{\gamma_1+\gamma_2}{2}\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{y^{k+2+\beta_1/2+\beta_2/2}} dy \\
\ll_k & N^{k+2} + N^{k+1} \sum_{\rho_1 \colon \gamma_1 > 0} N^{\beta_1/2} \gamma_1^{\beta_1/2-1/2} \sum_{\rho_2 \colon \gamma_2 > 0} N^{\beta_2/2} \gamma_2^{\beta_1/2-1/2} \int_1^{\infty} \frac{\exp\left(-\frac{\gamma_1+\gamma_2}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+2+\beta_1/2+\beta_2/2}} du \\
\asymp_k & N^{k+2} + N^{k+1} \sum_{\rho_1 \colon \gamma_1 > 0} N^{\beta_1/2} \gamma_1^{\beta_1/2-1/2} \sum_{\rho_2 \colon \gamma_2 > 0} \frac{N^{\beta_2/2} \gamma_2^{\beta_1/2-1/2}}{\left(\frac{\gamma_1+\gamma_2}{2}\right)^{k+1+\beta_1/2+\beta_2/2}}
\end{aligned}$$

where the last line follows from the same argument used in $\hat{A}_{4,2}^{+}$. So since

$$\left(\frac{\gamma_1}{2}\right)^{\beta_1/2}\left(\frac{\gamma_2}{2}\right)^{\beta_2/2} \leq \left(\frac{\gamma_1+\gamma_2}{2}\right)^{\beta_1/2+\beta_2/2}$$

we get

$$\hat{A}_{5,2}^{+} \ll N^{k+2} + N^{k+1} \sum_{\rho_1 \colon \gamma_1>0} N^{\beta_1/2}\gamma_1^{-1/2} \sum_{\rho_2 \colon \gamma_2>0} \frac{N^{\beta_2/2}\gamma_2^{-1/2}}{\left(\frac{\gamma_1+\gamma_2}{2}\right)^{k+1}}$$

and from the AM-GM inequality we get

$$\hat{A}_{5,2}^{+} \ll_k N^{k+2} + N^{k+1} \sum_{\rho_1 \colon \gamma_1>0} \frac{N^{\beta_1/2}}{\gamma_1^{k/2+1}} \sum_{\rho_2 \colon \gamma_2>0} \frac{N^{\beta_2/2}}{\gamma_2^{k/2+1}}$$

and so the convergence if $k>0$.
Now we have to consider the case $\gamma_1>0$ and $\gamma_2<0$. We have

$$\check{A}_{5,2}^{-} \ll \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2<0} |\gamma_2|^{\beta_2/2-1/2} \int_{-\infty}^{0} \frac{\exp\left(\frac{\gamma_2 \arctan(Ny)}{2} - \frac{\pi|\gamma_2|}{4}\right)}{|z|^{k+2+\beta_1/2+\beta_2/2}} |dz|$$

and using Lemma 3, Lemma 4 and the identity $\arctan(x)+\arctan(1/x) = -\pi/2$, $x<0$ we have

$$\begin{aligned}
\check{A}_{5,2}^{-} \ll_k & N^{k+2} \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1/2-1/2} \exp\left(-\frac{\pi}{8}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2<0} |\gamma_2|^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{8}|\gamma_2|\right) \\
& + \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2<0} |\gamma_2|^{\beta_2/2-1/2} \int_{-\infty}^{-1/N} \frac{\exp\left(-\frac{|\gamma_2|}{2}\left(\arctan(Ny)+\frac{\pi}{2}\right)\right)}{|y|^{k+2+\beta_{1/2}+\beta_2/2}} dy \\
\ll_k & N^{k+2} + N^{k+1} \sum_{\rho_1 \colon \gamma_1>0} N^{\beta_1/2}\gamma_1^{\beta_1/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2<0} N^{\beta_2/2} |\gamma_2|^{\beta_2/2-1/2} \int_{1}^{\infty} \frac{\exp\left(-\frac{|\gamma_2|}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+2+\beta_1/2+\beta_2/2}} \\
\ll_k & N^{k+2}
\end{aligned}$$

for $k>1/2$, from Lemma 3.

Now let us consider $y > 0$. We get

$$\begin{aligned}
\check{A}^{+}_{5,2} &\ll \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_0^\infty \frac{\exp\left(\frac{\gamma_1}{2}\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{|z|^{k+2+\beta_1/2+\beta_2/2}} |dz| \\
&\ll_k N^{k+2} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \exp\left(-\frac{\pi}{8}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \\
&\quad + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_{1/N}^\infty \frac{\exp\left(\frac{\gamma_1}{2}\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{y^{k+2+\beta_1/2+\beta_2/2}} dy \\
&\ll_k N^{k+2} + N^{k+2} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1/2-1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_1^\infty \frac{\exp\left(-\frac{\gamma_1}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+2+\beta_1/2+\beta_2/2}} du \\
&\ll_k N^{k+2}
\end{aligned}$$

from Lemma 3, for $k > 1/2$.
We proved the convergence so we obtain

$$\begin{aligned}
I_5 &= \frac{1}{8\pi i} \sum_{\rho_1} \Gamma\left(\frac{\rho_1}{2}\right) \sum_{\rho_2} \Gamma\left(\frac{\rho_2}{2}\right) \int_{(1/N)} e^{Nz} z^{-k-2-\rho_1/2-\rho_2/2} dz \\
&= \frac{N^{k+1}}{4} \sum_{\rho_1} \sum_{\rho_2} N^{\rho_1/2+\rho_2/2} \frac{\Gamma\left(\frac{\rho_1}{2}\right)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma\left(k+2+\rho_1/2+\rho_2/2\right)}.
\end{aligned}$$

5.7 Evaluation of I_6

We have to evaluate

$$I_6 = \frac{1}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \sum_{\rho_1} z^{-\rho_1} \Gamma(\rho_1) \sum_{\rho_2} z^{-\rho_2/2} \Gamma\left(\frac{\rho_2}{2}\right) \sum_{\rho_3} z^{-\rho_3/2} \Gamma\left(\frac{\rho_3}{2}\right) dz$$

so let us consider

$$A_{6,1} = \sum_{\rho_1} |\Gamma(\rho_1)| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-1}\right| \left|z^{-\rho_1}\right| \left|\sum_{\rho_2} z^{-\rho_2/2} \Gamma\left(\frac{\rho_2}{2}\right)\right| \left|\sum_{\rho_3} z^{-\rho_3/2} \Gamma\left(\frac{\rho_3}{2}\right)\right| |dz|,$$

and we assume, by symmetry, that $\gamma_1 > 0$. Let $y \le 0$. From (40) we have that

$$
\begin{aligned}
A_{6,1}^{-} \ll & N^{k+3}\sum_{\rho_1:\,\gamma_1>0} N^{\beta_1}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\int_{-1/N}^{0}\exp\left(\gamma_1\arctan\left(N\left|y\right|\right)\right)dy\\
& +\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\\
& \cdot\int_{-\infty}^{-1/N}\left|z\right|^{-k-1-\beta_1}\exp\left(\gamma_1\arctan\left(N\left|y\right|\right)\right)\left(N+\left|z\right|^{1/2}\log^2\left(2N\left|y\right|\right)\right)^2 dy\\
\ll & N^{k+3}+N^2\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\int_{-\infty}^{-1/N}\left|y\right|^{-k-1-\beta_1}dy\\
& +2N\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\int_{-\infty}^{-1/N}\left|y\right|^{-k-1/2-\beta_1}\log^2\left(2N\left|y\right|\right)dy\\
& +\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\int_{-\infty}^{-1/N}\left|y\right|^{-k-\beta_1}\log^4\left(2N\left|y\right|\right)dy\\
\ll_k & N^{k+3}
\end{aligned}
$$

for $k>1$.

Let $y>0$. We have

$$
\begin{aligned}
A_{6,1}^{+} \ll & N^2\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\int_{0}^{1/N}\exp\left(\gamma_1\arctan\left(Ny\right)-\frac{\pi}{2}\gamma_1\right)\frac{\left|dz\right|}{\left|z\right|^{k+1+\beta_1}}\\
& +\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\int_{1/N}^{\infty}\frac{\exp\left(\gamma_1\arctan\left(Ny\right)-\frac{\pi}{2}\gamma_1\right)\left(N+\left|z\right|^{1/2}\log^2\left(2N\left|y\right|\right)\right)^2}{\left|z\right|^{k+1+\beta_1}}dy.
\end{aligned}
$$

From Lemma 4 we have

$$
\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\int_{0}^{1/N}\exp\left(\gamma_1\arctan\left(Ny\right)-\frac{\pi}{2}\gamma_1\right)\frac{\left|dz\right|}{\left|z\right|^{k+1+\beta_1}}\ll_k N^{k+1}
$$

for $k>0$ so

$$
\begin{aligned}
A_{6,1}^{+} \ll & N^{k+3}+\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\int_{1/N}^{\infty}\frac{\exp\left(\gamma_1\arctan\left(Ny\right)-\frac{\pi}{2}\gamma_1\right)\left(N+\left|z\right|^{1/2}\log^2\left(2N\left|y\right|\right)\right)^2}{\left|z\right|^{k+1+\beta_1}}dy\\
\ll & N^{k+3}+N^2\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\int_{1/N}^{\infty}\exp\left(\gamma_1\arctan\left(Ny\right)-\frac{\pi}{2}\gamma_1\right)y^{-k-1-\beta_1}dy\\
& +2N\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\int_{1/N}^{\infty}\exp\left(\gamma_1\arctan\left(Ny\right)-\frac{\pi}{2}\gamma_1\right)\frac{\log^2\left(2Ny\right)}{y^{k+1/2+\beta_1}}dy\\
& +\sum_{\rho_1:\,\gamma_1>0}\gamma_1^{\beta_1-1/2}\int_{1/N}^{\infty}\exp\left(\gamma_1\arctan\left(Ny\right)-\frac{\pi}{2}\gamma_1\right)\frac{\log^4\left(2Ny\right)}{y^{k+\beta_1}}d
\end{aligned}
$$

and using the well known identity $\arctan\left(x\right)-\frac{\pi}{2}=-\arctan\left(\frac{1}{x}\right)$, $x>0$ and placing $Ny=u$ we get

$$
\begin{aligned}
A_{6,1}^{+} \ll & N^{k+3}+N^{k+2}\sum_{\rho_1:\,\gamma_1>0}N^{\beta_1}\gamma_1^{\beta_1-1/2}\int_{1}^{\infty}\exp\left(-\gamma_1\arctan\left(\frac{1}{u}\right)\right)u^{-k-1-\beta_1}dy\\
& +2N^{k+1/2}\sum_{\rho_1:\,\gamma_1>0}N^{\beta_1}\gamma_1^{\beta_1-1/2}\int_{1}^{\infty}\exp\left(-\gamma_1\arctan\left(\frac{1}{u}\right)\right)\frac{\log^2\left(2u\right)}{u^{k+1/2+\beta_1}}dy\\
& +N^{k-1}\sum_{\rho_1:\,\gamma_1>0}N^{\beta_1}\gamma_1^{\beta_1-1/2}\int_{1}^{\infty}\exp\left(-\gamma_1\arctan\left(\frac{1}{u}\right)\right)\frac{\log^4\left(2u\right)}{u^{k+\beta_1}}dy\\
\ll_k & N^{k+3}
\end{aligned}
$$

from Lemma 3, assuming $k>3/2$.
Now we have to study

$$
A_{6,2}=\sum_{\rho_1}\left|\Gamma\left(\rho_1\right)\right|\sum_{\rho_2}\left|\Gamma\left(\frac{\rho_2}{2}\right)\right|\int_{(1/N)}\left|e^{Nz}\right|\left|z^{-k-1}\right|\left|z^{-\rho_1}\right|\left|z^{-\rho_2/2}\right|\left|\sum_{\rho_3}z^{-\rho_3/2}\Gamma\left(\frac{\rho_3}{2}\right)\right|\left|dz\right|
$$

and, by symmetry, we can consider the cases $\gamma_1,\gamma_2>0$ or $\gamma_1>0$, $\gamma_2<0$. Let $\gamma_1,\gamma_2>0$ and $y\leq 0$. From (40) we have

$$
\begin{aligned}
\hat{A}_{6,2}^{-} \ll & N \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \int_{-1/N}^{0} \frac{|dz|}{|z|^{k+1+\beta_1+\beta_2/2}} \\
& + \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \int_{-\infty}^{-1/N} \frac{N + |y|^{1/2} \log^2(2N|y|)}{|y|^{k+1+\beta_1+\beta_2/2}} dy \\
\ll & N^{k+3} + \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \int_{-\infty}^{-1/N} \frac{1}{|y|^{k+1+\beta_1+\beta_2/2}} dy \\
& + \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \int_{-\infty}^{-1/N} \frac{\log^2(2N|y|)}{|y|^{k+1/2+\beta_1+\beta_2/2}} dy \\
\ll_k & N^{k+3}
\end{aligned}
$$

for $k > 1/2$.

Let $y > 0$. We have, using again (16) and (18), that

$$
\begin{aligned}
\hat{A}_{6,2}^{+} \ll & N \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \int_{0}^{1/N} \frac{\exp\left(\left(\gamma_1 + \frac{\gamma_2}{2}\right)\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2}} |dz| \\
& + \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \\
& \cdot \int_{1/N}^{\infty} \exp\left(\left(\gamma_1 + \frac{\gamma_2}{2}\right)\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \frac{\left(N + y^{1/2}\log^2(2Ny)\right)}{y^{k+1+\beta_1+\beta_2/2}} dy \\
\ll & N \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\gamma_1}{4}\right) \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\gamma_2}{8}\right) \int_{0}^{1/N} N^{k+1+\beta_1+\beta_2/2} dy \\
& + \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \\
& \cdot \int_{1/N}^{\infty} \exp\left(\left(\gamma_1 + \frac{\gamma_2}{2}\right)\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \frac{\left(N + y^{1/2}\log^2(2Ny)\right)}{y^{k+1+\beta_1+\beta_2/2}} d \\
\ll & N^{k+3} + N \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \int_{1/N}^{\infty} \frac{\exp\left(\left(\gamma_1 + \frac{\gamma_2}{2}\right)\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{y^{k+1+\beta_1+\beta_2/2}} dy \\
& + \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \int_{1/N}^{\infty} \frac{\exp\left(\left(\gamma_1 + \frac{\gamma_2}{2}\right)\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)\log^2(2Ny)}{y^{k+1/2+\beta_1+\beta_2/2}} dy
\end{aligned}
$$

and again from $\arctan(x) - \frac{\pi}{2} = -\arctan\left(\frac{1}{x}\right)$ and placing $Ny = u$ we get

$$
\hat{A}_{6,2}^{+} \ll N^{k+3} \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2:\,\gamma_2>0} \gamma_2^{\beta_2/2-1/2} \int_{1}^{\infty} \frac{\exp\left(-\left(\gamma_1 + \frac{\gamma_2}{2}\right)\arctan\left(\frac{1}{u}\right)\right)}{u^{k+1+\beta_1+\beta_2/2}} du
$$

$$+N^{k+3/2}\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\sum_{\rho_2\colon\gamma_2>0}\gamma_2^{\beta_2/2-1/2}\int_1^{\infty}\frac{\exp\left(-\left(\gamma_1+\frac{\gamma_2}{2}\right)\arctan\left(\frac{1}{u}\right)\right)\log^2(2u)}{u^{k+1/2+\beta_1+\beta_2/2}}du$$

and from the proof of Lemma 3 we have

$$\hat{A}^+_{6,2}\ll_k N^{k+3}\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\sum_{\rho_2\colon\gamma_2>0}\gamma_2^{\beta_2/2-1/2}\left(\gamma_1+\frac{\gamma_2}{2}\right)^{-k+1/2-\beta_1-\beta_2/2}$$

and observing that

$$\gamma_1^{\beta_1}\left(\frac{\gamma_2}{2}\right)^{\beta_2/2}\leq\left(\gamma_1+\frac{\gamma_2}{2}\right)^{\beta_1}\left(\gamma_1+\frac{\gamma_2}{2}\right)^{\beta_2/2}=\left(\gamma_1+\frac{\gamma_2}{2}\right)^{\beta_1/2+\beta_2/2}$$

we get

$$\begin{aligned}\hat{A}^+_{6,2}&\ll_k N^{k+3}\sum_{\rho_1\colon\gamma_1>0}\sum_{\rho_2\colon\gamma_2>0}\frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}(\gamma_1+\gamma_2)^{k-1/2}}\\&\ll_k N^{k+3}\sum_{\rho_1\colon\gamma_1>0}\frac{1}{\gamma_1^k}\sum_{\rho_2\colon 0<\gamma_2\leq\gamma_1}\frac{1}{\gamma_2^{1/2}}\\&\ll_k N^{k+3}\sum_{\rho_1\colon\gamma_1>0}\frac{\log(\gamma_1)}{\gamma_1^{k-1/2}}\end{aligned}$$

and so the convergence if $k>3/2$.

Let us assume that $\gamma_1>0$, $\gamma_2<0$ and $y\leq 0$. From (40), (18) and (16) we have

$$\begin{aligned}\check{A}^-_{6,2}\ll&\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\sum_{\rho_2\colon\gamma_2<0}|\gamma_2|^{\beta_2/2-1/2}\int_{-1/N}^{0}\frac{\exp\left(-\frac{|\gamma_2|}{2}\left(\arctan(Ny)+\frac{\pi}{2}\right)\right)|dz|}{|z|^{k+1+\beta_1+\beta_2/2}}\\&+N\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\sum_{\rho_2\colon\gamma_2<0}|\gamma_2|^{\beta_2/2-1/2}\int_{-\infty}^{-1/N}\frac{\exp\left(-\frac{|\gamma_2|}{2}\left(\arctan(Ny)+\frac{\pi}{2}\right)\right)|dz|}{|z|^{k+1+\beta_1+\beta_2/2}}\\&+\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\sum_{\rho_2\colon\gamma_2<0}|\gamma_2|^{\beta_2/2-1/2}\\&\cdot\int_{-\infty}^{-1/N}\frac{\exp\left(-\frac{|\gamma_2|}{2}\left(\arctan(Ny)+\frac{\pi}{2}\right)\right)\log^2(2N|y|)|dz|}{|z|^{k+1/2+\beta_1+\beta_2/2}}\\\ll&N^{k+3/2}\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\sum_{\rho_2\colon\gamma_2<0}|\gamma_2|^{\beta_2/2-1/2}\exp\left(-\frac{\pi}{8}|\gamma_2|\right)\\&+N^{k+5/2}\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\sum_{\rho_2}|\gamma_2|^{\beta_2/2-1/2}\int_1^{\infty}\frac{\exp\left(-\frac{|\gamma_2|}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+1+\beta_1+\beta_2/2}}du\\&+\sum_{\rho_1\colon\gamma_1>0}\gamma_1^{\beta_1-1/2}\exp\left(-\frac{\pi}{2}\gamma_1\right)\sum_{\rho_2\colon\gamma_2<0}|\gamma_2|^{\beta_2/2-1/2}\int_1^{\infty}\frac{\exp\left(-\frac{|\gamma_2|}{2}\arctan\left(\frac{1}{u}\right)\right)\log^2(2u)}{u^{k+1/2+\beta_1+\beta_2/2}}du\end{aligned}$$

where $u = Ny$. So by Lemma 3 we have, for $k > 1$, that

$$\breve{A}_{6,2}^{-} \ll_k N^{k+5/2}$$

If $y > 0$ we have essentially the same calculations exchanging the role of γ_1 and γ_2. We have

$$\begin{aligned}
\breve{A}_{6,2}^{+} \ll & \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_0^{1/N} \frac{\exp\left(\gamma_1 \left(\arctan(Ny) - \frac{\pi}{2}\right)\right)|dz|}{|z|^{k+1+\beta_1+\beta_2/2}} \\
& + N \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_{1/N}^{\infty} \frac{\exp\left(\gamma_1 \left(\arctan(Ny) - \frac{\pi}{2}\right)\right) dy}{y^{k+1+\beta_1+\beta_2/2}} \\
& + \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_{1/N}^{\infty} \frac{\exp\left(\gamma_1 \left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \log^2(2Ny)\, dy}{y^{k+1+\beta_1+\beta_2/2}} \\
& \ll N^{k+3/2} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{4}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \\
& + N^{k+5/2} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_1^{\infty} \frac{\exp\left(-\gamma_1 \arctan\left(\frac{1}{u}\right)\right)}{u^{k+1+\beta_1+\beta_2/2}} du \\
& + N^{k+2} \sum_{\rho_1 \colon \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 \colon \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_1^{\infty} \frac{\exp\left(-\gamma_1 \arctan\left(\frac{1}{u}\right)\right) \log^2(2u)}{u^{k+1/2+\beta_1+\beta_2/2}} du \\
& \ll_k N^{k+5/2}
\end{aligned}$$

from Lemma 3 for $k > 1$.
Now we have to consider

$$A_{6,3} = \sum_{\rho_1} |\Gamma(\rho_1)| \sum_{\rho_2} \left|\Gamma\left(\frac{\rho_2}{2}\right)\right| \sum_{\rho_3} \left|\Gamma\left(\frac{\rho_3}{2}\right)\right| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-1}\right| \left|z^{-\rho_1}\right| \left|z^{-\rho_2/2}\right| \left|z^{-\rho_3/2}\right| |dz| .$$

It is sufficient to consider the cases $\gamma_i > 0$, $i = 1, 2, 3$, $\gamma_1, \gamma_2 > 0$ and $\gamma_3 < 0$ and lastly $\gamma_1 > 0$, $\gamma_2, \gamma_3 < 0$. We will use the symbol $\bar{A}_{6,3}$ when we consider $A_{6,3}$ with the assumption $\gamma_i > 0$, $i = 1, 2, 3$, the symbol $\mathring{A}_{6,3}$ when we consider $A_{6,3}$ with the assumption $\gamma_1, \gamma_2 > 0$ and $\gamma_3 < 0$ and $\underline{A}_{6,3}$ when we consider $A_{6,3}$ with the assumption $\gamma_1 > 0$, $\gamma_2, \gamma_3 < 0$. From (40) we have

$$\begin{aligned}\bar{A}_{6,3}^{-} \ll & \sum_{\rho_1 :\, \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 :\, \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \sum_{\rho_3 :\, \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_3\right) \\ & \cdot \int_{-1/N}^{0} N^{k+1+\beta_1+\beta_2/2+\beta_3/2} dy \\ & + \sum_{\rho_1 :\, \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 :\, \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \sum_{\rho_3 :\, \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_3\right) \\ & \cdot \int_{-\infty}^{-1/N} |y|^{-k-1-\beta_1-\beta_2/2-\beta_3/2}\, dy \\ \ll_k & N^{k+2} + \sum_{\rho_1 :\, \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 :\, \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \sum_{\rho_3 :\, \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_3\right) \\ & \cdot \int_{1/N}^{\infty} y^{-k-1-\beta_1-\beta_2/2-\beta_3/2} dy \\ \ll_k & N^{k+2}\end{aligned}$$

for $k > 1$.

Let $y > 0$. From (18) and (19) we have

$$\begin{aligned}\bar{A}_{6,3}^{+} \ll & \sum_{\rho_1 :\, \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 :\, \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \sum_{\rho_3 :\, \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \\ & \cdot \int_{0}^{1/N} \exp\left(\left(\gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2}\right)\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) N^{k+1+\beta_1+\beta_2/2+\beta_3/2} dy \\ & + \sum_{\rho_1 :\, \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 :\, \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \sum_{\rho_3 :\, \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \\ & \cdot \int_{1/N}^{\infty} \frac{\exp\left(\left(\gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2}\right)\left(\arctan(Ny) - \frac{\pi}{2}\right)\right)}{y^{k+1+\beta_1+\beta_2/2+\beta_3/2}} dy \\ \ll & N^{k+2} \sum_{\rho_1 :\, \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi\gamma_1}{4}\right) \sum_{\rho_2 :\, \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi\gamma_2}{8}\right) \\ & \cdot \sum_{\rho_3 :\, \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \exp\left(-\frac{\pi\gamma_2}{8}\right) \\ & + N^{k+2} \sum_{\rho_1 :\, \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 :\, \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \\ & \cdot \sum_{\rho_3 :\, \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \int_{1}^{\infty} \frac{\exp\left(-\left(\gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2}\right)\arctan\left(\frac{1}{u}\right)\right)}{u^{k+1+\beta_1+\beta_2/2+\beta_3/2}} du\end{aligned}$$

and from the proof of the Lemma 3 we get

$$\bar{A}_{6,3}^{+} \ll N^{k+2} + N^{k+2} \sum_{\rho_1\colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2\colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \cdot \sum_{\rho_3\colon \gamma_3>0} \gamma_3^{\beta_3/2-1/2} \left(\gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2}\right)^{-k-\beta_1-\beta_2/2-\beta_3/2}$$

and observing that

$$\frac{\gamma_1^{\beta_1}\gamma_2^{\beta_2/2}\gamma_3^{\beta_3/2}}{2} \leq \left(\gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2}\right)^{\beta_1+\beta_2/2+\beta_3/2}$$

we get

$$\bar{A}_{6,3}^{+} \ll_k N^{k+2} + N^{k+2} \sum_{\rho_1\colon \gamma_1>0} \sum_{\rho_2\colon \gamma_2>0} \sum_{\rho_3\colon \gamma_3>0} \frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}\gamma_3^{1/2}\left(\gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2}\right)^k}$$

and from AM-GM inequality we get

$$\begin{aligned}\bar{A}_{6,3}^{+} \ll & N^{k+2} + N^{k+2} \sum_{\rho_1\colon \gamma_1>0} \frac{1}{\gamma_1^{k/3+1/2}} \sum_{\rho_2\colon \gamma_2>0} \frac{1}{\gamma_2^{k/3+1/2}} \sum_{\rho_3\colon \gamma_3>0} \frac{1}{\gamma_3^{k/3+1/2}} \\ \ll_k & N^{k+2}\end{aligned}$$

for $k > 3/2$.

Let $\gamma_1, \gamma_2 > 0$, $\gamma_3 < 0$ and $y \leq 0$. From (40) we have

$$\begin{aligned}\mathring{A}_{6,3}^{-} \ll & \sum_{\rho_1\colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2\colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \sum_{\rho_3\colon \gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \\ & \cdot \int_{-1/N}^{0} \frac{\exp\left(-\frac{|\gamma_3|}{2}\left(\arctan\left(Ny\right) + \frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} |dz| \\ & + \sum_{\rho_1\colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2\colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \sum_{\rho_3\colon \gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \\ & \cdot \int_{-\infty}^{-1/N} \frac{\exp\left(-\frac{|\gamma_3|}{2}\left(\arctan\left(Ny\right) + \frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} |dz| \\ \ll_k & N^{k+2} + N^{k+2} \sum_{\rho_1\colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2\colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \sum_{\rho_3\colon \gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \\ & \cdot \int_{1}^{\infty} \frac{\exp\left(-\frac{|\gamma_3|}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+1+\beta_1+\beta_2/2+\beta_3/2}} du\end{aligned}$$

and from the proof of Lemma 3 we get

$$\mathring{A}^{-}_{6,3} \ll_k N^{k+2} + \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{4}\gamma_2\right) \sum_{\rho_3 \colon \gamma_3<0} |\gamma_3|^{-k-1/2-\beta_1-\beta_2/2}$$
$$\ll_k N^{k+2}$$

for $k > 1/2$.
If $y > 0$ we have essentially the same calculations exchanging the role of γ_2,γ_1 and γ_3. We get

$$\begin{aligned}
\mathring{A}^{+}_{6,3} \ll & \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 \colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \sum_{\rho_3 \colon \gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{4}|\gamma_3|\right) \\
& \cdot \int_0^{1/N} \frac{\exp\left(\frac{\gamma_2+2\gamma_1}{2}\left(\arctan(Ny)-\frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} |dz| \\
& + \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 \colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \sum_{\rho_3 \colon \gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{4}|\gamma_3|\right) \\
& \cdot \int_{1/N}^{\infty} \frac{\exp\left(\frac{\gamma_2+2\gamma_1}{2}\left(\arctan(Ny)-\frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} |dz| \\
\ll_k & N^{k+2} + N^{k+2} \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 \colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \sum_{\rho_3 \colon \gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{4}|\gamma_3|\right) \\
& \cdot \int_1^{\infty} \frac{\exp\left(-\frac{\gamma_2+2\gamma_1}{2}\arctan\left(\frac{1}{u}\right)\right)}{u^{k+1+\beta_1+\beta_2/2+\beta_3/2}} du \\
\ll_k & N^{k+2} + \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 \colon \gamma_2>0} \gamma_2^{\beta_2/2-1/2} \\
& \cdot \sum_{\rho_3 \colon \gamma_3<0} |\gamma_3|^{-k-1/2-\beta_1-\beta_2/2} \exp\left(-\frac{\pi}{4}|\gamma_3|\right) \left(\frac{\gamma_2+2\gamma_1}{2}\right)^{-k-\beta_1-\beta_2/2-\beta_3/2}
\end{aligned}$$

again from the proof of Lemma 3 and now using again the AM-GM inequality and using the the bounds $0 < \beta_i < 1$, $i = 1, 2, 3$ we get

$$\begin{aligned}
\mathring{A}^{+}_{6,3} \ll_k & N^{k+2} \\
& + \sum_{\rho_1 \colon \gamma_1>0} \sum_{\rho_2 \colon \gamma_2>0} \sum_{\rho_3 \colon \gamma_3<0} \gamma_1^{-k-1/2} \gamma_2^{-k-1/2} |\gamma_3|^{-k-1/2} \exp\left(-\frac{\pi}{4}|\gamma_3|\right) \\
\ll_k & N^{k+2}.
\end{aligned}$$

Let $\gamma_2, \gamma_2 < 0$, $\gamma_1 > 0$ and $y < 0$. We have

$$
\begin{aligned}
\underline{A}_{6,3}^{-} \ll & \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2<0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3:\,\gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \\
& \cdot \int_{-1/N}^{0} \frac{\exp\left(-\frac{|\gamma_2|+|\gamma_3|}{2}\left(\arctan\left(Ny\right)+\frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}}\,|dz| \\
& + \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2<0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3:\,\gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \\
& \cdot \int_{-\infty}^{-1/N} \frac{\exp\left(-\left(\frac{|\gamma_2|+|\gamma_3|}{2}\right)\left(\arctan\left(Ny\right)+\frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}}\,|dz| \\
\ll & N^{k+2} + \sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \sum_{\rho_2:\,\gamma_2<0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3:\,\gamma_3<0} |\gamma_3|^{\beta_3/2-1/2} \\
& \cdot \int_{1}^{\infty} \frac{\exp\left(-\left(\frac{|\gamma_2|+|\gamma_3|}{2}\right)\arctan\left(\frac{1}{u}\right)\right)}{u^{k+1+\beta_1+\beta_2/2+\beta_3/2}}du \\
\ll_k & N^{k+2} \sum_{\rho_2:\,\gamma_2<0} \sum_{\rho_3:\,\gamma_3<0} |\gamma_2|^{\beta_2/2-1/2} |\gamma_3|^{\beta_3/2-1/2} \left(|\gamma_2|+|\gamma_3|\right)^{-k-\beta_2/2-\beta_3/2} \\
\ll_k & N^{k+2} \sum_{\rho_2:\,\gamma_2<0} |\gamma_2|^{-k-1/2} \sum_{\rho_3:\,\gamma_3<0} |\gamma_3|^{-k-1/2} \\
\ll_k & N^{k+2}
\end{aligned}
$$

using Lemma 3, for $k>1/2$.

Let $y>0$. Observing that

$$
-\left(\frac{|\gamma_2|+|\gamma_3|}{2}\right)\left(\arctan\left(Ny\right)+\frac{\pi}{2}\right) \leq -\left(\frac{|\gamma_2|+|\gamma_3|}{2}\right)\frac{\pi}{2}
$$

we have

$$\begin{aligned}\underline{A}_{6,3}^{+} \ll & \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \exp\left(-\frac{\pi}{4}\gamma_1\right) \sum_{\rho_2 \colon \gamma_2<0} \left|\gamma_2\right|^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{8}\left|\gamma_2\right|\right) \sum_{\rho_3 \colon \gamma_3<0} \left|\gamma_3\right|^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{8}\left|\gamma_3\right|\right) \\ & \cdot \int_0^{1/N} \frac{|dz|}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} \\ & + \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2 \colon \gamma_2<0} \left|\gamma_2\right|^{\beta_2/2-1/2} \exp\left(-\frac{\pi}{8}\left|\gamma_2\right|\right) \sum_{\rho_3 \colon \gamma_3<0} \left|\gamma_3\right|^{\beta_3/2-1/2} \exp\left(-\frac{\pi}{8}\left|\gamma_3\right|\right) \\ & \cdot \int_{1/N}^{\infty} \frac{\exp\left(\gamma_1\left(\arctan\left(Ny\right)-\frac{\pi}{2}\right)\right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} dy \\ & \ll_k N^{k+2} + N^{k+2} \sum_{\rho_1 \colon \gamma_1>0} \gamma_1^{\beta_1-1/2} \int_{1/N}^{\infty} \frac{\exp\left(-\gamma_1 \arctan\left(\frac{1}{u}\right)\right)}{|z|^{k+1+\beta_1}} du \\ & \ll_k N^{k+2}\end{aligned}$$

from Lemma 3 for $k > 1/2$.

Now we can exchange the integral with the series and get

$$\begin{aligned} I_6 &= \frac{1}{8\pi i} \sum_{\rho_1} \Gamma\left(\rho_1\right) \sum_{\rho_2} \Gamma\left(\frac{\rho_2}{2}\right) \sum_{\rho_3} \Gamma\left(\frac{\rho_3}{2}\right) \int_{(1/N)} e^{Nz} z^{-k-1-\rho_1-\rho_2/2-\rho_3/2} dz \\ &= \frac{N^k}{4} \sum_{\rho_1} \sum_{\rho_2} \sum_{\rho_3} \frac{N^{\rho_1+\rho_2/2+\rho_3/2} \Gamma\left(\rho_1\right) \Gamma\left(\frac{\rho_2}{2}\right) \Gamma\left(\frac{\rho_3}{2}\right)}{\Gamma\left(k+\rho_1+\rho_2/2+\rho_3/2\right)}. \end{aligned}$$

We have proved the following

Theorem 2. *Let N be a sufficient large integer. We have*

$$\sum_{n \leq N} r_{SP}\left(n\right) \frac{\left(N-n\right)^k}{\Gamma\left(k+1\right)} = M_1\left(N,k\right) + M_2\left(N,k\right) + M_3\left(N,k\right) + M_4\left(N,k\right) + O\left(N^{k+1}\right)$$

where

$$M_1(N,k) = \frac{N^{k+2}\pi}{4\Gamma(k+3)} \tag{41}$$

$$M_2(N,k) = \frac{N^{k+1}\pi}{4}\sum_{\rho} N^{\rho}\frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} - \frac{N^{k+3/2}\sqrt{\pi}}{2}\sum_{\rho} N^{\rho/2}\frac{\Gamma(\rho/2)}{\Gamma(k+5/2+\rho/2)} \tag{42}$$

$$\begin{aligned} M_3(N,k) =& \frac{N^{k+1/2}\sqrt{\pi}}{2}\sum_{\rho_1}\sum_{\rho_2} N^{\rho_1+\rho_2/2}\frac{\Gamma(\rho_1)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma(k+3/2+\rho_1+\rho_2/2)} \\ &+ \frac{N^{k+1}}{4}\sum_{\rho_1}\sum_{\rho_2} N^{\rho_1/2+\rho_2/2}\frac{\Gamma\left(\frac{\rho_1}{2}\right)\Gamma\left(\frac{\rho_2}{2}\right)}{\Gamma(k+2+\rho_1/2+\rho_2/2)} \end{aligned} \tag{43}$$

$$M_4(N,k) = \frac{N^k}{4}\sum_{\rho_1}\sum_{\rho_2}\sum_{\rho_3}\frac{N^{\rho_1+\rho_2/2+\rho_3/2}\Gamma(\rho_1)\Gamma\left(\frac{\rho_2}{2}\right)\Gamma\left(\frac{\rho_3}{2}\right)}{\Gamma(k+\rho_1+\rho_2/2+\rho_3/2)}, \tag{44}$$

for $k > 3/2$, where ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. If RH holds $M_4(N,k)$ can be incorporated in the error term.

6 The Circle method

The circle method was introduced in a paper of Hardy and Ramanjuan [14] and it is used to study additive problems. The method allows to turn an arithmetic problem in a problem that can be approached with real and complex analytic tools. To better explain how the method works, we illustrate the proof of the ternary Goldbach problem for sufficiently large numbers by Vinogradov [40]. We will use the Davenport [7] approach. Let be $N > 5$ and odd number and let us consider

$$R_3(N) = \sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3), \tag{45}$$

where

$$\Lambda(n) = \begin{cases} \log(p), & n = p^k \text{ for some integer } k \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

is the Von Mangoltd function. (45) is a weighed counting of the representations of N as sum of three primes powers. So we are not analyzing the original problem but this function is, for technical reason, more tractable and the error from to the original function to (45) is under control. Let us now consider the function

$$S(\alpha) = \sum_{m \leq N} \Lambda(m) e(m\alpha)$$

where $e(n) = e^{2\pi i n}$.
Due to the orthogonality of the complex exponential function, i.e.

$$\int_0^1 e(nx) e(-mx)\, dx = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}$$

we have the fundamental relation

$$R_3(N) = \int_0^1 S^3(\alpha) e(-N\alpha)\, d\alpha$$

which is also the N-th Fourier coefficient of the function $S^3(\alpha)$. It is possible to observe that $|S(\alpha)|$ has some "peaks" when α is near a rational number a/q with a "small" denominator (we will define rigorously these words). So the idea is to dissect the interval $[0, 1]$ in two parts that will call $\mathfrak{M}$ and $\mathfrak{m}$.
So we split the interval $[0, 1]$ using the so call Farey dissection

Definition (Farey fractions of order P). Let $P \geq 1$. The set of the Farey fractions is

$$\mathcal{F}(P) = \left\{ \frac{a}{q} : q \leq P,\ 1 \leq a \leq q,\ (a,q) = 1 \right\}.$$

Let us take $P = \log^B(N)$, where B will be chosen later, $Q = P/N$ and we consider now the following intervals

$$\mathfrak{M}(a,q) = \left[\frac{a}{q} - \frac{1}{Q}, \frac{a}{q} + \frac{1}{Q} \right]$$

where $a/q \in \mathcal{F}(P)$. Note that the these intervals are not overlapping since if we take $a_1/q_1,\ a_2/q_2 \in \mathcal{F}(P)$, $a_1/q_1 \neq a_2/q_2$ we have

$$\left| \frac{a_1}{q_1} - \frac{a_2}{q_2} \right| = \left| \frac{a_1 q_2 - a_2 q_1}{q_1 q_2} \right| \geq \frac{1}{q_1 q_2} \geq \frac{1}{P^2} > \frac{2}{Q} \tag{46}$$

since $\log^B(N) < N/2$ for a sufficiently large N. We define the major arc as

$$\mathfrak{M} = \bigcup_{q=1}^{P} \bigcup_{a=1}^{q}{}^{*} \mathfrak{M}(a,q)$$

where $*$ indicates the condition $(a,q) = 1$. It is not difficult to see that $\mathfrak{M} \subset \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right]$ so we define the minor arc as $\mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M}$.
Since

$$\begin{aligned} \int_0^1 S^3(\alpha)\, e(-N\alpha)\, d\alpha &= \int_{1/Q}^{1+1/Q} S^3(\alpha)\, e(-N\alpha)\, d\alpha \\ &= \int_{\mathfrak{m}} S^3(\alpha)\, e(-N\alpha)\, d\alpha + \int_{\mathfrak{M}} S^3(\alpha)\, e(-N\alpha)\, d\alpha \end{aligned}$$

because $S(\alpha)$ and $e(\alpha)$ have period 1, we will split the domain of integration in two parts: the major arc and the minor arc.

6.1 Major arc

From (46), we have that

$$\int_{\mathfrak{M}} S^3(\alpha)\, e(-N\alpha)\, d\alpha = \sum_{q \leq P} \sum_{a=1}^{q}{}^{*} \int_{\mathfrak{M}(a,q)} S^3(\alpha)\, e(-N\alpha)\, d\alpha \tag{47}$$

where again $*$ indicates the condition $(a,q)=1$. Let us consider $\alpha\in\mathfrak{M}(a,q)$, so $\alpha = a/q+\beta$, $|\beta|\leq 1$. From Davenport [7], pages 146-147, we have that

$$S(\alpha)=\frac{\mu(q)}{\phi(q)}T(\beta)+O\left(N\exp\left(-c\sqrt{\log(N)}\right)\right)$$

where $\mu(q)$ is the Möbius function, $\phi(q)$ is the Euler totient function, $T(\beta)=\sum_{n\leq N}e(n\beta)$ and $c>0$ is a positive constant. Hence

$$\begin{aligned}\int_{\mathfrak{M}(a,q)}S^{3}(\alpha)e(-N\alpha)\,d\alpha=&\frac{\mu(q)}{\phi^{3}(q)}e\left(-\frac{aN}{q}\right)\int_{-1/Q}^{1/Q}T^{3}(\beta)e(-N\beta)\,d\beta\\&+O\left(\frac{N^{3}}{Q}\exp\left(-c\sqrt{\log(N)}\right)\right)\end{aligned}\tag{48}$$

and so from (47)

$$\begin{aligned}\int_{\mathfrak{M}}S^{3}(\alpha)e(-N\alpha)\,d\alpha=&\sum_{q\leq P}\frac{\mu(q)}{\phi^{3}(q)}c_{q}(N)\int_{-1/Q}^{1/Q}T^{3}(\beta)e(-N\beta)\,d\beta\\&+O\left(N^{2}\exp\left(-c_{1}\sqrt{\log(N)}\right)\right)\end{aligned}$$

where $c_{q}(N)=\sum_{a=1}^{q^{*}}e\left(-\frac{aN}{q}\right)$ is the Ramanujan sum (for a reference see [32], page 110) and $c_1>0$ is a constant.
Now we observe that

$$T(\alpha)=\sum_{n\leq N}e(n\alpha)=\begin{cases}\frac{1-e((N+1)\alpha)}{1-e(\alpha)}=e\left(\frac{N\alpha}{2}\right)\frac{\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)}, & \alpha\notin\mathbb{Z}\\ N+1, & \alpha\in\mathbb{Z}\end{cases}$$

so obviously

$$T(\beta)\ll\min\left(N,|\beta|^{-1}\right);$$

hence we can immediately conclude that

$$\int_{1/Q}^{1-1/Q}\left|T^{3}(\beta)\right|d\beta\ll Q^{2}$$

so

$$\int_{-1/Q}^{1/Q} T^3(\beta)\, e(-N\beta)\, d\beta = \int_0^1 T^3(\beta)\, e(-N\beta)\, d\beta$$
$$- \int_{1/Q}^{1-1/Q} T^3(\beta)\, e(-N\beta)\, d\beta$$
$$= \int_0^1 T^3(\beta)\, e(-N\beta)\, d\beta$$
$$+O\left(Q^2\right).$$

Since we have extended the domain of integration to the whole interval $(0,1)$ we can see that

$$\int_0^1 T^3(\beta)\, e(-N\beta)\, d\beta = \sum_{\substack{m_1+m_2+m_3=n \\ m_i \geq 0,\, i=1,2,3}} 1 = \frac{(N-1)(N-2)}{2} = \frac{N^2}{2} + O(N)$$

and since

$$Q = \frac{N}{\log^B (N)}$$

we have that

$$\int_{-1/Q}^{1/Q} T^3(\beta)\, e(-N\beta)\, d\beta = \frac{N^2}{2} + O\left(\frac{N^2}{\log^{2B}(N)}\right).$$

To complete the estimation of (48) we have to study the following sum

$$\sum_{q \leq P} \frac{\mu(q)}{\phi^3(q)} c_q(N) = \sum_{q \geq 1} \frac{\mu(q)}{\phi^3(q)} c_q(N) - \sum_{q > P} \frac{\mu(q)}{\phi^3(q)} c_q(N);$$

now since it is possible to prove that

$$c_q(N) = \sum_{a=1}^{q}{}^{*}\, e\left(\frac{aN}{q}\right) = \mu\left(\frac{q}{(q,N)}\right) \frac{\phi(q)}{\phi(q/(q,N))},$$

we can easily conclude that

$$|c_q(N)| \leq \phi(q)$$

so

$$\sum_{q > P} \frac{\mu(q)}{\phi^3(q)} c_q(N) \ll \sum_{q > P} \frac{1}{\phi^2(q)} \ll \log^{1-B}(N)$$

(for the last inequality see [15], theorem 327). By the Euler product formula (see [1], chap.

11) we can conclude that

$$\sum_{q\geq1}\frac{\mu(q)}{\phi^3(q)}c_q(N)=\prod_{p|N}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right)=\mathfrak{S}_3(N)$$

where $\mathfrak{S}_3(N)$ is the singular series for the ternary Goldbach problem. It is interesting to note that $\mathfrak{S}_3(2n)=0$, which is consistent to the fact that we can not find an even number which is sum of three odd primes. So finally we have

$$\begin{aligned}\int_{\mathfrak{M}}S^3(\alpha)e(-N\alpha)\,d\alpha=&\left(\mathfrak{S}_3(N)+O\left(\log^{1-B}(N)\right)\right)\left(\frac{N^2}{2}+O\left(\frac{N^2}{\log^{2B}(N)}\right)\right)\\&+O\left(N^2\exp\left(-c\sqrt{\log(N)}\right)\right)\\=&\mathfrak{S}_3(N)\frac{N^2}{2}+O\left(N^2\log^{1-B}(N)\right).\end{aligned}$$

6.2 Minor Arc

We have to prove that the order of magnitude of the contribute of the minor arc is smaller that the order of major arc. We have that

$$\left|\int_{\mathfrak{m}}S^3(\alpha)e(-N\alpha)\,d\alpha\right|\leq\max_{\alpha\in\mathfrak{m}}|S(\alpha)|\int_0^1|S(\alpha)|^2\,d\alpha;$$

from the Parseval's identity and the PNT we have

$$\int_0^1|S(\alpha)|^2\,d\alpha=\sum_{m\leq N}\Lambda^2(m)\ll N\log(N)$$

and from Vinogradov's lemma (see [40]) we have

$$S(\alpha)\ll\left(\frac{N}{\sqrt{q}}+N^{4/5}+\sqrt{Nq}\right)\log^4(N)\ll N\log^{4-B/2}(N)$$

hence

$$\int_{\mathfrak{m}}S^3(\alpha)e(-N\alpha)\,d\alpha\ll N^2\log^{5-B/2}(N)$$

so setting $B=2(A+5)$ we finally get

Theorem (Vinogradov). *Let N be a sufficiently large integer. Then, for any fixed $A>0$, we have*

$$R_3(N)=\mathfrak{S}_3(N)\frac{N^2}{2}+O\left(N^2\log^{-A}(N)\right).$$

References

[1] T. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] B. C. Berndt, *Identities Involving the Coefficients of a Class of Dirichlet series. VIII*, Trans. Amer. Math. Soc. **201**, (1975), 247-261.

[3] F. Beukers, *Monodromy of A-hypergeometric functions*, Journal für die reine und angewandte Mathematik, Vol. **2016**, Issue 718, (2014), 183-206.

[4] M. Cantarini, *On the Cesàro average of the "Linnik numbers"*, accepted on Acta Arith.

[5] M. Cantarini, *On the Cesàro average of the numbers that can be written as a sum of a prime and two squares of primes*, submitted.

[6] K. Chandrasekharan and R. Narasimhan, *Hecke's Functional Equation and Arithmetical Identities*, Annals of Math. **74** (1961), 1-23.

[7] H. Davenport, *Multiplicative Number Theory*, third ed., Springer, 2000.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions,* Vol. 1. New York: Krieger, 1981

[9] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of integral transforms*, Vol. 1, McGraw-Hill, 1954.

[10] E. Freitag and R. Busam, *Complex analysis*, second ed., Springer-Verlag, 2009.

[11] D. A. Goldston, *On Hardy and Littlewood's contributions to the Goldbach conjecture*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori 1989), Univ. Salerno, Salerno, 1992, 115-155.

[12] G. H. Hardy and J. E. Littlewood, *Some problems of "Partitio numerorum"; III: On the expression of a number as sum of primes*, Acta Math. **44**, 1923, 1-70.

[13] G. H. Hardy and J. E. Littlewood, *Some problems of "Partitio numerorum"; V: A further contribution to the study of Goldbach's problem*, Proc. London Math. Soc., **22**, 1923, 46-56.

[14] G. H. Hardy and S. Ramanujan, *Asymptotic formula in combinatory analysis,* Proc. London Math. Soc. (2), **17**, 1918, 75-115.

[15] G. H. Hardy and E. M. Wright, *An introduction to the Theory of Numbers*, sixth ed., Oxford Science Publication, 2010.

[16] G. Harman and A. Kumchev, *On sums of squares of primes II*, J. Number Theory, **130** (2010), 1969-2002.

[17] H. Helfgott, *Major arcs for Goldbach's problem*, preprint. Available as http://arxiv.org/abs/1305.2897.

[18] H. Helfgott, *Minor arcs for Goldbach's problem*, preprint. Available as http://arxiv.org/abs/1205.5252.

[19] H. Helfgott, *The ternary Goldbach's conjecture is true*, preprint. Available as http://arxiv.org/abs/1312.7748.

[20] H. Helfgott and D. Platt, *Numerical verification of the ternary Goldbach*, To appear in Exp. Maths. Avaiable as http://arxiv.org/abs/1305.3062.

[21] L. K. Hua, *Some result in additive prime number theory*, Quart. J. Math. Oxford, **9** (1938), 68-80.

[22] J. Kaczorowski, A. Perelli, J. Pintz, *A note on the exceptional set for Goldbach's problem in short intervals*, Monatsh. Math., **116** (3-4) (1993), 275-282.

[23] A. Languasco, *Applications of some exponential sums on prime powers: a survey.* Proceedings of the "Terzo Incontro Italiano di Teoria dei Numeri", Scuola Normale Superiore, Pisa, 21-24 Settembre 2015, Rivista Matematica della Università di Parma, Vol. **7**, Number 1 (2016), 19-37.

[24] A. Languasco and A. Zaccagnini, *A Cesàro Average of Goldbach numbers*, Forum Mathematicum **27**.4 (2015), 1945–1960.

[25] A. Languasco and A. Zaccagnini, *A Cesàro Average of Hardy-Littlewood numbers*, J. Math. Anal. Appl. **401** (2013), 568–577.

[26] A. Languasco and A. Zaccagnini, *Sum of one prime and two squares of primes in short intervals*, J. Number Theory **159** (2016), 45–58.

[27] P. S. Laplace, *Théorie analytique des probabilités*, Courcier, 1812.

[28] L. Leung and M. Liu, *On generalized quadratic equations in three prime variables,* Monatsh. Math., **115** (1993), 133-167.

[29] H. Li, *Sums of one prime and two prime squares*, Acta Arith., **134** (2008), 1-9.

[30] Ju. V. Linnik, *A new proof of the Goldbach-Vinogradov theorem*, Rec. Math. [Mat. Sbornik] N.S. **19** (61) (1946), 3–8, (Russian).

[31] Ju. V. Linnik, *An asymptotic formula in an additive problem of Hardy-Littlewood,* Izv. Akad. Nauk SSSR Ser. Mat., Vol. **24**, Issue 25, (1960), 629–706 (Russian).

[32] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical theory*, Cambridge U. P., 2007.

[33] H. L. Montgomery and R. C. Vaughan, *The exceptional set of the Goldbach's conjecture*, Acta Math., **27**, 1975, 353-370.

[34] A. Perelli, J. Pintz, *On the exceptional set for Goldbach's problem in short intervals*, J. London Math. Soc. (2), **47** (1) (1993), 41-49.

[35] J. Pintz, *Recent results on the Goldbach conjecture,* Elementare und analytische Zahlentheorie, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, 20, Franz Steiner Verlag, Stuttgart 2006, 220-254.

[36] W. Schwarz, *Zur Darstellung von Zahlen durch Summen von Primzahlpotenzen. I. Dastellung hinreichend grosser Zahlen*, J. Reine Angew. Marh., **205** (1960/1961), 21-47.

[37] E. C. Titchmarsh, *The Theory of Functions*, second ed., Oxford U. P., 1988.

[38] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function*, second ed., Oxford U. P., 1986.

[39] I. M. Vinogradov, *Some theorems concerning the theory of primes*, Mat. Sb., N. S. **2**, 1937, 179-195 (Russian).

[40] I. M. Vinogradov, *The representation of an odd number as a sum of three primes*, Dokl. Akad. Nauk SSSR, **15**, 1937, 169-172 (Russian).

[41] M. Wang, *On the sum of a prime and two prime squares*, Acta Math. Sinica (Chin. Ser.), **47** (2004), 845-858.

[42] M. Wang and X. Meng, *The exceptional set in the two prime squares and a prime problem*, Acta Math. Sinica (Eng. Ser.), **22** (2006), 1329-1342.

[43] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, second ed., Cambridge U. P., 1966.

[44] L. Zhao, *The additive problem with one prime and two squares of primes*, J. Number Theory, **135** (2014), 8-27.

[45] A. Zygmund, *Trigonometric series, vol. I*, third ed., Cambridge Mathematical Library, 2002.