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Orthogonal stochastic duality from an algebraic point of view

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Introduction and motivations

About duality

Duality is a key property in the realm of interacting particle systems and more generally in the context of Markov processes. It is the aim of this thesis to explore the mathematical structure of systems with a dual process. In particular, the focus will be algebraic properties that provide (via the concept of symmetry) the explanation of a dual process. Why should one be interested in duality? Besides its own interesting mathematical structure, which will be indeed explained and investigated in this thesis, duality provides substantial information that can be inferred via a dual evolution. The class of processes that have a dual are amenable to several exact computations that find their origin in the simplification due to duality. We remark that duality is a rare property: only a few classes of processes are known to share this property.

The scheme would be to use stochastic duality and self-duality to analyze a complicated system in terms of a simpler one; duality can be used in several ways to simplify the analysis of the process under study. A partial list of examples of simplification includes:

1. *From continuous to discrete.* One can relate - via duality - Markov processes with a continuous state space to Markov processes with a discrete state space. We shall see an example of this simplification when we will discuss, for instance, the duality between the Brownian energy process (an interacting diffusion with state space $\mathbb{R}_+^N$, with $n \in \mathbb{N}$) and the symmetric inclusion process (an interacting particle system with state space $\mathbb{N}^N$).

2. *Absorbing boundaries.* One can put in a duality relation systems with given boundary conditions to simpler dual systems with absorbing boundary conditions. The classical examples (indeed one of the first example of duality known in the literature) is provided by a Brownian motion on the positive real line reflected in zero that is dual to a Brownian motion on $\mathbb{R}_+$ with absorbing boundary condition in zero. We will review this duality, as well as the analogous for random walkers, in Chapter 1.5. Another example, not discussed in this dissertation, is provided by the duality between the Brownian energy process with Ornstein-Uhlenbeck reservoirs and its dual, the symmetric inclusion process with absorbing reservoirs. Remark that while the stationary measure of the Brownian energy process with reservoirs is complicated and characterized by long-range correlations, the

stationary measure of the inclusion process with absorbing boundaries is much simpler, with a fraction of the particles absorbed in one reservoir and the remaining fraction absorbed in the other reservoir.

3. *From many to few.* Self-duality is a special case of duality, where the dual process is an independent copy of the original process. Even in this case the simplification coming from duality is substantial. For instance the n-point (with $n \in \mathbb{N}$) correlation functions can be described via only n dual particles. Thus the problem of computing correlations for an arbitrarily large (possible infinite) system is mapped to a problem for a finite system.

4. *Backward/forward evolution.* Often a duality relation is found between two processes, one of which evolves forward in time, the other evolving backward in time. The classical examples are several of the dualities found in mathematical population genetics models, i.e. the duality with a coalescence process for a sample of a multi-type population. We will review in Chapter 1.5 the duality between the Wright-Fisher diffusion and the block counting process of Kingman's coalescence.

Organization and contribution of this thesis

I shall describe hereafter the content and main results of my thesis. I developed a theory for stochastic duality using orthogonal polynomials as duality functions. The results apply to a large class of Markov processes with a common algebraic structure. As we shall see, it is this mathematical structure that is at the root of the orthogonal dualities.

The thesis is organized as follows. In Chapter 1 we give an introduction to duality theory, which is the main object of this work. Applications and advantages of having a dual process are recalled, as well as a bibliography on the subject. We then briefly review the theory of Markov semigroups and their generators which we will use later on. We go on with the notions of stochastic duality and duality functions. Here we shall present the first original result of this thesis, i.e. how to generate novel duality functions from known duality functions. We end the Chapter with some classical examples well known in the literature.

Chapter 2 is devoted to the description of the interacting particle systems for which we will prove duality. We present three continuous time jump processes (inclusion process, exclusion process, independent walkers), two interacting diffusions (Brownian momentum process, Brownian energy process) and a generalized Kipnis-Marchioro-Presutti model which arises from an instantaneous thermalization limit of the Brownian energy process. These classes of models have applications in different settings, e.g. non-equilibrium statistical mechanics, mathematical population genetics, wealth distribution in economics.

In Chapter 3 lies the second original results of this thesis: we show how dualities and self-duality relations can be achieved using orthogonal polynomials as duality and self-duality functions. We will use classical orthogonal polynomials. The orthogonality relation is with respect to the scalar product defined by the stationary measure of the processes. All the proofs follow the same idea: we write the action of the generators of the processes via explicit relations found using the hypergeometric properties of these polynomials. In particular, we are able to show three self-duality relations and three duality relations. The gain in having orthogonal polynomials is that they form an orthogonal base for a dense subspace of the Hilbert space for which they are orthogonal. Then one can approximate any function of the Hilbert space. We use Appendix A for a review of continuous and discrete hypergeometric polynomials and to show some results needed in Chapter 3.

The next goal is then to classify the orthogonal polynomial dualities into a general *algebraic scheme* recalled in Chapter 5. Before doing that, we use Chapter 4 for a condensed review of Lie algebras: it will be the Lie algebra representation theory that will help us in our aim. We will use that each generator of the process is naturally associated to a Lie algebra. In Chapter 5 we recall the algebraic scheme that allows to fit dualities and self-dualities into an algebraic approach. In this scheme duality arises as a change of representation and self-duality is derived from symmetries of the process generator. We review how this abstract scheme has been implemented to find dualities and self-duality relations with triangular self-duality functions or monomial duality functions.

Chapter 6 is then used to fit the new results of Chapter 3 into the algebraic scheme. This is also a recently developed result where further connections between stochastic duality and Lie algebra representation are established. In particular, we provide two theorems which are then used to retrieve orthogonal dualitities and self-dualities as a change of representation of the underlying Lie algebra. Besides the relations already presented in Chapter 3, this new approach allows to find a self-duality relation for a diffusion process via Bessel functions. We then explicitly find the symmetries that lead to Orthogonal self-duality in the same fashion of Chapter 5.

Last, we use Chapter 7 to introduce two asymmetric versions of the exclusion and inclusion process. For these two models classical duality functions are available and using the theory of Chapter 1 we show how to construct biorthogonal self-duality functions. However, we still lack a complete knowledge of (eventual) orthogonal dualities, possibly $q-$polynomials. Studying asymmetric processes is relevant in the context of non-equilibrium statistical mechanics, however, the extension of a duality relation from a symmetric to an asymmetric process is far from trivial.

Chapter 1

Stochastic duality

1.1 The importance of duality

Duality theory is a powerful tool to deal with Markov processes by which information on a given process can be extracted from another process, its *dual*. The link between the two processes is provided by a set of so-called *duality functions*, i.e. a set of observables that are functions of both processes and whose expectations, with respect to the two randomness, can be placed in a precise relation (provided in Definition 1.3 below).

It is worth to stress the relevance of a duality relation. Duality theory has been used in several contexts, we now give an overview of its implementations. Originally introduced for interacting particle systems in [63] and further developed in [50], the literature on stochastic duality covers nowadays a host of applications. A list of examples of systems that have been analyzed using duality includes: boundary driven and/or bulk driven models of transport [9, 37, 55, 64], heat conduction and derivation of Fourier law [31, 41], diffusive particle systems and their hydrodynamic limit [26, 50], asymmetric interacting particle systems scaling to KPZ equation [5, 10, 13, 22, 24, 35, 57, 58], six vertex models [12, 23], multispecies particle models [6, 7, 46–48], correlation inequalities [33] and mathematical population genetics [4, 15, 53, 66]. In all such different contexts it is used, in a way or another, the core simplifications described in the Introduction.

Besides applications, it is interesting to understand the mathematical structure behind duality. This goes back to classical works by Schütz and collaborators [60, 61], where the connection between stochastic duality and symmetries of quantum spin chains was pointed out. More recently, the works [16, 17, 32, 46, 48] further investigate this framework and provide an algebraic approach to Markov processes with duality starting from a Lie algebra in the symmetric case, and its quantum deformation in the asymmetric one.

The algebraic perspective starts from the hypothesis that the Markov generator is an element of the universal enveloping algebra of a Lie algebra. Then the derivation of a duality relation is based on two structural ideas:

i) duality can be seen as a change of representation of a Lie algebra: more precisely one moves between two equivalent representations and the *intertwiner* of those representations yields the duality function.

ii) self-duality is related to reversibility of the process and the existence of an algebra element that commutes with the generator of the process.

Therefore one has a constructive technique, in which duality functions arise from representation theory. Remarkably, this scheme can also be extended to quantum deformed algebras [16, 17].

Recently, an independent approach has established a connection between stochastic duality and the theory of special functions. In particular the works [8, 11, 28, 56] prove that for a large class of processes self-duality functions are provided by orthogonal polynomials and Bessel functions. In this dissertation, we will extensively investigate this route and fit these new duality and self-duality functions into the algebraic approach described above.

1.2 Preliminaries on semigroups of Markov processes

In this section we provide the minimal background on the theory of Markov processes that is necessary to define stochastic duality in the next section. For this part we follow [65].

We will assume in this section that the state space Ω is finite. This permits to introduce the concepts of Markov semigroups and Markov generators without the need to specify all the necessary domain/regularity properties of these operators. Our aim is rather to introduce them in the simplest possible setting to elucidate their probabilistic interpretation. We refer to [50] for the general theory of Markov processes where the assumption on a finite state space is removed.

The essence is that there is a natural correspondence between the two objects that will be defined now (semigroups and generators) that can be used to characterize a Markov process.

Let Ω be a finite set and $\mathscr{F}(\Omega)$ be the set of all functions $f : \Omega \to \mathbb{R}$.

Definition 1.1 (Markov semigroup). *A family $\{S_t\}_{t\geq 0}$ of linear operators on $\mathscr{F}(\Omega)$ is called Markov semigroup if the following conditions are satisfied:*

1. $S_0 = I$, *the identity operator on* $\mathscr{F}(\Omega)$.
2. $t \to S_t f$ *is right continuous for every* $f \in \mathscr{F}(\Omega)$.
3. $S_{t+s} = S_t S_s$, *for all* $s, t \geq 0$ *and all* $f \in \mathscr{F}(\Omega)$.
4. $S_t 1 = 1$, *for all* $t \geq 0$.
5. $S_t f \geq 0$ *for all non-negative* $f \in \mathscr{F}(\Omega)$.

As explained in [50], the importance of Markov semigroups consists in their correspondence with Markov processes. This correspondence is exhibited in the following theorem.

Theorem 1.1 (Correspondence between semigroups and Markov processes). *Suppose $\{S_t\}_{t\geq 0}$ is a Markov semigroup on $\mathscr{F}(\Omega)$, then there exists a unique Markov process $\{X_t\}_{t\geq 0}$ taking value in Ω such that*

$$S_t f(x) := \mathbb{E}_x f(X_t)$$

for all $f \in \mathscr{F}(\Omega)$, $x \in \Omega$ and $t \geq 0$. $\mathbb{E}_x$ denotes the expectation of the process started in $X_0 = x$.

One can show that, for each Markov semigroups $\{S_t\}_{t\geq 0}$, such a Markov process exists and is unique in distribution, given its initial distribution $P(X_0 \in \cdot)$.

Markov processes can also be characterized by their generators and the following theorem shows how to get the generator from the semigroup

Definition 1.2 (Markov generator). *A linear operator L on $\mathscr{F}(\Omega)$ with domain $\mathcal{D}(L) \subseteq \mathscr{F}(\Omega)$ is a Markov generator for the semigroup $\{S_t\}_{t\geq 0}$ if*

$$Lf = \lim_{t\to 0^+} \frac{S_t f - f}{t} \tag{1.1}$$

where the generator domain $\mathcal{D}(L)$ is the set of function on Ω for which the previous limit exists.

The above limit can formally be written as

$$Lf = \frac{d}{dt} S_t f\big|_{t=0} ,$$

and using Theorem 1.1 the above becomes

$$Lf = \frac{d}{dt}\mathbb{E}_x(f(X_t))\big|_{t=0} . \tag{1.2}$$

The following theorem shows how to get the semigroup from the generator in the finite dimensional context.

Theorem 1.2 (Correspondence between semigroup and generators). *Let Ω be a finite set. Let L be a Markov generator on $\mathscr{F}(\Omega)$ and $(S_t)_{t\geq 0}$ a Markov semigroup on $\mathscr{F}(\Omega)$, then these two object are related in the following way*

$$S_t f := e^{tL} f , \tag{1.3}$$

where the exponential is defined as

$$e^{tL} f = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} L^n \right) f .$$

which makes sense only if L is a bounded operator.

Remark 1.1. The relation (1.3) has to be interpreted as a formal relation when Ω is not a finite set. In particualr it is well defined only when L is a bounded operator. We refer to [50] for the case of a measurable metric space Ω.

The theorem providing the link between Markov generators and Markov semigroups in the general context where the state space $\mathscr{F}(\Omega)$ is the Banach space of continuous function on Ω with the sup norm $\| f \| = \sup_{x\in\Omega} | f(x) |$ is known in functional analysis as the Hille-Yosida theorem (see [50], Chapter 1).

For the sake of completeness we recall the backward and forward Kolmogorov equations. If $f \in \mathcal{D}(L)$, then $S_t f \in \mathcal{D}(L)$ and

$$\frac{d}{dt} S_t f = L S_t f \qquad \text{(backward)} \tag{1.4}$$

$$\frac{d}{dt} S_t f = S_t L f \qquad \text{(forward)} \tag{1.5}$$

1.3 Definitions of duality

Definition 1.3 (Duality of processes)**.** *Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two continuous time Markov processes with state spaces Ω and Ω^{dual} , respectively. We say that Y is* dual *to X with duality function $D : \Omega \times \Omega^{dual} \longmapsto \mathbb{R}$ if*

$$\mathbb{E}_x[D(X_t, y)] = \mathbb{E}_y[D(x, Y_t)] \ , \tag{1.6}$$

for all $(x, y) \in \Omega \times \Omega^{dual}$ and $t \geq 0$. In (1.6) *$\mathbb{E}_x$ (respectively $\mathbb{E}_y$) is the expectation w.r.t. the law of the X process initialized at x (respectively the Y process initialized at y). If X and Y are two independent copies of the same process, we say that Y is* self-dual *with self-duality function D.*

Note that self-duality can always be thought as a special case of duality where the dual process is an independent copy of the first one. The simplification of self-duality typically arises from the fact that in the copy process only a finite number of particles or variables is considered.

Recalling Theorem 1.1 one sees that a duality relation between two Markov processes is equivalent to a duality relation between their Markov semigroups, i.e.

$$\Big(S_t D(\cdot, y)\Big)(x) = \Big(S_t^{dual} D(x, \cdot)\Big)(y) \ , \ \text{for } t \geq 0 \tag{1.7}$$

where S_t denotes the semigroup of the original process X and S_t^{dual} the semigroup of the dual process Y.

Duality can be defined at the level of Markov generator in the following sense.

Definition 1.4 (Duality of generators)**.** *Let L and L^{dual} be generators of the two Markov processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$, respectively. We say that L^{dual} is* dual *to L with duality function $D : \Omega \times \Omega^{dual} \longrightarrow \mathbb{R}$ if*

$$[LD(\cdot, y)](x) = [L^{dual} D(x, \cdot)](y) \tag{1.8}$$

where we assume that both sides are well defined, i.e. $D(\cdot, y) \in \mathcal{D}(L)$ for all $y \in \Omega^{dual}$ and $D(x, \cdot) \in \mathcal{D}(L^{dual})$ for all $x \in \Omega$. In the case $L = L^{dual}$ we shall say that the process is self-dual *and the self-duality relation becomes*

$$[LD(\cdot, y)](x) = [LD(x, \cdot)](y) \ . \tag{1.9}$$

In equation (1.8) (resp. (1.9)) it is understood that L on the lhs acts on D as a function of the first variable x, while L^{dual} (resp. L) on the rhs acts on D as a function of the second variable y. Definition 1.4 is easier to work with, so we will usually work under the assumption that the notion of duality (resp. self-duality) is the one in equation (1.8) (resp. (1.9)).

Remark 1.2 (Countable state space)**.** If the original process $(X_t)_{t\geq 0}$ and the dual process $(Y_t)_{t\geq 0}$ are Markov processes with countable state space Ω and Ω^{dual} resp., then the duality relation is equivalent to

$$\sum_{x' \in \Omega} L(x, x')D(x', y) = \sum_{y' \in \Omega} L^{dual}(y, y')D(x, y') = \sum_{y' \in \Omega} (L^{dual})^T(y', y)D(x, y') \tag{1.10}$$

where L^T denotes the transposition of the generator L. In matrix notation (1.10) becomes

$$LD = D(L^{dual})^T \tag{1.11}$$

Once more, if $L^{dual} = L$ we obtain the corresponding identities for self-duality. In this context, the generator L is given by a matrix known as *rate matrix* such that

$$L(x, y) \geq 0 \qquad \text{and} \qquad \sum_y L(x, y) = 0 \ .$$

For $x \neq y$, we say that the process jumps from x to y with *rate* $L(x, y)$.

Remark 1.3 (Constant quantity)**.** If $D(x, y)$ is a duality function between two processes and the function $c : \Omega \times \Omega^{dual} \longrightarrow \mathbb{R}$ is constant under the dynamics of the two processes then $c(x, y)D(x, y)$ is also a duality function. We will always consider duality functions modulo the quantity $c(x, y)$. For instance, the interacting particle systems studied in Chapter 2 conserves the total number of particles and thus c is an arbitrary function of such conserved quantity.

When the state space Ω is infinite, the state on a site may diverge in a finite time (explosion). In other words, it may happen that some generators, although stochastic, do not allow to define a continuous time Markov process. As already remarked in the previous section, sufficient conditions for the existence of a Markov process with formal generator L are provided by the Hille-Yoshida theorem, see for instance [1, 40, 50, 65].

Obviously, the problem of existence is not present when the system is conservative and it is initialized with a finite number of particles. In this thesis we will not discuss the problem of existence for the interacting particle systems we will consider; however, we observe that, since in the dual process we always restrict to a finite number of dual particles then the duality relation in Definition 1.3 rigorously defines expectations on the left-hand side of equation (1.6) via the right-hand side.

Under suitable hypothesis the two notions of duality in Definition 1.3 and Definition 1.4 given above are equivalent, as explained in the following proposition (see also [15] and [38]).

Proposition 1.1 (Equivalence of duality definitions)**.** *Duality of processes always implies duality of their generators, the converse is true iff the processes' semigroups* $(S_t)_{t\geq 0}$ *and* $(S_t^{dual})_{t\geq 0}$*, respectively, are such that* $S_t D(x,\cdot) \in \mathcal{D}(L^{dual})$ *and* $S_t^{dual} D(\cdot, y) \in \mathcal{D}(L)$*.*

Proof. Let's first suppose the processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are dual to each other. Then equation (1.7) implies generators duality in equation (1.8).

The other implication follows from the uniqueness of the semigroup. Let L generates the semigroup S_t, then $f_t(x,y) = (S_t D(\cdot, y))(x)$ is the unique solution of the backward equation in (1.4), i.e.

$$\frac{d}{dt} f_t(x,y) = (L f_t(\cdot, y))(x) \tag{1.12}$$

with initial condition $f_0(x,y) = D(x,y)$ and for all $D(\cdot, y) \in \mathcal{D}(L)$. L is dual to L^{dual} through D by hypothesis, therefore it is implied that also $g_t(x,y) = \left(S_t^{dual} D(x,\cdot)\right)(y)$ solves equation (1.12) with the same initial value $g_0(x,y) = D(x,y)$. Indeed, since S_t^{dual} has generator L^{dual}, it follows from the Kolmogorov forward equation that

$$\begin{aligned}\frac{d}{dt} g_t(x,y) = \frac{d}{dt}\left(S_t^{dual} D(x,\cdot)\right)(y) = S_t^{dual}\left(L^{dual} D(x,\cdot)\right)(y) = \\ S_t^{dual}\left(L D(\cdot,y)\right)(x) = L\left(S_t^{dual} D(x,\cdot)\right)(y) = L g_t(x,y)\end{aligned}$$

where in the second last equation we used Fubini's theorem. Since equation (1.12) has an unique solution, it follows that $g_t(x,y) = f_t(x,y)$, i.e. $\left(S_t^{dual} D(x,\cdot)\right)(y) = (S_t D(\cdot, y))(x)$ □

We will always work under the assumption that the notion of duality in Definition 1.3 is equivalent to that given in Definition 1.4. In particular, it is easier and more convenient to work with equation (1.8).

It is one of our aims to show duality and self-duality as a consequence of a change of representation of a Lie algebra, so it will be convenient to extend the definition of duality for operators as well.

Definition 1.5 (Duality of operators)**.** *Let A and B be two generic operators of domain $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively, we say that A is dual to B if*

$$[AD(\cdot, y)](x) = [BD(x, \cdot)](y) \ , \tag{1.13}$$

where $D = D(x, y)$ is the duality function and $D(\cdot, y) \in \mathcal{D}(A)$, $D(x, \cdot) \in \mathcal{D}(B)$, respectively. If $B = A$ we will say that f is a self-duality function for the operator A.

Sometimes if D satisfies (1.13) with A and B different we will refer to it as an *intertwining function* between the two operators A and B. Moreover, it will be sometimes convenient to adopt the following notation

$$A_x D(x, y) := [AD(\cdot, y)](x)$$

to write that operator A acts on the first variable of D. The following basic example shows a duality relation between two operators, in this context the duality function plays the role of an intertwining between the two operators.

Example 1.1. Consider the two operators acting on a differentiable function $g : \mathbb{R} \to \mathbb{R}$ as follows

$$(Ag)(x) = xg(x) \qquad (Bg)(y) = \frac{\partial}{\partial y} g(y) \ .$$

Then $f(x, y) = e^{xy}$ is a duality function between operator A and B since

$$(Af(\cdot, y))(x) = xe^{xy} = (Bf(x, \cdot))(y) \ . \tag{1.14}$$

1.4 Duality and Self-duality functions via scalar product

This section is devoted to the first original results regarding (self-)duality functions of Markov processes. We will show that – in the setting of reversible processes – once a (self-)duality relation in the sense of Definition 1.4 is available then it is possible to generate new (self-)duality functions starting from those available from the initial relation. We restrict here to the setting of Markov processes with countable state space. We will need the notion of scalar product on some measure space $\mathsf{L}^2(\Omega, \mu)$, i.e.

$$\langle f, g \rangle_\mu = \sum_{x \in \Omega} f(x) g(x) \mu(x) \ .$$

We start with self-duality. Suppose that L is the generator of a Markov process with reversible measure μ, then L is a self-adjoint operator on $\mathsf{L}^2(\Omega, \mu)$ (see also Section 2.1.1 and 2.1.2), i.e. $L = L^*$, where L^* denotes the adjoint of L. Assume now that d_1 and d_2 are two self-duality function (not necessary two different ones) for L so that

$$Ld_i(\cdot, y)(x) = Ld_i(x, \cdot)(y) \qquad \text{for} \quad i = 1, 2 \quad \text{and} \quad (x, y) \in \Omega \times \Omega \ . \tag{1.15}$$

Next proposition shows that the scalar product of d_1 and d_2 is, by construction, a self-duality function.

Proposition 1.2 (New self-duality functions)**.** *If μ is a reversible measure for a generator L and if equation* (1.15) *holds, the function $D : \Omega \times \Omega \to \mathbb{R}$ given by*

$$D(x, y) = \langle d_1(x, \cdot), d_2(y, \cdot)\rangle_\mu \tag{1.16}$$

is a self-duality function for the generator L.

Proof. It will be convenient here to adopt the shorter notation already introduced in Section 1.3, for $(LD(\cdot, y))(x)$ we write $L_x D(x, y)$. Using this compact notation we first show a one-line proof of the proposition and we then give full details. The result follows from self-duality and self-adjointness, i.e.

$$\begin{aligned} L_x D(x, y) &= \langle L_x d_1(x, \cdot), d_2(y, \cdot)\rangle_\mu = \sum_z L_z d_1(x, z) d_2(y, z)\mu(x) = \sum_z d_1(x, z) L_z d_2(y, z)\mu(z) \\ &= \langle d_1(x, \cdot), L_y d_2(y, \cdot)\rangle_\mu = L_y D(x, y) \ . \end{aligned}$$

More explicitly, if $D(x, y) = \sum_z d_1(x, z)d_2(y, z)\mu(z)$ then,

$$(LD(\cdot, y))\,(x) = \sum_{x'} L(x, x')D(x', y) = \sum_{x'}\sum_z L(x, x')d_1(x', z)d_2(y, z)\mu(z) \ ,$$

since d_1 is a self-duality function for L,

$$\sum_{x'}\sum_z L(x, x')d_1(x', z)d_2(y, z)\mu(z) = \sum_{z'}\sum_z L(z, z')d_1(x, z')d_2(y, z)\mu(z) \ .$$

Since L is self-adjoint w.r.t. measure μ

$$\sum_{z'}\sum_z L(z, z')d_1(x, z)d_2(y, z')\mu(z) = \sum_{y'}\sum_z L(y, y')d_1(x, z)d_2(y', z)\mu(z) \ ,$$

which, using that d_2 is a self-duality function becomes

$$\sum_{y'} L(y, y')\sum_z d_1(x, z)d_2(y', z)\mu(z) = \sum_{y'} L(y, y')D(x, y') = (LD(x, \cdot))\,(y) \ .$$

□

In the context of duality, one would need d_1 to be a duality function between L and L^{dual} and d_2 to be a self-duality function of L^{dual}, i.e.

$$Ld_1(\cdot, y)(x) = L^{dual}d_1(x, \cdot)(y) \qquad \text{for} \quad (x, y) \in \Omega \times \Omega^{dual} \tag{1.17}$$

and

$$L^{dual}d_2(\cdot, y)(x) = L^{dual}d_2(x, \cdot)(y) \qquad \text{for} \quad (x, y) \in \Omega^{dual} \times \Omega^{dual} \ . \tag{1.18}$$

Moreover, if L^{dual} has reversible measure μ on $\mathsf{L}^2(\Omega^{dual}, \mu)$ then it is self-adjoint on $\mathsf{L}^2(\Omega^{dual}, \mu)$ and the following proposition holds.

Proposition 1.3 (New duality functions). *If μ is a reversible measure for a generator L^{dual} and if equations* (1.17) *and* (1.18) *hold, the function $D : \Omega \times \Omega^{dual} \to \mathbb{R}$ given by*

$$D(x, y) = \langle d_1(x, \cdot), d_2(y, \cdot)\rangle_\mu \tag{1.19}$$

is a duality function for L and L^{dual}.

Proof. The proof follows step by step the one of Proposition 1.2.

$$L_x D(x, y) = \langle L_x d_1(x, \cdot), d_2(y, \cdot)\rangle_\mu = \sum_z L_z^{dual} d_1(x, z) d_2(y, z)\mu(z) =$$
$$\sum_z d_1(x, z) L_z^{dual} d_2(y, z)\mu(z) = \langle d_1(x, \cdot), L_y^{dual} d_2(y, \cdot)\rangle_\mu = L_y^{dual} D(x, y) ,$$

where we use duality of L and L^{dual} in the second equality, then the self-adjointness of L^{dual} and finally the self-duality of L^{dual}. □

We now go back to the context of self-duality. The following proposition expands the result of Proposition 1.2. It turns out that when two self-duality functions, d and D, are in a relation via a scalar product with a third function F, then, assuming d to be a basis for $\mathsf{L}^2(\Omega, \mu)$, F must also be a self-duality function.

Proposition 1.4 (Basis and self-duality). *Assume that $\{n \mapsto d(x, n) \mid x \in \Omega\}$ is a basis of self-duality functions for $\mathsf{L}^2(\Omega, \mu)$ where μ is a reversible measure for the generator L. Let $F = F(z, n)$ be a function on $\Omega \times \Omega$ and define D by*

$$D(x, n) := \langle d(\cdot, x), F(\cdot, n)\rangle_\mu \ .$$

If D is self-duality function, so is F.

Proof. Using the short notation we have that

$$L_x D(x, n) = \langle L_x d(\cdot, x), F(\cdot, n)\rangle_\mu = \sum_{z\in\Omega} d(z, x) L_z F(z, n)\mu(z)$$

where we used that d is self-duality and that L is self-adjoint with respect to μ. On the other hand, since D is self-duality the above quantity must be equal to

$$L_n D(x, n) = \langle d(\cdot, x), L_n F(\cdot, n)\rangle_\mu = \sum_{z\in\Omega} d(z, x) L_n F(z, n)\mu(z) \ .$$

From the identity $L_x D(x, n) = L_n D(x, n)$, we have

$$\sum_{z\in\Omega} d(z, x) \left[L_z F(z, n) - L_n F(z, n)\right] \mu(z) = 0$$

and since d is a basis for $\mathsf{L}^2(\Omega, \mu)$, necessarily $L_z F(z, n) - L_n F(z, n) = 0$, i.e. F is also a self-duality function for L. □

How does the orthogonal property play a role? Of course not all self-duality functions built with this scalar product method turn out to be orthogonal. However, there is a sort of stability with respect to orthogonality property in the scalar product construction. More precisely, if we start with two *different* biorthogonal self-duality functions the scalar product construction yields novel biorthogonal self-duality functions that may happen to be *equal* and therefore orthogonal.

To state the next proposition, we will need the following result: if μ is a reversible measure for a Markov process with generator L, then $\frac{\delta_{x,y}}{\mu(x)}$ is a self-duality function (this is rigorously shown in Lemma 5.1).

Proposition 1.5 (Biorthogonal self-duality functions)**.** *Define D and $\widetilde{D}$ by*

$$D(x,n) = \langle d(x,\cdot), \widetilde{d}(\cdot,n)\rangle_\mu \qquad \widetilde{D}(x,n) = \langle \widetilde{d}(x,\cdot), d(\cdot,n)\rangle_\mu .$$

where both $\{d(x,m) : m \in \mathbb{N}\}$ and $\left\{\widetilde{d}(x,m) : m \in \mathbb{N}\right\}$ are basis for $\mathsf{L}^2(\Omega,\mu)$ for μ. Suppose d is self-duality function for a Markov process with reversible measure μ, such that

$$\langle d(\cdot,n), \widetilde{d}(\cdot,m)\rangle_\mu = \frac{\delta_{n,m}}{\mu(n)} .$$

Then D and $\widetilde{D}$ are biorthogonal self-duality functions, i.e.

$$\langle D(\cdot,m), \widetilde{D}(\cdot,n)\rangle_\mu = \frac{\delta_{m,n}}{\mu(m)} .$$

In particular, if $D = \widetilde{D}$ we have the orthogonality relations for D.

Proof. From Proposition 1.4, since d and $\dfrac{\delta_{n,m}}{\mu(n)}$ are self-duality functions, we have that $\widetilde{d}$ is also a self-duality function. From Proposition 1.2 we have that both D and $\widetilde{D}$ are self-duality functions since scalar product of self-dualities. Assuming now we can interchange the order of summation:

$$\begin{aligned}
\langle D(\cdot,m), \widetilde{D}(\cdot,n)\rangle_\mu &= \sum_x D(x,m)\widetilde{D}(x,n)\mu(x) \\
&= \sum_x \left(\sum_y d(x,y)\widetilde{d}(y,m)\mu(y) \right) \left(\sum_z \widetilde{d}(x,z) d(z,n)\mu(z) \right) \mu(x) \\
&= \sum_{y,z} \widetilde{d}(y,m) d(z,n)\mu(y)\mu(z) \sum_x d(x,y)\widetilde{d}(x,z)\mu(x) \\
&= \sum_{y,z} \mu(y)\mu(z)\widetilde{d}(y,m)d(z,n)\, \frac{\delta_{y,z}}{\mu(y)} \\
&= \sum_y \widetilde{d}(y,m)d(y,n)\mu(y) = \frac{\delta_{m,n}}{\mu(m)} .
\end{aligned}$$

□

We will see later on (Chapters 6 and 7) how to use this constructive approach to find orthogonal (self-)duality functions. Indeed, the fact that these new functions turn out to be orthogonal is not an immediate consequence of these propositions. It is something that has to be checked separately.

A last remark before closing this section. The scalar product method described in the propositions above could be useful to characterize the set of duality functions. This question was first asked in [53] where it was defined the concept of duality space, i.e. the subspace of all measurable functions on the configuration product space of two Markov processes for which the duality relation holds. In [53] the dimension of this space is computed for some simple systems and, as far as we know, a general answer is not available in the general case.

1.5 Simple examples of duality

We start by showing some examples and applications of duality. The first three examples will regard one of the oldest relation of duality, the well known *Siegmund duality* [62], which has the following duality function

$$D(x,y) = \mathbb{1}_{x \le y} \ .$$

This function turns out to be a duality function between reflected and absorbed random walk as well as reflected and absorbed Brownian motions. We first consider a continuous time random walk on the integers, here the stochastic duality relation can easily be proved in matrix notation. As an application, we will then use duality to characterize the stationary measure of the dual process. We then consider the analogous example on a discrete time setting, where the transition rate matrix is replaced by the transition matrix, which describes the probability of the transitions of the Markov chain. The same considerations about the stationary measure of the dual process can be done here. We then show Siegmund duality for Brownian motion, which was first observed by Lévy in 1948 [49].

Example 1.2 (Continuous time random walk)**.** This example of duality can easily be checked via the duality relation in matrix notation in the sense of equation (1.11) Let $X = (X_t)_{t \ge 0}$ be the random walk with state space in $\mathbb{N}_0 = \{0, 1, \ldots\}$ initialized as $X_0 = x$ that hops to the right with rate p and to the left with rate $q := 1 - p$. The site 0 is an absorption barrier, in the sense that once in zero the random walker stays there forever. This example of duality can easily be checked via the duality relation in matrix notation in the sense of equation (1.11). The generator of X can be written in the form of the following stochastic matrix

$$L^{abs} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ q & -1 & p & 0 & \cdots \\ 0 & q & -1 & p & \cdots \\ 0 & 0 & q & -1 & \cdots \\ & & & & \ddots \end{bmatrix} \ . \tag{1.20}$$

Its dual is still a random walk $Y = (Y_t)_{t\geq 0}$ initialized in $Y_0 = y$, with the same state space and reversed jump rates, i.e. it jumps to the right with rate q and to the left with rate p. Site 0 is a reflecting barrier, in the sense that, once in zero, the random walk is reflected with rate q to the site on the right. The rate matrix of Y is

$$L^{refl} = \begin{bmatrix} -q & q & 0 & 0 & \cdots \\ p & -1 & q & 0 & \cdots \\ 0 & p & -1 & q & \cdots \\ 0 & 0 & p & -1 & \cdots \\ & & & & \ddots \end{bmatrix} \tag{1.21}$$

Remark 1.4. The birth (resp. death) rates of the original process become the death (resp. birth) rates of the dual process, in term of element of matrices this is

$$\begin{aligned} L^{abs}(n, n+1) =& L^{refl}(n+1, n) = p \qquad \text{for } n \geq 1 \\ L^{abs}(n+1, n) =& L^{refl}(n, n+1) = q \qquad \text{for } n \geq 0 \,. \end{aligned}$$

The Siegmund duality function $D(x, y) = \mathbb{1}_{x\leq y}$ can be written in matrix notation as an upper triangular matrix D with all the non-zero elements equal to 1.

$$D = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \ddots \end{bmatrix} \tag{1.22}$$

Indeed, a simple calculation shows that a duality between stochastic matrices L^{abs} and L^{refl} with duality matrix in equation (1.22) holds as in (1.11), i.e.

$$L^{abs} D = D \left(L^{refl}\right)^T$$

since both sides are equal to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ q & -p & 0 & 0 & \cdots \\ 0 & q & -p & 0 & \cdots \\ 0 & 0 & q & -p & \cdots \\ & & & \ddots & \end{bmatrix} \tag{1.23}$$

We now show a simple application of duality. Given that X and Y are dual with Siegmund duality function in equation (1.22), then the duality relation in Definition 1.3 becomes

$$\mathbb{P}_x(X_t \leq y) = \mathbb{P}_y(Y_t \geq x) \,. \tag{1.24}$$

Suppose we want to characterize the stationary distribution of Y, i.e. we look for $\mathbb{P}_y(Y_\infty < x) = \lim_{t\to\infty} \mathbb{P}_y(Y_t < x)$. Assuming that $p > q$ and using duality one finds

$$\mathbb{P}_y(Y_\infty \geq x) = \mathbb{P}_x(X_\infty \leq y) = \left(\frac{q}{p}\right)^x . \tag{1.25}$$

The second identity can easily be found working with the absorbed random walker X. In the limit $t \to \infty$, X is either absorbed in zero or it goes to infinity, so the event $\{X_\infty \leq y\}$ is the same as $\{X_\infty = 0\}$ whose probability can be calculated conditioning on the first step. Indeed, let $u_x := \mathbb{P}(X_\infty = 0 \mid X_0 = x)$, then conditioning on the first step gives the difference equation

$$u_x = u_{x+1}p + u_{x-1}q ,$$

which is solved by $u_x = A_1 + A_2 \left(\frac{q}{p}\right)^x$. To find the constant we use the boundary conditions: the probability of being absorbed in zero, starting from zero is 1, i.e. $u_0 = 1$, which says that $A_1 + A_2 = 1$. On the other hand in order to $u_\infty = 0$ we get that, for $p > q$ it has to be verified that $A_1 = 0$, while for $p < q$ it has to be that $A_2 = 0$. And so it is now justified that $\mathbb{P}_x(X_\infty \leq y) = \left(\frac{q}{p}\right)^x$.

This one line computation in equation (1.25) shows that the stationary distribution of a reflected random walk Y is a geometric distribution, $Y_\infty \sim Geo\left(\frac{p-q}{p}\right)$.

Example 1.3 (Discrete time random walk). This example, which can be found in details in [53], shows that the considerations above are also true for two discrete time random walks, one with absorbing barrier and the other with a reflecting barrier. Using the fact that the two transition matrices, P^{abs} and P^{refl}, can be found from the rate matrices L^{abs} and L^{refl} adding the identity matrix Id, then one can easily verify the notion of duality in matrix notation as a consequence of the matrix duality of the previous example. To be more explicit, the transition matrix of the absorbed random walk is

$$P^{abs} = L^{abs} + Id = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \\ q & 0 & p & 0 & \cdots & \\ 0 & q & 0 & p & \cdots & \\ 0 & 0 & q & 0 & \cdots & \\ & & & & \ddots & \end{bmatrix} , \tag{1.26}$$

while the transition matrix of the reflected random walk is

$$P^{refl} = L^{refl} + Id = \begin{bmatrix} p & q & 0 & 0 & \cdots & \\ p & 0 & q & 0 & \cdots & \\ 0 & p & 0 & q & \cdots & \\ 0 & 0 & p & 0 & \cdots & \\ & & & & \ddots & \end{bmatrix} . \tag{1.27}$$

Indeed, one can easily check that

$$P^{abs} D = D(P^{refl})^T ,$$

where D is the Siegmund duality matrix in (1.22). In the same way as in the previous example one can easily characterized the stationary distribution of the reflected random walk via duality.

In [53] it is shown the case for two barriers, with finite state spaces $\Omega = \Omega^{dual} = [0, 1, \ldots, N]$ where site N is an absorbing barrier for the reflected random walk X while it is a reflecting barrier for the absorbed random walk Y.

Example 1.4 (Absorbed/Reflected Brownian motion)**.** The last Siegmund duality example we show has appeared in many different contexts such as queuing theory [3, 51], birth and death processes [39], interacting particle systems [21] and time reversal, by exchanging the entrance and exit laws for Markov processes [25]. It provides a duality relation between Brownian motion reflected at 0 and Brownian motion absorbed at 0 via the Siegmund duality function. In this setting $\Omega = \Omega^{dual} = [0, \infty)$ and $D(x, y) = \mathbb{1}_{x \le y}$. $(X_t)_{t \ge 0}$ is a Brownian motion started in $x > 0$ and reflected at 0 while $(Y_t)_{t \ge 0}$ is a Brownian motion started in $y > 0$ and absorbed in 0, i.e.

$$X_t = |x + B_t| \qquad \text{and} \qquad Y_t = \begin{cases} y + \tilde{B}_t & \text{if } t < \tau_0 \\ 0 & \text{if } t \ge \tau_0 \end{cases}$$

where B_t and $\tilde{B}_t$ are two independent standard Brownian motions and $\tau_0 = \inf \{t \ge 0 : Y_t = 0\}$ denotes the first time Y_t hits zero. With duality function $D(x, y) = \mathbb{1}_{x \le y}$ equation (1.6) becomes

$$\mathbb{P}_x(X_t \le y) = \mathbb{P}_y(Y_t \ge x) \ . \tag{1.28}$$

The duality relation (1.28) can be proved as follows. Consider first the left-hand side:

$$\mathbb{P}_x(X_t \le y) = \mathbb{P}(|x + B_t| \le y) = \mathbb{P}(B_t \le y - x) - \mathbb{P}(B_t \le -y - x) \ .$$

On the other hand, for the absorbed Brownian motion one has

$$\begin{aligned}
&\mathbb{P}_y(Y_t \ge x) = \mathbb{P}(\tilde{B}_t \ge x - y, \min_{s \le t} \tilde{B}_s \ge -y) = \\
&\mathbb{P}(\tilde{B}_t \ge x - y) - \mathbb{P}(\tilde{B}_t \ge x - y, \min_{s \le t} \tilde{B}_s \le -y) = \\
&\mathbb{P}(\tilde{B}_t \ge x - y) - \mathbb{P}(\tilde{B}_t \ge x - y \mid \min_{s \le t} \tilde{B}_s \le -y)\mathbb{P}(\min_{s \le t} \tilde{B}_s \le -y) = \\
&\mathbb{P}(\tilde{B}_t \ge x - y) - \mathbb{P}(\tilde{B}_t \le -x - y \mid \min_{s \le t} \tilde{B}_s \le -y)\mathbb{P}(\min_{s \le t} \tilde{B}_s \le -y) = \\
&\mathbb{P}(\tilde{B}_t \ge x - y) - \mathbb{P}(\tilde{B}_t \le -x - y, \min_{s \le t} \tilde{B}_s \le -y) = \\
&\mathbb{P}(\tilde{B}_t \ge x - y) - \mathbb{P}(\tilde{B}_t \le -x - y)
\end{aligned}$$

where the third equality follows from the fact that $\{\min_{s\leq t} \tilde{B}_s \leq -y\}$ is equivalent to $\{t > \tau_0\}$ and by the reflection principle of Brownian motion, the probability that $\tilde{B}_t$ is above $x - y$ when $t \geq \tau_0$ is just likely as to be below $x + y$. By symmetry of the normal distribution $\mathbb{P}(B_t \geq x - y) = \mathbb{P}(\tilde{B}_t \leq y - x)$ and so equality in (1.28) is achieved.

Another classical duality function is the so called *moment duality*,

$$D(x, y) = x^y \ .$$

If $x \in [0, 1]$ and $y \in \mathbb{N}$ using the duality relation in equation (1.6) it is possible to recover all the moments of X_t.

Example 1.5 (Wright-Fisher diffusion and Kingman coalescent)**.** Consider the Wright-Fisher diffusion process $X = (X_t)_{t\geq 0}$ on $\Omega = [0, 1]$ with generator

$$\left(L^{WF} f\right)(x) = \frac{x(x-1)}{2} \frac{d^2}{dx^2} f(x) \ . \tag{1.29}$$

X can be interpreted as the proportion of allele A in an infinite population of two alleles, (type A and a) that evolves under a "genetic drift ". Sites 0 and 1 are absorbing states, meaning that only one type will survive. It can be found starting from a Wright-Fisher process after considering an appropriate diffusive time-space rescaling. Its dual is a block counting process of Kingman coalescent type $Y = (Y_t)_{t\geq 0}$, which is a pure death process on $\mathbb{N}$ where transitions from y to $y - 1$ occur with coalescent rate $\binom{y}{2}$, i.e. the probability of choosing two individuals out of y. The generator of the dual process is thus

$$\left(L^K f\right)(y) = \frac{y(y-1)}{2} \left[f(y-1) - f(y)\right] \ . \tag{1.30}$$

Note that by definition of Y, $\lim_{t\to\infty} Y_t = 1$. It is easy to verify the generator duality in equation (1.8), i.e. $\left(L^{WF} D(\cdot, n)\right)(x) = \left(L^K D(x, \cdot)\right)(n)$ for the moment duality function $D(x, n) = x^n$. Duality here can be used to calculate the probability that allele A survives or heterozygosity, which is the probability that, choosing at random two individuals at time $t \geq 0$, they are of different type. Duality between the two processes reads

$$\mathbb{E}_x(X_t^y) = \mathbb{E}_y(x^{Y_t}) \ . \tag{1.31}$$

Then, setting $y = 1$ the expected value of the Wright-Fisher is

$$\mathbb{E}_x(X_t) = \mathbb{E}_1(x^{Y_t}) = x$$

where the first equality is due to duality while the second one, follows from the fact that the process Y is initialized in 1. By taking the limit of infinite time, we find

$$\mathbb{P}_x(X_\infty = 1) = \lim_{t\to\infty} \mathbb{E}_x(X_t) = x$$

showing that the probability that type a goes extinct equals the initial fraction of type A in the population. For $y = 2$, the second moment of X is

$$\mathbb{E}_x(X_t^2) = \mathbb{E}_2(x^{Y_t}) = x^2\mathbb{P}_2(Y_t = 2) + x\mathbb{P}_2(Y_t = 1) = x^2 e^{-t} + x(1 - e^{-t}) \ .$$

where the last equality follows from the fact that the transition from 2 to 1 take an exponential time of parameter 1, (see also [27], for rigorous proof).

With these results we can easily compute the expectation of $X_t(1 - X_t)$, which is the heterozygosity of the biallelic Wright-Fisher model.

$$\mathbb{E}_x\left(X_t(1 - X_t)\right) = x - (x^2 e^{-t} + x(1 - e^{-t})) = (1 - x)xe^{-t} \ .$$

As expected in the limit $t \to \infty$ heterozygosity converges to zero, i.e. all individuals will be either of type A or type a.

We will come back to this example in Chapter 5 and we will show how this duality can be understood as a change of representation of a Lie algebra.

Example 1.6. As last example we also recall that in the literature it exists the notion of *Laplace duality*, where the duality function takes the form $D(x, y) = e^{\lambda xy}$ for some $\lambda \in \mathbb{R}$ and $\Omega = \Omega^{dual} \subset \mathbb{R}$. We will discuss this example in detail in Section 5.3 where we use the change of representation technique to show that the Brownian motion initialized in x is dual to the multiplication with $e^{ty^2/2}$ and then to evaluate the generating function of the Brownian motion without any calculations.

Chapter 2

Models of interest

2.1 Preliminaries

The processes that we consider include interacting particle systems (exclusion process, inclusion process and independent random walkers process) where the quantity of interest is discrete, i.e. the number of particles, as well as interacting diffusions (Brownian momentum process, Brownian energy process) where continuous quantities (having the meaning of velocities or energy) are studied. We also consider redistribution models that are obtained from the previous one via a thermalization limit (e.g. Kipnis-Marchioro-Presutti processes). Each process is described via its generator, so for the sake of completeness we briefly recall the standard form of the generator of a jump process (with discontinuous trajectories and exponential time between jumps) and the generator of a diffusion (with continuous trajectories). Recall that a stationary distribution (also called invariant law) for a process is a probability distribution that remains unchanged as time progresses.

Definition 2.1 (Stationary mesure)**.** *Let $(X_t)_{t\geq 0}$ be a Markov process on the space Ω. A stationary measure for $(X_t)_{t\geq 0}$ is a probability measure μ on Ω such that if X_0 has distribution μ, then for all $t \geq 0$, X_t has distribution μ.*

2.1.1 Generator of jump processes

The generator of a Markov jump process $X = (X_t)_{t\geq 0}$ with a countable state space Ω has the form

$$(Lf)(x) = \sum_{y\in\Omega} c(x,y)\,[f(y) - f(x)]$$

where $c(x,y) \geq 0$ is the transition rate from configuration x to configuration y. Equivalently we can write

$$(Lf)(x) = \sum_{y\in\Omega} L(x,y)f(y)$$

where $L(x, y)$ is a matrix such that

$$L(x, y) = c(x, y) \geq 0 \tag{2.1}$$

$$L(x, x) = -\sum_{y \neq x} L(x, y) \leq 0 \; . \tag{2.2}$$

as a consequence $\sum_y L(x, y) = 0$. In the context of Chapter 1.2, let μ be a probability measure on Ω and S_t the semigroup associated to L as in Theorems 1.1 and 1.2, then $\mu_t := \mu S_t$ is the law of the process at time t which satisfies the forward equation in (1.5), i.e.

$$\frac{d}{dt}\mu_t = \mu_t L$$

which can also be written as

$$\frac{d}{dt}S_t = S_t L \; .$$

In this context a stationary measure, in the sense of Definition 2.1, can be defined as follows.

Definition 2.2 (Stationary measure). *A probability measure μ is a stationary measure if*

$$\mu = \mu S_t \quad \textit{for} \quad t \geq 0 \tag{2.3}$$

or equivalentely, if

$$\mu L = 0 \; . \tag{2.4}$$

Definition 2.3 (Reversible measure). *A probability measure μ is reversible if the detailed balance equations*

$$\mu(x)L(x, y) = \mu(y)L(y, x) \quad \textit{for} \quad x \neq y \in \Omega \tag{2.5}$$

are satisfied. In other words, the generator L is self-adjoint in $\mathsf{L}^2(\Omega, \mu)$.

Remark 2.1 (Reversibility implies invariance). It is easy to show that a reversible measure is always invariant. Indeed using equations in (2.1),

$$\begin{aligned}
\sum_{x \in \Omega} \mu(x)L(x, y) &= \sum_{x \neq y} \mu(x)L(x, y) + \mu(y)L(y, y) \\
&= \sum_{x \neq y} \mu(x)L(x, y) + \mu(y)\left(-\sum_{x \neq y} L(y, x)\right) \\
&= \sum_{x \neq y} \mu(x)L(x, y) - \mu(y)L(y, x) \\
&= 0 \; .
\end{aligned}$$

2.1.2 Generators of Markov diffusions

The generator of a diffusion process $X = (X_t)_t \geq 0$ with state space $\Omega = \mathbb{R}^N$ takes the form of a differential operator of the second order

$$(Lf)(x) = \sum_{i,j=1}^{N} a(x_i, x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) + \sum_{i=1}^{N} b(x_i) \frac{\partial}{\partial x_i} f(\mathbf{x}) \tag{2.6}$$

where the function f is twice differentiable, a is a symmetric matrix on $\mathbb{R}^N$, called diffusion matrix and b is a vector on $\mathbb{R}^N$ called the drift field. The diffusion matrix must be positive definite, i.e. for all $x \in \mathbb{R}^N$,

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j > 0 \qquad \text{for all } \xi \in \mathbb{R}^N \ .$$

We further assume that the law of the process $(X_t)_{t\geq 0}$ is absolutely continuous w.r.t the Lebesgue measure and we call $p_t(x, y)$ the probability density for the transition from x to y in a time t.

Example 2.1. The canonical example of a (one-dimensional) diffusion is the Brownian motion $\{W_t\}_{t\geq 0}$, whose generator is

$$L^{BMP} f(x) = \frac{1}{2}\frac{d^2 f}{dx^2}(x) \ .$$

It has transition density function

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \ ,$$

which allows to evaluated the probability that the Brownian motion started at x belongs to the set $A \subseteq \mathbb{R}$:

$$P_x(W_t \in A) = \int_A \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy \ .$$

Remark 2.2. There is a deep connection between the process with generator L in equation (2.6) and stochastic differential equations (SDE): the process $(X_t)_{t\geq 0}$ is the solution of the following SDE

$$dX_t = b(X_t)dt + a(X_t)dW_t \ .$$

Consider Definition 2.1 of a stationary measure. In this context it reads

$$\frac{d}{dt}\int_\Omega \mathbb{E}_x[f(X_t)]\mu(dx) = 0 \quad \text{for all } f \ .$$

Evaluating the time derivative at $t = 0$, in virtue of equation (1.2) we have

$$\int_\Omega Lf(x)\mu(dx) = 0 \quad \text{for all } f \ . \tag{2.7}$$

Equivalently, from the absolute continuity of μ one has $\mu(dx) = p(x)dx$, then μ is stationary if

$$L^* p = 0 \tag{2.8}$$

where L^* denotes the adjoint of L in $\mathsf{L}^2(\Omega, dx)$. The analogue of the detailed balance equation in (2.5) for a one-dimensional diffusion is

$$\mu(x) p_t(x,y) = \mu(y) p_t(y,x) \qquad \text{for all } x, y \in \Omega \text{ and } t \geq 0 \, .$$

Now multiplying by arbitrary functions $f(x)$ and $g(y)$ in the domain of the diffusion generator we obtain

$$\int \mu(x) f(x) \left(\int p(t,x,y) g(y) dy \right) dx = \int \mu(y) g(y) \left(\int p(t,y,x) f(x) dx \right) dy$$

observing that the inner integrals are

$$\mathbb{E}_x[g(X_t)] \qquad \text{and} \qquad \mathbb{E}_y[f(X_t)]$$

and differentiating with respect to t at $t = 0$ to obtain

$$\int f(x) \left(Lg(x) \right) \mu(x) dx = \int \left(Lf(y) \right) g(y) \mu(y) dy \, . \tag{2.9}$$

Definition 2.4 (Reversible stationary distribution)**.** *If equation* (2.9) *is satisfied for all functions f and g, then μ is called a reversible stationary distribution and we say that the diffusion with generator L is reversible with respect to μ.*

An equivalent way would be to say that L is self-adjoint with respect to μ, i.e.

$$\langle f, Lg \rangle = \langle Lf, g \rangle_\mu \, .$$

2.2 The Symmetric Exclusion Process, SEP(j)

The Symmetric Exclusion Processes (SEP) are a family of interacting particles processes on a generic graph, labeled by the parameter $j \in \mathbb{N}/2$. On the undirected and connected graph $G = (V, E)$ with $|V| = N$ sites (vertices) and edge set E, each site can have at most $2j$ particles, and jumps only occur when an edge exists between two sites. Jumps occur (in both edge directions) at rate proportional to the number of particles in the departure site times the number of holes in the arrival site. Usually in the literature the name Exclusion Process is referred to the case with parameter $j = 1/2$ (the case with hard core exclusion, i.e. each site can host at most one particle), whereas the generic case $j \in \mathbb{N}/2$ is named Partial Exclusion Process [60]; we shall not make in this thesis such a distinction, we will just use the name Exclusion process in a broad sense. The state space of the process is thus $\Omega = \{0, 1, \ldots, 2j\}^N$.

A particle configuration is denoted by $\mathbf{x} = (x_i)_{i \in V}$ where $x_i \in \{0, \ldots, 2j\}$ is interpreted as the number of particles at sites i. The process generator reads

$$L^{SEP(j)} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} x_i(2j - x_l) \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + (2j - x_i) x_l \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right] \tag{2.10}$$

where $f : \{0, 1, \ldots, 2j\}^N \to \mathbb{R}$ is a function in the domain of the generator and $\mathbf{x}^{i,l}$ denotes the particle configuration obtained from the configuration $\mathbf{x}$ by moving one particle from site i to site l:

$$x_k^{i,l} = \begin{cases} x_k & \text{if } k \neq \{i, l\} \\ x_i - 1 & \text{if } k = i \\ x_l + 1 & \text{if } k = l \end{cases} .$$

The following picture shows the dynamics of the SEP(j) on an N sites chain. By definition the

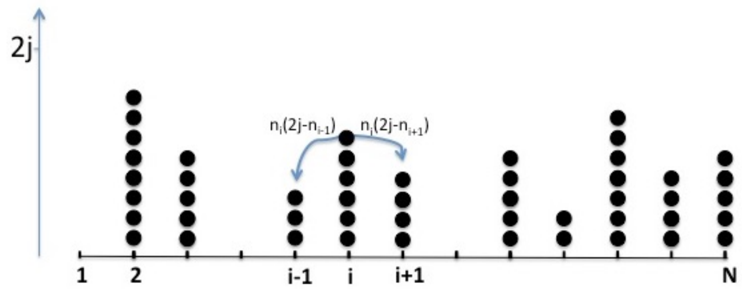

Figure 2.1: SEP dynamic

Exclusion process dynamics conserves the total number of particles.

Lemma 2.1 (Binomials are reversible measure of SEP)**.** *The reversible stationary measures μ_p^{SEP} are given by homogeneous product measures with marginals Binomial distributions with parameters $2j > 0$ and $p \in (0, 1)$. In essential,*

$$\mu_p^{SEP}(\boldsymbol{x}) = \prod_{k \in V} \rho_p^{SEP}(x_k) ,$$

where the marginal is

$$\rho_p^{SEP}(x) = \binom{2j}{x} p^x (1 - p)^{2j - x} , \qquad x \in \{0, 1, \ldots, 2j\} . \tag{2.11}$$

Proof. We need to show that μ_p^{SEP} satisfies the detailed balance equations in (2.5). Suppose, for example, a particle jumps from site i to site l with $(i, l) \in E$ and the rest of the configuration doesn't change, then the detailed balance for $1 \le x_i \le 2j$ and $0 \le x_l \le 2j - 1$ reads

$$x_i(2j - x_l)\rho(x_i)\rho(x_2) = (x_l + 1)(2j - x_i + 1)\rho(x_i - 1)\rho(x_l + 1) .$$

Separating x_i and x_l, we get

$$\frac{x_i \rho(x_i)}{(2j - x_i + 1)\rho(x_i - 1)} = \frac{(x_l + 1)\rho(x_l + 1)}{(2j - x_l)\rho(x_l)} \ .$$

These set of equations is solved by

$$\frac{(x+1)\rho(x+1)}{(2j-x)\rho(x)} = c \ , \qquad c > 0$$

so we have the recursion relation

$$\rho(x+1) = c\frac{2j - x}{x+1}\rho(x) \ . \tag{2.12}$$

Iteration of Equation (2.12) leads to

$$\rho(x) = c^x \frac{2j!}{x!(2j-x)!}\rho(0)$$

which we impose to be a probability measure, i.e. with total mass 1

$$\sum_{x=0}^{2j} \rho(x) = 1$$

and so we find the normalization $\rho(0) = \frac{1}{(1+c)^{2j}}$. By changing the parametrization of c in such a way that

$$c = \frac{p}{1-p}$$

one gets the usual binomial distribution of parameter $2j$ and $0 < p < 1$ in (2.11). □

2.3 The Symmetric Inclusion Process, SIP(k)

The Symmetric Inclusion Processes (SIP) are a family of Markov jump processes (labeled by a parameter $k > 0$) which can be defined in the same setting of before, a generic graph $G = (V, E)$. In this case the state space is unbounded so that each site can host an arbitrary number of particles, i.e. $\Omega = \mathbb{N}^{|V|}$. Jumps occur at rate proportional to the number of particles in the departure and the arrival sites, as the generator describes:

$$L^{SIP(k)} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} x_i(2k + x_l)\left[f(\mathbf{x}^{i,l}) - f(\mathbf{x})\right] + x_l(2k + x_i)\left[f(\mathbf{x}^{l,i}) - f(\mathbf{x})\right] \ . \tag{2.13}$$

From the expression of the generator it is immediate to see that the dynamics conserves the total number of particles. Moreover, because of the space homogeneity, the stationary reversible measure is given in product form, as the next lemma shows.

Lemma 2.2 (Negative binomials are reversible measures of SIP)**.** *The reversible invariant measures μ_p^{SIP} are given by the homogeneous product measures with marginals given by identical Negative Binomial distributions with parameters $2k > 0$ and $0 < p < 1$. In formula,*

$$\mu_p^{SIP}(\boldsymbol{x}) = \prod_{k \in V} \rho_p^{SIP}(x_k) \ ,$$

where

$$\rho_p^{SIP}(x) = \frac{\Gamma(2k+x)}{\Gamma(2k)x!} p^x (1-p)^{2k} \ , \qquad x \in \{0, 1, \ldots\} \ . \tag{2.14}$$

Proof. We need to show that μ_p^{SIP} satisfies the detailed balance equations in (2.5). Again suppose there is a jump from site i to site l with $(i,l) \in E$ and the rest of the configuration doesn't change, then the detailed balance equations reads

$$x_i(2k + x_l)\rho(x_i)\rho(x_l) = (x_l + 1)(2k + x_l - 1)\rho(x_i - 1)\rho(x_l + 1) \ .$$

Separating x_i and x_l, we get

$$\frac{x_i \rho(x_i)}{(2k + x_i - 1)\rho(x_i - 1)} = \frac{(x_l + 1)\rho(x_l + 1)}{(2k + x_l)\rho(x_l)}$$

that is solved by

$$\rho(x+1) = c \frac{2k + x}{x + 1} \rho(x) \qquad \text{for } c > 0 \ . \tag{2.15}$$

Iteration of equation (2.15) leads to

$$\rho(x) = c^x \frac{\Gamma(2k + x)}{\Gamma(x) x!} \rho(0)$$

which we impose to be a probability measure. The series $\sum_{x=0}^{\infty} \rho(x)$ converges for $|c| < 1$ and in particular, requiring

$$\sum_{x=0}^{\infty} \rho(x) = 1$$

we find the constant $\rho(0) = (1-c)^{2k}$. Calling $c = p$ with $0 < p < 1$ we find the negative binomial distribution of parameter $2k$ and p of equation (2.14). □

2.4 The Independent Random Walk, IRW

The Independent Random Walkers (IRW) is one of the simplest, yet non-trivial particle systems studied in the literature. It consists of independent particles that perform a symmetric continuous time random walk at rate 1 on the undirected connected graph $G = (V, E)$ with state space $\Omega = \mathbb{N}^{|V|}$. The generator is given by

$$L^{IRW} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} x_i \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + x_l \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right] \ . \tag{2.16}$$

From the expression of the generator it is immediate to see that the total number of particles is conserved. Detailed balance is satisfied by a product measure with marginals given by Poisson distributions.

Lemma 2.3 (Poisson are reversible measures for IRW)**.** *The reversible invariant measures μ^{IRW} are given by the homogeneous product measures with marginals given by identical Poisson distributions with parameter $\lambda > 0$, i.e.*

$$\mu^{IRW}(\boldsymbol{x}) = \prod_{k \in V} \rho^{IRW}(x_k)$$

where

$$\rho^{IRW}(x) = \frac{e^{-\lambda}\lambda^x}{x!}\,, \qquad\qquad x \in \mathbb{N}_0\ . \tag{2.17}$$

Proof. We need to show that μ^{IRW} satisfies the detailed balance equations in (2.5). Again suppose there is a jump from site i to site l with $(i,l) \in E$, then the detailed balance for $x_i \geq 1$ and $0 \leq x_l$ reads

$$x_i\rho(x_i)\rho(x_l) = (x_l+1)\rho(x_i-1)\rho(x_l+1)\ .$$

Separating x_i and x_l, we get

$$\frac{x_i\rho(x_i)}{\rho(x_i-1)} = \frac{(x_l+1)\rho(x_l+1)}{\rho(x_l)}$$

we have that

$$\rho(x+1) = c\frac{\rho(x)}{x+1} \qquad \text{for } c > 0\ . \tag{2.18}$$

Iteration of equation (2.18) leads to

$$\rho(x) = c^x\frac{\rho(x)}{x!}$$

which we impose to be a probability measure, so that the total mass function is 1

$$\sum_{x=0}^{\infty}\rho(x) = 1$$

and so, by convergence of the above series, we find the constant $\rho(0) = e^{-c}$. Letting $c = \lambda$, we find the Poisson distribution of parameter $\lambda > 0$. □

2.5 The Brownian Momentum Proces, BMP

The Brownian Momentum Process (BMP) is a Markov diffusion process introduced in [30]. On the undirected connected graph $G = (V,E)$ with N vertices and edge set E, the generator reads

$$L^{BMP}f(\mathbf{x}) = \sum_{\substack{1\leq i<l\leq N\\(i,l)\in E}}\left(x_i\frac{\partial f}{\partial x_l}(\mathbf{x}) - x_l\frac{\partial f}{\partial x_i}(\mathbf{x})\right)^2 \tag{2.19}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is a function in the domain of the generator. A configuration is denoted by $\mathbf{x} = (x_i)_{i\in V}$ where $x_i \in \mathbb{R}$ has to be interpreted as a particle momentum (or velocity if the mass is set to one). A peculiarity of this process regards its conservation law: if the process is started from the configuration $\mathbf{x}$ then $||\mathbf{x}||_2^2 = \sum_{i=1}^N x_i^2$ is constant during the evolution, i.e. the total kinetic energy is conserved.

Lemma 2.4 (Normal distribution is reversible for BMP)**.** *The stationary reversible measures of the BMP process are given by a family of product measures with marginals given by independent centered Gaussian random variables with variance* $\sigma^2 > 0$*.*

Proof. Consider the definition of stationary distribution for a diffusion in equation (2.8). Assuming that the stationary distribution is factorized and absolutely continuous, i.e. $\mu(dx, dy) = p(x)p(y)dxdy$ and using the expression of the generator, which is self-adjoint in $\mathsf{L}^2(\mathbb{R}^N, dx)$, then we are looking for functions that solves (see equation (2.8))

$$(x\partial_y - y\partial_x)^2 p(x)p(y) = 0$$

and since the right hand side must be equal to zero, we neglect the square on the left hand side and we find that

$$\frac{p'(x)}{xp(x)} = \frac{p'(y)}{yp(y)} \ .$$

This is solved by

$$\frac{p'(x)}{xp(x)} = c \ ,$$

integrating both sides

$$\begin{aligned}\int_0^y \frac{p'(x)}{p(x)}dx &= \int_0^y cxdx \\ \log\frac{p(y)}{p(0)} &= c\frac{y^2}{2} \\ p(y) &= p(0)e^{c\frac{y^2}{2}} \ .\end{aligned}$$

Imposing that p is a probability density, i.e. total area must be 1 then we find $p(0) = \frac{1}{\sqrt{2\pi\sigma^2}}$ and $c = -\frac{1}{\sigma^2}$. □

The BMP invariant measure is product of Gaussian distributions of mean 0 and free variance, which can always be fixed without loss of generality. Indeed, from now on we assume $\sigma^2 = 1$. To show that $\mu(dx, dy) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy$ is also a reversible distribution we would have to check equation (2.9). The easiest way to do this is to show that the BMP generator is self-adjoint with respect to $\mathsf{L}^2(\mathbb{R}^N, \mu(dx, dy))$. L^{BMP} is a combination of multiplication and derivative and so we start by finding their adjoint, clearly

$$\int f(x,y)(xg(x,y))\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy = \int (xf(x,y))g(x,y)\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy\ ,$$

meaning that x is self-adjoint, $x^* = x$. For the derivative, using integration by parts

$$\begin{aligned}\int f(x,y)\left(\partial_x g(x,y)\right)\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy &= -\int \partial_x\left(f(x,y)e^{-\frac{x^2}{2}}\right)g(x,y)\frac{e^{-\frac{y^2}{2}}}{2\pi}dxdy = \\ -\int \left(\partial_x f(x,y)\right)g(x,y)\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy &+ \int f(x,y)xg(x,y)\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy\ ,\end{aligned}$$

meaning that $\partial_x^* = -\partial_x + x$. We can now find the adjoint of L^{BMP}, i.e.

$$\begin{aligned}\left(L^{BMP}\right)^* &= (x\partial_y - y\partial_x)^*\,(x\partial_y - y\partial_x)^* \\ &= \left(\partial_y^* x^* - \partial_x^* y^*\right)\left(\partial_y^* x^* - \partial_x^* y^*\right) \\ &= ((-\partial_y + y)x - (-\partial_x + x)y)\,((-\partial_y + y)x - (-\partial_x + x)y) \\ &= (-x\partial_y + y\partial_x)\,(-x\partial_y + y\partial_x) = L^{BMP}\ .\end{aligned}$$

Showing that the BMP generator is self-adjoint in $\mathsf{L}^2(\mathbb{R}^N, \mu)$ means that the BMP process is reversible with respect to μ.

2.6 The Brownian Energy Process, BEP(k)

We now introduce a process, known as Brownian Energy Process with parameter k, BEP(k) in short notation, whose generator is

$$L^{BEP(k)}f(\mathbf{z}) = \sum_{\substack{1\le i<l\le N \\ (i,l)\in E}}\left[z_i z_l\left(\frac{\partial}{\partial z_i}f(\mathbf{z}) - \frac{\partial}{\partial z_l}f(\mathbf{z})\right)^2 - 2k(z_i - z_l)\left(\frac{\partial}{\partial z_i}f(\mathbf{z}) - \frac{\partial}{\partial z_l}f(\mathbf{z})\right)\right]\ , \tag{2.20}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is in the domain of the generator and $\mathbf{z} = (z_i)_{i\in V}$ denotes a configuration of the process with $z_i \in \mathbb{R}^+$ interpreted as a particle energy. The generator in (2.20) describes the evolution of a system in which the agents exchange their (kinetic) energies. It is easy to verify that the total energy of the system $\sum_{i\in V} x_i$ is conserved by the dynamic.

Lemma 2.5 (Gamma distributions are reversible for BEP(k)). *The stationary measures of the BEP(k) process are given by a product of independent Gamma distributions with shape parameter $2k$ and scale parameter θ, i.e. with Lebesgue probability mass function*

$$\rho^{BEP}(z) = \frac{z^{2k-1}e^{-\frac{z}{\theta}}}{\Gamma(2k)\theta^k}. \tag{2.21}$$

Without loss of generality we can set $\theta = 1$.

Proof. Since reversibility implies invariance we only check that factorized Gamma distributions

$$\rho^{BEP}(z_i)\rho^{BEP}(z_l) = \frac{z_i^{2k-1}e^{-z_i}}{\Gamma(2k)}\frac{z_l^{2k-1}e^{-z_l}}{\Gamma(2k)}$$

are reversible measures. As before, we check that $L^{BEP(k)} = \left(L^{BEP(k)}\right)^*$ where $*$ denotes the adjoint with respect to $\mathsf{L}^2(\mathbb{R}_+^N, \rho^{BEP})$. Clearly the multiblication by z_i is self-adjoint, $(z_i)^* = z_i$ while for the derivative, integration by parts leads to

$$\begin{aligned}
&\int f(z_i,z_l)\left(\partial_{z_i} g(z_i,z_l)\right)\frac{z_i^{2k-1}e^{-z_i}}{\Gamma(2k)}\frac{z_l^{2k-1}e^{-z_l}}{\Gamma(2k)}dz_i dz_l = \\
-&\int g(z_i,z_l)\left(\partial_{z_i}\frac{z_i^{2k-1}e^{-z_i}}{\Gamma(2k)}f(z_i,z_l)\right)\frac{z_l^{2k-1}e^{-z_l}}{\Gamma(2k)}dz_i dz_l = \\
&\int\left[-\partial_{z_i} f(z_i,z_l) + f(z_i,z_l) - \frac{2k-1}{x}f(z_i,z_l)\right]g(z_i,z_l)\frac{z_i^{2k-1}e^{-z_i}}{\Gamma(2k)}\frac{z_l^{2k-1}e^{-z_l}}{\Gamma(2k)}dz_i dz_l
\end{aligned}$$

from which we infer that $(\partial_{z_i})^* = -\partial_{z_i} + 1 - \frac{2k-1}{x}$. A long and straightforward computation shows that $\left(L^{BEP(k)}\right)^* = L^{BEP(k)}$. □

Remark 2.3. In [32] it was shown that the BEP(k) can be obtained from the BMP process once $4k \in \mathbb{N}$ vertical copies of the graph G are introduced. Under these circumstances denoting by $z_{i,\alpha}$ the momentum of the i^{th} particle at the α^{th} level, the kinetic energy per (vertical) site is

$$z_i = \sum_{\alpha=1}^{4k} x_{i,\alpha}^2 \,. \tag{2.22}$$

If we use the above change of variable in the generator of such BMP process on the ladder graph with $4k$ layers, the generator of the BEP(k) is revealed. In particular, setting

$$z_i = x_i^2 \tag{2.23}$$

which implies

$$\partial_{z_i} = \frac{1}{2x_i}\partial_{x_i} \tag{2.24}$$

$$\partial_{z_i}^2 = -\frac{1}{4x_i^2}\partial_{x_i} + \frac{1}{4x_i^2}\partial_{x_i}^2 \tag{2.25}$$

one finds the BMP generator L^{BMP} in (2.19) from the BEP generator $L^{BEP(k)}$ in (2.20) with $k = 1/4$.

2.7 The Kipnis-Marchioro-Presutti Process, KMP(k)

Before defining the process of this Section, we will introduce the concept of *instantaneous thermalization limit* of a process. An instantaneous thermalization regards a bond selected at random, where the quantity of interest (either the energy or the number of particles) is redistributed between the two sites according to the stationary measure of the original process at equilibrium on that bond, conditioning on the total amount of that quantity. More explicitly, we have that if L is the generator of the original process, then we defined the thermalized generator L^{th} as

$$(L^{th}f)(x) = \lim_{t\to\infty} \left(e^{tL} - 1\right) f(x) = \lim_{t\to\infty} \mathbb{E}_x f(X_t) - f(x) \ . \tag{2.26}$$

We now explain how a family of Kipnis-Marchioro-Presutti processes rises from a family of Brownian energy processes. The classical Kipnis-Marchioro-Presutti process (KMP) was first introduced by Kipnis, Marchioro and Presutti [41] in 1982 as a model of heat conduction that was solved by using a dual process. It is a stochastic model where a continuous non-negative variable (interpreted as energy) is uniformly redistributed among two sites on a lattice. A general version with parameter k, that we shall call KMP(k) was defined in [14], by considering a redistribution rule where a fraction p of the total energy is assigned to one particle and the remaining fraction $(1-p)$ to the other particle, with p a Beta$(2k, 2k)$ distributed random variable. Thus the case $k = 1/2$ corresponds to the original KMP model, with uniform redistribution between two sites.

In the KMP(k) model, the redistribution of $x_i + x_l$, on a bond (i, l) is done according to the stationary measure conditioned to the conservation of the total energy of the bond. If the two independent random variables X_i and X_l are distributed as Gamma with parameters $2k$ and θ, then the density function of one of them, say X_i, given that their sum is constant, $X_i + X_l = E$ is

$$f_{X_i|E}(x_i|E) = \frac{f_{X_i}(x_i) f_{X_l}(E - x_i)}{\int_{\mathbb{R}} f_{X_i}(v) f_{X_l}(E - v) dv} = \frac{x_i^{2k-1}(E - x_i)^{2k-1}}{\int_{\mathbb{R}} v^{2k-1}(E - v)^{2k-1} dv} \ ,$$

or equivalently, the random variable $p = \dfrac{X_i}{X_i + X_l}$ is distributed as a Beta of parameters ($2k$,$2k$). In [14] it was shown that KMP(k) is in turn related to the Brownian Energy Process with parameter k, as it can be obtained from the BEP(k) via an instantaneous thermalization limit as in equation (2.26). On the usual undirected and connected graph $G = (V, E)$ with $|V| = N$ sites and edge set E , the generator of the KMP(k) process is

$$L^{KMP(k)} f(\mathbf{x}) = \tag{2.27}$$

$$\sum_{\substack{1\le i<l\le N \\ (i,l)\in E}} \int_0^1 \left[f\left(x_1, \ldots, x_{i-1}, p(x_i + x_l), x_{i+1}, \ldots, x_{l-1}, (1-p)(x_i + x_l), \ldots, x_N\right) - f\left(\mathbf{x}\right)\right] \nu_{2k}(p) dp \tag{2.28}$$

where $\nu_{2k}(p)$ is the density function of the Beta distribution with parameters $(2k,\, 2k)$, i.e.

$$\nu_{2k}(p) = \frac{p^{2k-1}(1-p)^{2k-1}\Gamma(4k)}{\Gamma(2k)\Gamma(2k)}\ , \qquad p \in (0,1)\ . \tag{2.29}$$

Remark 2.4. As a consequence of the instantaneous thermalization, the KMP(k) process inherited from the BEP(k) process the same stationary and reversible measure. Moreover, as we will show in the next Chapter, duality and self-duality relations are conserved under this limit.

It will be useful to have the following conclusive remark regarding the shape of our generators.

Remark 2.5. All the above processes have state space $\Omega = \Omega_1 \times \cdots \times \Omega_N$, while the corresponding generators have the form

$$L = \sum_{i<j} L_{i,j}$$

where $L_{i,j}$ is an operator on $F(\Omega_i \times \Omega_j)$. This observation is crucial for our purpose, because it allows us to work with operators acting on functions of two variables, then the result can be extended to N variables.

Chapter 3

Orthogonal Dualities

3.1 Summary of the results

In this Chapter we prove three self-duality results between particles jump processes and two duality results between a diffusion and a jump process from which we will infer one last duality relation for processes of Kipnis-Marchioro-Presutti type. In particular, we try as ansatz for the duality and self-duality functions the hypergeometric polynomials (of continuous and discrete variables) which are orthogonal with respect to the reversible stationary measures of the processes introduced in the previous Chapter. It turns out that to have duality one has to make a (well defined) choice of the norm of these orthogonal polynomials, see also Remark 5.1.

The main idea is to show that the relation of (self-)duality between two generators in Definition 1.4 is verified using properties of the orthogonal polynomials such as their recurrence relations, the differential or difference equations as well as a raising operator equality. Once these relations are available then it is just a matter of (long!) computation to show that the (self-)duality relations holds.

The orthogonal dualities have been also proved in [56] with a totally different method relying on generating functions.

The orthogonal polynomials we use are some of those with hypergeometric structure. More precisely we consider classical orthogonal polynomials, both discrete and continuous, with the exception of discrete Hahn polynomials and continuous Jacobi polynomials. We will follow the definitions of orthogonal polynomials given in [42], see also Appendix A for a basic and non comprehensive theory on discrete and continuous polynomials as well as the definition of hypergeometric functions. The added value of linking duality functions to orthogonal polynomials lies on the fact that they constitute an orthogonal basis of the associated Hilbert space. Often in applications [13, 17, 37] some quantity of interest are expressed in terms of duality functions, for instance the current in interacting particle systems. This is then used in the study of the asymptotic properties and relevant scaling limits, see [?]. For these reasons it seems reasonable that having an orthogonal basis of polynomials should be useful in those analysis.

Similar results have been found in [56] with a generating function approach.

3.2 Self-duality results

We consider first self-duality relations for three discrete interacting particle systems. As already mentioned, in this case the dual process is an independent copy of the original one. Even if the original process and its dual are the same, a massive simplification occurs, namely the k-point correlation function of the original process can be expressed by duality in terms of only k dual particles. Thus a problem for many particles, possibly infinitely many in the infinite volume, may be studied via a finite number of dual walkers.

3.2.1 Self-duality for SEP(j)

The Symmetric Exclusion Process of parameter j introduced in Section 2.2 has marginal reversible measure the Binomial distribution, i.e.

$$\rho^{SEP}(x) = \binom{2j}{x} p^x (1-p)^{2j-x} \,, \qquad x \in \{0, 1, \ldots, 2j\} \,. \tag{3.1}$$

The polynomials orthogonal with respect to the Binomial distribution are the Krawtchouk polynomials $K_n(x)$ with parameter $2j$ [45]. In other words, they satisfy the orthogonality relation

$$\sum_{x=0}^{2j} K_n(x) K_m(x) \rho^{SEP}(x) = \delta_{n,m} d_n^2 \tag{3.2}$$

with norm in $\ell^2(\{0, 1, \ldots, 2j\}, \rho^{SEP})$ given by

$$d_n^2 = \frac{(2j-n)!n!}{(2j)!} \left(\frac{1-p}{p}\right)^n \,. \tag{3.3}$$

Their hypergeometric representation is

$$K_n(x) = {}_2F_1\left(\begin{matrix} -n, -x \\ -2j \end{matrix} \middle| \frac{1}{p}\right) \tag{3.4}$$

for $n = 0, 1, \ldots, 2j$. Krawtchouk polynomials are polynomial solutions of the finite difference equation (A.18)

$$x[K_n(x+1) - 2K_n(x) + K_n(x-1)] + \frac{2jp - x}{1-p}[K_n(x+1) - K_n(x)] + \frac{n}{1-p} K_n(x) = 0 \,. \tag{3.5}$$

As a consequence of the orthogonality they satisfy the three term recurrence relation (A.26)

$$x K_n(x) = -p(2j-n) K_{n+1}(x) + (n + 2jp - 2np) K_n(x) - (1-p) n K_{n-1}(x) \,. \tag{3.6}$$

Furthermore, the raising operator in (A.31) provides the relation

$$xK_n(x-1) + \frac{p}{1-p}(n+x-2j)K_n(x) = -(2j-n)\frac{p}{1-p}K_{n+1}(x) \ . \tag{3.7}$$

Next theorem shows how the Krawtchouk polynomials are self-duality functions for the symmetric exclusion process, the proof of the theorem relies on the structural properties of the Krawtchouk polynomials.

Theorem 3.1. *The SEP(j) with generator* (2.10) *is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i\in V} K_{n_i}(x_i) \tag{3.8}$$

where $K_n(x)$ denotes the Krawtchouk polynomial of degree n defined above.

Proof. We need to verify the self-duality relation in equation (1.9). Since the generator of the process is a sum of terms acting on two variables only, we shall verify the self-duality relation for two sites, say 1 and 2. We start by writing the action of the SEP(j) generator working on the duality function for these two sites:

$$\begin{aligned} L^{SEP(j)} D_{n_1}(x_1)D_{n_2}(x_2) &= x_1(2j-x_2)\left[D_{n_1}(x_1-1)D_{n_2}(x_2+1) - D_{n_1}(x_1)D_{n_2}(x_2)\right] \\ &+ (2j-x_1)x_2\left[D_{n_1}(x_1+1)D_{n_2}(x_2-1) - D_{n_1}(x_1)D_{n_2}(x_2)\right] \end{aligned} \tag{3.9}$$

rewriting this by factorizing site 1 and 2, i.e.

$$\begin{aligned} L^{SEP(j)} D_{n_1}(x_1)D_{n_2}(x_2) &= x_1D_{n_1}(x_1-1)(2j-x_2)D_{n_2}(x_2+1) - x_1D_{n_1}(x_1)(2j-x_2)D_{n_2}(x_2) \\ &+ (2j-x_1)D_{n_1}(x_1+1)x_2D_{n_2}(x_2-1) - (2j-x_1)D_{n_1}(x_1)x_2D_{n_2}(x_2) \end{aligned} \tag{3.10}$$

we see that we need an expression for the following terms:

$$xD_n(x) \ , \qquad xD_n(x-1) \ , \qquad (2j-x)D_n(x+1) \ . \tag{3.11}$$

To get those we first write the difference equation (3.5), the recurrence relation (3.6) and the raising operator equation (3.7) in terms of $D_n(x)$. Then the first term in (3.11) is simply obtained from the recurrence relation, whereas the second and third terms are provided by simple algebraic manipulations of the normalized raising operator equation and the normalized difference equation respectively. We get

$$xD_n(x) = -p(2j-n)D_{n+1}(x) + (n+2pj-2pn)D_n(x) - n(1-p)D_{n-1}(x) \tag{3.12}$$

$$xD_n(x-1) = -p(2j-n)D_{n+1}(x) + p(2j-2n)D_n(x) + npD_{n-1}(x) \tag{3.13}$$

$$(2j-x)D_n(x+1) = p(2j-n)D_{n+1}(x) + (1-p)(2j-2n)D_n(x) - \frac{n}{p}(1-p)^2D_{n-1}(x) \ . \tag{3.14}$$

These expressions can now be inserted into (3.10), which then reads:

$$
\begin{aligned}
&L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= [p(n_1 - 2j) D_{n_1+1}(x_1) + p(2j - 2n_1) D_{n_1}(x_1) + p n_1 D_{n_1-1}(x_1)] \times \\
&\left[p(2j - n_2) D_{n_2+1}(x_2) + (1-p)(2j - 2n_2) D_{n_2}(x_2) - \frac{n_2}{p}(1-p)^2 D_{n_2-1}(x_2) \right] \\
&+ [p(2j - n_1) D_{n_1+1}(x_1) - (n_1 + 2jp - 2pn_1) D_{n_1}(x_1) + (1-p) n_1 D_{n_1-1}(x_1)] \times \\
&[p(2j - n_2) D_{n_2+1}(x_2) - (n_2 + 2pj - 2pn_2) D_{n_2}(x_2) + n_2(1-p) D_{n_2-1}(x_2) + 2j D_{n_2}(x_2)] \\
&+ [p(n_2 - 2j) D_{n_2+1}(x_2) + p(2j - 2n_2) D_{n_2}(x_2) + p n_2 D_{n_2-1}(x_2)] \times \\
&\left[p(2j - n_1) D_{n_1+1}(x_1) + (1-p)(2j - 2n_1) D_{n_1}(x_1) - \frac{n_1}{p}(1-p)^2 D_{n_1-1}(x_1) \right] \\
&+ [p(2j - n_2) D_{n_2+1}(x_2) - (n_2 + 2pj - 2pn_2) D_{n_2}(x_2) + (1-p) n_2 D_{n_2-1}(x_2)] \times \\
&[p(2j - n_1) D_{n_1+1}(x_1) - (n_1 + 2pj - 2pn_1) D_{n_1}(x_1) + n_1(1-p) D_{n_1-1}(x_1) + 2j D_{n_1}(x_1)] \ .
\end{aligned}
$$

Working out the algebra, substantial simplifications are revealed in the above expression. A long but straightforward computation shows that only products of polynomials with degree $n_1 + n_2$ survive. In particular, after simplifications, one is left with

$$
\begin{aligned}
L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) &= n_1(2j - n_2) \left[D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2) \right] \\
&+ (2j - n_1) n_2 \left[D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2) \right]
\end{aligned}
$$

and the theorem is proved. □

3.2.2 Self-duality for SIP(k)

We introduced in Section 2.3 the symmetric inclusion process with parameter k and with reversible measures given by product of identical Negative Binomial distributions with parameters $2k > 0$ and $0 < p < 1$, i.e.

$$
\rho^{SIP}(x) = \binom{2k + x - 1}{x} p^x (1-p)^{2k} , \qquad x \in \{0, 1, \ldots\} . \tag{3.15}
$$

The polynomials that are orthogonal with respect to the Negative Binomial distribution are the Meixner polynomials $M_n(x)$ with parameter $2k$, first introduced in [52]. They satisfy the orthogonal relation

$$
\sum_{x=0}^{\infty} M_n(x) M_m(x) \rho^{SIP}(x) = \delta_{m,n} d_n^2 \tag{3.16}
$$

with norm in $\ell^2(\mathbb{N}_0, \rho^{SIP})$ given by

$$
d_n^2 = \frac{n! \Gamma(2k)}{p^n \Gamma(2k + n)} \tag{3.17}
$$

where $\Gamma(x)$ is the Gamma function. As hypergeoemtric function $M_n(x)$ is

$$M_n(x) = {}_2F_1\left(\begin{matrix} -n_i, -x_i \\ 2k \end{matrix} \middle| 1 - \frac{1}{p} \right) . \tag{3.18}$$

Meixner polynomials are solutions of the difference equation (A.18)

$$x\left[M_n(x+1) - 2M_n(x) + M_n(x-1)\right] + (2kp - x + xp)\left[M_n(x+1) - M_n(x)\right] \tag{3.19}$$
$$+\, n(1-p)M_n(x) = 0 . \tag{3.20}$$

As consequence of the orthogonality they satisfy the recurrence relation (A.26)

$$(p-1)xM_n(x) = p(2k+n)M_{n+1}(x) - (n + pn + 2kp)M_n(x) + nM_{n-1}(x) . \tag{3.21}$$

Furthermore the raising operator in equation (A.31) provides the identity

$$[p(n + 2k + x)]M_n(x) - xM_n(x-1) = p(2k+n)M_{n+1}(x) . \tag{3.22}$$

In analogy with the result for the Exclusion process it is possible to find a duality function for the Symmetric Inclusion Process in terms of the Meixner polynomials.

Theorem 3.2. *The SIP(k) is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i\in V} M_{n_i}(x_i) \tag{3.23}$$

where $M_n(x)$ is the Meixner polynomial of degree n.

Proof. As was done for the Exclusion Process we verify the self-duality relation in equation (1.9) for two sites, say 1 and 2. The action of the $SIP(k)$ generator working on the self-duality function for two sites is given by

$$\begin{aligned} L^{SIP(k)} D_{n_1}(x_1)D_{n_2}(x_2) &= x_1(2k+x_2)\left[D_{n_1}(x_1-1)D_{n_2}(x_2+1) - D_{n_1}(x_1)D_{n_2}(x_2)\right] \\ &+ (2k+x_1)x_2\left[D_{n_1}(x_1+1)D_{n_2}(x_2-1) - D_{n_1}(x_1)D_{n_2}(x_2)\right] . \end{aligned} \tag{3.24}$$

We rewrite this by factorizing site 1 and 2, i.e.

$$\begin{aligned} L^{SIP(k)} D_{n_1}(x_1)D_{n_2}(x_2) &= x_1D_{n_1}(x_1-1)(2k+x_2)D_{n_2}(x_2+1) - x_1D_{n_1}(x_1)(2k+x_2)D_{n_2}(x_2) \\ &+ (2k+x_1)D_{n_1}(x_1+1)x_2D_{n_2}(x_2-1) - (2k+x_1)D_{n_1}(x_1)x_2D_{n_2}(x_2) \end{aligned} \tag{3.25}$$

so that we now need an expression for the following terms:

$$xD_n(x) , \qquad xD_n(x-1) , \qquad (2k+x)D_n(x+1) . \tag{3.26}$$

To get those, we first write the difference equation (3.19), the recurrence relation (3.21) and the raising operator equation (3.22) in terms of $D_n(x)$. Then the first term in (3.26) is simply obtained from the recurrence relation, whereas the second and third terms are provided by simple algebraic manipulations of the normalized raising operator equation and the normalized difference equation. We have,

$$xD_n(x) = \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{n+p(n+2k)}{p-1}D_n(x) + \frac{n}{p-1}D_{n-1}(x) \tag{3.27}$$

$$xD_n(x-1) = \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{p}{p-1}(2k+2n)D_n(x) + \frac{p}{p-1}nD_{n-1}(x) \tag{3.28}$$

$$(2k+x)D_n(x+1) = \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{1}{p-1}(2k+2n)D_n(x) + \frac{1}{p-1}nD_{n-1}(x)\ . \tag{3.29}$$

These relations allow us to expand the SIP(k) generator in equation (3.25) as

$$\begin{aligned}
&L^{SIP}D_{n_1}(x_1)D_{n_2}(x_2) \\
&\quad= \left[\frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{p(2k+2n_1)}{p-1}D_{n_1}(x_1) + \frac{pn_1}{p-1}D_{n_1-1}(x_1)\right] \times \\
&\quad\left[\frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{2k+2n_2}{p-1}D_{n_2}(x_2) + \frac{n_2}{p-1}D_{n_2-1}(x_2)\right] \\
&\quad- \left[\frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{n_1+p(n_1+2k)}{p-1}D_{n_1}(x_1) + \frac{n_1}{p-1}D_{n_1-1}(x_1)\right] \times \\
&\quad\left[\frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{n_2+p(n_2+2k)}{p-1}D_{n_2}(x_2) + \frac{n_2}{p-1}D_{n_2-1}(x_2) + 2kD_{n_2}(x_2)\right] \\
&\quad+ \left[\frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{p(2k+2n_2)}{p-1}D_{n_2}(x_2) + \frac{pn_2}{p-1}D_{n_2-1}(x_2)\right] \times \\
&\quad\left[\frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{2k+2n_1}{p-1}D_{n_1}(x_1) + \frac{n_1}{p-1}D_{n_1-1}(x_1)\right] \\
&\quad- \left[\frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{n_2+p(n_2+2k)}{p-1}D_{n_2}(x_2) + \frac{n_2}{p-1}D_{n_2-1}(x_2)\right] \times \\
&\quad\left[\frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{n_1+p(n_1+2k)}{p-1}D_{n_1}(x_1) + \frac{n_1}{p-1}D_{n_1-1}(x_1) + 2kD_{n_1}(x_1)\right]\ .
\end{aligned}$$

At this point it is sufficient to notice that the coefficients of products of polynomials with degree different than n_1+n_2 are all zero, so that we are left with

$$\begin{aligned}
L^{SIP(k)}D_{n_1}(x_1)D_{n_2}(x_2) &= n_1(2k+n_2)\left[D_{n_1-1}(x_1)D_{n_2+1}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)\right] \\
&+ (2k+n_1)n_2\left[D_{n_1+1}(x_1)D_{n_2-1}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)\right]
\end{aligned}$$

and the theorem is proved. □

3.2.3 Self-duality for IRW

Recall the description of the process in Section 2.4, where the reversible measure has Poisson marginals

$$\rho^{IRW}(x) = \frac{e^{-\lambda}\lambda^x}{x!} , \qquad x \in \mathbb{N}_0 . \tag{3.30}$$

The polynomials orthogonal with respect to the Poisson distribution are the Charlier polynomials $C_n(x)$ [18] with parameter λ. In other words, Charlier polynomials satisfy the following orthogonal relation

$$\sum_{x=0}^{\infty} C_n(x)C_m(x)\rho^{IRW}(x) = \delta_{m,n}d_n^2 \tag{3.31}$$

where the norm in $\ell^2(\mathbb{N}_0, \rho^{IRW})$ is

$$d_n^2 = n!\lambda^{-n} . \tag{3.32}$$

They are solutions of the second order difference equation (A.18)

$$x\left[C_n(x+1) - 2C_n(x) + C_n(x-1)\right] + (\lambda - x)\left[C_n(x+1) - C_n(x)\right] + nC_n(x) = 0 . \tag{3.33}$$

As a consequence of the orthogonality, they satisfy the three term recurrence relation (A.26)

$$xC_n(x) = -\lambda C_{n+1}(x) + (n+\lambda)C_n(x) - nC_{n-1}(x) . \tag{3.34}$$

Furthermore, the raising operator has the form

$$\lambda C_n(x) - xC_n(x-1) = \lambda C_{n+1}(x) . \tag{3.35}$$

As the following theorem shows, the self-duality relation is given by the Charlier polynomials themselves.

Theorem 3.3. *The IRW is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i \in V} C_{n_i}(x_i) \tag{3.36}$$

where $C_n(x)$ is the Charlier polynomial of degree n.

Remark 3.1. Reading the Charlier polynomial as an hypergeometric function, the duality function then becomes

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} {}_2F_0\left(\begin{matrix} -n_i, -x_i \\ - \end{matrix}\middle| -\frac{1}{\lambda}\right) . \tag{3.37}$$

Proof. It is clear from (3.36) that the difference equations, the recurrence relations and the raising operator for $D_n(x)$ are respectively (3.33), (3.34) and (3.35), that we rewrite as:

$$D_n(x+1) = D_n(x) - \frac{n}{\lambda}D_{n-1}(x) \tag{3.38}$$

$$xD_n(x) = -\lambda D_{n+1}(x) + (n+\lambda)D_n(x) - nD_{n-1}(x) \tag{3.39}$$

$$xD_n(x-1) = \lambda D_n(x) - \lambda D_{n+1}(x). \tag{3.40}$$

As done before, we use the two particles IRW generator in (2.16) and the three equations above to check that the self-duality relation holds. We have

$$\begin{aligned}
L^{IRW} D_{n_1}(x_1) D_{n_2}(x_2) &= x_1 D_{n_1}(x_1-1) D_{n_2}(x_2+1) - x_1 D_{n_1}(x_1) D_{n_2}(x_2) \\
&+ D_{n_1}(x_1+1) x_2 D_{n_2}(x_2-1) - D_{n_1}(x_1) x_2 D_{n_2}(x_2) \\
&= [\lambda D_{n_1}(x_1) - \lambda D_{n_1+1}(x_1)] \left[D_{n_2}(x_2) - \frac{n_2}{\lambda} D_{n_2-1}(x_2) \right] \\
&- [-\lambda D_{n_1+1}(x_1) + (n_1+\lambda) D_{n_1}(x_1) - n_1 D_{n_1-1}(x_1)] [D_{n_2}(x_2)] \\
&+ \left[D_{n_1}(x_1) - \frac{n_1}{\lambda} D_{n_1+1}(x_2) \right] [\lambda D_{n_2}(x_2) - \lambda D_{n_2+1}(x_2)] \\
&- [D_{n_1}(x_1)] [-\lambda D_{n_2+1}(x_2) + (n_2+\lambda) D_{n_2}(x_2) - n_2 D_{n_2-1}(x_2)] \ .
\end{aligned}$$

After computing the products and suitable simplifications we get

$$\begin{aligned}
L^{IRW} D_{n_1}(x_1) D_{n_2}(x_2) \quad &= n_1 [D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)] \\
&+ n_2 [D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)] \ .
\end{aligned}$$

□

3.3 Duality results

We end the Chapter by showing two examples of duality: the initial process is an interacting diffusion, while the dual one is a jump process, which, in particular, turns out to be the SIP process introduced in Chapter 2.3. We also show an example of duality for a redistribution model of Kipnis-Marchioro-Presutti type.

3.3.1 Duality for the BMP

In Section 2.5 we have seen that the marginal reversible measure of the BMP process is

$$\rho^{BMP}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \ , \tag{3.41}$$

where, without loss of generality, we fix $\sigma = \frac{1}{2}$. The polynomials orthogonal with respect to ρ^{BMP} are the Hermite polynomials [54], i.e.

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) \rho^{BMP}(x) dx = \delta_{m,n} d_n^2 \tag{3.42}$$

with norm in $L^2(\mathbb{R}, \rho^{BMP})$ given by

$$d_n^2 = 2^n n! \ . \tag{3.43}$$

In terms of hypergeometric function, they are written as

$$H_n(x) = (2x)^n {}_2F_0\left(\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix} \middle| -\frac{1}{x^2} \right) . \tag{3.44}$$

Hermite polynomials are solutions of the second order differential equation (A.2)

$$H_n^{''}(x) - 2xH_n^{'}(x) + 2nH_n(x) = 0 . \tag{3.45}$$

As consequence of the orthogonality they satisfy the three term recurrence relation (A.11)

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 . \tag{3.46}$$

Furthermore, the raising operator is given by

$$2xH_n(x) - H_n^{'}(x) = H_{n+1}(x). \tag{3.47}$$

The following theorem has a similar version in [8] and it states the duality result between the Brownian momentum process and the Symmetric inclusion process of parameter $k = 1/4$ involving the Hermite polynomials. Indeed, the duality function is not a product of Hermite polynomials themselves, but it is provided with a suitable normalization of the Hermite polynomials of even degree.

Theorem 3.4. *The BMP process is dual to the SIP($\frac{1}{4}$) process through duality function*

$$D_{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i \in V} \frac{1}{(2n_i - 1)!!} H_{2n_i}(x_i) \tag{3.48}$$

where $H_{2n}(x)$ is the Hermite polynomial of degree $2n$.

Proof. Although the proof in [8] can be easily adapted to our case, we show here an alternative proof that follows our general strategy of using the structural properties of Hermite polynomials. It is sufficient, as before, to show the duality relation in equation (1.6) for sites 1 and 2. The action of the BMP generator on duality function reads

$$\begin{aligned} L^{BMP} D_{n_1}(x_1) D_{n_2}(x_2) &= (x_1 \partial_{x_2} - x_2 \partial_{x_1})^2 D_{n_1}(x_1) D_{n_2}(x_2) \\ &= x_1^2 D_{n_1}(x_1) D''_{n_2}(x_2) + D''_{n_1}(x_1) x_2^2 D_{n_2}(x_2) - x_1 D'_{n_1}(x_1) D_{n_2}(x_2) \\ &\quad - D_{n_1}(x_1) x_2 D'_{n_2}(x_2) - 2x_1 D'_{n_1}(x_1) x_2 D'_{n_2}(x_2) \end{aligned} \tag{3.49}$$

where we use $\partial_{x_i} = \frac{\partial}{\partial x_i}$. We now need the recurrence relation and the raising operator appropriately rewritten in term of the duality function in order to get suitable expression for

$$x^2 D_n(x), \qquad D''_n(x), \qquad xD'_n(x). \tag{3.50}$$

This can be done using

$$D_n(x) = \frac{1}{(2n-1)!!} H_{2n}(x) \tag{3.51}$$

so that

$$x^2 D_n(x) = \frac{1}{4}(2n+1)D_{n+1}(x) + \left(2n + \frac{1}{2}\right) D_n(x) + 2nD_{n-1}(x) \tag{3.52}$$

$$D_n^{''}(x) = 8nD_{n-1}(x) \tag{3.53}$$

$$xD_n^{'}(x) = 2nD_n(x) + 4nD_{n-1}(x) \tag{3.54}$$

where (3.52) is obtained from iterating twice the recurrence relation in (3.46), for equation (3.54) we combined (3.46) and (3.47) and then (3.53) is found from the differential equation (3.45) using (3.54). Proceeding with the substitution into the generator we find

$$\begin{aligned}
&L^{BMP} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= \left(\frac{1}{4}(2n_1+1)D_{n_1+1}(x_1) + \left(2n_1 + \frac{1}{2}\right) D_{n_1}(x_1) + 2n_1 D_{n_1-1}(x_1)\right) 8n_2 D_{n_2-1}(x_2) \\
&+ 8n_1 D_{n_1-1}(x_1) \left(\frac{1}{4}(2n_2+1)D_{n_2+1}(x_2) + \left(2n_2 + \frac{1}{2}\right) D_{n_2}(x_2) + 2n_2 D_{n_2-1}(x_2)\right) \\
&- \left(2n_1 D_{n_1}(x_1) + 4n_1 D_{n_1-1}(x_1)\right) D_{n_2}(x_2) - D_{n_1}(x_1) \left(2n_1 D_{n_2}(x_2) + 4n_2 D_{n_2-1}(x_2)\right) \\
&- 2\left(2n_1 D_{n_1}(x_1) + 4n_1 D_{n_1-1}(x_1)\right) \left(2n_2 D_{n_2}(x_2) + 4n_2 D_{n_2-1}(x_2)\right) \ .
\end{aligned}$$

Finally, after appropriate simplification of the terms whose degree is different from $n_1 + n_2$, we get

$$\begin{aligned}
L^{BMP} D_{n_1}(x_1) D_{n_2}(x_2) &= (2n_1+1)2n_2 \left[D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)\right] \\
&+ 2n_1(2n_2+1) \left[D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{2n_1}(x_1) D_{n_2}(x_2)\right] \\
&= L^{SIP} D_{n_1}(x_1) D_{n_2}(x_2)
\end{aligned}$$

which proves the theorem. □

3.3.2 Duality for BEP(k)

The Brownian energy process, introduced in Section 2.6, has as reversible measure products of Gamma distributions, whose density is

$$\rho^{BEP}(x) = \frac{x^{2k-1} e^{-x}}{\Gamma(2k)}. \tag{3.55}$$

The generalized Laguerre polynomials $L_n^{(2k-1)}(x)$ [54] are orthogonal with respect to ρ^{BEP}, i.e. they satisfy the orthogonal relation

$$\int_0^{+\infty} L_n^{(2k-1)}(x) L_m^{(2k-1)}(x) \rho^{BEP}(x) dx = \delta_{m,n} d_n^2 \tag{3.56}$$

with norm in $L^2(\mathbb{R}^+, \rho^{BEP})$ given by

$$d_n^2 = \frac{\Gamma(n+2k)}{n!\Gamma(2k)} \ . \tag{3.57}$$

They can be defined via their hypergeometric representation as

$$L_n^{(2k-1)}(x) = \frac{\Gamma(2k+n)}{n!\Gamma(2k)} {}_1F_1\left(\begin{matrix}-n\\2k\end{matrix}\middle| x\right) . \tag{3.58}$$

Generalized Laguerre polynomials are solutions of (A.2)

$$x\frac{d^2}{dx^2}L_n^{(2k-1)}(x) + (2k-x)\frac{d}{dx}L_n^{(2k-1)}(x) + nL_n^{(2k-1)}(x) = 0 \tag{3.59}$$

and they satisfy the recurrence relation (A.11)

$$xL_n^{(2k-1)}(x) = -(n+1)L_{n+1}^{(2k-1)}(x) + (2n+2k)L_n^{(2k-1)}(x) - (n+2k-1)L_{n-1}^{(2k-1)}(x) . \tag{3.60}$$

Furthermore, the raising operator in equation (A.16) is given by

$$(2k-x+n)L_n^{(2k-1)}(x) + x\frac{d}{dx}L_n^{(2k-1)}(x) = (n+1)L_{n+1}^{(2k-1)}(x) . \tag{3.61}$$

The duality relation for the Brownian energy process with parameter k is stated below.

Theorem 3.5. *The BEP(k) process and the SIP(k) process are dual via*

$$D_{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i\in V} \frac{n_i!\,\Gamma(2k)}{\Gamma(2k+n_i)} L_{n_i}^{(2k-1)}(x_i) = \prod_{i\in V} {}_1F_1\left(\begin{matrix}-n_i\\2k\end{matrix}\middle| x_i\right) \tag{3.62}$$

where $L_n^{(2k-1)}(x)$ *is the generalized Laguerre polynomial of degree* n*.*

Remark 3.2. The factor $\Gamma(2k)$ in equation (3.62) is not crucial to assess a duality relation, however, it allows to write the duality function as the hypergeometric function ${}_1F_1$.

Proof. As in the previous cases we notice that the proof can be shown for sites 1 and 2 only, in which case the generator of the BEP acts on

$$\begin{aligned}
L^{BEP(k)}D_{n_1}(x_1)D_{n_2}(x_2) = &\left[x_1x_2\left(\partial_{x_1}-\partial_{x_2}\right)^2 - 2k(x_1-x_1)(\partial_{x_1}-\partial_{x_2})\right] D_{n_1}(x_1)D_{n_2}(x_2)\\
=&(x_1\partial_{x_1}^2 + 2k\partial_{x_1})D_{n_1}(x_1)x_2D_{n_2}(x_2) + x_1D_{n_1}(x_1)(x_2\partial_{x_2}^2 + 2k\partial_{x_2})D_{n_2}(x_2)\\
&-x_1\partial_{x_1}D_{n_1(x_1)}(x_2\partial_{x_2}+2k)D_{n_2}(x_2) - (x_1\partial_{x_1}+2k)D_{n_1}(x_1)x_2\partial_{x_2}D_{n_2}(x_2) .
\end{aligned}$$

We seek an expression for

$$x\partial_x^2D_n + 2k\partial_xD_n, \qquad xD_n, \qquad x\partial_xD_n \tag{3.63}$$

that can easily be obtained rewriting (3.59), (3.60) and (3.61) for the duality function, using

$$D_n(x) = \frac{n!\,\Gamma(2k)}{\Gamma(2k+n)}L_n^{(2k-1)}(x) \tag{3.64}$$

so that, after simple manipulation

$$xD_n^{''}(x) + (2k - x)D_n^{'}(x) + nD_n(x) = 0 \tag{3.65}$$

$$xD_n(x) = -(n + 2k)D_{n+1}(x) + (2n + 2k)D_n(x) - nD_{n-1}(x) \tag{3.66}$$

$$xD_n^{'}(x) = nD_n(x) - nD_{n-1}(x) \ . \tag{3.67}$$

Note that plugging (3.67) into the difference equation (3.65), we get

$$xD_n^{''}(x) + 2kD_n^{'}(x) = -nD_{n-1}(x) \ .$$

Let's now use these information to write explicitly the BEP(k) generator.

$$\begin{aligned}
L^{BEP(k)}&D_{n_1}(x_1)D_{n_2}(x_2) = \\
&+ [-n_1D_{n_1-1}(x_1)]\,[-(2k+n_2)D_{n_2+1}(x_2) + (2n_2+2k)D_{n_2}(x_2) - n_2D_{n_2-1}(x_2)] \\
&+ [-(2k+n_1)D_{n_1+1}(x_1) + (2n_1+2k)D_{n_1}(x_1) - n_1D_{n_1-1}(x_1)]\,[-n_2D_{n_2-1}(x_2)] \\
&- [n_1D_{n_1}(x_1) - n_1D_{n_1-1}(x_1)]\,[(n_2+2k)D_{n_2}(x_2) - n_2D_{n_2-1}(x_2)] \\
&- [(n_1+2k)D_{n_1}(x_1) - n_1D_{n_1-1}(x_1)]\,[n_2D_{n_2}(x_2) - n_2D_{n_2-1}(x_2)] \ .
\end{aligned}$$

Expanding products in the above expression we find

$$\begin{aligned}
L^{BEP(k)}&D_{n_1}(x_1)D_{n_2}(x_2) \\
&= n_1D_{n_1-1}(x_1)(n_2+2k)D_{n_2+1}(x_2) + (n_1+2k)D_{n_1+1}(x_1)n_2D_{n_2-1}(x_2) \\
&+ n_1D_{n_1}(x_1)(n_2+2k)D_{n_2}(x_2) + (n_1+2k)D_{n_1+1}(x_1)n_2D_{n_2-1}(x_2) \\
&+ D_{n_1-1}(x_1)D_{n_2}(x_2)\,[-n_1(2n_2+2k) + n_1(n_2+2k) + n_1n_2] \\
&+ D_{n_1}(x_1)D_{n_2-1}(x_2)\,[-(2n_1+2k)n_1 + (n_1+2k)n_2 + n_1n_2] \\
&+ D_{n_1-1}(x_1)D_{n_2-1}(x_2)\,[n_1n_2 + n_1n_2 - n_1n_2 - n_1n_2] \ .
\end{aligned}$$

Noticing that the coefficients of the last three lines are zeros, we finally get

$$\begin{aligned}
L^{BEP(k)}D_{n_1}(x_1)D_{n_2}(x_2) &= n_1(n_2+2k)\,[D_{n_1-1}(x_1)D_{n_2+1}(x_2) - D_{n_1}(x_1)D_{n_2-1}(x_2)] \\
&+ (n_1+2k)n_2\,[D_{n_1+1}(x_1)D_{n_2-2}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)] \\
&= L^{SIP(k)}D_{n_1}(x_1)D_{n_2}(x_2)
\end{aligned} \tag{3.68}$$

where $L^{SIP(k)}$ works on the dual variables (n_1, n_2). □

3.3.3 Duality for the KMP(k)

The generalized Kipnis Marchioro Presutti procces introduced in Section 2.7 has a dual process with generator (see [14])

$$L^{\text{dual-}KMP(k)} f(\mathbf{n}) = \tag{3.69}$$
$$\sum_{\substack{1 \leq i < l \leq N \\ (i,l) \in E}} \sum_{r=0}^{n_i + n_{i+1}} \left[f\left(n_1, \ldots, n_{i-1}, r, n_{i+1}, \ldots, n_{l-1}, n_i + n_l - r, \ldots, n_N\right) - f\left(\mathbf{n}\right)\right] \mu_{2k}(r \mid n_i + n_{i+1})$$

where $\mu_{2k}(r|C)$ is the mass density function of the Beta Binomial distribution with parameters $(C,\, 2k,\, 2k)$, i.e.

$$\mu_{2k}(r \mid C) = \frac{\binom{2k+r-1}{r}\binom{2k+C-r-1}{C-r}}{\binom{4k+C-1}{C}}\,, \qquad r \in \{0, 1, \ldots, C\}\ . \tag{3.70}$$

This generator is the result of a thermalized limit of the SIP(k) (page 17 of [14]). The last theorem of this Chapter is stated below.

Theorem 3.6. *The KMP(k) is dual to dual-KMP(k)with duality function*

$$D_{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i \in V} \frac{n_i!\, \Gamma(2k)}{\Gamma(2k + n_i)} L_{n_i}^{(2k-1)}(x_i) \tag{3.71}$$

where $L_n^{(2k-1)}(x)$ *is the generalized Laguerre polynomial of degree* n*.*

Proof. As expected, the duality function is the same as the one for the BEP(k) and SIP(k). This shouldn't surprise since BEP(k) and SIP(k) are dual through duality function (3.71) and the thermalization limit doesn't affect the duality property. Indeed, considering two graph vertices, one has from [14]

$$L^{KMP(k)} f(x_1, x_2) = \lim_{t \to \infty} (e^{tL^{BEP(k)}} - I) f(x_1, x_2) \tag{3.72}$$

and

$$L^{\text{dual-}KMP(k)} f(n_1, n_2) = \lim_{t \to \infty} (e^{tL^{SIP(k)}} - I) f(n_1, n_2)\ . \tag{3.73}$$

Thus, combining the previous two equations and (3.68), the claim follows.

□

Chapter 4

Lie algebra representations

This Chapter is a non-comprehensive review of Lie algebras and representation theory: the expert reader could skip the whole Chapter without being affected. However, we would like to give a quick review of Lie algebra representation assuming no previous knowledge of the subject. We will only recall the main definitions and results which are needed in the context of stochastic duality. References on the subject and more details about Lie algebras and their representation can be found in [36]. In general a Lie algebra description starts with the introduction of the commutator also known as Lie bracket $[x, y] = xy - yx$ where the operation on the right hand side are the usual ones. Let's start by introducing what a Lie algebra is.

Definition 4.1 (Lie algebra)**.** *A finite dimensional linear space* $\mathfrak{g}$ *over a field* F *, together with an operation* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, denoted with the Lie bracket* $(x, y) \to [x, y]$ *of* x *and* y*, is called a Lie algebra over* F *if the following axioms are satisfied:*

- $(x, y) \to [x, y]$ *is bilinear* (4.1)
- $[x, y] = -[y, x]$ *(anti-commutative)* (4.2)
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity)* (4.3)

for all $x, y, z \in \mathfrak{g}$*.*

As field F we will usually assume it to be the set of complex numbers $\mathbb{C}$. The dimension of a Lie algebra is its dimension as a vector space over $\mathbb{C}$. We say that two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if there exists a vector space isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$, in this case ϕ is called an *isomorphism* for the two Lie algebras. A $*$-*structure* on a Lie algebra $\mathfrak{g}$ is a map $* : \mathfrak{g} \to \mathfrak{g}$ such that

- $(x^*)^* = x$
- $(ax + by)^* = \bar{a}x^* + \bar{b}y^*$
- $[x, y]^* = [y^*, x^*]$.

where $\bar{\cdot}$ stands for the complex conjugate. If V is equipped with an inner product $\langle \cdot, \cdot \rangle$ and $\cdot^*$ denotes the adjoint with respect to this inner product, i.e.

$$\langle Av, w \rangle = \langle v, A^* w \rangle$$

then $A \to A^*$ is an adjoint operation on $\mathfrak{g}$. If $\{x_1, x_2, \ldots x_n\}$ is a basis for $\mathfrak{g}$, then the Lie bracket on $\mathfrak{g}$ is uniquely characterized by the commutation relations

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ijk} x_k \quad \text{for} \quad 1 \leq i < j \leq n \; . \tag{4.4}$$

where c_{ijk} are said structure constants. The *center* of a Lie algebra $\mathfrak{g}$ is the set

$$\{x \in \mathfrak{g} : [x, a] = 0 \;\; \forall \; a \in \mathfrak{g}\} \; .$$

We say that the center is trivial if it contains only the zero element. The *conjugate* algebra of $\mathfrak{g}$ is $\bar{\mathfrak{g}}$ such that

$$[\bar{x}, \bar{y}] = [y, x] \; . \tag{4.5}$$

Definition 4.2 (Homomorphism)**.** *A Lie algebra homomorphism is a linear map* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *such that*

$$\phi\left([x, y]_{\mathfrak{g}_1}\right) = [\phi(x), \phi(x)]_{\mathfrak{g}_2} \, \forall \; x, y \in \mathfrak{g}_1 \; .$$

The homomorphism ϕ is unitary if it preserves the $*-$structure of the adjoint operation, i.e. $\phi(A^*) = \phi(A)^*$. If ϕ is invertible then its inverse is also a Lie algebra homomorphism and in this case we say that ϕ is a Lie algebra isomorphism.

If V is a finite dimensional vector space over F, we denote by $\mathfrak{gl}(V)$ the set of linear transformation $A : V \to V$, equipped with the bracket operation $[x, y] = xy - yx$ then $\mathfrak{gl}(V)$ becomes a Lie algebra over F since the axioms in equations (4.1), (4.2) and (4.3) are immediately satisfied. It is sometimes useful and convenient to fix a basis for V and identify $\mathfrak{gl}(V)$ with the set of all $n \times n$ matrices over F, denoted by $\mathfrak{gl}(n, F)$.

Definition 4.3 (Representation)**.** *A representation of* $\mathfrak{g}$ *is a pair* (ρ, V) *where* ρ *is a Lie algebra homomorphism and* V *is a vector space* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$*.*

A representation is *irreducible* if its only invariant subspaces are $W = V$ or $W = \{0\}$, where an invariant subspace of V is a linear subspace $W \subset V$ such that $xw \in W$ for all $w \in W$ and $x \in \mathfrak{g}$. A representation is *unitary* if the homomorhism ρ is unitary, it is *faithful* if ρ is an isomorphism to its image, $\rho(\mathfrak{g}) = \{\rho(x) : x \in \mathfrak{g}\}$. Let V, U be two representations of the same Lie algebra $\mathfrak{g}$, then an *intertwiner* of the two representations is a linear map $\phi : V \to U$ that preserves the structure of the representation, i.e.,

$$\phi(xv) = x\phi(v) \quad \forall \; x \in \mathfrak{g}, \; v \in V \; .$$

Definition 4.4 (Equivalent representations). *Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of $\mathfrak{g}$, they are said to be equivalent if there existx a nonsingular linear transformation $T : V_1 \to V_2$ with $\rho_2(x) = T\rho_1(x)T^{-1}$ for all $x \in \mathfrak{g}$.*

Proposition 4.1 (Schur's lemma).

1. *Let V and U be irreducible representations of the same Lie algebra and let $\phi : V \to U$ be an intertwiner. Then either $\phi = 0$ or ϕ is an isomorphism.*

2. *Let V be an irreducible representation of a same Lie algebra and let $\phi : V \to V$ be an intertwiner. Then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Using Schur's lemma it is possible to prove the following important corollary

Corollary 4.1. *Let (ρ_1, V_1) and (ρ_2, V_2) be isomorphic irreducible representations of some Lie algebra. Then there exists an intertwiner $\phi : V_1 \to V_2$ that is unique up to a multiplicative constant, such that*

$$\phi\rho_1(x) = \rho_2(x)\phi \ .$$

Proof. By assumption, V_1 and V_2 are isomorphic, so there exists an isomorphism $\phi : V_1 \to V_2$. Assume that $\psi : V_1 \to V_2$ is another intertwiner. Then $\phi^{-1} \circ \psi$ is an intertwiner from V_1 into itself, so by the second item of Schur's lemma,$\phi^{-1} \circ \psi = \lambda I$ and hence $\psi = \lambda\phi$. □

For an arbitrary Lie algebra $\mathfrak{g}$ it is always possible to construct its universal enveloping algebra.

Definition 4.5 (Universal enveloping algebra). *For every Lie algebra $\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$ is defined by the pair $(U(\mathfrak{g}), i)$, where U is an associative algebra with unity and $i : \mathfrak{g} \to U(\mathfrak{g})$ is a linear map satisfying*

$$i([x, y]) = i(x)i(y) - i(y)i(x)$$

for $x, y \in \mathfrak{g}$.

So any Lie algebra can naturally be embedded in an algebra, the universal enveloping algebra; existence and uniqueness of the pair $(U(\mathfrak{g}), i)$ is not hard to establish (see Chapter 5 of [36]). We now give three concrete example of Lie algebras and their representation given in terms of (unbounded) operators, we assume that the operators act on an appropriate dense subspace of the L^2-space. These representation will be recalled in Chapters 5 and 6 where we constructively show (self-)duality.

We now introduce three different algebras, namely the the Lie algebra $\mathfrak{su}(2)$, the Lie algebra $\mathfrak{su}(1, 1)$ and the the Heisenberg algebra. For each one we will present two different representations in the spirit of Definition 4.3. It will be useful to recall that the *coproduct* of Lie algebra elements X is denoted by $\Delta(X)$ and defined via the tensor product $\otimes$, as

$$\Delta(X) = 1 \otimes X + X \otimes 1 \tag{4.6}$$

and that it can be extended as an algebra homomorphism to the universal enveloping algebra.

4.1 The Lie algebra $\mathfrak{su}(2)$

The Lie algebra $\mathfrak{su}(2)$ is the three dimensional complex Lie algebra defined by commutators between its elements s_x, s_y and s_z:

$$[s_x, s_y] = 2is_z \qquad [s_y, s_z] = 2is_x \qquad [s_z, s_x] = 2is_y \; . \tag{4.7}$$

Conventionally this setting is equipped with a $*-$ structure, i.e. an adjoint operation that is defined by

$$s_x^* = s_x, \qquad s_y^* = s_y, \qquad s_z^* = s_z \; . \tag{4.8}$$

A faithful unitary representation of $\mathfrak{su}(2)$ is defined by matrices

$$S_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; . \tag{4.9}$$

Elements s_x, s_y and s_z can be mapped into some other operators which also generate the $\mathfrak{su}(2)$ Lie algebra. This new basis will be more convenient for our purpose and we will always refer to it as the actual basis of generators that generates the $\mathfrak{su}(2)$ algebra.

$$J^0 := \frac{1}{2}s_z \qquad J^+ := \frac{1}{2}(s_x + is_y) \qquad J^- := \frac{1}{2}(s_x - is_y) \; .$$

Using commutation relation in (4.7), one easily finds that J^0, J^+ and J^- satisfy the following commutation relations

$$[J^0, J^\pm] = \pm J^\pm \qquad \text{and} \qquad [J^+, J^-] = 2J^0 \; . \tag{4.10}$$

The $*-$ structure is defined by $(J^0)^* = J^0$, $(J^+)^* = J^-$ and $(J^-)^* = J^+$. The Casimir element is

$$\Omega = 2(J^0)^2 + J^+J^- + J^-J^+$$

which is central, i.e. commutes with all the algebra generators, is self-adjoint. It will be useful to have an expression for the coproduct of the Casimir:

$$\begin{aligned}
\Delta(\Omega) &= \Delta(2(J^0)^2) + \Delta(J^+J^-) + \Delta(J^-J^+) = 2\Delta(J^0)\Delta(J^0) + \Delta(J^+)\Delta(J^-) + \Delta(J^-)\Delta(J^+) \\
&= 2(1 \otimes J^0 + J^0 \otimes 1)^2 + (1 \otimes J^+ + J^+ \otimes 1)(1 \otimes J^- + J^- \otimes 1) \\
&+ (1 \otimes J^- + J^- \otimes 1)(1 \otimes J^+ + J^+ \otimes 1) \\
&= 1 \otimes \Omega + \Omega \otimes 1 + 4J^0 \otimes J^0 + 2J^+ \otimes J^+ + 2J^- \otimes J^- \; .
\end{aligned}$$

Discrete Representation

The action of the three generators on functions f in $\mathscr{F}(\mathbb{N})$, the set of all functions on $\mathbb{N}$, is given by

$$\begin{gathered}
(J^+f)(n) = nf(n-1) \qquad\qquad (J^-f)(n) = (2j-n)f(n+1) \\
(J^0f)(n) = (n-j)f(n)
\end{gathered} \tag{4.11}$$

where $f(-1) = f(2j+1) = 0$. The inner product is defined by

$$\langle f, g\rangle = \sum_{n=0}^{2j} f(n)g(n)w(n) \quad \text{where} \quad w(n) = \binom{2j}{n} \tag{4.12}$$

which conserve the the $*-$ structure and the action of the Casimir is basically a multiplication by a constant, $(\Omega f)(n) = 2j(j+1)f(n)$.

We end this Section finding a relation for the transpose of the operator J^-, it will be used later on in Chapter 6. Using the adjoint relation $(J^+)^* = J^-$ we have that

$$\begin{aligned}
\langle J^+ f, g\rangle &= \langle f, J^- g\rangle \\
\sum_n (J^+ f)(n)g(n)w(n) &= \sum_n f(n)(J^- g)(n)w(n) \\
\sum_{n,k} (J^+)_{n,k} f(k)g(n)w(n) &= \sum_{n,k} f(n)(J^-)_{n,k} g(k)w(n) \\
&= \sum_{n,k} f(k)(J^-)_{k,n} g(n)w(k)
\end{aligned}$$

since the above equality is true for every f and g, then

$$\begin{aligned}
(J^+)_{n,k} w(n) &= (J^-)_{n,k} w(k) \\
(J^-)_{k,n} &= w(k)^{-1}(J^+)_{n,k} w(n) \\
&= w(k)^{-1}((J^+)_{k,n})^T w(n)
\end{aligned}$$

So $J^- = w^{-1}(J^+)^T w = d(J^+)^T d^{-1}$ from which we infer

$$(J^+)^T = d^{-1}(J^-) \ , \tag{4.13}$$

where d is a diagonal matrix with elements $d = \frac{1}{w(n)}\delta_{k,n}$.

4.2 The Lie algebra $\mathfrak{su}(1,1)$

The Lie algebra $\mathfrak{su}(1,1)$ is defined by the following commutation relations between its elements t_x, t_y and t_z:

$$[t_x, t_y] = 2it_z \qquad [t_y, t_z] = -2it_x \qquad [t_z, t_x] = 2it_y \ . \tag{4.14}$$

Which are the same as those (4.7) except for the minus sign in the second commutator. It is customary to equip $\mathfrak{su}(1,1)$ with a $*-$ structure, i.e. an adjoint operation such that

$$t_x^* = t_x, \qquad t_y^* = t_y, \qquad t_z^* = t_z \ . \tag{4.15}$$

A faithful representation of $\mathfrak{su}(1,1)$ is defined by matrices

$$T_x := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T_y := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad T_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{4.16}$$

this representation doesn't satisfy the self-adjoint relations in (4.15) and hence do not define a unitary representation of $\mathfrak{su}(1,1)$. In fact, all unitary irreducible representation of $\mathfrak{su}(1,1)$ are infinite dimensional. Again, we will map generators t_x, t_y and t_z into a new set of generators defined as

$$K^0 := \frac{1}{2}t_x \qquad K^+ := \frac{1}{2}(t_y + it_y) \qquad K^- := \frac{1}{2}(t_y - it_z) \ .$$

Using the old commutation relations in (4.14) and the adjoint relation in (4.15) one finds that this new set of generators satisfy

$$[K^0, K^\pm] = \pm K^\pm \qquad \text{and} \qquad [K^-, K^+] = 2K^0 \ . \tag{4.17}$$

and the $*-$ structure is given by

$$(K^0)^* = K^0, \qquad (K^+)^* = K^-, \qquad (K^-)^* = K^+ \ .$$

The Casimir element is

$$\Omega = 2(K^0)^2 - K^+K^- - K^-K^+$$

which is self-adjoint, $\Omega = \Omega^*$ and it commutes with every element of the algebra. The coproduct of the Casimir is

$$\begin{aligned}\Delta(\Omega) &= \Delta(2(K^0)^2) - \Delta(K^+K^-) - \Delta(K^-K^+) = 2\Delta(K^0)\Delta(K^0) - \Delta(K^+)\Delta(K^-) - \Delta(K^-)\Delta(K^+) \\ &= 2(1\otimes K^0 + K^0 \otimes 1)^2 - (1\otimes K^+ + K^+\otimes 1)(1\otimes K^- + K^-\otimes 1) + \\ &- (1\otimes K^- + K^-\otimes 1)(1\otimes K^+ + K^+\otimes 1) \\ &= 1\otimes\Omega + \Omega\otimes 1 + 4K^0\otimes K^0 - 2K^+\otimes K^- - 2K^-\otimes K^+\end{aligned}$$

Discrete Representation

The action of the three generators on functions f in $\mathscr{F}(\mathbb{N})$ is given by

$$\begin{gathered}(K^+f)(n) = nf(n-1) \qquad\qquad (K^-f)(n) = (n+2k)f(n+1) \\ (K^0f)(n) = (n+k)f(n) \ .\end{gathered} \tag{4.18}$$

Equipping this setting with the inner product

$$\langle f, g\rangle = \sum_{n\in\mathbb{N}} f(n)g(n)w(n) \quad \text{where} \quad w(n) = \frac{\Gamma(2k+n)}{n!\Gamma(2k)} \tag{4.19}$$

the $*-$ structure holds. The action of the Casimir on function f in $\mathscr{F}(\mathbb{N})$ is basically a multiplication by a constant: $(\Omega f)(n) = 2k(k-1)f(n)$.

It will be useful to have a relation for the transpose of the operator as well, this can be done in the following way, using the adjoint relation $(K^+)^* = K^-$,

$$\begin{aligned}
\langle K^+ f, g\rangle &= \langle f, K^- g\rangle \\
\sum_n (K^+ f)(n) g(n) w(n) &= \sum_n f(n)(K^- g)(n) w(n) \\
\sum_{n,k} (K^+)_{n,k} f(k) g(n) w(n) &= \sum_{n,k} f(n)(K^-)_{n,k} g(k) w(n) \\
&= \sum_{n,k} f(k)(K^-)_{k,n} g(n) w(k).
\end{aligned}$$

This implies

$$\begin{aligned}
(K^+)_{n,k} w(n) &= (K^-)_{k,n} w(k) \\
(K^-)_{k,n} &= w(k)^{-1}(K^+)_{n,k} w(n) \\
&= w(k)^{-1}((K^+)_{k,n})^T w(n)
\end{aligned}$$

We call d the a diagonal matrix with elements $d_{k,n} = \frac{1}{w(n)}\delta_{k,n}$, so that we finally have $K^- = w^{-1}(K^+)^T w = d(K^+)^T d^{-1}$ from which we infer

$$(K^+)^T = d^{-1}(K^-)d \; . \tag{4.20}$$

Continuous Representation

A continuous representation of the three generators on functions f in $\mathscr{F}(\mathbb{R}^+)$ is given by

$$\begin{aligned}
(\mathcal{K}^+ f)(x) &= x f(x) \qquad\qquad (\mathcal{K}^- f)(x) = \left(x\frac{d^2}{dx^2} + 2k\frac{d}{dx}\right) f(x) \\
(\mathcal{K}^0 f)(x) &= \left(x\frac{d}{dx} + k\right) f(x) \; .
\end{aligned} \tag{4.21}$$

for $x \in \mathbb{R}^+$.

4.3 The Heisenberg algebra

The Heisenberg algebra is the Lie algebra with generators a and $a^\dagger$ such that

$$[a^\dagger, a] = 1 \; , \tag{4.22}$$

with $*-$structure $a^* = a^\dagger$.

Discrete Representation

The Heisenberg algebra has a representation on $\mathscr{F}(\mathbb{N})$ such that

$$(af)(n) = nf(n-1) \qquad (a^\dagger f)(n) = f(n+1) \tag{4.23}$$

This representation satisfies the $*-$structure with weight $w(n) = \dfrac{1}{n!}$. Later on, we will need the action of the transpose of a, namely a^T; to this end we will use the $*-$structure of the Heisenberg algebra.

$$\begin{aligned}
\langle af, g\rangle &= \langle f, a^\dagger g\rangle \\
\sum_n (af)(n)g(n)w(n) &= \sum_n f(n)(a^\dagger g)(n)w(n) \\
\sum_{n,k} (a)_{n,k} f(k)g(n)w(n) &= \sum_{n,k} f(n)(a^\dagger)_{n,k} g(k)w(n) \\
&= \sum_{n,k} f(k)(a^\dagger)_{k,n} g(n)w(k)\ .
\end{aligned}$$

Since the above holds for all functions f and g we get

$$(a)_{n,k} w(n) = (a^\dagger)_{k,n} w(k) \tag{4.24}$$
$$((a)_{k,n})^T = w(k)(a^\dagger)_{k,n} w(n)^{-1}\ . \tag{4.25}$$

Using the diagonal matrix d, with elements $d_{k,n} = \frac{1}{w(n)}\delta_{k,n}$, we have

$$a^T = d^{-1} a^\dagger d\ .$$

Continuous Representation

A conjugate continuous representation for the Heisenberg algebra can be preformed with operators A and $A^\dagger$, working on smooth functions $f : \mathbb{R} \to \mathbb{R}$ with compact support, defined as

$$(Af)(x) = \frac{d}{dx} f \qquad (A^\dagger f)(x) = x f(x) \tag{4.26}$$

sometimes known as the annihilation and creation operators. They satisfy the commutation relation

$$[A^\dagger, A] = -1\ . \tag{4.27}$$

Chapter 5

Algebraic approach

In the context of Markov processes duality and self-duality can be framed under two main structural categories which we will refer as *change of representation* in the context of duality and *symmetries* of the generator for self-duality. We will show the abstract theory in the next two Sections and we will show examples and applications in the last two ones. Reference and pioneering work on this can be found in [15] for the change of representation approach and [32] for the symmetries approach.

5.1 Change of representation and duality

In several cases, duality between two Markov generators arises from duality of operators which are some sort of building blocks (such as derivatives and multiplication operators) for the expression of the generators and therefore it is possible to consider these building blocks duality as duality between two representations of a Lie algebra. We will call such duality function intertwining function or intertwiner and the dual Lie algebra is known as the conjugate Lie algebra. The starting point would be to recognize the two representations of the Lie algebra, then, once an intertwining function has been found, one should retrieve the Markov generators as an element of the universal enveloping Lie algebra, if this is the case then duality between the two generators is a mere consequence of this procedure. Let $\mathfrak{g}$ be a Lie algebra with basis elements $\mathbf{x}_1, \dots, \mathbf{x}_n$ that satisfy commutation relations of the form

$$[\mathbf{x}_i, \mathbf{x}_j] = \sum_{k=1}^{n} c_{ijk} \mathbf{x}_k \quad (\text{for } i < j) \ .$$

Let $X_1, \dots, X_n$ on a linear space V and $Y_1, \dots, Y_n$ be linear operators on a discrete linear space W such that the following commutation relations are satisfied

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k \quad \text{and} \quad [Y_i, Y_j] = -\sum_{k=1}^{n} c_{ijk} Y_k$$

Then $X_1, \ldots, X_n$ is a representation on $\mathfrak{g}$ and $Y_1, \ldots, Y_n$ is a representation on the conjugate Lie algebra $\bar{\mathfrak{g}}$.

Theorem 5.1. *Let $\mathfrak{g}$ be a Lie algebra with basis elements $\{\mathbf{x}_i\}_{i=1}^n$ and $\{\mathbf{y}_i\}_{i=1}^n$ be basis elements of the conjugate Lie algebra $\bar{\mathfrak{g}}$. Then the following statements are verified*

- *$\{Y_i\}_{i=1}^n$ is a representation for $\bar{\mathfrak{g}}$ if and only if $\{Y_i^T\}_{i=1}^n$ is a representation for $\mathfrak{g}$.*
- *For $i = 1, \ldots, n$ X_i is dual to Y_i via D (which is invertible) if and only if $\{X_i\}_{i=1}^n$ and $\{Y_i^T\}_{i=1}^n$ are two irreducible equivalent representation of $\mathfrak{g}$.*

Proof. The first item is a simple computation, indeed

$$[Y_i^T, Y_j^T] = Y_i^T Y_j^T - Y_j^T Y_i^T = (Y_j Y_i)^T - (Y_i Y_j)^T = (Y_j Y_i - Y_i Y_j)^T = -[Y_i, Y_j]^T$$
$$= \left(\sum_k c_{ijk} Y_k \right)^T = \sum_k c_{ijk} Y_k^T \ .$$

For the second item, one implication is a consequence of Schur's Lemma, in Corollary 4.1, which states that there exist an invertible intertwiner $D : V \to W$ such that

$$X_i D = D Y_i^T \quad \text{for} \quad i = 1, \ldots, n \ .$$

On the other hand, using the duality hypothesis, since D is invertible we have

$$X_i = D Y_i^T D^{-1} \quad \text{for} \quad i = 1, \ldots, n$$

and it is easy to verify that they are representations of the same Lie algebra:

$$[X_i, X_j] = [D Y_i^T D^{-1}, D Y_j^T D^{-1}] = D[Y_i^T, Y_j^T] D^{-1} = D \sum_k c_{ijk} Y_k^T D^{-1} = \sum_k c_{ijk} X_k \ .$$

□

We think of X_i and Y_i as building blocks used to construct more elaborate operators. The following theorem (taken from [15], Theorem 2.1) says how these operators need to be constructed in order to preserve the duality property.

Theorem 5.2 (Combination of dual operators)**.** *Let X_1, Y_1 (resp. X_2, Y_2) be two operators dual to each other with the same duality function $D = D(x, y)$, such that for $i = 1, 2$, X_i work on a common domain $\mathcal{D}$ ($\mathcal{D} \subseteq \mathcal{D}(X_i)$ and Y_i work on $\mathcal{D}^{dual}$ ($\mathcal{D}^{dual} \subseteq \mathcal{D}(Y_i)$). Then for $c_1, c_2 \in \mathbb{R}$ we have*

1. *$c_1 X_1 + c_2 X_2$ is dual to $c_1 Y_1 + c_2 Y_2$, with the same duality function D.*
2. *$X_1 X_2$ is dual to $Y_2 Y_1$ with the same duality function D.*

More generally, we can say that

3. *Consider two collections of operators:* $\{X_i, i \in I\}$ *and* $\{Y_i, i \in I\}$*; if* X_i *is dual to* Y_i *for every* i *with duality function* D*, then, every element of the algebra generated by* $\{X_i, i \in I\}$ *is dual to an element of the algebra generated by* $\{Y_i, i \in I\}$.
More precisely $(X_{i_1})^{n_1} \cdots (X_{i_k})^{n_k}$ *is dual, with duality function* D*, to* $((Y_{i_k})^{n_k} \cdots (Y_{i_1})^{n_1}$ *for all* $n_1, \ldots, n_k \in \mathbb{N}$*. Moreover for constants* $\{c_i, i \in I\}$ *also* $\sum_{i\in I} c_i X_i$ *is dual to* $\sum_{i\in I} c_i Y_i$ *with duality function* D.

Proof.

1.
$$\begin{aligned}[c_1X_1 + c_2X_2]D(\cdot, y)(x) &= c_1X_1D(\cdot, y)(x) + c_2X_2D(\cdot, y)(x) \\ &= c_1Y_1D(x, \cdot)(y) + c_2Y_2D(x, \cdot)(y) \\ &= [c_1Y_1 + c_2Y_2]D(x, \cdot)(y).\end{aligned}$$

2.
$$\begin{aligned}X_1X_2D(\cdot, y)(x) &= X_1[X_2D(\cdot, y)(x)] \\ &= X_1Y_2D(x, \cdot)(y) \\ &= Y_2[X_1D(\cdot, y)(x)] \\ &= Y_2Y_1D(x, \cdot)(y).\end{aligned}$$

3. follows from *1.* and *2.*. □

Theorem 5.3 (Factorized duality function)**.** *Suppose we have operators* X_1 *dual to* Y_1 *with duality function* D_1 *and* X_2 *dual to* Y_2 *with duality function* D_2*; then* $X_1 \otimes X_2$ *is dual to* $Y_1 \otimes Y_2$*, with duality function* $D_1 \otimes D_2$*, where*

$$D_1 \otimes D_2(x_1, x_2; y_1, y_2) = D_1(x_1; y_1)D_2(x_2; y_2).$$

Proof.

$$\begin{aligned}&X_1 \otimes X_2[D_1 \otimes D_2(\cdot, \cdot; y_1, y_2)](x_1, x_2) \\ &= X_1X_2[D_1(\cdot; y_1)D_2(\cdot; y_2)](x_1, x_2) \\ &= [X_1D_1(\cdot; y_1)](x_1)[X_2D_2(\cdot; y_2)](x_2) \\ &= [Y_1D_1(x_1; \cdot)](y_1)[Y_2D_2(x_2; \cdot)](y_2) \\ &= Y_1Y_2[D_1(x_1; \cdot)D_2(x_2; \cdot)](y_1, y_2) \\ &= Y_1 \otimes Y_2[D_1 \otimes D_2(x_1, x_2; \cdot, \cdot)](y_1, y_2).\end{aligned}$$

□

The next theorem shows that once a duality relation for operators belonging to an algebra is available and these operators can be combined together to produce the generator of a Markov process, then the duality relation will carry over.

Corollary 5.1. *Assume that L can be written as a linear combination of finite products of operators X_i and that X_i and Y_i^T are two equivalent irreducible representations of the same algebra $\mathfrak{g}$. Then L^{dual} is a linear combination of a finite product of operators Y_i with reverse order and it is dual to L with duality function D (which is the invertible intertwiner of the two representations).*

Proof. The proof is straightforward from Theorem 5.1 and Theorem 5.2. □

Note that in general it is not guaranteed for L and/or L^{dual} to be Markov generators, however it can be verified separately.

Example 5.1. Suppose we are under the hypothesis of Corollary 5.1, then if L has the form

$$L = X_1 X_2 X_3 + X_4 X_5 \ ,$$

then L^{dual} becomes

$$L^{dual} = Y_3 Y_2 Y_1 + Y_5 Y_4 \ ,$$

5.2 Symmetries and self-duality

In this Section we recal a general scheme for constructing self-dualities of Markov processes whose generator is symmetric with respect to another operator.

Definition 5.1. *Let A and B be two matrices having the same dimension. We say that A is a symmetry of B if A commutes with B, i.e.*

$$[A, B] = AB - BA = 0 \ .$$

The main idea is that self-duality (in the context of Markov process with countable state space) can be recovered starting from a *trivial duality* which is based on reversible measures of the processes and then one can act with a symmetry of the model on this trivial self-duality to turn it into a non-trivial one. The following theorem formalizes this idea.

Theorem 5.4 (Symmetries and self-duality)**.** *Let d be a self-duality function of the generator L and let S be a symmetry of L, then $D = Sd$ is again a self-duality function for L.*

Proof. The proof follows from a straightforward computation

$$LD = LSd = SLd = SdL^T = DL^T$$

where the second identity follows from the fact that S and L commutes, while the third one is self-duality of the generator L with self-duality function d. □

If there is a description on the process generator in terms of a Lie algebra, then symmetries can be constructed using this algebraic structure. The two main elements of this thereon are the initial self-duality d and the symmetry operator S. In general, if the process has a reversible measure the self-duality d can easily be found starting from the reversibility

Lemma 5.1. *If the process associated to generator L has reversible measure μ, then the function*

$$d(x, y) = \frac{\delta_{x,y}}{\mu(x)}$$

is a self-duality function.

Proof. The proof follows from the reversibility of the measure μ. Since we are on a countable state space, we can check the notion of duality in matrix notation in equation (1.11). Namely,

$$Ld = dL^T$$

which expanded reads

$$\sum_{x'} L(x, x')d(x', y) = \sum_{y'} d(x, y')L^T(y', y) \ ,$$

once we substitute the expression of d

$$\sum_{x'} L(x, x')\frac{\delta_{x',y}}{\mu(y)} = \sum_{y'} \frac{\delta_{x,y'}}{\mu(x)} L^T(y', y) \ ,$$

the sum on the left hand side only survive for $x' = y$ while the one on the right hand side only for $y' = x$, i.e,

$$L(x, y)\frac{1}{\mu(y)} = L(y, x)\frac{1}{\mu(x)}$$

which is exactly the detailed balance equation in (2.5). □

We will usually refer to the diagonal self-duality function as trivial or cheap self-duality function.

One may now wonder how the operator S is found; for example, in [32] it is found using the expression of the process generator written in terms of the underlying algebra generators which turn out to be an element of the universal enveloping algebra. Then one should look for symmetries of this element which is central, i.e. it commutes with all the generator of the algebra. Whenever the process generator L can be written as the coproduct, defined in Chapter 4, of the Casimir Ω, then a symmetry of the Casimir can be extended via the coproduct as a symmetry of the generator as this lemma shows.

Lemma 5.2. *If S is a symmetry of the central element* C, *then $\Delta(S)$ is a symmetry for $\Delta(\mathrm{C})$.*

Proof. Starting from $[\mathrm{C}, S] = 0$ we want to show that $[\Delta(\mathrm{C}), \Delta(S)] = 0$.This follows from the fact that the coproduct is an algebra homomorphism.

$$[\Delta(\mathrm{C}), \Delta(S)] = \Delta(\mathrm{C})\Delta(S) - \Delta(S)\Delta(\mathrm{C}) = \Delta(\mathrm{C}S) - \Delta(\mathrm{C}) = \Delta(\mathrm{C}S - S\mathrm{C}) = 0 \ .$$

□

The last two Sections of this Chapter are used to show explicit examples of dualities and self-dualitites which are proved making use of the previous theorems.

5.3 Classical dualities

We start with the change of representation approach characterized in theorems of Section 5.1 which is used in order to derive some known dualities and describes how do they fit the abstract scheme. As a warm up we recall the last example of Section 1.5. Consider, to this end, the generators of the Heisenberg algebra described in equations (4.26). The Laplace duality is clearly an intertwining function between operator A acting on the x variable and $A^\dagger$ acting on the y variable, i.e. for $D(x, y) = e^{xy}$

$$(AD(\cdot, y))(x) = (A^\dagger D(x, \cdot))(y)$$

where the action of A and $A^\dagger$ are defined in equation (4.26). Using Corollary 5.1 it is also true that D is an intertwining function for a linear combination of operator, for example if we consider the second derivative we derive the generator of Brownian motion,

$$L = \frac{1}{2}A^2 = \frac{1}{2}\frac{d^2}{dx^2}$$

then the dual operator must be

$$L^{dual} = \frac{1}{2}(A^\dagger)^2 = \frac{1}{2}y^2 \ .$$

At this point, using duality it is immediate to evaluate the generating function of the Brownian motion $(X_t)_{t\geq 0}$ initialized at x

$$\mathbb{E}_x\left(e^{yX_t}\right) = \mathbb{E}\left(e^{y(x+W_t)}\right) = \mathbb{E}_y\left(e^{xY_t}\right) = e^{t\frac{y^2}{2}}e^{xy} \tag{5.1}$$

where in the first equality we used that X_t has the same distribution of $x + W_t$ for $(W_t)_{t\geq 0}$ the standard Brownian motion, the second one is due to duality and the last one follows from the fact that the "semigroup" generated by $\frac{y^2}{2}$ is the multiplication with $e^{t\frac{y^2}{2}}$.

Example 5.2 (Algebraic description of duality between Wright-Fisher diffusion and Kingman coalescent)**.** We go back now to the duality between the Wright-Fisher diffusion and the Kingman coalescent in Example 1.5. Using the continuous and discrete generators of the Heisenberg algebra we can write the generators of the two processes. In particular, we have that

$$L^{WF} = \frac{x(x-1)}{2}\frac{\partial^2}{\partial x^2} = \frac{1}{2}A^\dagger(1-A^\dagger)A^2 \tag{5.2}$$

and that

$$\left(L^K f\right)(n) = \frac{1}{2}n(n-1)\left[f(n-1)-f(n)\right] = \frac{1}{2}a^2(1-a^\dagger)a^\dagger \tag{5.3}$$

Note that the order of the Heisenberg generators is reversed for the two processes generator, as it should, from Theorem 5.2. The transposed operators of the discrete Heisenberg algebra generators are

$$(a^T f)(n) = (n+1)f(n+1) \qquad \left(\left(a^\dagger\right)^T f\right)(n) = f(n-1)\ ,$$

and they satisfy the same commutation relations as A and $A^\dagger$, namely

$$\left[a^T, \left(a^\dagger\right)^T\right] = 1\ .$$

Duality now follows from Corollary 5.1 and the fact that the moment duality function $D(x,n) = x^n$, is an intertwiner between the two representations, namely

$$\begin{aligned} AD\left(\cdot,n\right)(x) &= aD\left(x,\cdot\right)(n) \\ A^\dagger D\left(\cdot,n\right)(x) &= a^\dagger D\left(x,\cdot\right)(n)\ . \end{aligned}$$

The next example we show can be also fit into the scheme of Section 5.1, however we will also give another explanation which does not involve the conjugate algebra. It is the duality between SIP(k) and BEP(k) interpreted as a change of representation of the $\mathfrak{su}(1,1)$ Lie algebra.

Example 5.3 (Classical duality between BEP(k) and SIP(k))**.** Let's start by considering the BEP(k) generator defined in Section 2.6 working on two sites, i and j, for two sites only we have that $\mathbf{x} = (x_i, x_j)$; we then write it using the $\mathfrak{su}(1,1)$ Lie algebra generators in their continuous representation given in equation (4.21),

$$L^{BEP(k)} = x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2 - 2k(x_i - x_j)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) = \mathcal{K}_i^+\mathcal{K}_j^- + \mathcal{K}_i^-\mathcal{K}_j^+ - 2\mathcal{K}_i^0\mathcal{K}_j^0 + 2k^2\ . \tag{5.4}$$

A discrete representation for the conjugate $\mathfrak{su}(1,1)$ Lie algebra is

$$\begin{gathered} (K^+ f)(n) = (2k+n)f(n+1) \qquad (K^- f)(n) = nf(n-1) \\ (K^0 f)(n) = (n+k)f(n)\ . \end{gathered} \tag{5.5}$$

An easy computation shows that their transpose satisfy the usual $\mathfrak{su}(1,1)$ commutation relation

$$((K^+)^T f)(n) = (2k+n-1)f(n-1) \qquad ((K^-)^T f)(n) = (n+1)f(n+1)$$

$$((K^0)^T f)(n) = (n+k)f(n) \; .$$

Using the conjugate Lie algebra in (5.5), we can write SIP generator on two sites

$$\begin{aligned}\left(L^{SIP} f\right)(\mathbf{n}) &= n_i(2k+n_j)\left[f(\mathbf{n}^{i,j}) - f(\mathbf{n})\right] + n_j(2k+n_i)\left[f(\mathbf{n}^{j,i}) - f(\mathbf{n})\right] \\ &= \Big(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^0 K_j^0 + 2k^2\Big) f(\mathbf{n})\end{aligned}$$

where $\mathbf{n} = (n_i, n_j)$ and $\mathbf{n}^{i,j} = (n_i - 1, n_j + 1)$ is the hop of one particle from site i to site j. Note that here the order of all the products $K_i^+ K_j^-$ is not reversed, but this is not an issue since the operators K^a with $a \in \{+,-,0\}$ work on different sites and so they commutes. As in the previous example, the duality between BEP and SIP is a consequence of Corollary 5.1 and the fact that the classical duality function

$$D(x,n) = \frac{x^n \Gamma(2k)}{\Gamma(2k+n)} \tag{5.6}$$

is an intertwiner between the $\mathcal{K}^a$ and the K^a for $a \in \{+,-,0\}$.

As already mentioned the conjugate algebra in this case can be relegated in the backward due to the fact that $K_i^+ K_j^- = K_j^- K_i^+$. If instead of the conjugate algebra generators in (5.5), we use the discrete representation of the $\mathfrak{su}(1,1)$ Lie algebra in equations (4.18), then it is verified that

$$\begin{aligned}L^{BEP(k)} D(\cdot,n)(x) = \Big(\mathcal{K}_i^+ \mathcal{K}_j^- + \mathcal{K}_i^- \mathcal{K}_j^+ - 2\mathcal{K}_i^0 \mathcal{K}_j^0 + 2k^2\Big) D(\cdot,n)(x) = \\ \Big(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^0 K_j^0 + 2k^2\Big) D(x,\cdot)(n) = L^{SIP} D(x,\cdot)(n) \; .\end{aligned}$$

5.4 Classical self-dualities

Classical self-dualities for SEP(j), SIP(k) and IRW (see [14] where these dualities are summarized) can easily be found using Theorem 5.4, where in all the three cases the trivial self-duality function d is a diagonal matrix with the inverse of the reversible measure on the diagonal, i.e. $d_{x,y} = \frac{1}{\mu^{REV}(x)}\delta_{x,y}$, where $\mu^{REV}(x)$ is respectively $\rho^{SEP}(x)$ in equation (2.11), $\rho^{SIP}(x)$ in equation (2.14) and $\rho^{IRW}(x)$ in equation (2.17). Before going on with our examples, we remark that thanks to structure of all our generators remarked in 2.5 as well as the product structure of the (self-)duality function in Theorem 5.3 it is posssible to introduce the (self-)duality function in just one site and keep in mind that they factorize over the graph G. The classical self-duality functions found here with the aid of generator symmetries are

$$D(\mathbf{x},\mathbf{y})^{cl} = \prod_{i\in V} \frac{x!}{(x-y)!} \begin{cases} 1 & \text{for IRW} \\ \dfrac{(2j-y)!}{2j!} & \text{for SEP}(j) \\ \dfrac{\Gamma(2k)}{\Gamma(2k+y)} & \text{for SIP}(k) \end{cases} \tag{5.7}$$

Example 5.4 (Classical self-duality of SEP(j))**.** The $\mathfrak{su}(2)$ Lie algebra has Casimir given by

$$\Omega = 2(J^0)^2 + J^+J^- + J^-J^+$$

we recall for simplicity the coproduct of the Casimir

$$\Delta(\Omega) = 1\otimes\Omega + \Omega\otimes 1 + 4J^0\otimes J^0 + 2J^+\otimes J^+ + 2J^-\otimes J^-$$

On two sites the SEP generator (2.10) reads

$$L^{SEP} = J^+\otimes J^- + J^-\otimes J^+ + 2J^0\otimes J^0 + 2j^2$$

It is easy to see that

$$L^{SEP} = \frac{1}{2}\Delta(\Omega) - \frac{1}{2}\otimes\Omega - \Omega\otimes\frac{1}{2} + 2j^2$$

since $(\Omega f)(n) = 2j(j+1)f(n)$, then

$$\left(\frac{1}{2}\otimes\Omega + \Omega\otimes\frac{1}{2}\right) f(n_1)f(n_2) = 2j(j+1)f(n_1)f(n_2)$$

which means that, when acting on functions $f(n_1)f(n_2)$, the action of L^{SEP} is, up to a constant, equal to the action of $\Delta(\Omega)$, and so one has the SIP generator in terms of the coproduct of the Casimir element. This simple observation ensures that it is enough to look for a symmetry of the Casimir. A possible choice would be to take $S = J^+$, since Ω is a central elements, then $[\Omega, J^+] = 0$; this can be verified explicitly using the $\mathfrak{su}(2)$ algebra commutation relation. Indeed,

$$\begin{aligned}[\Omega, J^+] &= \Omega K^+ - J^+\Omega = 2(J^0)^2J^+ + J^+J^-J^+ + J^-(J^+)^2 - \big(2J^+(J^0)^2 + J^+J^-J^+ + J^-(J^+)^2\big) \\ &= 2(J^0)^2J^+ + J^-(J^+)^2 - 2J^+(J^0)^2 - (J^+)^2J^- \ .\end{aligned}$$

We now use the commutation relation of the $\mathfrak{su}(2)$ algebra in equation (4.10), in particular we substitute the following

$$\begin{aligned} J^0J^+ &= J^+ + J^+J^0 \quad & J^-J^+ &= -2J^0 + J^+J^- \\ J^+J^0 &= -J^+ + J^0J^+ \quad & J^+J^- &= 2J^0 + J^-J^+ \end{aligned}$$

so that we get

$$\begin{aligned}[\Omega, J^+] &= 2J^0(J^+ + J^+J^0) + (-2J^0 + J^+J^-)J^+ - 2(-J^+ + J^0J^+)J^0 - J^+(J^0 + J^-J^+) \\ &= 0\ .\end{aligned}$$

Using the fact that L^{SEP} is, up to a constant, equal to $\Delta(\Omega)$, and using Lemma 5.2, we have that the SEP generator has $\Delta(J^+) = 1 \otimes J^+ + J^+ \otimes 1$ as a symmetry. In order to have a factorized form of the self-duality function we consider the exponential of J^+ since if J^+ is a symmetry so is $(J^+)^i$ for every $i \in \mathbb{N}$, to be more precise we have that

$$\Delta(S) = e^{\Delta(J^+)} = \sum_{i=0}^{\infty} \frac{(J^+)^i}{i!}$$

will produce a factorized self-duality function for the symmetric inclusion process. To show that, we use the fact that the final expression for the duality function will have a product form, we can just find its expression for one site. In Lemma 2.1 we proved that the SEP reversible measure is the binomial distribution, neglecting constants and quantities which would not affect a duality relation (in the spirit of Remark 1.3). We consider as cheap self duality function

$$d^{ch}(x,y) = \frac{y!(2j-y)!}{2j!}\delta_{x,y} \tag{5.8}$$

Proposition 5.1 (Symmetry for the classical self-duality of SEP(j))**.** *The one site classical duality function of SEP in Equation* (5.7) *can be written in the form*

$$D^{cl} = Sd^{ch} \quad \textit{where} \quad S = e^{J^+}\ .$$

Proof. The proof is a straightforward calculation.

$$\begin{aligned}D^{cl}(x,y) &= \left(e^{J^+} d^{ch}(\cdot,y)\right)(x) = \sum_i \frac{1}{i!}(J^+)^i \frac{y!(2j-y)!}{2j!}\delta_{x,y} = \sum_i \frac{1}{i!}\frac{y!(2j-y)!}{2j!}\frac{x!}{(x-i)!}\delta_{x-i,y} \\ &= \frac{x!}{(x-y)!}\frac{(2j-y)!}{2j!}\ .\end{aligned}$$

□

We use now the same procedure to find the classical self-duality function for the symmetric inclusion process.

Example 5.5 (Classical self-duality of SIP(k))**.** The $\mathfrak{su}(1,1)$ Lie algebra has Casimir given by

$$\Omega = 2(K^0)^2 - K^+K^- - K^-K^+\ ,$$

we recall for simplicity the coproduct of the Casimir

$$\Delta(\Omega) = 1 \otimes \Omega + \Omega \otimes 1 + 4K^0 \otimes K^0 - 2K^+ \otimes K^- - 2K^- \otimes K^+$$

On two sites the SIP generator (2.13) is

$$L^{SIP} = K^+ \otimes K^- + K^- \otimes K^+ - 2K^0 \otimes K^0 + 2k^2$$

and so one has the SIP generator in term of the coproduct of the Casimir

$$L^{SIP} = -\frac{1}{2}\Delta(\Omega) - \frac{1}{2} \otimes \Omega - \Omega \otimes \frac{1}{2} + 2k^2$$

Again, the action of Ω on a function f on $\mathbb{N}$ is a multiplication by a constant, i.e. $(\Omega f)(n) = 2k(k-1)f(n)$, and so

$$\left(\frac{1}{2} \otimes \Omega + \Omega \otimes \frac{1}{2}\right) f(n_1)f(n_2) = 2k(k-1)f(n_1)f(n_2)$$

in other words, the action of L^{SIP} is, up to a constant, equal to the action of $\Delta(\Omega)$. This simple observation assures us that it is enough to look for a symmetry of the Casimir. A possible choice would be $S = K^+$, indeed,

$$\begin{aligned}[\Omega, K^+] &= \Omega K^+ - K^+\Omega = 2(K^0)^2K^+ - K^+K^-K^+ - K^-(K^+)^2 - \big(2K^+(K^0)^2 - (K^+)^2K^- - K^+K^-K^+\big) \\ &= 2(K^0)^2K^+ - K^-(K^+)^2 - 2K^+K^0 + (K^+)^2K^- \, .\end{aligned}$$

We now use the commutation relation of the $\mathfrak{su}(1,1)$ algebra in equation (4.17), in particular we substitute the following

$$\begin{aligned} K^0K^+ &= K^+ + K^+K^0 \quad & K^-K^+ &= 2K^0 + K^+K^- \\ K^+K^0 &= -K^+ + K^0K^+ \quad & K^+K^- &= 2K^0 + K^-K^+ \end{aligned}$$

so that we get

$$\begin{aligned}[\Omega, K^+] &= 2K^0(K^+ + K^+K^0) - (2K^0 + K^+K^-)K^+ - 2(-K^+ + K^0K^+)K^0 + K^+(-2K^0 + K^-K^+) \\ &= 0 \, .\end{aligned}$$

Using the fact that L^{SIP} is, up to a constant, $\Delta(\Omega)$ as well as Lemma 5.2 we have that the SIP generator has $\Delta(K^+) = 1 \otimes K^+ + K^+ \otimes 1$ as a symmetry. In order to have a factorized form of the self-duality function we consider the exponential of K^+ since if K^+ is a symmetry so is $(K^+)^n$ for every $n \in \mathbb{N}$, to be more precise we have that

$$\Delta(S) = e^{\Delta(K^+)}$$

will produce a factorized self-duality function for the symmetric inclusion process. To show that, we use the fact that the final expression for the duality function will be in product form, we can just find its expression for one site. In Lemma 2.2 we proved that the SIP reversible measure is the negative binomial distribution, neglecting constants and quantities which would not affect a duality relation (in the spirit of Remark 1.3), we consider as a cheap self-duality function

$$d^{ch}(x,y) = \delta_{x,y}\frac{x!\Gamma(2k)}{\Gamma(2k+x)} \tag{5.9}$$

Proposition 5.2 (Symmetry for the classical self-duality of SIP(k))**.** *The one site classical duality function of SIP in Equation* (5.7) *can be written in the form*

$$D^{cl} = Sd^{ch} \quad \textit{where} \quad S = e^{K^+} \, .$$

Proof. The proof is a straightforward calculation.

$$\begin{aligned} D^{cl}(x,y) &= \left(e^{K^+} d^{ch}(\cdot, y)\right)(x) = \sum_i \frac{1}{i!}(K^+)^i \frac{y!\Gamma(2k)}{\Gamma(2k+y)}\delta_{x,y} \\ &= \sum_i \frac{1}{i!} \frac{y!\Gamma(2k)}{\Gamma(2k+y)} \frac{x!}{(x-i)!}\delta_{x-i,y} = \frac{x!}{(x-y)!}\frac{\Gamma(2k)}{\Gamma(2k+y)} \, . \end{aligned}$$

□

Last, we show how to find the classical self-duality function for the independent random walk. In this context no Casimir is available and so we will not be using Lemma 5.2. The symmetry of the generator will be found by inspection of the generator itself, however, it should not surprise that it has the same form as in the previous two examples.

Example 5.6 (Classical self-duality of IRW)**.** The generator of the Independent Random Walk in (2.16) can be written for two sites in terms of the discrete generators of the Heisenberg algebra with representation in equation (4.23) as

$$L^{IRW} = (1 \otimes a - a \otimes 1)(a^\dagger \otimes 1 - 1 \otimes a^\dagger) \, .$$

The coproduct of a is a symmetry of L^{IRW}, indeed

$$\begin{aligned} [L^{IRW}, \Delta(a)] = L^{IRW}a - aL^{IRW} &= (1 \otimes a - a \otimes 1)(a^\dagger \otimes 1 - 1 \otimes a^\dagger)(a \otimes 1 + 1 \otimes a) \\ &\quad - (a \otimes 1 + 1 \otimes a)(1 \otimes a - a \otimes 1)(a^\dagger \otimes 1 - 1 \otimes a^\dagger) \end{aligned}$$

which turns out to be zero by doing the algebra. As in the two previous cases, we want a factorized self-duality function and so we chose the exponential of $\Delta(a)$. For one site, the classical self-duality function is given by the following Proposition

Proposition 5.3 (Symmetry for the classical self-duality of IRW)**.** *The one site classical duality function of IRW in Equation* (5.7) *can be written in the form*

$$D^{cl}(x,y) = \left(Sd^{ch}(\cdot, y)\right)(x) \quad \textit{where} \quad S = e^a \, .$$

Proof.

$$D^{cl}(x,y) = \left(e^a d^{ch}(\cdot, y)\right)(x) = \sum_k \frac{1}{k!}(a)^k y!\delta_{x,y} = \sum_k \frac{1}{k!} y! \frac{x!}{(x-k)!}\delta_{x-k,y} = \frac{x!}{(x-y)!} \, .$$

□

We conclude the Chapter with a final remark on the norm of orthogonal polynomials. In general orthogonal polynomials are uniquely defined once their L^2 norm is fixed or, equivalently, the leading coefficient is fixed. One may wonder how to chose the "right" normalization. One option would be to use the Gram–Schmidt method for orthogonalization [19]. Suppose we have duality and self-duality functions which are *non* orthogonal polynomials. Then it turns out that, using those as input for the Gram–Schmidt procedure, the method releases, as an output, the orthogognal polynomials with the suitable normalization.

Remark 5.1. First assume $\{P_n(x)\}_{n=0}^{\infty}$ is a polynomial sequence orthogonal to the measure $w(x)$ with respect to a scalar product on $\mathsf{L}^2(\Omega, w)$. Assume $v_n(x)$ is a certain duality function (the classical one), so using Gram–Schmidt we have

$$u_n = v_n - \sum_{k=0}^{n-1} \frac{\langle v_n, u_k \rangle}{\langle u_k, u_k \rangle} u_k.$$

We pulg in our ansatz $u_n(x) = b_n P_n(x)$ to get

$$b_n P_n(x) = v_n - \sum_{k=0}^{n-1} \frac{\langle v_n, b_k P_k(x) \rangle}{\langle b_k P_k(x), b_k P_k(x) \rangle} b_k P_k(x) = v_n - \sum_{k=0}^{n-1} \frac{\langle v_n, P_k(x) \rangle}{\langle P_k(x), P_k(x) \rangle} P_k(x) \ . \tag{5.10}$$

Consider the scalar product with $P_n(x)$ in both sides of the above equality to find

$$b_n \langle P_n(x), P_n(x) \rangle = \langle v_n(x), P_n(x) \rangle \ .$$

And so we have an explicit formula for the coefficient we want to find, i.e.

$$b_n = \frac{\langle v_n(x), P_n(x) \rangle}{\langle P_n(x), P_n(x) \rangle} \ .$$

The value of the coefficient b_n can now be evaluated case by case. The example below shows the computation for Laguerre polynomials: starting with the classical (monomial) duality functions, one gets the (normalized) orthogonal ones.

Example 5.7 (Orthogonal duality via Gram–Schmidt)**.** Recall from equation (5.6) that the classical duality function between BEP(k) and SIP(k) is

$$v_n(x) = x^n \frac{\Gamma(2k)}{\Gamma(2k+n)} \ .$$

Using the definition of hypergeometric function in equation (A.1) we can write the explicit formula for generalized Laguerre polynomials of parameter $2k-1$

$$L_n^{(2k-1)}(x) = \sum_{j=0}^{n} \frac{(-x)^j \Gamma(2k+n)}{\Gamma(2k+j)(n-j)!j!} = \sum_{j=0}^{n} \frac{(-1)^j \Gamma(2k+n)}{\Gamma(2k)(n-j)!j!} v_j(x) \ .$$

At this point we substitute in the formula for $b_n(x)$:

$$b_n = \frac{\langle v_n(x), L_n^{(2k-1)}(x)\rangle}{\langle L_n^{(2k-1)}(x), L_n^{(2k-1)}(x)\rangle} = \frac{\langle v_n(x), L_n^{(2k-1)}(x)\rangle}{\langle \sum_{j=0}^{n} \frac{(-1)^j\Gamma(2k+n)}{\Gamma(2k)(n-j)!j!} v_j(x), L_n^{(2k-1)}(x)\rangle}$$
$$= \frac{\langle v_n(x), L_n^{(2k-1)}(x)\rangle}{\frac{(-1)^n\Gamma(2k+n)}{n!\Gamma(2k)}\langle v_n(x), L_n(x)\rangle} = (-1)^n \frac{n!\Gamma(2k)}{\Gamma(2k+n)} \ .$$

So, a new duality function is given by

$$(-1)^n \frac{n!\Gamma(2k)}{\Gamma(2k+n)} L_n^{(2k-1)}(x) \ .$$

The other cases we have, work in the same way: We see that up to some conserved quantity, which are irrelevant to our purpose, it is possible to find the normalization for the new orthogonal dualities. It seems that the Gram–Schmidt procedure doesn't affect the duality property even if it is not clear why. One should check a posteriori, as we did, that the orthogonal functions found in this way are actually duality or self-duality functions.

Chapter 6

Orthogonal dualities from an algebraic outlook

In this Chapter we have two goals, one would be to fit the new duality relations established in Chapter 3 under the algebraic approach of Chapter 5, this is done in the spirit of works [34] and [29]. On the other hand we would also like to fill in the gap between *change of representation-duality* and *symmetries-self-dualities*. To this end we can show that both orthogonal duality and orthogonal self-duality can be understood via a change of representation of the appropriate Lie algebra.

6.1 Change of representation: orthogonal (self-)duality

The first part of this Section is taken from the work [34], so we refer to it for more details and technicalities. The main idea is that it is possible to write two Markov generators as combination of elements of the appropriate Universal enveloping algebra. This guaranties (self-)duality as long as one has two representations associated via an intertwining function. Given the structure of our generators of Chapter 2 it turns out that the duality functions are product of the intertwining functions, just like the previous results.

The main difference with the theory described in Section 5.1, where we show duality results via a change of representation, is that now we use two different representations of the very same Lie algebra. The concept of conjugate algebra is now replaced by a deeper use of the $*$-structure of the algebra. From [34] (Theorem 2.2) we have the following result.

Theorem 6.1. *Let $\Omega = \Omega_1 \times \ldots \times \Omega_N$ and $\Omega^{dual} = \Omega_1^{dual} \times \ldots \times \Omega_N^{dual}$. Consider the Lie algebra $\mathfrak{g}$ endowed with two unitarily equivalent $*$-representations which are intertwined by the function $d_i = d_i(x_i, y_i)$ with $(x_i, y_i) \in \Omega_i \times \Omega_i^{dual}$ in the following way*

$$[X^* d_i(\cdot, y_i)]\,(x_i) = [\mathcal{X} d_i(x_i, \cdot)]\,(y_i)\ , \tag{6.1}$$

where X and $\mathcal{X}$ are two representations for $\mathsf{X} \in \mathfrak{g}$. More precisely, the adjoint of the first representation is intertwined with the second one. Suppose that L and L^{dual} are self-adjoint operators on $\mathsf{L}^2(\Omega, \mu)$ and $\mathsf{L}^2(\Omega^{dual}, \nu)$ respectively, that can be written as the same element Y of the Universal enveloping algebra $U(\mathfrak{g})$ using the two representations above, respectively. Then L and L^{dual} are dual with duality function

$$D(x, y) = \prod_{i=1}^{N} d_i(x_i, y_i) , \qquad x = (x_1, \ldots, x_N) \in \Omega , \quad y = (y_1, \ldots, y_N) \in \Omega^{dual} .$$

Proof. The idea is to write the abstract element Y as $\mathsf{Y} = \sum \mathsf{Y}_{(1)} \otimes \cdots \otimes \mathsf{Y}_{(N)}$, where for $i = 1, \ldots, N$ $\mathsf{Y}_{(i)} \in U(\mathfrak{g})$, then since $L = Y$ and $L^{dual} = \mathcal{Y}$, we need to show that equation (1.8) is verified. Thanks to the product structure of D, it is enough to show that

$$[Y_i^* d(\cdot, y_i)](x_i) = [\mathcal{Y}_i d(x_i, \cdot)](y_i) .$$

Since $\mathsf{Y}_{(i)} = \mathsf{Y}_{i,1}\mathsf{Y}_{i,2} \cdots \mathsf{Y}_{i,k_i}$ for some $\mathsf{Y}_{i,j} \in \mathfrak{g}$, then thanks to equation (6.1) the result follows. □

A similar approach can be used in context of self-duality. Except for the Heisenberg algebra, the Lie algebras we will work with features a central element in the universal enveloping algebra, the Casimir Ω commutes with every other elements of the algebra. It is interesting to notice that, whenever the Casimir is available within the algebra, then the generator of the processes defined in Chapter 2 can be related to it via the coproduct Δ. More specifically, we will see that the generator of the process is, up to a constant, the coproduct of the Casimir $\Delta(\Omega)$.

We now move on to the following theorem which shows a connection between the generator of the process and self-duality. We will use it to show self-duality for our usual processes. Let $\mathcal{S}$ be a metric space, we denote by $\mathscr{F}(\mathcal{S})$ the space of real-valued functions on $\mathcal{S}$. We will also work with functions $f : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ and a linear operator $A : \mathcal{D}(A) \subset \mathscr{F}(\mathcal{S}) \to \mathscr{F}(\mathcal{S})$.

We will need the notion of intertwining function between two operators, after recalling its definition we present a basic example to clarify.

Definition 6.1 (Intertwining function). *The function $f : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is an intertwining function between operators A and B if the action of A on the first variable of f is equal to the action of B on the second variable, i.e. $(Af(\cdot, y))(x) = (Bf(x, \cdot))(y)$.*

Remark 6.1. As already introduced, sometimes it will be convenient to have a shorter notation: if $T : \mathscr{F}(\mathcal{S}) \to \mathscr{F}(\mathcal{S})$ is an operator and $f : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ a function, we write $T_x f$ for the function

$$(x, y) \mapsto [Tf(\cdot, y)](x)$$

and similarly for $T_y f$. In this notation $f(x, y)$ is an intertwining function between A and B if $A_x f = B_y f$.

Example 6.1. Consider the two operators acting on $g : \mathbb{R} \to \mathbb{R}$ defined as follows

$$(Ag)(x) = xg(x) \qquad (Bg)(y) = \frac{\partial}{\partial y} g(y) \ .$$

Then $f(x, y) = e^{xy}$ is an intertwining function between A and B since

$$(Af(\cdot, y))(x) = xe^{xy} = (Bf(x, \cdot))(y) \ .$$

For operators A and B we call a composition of the form

$$S(A, B) = A^{n_1} B^{n_2} A^{n_3} \dots A^{n_{k-1}} B^{n_k} \ , \qquad \text{for} \quad n_1, \dots, n_k \in \mathbb{N}_0$$

a string in A and B. If $S(A, B)$ is a string of this form, then the reverse string is

$$S^{rev}(A, B) = A^{n_k} B^{n_{k-1}} \dots A^{n_2} B^{n_1}$$

and this operation can be extended to linear combinations of strings: if

$$C = \sum_{i=1}^{k} c_i S_i(A, B) \ , \tag{6.2}$$

then

$$C^{rev} = \sum_{i=1}^{k} c_i S_i^{rev}(A, B) \ . \tag{6.3}$$

We are particularly interested in operators such that $C = C^{rev}$.

Theorem 6.2 (Intertwining functions, symmetries and self-duality)**.** *Let A and B be finite order difference or differential operators on $\mathscr{F}(\mathcal{S})$, and let $f : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be an intertwining function between A and B.*

1. *If f is symmetric, i.e., $f(x, y) = f(y, x)$, then f is an intertwining function between B and A.*

2. *Suppose that f is also an intertwining function between B and A, and C is a linear combination of strings in A and B such that $C(A, B) = C^{rev}(A, B)$. Then C is a self-dual operator with duality function f.*

Proof. For the first item, using the intertwiner hypothesis $(Af(\cdot, y))(x) = (Bf(x, \cdot))(y)$ and the symmetry of f, namely $f(x, y) = f(y, x)$, we will show that $(Af(x, \cdot))(y) = (Bf(\cdot, y))(x)$. First we show that

$$[Af(\cdot, y)](x) = [Af(y, \cdot)](x) \ . \tag{6.4}$$

In the discrete case, denoting by $a_{x,x'}$ the elements of the matrix associated to the operator A, we have

$$(Af(x, \cdot))(y) = \sum_{y'} a_{y,y'} f(x, y') = \sum_{y'} a_{y,y'} f(y', x) = (Af(\cdot, x))(y) \ ,$$

where we used the symmetry of the function f. In the continuous case for a first order differential operator $\partial_x = \dfrac{\partial}{\partial x}$

$$[\partial_x f(\cdot, y)](x) = \lim_{h\to 0} \frac{f(x+h, y) + f(x,y)}{h} = \lim_{h\to 0} \frac{f(y, x+h) + f(y,x)}{h} = [\partial_x f(y, \cdot)](x) \ .$$

For a finite order differential operator $A = \sum a_k(x)\partial_{x_1}^{k_1} \cdots \partial_{x_L}^{k_L}$, $x = (x_1, \ldots, x_L)$, this leads to (6.4) as in the previous case, assuming f is sufficiently smooth. Our initial hypothesis that the function f intertwines between the operator A and the operator B implies that

$$\left(Af(\cdot, x)\right)(y) = \left(Bf(y, \cdot)\right)(x) \ . \tag{6.5}$$

Identity (6.4) holds for the operator B as well, i.e.

$$\left(Bf(y, \cdot)\right)(x) = \left(Bf(\cdot, y)\right)(x) \ . \tag{6.6}$$

Combining together equations (6.4), (6.5) and (6.6) one proves that

$$\left(Af(x, \cdot)\right)(y) = \left(Bf(\cdot, y)\right)(x) \ ,$$

i.e., f is an intertwining function for B and A: in the notation of Remark 6.1 this is $B_x f = A_y f$.

For the second item observe that

$$(A^{n_1}B^{n_2})_x f = (A^{n_1})_x (A^{n_2})_y f = (A^{n_2})_y (A^{n_1})_x f = (A^{n_2}B^{n_1})_y f \ .$$

Iterating this procedure we get that

$$S(A,B)_x f = S^{rev}(A,B)_y f \ . \tag{6.7}$$

So now

$$C_x f = \sum_{j=1}^{k} c_j S_j(A,B)_x f = \sum_{j=1}^{k} c_j S_j^{rev}(A,B)_y f = C_y^{rev} f = C_y f \ ,$$

where the second identity comes from (6.7) and the fourth identity holds due to conditions on C. □

Example 6.2. Suppose, as in the previous theorem, that f is an intertwining function between operators A and B as well as between B and A, then examples of self-dual operators are

- $C_1 = AB$.
- $C_2 = [A,B]^2 = ABAB + BABA - AB^2A - BA^2B$.

Remark 6.2. In our applications the operator C will always be the generator of the processes. Typical examples to have in mind for $\mathscr{F}(\mathcal{S})$ are $\mathscr{C}(\mathcal{S})$, $\mathscr{C}_c(\mathcal{S})$, $\mathscr{C}_0(\mathcal{S})$ the sets of continuous real-valued functions on $\mathcal{S}$, continuous real-valued functions with compact support on $\mathcal{S}$, continuous real-valued functions on $\mathcal{S}$ going to zero at infinity.

Theorem 6.2 heavily relies on operators A and B so one may wonder how to construct them. In the majority of the cases the two operators A and B arise naturally from the structure of the Casimir element of the underlying algebra. The next lemma shows that, whenever the generator is (in terms of) the coproduct of the Casimir, A and B can be found as the coproduct of two other operators.

Lemma 6.1. *If the Casimir element Ω is a linear combination of strings in X and Y, i.e. $\Omega = h(X, Y)$, then $\Delta(\Omega) = h(\Delta(X), \Delta(Y))$. In particular, if $\Omega = \Omega^{rev}$, then $\Delta(\Omega) = \Delta(\Omega)^{rev}$.*

Proof. Consider the coproduct of Ω

$$\Delta(\Omega) = \Delta(h(X, Y)) = h(\Delta X, \Delta Y)$$

where the second equality follows from the fact that the coproduct is an algebra homomorphism. □

In the applications of the next Section, h turns out to be a polynomial of fourth degree. Moreover, anytime the process generator is, up to a constant, equal to the coproduct of the Casimir $L \sim \Delta(\Omega)$, it will be sufficient to look for operators X and Y for which the Casimir Ω is equal to Ω^{rev} instead of operators A and B for which the generator L is equal to L^{rev}. We end this Section showing that, once an intertwining function between two operators is available, it can be used to find an intertwining function for the coproduct of the two operators in the following way.

Lemma 6.2. *If X and Y are two Lie algebra elements acting on $\mathscr{F}(\mathcal{S})$ and $f(x, y)$ is an intertwining function between X and Y, then $f(x_1, y_1)f(x_2, y_2)$ intertwines $\Delta(X)$ with $\Delta(Y)$.*

Proof. Recalling that, for Lie algebra elements X the coproduct of X is defined as

$$\Delta(X) = 1 \otimes X + X \otimes 1 ,$$

then

$$[\Delta(X)f(\cdot, y_1)f(\cdot, y_2)](x_1, x_2) = f(x_1, y_1)[Xf(\cdot, y_2)](x_2) + [Xf(\cdot, y_1)](x_1)f(x_2, y_2)$$

which, using the intertwining hypothesis, becomes

$$f(x_1, y_1)[Yf(x_2, \cdot](y_2) + [Yf(x_1, \cdot](y_1)f(x_2, y_2) = [\Delta(Y)f(x_1, \cdot)f(x_2, \cdot)](y_1, y_2) .$$

□

6.1.1 Duality results

We use this Section for an application of Theorem 6.1 to prove the orthogonal duality relation between BEP(k) and SIP(k) processes. Recalling the generators of these two processes in Section 2.6 and Section 2.3 respectively one would like to find two representations of an abstract element Y belonging to the Universal enveloping $\mathfrak{su}(1,1)$ Lie algebra. We take $\mathsf{Y}_{i,l}$ as the coproduct of Casimir element of the $\mathfrak{su}(1,1)$ Lie algebra

$$\mathsf{Y}_{i,l} = 1 \otimes \Omega_l + \Omega_i \otimes 1 + 4\mathsf{K}^0{}_i \otimes \mathsf{K}^0{}_l - 2\mathsf{K}^+{}_i \otimes \mathsf{K}^-{}_l - 2\mathsf{K}^-{}_i \otimes \mathsf{K}^+{}_l \ .$$

So that, for appropriate representations $Y_{i,l}$ and $\mathcal{Y}_{i,l}$ we can find

$$L^{SIP} = \sum_{1\le i<l\le N} Y_{i,l} + 2k^2 \ , \tag{6.8}$$

while

$$L^{BEP} = \sum_{1\le i<l\le N} \mathcal{Y}_{i,l} + 2k^2 \ . \tag{6.9}$$

The first representation, $Y_{i,l}$, used in (6.8) is the standard discrete one introduced in equation (4.18), while we need to find the second one such that normalized Laguerre polynomials are intertwining functions. This is done in [34] (Lemma 4.12) using the recurrence relation and the differential equation for $d(x,n) = \frac{n!\Gamma(2k)}{\Gamma(2k+n)} L_n(x)$, where $L_n(x)$ is the generalized Laguerre polynomial in x of degree n and parameter $2k-1$. It turns out that the second (continuous) representation is

$$\begin{aligned} &(\mathcal{K}^+ f)(x) = -x\frac{\partial}{\partial x} f(x) - \frac{2k-x}{2} f(x) \qquad\qquad (\mathcal{K}^- f)(x) = -\frac{1}{2} i x f(x) \\ &(\mathcal{K}^0 f)(x) = 2ix\frac{\partial^2}{\partial x^2} f(x) + 2i(2k-x)\frac{\partial}{\partial x} f(x) + \frac{i}{2}(4k-x) f(x) \ . \end{aligned} \tag{6.10}$$

It is simple to verify that $\mathcal{K}^+$, $\mathcal{K}^-$ and $\mathcal{K}^0$ satisfy the $\mathfrak{su}(1,1)$ commutation relations in equation (4.17) and that L^{BEP} can easily be found using this representation as in equation (6.9). Finally, Theorem 6.1 gives duality between symmetric inclusion process SIP(k) and Brownian energy process BEP(k).

Theorem 6.3. *Generators L^{SIP} and L^{BEP} are dual with duality function*

$$D(x,n) = \frac{n!\Gamma(2k)}{\Gamma(2k+n)} L_n(x) \ .$$

6.1.2 Self-duality results

This Section includes five sub-sections where detailed examples are provided: in each sub-section the natural Lie algebra is recalled from Chapter 4. Each paragraph ends with a theorem where the statement of a self-duality relation is proven via Theorem 6.2.

Remark 6.3. The generators in Chapter 2 were defined in the most general setting, i.e. on an undirected and connected graph G. However, noticing that our generators only acts on two (connected) variables at time, without loss of generality we can restrict the setting on two sites.

Self-duality of SEP with Krawtchouk polynomials

Recall the $\mathfrak{su}(2)$ Lie algebra introduced in Chapter 4.1 where the coproduct of the Casimir is

$$\Delta(\Omega) = 1 \otimes \Omega + \Omega \otimes 1 + 4J^0 \otimes J^0 + 2J^+ \otimes J^- + 2J^- \otimes J^+$$

The SEP generator (in equation (2.10)) on two sites is

$$L^{SEP} = J^+ \otimes J^- + J^- \otimes J^+ + 2J^0 \otimes J^0 + 2j^2 \ .$$

It is easy to see that

$$L^{SEP} = \frac{1}{2}\Delta(\Omega) - \frac{1}{2} \otimes \Omega - \Omega \otimes \frac{1}{2} + 2j^2$$

since $(\Omega f)(n) = 2j(j+1)f(n)$, then

$$\left(\frac{1}{2} \otimes \Omega + \Omega \otimes \frac{1}{2}\right) f(n_1)f(n_2) = 2j(j+1)f(n_1)f(n_2)$$

which means that, when acting on functions $f(n_1)f(n_2)$, the action of L^{SEP} is, up to a constant, equal to the action of $\Delta(\Omega)$.

The idea now (see [43] and [34]) is to look for eigenfunctions of $X_p = J^+ + J^- - a(p)J^0$ with an appropriate choice of $a(p)$. Let's start from the three term recurrence relation for the symmetric *Krawtchouk polynomials*, defined via the hypergeometric function as

$$K_n(x) := {}_2F_1\left(\begin{matrix} -n, -x \\ -2j \end{matrix} \,\middle|\, \frac{1}{p}\right) \qquad n, x \in \mathbb{N}_{2j} \ .$$

The three term recurrence relation for $K_n(x)$ is

$$-xK_n(x) = p(2j-n)K_{n+1}(x) - (2jp - 2np + n)K_n(x) + n(1-p)K_{n-1}(x) \ .$$

We want to read this identity as an eigenvalue equation for X_p with Krawtchouk polynomials as eigenfunctions. We set

$$a(p) = \frac{(1-2p)}{[p(1-p)]^{1/2}} \ ,$$

so that $k(x,n) = \left(\frac{p}{1-p}\right)^{\frac{1}{2}(n+x)} K_n(x)$ is a symmetric (in n and x) eigenfunction of the operator X_p , i.e.

$$\left(X_p k(x,\cdot)\right)(n) = \lambda(x)k(x,n)$$

where $\lambda(x) = -\dfrac{x-j}{[p(1-p)]^{1/2}}$ is the eigenvalue. Define

$$J_p = -[p(1-p)]^{1/2} X_p,$$

then $k(x,n)$ is of course also an eigenfunction of J_p: $(J_p k(x,\cdot))(n) = (x-j)k(x,n)$. Comparing this with the action of J^0, we have

$$(J_p k(x,\cdot))(n) = (J^0 k(\cdot,n))(x), \tag{6.11}$$

i.e., k is an intertwining function between J^0 and J_p. We have now worked everything out in order to prove the following theorem.

Theorem 6.4. *The symmetric exclusion process is self-dual with duality function $k(x_1,n_1)k(x_2,n_2)$.*

Proof. The statement of the theorem follows from Theorem 6.2. First, by Lemma 6.2 $k(x_1,n_1)k(x_2,n_2)$ is an intertwining function between $\Delta(J^0)$ and $\Delta(J_p)$ because of equation (6.11). Moreover, $k(x_1,n_1)k(x_2,n_2)$ is symmetric in (x_1,x_2) and (n_1,n_2), so by the first item of Theorem 6.2 it is also an intertwining function between $\Delta(J_p)$ and $\Delta(J^0)$. It is left to show that $L(\Delta(J^0),\Delta(J_p)) = L^{rev}(\Delta(J^0),\Delta(J_p))$ where L is the generator of the SEP process. Using Lemma 6.1, we can just check that $\Omega = \Omega^{rev}$ with respect to J^0 and J_p. Indeed, using the following identities

$$J^- + J^+ = X_p + aJ^0 \qquad J^- - J^+ = [X_p, J^0] \tag{6.12}$$

we have

$$\begin{aligned} 2\Omega &= 4(J^0)^2 + 2J^+J^- + 2J^-J^+ = 4(J^0)^2 + (J^- + J^+)^2 - (J^- - J^+)^2 \\ &= 4(J^0)^2 + (X_p + a(J^0)^2 - \left([X_p, J^0]\right)^2 \\ &= 4(J^0)^2 + \left(-\frac{1}{\sqrt{p}\sqrt{1-p}}J_p + aJ^0\right)^2 - \left(\left[-\frac{1}{\sqrt{p}\sqrt{1-p}}H_p, J^0\right]\right)^2 . \end{aligned}$$

From $4 + a^2 = \frac{1}{p(1-p)}$ we obtain

$$\Omega = \frac{1}{2p(1-p)}((J^0)^2 + J_p^2) - \frac{1-2p}{2p(1-p)}(J^0 J_p - J_p J^0) + \frac{1}{2p(1-p)}[J^0, J_p]^2,$$

from which we can read off that $\Omega = \Omega^{rev}$. By Theorem 6.2 the SEP generator is self-dual with duality $k(x_1,n_1)k(x_2,n_2)$.

□

The $\mathfrak{su}(2)$ algebra is generated by J^0 and J_p for which we have a representation on the n variable as well as a representation on the x variable. Using equations in (6.12) operators J^+ and J^- can also be realised as operators on the x variable, producing a different representation of the $\mathfrak{su}(2)$ algebra. It is important to stress that in this case the Krawtchouk polynomials are not intertwining as for the operators J^0 and J_p. Indeed, some straightforward but long computations would show that the operators J^+ and J^- have a tri-diagonal representation on the x variable, which we identify with the subscript p

$$\begin{aligned} \left[J_p^- k(\cdot,n)\right](x) &= \frac{4p-1}{4}(2j-x)k(x+1,n) - 2\sqrt{p(1-p)}(x-j)k(x,n) + \frac{4p-3}{4}xk(x-1,n) \\ \left[J_p^+ k(\cdot,n)\right](x) &= \frac{4p-3}{4}(2j-x)k(x+1,n) - 2\sqrt{p(1-p)}(x-j)k(x,n) + \frac{4p-1}{4}xk(x-1,n) . \end{aligned}$$

One could also upgrade the representation for function of only one variable, the x one.

Self-duality of SIP with Meixner polynomials

The $\mathfrak{su}(1,1)$ algebra is the Lie algebra generated introduced in Chapter 4.2, with Casimir

$$\Omega = 2(K^0)^2 - K^+K^- - K^-K^+ \ ,$$

we recall for simplicity the coproduct of the Casimir

$$\Delta(\Omega) = 1 \otimes \Omega + \Omega \otimes 1 + 4K^0 \otimes K^0 - 2K^+ \otimes K^- - 2K^- \otimes K^+ \ .$$

On two sites the SIP generator (2.13) is

$$L^{SIP} = K^+ \otimes K^- + K^- \otimes K^+ - 2K^0 \otimes K^0 + 2k^2 \ ,$$

and so one has the SIP generator in term of the coproduct of the Casimir

$$L^{SIP} = -\frac{1}{2}\Delta(\Omega) - \frac{1}{2} \otimes \Omega - \Omega \otimes \frac{1}{2} + 2k^2 \ .$$

Again, the action of Ω on a function of f on $\mathbb{N}$ is a multiplication by a constant with respect to n, meaning that $(\Omega f)(n) = 2k(k-1)f(n)$, and so

$$\left(\frac{1}{2} \otimes \Omega + \Omega \otimes \frac{1}{2}\right) f(n_1)f(n_2) = 2k(k-1)f(n_1)f(n_2)$$

in other words, the action of L^{SIP} is, up to a constant, equal to the action of $\Delta(\Omega)$.

Consider the symmetric *Meixner polynomials* defined in Section 3.2.2

$$M_n(x) := {}_2F_1\left(\begin{matrix}-n, -x\\ 2k\end{matrix}\,\middle|\, 1 - \frac{1}{c}\right) \qquad x, n \in \mathbb{N}_0 \ .$$

The three term recurrence relation for the Meixner polynomials is

$$(c-1)xM_n(x) = c(n+2k)M_{n+1}(x) - (n + nc + 2kc)M_n(x) + nM_{n-1}(x) \ .$$

Let's define $X_c := K^+ + K^- - a(c)K^0$ with $a(c) = \frac{(1+c)}{\sqrt{c}}$, for which the function $m(x,n) = c^{\frac{1}{2}(x+n)}M_n(x; 2k, c)$ is an eigenfunction, namely

$$\left(X_c m(x,\cdot)\right)(n) = \frac{(c-1)}{\sqrt{c}}(x+k)m(x,n) \ .$$

Calling $K_c = \dfrac{\sqrt{c}}{(c-1)}X_c$ we have

$$\left(K_c m(x,\cdot)\right)(n) = (x+k)m(n,x) = \left(K^0 m(\cdot,n)\right)(x) \ ,$$

so that $m(x,n)$ is an intertwining function between K^0 and K_c. We have now all the ingredients to prove self-duality for the SIP process.

Theorem 6.5. *The symmetric inclusion process is self-dual with duality function* $m(x_1, n_1)m(x_2, n_2)$.

Proof. The proof is analogous to the proof of Theorem 6.4, note that in this case the expression for the Casimir as function of K^0 and K_c becomes

$$\Omega = -\frac{(c-1)^2}{2c}((K^0)^2 + K_c^2) + \frac{1-c^2}{2c}(K^0 K_c + K_c K^0) + \frac{(c-1)^2}{2c}[K_c, K^0]^2 .$$

□

The realisation of K^+ and K^- as operator on the x variable can be found as in the previous Section, and it turns out that

$$\left[K_c^+ m(n,\cdot)\right](x) = \frac{1}{c-1}(2k+x)m(n,x+1) + \frac{2\sqrt{c}}{c-1}(x+k)m(n,x) + \frac{c}{c-1}xm(n,x-1)$$
$$\left[K_c^- m(n,\cdot)\right](x) = -\frac{c}{c-1}(2k+x)m(n,x+1) + \frac{2\sqrt{c}}{c-1}(x+k)m(n,x) - \frac{1}{c-1}xm(n,x-1)$$

where we stress again that the Meixner polynomials are not a change of representation between K^+ (resp. K^-) and K_c^+ (resp. K_c^-) as it happens between K^0 and K_c.

Self-duality for BEP and Bessel functions

In this Section we show self-duality for the BEP process of parameter k. It is the first time we show self-duality for a continuous process, the result is not available using the techniques of Chapter 3 but it could be found here [56] via a generating function approach and here [34] using a change of representation argument. Consider the BEP(k) process defined in Section 2.6. We use the $\mathfrak{su}(1,1)$ Lie algebra of the previous Section with a continuous representation, which has been already introduced in [34] so we recall here what we need. Generators H, E and F are defined on $\mathsf{L}^2(\mathbb{R}^+, \mu_k)$, with $\mu_k = \dfrac{z^{2k-1}e^{-z}}{\Gamma(2k)}$, $k > 0$, and act on functions $f(z)$ as

$$\begin{aligned}
(Hf)(z) &:= (-2z\partial_z - (2k-z))\, f(z) \\
(Ef)(z) &:= -\frac{1}{2} iz f(z) \\
(Ff)(z) &:= \left(-2iz\partial_z^2 - 2i(2k-z)\partial_z + \frac{i}{2}(4k-z)\right) f(z)
\end{aligned} \tag{6.13}$$

where $\partial_z := \dfrac{\partial}{\partial z}$. Note that in this case the $*$-structure is $H^* = -H$, $E^* = -E$ and $F^* = -F$ and the Casimir Ω is still self-adjoint. The BEP generator (in equation (2.20)) on two sites is

$$L^{BEP} = -\frac{1}{2}\left(\Delta(\Omega) - 1\otimes\Omega - \Omega\otimes 1\right) + 2k^2.$$

Bessel functions of the first kind are defined in terms of hypergeometric functions as

$$J_\nu(z) := \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix}\middle| -\frac{z^2}{4}\right) \qquad \nu > -1 .$$

They are solutions of the second order differential equation

$$-\partial_z^2 J_\nu(z) - \frac{1}{z}\partial_z J_\nu(z) + \frac{\nu^2}{z^2} J_\nu(z) = J_\nu(z) \ .$$

From the differential equation above, one infers that $J_\nu(zw)$ is an eigenfunction for the second order operator T with eigenvalue w^2, as in [34] we have

$$T = -\partial_z^2 - \frac{1}{z}\partial_z + \frac{\nu^2}{z^2} \qquad T J_\nu(zw) = w^2 J_\nu(zw) \ . \tag{6.14}$$

Consider now the action of operator F on the z variable of $J_\nu(zw)$ as in equation (6.13), using the second order differential equation for $J_\nu(zw)$ in (6.14) we can find that the eigenfunctions of F are given in terms of Bessel functions $J_{(2k-1)}(\sqrt{zw})$, which are solutions of the following second order differential equation

$$-2z\partial_z^2 J_{(2k-1)}(\sqrt{zw}) - 2\partial_z J_{2k-1}(\sqrt{zw}) + \frac{(2k-1)^2}{2z} J_{(2k-1)}(\sqrt{zw}) = J_{(2k-1)}(\sqrt{zw}) \ .$$

Consider ([34], Lemma 4.16) the function defined as follows,

$$J(z,w) = e^{\frac{1}{2}(z+w)}(zw)^{-k+\frac{1}{2}} J_{2k-1}(\sqrt{zw}) \ , \tag{6.15}$$

then $J(z,w)$ is an eigenfunction for F with eigenvalue $\frac{1}{2}iw$, i.e.

$$(FJ(\cdot,w))(z) = \frac{1}{2} iwJ(z,w) = (-EJ(z,\cdot))(w) \ .$$

We see that $J(z,w)$ is an intertwining function between operators F and $-E$. Given the symmetry if J, i.e. $J(z,w) = J(w,z)$, we could also obtain the action of E on the w variable.

Theorem 6.6. *The Brownian energy process is self-dual with duality function* $J(z_1,w_1)J(z_2,w_2)$.

Proof. The proof is analogous to the proof of Theorem 6.4 where the intertwined operators are F and $-E$, and the Casimir is

$$\Omega = \frac{1}{2}H^2 + EF + FE = \frac{1}{2}[-E,F]^2 - (-E)F - F(-E) \ ,$$

which is equal to Ω^{rev}. □

Operators E, F and H act on variables z and w in the following way for duality functions $J(z,w;k)$.

$$
\begin{aligned}
(HJ(\cdot, w; k))(z) &:= (-2z\partial_z - (2k - z))\, J(z, w; k) \\
&= (-2w\partial_w - (2k - w))\, J(z, w; k) \\
&= (HJ(z, \cdot; k))(w) \\
(FJ(\cdot, w; k))(z) &:= \left(-2iz\partial_z^2 - 2i(2k - z)\partial_z + \frac{i}{2}(4k - z)\right) J(z, w; k) \\
&= \frac{1}{2} iwJ(z, w; k) \\
&= (-EJ(z, \cdot; k))(w) \\
(EJ(\cdot, w; k))(z) &:= -\frac{1}{2} izJ(z, w; k) \\
&= \left(2iw\partial_w^2 + 2i(2k - w)\partial_w - \frac{i}{2}(4k - w)\right) J(z, w; k) \\
&= (-FJ(z, \cdot; k))(w) \ .
\end{aligned}
$$

It is immediate to notice that, with these two continuous representations, our self-duality functions are indeed intertwining functions between F and $-E$ and E with $-F$.

A change of variable for Bessel functions and self-duality for the Brownian momentum process

The idea of this Section is to obtain the self-duality of the BMP process as a consequence of the change of variable highlighted in Remark 2.3. The representation (6.13) will provide a new representation for the action of the Lie algebra generators

$$
\begin{aligned}
\left(\tilde{H} f\right)(x) &:= \left(-x\partial_x - \left(\frac{1}{2} - x^2\right)\right) f(x) \\
\left(\tilde{E} f\right)(x) &:= -\frac{1}{2} ix^2 f(x) \\
\left(\tilde{F} f\right)(x) &:= \left(-\frac{i}{2}\partial_x^2 + ix\partial_x + \frac{i}{2}(1 - x^2)\right) f(x) \ .
\end{aligned}
\tag{6.16}
$$

In equation (6.15) we set $z = x^2$, $w = y^2$ and $k = \frac{1}{4}$ so that the candidate BMP self-duality function becomes

$$
\tilde{J}(x, y) = e^{\frac{1}{2}(x^2 + y^2)} (|xy|)^{\frac{1}{2}} J_{-1/2}(xy) = e^{\frac{1}{2}(x^2 + y^2)} \sqrt{\frac{2}{\pi}} \cos(xy) \ ,
$$

where the second identity follows from the fact that, for fixed parameter $\nu = -1/2$, Bessel functions assume the simple form of

$$
J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x) \ .
$$

Theorem 6.7. *The Brownian momentum process is self-dual with self-duality function* $\tilde{J}(x_1, y_1)\tilde{J}(x_2, y_2)$.

Proof. Given the $\mathfrak{su}(1,1)$ algebra representation in (6.16), one could argue similarly as in Theorem 6.6 to prove that $\tilde{J}(x, y)$ is indeed a self-duality function for the BMP process. However, we follow another analogous path. We show here that the self-duality for the BMP process can be obtained from the self-duality of the BEP via the change of variable in Remark 2.3.

For the invertible operator $V : \mathsf{L}^2(\mathbb{R}^+, \mu_{1/4}) \to \mathsf{L}^2_e(\mathbb{R}, \frac{e^{-x^2}}{\sqrt{\pi}})$ given by $(Vf)(x) = f(x^2)$ and where L^2_e is the L^2 space of even functions, we have

$$\begin{aligned}\tilde{H} &= VHV^{-1}\\ \tilde{E} &= VEV^{-1}\\ \tilde{F} &= VFV^{-1} \,.\end{aligned}$$

One can now easily check that

$$L^{BMP} = VL^{BEP(1/4)}V^{-1} \,. \tag{6.17}$$

At this point we indicate with $D^{1/4}(z, w)$ and $D(x, y)$ the self-duality functions of the BEP process with $k = 1/4$ and the BMP process respectively, so that the following relation holds

$$D(x, y) = \left(V_x V_y D^{1/4}\right)(x, y) = D^{1/4}(x^2, y^2) \,, \tag{6.18}$$

where we use the notation of Remark 6.1. For the generators this gives

$$\begin{aligned}L_x^{BMP} D &= L_x^{BMP} V_x V_y D^{1/4}\\ &= V_x L_x^{BEP(1/4)} V_y D^{1/4}\\ &= V_x V_y L_y^{BEP(1/4)} D^{1/4}\\ &= V_x L_y^{BMP} V_y D^{1/4}\\ &= L_y^{BMP} D \,.\end{aligned}$$

Here we used that operators acting on x commute with operators acting on y, the first and last equalities are true in virtue of equation (6.18), the second and fourth ones both come from equation (6.17), and the third one is the self-duality of the BEP(1/4) process in Theorem 6.6. □

As before, it is an easy computation to find the realisation of operators E, F and H on function of x and y acting on $\tilde{J}$.

$$\begin{aligned}
\left(H\tilde{J}(\cdot, y; k)\right)(x) &:= \left(-x\partial_x - \left(\frac{1}{2} - x^2\right)\right)\tilde{J}(x, y; k) \\
&= \left(-y\partial_y - \left(\frac{1}{2} - y^2\right)\right)\tilde{J}(x, y; k) \\
&= \left(H\tilde{J}(x, \cdot; k)\right)(y) \\
\left(F\tilde{J}(\cdot, y; k)\right)(x) &:= \left(-\frac{i}{2}\partial_x^2 + ix\partial_x + \frac{i}{2}(1 - x^2)\right)\tilde{J}(x, y; k) = \frac{1}{2}iy^2\tilde{J}(x, y; k) \\
&= \left(-E\tilde{J}(x, \cdot; k)\right)(y) \\
\left(E\tilde{J}(\cdot, y; k)\right)(x) &:= -\frac{1}{2}ix^2\tilde{J}(x, y; k) \\
&= \left(\frac{i}{2}\partial_y^2 - iy\partial_y - \frac{i}{2}(1 - y^2)\right)\tilde{J}(x, y; k) \\
&= \left(-F\tilde{J}(x, \cdot; k)\right)(y) \ .
\end{aligned}$$

See that our self-duality functions are indeed intertwining functions between F (resp. E) and E(resp. $-F$).

Self-duality of IRW and Charlier polynomials

Last paragraph of this Section is dedicated to independent random walkers, in this case no Casimir element is available and so we will show that conditions of Theorem 6.2 are satisfied following a different route. Since it will be easier to work with another representation of the Heisenberg algebra, we will introduce a new representation. Taking $\lambda = 1$ and the operator Z to be the identity we would find the Lie representation presented in Section 4.3. However, we will now have that the Heisenberg algebra has generators a, $a^\dagger$ and Z satisfying the relations

$$[a, Z] = [a^\dagger, Z] = 0 \qquad [a^\dagger, a] = Z \ .$$

The $*-$structure is defined by $a^* = a^\dagger$, and $Z^* = Z$. In this setting, a, $a^\dagger$ and Z are operators on $l^2(\mathbb{N}_0, \mu)$ where the scalar product is defined by

$$\langle f, g\rangle_\mu = \sum_{n\in\mathbb{N}} f(n)g(n)\mu(n)$$

and $\mu(n) = \dfrac{\lambda^n}{n!}$ for $\lambda > 0$. Generators a, $a^\dagger$ and Z act on functions $f(n)$ on $\mathbb{N}_0$ as

$$\begin{aligned}
(af)(n) &:= nf(n-1) \\
(a^\dagger f)(n) &:= \lambda f(n+1) \\
(Zf)(n) &:= \lambda f(n) \ .
\end{aligned} \tag{6.19}$$

In this representation the independent random walk generator (in equation (2.16)), up to constant λ, on two sites is

$$L^{IRW} = (1 \otimes a - a \otimes 1)(a^\dagger \otimes 1 - 1 \otimes a^\dagger) = a^\dagger \otimes a - 1 \otimes aa^\dagger - aa^\dagger \otimes 1 + a \otimes a^\dagger \ . \tag{6.20}$$

We remark that since no Casimir element is available we will not be looking for a function for which the Casimir is reversible, this time we will search directly for operators for which L is equal to L^{rev}. To this end let's define the operator $X := Z - a^\dagger$ and we notice that the Heisenberg algebra is generated by a and X, since using $Z = [a, X]$ and $a^\dagger = -X + [a, Z]$, the generator of the IRW process in (6.20) becomes

$$L^{IRW} = -X \otimes a + 1 \otimes aX + aX \otimes 1 - a \otimes X \ . \tag{6.21}$$

As done before, it is time to introduce our candidate self-duality functions: the *Charlier polynomials* are defined by

$$C_n(x) = {}_2F_0 \left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{\lambda} \right), \qquad x, n \in \mathbb{N}_0$$

they are clearly symmetric in x and n. They satisfy the three term recurrence relation

$$-xC_n(x) = \lambda C_{n+1}(x) - (n + \lambda)C_n(x) + nC_{n-1}(x) \ ,$$

and the following forward shift relation

$$xC_n(x-1) = \lambda C_n(x) - \lambda C_{n+1}(x) \ . \tag{6.22}$$

We conclude this Section with the proof of the next theorem, by giving operators A and B such that the hypothesis of Theorem 6.2 are satisfied.

Theorem 6.8. *The independent random walk process is self-dual with self-duality function* $C(x_1, n_1)C(x_2, n_2)$*, where* $C(x, n) = C_n(x)$

Proof. First, let's show that item one of Theorem 6.2 is satisfied. From the definition of the Charlier polynomials we have that $C(n, x) = C(x, n)$, so that $C(x_1, n_1)C(x_2, n_2)$ is symmetric in (x_1, x_2) and (n_1, n_2). Define $A = a \otimes 1 - 1 \otimes a$ and $B = X \otimes 1 - 1 \otimes X$, then $C(x_1, n_1)C(x_2, n_2)$ is an intertwining function for A and B. Indeed, for one site

$$\begin{aligned} (XC(\cdot, x))(n) &= \left(\left(Z - a^\dagger \right) C(\cdot, x) \right)(n) \\ &= \lambda \left(C(n, x) - C(n+1, x) \right) \\ &= xC(n, x-1) \\ &= (aC(n, \cdot))(x) \end{aligned}$$

where the second equality follows immediately from (6.19) and the third one follows from equation (6.22). For two sites it simply becomes

$$\begin{aligned}(BC(x_1,\cdot)C(x_2,\cdot))\,(n_1,n_2) &= (XC(x_1,\cdot))(n_1)C(x_2,n_2) - C(x_1,n_1)(XC(x_2,\cdot)(n_2)\\ &= (aC(\cdot,n_1)(x_1)C(x_2,n_2) - C(x_1,n_1)(XC(\cdot,n_2))(x_2)\\ &= (AC(\cdot,n_1)C(\cdot,n_2,))\,(x_1,x_2)\ .\end{aligned}$$

For the second item one can check from (6.21) that the generator of the IRW process is given by $L^{IRW} = AB$, which is equal to L^{rev}. □

Operators a, $a^\dagger$ and Z act on variables n and x in the following way for Charlier polynomials.

$$\begin{aligned}(aC(x,\cdot))\,(n) &:= nC(x,n-1) = \lambda C(x,n) - \lambda C(x+1,n) = \left(\left(Z-a^\dagger\right)C(\cdot,n)\right)(x)\\ \left(a^\dagger C(x,\cdot)\right)(n) &:= \lambda C(x,n+1) = \lambda C(x,n) - xC(x-1,n) = ((Z-a)\,C(\cdot,n))\,(x)\\ (ZC(x,\cdot))\,(n) &:= \lambda C(x,n) = (ZC(\cdot,n))\,(x)\ .\end{aligned}$$

6.2 Symmetries leading to orthogonal self-duality

In this Section we explicitly show who are the operator symmetries $\tilde{S}$, given in terms of the underling Lie algebra generators which allows to recover the orthogonal duality functions. It is important to mention that, since we start from a (trivial) duality which is orthogonal then the operator $\tilde{S}$ that produces the orthogonal duality must be unitary. Recall that a unitary operator is a linear operator such that the inverse is its adjoint, i.e.

$$UU^* = U^*U = I\ .$$

As a consequence of this, we will have that the norm of the cheap duality function must be the same as the norm of the orthogonal duality function, since, for some appropriate Hilbert space

$$\| D^{orthogonal} \| = \langle \tilde{S}d^{cheap}, \tilde{S}d^{cheap}\rangle = \| d^{cheap} \|\ .$$

6.2.1 Krawtchouk polynomials

Standard Krawtchouk polynomials are defined as

$$K_{j,p}(x,y) = K(x,y) = {}_2F_1\left(\begin{matrix}-x,-y\\ -2j\end{matrix}\,\middle|\,\frac{1}{p}\right) = \sum_{i=0}^{x\wedge y}\frac{(-1)^i}{i!}\frac{x!}{(x-i)!}\frac{y!}{(y-i)!}\frac{(2j-i)!}{2j!}\left(\frac{1}{p}\right)^i\ . \tag{6.23}$$

Since the scalar product of two self-duality functions is still a self-duality function (Proposition 1.2), we define the scalar self-duality function as

$$\begin{aligned}D^{sca}_{ab}(x,y) &:= \langle D^{cl}_a(x,\cdot), D^{cl}_b(y,\cdot)\rangle_w =\\ &\sum_i D^{cl}(x,i)(ab)^i D^{cl}(y,i)w_(i)\end{aligned}$$

where $w(n) = \dfrac{2j!}{(2j-n)!n!}$. We choose to set, for example $a = 1$ and $b = -\frac{1}{p}$ (this choice is not unique), so we have that $K(x, y) = D^{sca}_{-1/p}(x, y)$. In matrix notation the relation becomes

$$K = D^{cl}(d^{ch})^{-1}(D^{cl}_{-1/p})^T = e^{J^+}d^{ch}(d^{ch})^{-1}(e^{J^+}d^{ch}_{-1/p})^T = e^{J^+}d^{ch}_{-1/p}(e^{J^+})^T \ , \tag{6.24}$$

which is clearly symmetric. Using the transpose equation for J^- as in (4.13) we have $(J^+)^T = d^{-1}(J^-)d$. Substitution into K produces

$$K = e^{J^+}d^{ch}_{-1/p}(d^{ch})^{-1}e^{J^-}d^{ch} \ . \tag{6.25}$$

Setting $(N)_{x,y} = \left(-\frac{1}{p}\right)^y \delta_{x,y}$ it is clear that

$$K = e^{J^+}Ne^{J^-}d^{ch} = e^{J^+}e^{-J^-}Nd^{ch} \tag{6.26}$$

and N anti-commutes with J^- with factor p, i.e. $NJ^- = -pJ^-N$ which implies that $Ne^{J^-} = e^{-pJ^-}N$ so we can write the orthogonal self-duality function in matrix notation as

$$K = e^{J^+}e^{-pJ^-}d^{ch}_{-1/p} \ , \tag{6.27}$$

since $Nd^{ch} = d^{ch}_{-1/p}$ by definition of the self-duality with parameter. We now identify the symmetry as

$$\tilde{S} = e^{J^+}e^{-pJ^-} \ . \tag{6.28}$$

The next lemma shows via generating functions that $\tilde{S}d^{ch}_{-1/p}$ is a Krawtchouk polynomial. This can be done showing that the generating function of Krawtchouk polynomials has the same form of the generating function of $\tilde{S}d^{ch}_{-1/p}$. It will be convenient to define the generating function G as in [42] (formula 9.11.10), i.e. the generating function of $g(n)$ is

$$(G\ g)\ (t) = \sum_{n=0}^{2j} g(n)\binom{2j}{n}t^n \ ,$$

which, for Krawtchouk polynomials $K_n(x)$ of parameters p and $2j$ is

$$\sum_{n=0}^{2j} K_n(x)\binom{2j}{n}t^n = \left(1 - \frac{(1-p)}{p}t\right)^x (1+t)^{2j-x} \ . \tag{6.29}$$

Lemma 6.3. *The generating function of $\tilde{S}d^{ch}_{-1/p}$, i.e.*

$$G(\tilde{S}d^{ch}_{-1/p}) = \sum_{n=0}^{2j} e^{J^+}e^{-pJ^-}d^{ch}_{-1/p}\binom{2j}{n}t^n$$

is the generating function of Krawtchouk polynomials in equation (6.29).

Proof. The idea is to evaluate $G\left(e^{J^+}e^{-pJ^-}d^{ch}_{-1/p}\right)$ which can be done in a simple way using the fact that G, understood as operator acting on functions of the n variable commutes with operators $J^{\pm}$ in the following way.

$$\begin{aligned}
GJ^+f(t) &= \sum_{n=0}^{2j} nf(n-1)\binom{2j}{n}t^n \\
&= 2j\ t\sum_{n=0}^{2j} f(n)\binom{2j}{n}t^n - t^2\sum_{n=0}^{2j} f(n)\binom{2j}{n}nt^{n-1} \\
&= \left(2j\ t - t^2\frac{\partial}{\partial t}\right)Gf(n) \\
&= \mathcal{J}^+Gf(t)\ ,
\end{aligned}$$

this implicitly defines the operator $\mathcal{J}^+$ which acts on function of the t variable as

$$\mathcal{J}^+ := 2j\ t - t^2\frac{\partial}{\partial t}\ .$$

Similarly,

$$GJ^-f(n) = \sum_{n=0}^{2j}(2j-n)f(n+1)\binom{2j}{n}t^n = \sum_{n=0}^{2j} f(n)\binom{2j}{n}t^{n-1} = \left(\frac{\partial}{\partial t}\right)Gf(n) = \mathcal{J}^-Gf(n)\ ,$$

so the operator $\mathcal{J}^-$ is a first derivative, defined as

$$\mathcal{J}^-g(t) := \frac{\partial g}{\partial t}(t)\ .$$

The action of the exponentials on functions of the t variable, can be found as follows. Clearly, using Taylor expansion

$$\left(e^{-p\mathcal{J}^-}\phi\right)(t) = e^{-p\frac{\partial}{\partial t}}\phi(t) = \phi(t-p)$$

since both sides are equal to $\sum_n \dfrac{\phi^{(n)}(t)}{n!}(-p)^n$. The action of the other exponential operator can be found solving a partial differential equation. Set

$$\psi(t,\gamma) = e^{\gamma\left(-t^2\frac{\partial}{\partial t}+2j\ t\right)}f(t) = e^{\gamma\mathcal{K}^+}f(t)$$

which is solution of

$$\frac{\partial}{\partial t}\psi(t,\gamma) = \left(t^2\frac{\partial}{\partial t} + 2j\ t\right)\psi(t,\gamma)\ .$$

with the aid of Feynman-Kac formula one finds

$$\begin{aligned}
\psi(t,\gamma) &= e^{\gamma\mathcal{J}^+}f(t) = e^{\gamma\left(2j\ t-t^2\frac{\partial}{\partial t}\right)}f(t) \\
&= e^{\int_0^\gamma 2j\frac{t}{1+st}ds}f\left(\frac{t}{1+\gamma t}\right) \\
&= (1+\gamma t)^{2j}f\left(\frac{t}{1+\gamma t}\right)
\end{aligned}$$

we get the wanted result for $\gamma = 1$, i.e.

$$e^{\jmath^+} f(t) = (1+t)^{2j} f\left(\frac{t}{1+t}\right) \ .$$

We now use the commutative property of operators $J^{\pm}$ and $\jmath^{\pm}$ with G so that we just need to calculate the rhs of

$$G\left(e^{J^+}e^{-pJ^-}d^{ch}_{-1/p}\right) = e^{\jmath^+}e^{-p\jmath^-}G\left(d^{ch}_{-1/p}\right) \ .$$

The generating function of the cheap self-duality function is

$$G\left(d^{ch}_{-1/p}(x,\cdot)\right)(t) = \sum_{n=0}^{2j} \frac{n!(2j-n)!}{2j!}\left(-\frac{1}{p}\right)^n \delta_{x,n}\binom{2j}{n}t^n = \left(-\frac{t}{p}\right)^x$$

and so

$$\begin{aligned} e^{\jmath^+}e^{-p\jmath^-}G\left(d^{ch}_{-1/p}\right) &= e^{\jmath^+}e^{-p\jmath^-}\left(-\frac{t}{p}\right)^x = e^{\jmath^+}\left(-\frac{t-p}{p}\right)^x \\ &= (1+t)^{2j}\left(-\frac{\frac{t}{1+t}-p}{p}\right)^x = (1+t)^{2j-x}\left(1-\frac{1-p}{p}t\right)^x \ , \end{aligned}$$

which is exactly the generating function of Krawtchouk polynomials. □

6.2.2 Meixner polynomials

Standard Meixner polynomials are defined as

$$M_{k,p}(x,y) = M(x,y) = {}_2F_1\left(\left.\begin{matrix}-x,-y\\2k\end{matrix}\right| 1-\frac{1}{p}\right) = \sum_{i=0}^{x\wedge y}\frac{1}{i!}\frac{x!}{(x-i)!}\frac{y!}{(y-i)!}\frac{\Gamma(2k)}{\Gamma(2k+i)}\left(\frac{p-1}{p}\right)^j \ . \tag{6.30}$$

Consider the scalar product self-duality w.r.t $w(n) = \dfrac{\Gamma(2k+n)}{\Gamma(2k)}$

$$\begin{aligned} D^{sca}_{ab}(x,y) &:= \langle D^{cl}_a(x,\cdot), D^{cl}_b(y,\cdot)\rangle_w = \\ &\sum_i D^{cl}(x,i)D^{cl}(y,i)(ab)^i_w(i) \ . \end{aligned}$$

Setting $a = 1$ and $b = \frac{p-1}{p}$ we have that

$$D^{sca}_{\frac{p-1}{p}}(x,y) = M(x,y) \ , \tag{6.31}$$

so we can write the Meixner polynomials in terms of classical self-duality functions as a scalar product, i.e.

$$\begin{aligned} M(x,y) &:= \langle D^{cl}(x,\cdot), D^{cl}_{\frac{p-1}{p}}(y,\cdot)\rangle_w = \\ &\sum_i D^{cl}(x,i)D^{cl}_{\frac{p-1}{p}}(y,i)w(i) \ . \end{aligned}$$

In matrix notation the relation becomes

$$M = D^{cl}(d^{ch})^{-1}(D^{cl}_{\frac{p-1}{p}})^T = e^{K^+}d^{ch}(d^{ch})^{-1}d^{ch}_{\frac{p-1}{p}}(e^{K^+}d^{ch}_{\frac{p-1}{p}})^T = e^{K^+}d^{ch}_{\frac{p-1}{p}}(e^{K^+})^T \tag{6.32}$$

which is clearly symmetric. Using equation (4.20), i.e. $(K^+)^T = d^{-1}(K^-)d$ and subbing $(K^+)^T$ into M we find

$$M = e^{K^+}d^{ch}_{\frac{p-1}{p}}(e^{K^+})^T = e^{K^+}d^{ch}_{\frac{p-1}{p}}(d^{ch})^{-1}e^{K^-}d^{ch} \tag{6.33}$$

letting $N(x,y) = \left(\frac{p-1}{p}\right)^x \delta_{x,y}$ we have

$$M = e^{K^+}Ne^{K^-}d^{ch} \; . \tag{6.34}$$

As before it is easy to show that $NK^- = \frac{p}{p-1}K^-N$ and seeing the operator N as a factor of the cheap self-duality function

$$M = e^{K^+}e^{\frac{p}{p-1}K^-}d^{ch}_{\frac{p-1}{p}} \; , \tag{6.35}$$

where we identify the symmetry as $\tilde{S} = e^{K^+}e^{\frac{p}{p-1}K^-}$. This computation shows that also orthogonal duality can be viewed as the product of the symmetry $\tilde{S}$ and the cheap duality of parameter $\frac{p-1}{p}$.

As done before, one can check independently that the $\tilde{S}d^{ch}_{\frac{p-1}{p}}$ are Meixner polynomials using their generating function as in [42] (formula 9.10.11), where the generating function of $g(n)$ is defined as

$$\left(Gg\left(n\right)\right)(t) = \sum_{n=0}^{\infty} g(n)(2k)_n \frac{t^n}{n!} \; .$$

For Meixner polynomials $M_n(x)$ of parameters p and $2k$ the generating function can be evaluated and it is

$$\sum_{n=0}^{\infty} M_n(x)(2k)_n \frac{t^n}{n!} = \left(1 - \frac{t}{p}\right)^x (1-t)^{-2k-x} \; . \tag{6.36}$$

Lemma 6.4. *The generating function of $\tilde{S}d^{ch}_{\frac{p-1}{p}}$ coincides with the generating function of Meixner polynomials in equation* (6.36)*.*

Proof. The proof follows the same idea of the one before; instead of evaluating $G\left(e^{J^+}e^{-pJ^-}d^{ch}_{\frac{p-1}{p}}\right)$ we find continuous operators $\mathcal{K}^{\pm}$ as

$$GK^+f(n) = \sum_{n=0}^{\infty} nf(n-1)(2k)_n \frac{t^n}{n!} = \left(t^2\frac{\partial}{\partial t} + 2k\ t\right)Gf(n) = \mathcal{K}^+Gf(n) \; ,$$

from which we infer

$$\mathcal{K}^+ := t^2\frac{\partial}{\partial t} + 2k\ t \; .$$

Similarly,

$$GK^- f(n) = \sum_{n=0}^{\infty} (2k+n) f(n+1) (2k)_n \frac{t^n}{n!} = \left(\frac{\partial}{\partial t}\right) Gf(n) = \mathcal{K}^- Gf(n) \ ,$$

so the operator $\mathcal{K}^-$ is a first derivative, defined by

$$\mathcal{K}^- := \frac{\partial}{\partial t} \ .$$

As before, using Taylor expansion

$$\left(e^{\frac{p}{p-1}\mathcal{K}^-}\phi\right)(t) = e^{\frac{p}{p-1}\frac{\partial}{\partial t}}\phi(t) = \phi\left(t + \frac{p}{p-1}\right) \ .$$

While the other operator is

$$e^{\mathcal{K}^+} f(t) = \frac{1}{(1-t)^{2k}} f\left(\frac{t}{1-t}\right) \ ,$$

found solving

$$\frac{\partial}{\partial \gamma}\psi(t,\gamma) = \left(t^2 \frac{\partial}{\partial t} + 2k\ t\right)\psi(t,\gamma)$$

We now use the commutative property of operators $K^{\pm}$ and $\mathcal{K}^{\pm}$ with G to justify

$$G\left(e^{K^+} e^{\frac{p}{p-1}K^-} d^{ch}_{\frac{p-1}{p}}\right) = e^{\mathcal{K}^+} e^{\frac{p}{p-1}\mathcal{K}^-} G\left(d^{ch}_{\frac{p-1}{p}}\right) \ .$$

The generating function of the cheap self-duality function is

$$G\left(d^{ch}_{\frac{p-1}{p}}(x,\cdot)\right)(t) = \sum_{n=0}^{\infty} \frac{n!\Gamma(2k)}{\Gamma(2k+n)} \left(\frac{p-1}{p}\right)^n \delta_{x,n} (2k)_n \frac{t^n}{n!} = \left(\frac{p-1}{p} t\right)^x$$

and so

$$e^{\mathcal{K}^+} e^{\frac{p}{p-1}\mathcal{K}^-} G\left(d^{ch}_{\frac{p-1}{p}}\right) = e^{\mathcal{K}^+} e^{\frac{p}{p-1}\mathcal{K}^-} \left(\frac{p-1}{p} t\right)^x = e^{\mathcal{K}^+} \left(\frac{p-1}{p}\left(t + \frac{p}{p-1}\right)\right)^x$$

$$= e^{\mathcal{K}^+}\left(\frac{p-1}{p} t + 1\right)^x = \frac{1}{(1-t)^{2k}}\left(\frac{p-1}{p}\frac{t}{1-t} + 1\right)^x = (1-t)^{-2k-x}\left(\frac{p-t}{p}\right)^x$$

which is exactly the generating function of Meixner polynomials. □

6.2.3 Charlier polynomials

Standard Charlier polynomials are defined as

$$C^{\lambda}(x,y) = {}_2F_0\left(\begin{matrix} -x, -y \\ - \end{matrix} \middle| -\frac{1}{\lambda}\right) = \sum_{k=0}^{x \wedge y} \frac{(-1)^k}{k!} \frac{x!}{(x-k)!} \frac{y!}{(y-k)!} \frac{1}{\lambda^k} \ . \qquad (6.37)$$

Using again the ideas of Section 1.4, i.e. the fact that the scalar product of (self-)dualities is still a (self-)duality, we define the following scalar self-duality function with respect to $w(n) = \frac{1}{\lambda^n}$

$$\begin{aligned} D^{sca}_{ab}(x,y) := \langle D^{cl}_a(x,\cdot), D^{cl}_b(y,\cdot)\rangle_w = \\ \sum_k D^{cl}_a(x,k) D^{cl}_b(y,k)\mu(k) = \\ \sum_k \frac{x!}{(x-k)!}a^k \frac{y!}{(y-k)!} b^k \frac{1}{k!} = \\ \sum_{k=0}^{x\wedge y} \frac{(ab)^k}{k!} \frac{x!}{(x-k)!} \frac{y!}{(y-k)!} \end{aligned}$$

Setting $ab = -\frac{1}{\lambda}$ so that, for example, we can choose $a = 1$ and $b = -\frac{1}{\lambda}$ we have that $C^\lambda(x,y) = D^{sca}_{-1/\lambda}(x,y)$ In matrix notation the relation becomes

$$C^\lambda = D^{cl}_1 d^{ch}_{-1} (D^{-1/\lambda}_{cl})^T = e^a d^{ch} (d^{ch})^{-1} d^{ch}_{-1/\lambda} (e^a)^T = e^a d^{ch}_{-1/\lambda} (e^a)^T \tag{6.38}$$

which is clearly symmetric. Using that $a^T = (d^{ch})^{-1} a^\dagger d^{ch}$ as shown in (4.24) we get that

$$\tilde{C}^\lambda = e^a d^{ch}_{-1/\lambda} (e^a)^T = e^a d^{ch}_{-1/\lambda} (d^{ch})^{-1} e^{a^\dagger} d^{ch}\ . \tag{6.39}$$

Recalling the definition of d^{ch}_a, we see that $d^{ch}_{-1/\lambda}(d^{ch})^{-1} = \left(-\frac{1}{\lambda}\right)^y$ and calling $(N)_{x,y} = \left(-\frac{1}{\lambda}\right)^x \delta_{x,y}$ we have that

$$\tilde{C}^\lambda = e^a N e^{a^\dagger} d^{ch}\ . \tag{6.40}$$

At this stage we could already identify the symmetry, however, we rather have a symmetry only in terms of the generators of the Heisenberg algebra. To this end, note that

$$Na^\dagger = -\lambda a^\dagger N$$

since

$$(Na^\dagger f)(n) = Nf(n+1) = \left(-\frac{1}{\lambda}\right)^n f(n+1)$$

and

$$(-\lambda a^\dagger N f)(n) = -\lambda a^\dagger \left(-\frac{1}{\lambda}\right)^n f(n) = -\lambda \left(-\frac{1}{\lambda}\right)^{n+1} f(n+1) = \left(-\frac{1}{\lambda}\right)^n f(n+1)\ .$$

This implies that $Ne^{a^\dagger} = e^{-\lambda a^\dagger} N$, and so we can finally write the orthogonal self-duality function as

$$\tilde{C}^\lambda = e^a e^{-\lambda a^\dagger} N d^{ch} = e^a e^{-\lambda a^\dagger} d^{ch}_{-1/\lambda}\ , \tag{6.41}$$

where the last identity follows from the fact that acting with the operator N will just produce the factor $-\frac{1}{\lambda}$ in the cheap duality. It is easy now to identify the symmetry that, applied to the cheap duality $d^{ch}_{-1/\lambda}$, produces the Charlier orthogonal polynomials. Indeed,

$$\tilde{S} = e^{a} e^{-\lambda a^{\dagger}} \ . \tag{6.42}$$

We conclude the Chapter showing that the generating function of $\tilde{S} d^{ch}_{-1/\lambda}$ coincides with the generating function of the Charlier polynomials as defined in [42] (formula 9.14.11) in the following way

$$(Gg\,(n))\,(t) = \sum_{n=0}^{\infty} g(n) \frac{t^n}{n!} \ ,$$

For Charlier polynomials $C_n(x)$ of parameter λ the generating function is

$$\sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} = e^t \left(1 - \frac{t}{\lambda}\right)^x \ . \tag{6.43}$$

Lemma 6.5. *The generating function of* $\tilde{S} d^{ch}_{-1/\lambda}$ *coincides with the generating function of Charlier polynomials in equation* (6.43).

Proof. The proof follows the same idea of the previous two. We need to evaluate $e^{a} e^{-\lambda a^{\dagger}} G\left(d^{ch}_{-1/\lambda}\right)$ where a and $a^{\dagger}$ are found as before:

$$(Ga\ f)(n) \sum_{n=0}^{\infty} n f(n-1) \frac{t^n}{n!} = tGf(n) = aGf(n)$$

so we get

$$a := t \ ,$$

and of course

$$e^{a} \phi(t) = e^{t} \phi(t) \ .$$

While

$$(Ga^{\dagger}\ f)(n) \sum_{n=0}^{\infty} f(n+1) \frac{t^n}{n!} = \frac{\partial}{\partial t} Gf(n) = a^{\dagger} Gf(n)$$

which allows us to infer that

$$a^{\dagger} := \frac{\partial}{\partial t} \ ,$$

and that

$$e^{-\lambda a^{\dagger}} \phi(t) = e^{-\lambda \frac{\partial}{\partial t}} \phi(t) = \phi(t - \lambda) \ .$$

The generating function of the cheap self-duality function is

$$G d^{ch}_{-1/\lambda} = \left(-\frac{t}{\lambda}\right)^x$$

and so we can easily find that

$$e^a e^{-\lambda a^\dagger} G\left(d^{ch}_{-1/\lambda}\right) = e^a e^{-\lambda a^\dagger} \left(-\frac{t}{\lambda}\right)^x = e^a \left(-\frac{t-\lambda}{\lambda}\right)^x = e^t \left(1 - \frac{t}{\lambda}\right)^x$$

which is exactly the generating function of Charlier polynomials. □

Chapter 7

Perspective on the asymmetric processes

7.1 Overview of the processes

This last Chapter is contains some (work in progress) results regarding the asymmetric versions of the exclusion process introduced in Section 2.2 and the inclusion process introduced in Section 2.3. Being our aim to study dualities, we will work with the asymmetric version of these models that have the algebraic structure leading to duality. The asymmetric simple exclusion process is introduced in [17], while the asymmetric simple inclusion process is introduced in [16]. Now particles move asymmetrically with a bias determined by a parameter $q \in (0,1)$, when the limit $q \to 1$ is considered one recovers the symmetric versions studied in Chapter 2. From the algebraic point of view when passing from symmetric to asymmetric processes, one has to change from the original Lie algebra to the corresponding *deformed quantum Lie algebra*, where the deformation parameter q is related to the asymmetry. In the context of the standard ASEP with hard core exclusion, this was first observed by Schütz [60, 61]. Some dualities for these models have been shown in [16, 17], in particular self-duality functions loose their product structure, which is indeed replaced by a nested structure that we will recall below. These self-dualities in the limit $q \to 1$ degenerate to the classical (triangular) self-dualitites with product structure of the asymmetric cases. Orthogonal self-dualitites for these asymmetric models are not known yet, and this is the question that is preliminary explored in this Chapter. Let's now introduce the two processes via their generators, for a detail description of the processes, their algebraic construction via quantum Hamiltonian and its symmetries as well as their properties, such as self-duality results with classical self-duality functions, we refer to the papers [16, 17] mentioned above.

In analogy of what done in Chapter 2 we define our processes on the undirected and connected graph $G = (V, E)$ with $|V| = N$ sites (vertices) and edge set E. Moreover, the asymmetry introduced will not affect the shape of the generators in the sense that their action is read on two sites only and then summed up to N. The following definitions regarding q numbers are required

Definition 7.1 (q-number). *For $q \in (0,1)$ and $n \in \mathbb{N}$ we introduce the q-number*

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

which satisfy $\lim_{q\to 1}[n]_q = n$.

The first four q-numbers are $[0]_q = 0$, $[1]_q = 1$, $[2]_q = q+q^{-1}$ and $[0]_q = q^2+1+q^{-2}$. It is also possible to define the analogue of the factorial, the binomial coefficient and the Pochhammer symbol.

Definition 7.2 (q-factorial, q-binomial and q-Pochhammer). *We define the q-factorial*

$$[n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q \qquad n \in \mathbb{N}$$

the q-binomial coefficient as

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad n,k \in \mathbb{N},\ k \leq n\ ,$$

last, we define the q-Pochhammer symbol by

$$(a;q)_m = (1-a)(1-aq)\cdots(1-aq^{m-1}) \qquad m \in \mathbb{N},\ a \in \mathbb{R}\ .$$

Later on we will use the q-version of the Newton formula, which we recall

$$\sum_{\kappa=0}^{N} \binom{N}{\kappa}_q q^{\kappa N} \left(tq^{-1}\right)^{\kappa} = \prod_{j=1}^{N} (1 + tq^{2(j-1)})\ . \tag{7.1}$$

7.1.1 Asymmetric Exclusion Process

The asymmetric exclusion process of parameters q and $2j$, ASEP(q,j), is a particle system process where particle jumps (in both directions) to nearest neighbors sites according to rates described by the generator. Each site can accommodate up to $2j$ particles, if this number is reached, the jump is forbidden.

Definition 7.3 (ASEP(q,j) process). *Let $q \in (0,1)$ and $j \in \mathbb{N}/2$, then the ASEP(q,j) generator has the form*

$$L^{ASEP(q,j)} f(\mathbf{x}) = \sum_{\substack{1\leq i<l\leq N\\ (i,l)\in E}} q^{x_i - x_l - (2j+1)} [x_i]_q [2j - x_l]_q \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + \tag{7.2}$$

$$q^{x_i - x_l + (2j+1)} [2j - x_i]_q [x_l]_q \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right]$$

where x_i is the number of particles in site $i \in V$ and $\boldsymbol{x}^{i,l}$ denotes the particle configuration that is obtained from $\mathbf{x}$ by moving a particle from site i to site l. Moreover, $f : \{0,1,\ldots,2j\}^N \to \mathbb{R}$ is a function in the domain of the generator.

The reversible product measure μ_α is (see Thereom 3.1 of [17])

$$\mathbb{P}_{\mu_\alpha}(x_i = n) = \frac{\alpha^n}{Z_i^{(\alpha)}} \binom{2j}{n}_q \cdot q^{-4jin} q^{2j(1+n)} , \qquad n \in \mathbb{N} \tag{7.3}$$

where

$$Z_i^{(\alpha)} = \sum_{n=0}^{2j} \binom{2j}{n}_q \alpha^n q^{2n(1+j-2ji)}$$

is a normalizing constant.

7.1.2 Asymmetric Inclusion Process

The asymmetric inclusion process of parameters q and $2k$, ASIP(q,k), is a particle system process where particle jumps (in both directions) to nearest neighbors sites. It can be considered as the asymmetric analogue of the SIP(k) process.

Definition 7.4 (ASIP(q, k) process)**.** *Let $q \in (0, 1)$ and $k \in \mathbb{N}/2$, then the ASIP(q,k) is specified by the following generator*

$$L^{ASIP(q,k)} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} q^{x_i - x_l + (2k-1)} [x_i]_q [2k + x_l]_q \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + \tag{7.4}$$
$$q^{x_i - x_l - (2k+1)} [2k + x_i]_q [x_l]_q \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right]$$

where x_i is the number of particles in site $i \in V$ and $\boldsymbol{x}^{i,l}$ denotes the particle configuration that is obtained from $\mathbf{x}$ by moving a particle from site i to site l. Moreover, $f : \mathbb{N}^N \to \mathbb{R}$ is a function in the domain of the generator.

The product reversible measure μ_α is (see Thereom 2.1 of [16])

$$\mathbb{P}_{\mu_\alpha}(x_i = n) = \frac{\alpha^n}{Z_i^{(\alpha)}} \binom{n + 2k - 1}{n}_q \cdot q^{4kin} \qquad n \in \mathbb{N} \tag{7.5}$$

where

$$Z_i^{(\alpha)} = \sum_{n=0}^{\infty} \binom{n + 2k - 1}{n}_q \alpha^n q^{4kin}$$

is a normalizing constant.

7.2 Perspective and open problems

We devote the rest of this thesis to some results that are still work in progress. The idea would be to implement the theory of Section 1.4 to produce orthogonal self-duality functions for the asymmetric processes starting from their classical self-duality functions. Remember that these new functions are biorthoognal by construction. We recall the classical self-dualtiy functions for ASEP(q,j) and ASIP(q,k) as well as their reversible measures as in [16,17] and then apply Proposition 1.5. We mention that we multiply for some constants (to the dual variable) to the functions introduced [16,17], this will bring some advantages in the computation of the scalar product, while the self-duality property is not affected, see Remark 1.3. Since the generators do not depend on these constants (namely λ_1 and λ_2) we will chose them in a convenient way. Setting these constants equal to one, one recovers the standard classical self-duality functions.

7.2.1 Biorthogonal self-duality functions for ASEP(q,j)

For the asymmetric exclusion process of parameter q and j we have that $d^{cl}_{\lambda_1}$ and $d^{cl}_{\lambda_2}$ are both self-duality functions (See Theorem 3.2 of [17]), where

$$d^{cl}_{\lambda_1}(x,n) = \prod_{i=1}^{L} Z_i^{(\alpha)} \cdot \frac{\binom{x_i}{n_i}_q}{\binom{2j}{n_i}_q} \cdot q^{x_i\left[2\sum_{m=1}^{i-1} n_m + n_i\right] + 4jin_i} \lambda_1^{n_i} \cdot \mathbf{1}_{n_i \le x_i} \;, \tag{7.6}$$

and

$$d^{cl}_{\lambda_2}(x,n) = \prod_{i=1}^{L} Z_i^{(\alpha)} \cdot \frac{\binom{x_i}{n_i}_q}{\binom{2j}{n_i}_q} \cdot q^{-n_i\left[2\sum_{m=1}^{i-1} x_m + x_i\right] + 4kin_i} \lambda_2^{n_i} \cdot \mathbf{1}_{n_i \le x_i} \;. \tag{7.7}$$

The following lemma shows that $d^{cl}_{\lambda_1}$ and $d^{cl}_{\lambda_2}$ are biorthogonal for an appropriate choice of the parameters λ_1 and λ_2.

Lemma 7.1. *Choosing* $\lambda_1 = -\frac{q}{\alpha}$ *and* $\lambda_2 = -\frac{1}{q\alpha}$ *we have*

$$\langle d^{cl}_{-\frac{q}{\alpha}}(x,\cdot), d^{cl}_{-\frac{1}{q\alpha}}(\cdot,n)\rangle_{\mu(\alpha)} = \frac{\delta_{x,n}}{\mu_\alpha(n)} \;. \tag{7.8}$$

Proof.

$$\langle d_{\lambda_1}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_{\mu(\alpha)} = \sum_{y_1=n_1}^{x_1} \cdots \sum_{y_L=n_L}^{x_L} \prod_{i}^{L} \frac{\binom{x_i}{y_i}_q}{\binom{2j}{y_i}_q} \cdot q^{x_i\left[2\sum_{m=1}^{i-1} y_m + y_i\right] + 4jiy_i} \lambda_1^{y_i} \cdot$$

$$\cdot \frac{\binom{y_i}{n_i}_q}{\binom{2j}{n_i}_q} \cdot q^{-n_i\left[2\sum_{m=1}^{i-1} y_m + y_i\right] + 4jin_i} \lambda_2^{n_i} \cdot \frac{\alpha^{y_i}}{Z_i^{(\alpha)}} \binom{2j}{y_i}_q \cdot q^{-4jiy_i} \cdot \left(Z_i^{(\alpha)}\right)^2$$

$$= q^{4j\sum_i^L in_i} \cdot \lambda_2^{\sum_i^L n_i} \cdot \prod_i^L Z_i^{(\alpha)} \cdot \sum_{y_1=n_1}^{x_1} \cdots \sum_{y_L=n_L}^{x_L} \prod_i^L \frac{\binom{y_i}{n_i}_q \binom{x_i}{y_i}_q}{\binom{2j}{n_i}_q} \cdot q^{(x_i-n_i)\left[2\sum_{m=1}^{i-1} y_m\right]} q^{y_i(x_i-n_i)} \, (\alpha\lambda_1)^{y_i} \;.$$

Let's call $c(x,n)$ what does not depend on y_i, so that

$$c(x,n) := \prod_{i=1}^{L} \frac{\binom{x_i}{n_i}_q}{\binom{2j}{n_i}_q} q^{4j\sum_{i=1}^L in_i} \lambda_2^{\sum_{i=1}^L n_i} Z_i^{(\alpha)} \;.$$

where we have used that
$$\binom{y_i}{n_i}_q \binom{x_i}{y_i}_q = \binom{x_i}{n_i}_q \binom{x_i - n_i}{y_i - n_i}_q \, .$$
We now have reduced to
$$\langle d_{\lambda_1}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_\mu = c(x,n) \prod_{i=1}^{L} \sum_{y_i=n_i}^{x_i} \binom{x_i-n_i}{y_i-n_i}_q q^{2(x_i-n_i)\sum_{m=1}^{i-1} y_m} \, q^{y_i(x_i-n_i)} \, (\alpha\lambda_1)^{y_i}$$
$$= c(x,n) \prod_{i=1}^{L} \sum_{y_i=n_i}^{x_i} \binom{x_i-n_i}{y_i-n_i}_q q^{2y_i \sum_{m=i+1}^{L}(x_m-n_k)} \, q^{y_i(x_i-n_i)} \, (\alpha\lambda_1)^{y_i}$$
$$= c(x,n) \prod_{i=1}^{L} \sum_{z_i=0}^{x_i-n_i} \binom{x_i-n_i}{z_i}_q q^{2z_i \sum_{m=i+1}^{L}(x_m-n_k)} q^{2n_i \sum_{m=i+1}^{L}(x_m-n_k)} \, q^{z_i(x_i+n_i)} q^{n_i(x_i-n_i)} \, (\alpha\lambda_1)^{z_i+n_i}$$
where we used the change of variable $y_i - n_i = z_i$ in the last identity. Calling $\tilde{c}(x,n)$ what does not depend on z_i, i.e.
$$\tilde{c}(x,n) := c(x,n) \prod_{i=1}^{L} \left(q^{2\sum_{m=i+1}^{L}(x_m-n_k)} q^{x_i-n_i} \, \alpha\lambda_1 \right)^{n_i} \, .$$
We now have that
$$\langle d_{\lambda_1}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_{\mu(\alpha)} = \tilde{c}(x,n) \prod_{i=1}^{L} \sum_{z_i=0}^{x_i-n_i} \binom{x_i-n_i}{z_i}_q \left(q^{2\sum_{m=i+1}^{L}(x_m-n_m)} \, \alpha\lambda_1 \right)^{z_i} \, q^{z_i(x_i-n_i)} \, .$$
Using the Newton formula for q-coefficients as in equation (7.1), we get
$$\langle d_{\lambda_1}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_{\mu(\alpha)} = \tilde{c}(x,n) \prod_{i=1}^{L} \prod_{r=1}^{x_i-n_i} \left(1 + q^{2r-1} \, q^{2\sum_{m=i+1}^{L}(x_m-n_m)} \alpha\lambda_1 \right) \, .$$
Let us choose $\lambda_1 = -\frac{q^{-1}}{\alpha}$. Suppose $n_L \neq x_L$ then the L-th term in the product is equal to $\prod_{r=1}^{x_L-n_L} \left(1 - q^{2(r-1)}\right)^{x_L-n_L}$ that contains the term $r=1$ that is equal to 0. Suppose now that $n_L - x_L = \cdots = n_{i+1} - x_{i+1} = 0$ and $n_i - x_i \neq 0$, then the i-th term in the product is equal to
$$\prod_{r=1}^{x_i-n_i} \left(1 - q^{2(r-1)} \right) = 0 \, .$$
Then
$$\langle d_{-\frac{q^{-1}}{\alpha}}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle = \tilde{c}(n,n) \, \delta_{x,n}$$
where
$$\tilde{c}(n,n) = \prod_i \frac{Z_i^{(\alpha)}}{\binom{2j}{n_i}_q} q^{4jin_i} \cdot \left(-\frac{\lambda_2}{q} \right)^{n_i} = \frac{1}{\mu_\alpha(n)}$$
choosing $\lambda_2 = -\frac{q}{\alpha}$ we get the result. □

Now we can apply Proposition 1.5 with

$$d(x,n) = d_{-\frac{q^{-1}}{\alpha}}(n,x) \qquad \tilde{d}(x,n) = d_{-\frac{q}{\alpha}}(x,n)\ ,$$

then

$$D(x,n) := \langle d(x,\cdot), \tilde{d}(\cdot,n)\rangle \qquad \tilde{D}(x,n) := \langle \tilde{d}(x,\cdot), d(\cdot,n)\rangle$$

are duality functions and are biorthogonal, i.e.

$$\langle D(\cdot,m), \tilde{D}(\cdot,n)\rangle_c = \delta_v(m,n)\ .$$

7.2.2 Biorthogonal self-duality functions for ASIP(q,k)

For the asymmetric inclusion process of parameter q and k we have that $d^{cl}_{\lambda_1}$ and $d^{cl}_{\lambda_2}$ are both self-duality (See Theorem 5.1 of [16]), where

$$d^{cl}_{\lambda_1}(x,n) = \prod_{i=1}^{L} Z_i^{(\alpha)} \cdot \frac{\binom{x_i}{n_i}_q}{\binom{y_i+2k-1}{n_i}_q} \cdot q^{x_i\left[2\sum_{m=1}^{i-1} n_m + n_i\right]-4kin_i}\, \lambda_1^{n_i} \cdot \mathbf{1}_{n_i\le x_i} \tag{7.9}$$

where we used that $\sum_i n_i \left[2\sum_{m=1}^{i-1} n_m + n_i\right] = const.$ Moreover

$$d^{cl}_{\lambda_2}(x,n) = \prod_{i=1}^{L} Z_i^{(\alpha)} \cdot \frac{\binom{x_i}{n_i}_q}{\binom{y_i+2k-1}{n_i}_q} \cdot q^{-n_i\left[2\sum_{m=1}^{i-1} x_m + x_i\right]-4kin_i}\, \lambda_2^{n_i} \cdot \mathbf{1}_{n_i\le x_i}\ . \tag{7.10}$$

Note that $d^{cl}_{\lambda_1}$ and $d^{cl}_{\lambda_2}$ are actually the same modulo a multiplicative constant, i.e.

$$\prod_i q^{x_i[2\sum_{m<i} n_m + n_i]} = q^C \cdot \prod_i q^{-n_i[2\sum_{m<i} x_m + x_i]}, \qquad C = \sum_i x_i \cdot \sum_i n_i\ .$$

In analogy of what proved in Lemma 7.1, we can show that d_1 and d_2 are biorthogonal for an appropriate choice of the parameters λ_1 and λ_2.

Lemma 7.2. *Choosing $\lambda_1 = -\frac{q^{-1}}{\alpha}$ and $\lambda_2 = -\frac{q}{\alpha}$ we have*

$$\langle d^{cl}_{-\frac{1}{\alpha q}}(x,\cdot), d^{cl}_{-\frac{q}{\alpha}}(\cdot,n)\rangle_{\mu(\alpha)} = \frac{\delta_{x,n}}{\mu_\alpha(n)}\ . \tag{7.11}$$

Proof.

$$\begin{aligned}
&\langle d_{\lambda_1}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_{\mu(\alpha)} = \\
&= \sum_{y_1=n_1}^{x_1}\cdots\sum_{y_L=n_L}^{x_L} \prod_i \frac{\binom{x_i}{y_i}_q}{\binom{y_i+2k-1}{y_i}_q} \cdot q^{x_i\left[2\sum_{m=1}^{i-1} y_m + y_i\right]-4kiy_i}\, \lambda_1^{y_i} \cdot \\
&\cdot \frac{\binom{y_i}{n_i}_q}{\binom{n_i+2k-1}{n_i}_q} \cdot q^{-n_i\left[2\sum_{m=1}^{i-1} y_m + y_i\right]-4kin_i}\, \lambda_2^{n_i} \cdot \frac{\alpha^{y_i}}{Z_i^{(\alpha)}} \binom{y_i+2k-1}{y_i}_q \cdot q^{4kiy_i} \cdot \left(Z_i^{(\alpha)}\right)^2 \\
&= q^{-4k\sum_i in_i} \cdot \lambda_2^{\sum_i n_i} \cdot \prod_i Z_i^{(\alpha)} \cdot \sum_{y_1=n_1}^{x_1}\cdots\sum_{y_L=n_L}^{x_L} \prod_i \frac{\binom{y_i}{n_i}_q \binom{x_i}{y_i}_q}{\binom{n_i+2k-1}{n_i}_q} \cdot q^{(x_i-n_i)\left[2\sum_{m=1}^{i-1} y_m + y_i\right]}\, (\alpha\lambda_1)^{y_i}
\end{aligned}$$

where

$$\binom{y_i}{n_i}_q \binom{x_i}{y_i}_q = \binom{x_i}{n_i}_q \binom{x_i - n_i}{y_i - n_i}_q$$

then, for

$$c(x,n) := \prod_i \frac{\binom{x_i}{n_i}_q}{\binom{n_i+2k-1}{n_i}_q} q^{-4k\sum_i i n_i} \cdot \lambda_2^{\sum_i n_i} \cdot \prod_i Z_i^{(\alpha)}$$

we find

$$\langle d_{\lambda_1}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_{\mu(\alpha)} = c(x,n) \cdot \sum_{y_1=n_1}^{x_1} \dots \sum_{y_L=n_L}^{x_L} \prod_i \binom{x_i - n_i}{y_i - n_i}_q \cdot q^{(x_i-n_i)\left[2\sum_{m=1}^{i-1} y_m + y_i\right]} (\alpha\lambda_1)^{y_i}$$

$$= c(x,n) \cdot \sum_{y_1=n_1}^{x_1} \dots \sum_{y_L=n_L}^{x_L} \prod_i \binom{x_i - n_i}{y_i - n_i}_q \cdot q^{2y_i \sum_{m=i+1}^{L}(x_m - n_m)} q^{(x_i-n_i)y_i} (\alpha\lambda_1)^{y_i}$$

$$= \tilde{c}(x,n) \cdot \prod_i \sum_{z_i=0}^{x_i-n_i} \binom{x_i - n_i}{z_i}_q \cdot \left(q^{2\sum_{m=i+1}^{L}(x_m-n_m)} \cdot \alpha\lambda_1\right)^{z_i} q^{(x_i-n_i)z_i}$$

with the convention $\sum_{m=i+1}^{L}(x_m - n_m) = 0$ for $i = L$ and with

$$\tilde{c}(x,n) = c(x,n) \cdot \prod_i \left(q^{2\sum_{m=i+1}^{L}(x_m-n_m)+(x_i-n_i)} \cdot \alpha\lambda_1\right)^{n_i} .$$

We use the Newton formula in equation (7.1) that yields

$$\langle d_{\lambda_1}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_{\mu(\alpha)} = \tilde{c}(x,n) \cdot \prod_{i=1}^{L} \prod_{j=1}^{x_i-n_i} \left(1 + q^{2j-1} q^{2\sum_{m=i+1}^{L}(x_m-n_m)} \cdot \alpha\lambda_1\right) .$$

Let us choose $\lambda_1 = -\frac{q^{-1}}{\alpha}$. Suppose $n_L \neq x_L$ then the L-th term in the product is equal to $\prod_{j=1}^{x_L-n_L} \left(1 - q^{2(j-1)}\right)^{x_L-n_L}$ that contains the term $j = 1$ that is equal to 0. Suppose now that $n_L - x_L = \cdots = n_{i+1} - x_{i+1} = 0$ and $n_i - x_i \neq 0$, then the i-th term in the product is equal to

$$\prod_{j=1}^{x_i-n_i} \left(1 - q^{2(j-1)}\right) = 0 .$$

Then

$$\langle d_{-\frac{q^{-1}}{\alpha}}(x,\cdot), d_{\lambda_2}(\cdot,n)\rangle_{\mu(\alpha)} = \tilde{c}(n,n)\, \delta_{x,n}$$

$$\tilde{c}(n,n) = \prod_i \frac{Z_i^{(\alpha)}}{\binom{n_i+2k-1}{n_i}_q} q^{-4kin_i} \cdot \left(-\frac{\lambda_2}{q}\right)^{n_i} = \frac{1}{\mu_\alpha(n)}$$

choosing $\lambda_2 = -\frac{q}{\alpha}$ we get the result. □

Now we can apply Proposition 1.5 with

$$d(x,n) = d_{-\frac{q^{-1}}{\alpha}}(n,x) \qquad \tilde{d}(x,n) = d_{-\frac{q}{\alpha}}(x,n) \ ,$$

then

$$D(x,n) := \langle d(x,\cdot), \tilde{d}(\cdot,n)\rangle \qquad \tilde{D}(x,n) := \langle \tilde{d}(x,\cdot), d(\cdot,n)\rangle$$

are duality functions and are biorthogonal, i.e.

$$\langle D(\cdot,m), \tilde{D}(\cdot,n)\rangle_c = \delta_v(m,n) \ .$$

For both ASEP(q, j) and ASIP(q,k) future work will concern orthogonality result. In particular, for each case we only have that the two novel self-duality functions are biorthogonal by construction, one should check by hand if they are actually the same function, possibly Quantum q-Krawtchouk polynomials for the exclusion (see Section 14.14 of [42]) and q- Meixner for the inclusion (see Section 14.13 of [42]). Moreover, we also wonder if there is a constructive approach, in the spirit of the one of Section 1.4 which establishes orthogonal polynomials as duality functions by construction.

Appendix A

Basics on hypergeometric orthogonal polynomials

In this Section we give a quick overview of the continuous and the discrete hypergeometric polynomials (see [20, 42, 54, 59]) by reviewing some of their structural properties that will be used in the following.

We start by recalling that the hypergeometric orthogonal polynomials arise from an hypergeometric equation, whose solution can be written in terms of an hypergeometric function ${}_rF_s$.

Definition A.1 (Hypergeometric function)**.** *The hypergeometric function is defined by the series*

$$ {}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k}\frac{x^k}{k!} \tag{A.1} $$

where $(a)_k$ denotes the Pochhammer symbol defined in terms of the Gamma function as

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. $$

Remark A.1. Whenever one of the numerator parameter a_j is a negative integer $-n$, the hypergeometric function ${}_rF_s$ is a finite sum up to n, i.e. a polynomial in x of degree n.

The continuous case. Consider the hypergeometric differential equation

$$ \sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0 \tag{A.2} $$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree respectively and λ is a constant. A peculiarity of the hypergeometric equation is that, for all n, $y^{(n)}(x)$, i.e. the n^{th} derivative of a solution $y(x)$, also solves an hypergeometric equation, namely

$$ \sigma(x)y^{(n+2)}(x) + \tau_n(x)y^{(n+1)}(x) + \mu_n y^{(n)}(x) = 0 \tag{A.3} $$

with

$$\tau_n(x) = \tau(x) + n\sigma^{'}(x) \tag{A.4}$$

and

$$\mu_n = \lambda + n\tau^{'} + \frac{1}{2}n(n-1)\sigma^{''} . \tag{A.5}$$

We concentrate on a specific family of solutions: for each $n \in \mathbb{N}$, let $\mu_n = 0$, so that

$$\lambda = \lambda_n = -n\tau^{'} - \frac{1}{2}n(n-1)\sigma^{''} \tag{A.6}$$

and equation (A.3) has a particular solution given by $y^{(n)}(x)$ constant. This implies that $y(x)$ is a polynomial of degree n, called *polynomial of hypergeometric type* (see Remark A.1) and denoted by $p_n(x)$. In the following we will assume that those polynomials are of the form

$$p_n(x) = a_n x^n + b_n x^{n-1} + \dots \qquad a_n \neq 0 . \tag{A.7}$$

It is well known [54] that polynomials of hypergeometric type satisfy the orthogonality relation

$$\int_a^b p_n(x)p_m(x)\rho(x)dx = \delta_{n,m}d_n^2(x) \tag{A.8}$$

for some (premarkibly infinite) constants a and b and where the function $\rho(x)$ satisfies the differential equation

$$(\sigma\rho)^{'} = \tau\rho . \tag{A.9}$$

The sequence d_n^2 can be written in terms of $\sigma(x), \rho(x)$ and a_n as

$$d_n^2 = \frac{(a_n n!)^2}{\prod_{k=0}^{n-1}(\lambda_n - \lambda_k)} \int_a^b (\sigma(x))^n \rho(x)dx . \tag{A.10}$$

As a consequence of the orthogonal property the polynomials of hypergeometric type satisfy a three terms recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \tag{A.11}$$

where

$$\alpha_n = c_{n+1,n} \qquad \beta_n = c_{n,n} \qquad \gamma_n = c_{n-1,n} \tag{A.12}$$

with

$$c_{k,n} = \frac{1}{d_k^2}\int_a^b p_k(x)xp_n(x)\rho(x)dx . \tag{A.13}$$

The coefficients $\alpha_n, \beta_n, \gamma_n$ can be expressed in terms of the squared norm d_n^2 and the leading coefficients a_n, b_n in (A.7) as [54]

$$\alpha_n = \frac{a_n}{a_{n+1}} \qquad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \qquad \gamma_n = \frac{a_{n-1}}{a_n}\frac{d_n^2}{d_{n-1}^2} . \tag{A.14}$$

Finally, we will use the raising operator R that, acting on the polynomials $p_n(x)$, provides the polynomials of degree $n+1$. Such an operator is obtained from the Rodriguez formula, which provides an explicit form for polynomials of hypergeometric type

$$p_n(x) = \frac{B_n(\sigma^n(x)\rho(x))^{(n)}}{\rho(x)} \qquad \text{with} \qquad B_n = \frac{a_n}{\prod_{k=0}^{n-1}\left(\tau' + \frac{n+k-1}{2}\sigma''\right)} \ . \tag{A.15}$$

The expression of the raising operator (see eq. 1.2.13 in [54]) reads

$$\mathrm{R}p_n(x) = r_n p_{n+1}(x) \tag{A.16}$$

where

$$\mathrm{R}p_n(x) = \lambda_n \tau_n(x) p_n(x) - n\sigma(x)\tau_n^{'} p_n^{'}(x) \qquad \text{and} \qquad r_n = \lambda_n \frac{B_n}{B_{n+1}} \ . \tag{A.17}$$

Remark that the raising operator increases the degree of the polynomial by one, similarly to the so-called backward shift operator [42]. However the raising operator in (A.16) does not change the parameters involved in the function ρ, whereas the backward operator increases the degree and lowers the parameters [44].

The discrete case. Everything discussed for the continuous case has a discrete analog, where the derivatives are replaced by the discrete difference derivatives. In particular it is worth mentioning that

$$\Delta f(x) = f(x+1) - f(x) \qquad \text{and} \qquad \nabla f(x) = f(x) - f(x-1) \ .$$

The corresponding hypergeometric differential equation (A.2) is the discrete hypergeometric difference equation

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0 \tag{A.18}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of second and first degree respectively, λ is a constant. The differential equation solved by the n^{th} discrete derivative of $y(x)$, $y^{(n)}(x) := \Delta^n y(x)$, is the solution of another difference equation of hypergeometric type

$$\sigma(x)\Delta\nabla y^{(n)}(x) + \tau_n \Delta y^{(n)}(x) + \mu_n y^{(n)}(x) = 0 \tag{A.19}$$

with

$$\tau_n(x) = \tau(x+n) + \sigma(x+n) - \sigma(x) \tag{A.20}$$

and

$$\mu_n = \lambda + n\tau^{'} + \frac{1}{2}n(n-1)\sigma^{''} \ . \tag{A.21}$$

If we impose $\mu_n = 0$, then

$$\lambda = \lambda_n = -n\tau^{'} - \frac{1}{2}n(n-1)\sigma^{''} \tag{A.22}$$

and $y^{(n)}(x)$ is a constant solution of equation (A.19). Under these conditions, $y(x)$, solution of (A.18), is a polynomial of degree n, called *discrete polynomial of hypergeometric type* (see remark A.1) and denoted by $p_n(x)$.
The derivation of the orthogonal property is done in a similar way than the one for the continuous case where the integral is replaced by a sum

$$\sum_{x=a}^{b-1} p_n(x)p_m(x)\rho(x) = \delta_{n,m}d_n^2 \tag{A.23}$$

constants a and b can be either finite or infinite and the function $\rho(x)$ is solution of

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x) \ . \tag{A.24}$$

The sequence d_n^2 can be written in terms of $\sigma(x), \rho(x)$ and a_n as

$$d_n^2 = \frac{(a_n n!)^2}{\prod_{k=0}^{n-1}(\lambda_n - \lambda_k)} \sum_{x=a}^{b-n-1} \left(\rho(x+n) \prod_{k=1}^{n} \sigma(x+k) \right) \ . \tag{A.25}$$

As a consequence of the orthogonal property, the discrete polynomials of hypergeometric type satisfy a three terms recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \tag{A.26}$$

where

$$\alpha_n = c_{n+1,n} \qquad \beta_n = c_{n,n} \qquad \gamma_n = c_{n-1,n} \tag{A.27}$$

with

$$c_{k,n} = \frac{1}{d_k^2} \sum_{x=a}^{b} p_k(x) x p_n(x) \ . \tag{A.28}$$

The coefficients $\alpha_n, \beta_n, \gamma_n$ can be expressed in terms of the squared norm d_n^2 and the leading coefficients a_n, b_n in (A.7) as [54]

$$\alpha_n = \frac{a_n}{a_{n+1}} \qquad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \qquad \gamma_n = \frac{a_{n-1}}{a_n}\frac{d_n^2}{d_{n-1}^2} \ . \tag{A.29}$$

The discrete Rodriguez formula

$$p_n(x) = \frac{B_n}{\rho(x)} \nabla^n \left[\rho(x+n) \prod_{k=1}^{n} \sigma(x+k) \right] \qquad \text{with} \qquad B_n = \frac{a_n}{\prod_{k=0}^{n-1} \left(\tau' + \frac{n+k-1}{2}\sigma'' \right)} \tag{A.30}$$

leads to an expression for the discrete raising operator R (see eq. 2.2.10 in [54])

$$\mathrm{R}p_n(x) = r_n p_{n+1}(x) \tag{A.31}$$

where

$$\mathrm{R}p_n(x) = \left[\lambda_n \tau_n(x) - n\tau_n^{'}\sigma(x)\nabla\right] p_n(x) \qquad\qquad r_n = \lambda_n \frac{B_n}{B_{n+1}}\ . \tag{A.32}$$

We remark again that the raising operator shouldn't be confused with the backward shift operator in [44] which changes the value of parameters of the distribution ρ.

References on these are [42] for the discrete polynomials and [2] for the Bessel functions.

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