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CICLO XXXI

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Estimates for a family of exponential sums with modular coefficients

Settore Scientifico Disciplinare MAT/05

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Of course it is happening inside your head, Harry, but why on earth should that mean that it is not real?

J.K. Rowling, Harry Potter and the Deathly Hallows

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Contents

1 Introduction

We denote with the term exponential sum every sum S which can be written as

$$S = \sum_{a \le n \le b} w_n e^{2\pi i x_n} \tag{1.1}$$

where $1 \le a < b < +\infty$ and $\{w_n\}$, $\{x_n\}$ are two sequences of complex and real numbers respectively. From now on we will use the standard notation $e(\theta) = e^{2\pi i\theta}$ so that the sum above becomes $S = \sum_{a \le n \le b} w_n e(x_n)$. Studying exponential sums is a central point in Analytic Number Theory since they can be encountered approaching a large variety of problems; the main question one may want to answer is how large the modulus of S is in terms of the range of summation $[a,b]$ and, naturally, of the sequences $\{x_n\}$ and $\{w_n\}$. An example of application comes from the study of the Riemann zeta-function $\zeta(s)$; if we write $s = \sigma + it$, then

$$\zeta(s) = \sum_{n \le N} n^{-s} + \frac{N^{1-s}}{s-1} + O\left(N^{-\sigma}\right) \tag{1.2}$$

holds in a suitable region of the complex plane; it is immediate to see that an estimate for $\zeta(s)$ reduces to the study of the sum

$$\sum_{n \le N} n^{-it} = \sum_{n \le N} e\left(-\frac{t}{2\pi} \log n\right).$$

This is an example of a Weyl sum, a family of exponential sums which are obtained by taking $w_n \equiv 1$ and $x_n = f(n)$ in (1.1), with $f(x)$ a real-valued function over $[a,b]$ satisfying suitable hypotheses of regularity. One of the first examples of application of exponential sums was given by Weyl (see [45] and [46] for his original works) who established a famous criterion for the equidistribution of sequences modulo 1; in the twenties he developed a general method to estimate exponential sums which is extremely effective when $f(n) = P(n)$ with $P(x)$ a real polynomial of any degree. In the Appendix we recall his classical theorem and we prove a slight modification of it, which is actually really useful in our applications. Amongst the other fundamental contributions, we recall the work of van der Corput (see [40] and [41]), and Vinogradov (see [43]). More recently Vaughan [42] developed a new method which has important applications to the study of exponential sums when the weights w_n in (1.1) define an arithmetic function related to an L-function; the classical example is $x_n = \Lambda(n)$, the von Mangoldt function, whose generating function is the logarithmic derivative of $\zeta(s)$.

Let us consider the case $w_n = \mu(n)$ with $\mu(n)$ the Möbius function; the study of this kind of sums lies in the intersection between the study of exponential sums and another interesting problem usually referred to as *Möbius Randomness Law*. In Chapter 2 we will describe this phenomenon more precisely; here we content ourselves to say that we expect the sum

$$\sum_{n\leq x} \mu(n)\xi(n) \tag{1.3}$$

to be relatively small (for a large class of sequences $\{\xi(n)\}$) because of the random changes of sign of $\mu(n)$. If we take $\xi(n) = e(P(n))$ with $P(t)$ a real polynomial we get the expression

$$\sum_{n\leq x} \mu(n)e(P(n)) \tag{1.4}$$

which has been studied by Davenport (in the linear case) and Hua (when the degree of the polynomial is greater than one). A theorem collecting their results can be found at the beginning of Chapter 4. In our work we study a family of exponential sums which is apparently close to (1.4), but which actually requires a sharper set of tools to be studied; what we do is to replace $\mu(n)$ with $\mu_f(n)$, the equivalent of the Möbius function for the L-function associated to a cuspidal eigenform f. The definition and some relevant properties of this function can be found in Chapter 3. The result we prove is the following.

Theorem. *Let $P(t)$ be a real polynomial of degree $k \geq 3$; then,*

$$S_f(x;P) = \sum_{n\leq x} \mu_f(n)e(P(n)) \ll_k \frac{x}{\log^{1/4} x} \log\log x.$$

We consider polynomials of degree at least 3 because, for smaller degrees, a much stronger result can be produced; we recall the result for the linear and quadratic case in Theorem 4.4 and Theorem 4.8. We note here that the estimates were essentially already known or easily deducible from the works of Fouvry & Ganguly [11] and Hou & Lü [16], with the latter based on previous important work of Pitt [34]. The basic idea in both the situations is to apply Diophantine approximation to the leading coefficient of the polynomial and to approach the problem in two different ways according to the size of the denominator, say q, of the approximation. When q is *small* we use the analytic properties of the L-function generating $\mu_f(n)$, while when q is *large* we apply the Vaughan identity. In the latter case, while dealing with the so called Type II sums, it is necessary to have a good estimate for the quantity

$$\sum_{n\leq x} a_f(n)e(P(n)) \tag{1.5}$$

where $a_f(n)$ is the sequence of normalized Fourier coefficients of f and, as a multiplicative function, the inverse of $\mu_f(n)$ with respect to the Dirichlet convolution. The main reason we can not apply the same technique described above to polynomials of higher degree is that estimates for the

sum in (1.5) are not known and it seems quite a difficult problem to prove them for higher degree polynomials along the lines in the paper of Pitt. In our theorem we apply Diophantine approximation to all the coefficients of the polynomial and, again, we proceed differently according to the size of the denominators; the main difference is that, when at least one of them is *large*, we can not use the Vaughan identity, because the saving for the sum in (1.5) is not good enough to produce an interesting result. Instead, we use a direct approach, namely our Theorem 3.7. This is an extension of a result of Bourgain, Sarnak and Ziegler [3] which is a standard tool for estimating sums as in (1.3) when $\xi(n)$ is bounded. Their Theorem works even replacing $\mu(n)$ with any multiplicative bounded function, but it can not be applied to $\mu_f(n)$ which is not bounded. In our version, we both weaken the hypotheses on the multiplicative function and we make the result quantitative.
Finally, we give another application of Theorem 3.7 proving an estimate for the sum

$$S'_f(x,P) := \sum_{n\le x} a_f(n)e(P(n)).$$

We are able to prove the following result.

Theorem. *Let $P(t)$ be a real polynomial of degree $k \ge 3$; then,*

$$S'_f(x,P) \ll_k \frac{x}{\log^{1/2} x}\log\log x.$$

For a description of the contents of each chapter we refer to the brief introductions at the beginning of each of them.

Finally, we would like to note that a stronger version of the results proved in Theorem 3.7 and hence in Theorem 4.11 and Theorem 4.14 can be found in the work [5] (in preparation) by the author and the supervisors of this thesis.

2 The Sarnak Conjecture

In this chapter we collect some well-known results about the Möbius function; after recalling its definition and some of the properties which give this function a central role in Analytic Number Theory, we will focus on what is usually known as *Möbius Randomness Law*. In the second section, we will focus on a conjecture formulated by Sarnak in 2010.

2.1 The Möbius Randomess Law

Definition 2.1. *We define the Möbius function* $\mu : \mathbb{N} \to \{-1, 0, 1\}$ *as the multiplicative function such that*

$$\mu(p^e) = \begin{cases} -1 & e = 1, \\ 0 & e \geq 2 \end{cases}$$

for every prime p. In particular, if n is a square-free positive integer which is the product of k distinct primes, then $\mu(n) = (-1)^k$.

One of the main reasons for the importance of the Möbius function in Number Theory is its connection with the Riemann zeta-function. We recall some basic properties of this function; for a proof of the following results and a deeper analysis about its properties we refer to Davenport, [8] and Titchmarsh, [39].

Theorem 2.2. *Let* $\zeta(s)$ *be the Riemann zeta-function defined for* $\Re s > 1$ *as the Dirichlet series*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

The function $\zeta(s)$ *can be continued to a meromorphic function with a simple pole at* $s = 1$ *and satisfies the functional equation*

$$\Lambda(s) = \Lambda(1 - s) \tag{2.1}$$

where

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2.2}$$

and $\Gamma(s)$ is the Gamma function.

In the region of absolute convergence $\zeta(s)$ con be written as the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Finally, in the same half plane, $\zeta(s)^{-1}$ can be written as a Dirichlet series with coefficient $\mu(n)$, i.e.

$$\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s}. \tag{2.3}$$

A central problem in Analytic Number Theory is to determine the distribution of the zeros of $\zeta(s)$; it is well-known that $\zeta(s)$ has no zeros for $\Re s \geq 1$ and that the only zeros in the half plane $\Re s \leq 0$ coincide with the negative even integers (the *trivial zeros*). Moreover, it can be proved that in the so called critical strip $0 < \Re s < 1$ the function $\zeta(s)$ has infinitely many zeros (the *non-trivial zeros*) which are symmetric with respect to the real axis and the complex line of real part $1/2$. The most important Conjecture in Analytic Number Theory about the position of the non-trivial zeros of $\zeta(s)$ is the Riemann Hypothesis.

Conjecture 2.3 (RH)**.** *All the non-trivial zeros of the Riemann zeta-function are on the complex line of real part* $1/2$*.*

There exist several conditions which are equivalent to the Riemann Hypothesis; we are particularly interested in one involving $\mu(n)$. Starting from the identity (2.3), by partial summation we easily get for $\Re s > 1$

$$\frac{1}{\zeta(s)} = s\int_1^{+\infty} \frac{M(t)}{t^{s+1}} dt \tag{2.4}$$

where

$$M(t) = \sum_{n\leq t} \mu(n)$$

is the Mertens function. It is easy to see that, if the estimate $M(t) \ll t^{\alpha}$ with $\alpha \in (1/2, 1)$ held, then the integral on the right-hand side of (2.4) would define a holomorphic continuation of $\zeta(s)^{-1}$ to the half plane with real part greater than α. This would imply that $\zeta(s)$ has no zeros in that region and, by symmetry, also in $0 < \Re s < 1-\alpha$. This leads to the conclusion that if we could prove that the estimate

$$M(t) \ll_{\varepsilon} t^{1/2+\varepsilon} \tag{2.5}$$

holds for every $\varepsilon > 0$, then this would imply the truth of the Riemann Hypothesis. Indeed, the two conditions are equivalent (for example, see [39], § 14.25).

It is well-known that the Prime Number Theorem is equivalent to $M(x) = o(x)$ as $x \to +\infty$; the cancellation in the sum is essentially due to the changes of sign of the Möbius function. Indeed, we can interpret condition (2.5) from a probabilistic point of view. Let us recall that a symmetric simple random walk on $\mathbb{Z}$ is a stochastic sequence $\{S_n\}$ with $S_0 = 0$ and $S_N = \sum_{n \leq N} X_n$ where $\{X_n\}$ are independent and identically distributed random variables taking values $+1, -1$ each with probability $1/2$. It can be proved that

$$\lim_{N \to +\infty} \frac{\mathbb{E}(|S_N|)}{N^{1/2}} = \left(\frac{2}{\pi}\right)^{1/2} \tag{2.6}$$

where $\mathbb{E}(X)$ is the expected value of the random variable X (for definitions and more general results on stochastic sequences see, for example, Norris' book [31]).
Now, let us consider a "walker" on $\mathbb{Z}$ who starts at 0 and behaves at the n-th step in this way:

- he moves a unit to the left if $\mu(n) = -1$;
- he moves a unit to the right if $\mu(n) = 1$;
- he does not move if $\mu(n) = 0$.

Then, the Riemann Hypothesis in its equivalent form (2.5), compared to (2.6), suggests that $M(x)$ behaves similarly to a simple random walk on $\mathbb{Z}$. In this sense, we may say that the changes of sign of $\mu(n)$ behave randomly.

An interesting aspect related to this sort of randomness of μ is the *Möbius Randomness Law* which asserts that for every reasonable bounded sequence $\xi(n)$, the twisted sum $S(x, \xi)$ defined as

$$S(x, \xi) := \sum_{n \leq x} \mu(n)\xi(n) \tag{2.7}$$

is small due to the random changes of sign of $\mu(n)$.

Definition 2.4. *We say that the Möbius function is orthogonal to the bounded sequence* $\{\xi(n)\}$ *if*

$$S(x, \xi) = o(x)$$

as x tends to infinity.

Before focusing on what "reasonable" may mean we remark that the importance of studying this kind of sums is due to their connection with related sums over primes. Recalling the identity

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d \quad \forall n \geq 1$$

where $\Lambda(n)$ is the von Mangoldt function, one can write

$$\sum_{n \leq x} \Lambda(n)\xi(n) = -\sum_{d \leq x} \mu(d) \log d \sum_{\substack{m \leq x \\ m \equiv 0 \bmod d}} \xi(m) = -\sum_{d \leq x} \mu(d)\Xi_d(x),$$

say. Now, if we assume that a reasonable sequence $\xi(n)$ produces a reasonable sequnce $\Xi_d(x)$, then the Möbius Randomness Law would allow us to assume that the contribution for large d's in the sum above is small, so that we can approximate well the sum truncating it at some height D smaller than x. Dealing with smaller d's could be useful in order to find an asymptotic behaviour for the functions $\Xi_d(x)$ and this could be used to find, at least heuristically, the behaviour of the initial sum. For more details about the results we may expect from this heuristics see, for example, Iwaniec & Kowalski [19], § 13.1.

Now, let us return to the Möbius Randomness Law; to understand which kind of sequences may be "reasonable", we consider two examples. First let us take $\xi(n) \equiv 1$; the sum $S(x,\xi)$ coincides with the Mertens function $M(x)$, hence we can conclude that $\mu(n)$ is orthogonal to the sequence identically equal to 1 (and, in general, to any constant sequence). On the other hand, if we choose $\xi(n) = \mu(n)$, the function $S(x,\mu)$ counts the number of positive square-free integers up to x and since this set has positive density $\zeta(2)^{-1}$ we have

$$S(x,\mu) \sim \frac{1}{\zeta(2)}x$$

as $x \to +\infty$; in particular, this means that the Möbius function is not orthogonal to itself. As a conclusion, we expect that if $\xi(n)$ is too similar to $\mu(n)$ there can not be much cancellation in $S(x,\xi)$ so that, to have orthogonality, we may consider as "reasonable", sequences which are not "random" enough to approximate the Möbius function.
In the next section we will see how a definition of complexity for a sequence can be given in the context of dynamical systems and how this definition is helpful to identify a large class of sequences for which we expect to have orthogonality.

2.2 The Sarnak Conjecture

We follow the work and notations of Furstenberg (see [12], but also Glasner's book [13]).

Definition 2.5. *A flow F is an abstract dynamical system $F = (X,T)$ where X is a compact metric topological space and T is a continuous map from X to X. We say the a sequence $\{\xi(n)\}$ is observed or realized in F if there is a point $x \in X$ and a continuous function f from X to $\mathbb{C}$ such that*

$$\xi(n) = f(T^n(x)) \quad \forall n \geq 1.$$

Essentially, we are considering sequences which are the image through a continuous function of the orbits in X generated by the trivial action of T. Since X is compact we immediately deduce that any sequence observable in X is bounded; vice-versa, if $\xi(n)$ is bounded, then it is contained in a compact subset K of $\mathbb{C}$ and then it can be observed in the flow $F = (X,T)$ where $X = K^{\mathbb{N}}$ with the product topology and T is the shift map defined by $T((x_1,x_2,\ldots)) = (x_2,x_3,\ldots)$. In particular, we can observe the sequence $\xi(n)$ by taking the point $x = (\xi(1),\xi(2),\xi(3),\ldots) \in X$ and, as function f, the projection over the first coordinate, i.e. $f((x_1,x_2,\ldots)) = x_1$.

As anticipated in the previous section, we want to measure the complexity of a sequence by the complexity of the simplest flow in which it can be observed. The basic measure of the complexity of a flow F is its topological entropy $h(F)$ which we define in the following lines.

Definition 2.6. *Let $F = (X,T)$ be a flow; for every $\varepsilon > 0$ and $n \geq 1$ we say that a set $E \subset X$ is a (n,ε)-spanning set if*

$$\forall x \in X,\ \exists y \in E \ \ s.t.\ d(T^i(x),T^i(y)) < \varepsilon,\ \forall i \in \{0,\dots,n-1\}$$

where $d(\cdot,\cdot)$ is the distance defined on X. Let $r(\varepsilon,n)$ be the minimal cardinality of a (n,ε)-spanning set, which is finite by compactness. Then, for every $\varepsilon > 0$ we define

$$\overline{h}(\varepsilon) = \limsup_{n\to+\infty} \frac{\log r(\varepsilon,n)}{n}.$$

Finally we define the topological entropy $h(F)$ of F as

$$h(F) = \sup_{\varepsilon>0} \overline{h}(\varepsilon).$$

This definition can be found in Bowen [4]; we refer to Adler, Konheim & McAndrew [1] for a more general one based only on open coverings and hence valid even in non-metric spaces. The two definitions are equivalent in metric spaces and we refer to both the works cited for some basic properties. We just remark that the non-negative number $h(F)$ is essentially, by definition, a measure of the exponential growth rate of the number of orbits in the flow F. We are interested in flows with low complexity.

Definition 2.7. *We say that a flow F is deterministic if $h(F) = 0$. A sequence $\xi(n)$ is said to be deterministic if it can be observed in a deterministic flow.*

In 2010, Sarnak [35] formulated a conjecture on the behaviour of the sum $S(x,\xi)$ defined in (2.7) when $\xi(n)$ is deterministic.

Conjecture 2.8 (Sarnak)**.** *Let $F = (X,T)$ be a deterministic flow and $\xi(n)$ any observable sequence in F; then*

$$S(x,\xi) = o(x)$$

as $x \to +\infty$, i.e. μ is orthogonal to the sequence $\xi(n)$.

Sarnak's Conjecture asserts, in particular, that any deterministic sequence is "reasonable" in the context of the Möbius Randomness Law.
A first interesting question is if $\mu(n)$ is deterministic; if it were, the Conjecture would be obviously false, but Sarnak himself proved in [35] that $\mu(n)$ is not deterministic by showing that the simplest possible flow in which it can be observed has positive topological entropy.
We briefly recall some cases in which the Conjecture has been proved; we remark that the first three example below are results proved far before the formulation of the Conjecture, but they have a simple interpretation in the context of flows.

i) When F is a single point (i.e. when X is), then every observable sequence is a constant sequence and orthogonality is equivalent to the Prime Number Theorem, as already stated.

ii) More generally, if F is finite (i.e. if X is), then F is said to be periodic and orthogonality is equivalent to the quantitative Dirichlet Theorem on primes in progressions.

iii) If $F = (X, T_\alpha)$ where $X = \mathbb{R}/\mathbb{Z}$ is the one dimensional torus, $\alpha \in X$ and T_α is the translation $x \mapsto x + \alpha$; the result was proven by Davenport [7], but it can be easily generalized to any Kronecker flow, i.e. a flow $F = (X, T_a)$ where X is a compact abelian group, $a \in X$ and $T_a(x) = ax$ for all $x \in X$.

iv) If F is a horocycle flow the truth of the Conjecture was proved by Bourgain, Sarnak and Ziegler in [3]; in the next chapter we will analyse and generalise a powerful tool they developed to prove this result.

There are many other situations in which Sarnak's conjecture has been proved; amongst the most recent results we recall the works of Green & Tao [14], Bourgain [2], Liu & Sarnak [22], and Müllner [29].

We conclude this chapter by introducing an even stronger Conjecture formulated by Chowla [6] on the correlations of the Möbius function.

Conjecture 2.9. *Let $0 \le a_1 < a_2 < \cdots < a_t$ be integers and $k_i \in \{1,2\}$ for $i = 1,\dots,t$ not all even; then*

$$\sum_{n \le x} \mu(n+a_1)^{k_1} \mu(n+a_2)^{k_2} \cdots \mu(n+a_t)^{k_t} = o(x)$$

as $x \to +\infty$.

We remark that when $t = 1$ the Conjecture is proved since we are considering essentially the Mertens function, but very little is known when $t \ge 2$; recently, Matomäki, Radziwiłł and Tao [26] proved an average version of the Conjecture in short intervals for $t = 2$ (see also the papers [25] and [27] by the same authors for further progresses on this topic). A logarithmically averaged version of the Conjecture has been successfully studied by Tao [37] for $t = 2$ and by Tao and Teräväinen [38] for t odd.

A proof of the fact that Chowla's Conjecture implies the Sarnak Conjecture can be found in El Abdalaoui, Kułaga-Przymus, Lemańczyk & de la Rue [9]; we only note that the authors prove the implication in a very general case, namely replacing the Möbius function with any arithmetic function taking values in $\{-1, 0, +1\}$.

3 Modular Forms

In this chapter we want to extend some ideas and problems we investigated in the previous chapter. In the first section we recall some well-known result about holomorphic cuspidal forms and L-functions attached to them; this will lead us to introduce a family of arithmetic functions which are strictly related to the Möbius function. As we have done in Section 2.1 we introduce a class of twisted sums involving these functions and we see how the definition of orthogonality transfers in this context. The latter part of the chapter will be used to present a generalization of a result by Bourgain, Sarnak and Ziegler (see Theorem 3.6) and to prove that it is suitable to be used for studying the new sums introduced.

For an introduction and more general results about modular forms we refer to Iwaniec [18].

3.1 Basic Facts

Let f be normalized cuspidal form of weight d for the full modular group which is an eigenfunction for every Hecke operator. From now on, we think f as fixed. Let $\{a_f(n)n^{(d-1)/2}\}$ be the sequence of its Fourier coefficients. We define the L-function $L(s,f)$ associated to f for $\Re s > 1$ as

$$L(s,f) = \sum_{n\geq 1} \frac{a_f(n)}{n^s}. \tag{3.1}$$

Lemma 3.1. *The series $L(s,f)$ is absolutely convergent for $\Re s > 1$ and it can be continued to an entire function which satisfies the functional equation*

$$\Lambda(s) = i^d \Lambda(1-s) \tag{3.2}$$

where

$$\Lambda(s) = (2\pi)^{-s}\Gamma\Big(s + \frac{d-1}{2}\Big)L(s,f). \tag{3.3}$$

The coefficients a_f are real-valued, multiplicative and for every prime p there exists a complex number $\alpha_f(p)$ with $|\alpha_f(p)| = 1$ such that

$$a_f(p) = \alpha_f(p) + \overline{\alpha}_f(p); \tag{3.4}$$

moreover, they satisfy the recursive formula

$$a_f(p^{l+1}) = a_f(p)a_f(p^l) - a_f(p^{l-1}) \tag{3.5}$$

for every prime p and for every $l \geq 1$. As a consequence, the function $L(s,f)$ can be written, for $\Re s > 1$, as the Euler product

$$L(s,f) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha}_f(p)}{p^s}\right)^{-1}. \tag{3.6}$$

Proof. See Iwaniec [18], in particular Chapters from 5 to 7. □

In the half plane of absolute convergence we can express $L(s,f)^{-1}$ as a Dirichlet series; we define $\mu_f(n)$ as the sequence generated by $L(s,f)^{-1}$, i.e.

$$L(s,f)^{-1} = \sum_{n\geq 1} \frac{\mu_f(n)}{n^s}. \tag{3.7}$$

In the following Lemma we collect some results and estimates for the coefficients $a_f(n)$ and $\mu_f(n)$.

Lemma 3.2. *Let $a_f(n)$ and $\mu_f(n)$ be as above and $b(n)$ be either of the two functions; then*

1. $|b(n)| \leq d(n)$ *where $d(n)$ is the divisor function;*

2. $$\sum_{n\leq x} |b(n)| \ll_f \frac{x}{\log^\delta x}; \tag{3.8}$$

   where $\delta = 1/16$;

3. *there exists a positive constant c_f depending on f such that*

   $$\sum_{n\leq x} |a_f(n)|^2 = c_f x + \mathscr{O}(x^{3/5}) \tag{3.9}$$

   and, as a consequence,

   $$\sum_{n\leq x} |\mu_f(n)|^2 \ll_f x. \tag{3.10}$$

Proof. From the Euler product for $L(s,f)$ we get

$$L(s,f)^{-1} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right) \left(1 - \frac{\overline{\alpha}_f(p)}{p^s}\right)$$

which leads to the following expression for $\mu_f(n)$:

$$\mu_f(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k a_f(p_1 \cdots p_k) & \text{if } n = p_1 \cdots p_k (p_{k+1} \cdots p_r)^2 \quad p_i \neq p_j; \\ 0 & \text{elsewhere.} \end{cases} \tag{3.11}$$

We remark that the function $\mu_f(n)$ is supported on cube-free integers.

From (3.6) we can write a_f as the Dirichlet convolution $\alpha * \overline{\alpha}$ where α is a completely multiplicative function such that $\alpha(p) = \alpha_f(p)$; recalling that $|\alpha_f(p)| = 1$, we immediately have

$|a_f(n)| \leq d(n)$. From (3.11) it follows that the same bound holds for $\mu_f(n)$ too.

The bound (3.8) for $a_f(n)$ is proved by Elliott, Moreno and Shahidi in [10]; for $\mu_f(n)$, using (3.11), we have that

$$\begin{aligned}
\sum_{n\leq x} |\mu_f(n)| &= \sum_{(p_1\cdots p_k)^2\leq x} \ \sum_{p_{k+1}\cdots p_r \leq x/(p_1p_2\cdots p_k)^2} |a_f(p_{k+1}\cdots p_r)| \leq \sum_{d^2\leq x}\sum_{n\leq x/d^2} |a_f(n)| \\
&= \sum_{d^2\leq x/4}\sum_{n\leq x/d^2} |a_f(n)| + \sum_{x/4<d^2\leq x}\sum_{n\leq x/d^2} |a_f(n)| \\
&\ll \sum_{d^2\leq x/4}\sum_{n\leq x/d^2} |a_f(n)| + x^{1/2} \\
&\ll \sum_{d^2\leq x/4} \frac{x}{d^2}\log^{-\delta}(x/d^2) + x^{1/2} = S(x) + x^{1/2},
\end{aligned}$$

say. Now

$$S(x) \ll \int_1^{x^{1/2}/2} \frac{x}{t^2}\log^{-\delta}(x/t^2)\,dt$$

and with the change of variable $y = x^{1/2}/t$ we have

$$\begin{aligned}
S(x) &\ll x^{1/2}\int_2^{x^{1/2}} \frac{dy}{\log^\delta y^2} = x^{1/2}\left(\int_2^{x^{1/4}} + \int_{x^{1/4}}^{x^{1/2}}\right)\left(\frac{dy}{\log^\delta y^2}\right) \\
&\ll x^{1/2}\left(x^{1/4} + \frac{x^{1/2}}{\log^\delta x^{1/2}}\right) \ll \frac{x}{\log^\delta x}.
\end{aligned}$$

Collecting the results we get

$$\sum_{n\leq x} |\mu_f(n)| \ll \frac{x}{\log^\delta x}. \tag{3.12}$$

Finally, the asymptotic in (3.9) is a result due to Rankin and Selberg (see [18], Chapter 13) and with the same technique we used to obtain (3.12) we easily deduce (3.10). □

3.2 Twisted sums of $\mu_f(n)$

It is easy to establish an analogy between the Möbius function μ and μ_f; summarizing, we may say that the μ_f plays for $L(s,f)$ the role which μ plays for the Riemann zeta-function. As we have already remarked in the previous section, the function $L(s,f)$ shares several properties (and open conjectures) with $\zeta(s)$; in particular, one can formulate an equivalent version of the Riemann Hypothesis for the class of L-function $L(s,f)$.

Theorem 3.3. *Let $L(s,f)$ be as in* (3.1)*; then the following conditions are equivalent:*

- *all the zeros of $L(s,f)$ in the critical strip $0 < \Re s < 1$ lie on the complex line $\Re s = 1/2$;*
- *for every $\varepsilon > 0$*

$$M_f(x) := \sum_{n\leq x} \mu_f(n) \ll_\varepsilon x^{1/2+\varepsilon}$$

as $x \to +\infty$.

The equivalence comes from identities which are completely analogous to what we have seen for $M(x)$ and $\zeta(s)$ in Chapter 1 (we refer, again, to Iwaniec & Kowalski [19] for more details and proofs of this facts).

Since we conjecture that the same estimate should hold for both the Mertens function $M(x)$ and its modular analogue $M_f(x)$, it is natural to ask if the properties of randomness expected for $\mu(n)$ are still valid for $\mu_f(n)$. In particular, we introduce a new version of the sums introduced in (2.7).

Definition 3.4. *Given a bounded sequence $\xi(n)$ we define the sum $S_f(x,\xi)$ for $x \geq 1$ as*

$$S_f(x,\xi) := \sum_{n\leq x} \mu_f(n)\xi(n). \tag{3.13}$$

Aiming to extend the notion of orthogonality which we have defined for the Möbius function, we note that the condition

$$S_f(x,\xi) = o(x)$$

is trivially satisfied by any bounded sequence $\xi(n)$; this is due to the estimate (3.8) that we have proved for $\mu_f(n)$. To replace the condition above with a more reasonable one, we would like to know the asymptotic behaviour of the sum

$$\sum_{n\leq x} |\mu_f(n)|.$$

It is easy to prove that, unconditionally,

$$\frac{x}{\log x} \ll_f \sum_{n\leq x} |\mu_f(n)| \ll_f \frac{x}{\log^{1/16} x};$$

the upper bound is part of Lemma 3.2 and the lower bound comes from

$$\sum_{n\leq x} |\mu_f(n)| \geq \sum_{p\leq x} |\mu_f(p)| = \sum_{p\leq x} |a_f(p)| \gg \frac{x}{\log x}$$

(for the last estimate see Fouvry & Ganguly [11], Proposition 3.1). However, to find the correct asymptotic, we need to assume a quantitative version of the Sato-Tate conjecture for cuspidal eigenforms. We recall briefly that the Conjecture asserts that the sequence $\{\theta_f(p)\} \subseteq [0,\pi]$ defined by the identity

$$a_f(p) = 2\cos\theta_f(p)$$

is uniformly distributed with respect to the Sato-Tate measure

$$\frac{2}{\pi}\sin^2\theta\, d\theta.$$

In [10] Elliot, Moreno and Shahidi find the asymptotic behaviour of the sum

$$\sum_{n\leq x} |a_f(n)|;$$

in particular, assuming that as $x \to \infty$,

$$\sum_{\substack{p \le x \\ \theta_f(p) \le \alpha}} 1 = \frac{2}{\pi} \int_0^{\alpha} \sin^2 \theta \, d\theta \cdot \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \tag{3.14}$$

uniformly in $\alpha \in [0, \pi]$, they prove that there is a positive constant c_f such that

$$\sum_{n \le x} |a_f(n)| \sim c_f \frac{x}{\log^{\delta} x} \tag{3.15}$$

where $\delta = 1 - 8/(3\pi) \simeq 0.15$.

We now want to prove that replacing $|a_f(n)|$ with $|\mu_f(n)|$ the same asymptotic holds with possibly a different constant $c'_f > 0$. We begin recalling that $\mu_f(p) = -a_f(p)$ and $\mu_f(p^2) = 1$ for every prime p. The basic idea is to prove that the functions

$$F(s) = \sum_{n \ge 1} \frac{|a_f(n)|}{n^s} \quad \text{and}$$
$$H(s) = \sum_{n \ge 1} \frac{|\mu_f(n)|}{n^s} = \prod_p \left(1 + \frac{|a_f(p)|}{p^s} + \frac{1}{p^{2s}}\right)$$

differ by a factor which is holomorphic in the half plane $\Re s > 1/2$; to do this we study the ratio $(F/G)(s)$. We set $\beta(n) = |a_f(n)|$ and $\gamma(n) = |\mu_f(n)|^{-1^*}$, where -1^* denotes the inverse with respect to the Dirichlet convolution. The function $(F/G)(s)$ can be expressed for $\Re s > 1$ as a Dirichlet series with coefficients $\eta(n)$ defined by

$$\eta(n) = (\beta * \gamma)(n).$$

It is easy to see that the modulus of the function $\eta(n)$ does not grow too rapidly; we already know from Lemma 3.2 that $\beta(n) \le d(n)$ and the same estimate holds also for $\gamma(n)$ since

$$H(s)^{-1} = \prod_p \left(1 + \frac{\varepsilon_p \alpha_f(p))}{p^s}\right)^{-1} \left(1 + \frac{\varepsilon_p \overline{\alpha}_f(p))}{p^s}\right)^{-1}$$

where $\varepsilon_p = \text{sign}(a_f(p))$. From the definition of Dirichlet convolution we conclude that

$$|\eta(n)| \le \sum_{h|n} |\beta(h)||\gamma(n/h)| \le \sum_{h|n} d(h)d(n/h) = (d * d)(n) = d_4(n)$$

with $d_l(n)$ the generalized divisor function which satisfies $d_l(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$ (see, for example, Linnik [21], Chapter 1). Moreover, since $\gamma(p) = -|\mu_f(p)| = -|a_f(p)|$ we have that

$$\eta(p) = \beta(p) + \gamma(p) = |a_f(p)| - |a_f(p)| = 0$$

for every prime p. Collecting these results we can write the ratio $g(s) := (F/G)(s)$ as the Euler product

$$g(s) = \prod_p \left(1 + \frac{\eta(p^2)}{p^{2s}} + \frac{\eta(p^3)}{p^{3s}} + \cdots\right)$$

which is convergent for $\Re s > 1/2$. Moreover, it's easy to see that each p-factor in the Euler product of $g(s)$ does not vanish at $s = 1$, so that $g(1) \neq 0$. Then, the asymptotic for $\sum_{n\leq x} |\mu_f(n)|$ follows from (3.15) and from the trivial identity $F(s) = g(s)H(s)$.

Even if the asymptotic has been obtained conditionally, it can be used to extend properly the definition of orthogonality we have introduced for the Möbius function.

Definition 3.5. *We say that the function $\mu_f(n)$ is orthogonal to the bounded sequence $\{\xi(n)\}$ if*

$$S_f(x,\xi) = o\left(\frac{x}{\log^\delta x}\right) \qquad \text{as } x \to +\infty,$$

where $\delta = 1 - 8/(3\pi)$.

In Chapter 4 we will show a family of sequences for which we can prove orthogonality; to make an example here, we note that if we consider a constant or a periodic sequence, then orthogonality follows from the equivalent of the Prime Number Theorem in progressions valid for modular cuspidal coefficients. For a precise statement see Lemma 4.3 in the next chapter.

In the next section we develop an extremely useful tool for studying orthogonality properties of $\mu_f(n)$.

3.3 A generalisation of a result of Bourgain, Sarnak & Ziegler

The Vinogradov's bilinear method and its modern variants such as the Vaughan identity are standard tools which can be successfully used to estimate the sums $\sum_{n\leq x} \mu(n)F(n)$ in terms of Type I and Type II sums (see, for example, Iwaniec & Kowalski [19]). In [3], Bourgain, Sarnak and Ziegler prove the following result which can be considered another modern version of Vinogradov's method.

Theorem 3.6. *Let $F : \mathbb{N} \to \mathbb{C}$ with $|F| \leq 1$ and let ν be a multiplicative function with $|\nu| \leq 1$. Let $\tau > 0$ be a small parameter and assume that for all primes $p_1 \neq p_2$ with $p_1, p_2 \leq \exp(1/\tau)$ we have*

$$\left|\sum_{m\leq M} F(p_1 m)\overline{F(p_2 m)}\right| \leq \tau M \tag{3.16}$$

for M large enough. Then, for x large enough,

$$\sum_{n\leq x} \nu(n)F(n) \ll x\tau^{1/2}\log\frac{1}{\tau}. \tag{3.17}$$

We remark that the control over the sums in (3.16) (which occur in the estimate of Type II sums) is required to hold uniformly only for primes in a bounded range, depending on τ (and not on x), as M tends to infinity. This is enough to prove that the sum in (3.17) is $o(x)$, but it gives us no rates. This powerful tool has been used to prove the orthogonality of the Möbius function in several situations, see for example the works of Bourgain, Sarnak & Ziegler, [3], Liu & Sarnak [22] and Wang [44].

We can't use this Theorem directly to deal with sums involving $\mu_f(n)$ essentially for two reasons:

1. the functions $\mu_f(n)$ are not bounded;

2. as pointed out at the end of the previous section, to get a non-trivial estimate for $S_f(x,F)$ we need to go below $x\log^{-\delta} x$.

We present here a new version of the theorem in which we both weaken the condition required for the multiplicative function and produce a quantitative version of the result.

Theorem 3.7. *Let $F(n)$ and $a(n)$ be two arithmetic functions with $|F(n)|\le 1$ and $a(n)$ multiplicative satisfying*

(i) $|a(p)|\le K$ *for all primes p and some $K>0$;*

(ii)

$$\sum_{n\le x}|a(n)|^2 \ll x;$$

(iii)

$$\sum_{\substack{n\le x\\(n,P)=1}}|a(n)|^2 \ll \frac{\phi(P)}{P}x$$

with $P=\prod_{y<p\le z} p$ *and* $1\ll y(x)<z(x)\ll \exp(\log x/\log\log x)$.

Let $T=T(x)\ge 1$ be a positive parameter which tends to infinity when $x\to+\infty$ and which satisfies

$$T\le \log x; \tag{3.18}$$

assume that for all primes $p_1\ne p_2$, with $p_1,p_2\le \exp(T\log^{-1}T)$ we have

$$\sum_{m\le x} F(p_1 m)\overline{F(p_2 m)} \ll \frac{x}{T} \tag{3.19}$$

for x large enough. Then

$$\sum_{n\le x} a(n)F(n) \ll \frac{\log T}{T^{1/2}}x \tag{3.20}$$

where the implied constant does not depend on T.

We make a remark on the range $1\ll y(x)<z(x)\ll \exp(\log x/\log\log x)$ in condition *(iii)*; a posteriori, this is the minimal condition we have to require to make the proof work with our choice of the parameters A and B in (3.39).

Proof. Following the idea of the proof in [3], we want to decompose the set $I_x:=[1,x]\cap\mathbb{Z}$ as $I_x=\mathcal{N}_1\cup\mathcal{N}_2$ with $\mathcal{N}_1\cap\mathcal{N}_2=\emptyset$ and such that:

- integers in $\mathcal{N}_1$ have a good factorisation; in particular, they have unique prime factors in suitable intervals;
- the cardinality of the set $\mathcal{N}_2$ is small (depending on T).

Let $A = A(T), B = B(T)$ be two integers to be decided later satisfying

$$A(T) \to +\infty \text{ as } T \to +\infty \quad \text{and} \quad 1 \ll A < B \ll x^{\delta} \tag{3.21}$$

for some small positive δ. We define the sequence

$$A_j := (A+j)^2 \qquad \text{for } j = 0, \dots, J \tag{3.22}$$

where $J = B - A$; clearly we have $A_0 = A^2$ and $A_J = B^2$. For every integer j such that $1 \le j \le J$ let us consider the sets

$$P_j = \left\{ p \text{ prime } : p \in (A_{j-1}, A_j] \right\}$$

and

$$Q_j = \left\{ m \in \left[1, \frac{x}{A_j}\right] : m \text{ has no prime factor in } \bigcup_{1 \le i \le j} P_i \right\}.$$

We remark that the sets P_j form a decomposition of the set of primes in the interval $(A^2, B^2]$ and the product sets $P_j Q_j$, where $1 \le j \le J$ are pairwise disjoint and contained in I_x.
We define

$$\mathcal{N}_1 := \bigcup_{1 \le j \le J} P_j Q_j = \bigcup_{1 \le j \le J} \left\{ pm \mid p \in P_j, m \in Q_j \right\}$$

and we want to estimate the sum

$$S_1 = \sum_{n \in \mathcal{N}_1} a(n) F(n).$$

Since $P_j \cap Q_j = \emptyset$, the sets $P_j \times Q_j$ and $P_j Q_j$ are in bijection; moreover, for $p \in P_j$ and $m \in Q_j$ we have $(p, m) = 1$; then using the multiplicativity of the function $a(n)$, by a trivial computation we get

$$\begin{aligned} S_1 = \sum_{n \in \mathcal{N}_1} a(n) F(n) &= \sum_{1 \le j \le J} \sum_{pm \in P_j Q_j} a(pm) F(pm) \\ &= \sum_{1 \le j \le J} \sum_{m \in Q_j} a(m) \sum_{p \in P_j} a(p) F(pm) \\ &= \sum_{1 \le j \le J} S_{1,j}, \end{aligned}$$

say. Now, we want to estimate $S_{1,j}$ for $1 \le j \le J$. We begin applying the Cauchy inequality, i.e.

$$|S_{1,j}| \le \left(\sum_{m \in Q_j} |a(m)|^2 \right)^{1/2} \left(\sum_{m \in Q_j} \left| \sum_{p \in P_j} a(p) F(pm) \right|^2 \right)^{1/2}. \tag{3.23}$$

Observe that the set Q_j can be written as

$$Q_j = \left\{ m \in \left[1, \frac{x}{A_j}\right] : (m, \mathfrak{p}_j) = 1 \right\}$$

where $\mathfrak{p}_j$ is the product of all primes in $\bigcup_{1\leq i\leq j} P_i$, i.e.

$$\mathfrak{p}_j = \prod_{\substack{A_0 < p \leq A_j \\ p \text{ prime}}} p.$$

Using standard sieve methods (see, for example, Halberstam & Richert, [15], Theorem 2.1) it can be proved that, as x tends to infinity,

$$|\{n \leq x, (n,p) = 1, \forall p \in P\}| \gg x \prod_{p\in P}\left(1 - \frac{1}{p}\right)$$

assuming that $P \subset [1, x^{\delta'}]$ for some small $\delta' > 0$ (we are not interested in the optimal choice, we simply note that we may take $\delta' = 1/6$). From (3.21) we have that $A_j \ll x^{2\delta}$ and $x/A_j \gg x^{1-2\delta}$, hence assuming δ sufficiently small we conclude that

$$|Q_j| \gg \frac{\phi(\mathfrak{p}_j)}{\mathfrak{p}_j}\frac{x}{A_j} \tag{3.24}$$

as $x \to +\infty$. Under much weaker hypotheses (see [15], Theorem 3.5) we also have

$$|Q_j| \ll \frac{\phi(\mathfrak{p}_j)}{\mathfrak{p}_j}\frac{x}{A_j}. \tag{3.25}$$

Assuming A_0 and A_J large enough we can use condition (iii) (again, we will see that the hypotheses are satisfied by the choice of A and B) and (3.25) to get

$$\sum_{m\in Q_j} |a(m)|^2 \ll |Q_j|$$

and so

$$|S_{1,j}| \ll |Q_j|^{1/2}\left(\sum_{m\leq x/A_j}\left|\sum_{p\in P_j} a(p)F(pm)\right|^2\right)^{1/2}$$

where we replaced the condition $m \in Q_j$ in (3.23) with $m \leq x/A_j$ (which is trivially weaker) in the second factor.

Computing the square of the norm, using condition (i) and rearranging the sum we finally have

$$\begin{aligned}|S_{1,j}| &\ll |Q_j|^{1/2}\left(\sum_{m\leq x/A_j}\sum_{p_1,p_2\in P_j} a(p_1)F(p_1m)\overline{a(p_2)F(p_2m)}\right)^{1/2}\\ &\ll |Q_j|^{1/2}\left(\sum_{p_1,p_2\in P_j}\left|\sum_{m\leq x/A_j} F(p_1m)\overline{F(p_2m)}\right|\right)^{1/2}.\end{aligned}$$

We can estimate the inner sum using the hypothesis (3.19), but we need to handle the diagonal case separately; so, for $p_1 = p_2$

$$\sum_{p_1 \in P_j} \sum_{m \le x/A_j} |F(p_1 m)|^2 \le \frac{|P_j|}{A_j} x. \tag{3.26}$$

For non-diagonal elements we have

$$\sum_{p_1 \ne p_2 \in P_j} \left| \sum_{m \le x/A_j} F(p_1 m) \overline{F(p_2 m)} \right| \le \frac{1}{T} \frac{|P_j|^2}{A_j} x. \tag{3.27}$$

We remark that, in order to apply the hypothesis, we need that the primes involved satisfy the condition $p_1, p_2 \le \exp(T \log^{-1} T)$; it is easy to see that our choice of A and B in (3.39) is coherent with this condition.

From (3.26) and (3.27) we have

$$|S_{1,j}| \ll |Q_j|^{1/2} \left(\left(|P_j| + \frac{|P_j|^2}{T} \right) \frac{x}{A_j} \right)^{1/2}$$

and, summing over $1 \le j \le J$,

$$|S_1| \ll x^{1/2} \sum_{1 \le j \le J} \left(\frac{|P_j Q_j|^{1/2}}{A_j^{1/2}} + T^{-1/2} |Q_j|^{1/2} \frac{|P_j|}{A_j^{1/2}} \right). \tag{3.28}$$

To complete the estimate of S_1 we consider the sums:

$$S_D = \sum_{1 \le j \le J} \frac{|P_j Q_j|^{1/2}}{A_j^{1/2}} \quad \text{and} \quad S_{ND} = \sum_{1 \le j \le J} |Q_j|^{1/2} \frac{|P_j|}{A_j^{1/2}}.$$

For the first one, applying the Cauchy inequality, we have

$$\begin{aligned} S_D &\le \left(\sum_{1 \le j \le J} |P_j Q_j| \right)^{1/2} \left(\sum_{1 \le j \le J} \frac{1}{A_j} \right)^{1/2} \\ &\ll x^{1/2} \left(\sum_{A^2 \le n^2 \le B^2} \frac{1}{n^2} \right)^{1/2} \\ &\ll \frac{x^{1/2}}{A^{1/2}} \end{aligned} \tag{3.29}$$

where we used the fact that the sets $P_j Q_j$ are pairwise disjoint and contained in I_x.

To deal with S_{ND} we need an estimate for the cardinality of P_j; an application of the Brun-Titchmarsh inequality (see [15], Theorem 3.7) to the interval $(u, u+w)$ with $u = A_{j-1}$ and $w = A_j - A_{j-1} = 2A + 2j - 1$ gives

$$|P_j| \ll \frac{w}{\log w} \ll \frac{A_j^{1/2}}{\log A_j}. \tag{3.30}$$

Then, using the estimates (3.25) and (3.30) we get

$$S_{ND} \ll \sum_{1 \le j \le J} \left(\frac{\phi(\mathfrak{p}_j)}{\mathfrak{p}_j} \frac{x}{A_j} \right)^{1/2} \frac{1}{\log A_j}.$$

Using Mertens' formula and assuming A_0 (and hence A_j) sufficiently large we deduce that

$$\frac{\phi(\mathfrak{p}_j)}{\mathfrak{p}_j} = \prod_{\substack{A_0<p\leq A_j \\ p \text{ prime}}} \left(1-\frac{1}{p}\right) \sim \frac{\log A_0}{\log A_j}$$

and then

$$\begin{aligned} S_{ND} &\ll x^{1/2}\log^{1/2} A \sum_{1\leq j\leq J} \frac{1}{A_j^{1/2}\log^{3/2} A_j} \ll x^{1/2}\log^{1/2} A \sum_{1\leq j\leq J} \frac{1}{A_j^{1/2}(\log^{3/2} A_j)} \\ &\ll x^{1/2}\log^{1/2} A \sum_{A<n\leq B} \frac{1}{n\log^{3/2} n} \ll x^{1/2}. \end{aligned} \tag{3.31}$$

Collecting the results in (3.29) and (3.31) and inserting them in (3.28) we obtain

$$|S_1| \ll \left(A^{-1/2}+T^{-1/2}\right)x. \tag{3.32}$$

Now, we need to estimate S_2, the sum over the set $\mathscr{N}_2$ defined as

$$\mathscr{N}_2 := I_x \setminus \mathscr{N}_1 = I_x \setminus \bigcup_{1\leq j\leq J} P_j Q_j.$$

To do this we will find an estimate for the cardinality of the set $\mathscr{N}_2$ and then we will use condition (ii) to conclude via the Cauchy inequality that

$$\begin{aligned} |S_2| = \left|\sum_{n\in\mathscr{N}_2} a(n)F(n)\right| &\leq |\mathscr{N}_2|^{1/2}\left(\sum_{n\leq x}|a(n)F(n)|^2\right)^{1/2} \\ &\leq |\mathscr{N}_2|^{1/2}\left(\sum_{n\leq x}|a(n)|^2\right)^{1/2} \\ &\ll |\mathscr{N}_2|^{1/2}x^{1/2}. \end{aligned} \tag{3.33}$$

For $1 \leq j \leq J$ let us consider the sets

$$H_j = \left\{ n\in I_x : n \text{ has a single divisor in } P_j \text{ and no divisors in } \bigcup_{1\leq i<j} P_i \right\}.$$

The product set P_jQ_j is contained in H_j and we have the trivial inclusion

$$(S_j \setminus P_jQ_j) \subset P_jR_j$$

where

$$R_j = \left\{ m \in \left[\frac{x}{A_j}, \frac{x}{A_{j-1}}\right] : (m,\mathfrak{p}_j)=1 \right\}.$$

The cardinality of the set R_j satisfies

$$|R_j| \ll \frac{\phi(\mathfrak{p}_j)}{\mathfrak{p}_j}\frac{x}{A_j^{3/2}}.$$

Summing over j and using (3.30), we get

$$\sum_{1\le j\le J} \left|H_j \setminus P_jQ_j\right| \le \sum_{1\le j\le J} \frac{\phi(\mathfrak{p}_j)}{\mathfrak{p}_j}\frac{|P_j|}{A_j^{3/2}}x$$
$$\ll x\log A \sum_{1\le j\le J}\frac{1}{A_j\log^2 A_j} \ll \frac{x}{A\log A}. \tag{3.34}$$

The sets H_j are disjoint and included in the set H defined as

$$H = \{n \in I_x : n \text{ has a prime factor in } (A_0, A_J)\};$$

moreover

$$H \setminus \bigcup_{1\le j\le J} H_j \quad \subset \bigcup_{1\le j\le J}\left\{n\in I_x : n \text{ has at least two prime divisors in } P_j\right\}.$$

Then

$$\left|H \setminus \bigcup_{1\le j\le J} H_j\right| \ll \sum_{1\le j\le J}\sum_{p_1,p_2\in P_j}\frac{x}{p_1p_2} \le x \sum_{1\le j\le J}\left(\frac{|P_j|}{A_j}\right)^2$$
$$\ll x\sum_{1\le j\le J}\frac{1}{A_j\log^2 A_j} \ll \frac{x}{A\log^2 A}. \tag{3.35}$$

Finally, observe that

$$|I_x \setminus H| \ll x \prod_{\substack{A_0<p\le A_J\\ p \text{ prime}}}\left(1-\frac{1}{p}\right)$$

(again from [15], Theorem 3.5); we can suppose A_0 and A_J to be large enough to use the Mertens formula to estimate the product on the right side above

$$\prod_{\substack{A_0<p\le A_J\\ p \text{ prime}}}\left(1-\frac{1}{p}\right) \sim \frac{\log A_0}{\log A_J} = \frac{\log A}{\log B}$$

and conclude that

$$|I_x \setminus H| \ll \frac{\log A}{\log B}x. \tag{3.36}$$

Collecting the results in (3.34), (3.35) and (3.36) we can say that

$$|\mathscr{N}_2| \le \sum_{1\le j\le J}\left|H_j\setminus P_jQ_j\right| + \left|H\setminus \bigcup_{1\le j\le J} H_j\right| + |I_x\setminus H|$$
$$\ll \left(\frac{1}{A\log A}+\frac{\log A}{\log B}\right)x.$$

From (3.33) we get

$$|S_2| \ll \left(\frac{1}{A\log A}+\frac{\log A}{\log B}\right)^{1/2} x. \tag{3.37}$$

Collecting the results in (3.32) and (3.37) we conclude

$$\sum_{n\le x} a(n)F(n) = S_1 + S_2 \ll \left(\frac{1}{A^{1/2}}+\frac{1}{T^{1/2}}+\left(\frac{\log A}{\log B}\right)^{1/2}\right)x. \tag{3.38}$$

Now we have to choose the parameters A and B; we take

$$A = T \qquad \text{and} \qquad B = A^{T/\log^2 T}. \tag{3.39}$$

Making this choice in (3.38) the theorem follows. □

We remark that the condition (3.18) on the parameter T sets a lower bound for the cancellation we can obtain applying this theorem; essentially, we can not go below the square root of $\log x$.

3.4 Modular coefficients satisfy the hypotheses of Theorem 3.7

In this section we prove that the coefficients $a_f(n)$ (and then $\mu_f(n)$) satisfy the hypotheses of Theorem 3.7.

Conditions *(i)* and *(ii)* follow immediately from Lemma 3.2. To show that condition *(iii)* is satisfied, we need to prove that

$$\sum_{\substack{n\leq x\\(n,P)=1}} |a_f(n)|^2 \ll \frac{\phi(P)}{P}x \tag{3.40}$$

for every integer P that can be written as

$$P = \prod_{y<p\leq z} p \tag{3.41}$$

with $1 \ll y(x) < z(x) \ll \exp(\log x/\log\log x)$. Indeed, it is not necessary to assume this bound for $z(x)$; our proof is still valid under the hypothesis $1 \ll y(x) < z(x) \leq x$.
The main tool we will use is a Theorem proven by Shiu in [36] and generalized by Nair in [30]. Their result can be used to obtain good estimates for sums of the form $\sum_{n\leq x} F(a_n)$, where the function F and the sequence a_n satisfy suitable conditions. Since we need to use it in a very particular case, it could be useful to recall briefly the properties required for the function F and consider a simpler version of the result. Where possible, we will try to use the same notations as in [30].

We begin recalling the definition of the class M of arithmetic functions F satisfying the following axioms:

- F is multiplicative, real-valued and non-negative;
- there is a constant $A_0 > 0$ such that, for every prime p,

$$F(p^l) \leq A_0^l \quad \forall l \in \mathbb{N}; \tag{3.42}$$

- for every $\varepsilon > 0$ there is a constant $A_1 = A_1(\varepsilon) > 0$ such that

$$F(n) \leq A_1 n^{\varepsilon} \quad \forall n \in \mathbb{N}. \tag{3.43}$$

An example of a function belonging to M is the divisor function $d(n)$.

Corollary 3.8. *Let $F \in M$ and x be sufficiently large; then*

$$\sum_{n\le x} F(n) \ll x \prod_{p\le x}\left(1-\frac{1}{p}\right)\exp\left(\sum_{p\le x}\frac{F(p)}{p}\right) \tag{3.44}$$

and the implied constant depends on F only through the constants A_0 and A_1 defined in (3.42) *and* (3.43) *respectively.*

Proof. This can be easily deduced from the main Theorem in [30] by taking $P(t)=t$, $y=x$. We remark that the function $\rho(m)$, defined as the number of solution modulo m of the congruence $P(n)\equiv 0 \bmod m$, is identically equal to 1 with our choice of the polynomial P. □

Now, we define the function F as

$$F(n)=\begin{cases}|a_f(n)|^2 & (n,P)=1\\ 0 & (n,P)>1.\end{cases}$$

We remark that, since P depends on x, also F does; it follows immediately from the definition that

$$\sum_{\substack{n\le x\\ (n,P)=1}} |a_f(n)|^2 = \sum_{n\le x} F(n).$$

First, we will show that $F\in M$, then we will use Corollary 3.8 to prove that (3.40) holds. Obviously, F is real valued and non-negative; to show that it is also multiplicative let us consider $n,m\in\mathbb{N}$ with $(n,m)=1$. If both n and m are coprime with P, then also their product is coprime with P and using the multiplicativity of a_f we have

$$F(nm)=|a_f(nm)|^2=|a_f(n)|^2|a_f(m)|^2=F(n)F(m).$$

On the other hand, if at least one between n and m is not coprime with P, then both $F(nm)$ and $F(n)F(m)$ are equal to 0 and $F(nm)=F(n)F(m)$ trivially. Thus, we have proved that F is multiplicative (for every possible choice of P). Now, we deal with the inequalities in (3.42) and (3.43). The first one is a consequence of the recursive formula (3.5), in particular it follows from $|a_f(p)|\le 2$ that $|a_f(p^l)|\le 2^l$ for every $l\in\mathbb{N}$. Then, for the function F we can conclude that

$$F(p^l)\le |a_f(p^l)|^2\le 4^l$$

for every $l\in\mathbb{N}$. For the second inequality, we recall that $|a_f(n)|\le d(n)$; then,

$$F(n)\le |a_f(n)|^2\le d(n)^2 \ll_\varepsilon n^\varepsilon \tag{3.45}$$

for every $\varepsilon>0$. Thus, we have proved that the function F (for every possible P) belongs to M; moreover we can take $A_0=4$ and the constant $A_1(\varepsilon)$, which is implicit in (3.45), can be chosen independently from P; this is crucial, since it means that the implicit constant in (3.44) does not depend on P and hence on x. Applying the Corollary to F we obtain

$$\sum_{n\le x} F(n) \ll x\prod_{p\le x}\left(1-\frac{1}{p}\right)\exp\left(\sum_{p\le x}\frac{F(p)}{p}\right) = x\prod_{p\le x}\left(1-\frac{1}{p}\right)\exp\left(\sum_{p\nmid P}\frac{|a_f(p)|^2}{p}\right). \qquad (3.46)$$

We need to estimate the exponential factor; observing that for $t\in[0,2]$ one can use $\exp(t) = 1+t+O(t^2)$, we write

$$\exp\left(\sum_{p\nmid P}\frac{|a_f(p)|^2}{p}\right) = \prod_{p\nmid P}\exp\left(\frac{|a_f(p)|^2}{p}\right) = \prod_{p\nmid P}\left(1+\frac{|a_f(p)|^2}{p}+O\left(\frac{|a_f(p)|^4}{p^2}\right)\right).$$

Since $|a_f(p)|^4$ is bounded, we conclude that

$$\begin{aligned}\exp\left(\sum_{p\nmid P}\frac{|a_f(p)|^2}{p}\right) &\ll \prod_{p\nmid P}\left(1+\frac{|a_f(p)|^2}{p}\right) = \prod_{p\le x}\left(1+\frac{|a_f(p)|^2}{p}\right)\prod_{p\mid P}\left(1+\frac{|a_f(p)|^2}{p}\right)^{-1}\\ &\ll \prod_{p\le x}\left(1+\frac{|a_f(p)|^2}{p}\right)\prod_{p\mid P}\left(1-\frac{|a_f(p)|^2}{p}\right) \qquad (3.47)\end{aligned}$$

where we used that $(1+t)^{-1} = 1-t+O(t^2)$ for $|t|\le 1/2$. We remark that, by definition of P in (3.41), we can assume its least prime factor to be arbitrarily large, as $x\to+\infty$.
The next step is to prove that

$$\prod_{p\le t}\left(1\pm\frac{|a_f(p)|^2}{p}\right) \ll \prod_{p\le t}\left(1\pm\frac{1}{p}\right) \qquad (3.48)$$

as $t\to+\infty$; we will do the computation only in the case with the minus sign, since the case with plus sign behaves in the same way. We begin computing the logarithm of the ratio of the two products in (3.48), i.e.

$$\begin{aligned}\log\left[\prod_{p\le t}\left(1-\frac{|a_f(p)|^2}{p}\right)\left(1-\frac{1}{p}\right)^{-1}\right] &= \sum_{p\le x}\left[\log\left(1-\frac{|a_f(p)|^2}{p}\right)-\log\left(1-\frac{1}{p}\right)\right]\\ &= \sum_{p\le t}\frac{1-|a_f(p)|^2}{p}+O(1).\end{aligned}$$

By partial summation, the sum over the primes can be written as

$$\sum_{p\le t}\frac{1-|a_f(p)|^2}{p} = \frac{R(t)}{t}+\int_2^t\frac{R(w)}{w^2}dw \qquad (3.49)$$

where $R(t) = \pi(t)-\sum_{p\le t}|a_f(p)|^2$. Using the Prime Number Theorem and the equivalent version for the Rankin-Selberg L-function (see, for example [23], Corollary 1.2 and Lemma 5.1) we have that there exist two positive constants c_1, c_2 such that

$$R(w) = \mathrm{li}(w)+O\left(we^{-c_1\sqrt{\log w}}\right)-\left[\mathrm{li}(w)+O\left(we^{-c_2\sqrt{\log w}}\right)\right] = O\left(we^{-c\sqrt{\log w}}\right)$$

with $c = \min(c_1, c_2)$ and as $w \to +\infty$. Inserting this estimate in (3.49) we find that

$$\sum_{p \le t} \frac{1 - |a_f(p)|^2}{p} \ll e^{-c\sqrt{\log t}} + \int_2^t \frac{e^{-c\sqrt{\log w}}}{w} dw \ll 1.$$

In conclusion, we showed that

$$\log \left[\prod_{p \le t} \left(1 - \frac{|a_f(p)|^2}{p}\right) \left(1 - \frac{1}{p}\right)^{-1} \right] \ll 1$$

and hence the claim in (3.48) is proved. As an immediate consequence, we have that

$$\begin{aligned}\prod_{p|P}\left(1-\frac{|a_f(p)|^2}{p}\right) &= \prod_{y<p\le z}\left(1-\frac{|a_f(p)|^2}{p}\right) = \prod_{p\le z}\left(1-\frac{|a_f(p)|^2}{p}\right)\prod_{p\le y}\left(1-\frac{|a_f(p)|^2}{p}\right)^{-1} \\ &\ll \prod_{p\le z}\left(1-\frac{1}{p}\right)\prod_{p\le y}\left(1+\frac{1}{p}\right) \ll \prod_{y<p\le z}\left(1-\frac{1}{p}\right) \\ &= \prod_{p|P}\left(1-\frac{1}{p}\right) = \frac{\phi(P)}{P}. \end{aligned} \tag{3.50}$$

Inserting the estimate (3.47) in (3.46) and using (3.48) and (3.50) we can finally conclude that

$$\begin{aligned}\sum_{n\le x} F(n) &\ll x\prod_{p\le x}\left(1-\frac{1}{p}\right)\prod_{p\le x}\left(1+\frac{|a_f(p)|^2}{p}\right)\prod_{p|P}\left(1-\frac{|a_f(p)|^2}{p}\right) \\ &\ll x\prod_{p\le x}\left(1-\frac{1}{p}\right)\prod_{p\le x}\left(1+\frac{1}{p}\right)\frac{\phi(P)}{P} \\ &\ll \frac{\phi(P)}{P}x. \end{aligned}$$

By applying the same technique we used in Lemma 3.2 to deduce (3.12) we can easily say that the same estimate holds replacing $a_f(n)$ with $\mu_f(n)$. Hence, we have proved that both the coefficients $a_f(n)$ and $\mu_f(n)$ satisfy condition *(iii)* of Theorem 3.7.

4 Polynomial Orthogonality

In this chapter we consider a particular case of the sums defined in Section 3.2 as we take $\xi(n) = e(P(n))$ with $P(t) \in \mathbb{R}[t]$ a real polynomial of degree $k \geq 1$; we will use the notation

$$S_f(x,P) = \sum_{n \leq x} \mu_f(n) e(P(n)). \tag{4.1}$$

In Chapter 2 we recalled that the Möbius function is orthogonal to the sequences $e(\alpha n)$ with α any real number, that is, to $e(P(n))$ with P a polynomial of degree 1. A much more general result has been proved by Hua [17] who extended the orthogonality to sequences with polynomials of any degree; collecting the result in Davenport [7] and Hua we have the following Theorem.

Theorem 4.1. *Let ν be a positive integer, let $0 \leq l < \nu$ and let $P(t)$ be a real polynomial of degree $k > 0$; then, for every $A > 0$,*

$$\sum_{\substack{n \leq x \\ n \equiv l \bmod \nu}} \mu(n) e(P(n)) \ll_{k,A} \frac{x}{\log^A x} \tag{4.2}$$

as $x \to +\infty$.

The results we are able to prove for $S_f(x,P)$ heavily depend on the degree k of P. If $k \leq 2$, orthogonality in a form similar to (4.2) can be proved; this is done in the first section. In the second and in the third one we deal with polynomials of degree greater than two; we are still able to prove orthogonality, but the techniques used provide us a weaker estimate. We first reduce to consider monomials and then we extend the result to any polynomial of degree $k \geq 3$.

4.1 Polynomials of degree 1 and 2

We start from polynomial of degree 1, i.e. $P(x) = \alpha x$ with $\alpha \in \mathbb{R}$; we can assume the constant term to be equal to zero since the modulus of the sum $S_f(x,P)$ does not depend on it. Then (4.1) becomes

$$S_f(x,P) = S_f(x,\alpha) = \sum_{n \leq x} \mu_f(n) e(n\alpha) \tag{4.3}$$

and since $e(x)$ is a 1-periodic function we consider $\alpha \in \mathbb{T}$. We will need the following lemmas.

Lemma 4.2. *Let $\alpha \in \mathbb{R}$ and $a, q \in \mathbb{Z}$ such that $(a,q) = 1$ and $|\alpha - (a/q)| \leq 1/q^2$. Then*

$$S_f(x,\alpha) \ll_f \left(xq^{-1/2} + x^{1/2}q^{1/2} + x^{5/6}\right)\log^r x \tag{4.4}$$

for some $r > 0$.

This result was proven by Perelli in [32].

Lemma 4.3. *Let χ be any Dirichlet character of conductor q. Then there is an absolute constant $c > 0$ such that*

$$\sum_{n\leq x} \mu_f(n)\chi(n) \ll_f q^{1/2} x \exp\left(-c\sqrt{\log x}\right). \tag{4.5}$$

A proof of this can be found as a partial result of Theorem 4.1 in Fouvry & Ganguly [11]. We remark that in this paper the authors study the sum

$$\sum_{n\leq x} \mu(n)a_f(n)e(\alpha n) \tag{4.6}$$

which is really close to $S_f(x,\alpha)$, since the multiplicative functions $\mu_f(n)$ and $\mu(n)a_f(n)$ coincide over the primes. The bound they find for the sum in (4.6) is the same we have for $S_f(x,\alpha)$ in (4.7) and we expect that one can obtain one estimate from the other with no much effort; however, we prefer to give an independent proof of Theorem 4.4 using the same technique as in [11]. The main purpose is to show in this simpler environment the ideas we will use also in Theorems 4.8, 4.10 and 4.11.

Theorem 4.4. *There is a constant $c > 0$ such that for all $\alpha \in \mathbb{T}$*

$$S_f(x,\alpha) \ll_f x\exp\left(-c\sqrt{\log x}\right) \tag{4.7}$$

with the implied constant depending only on the cuspidal form f. In particular, the estimate is uniform in α.

Proof. Let $Q = Q(x) > 1$ be a parameter to be decided later; given $\alpha \in \mathbb{T}$, by Dirichlet's approximation theorem, there is always a rational number a/q with $(a,q) = 1$ satisfying the conditions

$$1 \leq q \leq Q \quad \text{and} \quad \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{qQ}. \tag{4.8}$$

We remark that, since Q is chosen as a function of x, also a and q will depend on x. We need two different arguments according to the size of q; for q *small* we use Lemma 4.3, for q *large* we use Lemma 4.2. We deal first the case when q is small, namely when $q \leq \exp(\bar{c}\sqrt{\log x})$ for some positive constant $\bar{c}$ which will be specified later. The basic idea is to approximate $S_f(x,\alpha)$ with $S_f(x,a/q)$, to split the sum in the residue classes modulo q and then to apply Lemma 4.3. By partial summation, we have

$$S_f(x,\alpha) = \sum_{n\leq x} \mu_f(n)e(n(\alpha - a/q))e(na/q)$$

$$\ll |S_f(x,a/q)| + \int_1^x \left| \left(\alpha - \frac{a}{q} \right) S_f(t,a/q) \right| dt + 1. \tag{4.9}$$

To estimate (4.9) it is sufficient to study the sum $S_f(x,a/q)$; using the periodicity of the exponential function we can write

$$S_f(x,a/q) = \sum_{b \bmod q} e\left(\frac{ab}{q}\right) \sum_{\substack{n\le x \\ n\equiv b \bmod q}} \mu_f(n). \tag{4.10}$$

The next step in to express the condition $n \equiv b \bmod q$ using the orthogonality of Dirichlet characters, but to do this we need to ensure the class and the modulus to be coprime. To do this we write

$$d = (b,q), \quad b_1 = b/d \quad \text{and} \quad q_1 = q/d$$

so that

$$\sum_{\substack{n\le x \\ n\equiv b \bmod q}} \mu_f(n) = \sum_{\substack{n\le x/d \\ n\equiv b_1 \bmod q_1}} \mu_f(dn). \tag{4.11}$$

Since the function μ_f is supported on the cube-free integers we can assume that d is cube-free, otherwise we would have $\mu_f(dn) = 0$ for every n. So we can write

$$d = \prod_{i=1}^{s} p_i \prod_{j=1}^{t} q_j^2 = PQ^2, \tag{4.12}$$

say, where the primes p_i, q_j are all distinct. We want to use the multiplicativity of function μ_f, but it is possible that n and d in (4.11) are not coprime; let δ be their greater common divisor. We observe that if δ divides Q then dn in not cube-free and $\mu_f(dn) = 0$; so we can assume that δ divides P and not Q. We rearrange the sum in (4.11) collecting the integers $n \le x/d$ such that $(n,d) = \delta$ and the summing over δ dividing P. Writing $n = \delta m$ we have

$$\sum_{\substack{n\le x \\ n\equiv b_1 \bmod q_1}} \mu_f(dn) = \sum_{\delta | P} \sum_{\substack{m\le x/d\delta \\ \delta m\equiv b_1 \bmod q_1 \\ (m,P)=1}} \mu_f(d\delta m).$$

Let us consider now the congruence $\delta m \equiv b_1 \bmod q_1$. There are essentially two possibilities: if δ and q_1 are coprime we can rewrite it as $m \equiv b_\delta \bmod q_1$ where $b_\delta = b_1 \delta^{-1} \bmod q_1$; otherwise, if δ and q_1 are not coprime, the congruence has no solution, since b_1 and q_1 are coprime by definition. So we can write

$$\sum_{\delta | P} \sum_{\substack{m\le x/d\delta \\ \delta m\equiv b_1 \bmod q_1 \\ (m,P)=1}} \mu_f(d\delta m) = \sideset{}{^*}\sum_{\delta | P} \sum_{\substack{m\le x/d\delta \\ m\equiv b_\delta \bmod q_1 \\ (m,d)=1}} \mu_f(d\delta m) \tag{4.13}$$

where $\sum^*$ means that the sum in taken over δ such that $(\delta, q_1) = 1$. Finally, we replaced the condition $(n,P) = 1$ with $(n,d) = 1$ because, as already remarked, $(m,Q) > 1$ implies $\mu_f(d\delta n) = 0$. Using the multiplicativity of $\mu_f(n)$ we find from (4.13) that

$$\sideset{}{^*}\sum_{\delta | P} \sum_{\substack{n\le x/d\delta \\ n\equiv b_\delta \bmod q_1 \\ (n,d)=1}} \mu_f(d\delta n) = \sideset{}{^*}\sum_{\delta | P} \mu_f(d\delta) \sum_{\substack{n\le x/d\delta \\ n\equiv b_\delta \bmod q_1}} \mu_f(n)\chi_d(n)$$

where χ_d is the principal character modulo d. Using the orthogonality of Dirichlet's characters we can write the sum above as

$$\sum_{\delta|P}{}^{*} \frac{\mu_f(d\delta)}{\phi(q_1)} \sum_{\chi \bmod q_1} \overline{\chi}(b_\delta) \sum_{n\le x/d\delta} \mu_f(n)(\chi_d\chi)(n) \tag{4.14}$$

where $\chi_d\chi$ is a character of conductor $\le q$. We can now apply Lemma 4.3 to the inner sum in (4.14); we find that

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv b \bmod q}} \mu_f(n) &\ll_f \sum_{\delta|P}{}^{*} \frac{|\mu_f(d\delta)|}{\phi(q_1)} \sum_{\chi \bmod q_1} q^{1/2}\frac{x}{d\delta}\exp\left(-c\sqrt{\log(x/d\delta)}\right)\\ &\ll_f q^{1/2} x \exp\left(-c\sqrt{\log(x/q^2)}\right)\sum_{\delta|P}{}^{*}\frac{|\mu_f(d\delta)|}{d\delta} \end{aligned}\tag{4.15}$$

where we used the trivial estimate $d\delta \le q^2$. Since $\mu_f(d\delta) \ll (d\delta)^{1/2}$ we trivially have that

$$\sum_{\delta|P}{}^{*}\frac{|\mu_f(d\delta)|}{d\delta} \ll d^{-1/2}\sum_{\delta\le P}\delta^{-1/2} \ll \left(\frac{P}{d}\right)^{1/2} \ll 1$$

and inserting estimate (4.15) in (4.10) we have

$$S_f(x,a/q) \ll_f q^{3/2} x \exp\left(-c\sqrt{\log(x/q^2)}\right). \tag{4.16}$$

Finally, from (4.9), we deduce

$$S_f(x,\alpha) \ll_f q^{1/2} x \exp\left(-c\sqrt{\log(x/q^2)}\right)\left(q+\frac{x}{Q}\right). \tag{4.17}$$

Now, we set $Q = x\exp\left(-c/3\sqrt{\log x}\right)$ where c is the same constant appearing in (4.17) and we assume that $q \le x/Q = \exp\left(c/3\sqrt{\log x}\right)$; it follows immediately from (4.17) that

$$\begin{aligned}S_f(x,\alpha) &\ll_f x\exp\left(\frac{c}{2}\sqrt{\log x}\right)\exp\left(-c\sqrt{\log x - \frac{2c}{3}\sqrt{\log x}}\right)\\ &\ll_f x\exp\left(-c'\sqrt{\log x}\right)\end{aligned}\tag{4.18}$$

for some suitable $c' > 0$. Then, we have proved that the sum $S_f(x,\alpha)$ satisfies the estimate (4.18) when q is small. When q is large, i.e. when it satisfies $\exp\left(c/3\sqrt{\log x}\right) < q \le Q$, we can apply Lemma 4.2. In particular, the range in which q lies allows us to estimate both $xq^{-1/2}$ and $x^{1/2}q^{1/2}$ with $x\exp(-c/6\sqrt{\log x})$; we conclude that there exist a positive constant c'' such that

$$S_f(x,\alpha) \ll_f x\exp\left(-c''\sqrt{\log x}\right). \tag{4.19}$$

Collecting the results in (4.18) and (4.19) the theorem is proved. □

Corollary 4.5. *Let q be a positive integer and $0 \le a < q$; then*

$$\sum_{\substack{n\le x\\ n\equiv b \bmod q}} \mu_f(n) \ll_f x\exp(-c\sqrt{\log x})$$

with the implied constant depending only on f and c the same positive constant appearing in Theorem 4.4.

Proof. Using the orthogonality properties of the exponential function $e(x)$, we can write the sum as

$$\sum_{\substack{n\le x\\ n\equiv b \bmod q}} \mu_f(n) = \frac{1}{q}\sum_{h \bmod q} e(-hb/q)\sum_{n\le x}\mu_f(n)e(nh/q).$$

The Corollary follows immediately applying the estimate of Theorem 4.4. □

We now consider polynomials of degree 2; the technique used will be essentially the same as in the linear case. We remark that in [16] Hou and Lü found the same estimates for the sums

$$\sum_{n\le x}\Lambda_f(n)e(P(n))$$

with $P(t)$ a polynomial of degree 2 and $\Lambda_f(n)$ defined by

$$\frac{L'(s,f)}{L(s,f)} = \sum_{n\ge 1}\frac{\Lambda_f(n)}{n^s}, \quad \Re s > 1.$$

We need the following two lemmas.

Lemma 4.6. *Let $\alpha,\beta \in \mathbb{T}$; then, for any $\varepsilon > 0$,*

$$\sum_{n\le x} a_f(n)e(\alpha n^2+\beta n) \ll_\varepsilon N^{\frac{7}{8}+\varepsilon}$$

This is a result due to Liu and Ren (see [24]) which improves the first non-trivial estimate proved by Pitt [34].

Lemma 4.7. *Let $\{h(n)\},\{k(n)\}$ be two sequences of complex numbers satisfying*

$$\sum_{m\sim M}|h(n)|^2 \ll M\log^{b_1} M \quad \textit{and} \quad \sum_{m\sim N}|k(n)|^2 \ll N\log^{b_2} N$$

for $M,N\ge 2$ and for some $b_1,b_2>0$ (we write $n\sim N$ for $N/2<n\le N$). Then, there exists a positive constant $c>0$ such that for every $\alpha,\beta\in\mathbb{T}$

$$\sum_{\substack{n\sim N, m\sim M\\ nm\sim x}} h(n)k(m)e(\alpha(nm)^2+\beta nm) \ll x\left(\frac{1}{q}+\frac{1}{M}+\frac{1}{N^4}+\frac{q}{x^2}\right)^{\frac{1}{8}}\log^c x$$

where $q\ge 1$ satisfies

$$\left|\alpha-\frac{a}{q}\right| \le \frac{1}{q^2}$$

for some integer a such that $(a,q)=1$.

A proof of this result can be found in [16].

Theorem 4.8. *There is a constant $c>0$ such that for all polynomials $P(x)\in\mathbb{T}[x]$ of degree 2*

$$S_f(x,P) \ll_f x\exp\left(-c\sqrt{\log x}\right) \tag{4.20}$$

with the implied constant depending only on f.

Proof. Let us write $P(x) = \alpha x^2 + \beta x$ and let Q, q and a be as in (4.8), i.e. such that

$$1 \le q \le Q \quad \text{and} \quad \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qQ}. \tag{4.21}$$

Moreover, we set

$$Q = x^2 \exp\left(-C\sqrt{\log x}\right) \tag{4.22}$$

with $C > 0$ a constant to be decided later. As in the case of degree 1 polynomials we proceed in two different ways according to the size of $q = q(x)$. If $q \le \exp\left(C\sqrt{\log x}\right)$ we have, as in (4.9),

$$\begin{aligned} S_f(x,P) &= \sum_{n\le x} \mu_f(n) e(n^2 a/q + \beta n) e(n^2(\alpha - a/q)) \\ &\ll |S_f(x,P_q)| + \int_1^x \left| t\left(\alpha - \frac{a}{q}\right) S_f(t,P_q)\right| dt + 1 \end{aligned} \tag{4.23}$$

where $P_q(x) = (a/q)x^2 + \beta x$. To estimate $S_f(x,P_q)$ we write

$$\begin{aligned} S_f(x,P_q) &= \sum_{n\le x} \mu_f(n) e(n^2(a/q) + \beta n) \\ &= \sum_{b \bmod q} e\left(\frac{ab^2}{q}\right) \sum_{\substack{n\le x \\ n\equiv b \bmod q}} \mu_f(n) e(\beta n) \\ &= \frac{1}{q} \sum_{b \bmod q} \sum_{d \bmod q} e\left(\frac{ab^2 - db}{q}\right) \sum_{n\le x} \mu_f(n) e\left((\beta + d/q)n\right) \end{aligned} \tag{4.24}$$

where we used the orthogonality properties of the exponential function. Then,

$$S_f(x,P_q) \ll \frac{1}{q} \sum_{b \bmod q} \sum_{d \bmod q} \left| e\left(\frac{ab^2 - db}{q}\right)\right| \left|\sum_{n\le x} \mu_f(n) e\left((\beta + d/q)n\right)\right|$$

and we can use Theorem 4.4 to conclude that

$$S_f(x,P_q) \ll qx\exp(-c\sqrt{\log x}). \tag{4.25}$$

Now we choose $C = c/2$ in (4.22); inserting (4.25) in (4.23) and recalling that $q \le x^2/Q = \exp(C\sqrt{\log x})$ we find that

$$\begin{aligned} S_f(x,P) &\ll qx\exp(-c\sqrt{\log x})\left(1 + \frac{x^2}{qQ}\right) = x\exp(-c\sqrt{\log x})\left(q + \frac{x^2}{Q}\right) \\ &\ll x\exp(-\frac{c}{2}\sqrt{\log x}). \end{aligned} \tag{4.26}$$

Now we need to deal the case $\exp(C\sqrt{\log x}) < q \le Q$; we will use Vaughan's identity as well as Lemma 4.6 and Lemma 4.7.

By taking $\xi(n) = e(\alpha n^2 + \beta n)$ in (A.1) we get

$$S_f(x,P) = 2S_1 - S_2 - S_3$$

where

$$S_1 = \sum_{n\le Y} \mu_f(n)e(\alpha n^2+\beta n),$$
$$S_2 = \sum_{n_1\le Y^2} A_f(n_1) \sum_{n_2\le x/n_1} a_f(n_2)e(\alpha(n_1n_2)^2+\beta n_1n_2),$$
$$S_3 = \sum_{Y<n_1\le x/Y}\ \sum_{Y<n_2<x/n_1} \mu_f(n_1)B_f(n_2)e(\alpha(n_1n_2)^2+\beta n_1n_2)$$

and

$$A_f(n) = \sum_{\substack{n_1n_2=n\\ n_i\le Y}} \mu_f(n_1)\mu_f(n_2), \qquad B_f(n) = \sum_{\substack{n_1n_2=n\\ n_2\le Y}} a_f(n_1)\mu_f(n_2).$$

We have that $S_1 \ll_f Y$ from Lemma 3.2 and $S_2 \ll_f x^{7/8+\varepsilon}Y^2$ from Lemma 4.6. To estimate S_3 we can use Lemma 4.7; we begin writing the sum as a linear combination of $\mathscr{O}(\log^2 x)$ terms of the form

$$\sum_{\substack{m\sim M, n\sim N'\\ nm\sim N''}} \mu_f(n)B_f(m)e(\alpha(nm)^2+\beta nm) \tag{4.27}$$

with

$$Y<M<\frac{2x}{Y},\quad Y<N'<\frac{2x}{Y},\quad Y^2<N''<2x,\quad MN'\asymp N''.$$

Each of these sums satisfies the hypotheses of the Lemma; in fact, from Lemma 3.2, we have

$$\sum_{n\sim N} |\mu_f(n)|^2 \ll N,$$

and

$$|B(n)| \le \sum_{n_1n_2=n} d(n_1)d(n_2) = (d*d)(n) = d_4(n);$$

which implies that

$$\sum_{n\sim N} |B_f(n)|^2 \le \sum_{n\sim N} d_4(n)^2 \ll N\log^7 N$$

(see Linnik's book [21], Chapter 1 for the properties of generalised divisor function). Hence, there is a constant $h>0$ such that

$$S_3 \ll_f \left(x\left(\frac{1}{q}+\frac{q}{x^2}\right)^{1/8} + \frac{x}{Y^{1/8}}\right)\log^h x.$$

Recalling that we are in the case $\exp(C\sqrt{\log x}) < q \le x^2\exp(-C\sqrt{\log x})$ and making the choice $Y=\exp(C\sqrt{\log x})$ we conclude that

$$S_3 \ll_f x\exp(-c'\sqrt{\log x})$$

for some positive constant c'; collecting the estimates we find that

$$S_f(x,P) \ll_f x\exp(-c'\sqrt{\log x}). \tag{4.28}$$

The theorem follows from (4.26) and (4.28). □

Corollary 4.9. *Let $P(t)$ be any real polynomial of degree less than or equal to 2; then, for any positive integer ν and $0 \leq a < \nu$,*

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod \nu}} \mu_f(n) e(P(n)) \ll x \exp(-c\sqrt{\log x})$$

with $c > 0$ the same constant appearing in Theorem 4.8.

Proof. As we have done in Corollary 4.5 we use the orthogonality properties of $e(x)$ to write

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \mu_f(n) e(P(n)) = \frac{1}{q} \sum_{h \bmod q} e(-ah/q) \sum_{n \leq x} \mu_f(n) e(P(n) + nh/q).$$

Now we can apply Theorem 4.8 to every polynomial $P_h(t) = P(t) + (h/q)t$ and the Corollary is proved. □

4.2 Monomial of degree $k \geq 3$

We want to study what happens when we increase the degree of the polynomial. The techniques we used in the previous section can not be applied here, because it would be crucial to have strong estimates for the sums

$$\sum_{n \leq x} a_f(n) e(P(n))$$

which are not known if the degree of P is greater than 2. We begin by considering monomials of degree $k \geq 3$.

Theorem 4.10. *Let $S_f(x, \alpha)$ be defined as*

$$S_f(x, \alpha; k) = \sum_{n \leq x} \mu_f(n) e(\alpha n^k)$$

where $x \geq 1$, $k \geq 3$ and $\alpha \in \mathbb{T}$. Then,

$$S_f(x, \alpha; k) \ll_k \frac{x}{\log^{1/4} x} \log\log x$$

with the implied constant uniform in α.

Proof. As in the proof of Theorem 4.8 we consider $Q = Q(x)$ and, in analogy, we set it as $Q = x^k \exp\left(-c_Q\sqrt{\log x}\right)$ with $c_Q > 0$ to be decided later. Let q and a be as in (4.8), i.e. such that

$$1 \leq q \leq Q \quad \text{and} \quad \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{qQ}. \tag{4.29}$$

Again, the proof requires two different arguments according to the size of q. First we assume that q is not too large, in particular $q \leq \exp\left(c_q\sqrt{\log x}\right)$ for some $c_q > 0$. As in (4.9) we find

$$S_f(x, \alpha; k) \ll |S_f(x, a/q; k)| + k \int_1^x |S_f(x, a/q; k)| \left|\alpha - \frac{a}{q}\right| t^{k-1} dt. \tag{4.30}$$

To estimate $S_f(x,a/q;k)$ we compute

$$S_f(x,a/q;k) = \sum_{b=1}^{q} e\left(\frac{a}{q}b^k\right) \sum_{\substack{n \leq x \\ n \equiv b \bmod q}} \mu_f(n) \tag{4.31}$$

and using Corollary 4.5 we conclude that there exists a positive constant c such that

$$S_k(x,a/q;k) \ll qx\exp\left(-c\sqrt{\log x}\right).$$

Thus, from above, we have

$$\begin{aligned} S_f(x,\alpha;k) &\ll_k qx\exp\left(-c\sqrt{\log x}\right)\left(1+\int_1^x \frac{t^{k-1}}{qQ}dt\right) \\ &\ll_k x\exp\left(-c\sqrt{\log x}\right)\left(\exp\left(c_q\sqrt{\log x}\right)+\exp\left(c_Q\sqrt{\log x}\right)\right) \\ &\ll_k x\exp\left((-c+c')\sqrt{\log x}\right) \end{aligned} \tag{4.32}$$

where $c' = \max(c_q, c_Q)$. We choose c_q and c_Q such that $c' - c < 0$.

Let us suppose, now, that $\exp(c_q\sqrt{\log x}) < q \leq Q$; we want to use Theorem 3.7 with $T = c_T\sqrt{\log x}$ and $c_T > 0$ to be decided later. We remark that $T^{1/2} \asymp \log^{1/4} x$ will be, essentially, the saving over the trivial bound that we obtain by the application of the theorem (see (4.36)). To apply Theorem 3.7 we need to estimate the sum

$$S_{1,2} := \sum_{n \leq x} e\left(\alpha\left(p_1^k - p_2^k\right)n^k\right)$$

for every pair of distinct primes $p_2 < p_1 \leq \exp\left(c_T\sqrt{\log x}\right)$.

We want to apply Lemma A.2 to the polynomial $S(t) = \alpha(p_1^k - p_2^k)t^k$, hence we need to find a good approximation for the leading coefficient $\alpha(p_1^k - p_2^k)$. The inequality $|\alpha - a/q| \leq (qQ)^{-1}$ implies immediately that

$$\left|\alpha(p_1^k - p_2^k) - \frac{a(p_1^k - p_2^k)}{q}\right| \leq \frac{(p_1^k - p_2^k)}{qQ}. \tag{4.33}$$

To ensure that $p_1^k - p_2^k$ are q are coprime we divide them by their greatest common divisor, so let $P'_{1,2}$ and $q'_{1,2}$ be defined as

$$r_{1,2} = (p_1^k - p_2^k, q), \quad p_1^k - p_2^k = r_{1,2}P'_{1,2} \quad \text{and} \quad q = r_{1,2}q'_{1,2};$$

inequality (4.33) becomes

$$\left|\alpha(p_1^k - p_2^k) - \frac{aP'_{1,2}}{q'_{1,2}}\right| \leq \frac{P'_{1,2}}{q'_{1,2}Q} \leq \frac{P'_{1,2}}{{q'_{1,2}}^2}. \tag{4.34}$$

Then, for every $Z > 0$, we have

$$S_{1,2} \ll_k x\left(\frac{P'_{1,2}Z}{q'_{1,2}} + \frac{P'_{1,2}Z}{x}\log q'_{1,2} + P'_{1,2}Z\frac{q'_{1,2}}{x^k}\log q'_{1,2} + \frac{\log^{b(k)} x}{Z}\right)^{\kappa} \tag{4.35}$$

with $\kappa = 2^{1-k}$; we take $Z = \exp\left(c_Z\sqrt{\log x}\right)$ form some $c_Z > 0$ and we remark that

$$\frac{P'_{1,2}}{q'_{1,2}} = \frac{p_1^k - p_2^k}{q} \le \exp\left((kc_T - c_q)\sqrt{\log x}\right), \quad P'_{1,2} \le p_1^k - p_2^k \le \exp\left(kc_T\sqrt{\log x}\right), \quad q'_{1,2} \le q \le Q.$$

Inserting these estimate in (4.35) we find that

$$\begin{aligned} S_{1,2} \ll_k x \Bigg(& \exp\left((c_Z + kc_T - c_q)\sqrt{\log x}\right) \\ & + \exp\left((c_Z + kc_T)\sqrt{\log x}\right)\frac{\log x}{x} \\ & + \exp\left((kc_T + c_Z - c_Q)\sqrt{\log x}\right)\log x \\ & + \frac{\log^{b(k)} x}{\exp\left(c_Z\sqrt{\log x}\right)}\Bigg)^{\kappa}. \end{aligned}$$

Since we can choose c_T and c_Z such that

$$\begin{cases} c_Z + kc_T - c_q < 0 \\ kc_T + c_Z - c_Q < 0 \end{cases}$$

we conclude that

$$S_{1,2} \ll_k x\exp\left(-c''\sqrt{\log x}\right).$$

for some positive constant $c'' = c''(k)$. Since the hypotheses of Theorem 3.7 are satisfied we can conclude that, for $\exp(c_q\sqrt{\log x}) < q \le Q$,

$$S_f(x,\alpha;k) \ll_k \frac{x}{\log^{1/4} x}\log\log x. \tag{4.36}$$

The Theorem follows from (4.32) and (4.36). □

4.3 Generic polynomial of degree $k \ge 3$

In this section we prove that the same result that holds for monomials can be extended to any polynomial of degree greater than two. So, let us assume that $P(n) = \alpha_k n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_1 n$ with $\alpha_j \in \mathbb{T}$ for $j = 1, \ldots, k$ and $k \ge 3$.

Theorem 4.11. *Let $P(t)$ be as above; then,*

$$S_f(x;P) \ll_k \frac{x}{\log^{1/4} x}\log\log x. \tag{4.37}$$

Proof. Let us set $Q' = x^k\exp\left(-c_{Q'}\sqrt{\log x}\right)$ and $Q = x^k\exp\left(-c_Q\sqrt{\log x}\right)$; by Diophantine approximation there exist $a_i, q_i \in \mathbb{Z}$ with $(a_i, q_i) = 1$ such that

$$\begin{aligned} &\left|\alpha_k - \frac{a_k}{q_k}\right| \le \frac{1}{q_k Q'} \\ &\left|\alpha_i - \frac{a_i}{q_i}\right| \le \frac{1}{q_i Q} \qquad i = 2, \ldots, k-1. \end{aligned}$$

We have to consider several cases:

- in *Case A* we will assume that all the denominators q_j are small and we will proceed using partial summation as in the previous theorems;
- in *Case B* we will assume that q_k is large and we will use the result of Theorem 4.10;
- in *Case C* we will consider the last scenario with q_k small and q_t large for some $2 \leq t < k$ and we will use a combination of techniques used in the first two cases.

To determine when the denominators q_j are small or large we will introduce a set of constants c_j that will be chosen at the end of the proof; we remark here that they will depend only on the degree k and on the constant c of Corollary 4.9.

Finally we note that we do not approximate the coefficient α_1; this ensures us that the polynomial H defined in (4.44) has degree at least 2, which is crucial. On the other hand, the estimate valid for polynomials of degree 1, see Theorem 4.4 and its corollaries, is strong enough to be used in (4.39) to get a suitable bound.

Case A

We assume $q_i \leq \exp\left(c_i\sqrt{\log x}\right)$ for $2 \leq i \leq k$ and for some constants $c_i > 0$ (to be decided later). Then by partial summation we have

$$S_f(x,P) \ll_k \left|S_f(x,\overline{P})\right| + \int_1^x \left|S_f(y,\overline{P})\right| \left|\alpha_k - \frac{a_k}{q_k}\right| y^{k-1} dy + \exp\left(c_Q\sqrt{\log x}\right) \tag{4.38}$$

where $\overline{P}(n) = (a_k/q_k)n^k + \ldots + (a_2/q_2)n^2 + \alpha_1 n$. Now, let $q = \mathrm{lcm}\,(q_2,\ldots,q_k)$,

$$S_f(x,\overline{P}) = \sum_{b=1}^{q} e\left(\overline{P}(b)\right) \sum_{\substack{n\leq x\\ n\equiv b \bmod q}} \mu_f(n)e(\alpha n) \ll qx\exp\left(-c\sqrt{\log x}\right) \tag{4.39}$$

where c is the constant in Corollary 4.9. Inserting this estimate in (4.38) we have

$$\begin{aligned} S_f(x,P) &\ll_k qx\exp\left(-c\sqrt{\log x}\right)\left(1+\exp\left(c_{Q'}\sqrt{\log x}\right)\right) \\ &\ll_k x\exp\left(-\left(c - c_{Q'} - \sum_{i=2}^{k} c_i\right)\sqrt{\log x}\right). \end{aligned} \tag{4.40}$$

We will assume that

$$c - \left(c_{Q'} + \sum_{i=2}^{k} c_i\right) > 0. \tag{4.41}$$

Case B

We assume that $q_k > \exp\left(c_k\sqrt{\log x}\right)$; we can proceed as in the second part of the proof of Theorem 4.10, since the technique used there requires hypothesis only on the leading coefficient. So we can conclude that

$$S_f(x,P) \ll_k \frac{x}{\log^{1/4} x} \log\log x. \tag{4.42}$$

Case C

Let t be the largest index, $2 \leq t \leq k-1$ such that $q_t > \exp\left(c_t\sqrt{\log x}\right)$; as in the monomial case we want to use Theorem 3.7 with $T = c_T\sqrt{\log x}$ and $c_T > 0$ to be fixed later. As already stated in Theorem 4.10 this choice will lead to a saving which is essentially $T^{1/2} \asymp \log^{1/4} x$. We need to estimate the sum

$$S_{1,2}(x) := \sum_{n\le x} e\left(\sum_{i=1}^{k} \alpha_i \left(p_1^i - p_2^i\right) n^i\right)$$

for every pair of distinct primes $p_2 < p_1 \le \exp\left(c_T\sqrt{\log x}\right)$. A direct application of Lemma A.2 wouldn't give any interesting result, so we first need to transform the sum $S_{1,2}$. By partial summation we get

$$S_{1,2}(x) \ll_k \left|S'_{1,2}(x)\right| + \int_1^x \left|S'_{1,2}(y)\right| \left|\alpha_k - \frac{a_k}{q_k}\right| \left(p_1^k - p_2^k\right) y^{k-1} dy + \exp\left(c_Q\sqrt{\log x}\right) \tag{4.43}$$

where

$$\begin{aligned} S'_{1,2}(x) &:= \sum_{n\le x} e\left(\sum_{i=1}^{t} \alpha_i \left(p_1^i - p_2^i\right) n^i + \sum_{i=t+1}^{k} \frac{a_i}{q_i} \left(p_1^i - p_2^i\right) n^i\right) \\ &= \sum_{n\le x} e\left(H(n) + \sum_{i=t+1}^{k} \frac{a_i}{q_i} \left(p_1^i - p_2^i\right) n^i\right), \end{aligned} \tag{4.44}$$

say. Let $\overline{q}$ be the $\operatorname{lcm}(q_{t+1}, \dots, q_k)$, then we write

$$\begin{aligned} S'_{1,2}(x) &= \sum_{b=1}^{\overline{q}} e\left(\sum_{i=t+1}^{k} \frac{a_i}{q_i} \left(p_1^i - p_2^i\right) b^i\right) \sum_{\substack{n\le x \\ n\equiv b \bmod \overline{q}}} e\left(H(n)\right) \\ &= \frac{1}{\overline{q}} \sum_{b=1}^{\overline{q}} e\left(\sum_{i=t+1}^{k} \frac{a_i}{q_i} \left(p_1^i - p_2^i\right) b^i\right) \sum_{h=1}^{\overline{q}} e\left(\frac{-hb}{\overline{q}}\right) \sum_{n\le x} e\left(H(n) + \frac{hn}{\overline{q}}\right). \end{aligned} \tag{4.45}$$

The polynomial $H(n) + hn/\overline{q}$ has degree $t \ge 2$ and leading coefficient $\alpha_t(p_1^t - p_2^t)$; now we apply Lemma A.2 to the sum

$$R(x) := \sum_{n\le x} e\left(H(n) + \frac{hn}{\overline{q}}\right).$$

By definition, inequalities (4.33) and (4.34) hold with q and k replaced by q_t and t respectively; hence we can conclude as in (4.35) that, for any $Z > 0$,

$$R(x) \ll_k x\left(\frac{(p_1^t - p_2^t)Z}{q_t} + \frac{(p_1^t - p_2^t)Z}{x}\log q_t + (p_1^t - p_2^t)Z\frac{q_t}{x^k}\log q_t + \frac{\log^{b(k)} x}{Z}\right)^{\kappa}$$

with $\kappa = 2^{1-t}$.

Setting $Z = \exp\left(\rho_t\sqrt{\log x}\right)$ and recalling that

$$\begin{aligned} &(p_1^t - p_2^t) \le \exp\left(tc_T\sqrt{\log x}\right) \\ &q_t > \exp\left(c_t\sqrt{\log x}\right) \\ &q_t \le x^k \exp\left(-c_Q\sqrt{\log x}\right) \end{aligned}$$

we can conclude that

$$R(x) \ll_k x\exp\left(-c''_t\sqrt{\log x}\right)$$

for any $c_t'' > 0$ satisfying the conditions

$$2^{t-1}c_t'' < \begin{cases} c_Q - tc_T - \rho_t \\ c_t - tc_T - \rho_t \\ \rho_t \end{cases} \tag{4.46}$$

We assume $c_Q > c_t$ for every $t \in \{2, \dots, k\}$ so that the first condition in (4.46) can be removed. Inserting the estimate for $R(x)$ in (4.45) and (4.43) we have

$$S'_{1,2}(x) \ll_k \overline{q}x \exp\left(-c_t'' \sqrt{\log x}\right)$$

and

$$\begin{aligned} S_{1,2}(x) &\ll_k \overline{q}x \exp\left(-c'' \sqrt{\log x}\right)\left(1 + \exp\left((c_{Q'} + kc_T)\sqrt{\log x}\right)\right) \\ &\ll_k x \exp\left(-\left(c_t'' - c_{Q'} - kc_T - \sum_{i=t+1}^{k} c_i\right)\sqrt{\log x}\right). \end{aligned}$$

If we assume that

$$c_t'' - c_{Q'} - kc_T - \sum_{i=t+1}^{k} c_i > 0, \tag{4.47}$$

then the hypotheses of Theorem 3.7 are satisfied and we can conclude that

$$S_f(x;P) \ll_k \frac{x}{\log^{1/4} x} \log\log x. \tag{4.48}$$

To complete the analysis of Case C, we need to verify that the system of conditions (4.41), (4.46) and (4.47) admits a solution; this is done in Lemma 4.12.
Then, the Theorem follows from the results in (4.40), (4.42) and (4.48). □

We remark that the estimate (4.37) depends on the degree of the polynomial, but is uniform in the coefficients; hence, as we did in Corollary 4.9, we can extend the result to any arithmetic progression, i.e.

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod \nu}} \mu_f(n) e(P(n)) \ll_k \frac{x}{\log^{1/4} x} \log\log x$$

where ν is a positive integer and $0 \leq a < \nu$.

To conclude the proof of Theorem 4.11 we need the following fact.

Lemma 4.12. *Let k be an integer greater than two and $c > 0$ a given real constant; the system of linear inequalities*

$$\begin{cases} \alpha, \beta_t, \gamma_t, \delta, \lambda_t > 0 & 2 \leq t \leq k \\ \alpha + \sum_{j=2}^{k} \beta_j < c & \\ 2^{t-1}\gamma_t < \min \begin{cases} \beta_t - t\delta - \lambda_t \\ \lambda_t \end{cases} & 2 \leq t \leq k-1 \\ \gamma_t - \alpha - k\delta - \sum_{j=t+1}^{k} \beta_j > 0 & 2 \leq t \leq k-1 \end{cases} \tag{4.49}$$

admits infinitely many solutions.

Proof. Since all the inequalities are strict, it is enough to prove the existence of a solution to conclude that there are infinitely many.

It is easy to see that the second and the third set of conditions imply that $\beta_2 > \beta_3 > \ldots > \beta_k$. We begin choosing $\lambda_t = \beta_t/2$, so that the system becomes

$$\begin{cases} \alpha + \sum_{j=2}^{k} \beta_j < c \\ 2^{t-1}\gamma_t < \beta_t/2 - t\delta & 2 \le t \le k-1 \\ \gamma_t - \alpha - k\delta - \sum_{j=t+1}^{k} \beta_j > 0 & 2 \le t \le k-1. \end{cases} \tag{4.50}$$

Assuming $t\delta < \gamma_t/4$, we have that

$$\gamma_t = \frac{\beta_t}{2^{t+1}}$$

satisfies the second inequality, so that the we can consider a new system

$$\begin{cases} \alpha + \sum_{j=2}^{k} \beta_j < c \\ \beta_t/2^{t+1} - \alpha - k\delta - \sum_{j=t+1}^{k} \beta_j > 0 & 2 \le t \le k-1. \end{cases} \tag{4.51}$$

By considering the last inequality for $t = k-1$, we get

$$\frac{\beta_{k-1}}{2^k} - \alpha - k\delta - \beta_k > 0$$

which is satisfied by taking

$$\alpha = k\delta = \beta_k = \frac{\beta_{k-1}}{2^{k+4}}$$

We remark that, with this choice, $t\delta < k\delta = \beta_{k-1}/2^{k+4} < \gamma_t/4$. Now, by considering $t = k-1$ we were able to express β_k as a function of β_{k-1}; we can reiterate the process by taking $t = k-2, k-3, \ldots, 2$, so that we can establish a condition involving β_t and β_{t+1}. In particular we can choose

$$\beta_t = \frac{\beta_{t-1}}{2^{t+4}}$$

for $3 \le t \le k-2$. Finally, we need to determine a value for β_2 in order to satisfy the last condition to be considered, which is

$$\alpha + \sum_{j=2}^{k} \beta_j < c.$$

With the choices made so far, it is sufficient to take $\beta_2 = c/2$.

We conclude remarking that in the solution we found all the variables can be expressed as a ratio with numerator c and denominator a suitable power of 2 with exponent depending on k and not exceeding $k(k+1)$. □

In the proof of Theorem 4.11 we use this Lemma with:

$$\begin{cases} \alpha = c_{Q'}, \\ \beta_t = c_t \qquad 2 \leq t \leq k, \\ \gamma_t = c''_t, \\ \delta = c_T, \\ \lambda_t = \rho_t \qquad 2 \leq t \leq k-1. \end{cases}$$

In the last part of this chapter we will show that the estimate for $S_f(x,P)$ proved in Theorem 4.11 is also valid for the sum $S'_f(x,P)$ defined as

$$S'_f(x,P) = \sum_{n \leq x} a_f(n) e(P(n)).$$

Again, we consider polynomials of degree $k \geq 3$, since for lower degrees much stronger bounds can be proved. For polynomials of degree at most 2 we have already recalled in Lemma 4.6 that the estimate

$$S'_f(x,P) \ll_\varepsilon x^{7/8+\varepsilon} \tag{4.52}$$

holds for every $\varepsilon > 0$. Actually, when we consider linear polynomial, the exponent in (4.52) can be lowered.

Lemma 4.13. *For every $\alpha \in \mathbb{R}$*

$$\sum_{n \leq x} a_f(n) e(\alpha n) \ll_f x^{1/2} \log x. \tag{4.53}$$

This was originally proved by Wilton [47] and then the estimate was improved by Jutila [20] who removed the factor $\log x$; it is known that the exponent $1/2$ in (4.53) is optimal if one requires uniformity in α. Moreover, using this uniformity, we immediately deduce that the same estimate in (4.53) holds if we consider the sum over any arithmetic progression.

Now, let us adapt the proof of Theorem 4.11 to deal with the sum $S'_f(x,P)$; in Case A we can just do the same computations with $\mu_f(n)$ replaced by $a_f(n)$ and use Lemma 4.13 instead of Corollary 4.9 in (4.39). Then we get

$$S'_f(x,P) \ll x^{1/2} \exp\left(C\sqrt{\log x}\right)$$

for some positive constant C; this is much stronger than what we need and we do not even need to consider condition (4.41).

In Case B and Case C the proof does not need any modification because, as we already stated, both $a_f(n)$ and $\mu_f(n)$ satisfy the hypothesis of Theorem 3.7. Hence, we have proved that

$$S'_f(x;P) \ll_k \frac{x}{\log^{1/4} x)} \log\log x.$$

In fact, we are able to improve this result applying the same ideas of Theorem 4.11 with a different setting.

Theorem 4.14. *Let $P(t)$ be a real polynomial of degree $k \geq 3$; then,*

$$S'_f(x;P) \ll_k \frac{x}{\log^{1/2} x} \log\log x. \tag{4.54}$$

Proof. We just give a sketch of the proof, since the computations needed are formally the same as in Theorem 4.11. In particular, what we do is to replace any occurrence of $\sqrt{\log x}$ with $\log x$.

We begin with Diophantine approximation; the original setting $Q' = x^k \exp\left(-c_{Q'}\sqrt{\log x}\right)$ and $Q = x^k \exp\left(-c_Q\sqrt{\log x}\right)$ becomes $Q' = x^k x^{-c_{Q'}}$ and $Q = x^k x^{-c_Q}$. Let $a_i, q_i \in \mathbb{Z}$ with $(a_i, q_i) = 1$ be such that

$$\begin{aligned} \left|\alpha_k - \frac{a_k}{q_k}\right| &\leq \frac{1}{q_k Q'} \\ \left|\alpha_i - \frac{a_i}{q_i}\right| &\leq \frac{1}{q_i Q} \qquad i = 2, \ldots, k-1. \end{aligned}$$

In Case A, the new condition on the denominators becomes $q_j \leq x^{c_j}$ for $2 \leq j \leq k$ and for some positive small constant c_j; applying partial summation and Lemma 4.13 in the version improved by Jutila we get

$$S'_f(x,P) \ll_k q x^{1/2} x^{c_{Q'}} \leq x^{\delta} \tag{4.55}$$

where $q = \mathrm{lcm}\,(q_2, \ldots, q_k)$ and

$$\delta = \frac{1}{2} + c_{Q'} + \sum_{j=2}^{k} c_j.$$

The condition (4.41) is naturally replaced by $\delta < 1$, i.e.

$$\frac{1}{2} - \left(c_{Q'} + \sum_{j=2}^{k} c_j\right) > 0.$$

In Case B and Case C we can see where the improvement in (4.54) comes from. In both cases we set $T = c_T \log x$ for some positive small constant c_T; applying Theorem 3.7 with this choice we essentially get a saving of the order of $\log^{1/2} x$ instead of $\log^{1/4} x$ which was obtained with the setting $T = c_T\sqrt{\log x}$

Finally, we remark that, since the computations are formally the same, the system of inequalities that we find here is exactly the same as in (4.41), (4.46) and (4.47) of Theorem 4.11, but with the constant c replaced by $1/2$. □

A Appendix

A.1 Vaughan's Identity

We recall here the analogue of Vaughan's identity for $L(s,f)^{-1}$. We start from the identity

$$L(s,f)^{-1} = 2G(s) - L(s,f)G(s)^2 + \left(L(s,f)^{-1} - G(s)\right)(1 - L(s,f)G(s)) \tag{A.1}$$

valid in the region of the complex plane where $L(s,f), L(s,f)^{-1}$ and $G(s)$ are defined. Let $\{\xi(n)\}$ be a sequence of complex numbers and $Y > 1$; by taking

$$G(s) = \sum_{n \le Y} \frac{\mu_f(n)}{n^s},$$

and considering (A.1) for $\Re s > 1$ we conclude that

$$\sum_{n \le x} \mu_f(n)\xi(n) = 2S_1 - S_2 - S_3$$

where

$$\begin{aligned} S_1 &= \sum_{n \le Y} \mu_f(n)\xi(n), \\ S_2 &= \sum_{n_1 \le Y^2} A_f(n_1) \sum_{n_2 \le x/n_1} a_f(n_2)\xi(n_1 n_2), \\ S_3 &= \sum_{Y < n_1 \le x/Y} \sum_{Y < n_2 < x/n_1} \mu_f(n_1) B_f(n_2) \xi(n_1 n_2) \end{aligned}$$

and

$$A_f(n) = \sum_{\substack{n_1 n_2 = n \\ n_i \le Y}} \mu_f(n_1)\mu_f(n_2), \qquad B_f(n) = \sum_{\substack{n_1 n_2 = n \\ n_2 \le Y}} a_f(n_1)\mu_f(n_2).$$

A.2 Exponential sums with polynomials

Lemma A.1. *Let c be a positive constant, $\alpha \in \mathbb{T}$ and consider $a, q \in \mathbb{N}, (a,q) = 1$ such that*

$$\left|\alpha - \frac{a}{q}\right| \le \frac{c}{q^2}.$$

Then, for any $H, N \ge 1$,

$$\sum_{n=1}^{H} \min\left(N, \frac{1}{\|\alpha n\|}\right) \ll c\left(\frac{NH}{q} + H\log q + N + q\log q\right)$$

where $\|z\|$ is the distance from z to the nearest integer.

Proof. Let $M \geq 1$ be an integer; for $y \in \mathbb{R}$ we define the function $S(y)$ as

$$S(y) = \left| \left\{ M < n \leq M + q \ \mid \ \|n\alpha - y\| \leq \frac{1}{2q} \right\} \right|.$$

For every integer m in $[1, q/2]$, if n is counted in $S(m/q)$ (or in $S((q-m)/q)$), from the definition we have that

$$\|n\alpha\| > \frac{m}{q} - \frac{1}{2q} \gg \frac{m}{q};$$

then

$$\sum_{n=M+1}^{M+q} \min\left(N, \frac{1}{\|\alpha n\|}\right) \ll N \cdot S(0) + \sum_{m=1}^{q/2} \frac{q}{m} \left(S\left(\frac{m}{q}\right) + S\left(\frac{q-m}{q}\right)\right). \tag{A.2}$$

Now we need to prove that $S(y) \ll c$; with a change of variables we write

$$S(M\alpha + \nu) = \left| \left\{ 1 \leq m \leq q \ \mid \ \|m\alpha - \nu\| \leq \frac{1}{2q} \right\} \right|.$$

If $\alpha = a/q + \delta$ with $|\delta| \leq c/q^2$ then the condition above implies that

$$\left\| m\frac{a}{q} - \nu \right\| \leq \|m\alpha - \nu\| + \|m\delta\| \leq \frac{1}{2q} + \frac{c}{q} \leq \frac{2c}{q}.$$

Since in the interval $\left[\nu - \frac{2c}{q}, \nu + \frac{2c}{q}\right]$ there are at most $4c+1$ fractions of the form r/q and ma/q must be one of them, we can conclude that $S(M\alpha + \nu) \ll c$ uniformly in ν which implies that $S(y) \ll c$ uniformly in y.

Inserting this estimate in (A.2) we have

$$\sum_{n=M+1}^{M+q} \min\left(N, \frac{1}{\|\alpha n\|}\right) \ll c\,(N + q \log q).$$

Applying this estimate to $(H/q + 1)$ blocks of length $\leq q$ the Lemma is proved. □

The following Lemma is a generalization of a well-known result due to Weyl (see, for example, Montgomery [28], Chapter 3, Theorem 2) which asserts that, given a real valued polynomial $P(t)$ of degree $k \geq 1$ with leading coefficient α, the estimate

$$\sum_{n \leq x} e(P(n)) \ll_k x^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{x} + \frac{q}{x^k} \log q \right)^{\kappa}$$

holds for $\kappa = 2^{1-k}$ and a, q coprime integers such that $|\alpha - a/q| \leq q^{-2}$. We are able to weaken the Diophantine condition (as we did in the previous Lemma) and, more importantly, we remove the factor x^{ε}; to do this we rely on a technique used by Perelli and Zaccagnini in [33].

Lemma A.2. *Let α, a, q and c be as in the previous Lemma and $P(t) = \alpha t^k + \alpha_{k-1} t^{k-1} + \ldots + \alpha_1 t$ be a polynomial of degree $k \geq 1$; then, for any $Z > 0$, we have*

$$S := \sum_{n \leq x} e(P(n)) \ll_k x \left(\frac{cZ}{q} + \frac{cZ}{x} \log q + cZ \frac{q}{x^k} \log q + \frac{\log^{b(k)} x}{Z} \right)^{\kappa} \tag{A.3}$$

where $b(k) = k^2 - 2k$ and $\kappa = 2^{1-k}$.

Proof. Applying the standard Weyl method to deal with exponential sums we get

$$|S|^{2^{k-1}} \ll_k x^{2^{k-1}-1} + x^{2^{k-1}-k} \sum_{h_1,h_2,\dots,h_{k-1}} \min\left(x, \frac{1}{\|k!h_1h_2\cdots h_{k-1}\alpha\|}\right) \tag{A.4}$$

where each h_j satisfies the condition $1 \le h_j \le x$. Since $k!h_1h_2\cdots h_{k-1} \in \left[1, k!x^{k-1}\right]$ we have

$$S' := \sum_{h_1,h_2,\dots,h_{k-1}} \min\left(x, \frac{1}{\|k!h_1h_2\cdots h_{k-1}\alpha\|}\right) \le \sum_{h\le k!x^{k-1}} d_{k-1}(h) \min\left(x, \frac{1}{\|h\alpha\|}\right). \tag{A.5}$$

Now, let us consider the set H^+ of the integers h up to $k!x^{k-1}$ such that $d_{k-1}(h) > Z$ and H^- its complementary set. Applying Lemma A.1 to H^- we have

$$\begin{aligned}\sum_{h\in H^-} d_{k-1}(h) \min\left(x, \frac{1}{\|h\alpha\|}\right) &\le Z \sum_{h\le k!x^{k-1}} \min\left(x, \frac{1}{\|h\alpha\|}\right) \\ &\ll cZ\left(\frac{x^k}{q} + x^{k-1}\log q + x + q\log q\right)\end{aligned}$$

while for H^+ we write

$$\begin{aligned}\sum_{h\in H^+} d_{k-1}(h) \min\left(x, \frac{1}{\|h\alpha\|}\right) &\le \frac{1}{Z} \sum_{h\le k!x^{k-1}} d_{k-1}(h)^2 \min\left(x, \frac{1}{\|h\alpha\|}\right) \\ &\le \frac{x}{Z} \sum_{h\le k!x^{k-1}} d_{k-1}(h)^2 \ll_k \frac{x^k}{Z}\log^{b(k)} x,\end{aligned}$$

where $b(k) = k^2 - 2k$ (for the properties of the generalized divisor function we refer to Linnik's book, [21], Chapter 1). Collecting the results and inserting them in (A.5) we get

$$S' \ll_k x^k \left(\frac{cZ}{q} + \frac{cZ}{x}\log q + cZ\frac{q}{x^k}\log q + \frac{\log^{b(k)} x}{Z}\right).$$

Finally, the Lemma follows from (A.4). □

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