Rota-Baxter operators on BiHom-associative algebras and related structures

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Abstract

The purpose of this paper is to study Rota-Baxter operators for BiHom-associative algebras. Moreover, we introduce and discuss the properties of the notions of BiHom-(tri)dendriform algebra, BiHom-Zinbiel algebra and BiHom-quadri-algebra. We construct the free Rota-Baxter BiHom-associative algebra and present some observations about categories and functors related to Rota-Baxter structures.

Keywords: Rota-Baxter operator; BiHom-associative algebra; BiHom-dendriform algebra

Introduction

The first instance of Hom-type algebras appeared in the Physics literature when looking for quantum deformations of some algebras of vector fields, like Witt and Virasoro algebras, in connection with oscillator algebras ([1, 21]). A quantum deformation consists of replacing the usual derivation by a σ-derivation. It turns out that the algebras obtained in this way no longer satisfy the Jacobi identity but they satisfy a modified version involving a homomorphism. These algebras were called Hom-Lie algebras and studied by Hartwig, Larsson and Silvestrov in [20, 22]. The Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced in [30], where it is shown that the commutator bracket defined by the multiplication in a Hom-associative algebra leads naturally to a Hom-Lie algebra. The adjoint functor from the category of Hom-Lie algebras to the category of Hom-associative algebras and the enveloping algebra were constructed in [37]. A categorical approach to Hom-type algebras was considered in [9]. A generalization has been given in [18], where a construction of a Hom-category including a group action led to concepts of BiHom-type algebras. Hence, BiHom-associative algebras and BiHom-Lie algebras, involving two linear maps (called structure maps), were introduced. The main axioms for these types of algebras (BiHom-associativity, BiHom-skew-symmetry and BiHom-Jacobi condition) were dictated by categorical considerations.

Rota-Baxter operators first appeared in G. Baxter's work in probability and the study of fluctuation theory ([7]). Afterwards, Rota-Baxter algebras were intensively studied by G. C. Rota ([34, 35]) in connection with combinatorics. Rota-Baxter operators have appeared in a wide range of areas in pure and applied mathematics, for example under the name *multiplicativity constraint* in the work of A. Connes and D. Kreimer ([10]) about their Hopf algebra approach to renormalization of quantum field theory. This seminal work was followed by an important development of the theory of Rota-Baxter algebras and their connections to other algebraic structures, see for example [2, 4, 5, 11, 12, 14, 15, 16, 19]. In the context of Lie algebras, Rota-Baxter operators were introduced independently by Belavin and Drinfeld ([8]) and Semenov-Tian-Shansky ([36]), in connection with solutions of the (modified) classical Yang-Baxter equation. Motivated by K-theory, Loday ([24]) introduced dendriform algebras, that dichotomize an associative multiplication. It turns out that dendriform algebras are connected to several areas in mathematics and physics. Moreover, Rota-Baxter algebras are related to dendriform algebras via a pair of adjoint functors ([11, 13]).

Rota-Baxter operators on associative or Lie algebras have been intensively studied. Recently, Rota-Baxter operators on other types of algebras started to be investigated. For instance, Rota-Baxter operators on flexible, alternative, Leibniz and Malcev algebras have been studied in [17] and [27], while Rota-Baxter operators on Hom-type algebras have been studied in [28, 31]. We begin here the study of Rota-Baxter operators on BiHom-associative algebras. The main question we wanted to answer is the following. Let $(A, \cdot, \alpha, \beta)$ be a BiHom-associative algebra, $R : A \to A$ a Rota-Baxter operator of weight λ and define a new multiplication on A by $a * b = R(a) \cdot b + a \cdot R(b) + \lambda a \cdot b$, for all $a, b \in A$; then under what circumstances is $(A, *, \alpha, \beta)$ a BiHom-associative algebra? As we will see, a sufficient condition is that R commutes with α and β. However, along the way, we will make a detailed study of various structures that may be related to Rota-Baxter operators on BiHom-associative algebras.

The paper is organized as follows. In Section 1, we recall the definitions of Rota-Baxter operators, different types of algebras (dendriform, Zinbiel, tridendriform, quadri-algebra) and the notion of BiHom-associative algebra. Moreover, we describe the construction called the Yau twist. In Section 2, we introduce the notions of BiHom-(tri)dendriform algebra, BiHom-Zinbiel algebra and BiHom-quadri-algebra. We discuss their properties, Yau twists and some basic constructions. In Section 3, we deal with Rota-Baxter structures for BiHom-associative algebras. We provide several concrete examples (obtained by a computer algebra system) of Rota-Baxter operators on some BiHom-associative algebras, then we discuss some general properties and describe some key constructions. We construct, in Section 4, the free Rota-Baxter BiHom-associative algebra and give some observations about categories and functors related to Rota-Baxter structures. Finally, a connection between Rota-Baxter operators on BiHom-associative algebras and so-called weak BiHom-pseudotwistors is provided in Section 5.

1 Preliminaries

We work over a base field $\Bbbk$. All algebras, linear spaces etc. will be over $\Bbbk$; unadorned $\otimes$ means $\otimes_{\Bbbk}$. We denote by ${}_{\Bbbk}\mathcal{M}$ the category of linear spaces over $\Bbbk$. Unless otherwise specified, the algebras (associative or not) that will appear in what follows are *not* supposed to be unital, and a multiplication $\mu : A \otimes A \to A$ on a linear space A is denoted by juxtaposition: $\mu(v \otimes v') = vv'$. For the composition of two maps f and g, we write either $g \circ f$ or simply gf. For the identity map on a linear space A we use the notation id_A.

Definition 1.1 *Let A be a linear space and $\mu : A \otimes A \to A$, $\mu(x \otimes y) = xy$, for all $x, y \in A$, a linear multiplication on A and let $\lambda \in \Bbbk$. A Rota-Baxter operator of weight λ for (A, μ) is a linear map $R : A \to A$ satisfying the so-called Rota-Baxter condition*

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy), \quad \forall\, x, y \in A. \tag{1.1}$$

In this case, if we define on A a new multiplication by $x * y = xR(y) + R(x)y + \lambda xy$, for all $x, y \in A$, then $R(x * y) = R(x)R(y)$, for all $x, y \in A$, and R is a Rota-Baxter operator of weight λ for $(A, *)$. If (A, μ) is associative then $(A, *)$ is also associative.

Definition 1.2 *([24]) A dendriform algebra is a triple $(A, \prec, \succ)$ consisting of a linear space A and two linear operations $\prec, \succ : A \otimes A \to A$ satisfying the conditions (for all $x, y, z \in A$):*

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z), \tag{1.2}$$
$$(x \succ y) \prec z = x \succ (y \prec z), \tag{1.3}$$
$$x \succ (y \succ z) = (x \prec y + x \succ y) \succ z. \tag{1.4}$$

A morphism $f : (A, \prec, \succ) \to (A', \prec', \succ')$ of dendriform algebras is a linear map $f : A \to A'$ satisfying $f(x \prec y) = f(x) \prec' f(y)$ and $f(x \succ y) = f(x) \succ' f(y)$, for all $x, y \in A$.

A dendriform algebra $(A, \prec, \succ)$ is called commutative if $x \prec y = y \succ x$, for all $x, y \in A$.

Definition 1.3 *([24]) A (right) Zinbiel algebra is an algebra (A, μ) for which*

$$(xy)z = x(yz) + x(zy), \quad \forall\, x, y, z \in A. \tag{1.5}$$

By [24], Zinbiel algebras are exactly commutative dendriform algebras, in the following sense. If (A, μ) is a Zinbiel algebra and we define $x \prec y = xy$ and $x \succ y = yx$, for $x, y \in A$, then $(A, \prec, \succ)$ is a commutative dendriform algebra. Conversely, if $(A, \prec, \succ)$ is a commutative dendriform algebra and we define $\mu(x \otimes y) = x \prec y$, for $x, y \in A$, then (A, μ) is a Zinbiel algebra.

Definition 1.4 *([25]) A tridendriform algebra is a 4-tuple $(A, \prec, \succ, \cdot)$, where A is a linear space and $\prec, \succ, \cdot : A \otimes A \to A$ are linear operations satisfying the conditions (for all $x, y, z \in A$):*

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z + y \cdot z), \tag{1.6}$$
$$(x \succ y) \prec z = x \succ (y \prec z), \tag{1.7}$$
$$x \succ (y \succ z) = (x \prec y + x \succ y + x \cdot y) \succ z, \tag{1.8}$$
$$x \cdot (y \succ z) = (x \prec y) \cdot z, \tag{1.9}$$
$$x \succ (y \cdot z) = (x \succ y) \cdot z, \tag{1.10}$$
$$x \cdot (y \prec z) = (x \cdot y) \prec z, \tag{1.11}$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z. \tag{1.12}$$

A morphism $f : (A, \prec, \succ, \cdot) \to (A', \prec', \succ', \cdot')$ of tridendriform algebras is a linear map $f : A \to A'$ satisfying $f(x \prec y) = f(x) \prec' f(y)$, $f(x \succ y) = f(x) \succ' f(y)$ and $f(x \cdot y) = f(x) \cdot' f(y)$, for all $x, y \in A$.

Definition 1.5 *([2]) A quadri-algebra is a 5-tuple $(Q, \nwarrow, \swarrow, \nearrow, \searrow)$ consisting of a linear space Q and four linear maps $\nwarrow, \swarrow, \nearrow, \searrow : Q \otimes Q \to Q$ satisfying the axioms below (1.18)-(1.22) (for all $x, y, z \in Q$). In order to state them, consider the following operations:*

$$x \succ y := x \nearrow y + x \searrow y \tag{1.13}$$
$$x \prec y := x \nwarrow y + x \swarrow y, \tag{1.14}$$
$$x \vee y := x \searrow y + x \swarrow y, \tag{1.15}$$
$$x \wedge y := x \nearrow y + x \nwarrow y, \tag{1.16}$$

$$\begin{aligned} x * y : &= x \searrow y + x \nearrow y + x \swarrow y + x \nwarrow y \\ &= x \succ y + x \prec y = x \vee y + x \wedge y. \end{aligned} \tag{1.17}$$

The axioms are

$$(x \nwarrow y) \nwarrow z = x \nwarrow (y * z), \; (x \nearrow y) \nwarrow z = x \nearrow (y \prec z), \tag{1.18}$$
$$(x \wedge y) \nearrow z = x \nearrow (y \succ z), \; (x \swarrow y) \nwarrow z = x \swarrow (y \wedge z), \tag{1.19}$$
$$(x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), \; (x \vee y) \nearrow z = x \searrow (y \nearrow z), \tag{1.20}$$
$$(x \prec y) \swarrow z = x \swarrow (y \vee z), \; (x \succ y) \swarrow z = x \searrow (y \swarrow z), \tag{1.21}$$
$$(x * y) \searrow z = x \searrow (y \searrow z). \tag{1.22}$$

A morphism $f : (Q, \nwarrow, \swarrow, \nearrow, \searrow) \to (Q', \nwarrow', \swarrow', \nearrow', \searrow')$ of quadri-algebras is a linear map $f : Q \to Q'$ such that $f(x \nearrow y) = f(x) \nearrow' f(y), f(x \searrow y) = f(x) \searrow' f(y), f(x \nwarrow y) = f(x) \nwarrow' f(y)$ and $f(x \swarrow y) = f(x) \swarrow' f(y)$, for all $x, y \in Q$. As a consequence, we also have $f(x \succ y) = f(x) \succ' f(y)$, $f(x \prec y) = f(x) \prec' f(y)$, $f(x \vee y) = f(x) \vee' f(y)$, $f(x \wedge y) = f(x) \wedge' f(y)$ and $f(x * y) = f(x) *' f(y)$, for all $x, y \in Q$.

Definition 1.6 *([18]) A BiHom-associative algebra over $\mathbb{k}$ is a 4-tuple (A, μ, α, β), where A is a $\mathbb{k}$-linear space, $\alpha : A \to A$, $\beta : A \to A$ and $\mu : A \otimes A \to A$ are linear maps, with notation $\mu(x \otimes y) = xy$, for all $x, y \in A$, satisfying the following conditions, for all $x, y, z \in A$:*

$$\alpha \circ \beta = \beta \circ \alpha, \tag{1.23}$$
$$\alpha(xy) = \alpha(x)\alpha(y) \text{ and } \beta(xy) = \beta(x)\beta(y), \quad \text{(multiplicativity)} \tag{1.24}$$
$$\alpha(x)(yz) = (xy)\beta(z). \quad \text{(BiHom-associativity)} \tag{1.25}$$

We call α and β (in this order) the structure maps of A.

A morphism $f : (A, \mu_A, \alpha_A, \beta_A) \to (B, \mu_B, \alpha_B, \beta_B)$ of BiHom-associative algebras is a linear map $f : A \to B$ such that $\alpha_B \circ f = f \circ \alpha_A$, $\beta_B \circ f = f \circ \beta_A$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

If (A, μ) is an associative algebra, where $\mu : A \otimes A \to A$ is the multiplication, and $\alpha, \beta : A \to A$ are commuting algebra endomorphisms, then $(A, \mu \circ (\alpha \otimes \beta), \alpha, \beta)$ is a BiHom-associative algebra, denoted by $A_{(\alpha,\beta)}$ and called the Yau twist of (A, μ).

More generally, let $(D, \mu, \tilde{\alpha}, \tilde{\beta})$ be a BiHom-associative algebra and $\alpha, \beta : D \to D$ two multiplicative linear maps such that any two of the maps $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ commute. Then $(D, \mu \circ (\alpha \otimes \beta), \tilde{\alpha} \circ \alpha, \tilde{\beta} \circ \beta)$ is also a BiHom-associative algebra, denoted by $D_{(\alpha,\beta)}$.

Definition 1.7 *Let $(A, \mu_A, \alpha_A, \beta_A)$ be a BiHom-associative algebra and (M, α_M, β_M) a triple where M is a linear space and $\alpha_M, \beta_M : M \to M$ are commuting linear maps.*
(i) (M, α_M, β_M) is a left A-module if we have a linear map $A \otimes M \to M$, $a \otimes m \mapsto a \cdot m$, such that $\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m)$, $\beta_M(a \cdot m) = \beta_A(a) \cdot \beta_M(m)$ and

$$\alpha_A(a) \cdot (a' \cdot m) = (a \cdot a') \cdot \beta_M(m), \quad \forall\ a, a' \in A,\ m \in M. \tag{1.26}$$

(ii) (M, α_M, β_M) is a right A-module if we have a linear map $M \otimes A \to M$, $m \otimes a \mapsto m \cdot a$, such that $\alpha_M(m \cdot a) = \alpha_M(m) \cdot \alpha_A(a)$, $\beta_M(m \cdot a) = \beta_M(m) \cdot \beta_A(a)$ and

$$\alpha_M(m) \cdot (a \cdot a') = (m \cdot a) \cdot \beta_A(a'), \quad \forall\ a, a' \in A,\ m \in M. \tag{1.27}$$

(iii) If (M, α_M, β_M) is a left and right A-module, then M is called an A-bimodule if

$$\alpha_A(a) \cdot (m \cdot a') = (a \cdot m) \cdot \beta_A(a'), \quad \forall\ a, a' \in A,\ m \in M. \tag{1.28}$$

If (A, μ, α, β) is a BiHom-associative algebra, then (A, α, β) is an A-bimodule, with actions defined by $a \cdot m = am$ and $m \cdot a = ma$, for all $a, m \in A$.

Similarly to the classical (associative) case and to the Hom-case proved in [29], one can characterize bimodules in terms of split null extensions.

2 BiHom-(tri)dendriform algebras and BiHom-quadri-algebras

In this section, we introduce the notions of BiHom-dendriform algebra, BiHom-tridendriform algebra and BiHom-quadri-algebra, generalizing the Hom-type structures given in [3, 28, 31]. Moreover, we provide some key constructions.

Definition 2.1 *A BiHom-dendriform algebra is a 5-tuple $(A, \prec, \succ, \alpha, \beta)$ consisting of a linear space A and linear maps $\prec, \succ : A \otimes A \to A$ and $\alpha, \beta : A \to A$ satisfying the conditions*

$$\alpha \circ \beta = \beta \circ \alpha, \tag{2.1}$$
$$\alpha(x \prec y) = \alpha(x) \prec \alpha(y),\ \ \alpha(x \succ y) = \alpha(x) \succ \alpha(y), \tag{2.2}$$
$$\beta(x \prec y) = \beta(x) \prec \beta(y),\ \ \beta(x \succ y) = \beta(x) \succ \beta(y), \tag{2.3}$$
$$(x \prec y) \prec \beta(z) = \alpha(x) \prec (y \prec z + y \succ z), \tag{2.4}$$
$$(x \succ y) \prec \beta(z) = \alpha(x) \succ (y \prec z), \tag{2.5}$$
$$\alpha(x) \succ (y \succ z) = (x \prec y + x \succ y) \succ \beta(z), \tag{2.6}$$

for all $x, y, z \in A$. We call α and β (in this order) the structure maps of A.

A morphism $f : (A, \prec, \succ, \alpha, \beta) \to (A', \prec', \succ', \alpha', \beta')$ of BiHom-dendriform algebras is a linear map $f : A \to A'$ satisfying $f(x \prec y) = f(x) \prec' f(y)$ and $f(x \succ y) = f(x) \succ' f(y)$, for all $x, y \in A$, as well as $f \circ \alpha = \alpha' \circ f$ and $f \circ \beta = \beta' \circ f$.

A BiHom-dendriform algebra $(A, \prec, \succ, \alpha, \beta)$ is called commutative if

$$\beta(x) \prec \alpha(y) = \beta(y) \succ \alpha(x), \quad \forall\, x, y \in A. \tag{2.7}$$

Proposition 2.2 *Let $(A, \prec, \succ)$ be a dendriform algebra and $\alpha, \beta : A \to A$ two commuting dendriform algebra endomorphisms. Define $\prec_{(\alpha,\beta)}, \succ_{(\alpha,\beta)} : A \otimes A \to A$ by*

$$x \prec_{(\alpha,\beta)} y = \alpha(x) \prec \beta(y) \quad \text{ and } \quad x \succ_{(\alpha,\beta)} y = \alpha(x) \succ \beta(y),$$

for all $x, y \in A$. Then $A_{(\alpha,\beta)} := (A, \prec_{(\alpha,\beta)}, \succ_{(\alpha,\beta)}, \alpha, \beta)$ is a BiHom-dendriform algebra, called the Yau twist of A. Moreover, assume that $(A', \prec', \succ')$ is another dendriform algebra and $\alpha', \beta' : A' \to A'$ are two commuting dendriform algebra endomorphisms and $f : A \to A'$ is a morphism of dendriform algebras satisfying $f \circ \alpha = \alpha' \circ f$ and $f \circ \beta = \beta' \circ f$. Then $f : A_{(\alpha,\beta)} \to A'_{(\alpha',\beta')}$ is a morphism of BiHom-dendriform algebras.

Proof. We only prove (2.4) and leave the rest to the reader. By using the formulae for $\prec_{(\alpha,\beta)}$ and $\succ_{(\alpha,\beta)}$ and the fact that α and β are commuting dendriform algebra endomorphisms, one can compute, for all $x, y, z \in A$:

$$\begin{aligned}
(x \prec_{(\alpha,\beta)} y) \prec_{(\alpha,\beta)} \beta(z) &= (\alpha^2(x) \prec \alpha\beta(y)) \prec \beta^2(z),\\
\alpha(x) \prec_{(\alpha,\beta)} (y \prec_{(\alpha,\beta)} z) &= \alpha^2(x) \prec (\alpha\beta(y) \prec \beta^2(z)),\\
\alpha(x) \prec_{(\alpha,\beta)} (y \succ_{(\alpha,\beta)} z) &= \alpha^2(x) \prec (\alpha\beta(y) \succ \beta^2(z)).
\end{aligned}$$

Thus, (2.4) follows from (1.2) applied to the elements $\alpha^2(x)$, $\alpha\beta(y)$, $\beta^2(z)$. $\square$

Remark 2.3 *More generally, let $(A, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra and $\tilde{\alpha}, \tilde{\beta} : A \to A$ two morphisms of BiHom-dendriform algebras such that any two of the maps $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ commute. Define new multiplications on A by $x \prec' y = \tilde{\alpha}(x) \prec \tilde{\beta}(y)$ and $x \succ' y = \tilde{\alpha}(x) \succ \tilde{\beta}(y)$, for all $x, y \in A$. Then one can prove that $(A, \prec', \succ', \alpha \circ \tilde{\alpha}, \beta \circ \tilde{\beta})$ is a BiHom-dendriform algebra.*

We introduce now the BiHom-analogue of Zinbiel algebras.

Definition 2.4 *A BiHom-Zinbiel algebra is a 4-tuple $(A, \cdot, \alpha, \beta)$ where $(A, \cdot)$ is an algebra, $\alpha, \beta : A \to A$ are commuting algebra maps and the following relations are satisfied, for all $x, y, z \in A$:*

$$(x \cdot \beta(y)) \cdot \alpha\beta(z) = \alpha(x) \cdot (\beta(y) \cdot \alpha(z)) + \alpha(x) \cdot (\beta(z) \cdot \alpha(y)), \tag{2.8}$$

$$(\beta(y) \cdot \alpha(x)) \cdot \alpha\beta(z) = (\beta(y) \cdot \beta(z)) \cdot \alpha^2(x). \tag{2.9}$$

We can obtain examples of BiHom-Zinbiel algebras by twisting Zinbiel algebras:

Proposition 2.5 *Let (A, μ) be a Zinbiel algebra and $\alpha, \beta : A \to A$ two commuting algebra maps. Define a new multiplication on A by $x \cdot y = \alpha(x)\beta(y)$, for all $x, y \in A$. Then $(A, \cdot, \alpha, \beta)$ is a BiHom-Zinbiel algebra, called the Yau twist of A.*

Proof. For $x, y, z \in A$ we compute:

$$\begin{aligned}
(x \cdot \beta(y)) \cdot \alpha\beta(z) &= (\alpha^2(x)\alpha\beta^2(y))\alpha\beta^2(z)\\
&\overset{(1.5)}{=} \alpha^2(x)(\alpha\beta^2(y)\alpha\beta^2(z)) + \alpha^2(x)(\alpha\beta^2(z)\alpha\beta^2(y))\\
&= \alpha^2(x)\beta(\alpha\beta(y)\alpha\beta(z)) + \alpha^2(x)\beta(\alpha\beta(z)\alpha\beta(y))\\
&= \alpha^2(x)\beta(\beta(y) \cdot \alpha(z)) + \alpha^2(x)\beta(\beta(z) \cdot \alpha(y))\\
&= \alpha(x) \cdot (\beta(y) \cdot \alpha(z)) + \alpha(x) \cdot (\beta(z) \cdot \alpha(y)),
\end{aligned}$$

$$\begin{aligned}
(\beta(y) \cdot \alpha(x)) \cdot \alpha\beta(z) &= (\alpha^2\beta(y)\alpha^2\beta(x))\alpha\beta^2(z)\\
&\overset{(1.5)}{=} (\alpha^2\beta(y)\alpha\beta^2(z))\alpha^2\beta(x)\\
&= \alpha(\alpha\beta(y)\beta^2(z))\beta(\alpha^2(x)) = (\beta(y) \cdot \beta(z)) \cdot \alpha^2(x),
\end{aligned}$$

finishing the proof. $\square$

Remark 2.6 *An immediate consequence of (2.8) is that*

$$(x \cdot \beta(y)) \cdot \alpha\beta(z) = (x \cdot \beta(z)) \cdot \alpha\beta(y), \quad \forall\, x, y, z \in A. \tag{2.10}$$

Remark 2.7 *In a BiHom-Zinbiel algebra $(A, \cdot, \alpha, \beta)$ with bijective α and β, the relation (2.9) is superfluous (being a consequence of (2.8)). Indeed, we can compute:*

$$\begin{aligned}
(\beta(y) \cdot \alpha(x)) \cdot \alpha\beta(z) \quad &= \quad (\beta(y) \cdot \beta(\alpha\beta^{-1}(x))) \cdot \alpha\beta(z) \\
&\overset{(2.10)}{=} \quad (\beta(y) \cdot \beta(z)) \cdot \alpha\beta(\alpha\beta^{-1}(x)) = (\beta(y) \cdot \beta(z)) \cdot \alpha^2(x), \quad \textit{q.e.d.}
\end{aligned}$$

Proposition 2.8 *Let $(A, \prec, \succ, \alpha, \beta)$ be a commutative BiHom-dendriform algebra. Define the operation $x \cdot y = x \prec y$, for all $x, y \in A$. Then $(A, \cdot, \alpha, \beta)$ is a BiHom-Zinbiel algebra.*

Proof. We compute:

$$\begin{aligned}
(x \cdot \beta(y)) \cdot \alpha\beta(z) \quad &= \quad (x \prec \beta(y)) \prec \alpha\beta(z) \\
&\overset{(2.4)}{=} \quad \alpha(x) \prec (\beta(y) \prec \alpha(z)) + \alpha(x) \prec (\beta(y) \succ \alpha(z)) \\
&\overset{(2.7)}{=} \quad \alpha(x) \prec (\beta(y) \prec \alpha(z)) + \alpha(x) \prec (\beta(z) \prec \alpha(y)) \\
&= \quad \alpha(x) \cdot (\beta(y) \cdot \alpha(z)) + \alpha(x) \cdot (\beta(z) \cdot \alpha(y)),
\end{aligned}$$

$$\begin{aligned}
(\beta(y) \cdot \alpha(x)) \cdot \alpha\beta(z) \quad &= \quad (\beta(y) \prec \alpha(x)) \prec \alpha\beta(z) \\
&\overset{(2.7)}{=} \quad (\beta(x) \succ \alpha(y)) \prec \alpha\beta(z) \\
&\overset{(2.5)}{=} \quad \alpha\beta(x) \succ (\alpha(y) \prec \alpha(z)) = \beta(\alpha(x)) \succ \alpha(y \prec z) \\
&\overset{(2.7)}{=} \quad \beta(y \prec z) \prec \alpha^2(x) = (\beta(y) \cdot \beta(z)) \cdot \alpha^2(x),
\end{aligned}$$

finishing the proof. □

The converse holds in the case of bijective structure maps.

Proposition 2.9 *Let $(A, \cdot, \alpha, \beta)$ be a BiHom-Zinbiel algebra such that α and β are bijective. Define new operations on A by $x \prec y = x \cdot y$ and $x \succ y = \beta\alpha^{-1}(y) \cdot \alpha\beta^{-1}(x)$, for all $x, y \in A$. Then $(A, \prec, \succ, \alpha, \beta)$ is a commutative BiHom-dendriform algebra.*

Proof. Obviously, A is commutative since $\beta(x) \succ \alpha(y) = \beta(y) \cdot \alpha(x) = \beta(y) \prec \alpha(x)$. Clearly, α and β are multiplicative with respect to $\prec$ and $\succ$. Now we prove (2.4), (2.5) and (2.6):

$$\begin{aligned}
(x \prec y) \prec \beta(z) \quad &= \quad (x \cdot y) \cdot \beta(z) = (x \cdot \beta(\beta^{-1}(y))) \cdot \alpha\beta(\alpha^{-1}(z)) \\
&\overset{(2.8)}{=} \quad \alpha(x) \cdot (y \cdot z) + \alpha(x) \cdot (\beta\alpha^{-1}(z) \cdot \alpha\beta^{-1}(y)) \\
&= \quad \alpha(x) \prec (y \prec z) + \alpha(x) \prec (y \succ z),
\end{aligned}$$

$$\begin{aligned}
(x \succ y) \prec \beta(z) \quad &= \quad (\beta\alpha^{-1}(y) \cdot \alpha\beta^{-1}(x)) \cdot \beta(z) \\
&= \quad (\beta(\alpha^{-1}(y)) \cdot \alpha(\beta^{-1}(x))) \cdot \alpha\beta(\alpha^{-1}(z)) \\
&\overset{(2.9)}{=} \quad (\beta\alpha^{-1}(y) \cdot \beta\alpha^{-1}(z)) \cdot \alpha^2\beta^{-1}(x)
\end{aligned}$$

$$= \quad \beta\alpha^{-1}(y \prec z) \cdot \alpha\beta^{-1}(\alpha(x)) = \alpha(x) \succ (y \prec z),$$

$$\begin{aligned}
\alpha(x) \succ (y \succ z) &= \alpha(x) \succ (\beta\alpha^{-1}(z) \cdot \alpha\beta^{-1}(y)) = (\beta^2\alpha^{-2}(z) \cdot y) \cdot \alpha^2\beta^{-1}(x) \\
&= (\beta^2\alpha^{-2}(z) \cdot \beta(\beta^{-1}(y))) \cdot \alpha\beta(\alpha\beta^{-2}(x)) \\
&\overset{(2.8)}{=} \beta^2\alpha^{-1}(z) \cdot (y \cdot \alpha^2\beta^{-2}(x)) + \beta^2\alpha^{-1}(z) \cdot (\alpha\beta^{-1}(x) \cdot \alpha\beta^{-1}(y)) \\
&= \beta^2\alpha^{-1}(z) \cdot (\beta\alpha^{-1}(\alpha\beta^{-1}(y)) \cdot \alpha\beta^{-1}(\alpha\beta^{-1}(x))) \\
&\quad +\beta^2\alpha^{-1}(z) \cdot \alpha\beta^{-1}(x \prec y) \\
&= \beta^2\alpha^{-1}(z) \cdot (\alpha\beta^{-1}(x) \succ \alpha\beta^{-1}(y)) + \beta^2\alpha^{-1}(z) \cdot \alpha\beta^{-1}(x \prec y) \\
&= \beta^2\alpha^{-1}(z) \cdot \alpha\beta^{-1}(x \succ y) + \beta^2\alpha^{-1}(z) \cdot \alpha\beta^{-1}(x \prec y) \\
&= (x \succ y) \succ \beta(z) + (x \prec y) \succ \beta(z),
\end{aligned}$$

finishing the proof. □

Definition 2.10 *A BiHom-tridendriform algebra is a 6-tuple $(A, \prec, \succ, \cdot, \alpha, \beta)$, where A is a linear space and $\prec, \succ, \cdot : A \otimes A \to A$ and $\alpha, \beta : A \to A$ are linear maps satisfying*

$$\alpha \circ \beta = \beta \circ \alpha, \tag{2.11}$$
$$\alpha(x \prec y) = \alpha(x) \prec \alpha(y),\ \alpha(x \succ y) = \alpha(x) \succ \alpha(y),\ \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \tag{2.12}$$
$$\beta(x \prec y) = \beta(x) \prec \beta(y),\ \beta(x \succ y) = \beta(x) \succ \beta(y),\ \beta(x \cdot y) = \beta(x) \cdot \beta(y), \tag{2.13}$$
$$(x \prec y) \prec \beta(z) = \alpha(x) \prec (y \prec z + y \succ z + y \cdot z), \tag{2.14}$$
$$(x \succ y) \prec \beta(z) = \alpha(x) \succ (y \prec z), \tag{2.15}$$
$$\alpha(x) \succ (y \succ z) = (x \prec y + x \succ y + x \cdot y) \succ \beta(z), \tag{2.16}$$
$$\alpha(x) \cdot (y \succ z) = (x \prec y) \cdot \beta(z), \tag{2.17}$$
$$\alpha(x) \succ (y \cdot z) = (x \succ y) \cdot \beta(z), \tag{2.18}$$
$$\alpha(x) \cdot (y \prec z) = (x \cdot y) \prec \beta(z), \tag{2.19}$$
$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z), \tag{2.20}$$

for all $x, y, z \in A$. We call α and β (in this order) the structure maps of A.

A morphism $f : (A, \prec, \succ, \cdot, \alpha, \beta) \to (A', \prec', \succ', \cdot', \alpha', \beta')$ of BiHom-tridendriform algebras is a linear map $f : A \to A'$ satisfying $f(x \prec y) = f(x) \prec' f(y)$, $f(x \succ y) = f(x) \succ' f(y)$ and $f(x \cdot y) = f(x) \cdot' f(y)$, for all $x, y \in A$, as well as $f \circ \alpha = \alpha' \circ f$ and $f \circ \beta = \beta' \circ f$.

A BiHom-dendriform algebra $(A, \prec, \succ, \alpha, \beta)$ is also a BiHom-tridendriform algebra, with the same operations $\prec$, $\succ$ and the same maps α, β and with the operation $\cdot$ defined by $x \cdot y = 0$ for all $x, y \in A$. Conversely, we have:

Proposition 2.11 *Let $(A, \prec, \succ, \cdot, \alpha, \beta)$ be a BiHom-tridendriform algebra. Then $(A, \prec', \succ', \alpha, \beta)$ is a BiHom-dendriform algebra, where the new operations $\prec'$ and $\succ'$ are defined by $x \prec' y = x \prec y + x \cdot y$ and $x \succ' y = x \succ y$, for all $x, y \in A$.*

Proof. The relations (2.1)-(2.3) for $\prec'$ and $\succ'$ are obvious. We check (2.4)-(2.6):

$$\begin{aligned}
(x \prec' y) \prec' \beta(z) &= (x \prec y + x \cdot y) \prec' \beta(z) \\
&= (x \prec y + x \cdot y) \prec \beta(z) + (x \prec y + x \cdot y) \cdot \beta(z)
\end{aligned}$$

$$
\begin{aligned}
&= (x \prec y) \prec \beta(z) + (x \cdot y) \prec \beta(z) \\
&\quad +(x \prec y) \cdot \beta(z) + (x \cdot y) \cdot \beta(z) \\
&\overset{(2.14),\,(2.17),\,(2.19),\,(2.20)}{=} \alpha(x) \prec (y \prec z + y \succ z + y \cdot z) + \alpha(x) \cdot (y \prec z) \\
&\quad +\alpha(x) \cdot (y \succ z) + \alpha(x) \cdot (y \cdot z) \\
&= \alpha(x) \prec (y \prec z + y \succ z + y \cdot z) \\
&\quad +\alpha(x) \cdot (y \prec z + y \succ z + y \cdot z) \\
&= \alpha(x) \prec' (y \prec z + y \succ z + y \cdot z) \\
&= \alpha(x) \prec' (y \prec' z + y \succ' z),
\end{aligned}
$$

$$
\begin{aligned}
(x \succ' y) \prec' \beta(z) &= (x \succ y) \prec \beta(z) + (x \succ y) \cdot \beta(z) \\
&\overset{(2.15),\,(2.18)}{=} \alpha(x) \succ (y \prec z) + \alpha(x) \succ (y \cdot z) \\
&= \alpha(x) \succ (y \prec z + y \cdot z) = \alpha(x) \succ' (y \prec' z),
\end{aligned}
$$

$$
\begin{aligned}
\alpha(x) \succ' (y \succ' z) &= \alpha(x) \succ (y \succ z) \\
&\overset{(2.16)}{=} (x \prec y + x \succ y + x \cdot y) \succ \beta(z) \\
&= (x \prec' y + x \succ' y) \succ' \beta(z),
\end{aligned}
$$

finishing the proof. $\square$

Proposition 2.12 *Let $(A, \prec, \succ, \cdot)$ be a tridendriform algebra and $\alpha, \beta : A \to A$ two commuting tridendriform algebra endomorphisms. Define $\prec_{(\alpha,\beta)}, \succ_{(\alpha,\beta)}, \cdot_{(\alpha,\beta)} : A \otimes A \to A$ by*

$$x \prec_{(\alpha,\beta)} y = \alpha(x) \prec \beta(y), \quad x \succ_{(\alpha,\beta)} y = \alpha(x) \succ \beta(y), \quad x \cdot_{(\alpha,\beta)} y = \alpha(x) \cdot \beta(y),$$

for all $x, y \in A$. Then $A_{(\alpha,\beta)} := (A, \prec_{(\alpha,\beta)}, \succ_{(\alpha,\beta)}, \cdot_{(\alpha,\beta)}, \alpha, \beta)$ is a BiHom-tridendriform algebra, called the Yau twist of A. Moreover, assume that $(A', \prec', \succ', \cdot')$ is another tridendriform algebra and $\alpha', \beta' : A' \to A'$ are two commuting tridendriform algebra endomorphisms and $f : A \to A'$ is a morphism of tridendriform algebras satisfying $f \circ \alpha = \alpha' \circ f$ and $f \circ \beta = \beta' \circ f$. Then $f : A_{(\alpha,\beta)} \to A'_{(\alpha',\beta')}$ is a morphism of BiHom-tridendriform algebras.

Proof. Similar to the proof of Proposition 2.2 and left to the reader. $\square$

Remark 2.13 *More generally, one can prove that, if $(A, \prec, \succ, \cdot, \alpha, \beta)$ is a BiHom-tridendriform algebra and $\tilde{\alpha}, \tilde{\beta} : A \to A$ are two morphisms of BiHom-tridendriform algebras such that any two of the maps $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ commute, and we define new multiplications on A by $x \prec' y = \tilde{\alpha}(x) \prec \tilde{\beta}(y)$, $x \succ' y = \tilde{\alpha}(x) \succ \tilde{\beta}(y)$ and $x \cdot' y = \tilde{\alpha}(x) \cdot \tilde{\beta}(y)$, for all $x, y \in A$, then $(A, \prec', \succ', \cdot', \alpha \circ \tilde{\alpha}, \beta \circ \tilde{\beta})$ is a BiHom-tridendriform algebra.*

Proposition 2.14 *Let $(A, \prec, \succ, \cdot, \alpha, \beta)$ be a BiHom-tridendriform algebra. Define a multiplication $* : A \otimes A \to A$ by $x * y = x \prec y + x \succ y + x \cdot y$, for all $x, y \in A$. Then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra.*

Proof. Obviously we have $\alpha(x * y) = \alpha(x) * \alpha(y)$ and $\beta(x * y) = \beta(x) * \beta(y)$, for all $x, y \in A$, by using (2.12) and (2.13). Now we compute, for $x, y, z \in A$:

$$\alpha(x) * (y * z) = \alpha(x) * (y \prec z) + \alpha(x) * (y \succ z) + \alpha(x) * (y \cdot z)$$

$$
\begin{aligned}
&= && \alpha(x) \prec (y \prec z) + \alpha(x) \succ (y \prec z) + \alpha(x) \cdot (y \prec z) \\
& && +\alpha(x) \prec (y \succ z) + \alpha(x) \succ (y \succ z) + \alpha(x) \cdot (y \succ z) \\
& && +\alpha(x) \prec (y \cdot z) + \alpha(x) \succ (y \cdot z) + \alpha(x) \cdot (y \cdot z) \\
&= && \alpha(x) \prec (y \prec z + y \succ z + y \cdot z) + \alpha(x) \succ (y \prec z) \\
& && +\alpha(x) \succ (y \succ z) + \alpha(x) \cdot (y \prec z) + \alpha(x) \cdot (y \succ z) \\
& && +\alpha(x) \succ (y \cdot z) + \alpha(x) \cdot (y \cdot z) \\
&\overset{(2.14)-(2.16)}{=} && (x \prec y) \prec \beta(z) + (x \succ y) \prec \beta(z) + (x \prec y) \succ \beta(z) \\
& && +(x \succ y) \succ \beta(z) + (x \cdot y) \succ \beta(z) + \alpha(x) \cdot (y \prec z) \\
& && +\alpha(x) \cdot (y \succ z) + \alpha(x) \succ (y \cdot z) + \alpha(x) \cdot (y \cdot z) \\
&\overset{(2.17)-(2.20)}{=} && (x \prec y) \prec \beta(z) + (x \succ y) \prec \beta(z) + (x \prec y) \succ \beta(z) \\
& && +(x \succ y) \succ \beta(z) + (x \cdot y) \succ \beta(z) + (x \cdot y) \prec \beta(z) \\
& && +(x \prec y) \cdot \beta(z) + (x \succ y) \cdot \beta(z) + (x \cdot y) \cdot \beta(z) \\
&= && (x \prec y + x \succ y + x \cdot y) \prec \beta(z) \\
& && +(x \prec y + x \succ y + x \cdot y) \succ \beta(z) \\
& && +(x \prec y + x \succ y + x \cdot y) \cdot \beta(z) \\
&= && (x * y) \prec \beta(z) + (x * y) \succ \beta(z) + (x * y) \cdot \beta(z) = (x * y) * \beta(z),
\end{aligned}
$$

finishing the proof. $\square$

Corollary 2.15 *Let $(A, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra. Define a multiplication $* : A \otimes A \to A$ by $x * y = x \prec y + x \succ y$. Then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra.*

Similarly to the characterization of dendriform algebras in terms of bimodules (see for instance [6], Section 6.3) and by using Corollary 2.15, we obtain the following characterization of BiHom-dendriform algebras:

Proposition 2.16 *Let A be a linear space, $\prec, \succ: A \otimes A \to A$ linear maps and $\alpha, \beta : A \to A$ two commuting linear maps that are multiplicative with respect to $\prec$ and $\succ$. Define $a * b = a \prec b + a \succ b$, for all $a, b \in A$. Then $(A, \prec, \succ, \alpha, \beta)$ is a BiHom-dendriform algebra if and only if $(A, *, \alpha, \beta)$ is a BiHom-associative algebra and (A, α, β) is a bimodule over $(A, *, \alpha, \beta)$, with actions $a \cdot m = a \succ m$ and $m \cdot a = m \prec a$, for all $a, m \in A$.*

We introduce now the BiHom version of quadri-algebras (for the Hom version see [3]).

Definition 2.17 *A BiHom-quadri-algebra is a 7-tuple $(Q, \nwarrow, \swarrow, \nearrow, \searrow, \alpha, \beta)$ consisting of a linear space Q and linear maps $\nwarrow, \swarrow, \nearrow, \searrow: Q \otimes Q \to Q$ and $\alpha, \beta : Q \to Q$ satisfying the axioms below (2.26)-(2.35) (for all $x, y, z \in Q$). To state them, consider the following operations:*

$$x \succ y := x \nearrow y + x \searrow y \tag{2.21}$$
$$x \prec y := x \nwarrow y + x \swarrow y, \tag{2.22}$$
$$x \vee y := x \searrow y + x \swarrow y, \tag{2.23}$$
$$x \wedge y := x \nearrow y + x \nwarrow y, \tag{2.24}$$

$$x * y : \; = \; x \searrow y + x \nearrow y + x \swarrow y + x \nwarrow y$$

$$= \quad x \succ y + x \prec y = x \vee y + x \wedge y. \tag{2.25}$$

The axioms are

$$\alpha \circ \beta = \beta \circ \alpha, \tag{2.26}$$
$$\alpha(x \nearrow y) = \alpha(x) \nearrow \alpha(y), \quad \alpha(x \searrow y) = \alpha(x) \searrow \alpha(y), \tag{2.27}$$
$$\alpha(x \nwarrow y) = \alpha(x) \nwarrow \alpha(y), \quad \alpha(x \swarrow y) = \alpha(x) \swarrow \alpha(y), \tag{2.28}$$
$$\beta(x \nearrow y) = \beta(x) \nearrow \beta(y), \quad \beta(x \searrow y) = \beta(x) \searrow \beta(y) \tag{2.29}$$
$$\beta(x \nwarrow y) = \beta(x) \nwarrow \beta(y), \quad \beta(x \swarrow y) = \beta(x) \swarrow \beta(y) \tag{2.30}$$
$$(x \nwarrow y) \nwarrow \beta(z) = \alpha(x) \nwarrow (y * z), \quad (x \nearrow y) \nwarrow \beta(z) = \alpha(x) \nearrow (y \prec z), \tag{2.31}$$
$$(x \wedge y) \nearrow \beta(z) = \alpha(x) \nearrow (y \succ z), \quad (x \swarrow y) \nwarrow \beta(z) = \alpha(x) \swarrow (y \wedge z), \tag{2.32}$$
$$(x \searrow y) \nwarrow \beta(z) = \alpha(x) \searrow (y \nwarrow z), \quad (x \vee y) \nearrow \beta(z) = \alpha(x) \searrow (y \nearrow z), \tag{2.33}$$
$$(x \prec y) \swarrow \beta(z) = \alpha(x) \swarrow (y \vee z), \quad (x \succ y) \swarrow \beta(z) = \alpha(x) \searrow (y \swarrow z), \tag{2.34}$$
$$(x * y) \searrow \beta(z) = \alpha(x) \searrow (y \searrow z). \tag{2.35}$$

A morphism $f : (Q, \nwarrow, \swarrow, \nearrow, \searrow, \alpha, \beta) \to (Q', \nwarrow', \swarrow', \nearrow', \searrow', \alpha', \beta')$ *of BiHom-quadri-algebras is a linear map* $f : Q \to Q'$ *satisfying* $f(x \nearrow y) = f(x) \nearrow' f(y), f(x \searrow y) = f(x) \searrow' f(y), f(x \nwarrow y) = f(x) \nwarrow' f(y)$ *and* $f(x \swarrow y) = f(x) \swarrow' f(y)$*, for all* $x, y \in Q$*, as well as* $f \circ \alpha = \alpha' \circ f$ *and* $f \circ \beta = \beta' \circ f$*.*

Proposition 2.18 *Let* $(Q, \nwarrow, \swarrow, \nearrow, \searrow)$ *be a quadri-algebra and* $\alpha, \beta : Q \to Q$ *two commuting quadri-algebra endomorphisms. Define* $\searrow_{(\alpha,\beta)}, \nearrow_{(\alpha,\beta)}, \swarrow_{(\alpha,\beta)}, \nwarrow_{(\alpha,\beta)} : Q \otimes Q \to Q$ *by*

$$x \searrow_{(\alpha,\beta)} y = \alpha(x) \searrow \beta(y), \qquad x \nearrow_{(\alpha,\beta)} y = \alpha(x) \nearrow \beta(y),$$
$$x \swarrow_{(\alpha,\beta)} y = \alpha(x) \swarrow \beta(y), \qquad x \nwarrow_{(\alpha,\beta)} y = \alpha(x) \nwarrow \beta(y),$$

for all $x, y \in Q$*. Then* $Q_{(\alpha,\beta)} := (Q, \nwarrow_{(\alpha,\beta)}, \swarrow_{(\alpha,\beta)}, \nearrow_{(\alpha,\beta)}, \searrow_{(\alpha,\beta)}, \alpha, \beta)$ *is a BiHom-quadri-algebra, called the Yau twist of Q. Moreover, assume that* $(Q', \nwarrow', \swarrow', \nearrow', \searrow')$ *is another quadri-algebra and* $\alpha', \beta' : Q' \to Q'$ *are two commuting quadri-algebra endomorphisms and* $f : Q \to Q'$ *is a morphism of quadri-algebras satisfying* $f \circ \alpha = \alpha' \circ f$ *and* $f \circ \beta = \beta' \circ f$*. Then* $f : Q_{(\alpha,\beta)} \to Q'_{(\alpha',\beta')}$ *is a morphism of BiHom-quadri algebras.*

Proof. We only prove (2.31) and leave the rest to the reader. We denote by $x \succ_{(\alpha,\beta)} y := x \nearrow_{(\alpha,\beta)} y + x \searrow_{(\alpha,\beta)} y$, $x \prec_{(\alpha,\beta)} y := x \nwarrow_{(\alpha,\beta)} y + x \swarrow_{(\alpha,\beta)} y$, $x \vee_{(\alpha,\beta)} y := x \searrow_{(\alpha,\beta)} y + x \swarrow_{(\alpha,\beta)} y$, $x \wedge_{(\alpha,\beta)} y := x \nearrow_{(\alpha,\beta)} y + x \nwarrow_{(\alpha,\beta)} y$ and $x *_{(\alpha,\beta)} y := x \searrow_{(\alpha,\beta)} y + x \nearrow_{(\alpha,\beta)} y + x \swarrow_{(\alpha,\beta)} y + x \nwarrow_{(\alpha,\beta)} y$, for all $x, y \in Q$. It is easy to get $x \succ_{(\alpha,\beta)} y = \alpha(x) \succ \beta(y), x \prec_{(\alpha,\beta)} y = \alpha(x) \prec \beta(y), x \vee_{(\alpha,\beta)} y = \alpha(x) \vee \beta(y), x \wedge_{(\alpha,\beta)} y = \alpha(x) \wedge \beta(y)$ and $x *_{(\alpha,\beta)} y = \alpha(x) * \beta(y)$ for all $x, y \in Q$. By using the fact that α and β are two commuting quadri-algebra endomorphisms, one can compute, for all $x, y, z \in Q$:

$$(x \nwarrow_{(\alpha,\beta)} y) \nwarrow_{(\alpha,\beta)} \beta(z) = (\alpha^2(x) \nwarrow \alpha\beta(y)) \nwarrow \beta^2(z),$$
$$(x \nearrow_{(\alpha,\beta)} y) \nwarrow_{(\alpha,\beta)} \beta(z) = (\alpha^2(x) \nearrow \alpha\beta(y)) \nwarrow \beta^2(z),$$
$$\alpha(x) \nwarrow_{(\alpha,\beta)} (y *_{(\alpha,\beta)} z) = \alpha^2(x) \nwarrow (\alpha\beta(y) * \beta^2(z)),$$
$$\alpha(x) \nearrow_{(\alpha,\beta)} (y \prec_{(\alpha,\beta)} z) = \alpha^2(x) \nearrow (\alpha\beta(y) \prec \beta^2(z)).$$

Thus, (2.31) follows from (1.18) applied to the elements $\alpha^2(x)$, $\alpha\beta(y)$, $\beta^2(z)$. $\square$

Remark 2.19 *More generally, let* $(Q, \nwarrow, \swarrow, \nearrow, \searrow, \alpha, \beta)$ *be a BiHom-quadri-algebra and* $\tilde{\alpha}, \tilde{\beta}: Q \to Q$ *two morphisms of BiHom-quadri-algebras such that any two of the maps* $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ *commute. Define new multiplications on* Q *by* $x \nearrow' y = \tilde{\alpha}(x) \nearrow \tilde{\beta}(y)$, $x \searrow' y = \tilde{\alpha}(x) \searrow \tilde{\beta}(y)$, $x \nwarrow' y = \tilde{\alpha}(x) \nwarrow \tilde{\beta}(y)$ *and* $x \swarrow' y = \tilde{\alpha}(x) \swarrow \tilde{\beta}(y)$*. Then* $(Q, \nwarrow', \swarrow', \nearrow', \searrow', \alpha \circ \tilde{\alpha}, \beta \circ \tilde{\beta})$ *is a BiHom-quadri-algebra.*

Remark 2.20 *Let* $(Q, \nwarrow, \swarrow, \nearrow, \searrow, \alpha, \beta)$ *be a BiHom-quadri-algebra. From (2.31)-(2.35) we obtain, for all* $x, y, z \in Q$:

$$(x \prec y) \prec \beta(z) = \alpha(x) \prec (y * z), \; (x \succ y) \prec \beta(z) = \alpha(x) \succ (y \prec z),$$
$$(x * y) \succ \beta(z) = \alpha(x) \succ (y \succ z).$$

Thus, $(Q, \prec, \succ, \alpha, \beta)$ *is a BiHom-dendriform algebra. By analogy with [2], we denote it by* Q_h *and call it the horizontal BiHom-dendriform algebra associated to* Q.

Also from (2.31)-(2.35) we obtain, for all $x, y, z \in Q$:

$$(x \wedge y) \wedge \beta(z) = \alpha(x) \wedge (y * z), \; (x \vee y) \wedge \beta(z) = \alpha(x) \vee (y \wedge z),$$
$$(x * y) \vee \beta(z) = \alpha(x) \vee (y \vee z).$$

Thus, $(Q, \wedge, \vee, \alpha, \beta)$ *is a BiHom-dendriform algebra, which, again by analogy with [2], is denoted by* Q_v *and is called the vertical BiHom-dendriform algebra associated to* Q.

From Corollary 2.15 we immediately obtain:

Corollary 2.21 *Let* $(Q, \nwarrow, \swarrow, \nearrow, \searrow, \alpha, \beta)$ *be a BiHom-quadri-algebra. Then* $(Q, *, \alpha, \beta)$ *is a BiHom-associative algebra, where* $x * y = x \searrow y + x \nearrow y + x \swarrow y + x \nwarrow y$, *for all* $x, y \in Q$.

Just as in the classical situation in [2], the tensor product of two BiHom-dendriform algebras becomes naturally a BiHom-quadri-algebra.

Proposition 2.22 *Let* $(A, \prec, \succ, \alpha, \beta)$ *and* $(B, \prec', \succ', \alpha', \beta')$ *be two BiHom-dendriform algebras. On the linear space* $A \otimes B$ *define bilinear operations by (for all* $a_1, a_2 \in A$ *and* $b_1, b_2 \in B$*):*

$$(a_1 \otimes b_1) \nwarrow (a_2 \otimes b_2) = (a_1 \prec a_2) \otimes (b_1 \prec' b_2),$$
$$(a_1 \otimes b_1) \swarrow (a_2 \otimes b_2) = (a_1 \prec a_2) \otimes (b_1 \succ' b_2),$$
$$(a_1 \otimes b_1) \nearrow (a_2 \otimes b_2) = (a_1 \succ a_2) \otimes (b_1 \prec' b_2),$$
$$(a_1 \otimes b_1) \searrow (a_2 \otimes b_2) = (a_1 \succ a_2) \otimes (b_1 \succ' b_2).$$

Then $(A \otimes B, \nwarrow, \swarrow, \nearrow, \searrow, \alpha \otimes \alpha', \beta \otimes \beta')$ *is a BiHom-quadri-algebra.*

Proof. The relations (2.26)-(2.30) are obvious. We denote by $\succ_\otimes, \prec_\otimes, \vee_\otimes, \wedge_\otimes, *_\otimes$ the operations defined on $A \otimes B$ by (2.21)-(2.25) corresponding to the operations $\nwarrow, \swarrow, \nearrow, \searrow$ defined above. One can easily see that, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

$$(a_1 \otimes b_1) \prec_\otimes (a_2 \otimes b_2) = (a_1 \prec a_2) \otimes (b_1 *' b_2),$$
$$(a_1 \otimes b_1) \succ_\otimes (a_2 \otimes b_2) = (a_1 \succ a_2) \otimes (b_1 *' b_2),$$
$$(a_1 \otimes b_1) \wedge_\otimes (a_2 \otimes b_2) = (a_1 * a_2) \otimes (b_1 \prec' b_2),$$
$$(a_1 \otimes b_1) \vee_\otimes (a_2 \otimes b_2) = (a_1 * a_2) \otimes (b_1 \succ' b_2),$$
$$(a_1 \otimes b_1) *_\otimes (a_2 \otimes b_2) = (a_1 * a_2) \otimes (b_1 *' b_2).$$

Now we prove (2.31) and leave the rest to the reader:

$$[(a_1 \otimes b_1) \nwarrow (a_2 \otimes b_2)] \nwarrow (\beta \otimes \beta')(a_3 \otimes b_3)$$

$$\begin{aligned}
&= [(a_1 \prec a_2) \otimes (b_1 \prec' b_2)] \nwarrow (\beta(a_3) \otimes \beta'(b_3))\\
&= ((a_1 \prec a_2) \prec \beta(a_3)) \otimes ((b_1 \prec' b_2) \prec' \beta'(b_3))\\
&\overset{(2.4)}{=} (\alpha(a_1) \prec (a_2 * a_3)) \otimes (\alpha'(b_1) \prec' (b_2 *' b_3))\\
&= (\alpha(a_1) \otimes \alpha'(b_1)) \nwarrow [(a_2 * a_3) \otimes (b_2 *' b_3)]\\
&= (\alpha \otimes \alpha')(a_1 \otimes b_1) \nwarrow [(a_2 \otimes b_2) *_{\otimes} (a_3 \otimes b_3)],
\end{aligned}$$

$$[(a_1 \otimes b_1) \nearrow (a_2 \otimes b_2)] \nwarrow (\beta \otimes \beta')(a_3 \otimes b_3)$$

$$\begin{aligned}
&= [(a_1 \succ a_2) \otimes (b_1 \prec' b_2)] \nwarrow (\beta(a_3) \otimes \beta'(b_3))\\
&= ((a_1 \succ a_2) \prec \beta(a_3)) \otimes ((b_1 \prec' b_2) \prec' \beta'(b_3))\\
&\overset{(2.4),(2.5)}{=} (\alpha(a_1) \succ (a_2 \prec a_3)) \otimes (\alpha'(b_1) \prec' (b_2 *' b_3))\\
&= (\alpha(a_1) \otimes \alpha'(b_1)) \nearrow [(a_2 \prec a_3) \otimes (b_2 *' b_3)]\\
&= (\alpha \otimes \alpha')(a_1 \otimes b_1) \nearrow [(a_2 \otimes b_2) \prec_{\otimes} (a_3 \otimes b_3)],
\end{aligned}$$

finishing the proof. $\square$

3 Rota-Baxter operators

In this section, we study Rota-Baxter structures in the case of BiHom-type algebras. The definition of a Rota-Baxter operator for a BiHom-type algebra is exactly the same as in Definition 1.1. Notice that it only uses the multiplication but not the structure maps. We first construct some examples, then provide some constructions and properties.

Example 3.1 *We consider the following 2-dimensional BiHom-associative algebra (introduced in [18]), where the multiplication and the structure maps α, β are defined, with respect to a basis $\{e_1, e_2\}$, by*

$$\begin{aligned}
&\mu(e_1, e_1) = e_1, && \mu(e_1, e_2) = be_1 + (1-a)e_2,\\
&\mu(e_2, e_1) = \frac{b(1-a)}{a}e_1 + ae_2, && \mu(e_2, e_2) = \frac{b}{a}e_2,\\
&\alpha(e_1) = e_1, && \alpha(e_2) = \frac{b(1-a)}{a}e_1 + ae_2,\\
&\beta(e_1) = e_1, && \beta(e_2) = be_1 + (1-a)e_2,
\end{aligned}$$

where a, b are parameters in $\Bbbk$ with $a \neq 0$.

- *They carry Rota-Baxter operators R of weight 0, defined with respect to the basis by*

$$R(e_1) = 0, \quad R(e_2) = re_1,$$

or

$$R(e_1) = r_1e_1 + r_2e_2, \quad R(e_2) = -\frac{r_1^2}{r_2}e_1 - r_1e_2.$$

- *They carry the following Rota-Baxter operators R of weight 1, defined with respect to the basis by*

$$R(e_1) = -e_1, \quad R(e_2) = re_1,$$

or

$$R(e_1) = 0, \quad R(e_2) = re_1 - e_2,$$

or

$$R(e_1) = -e_1, \quad R(e_2) = -e_2,$$

or

$$R(e_1) = r_1e_1 + r_2e_2, \quad R(e_2) = -\frac{r_1(r_1+1)}{r_2}e_1 - (r_1+1)e_2.$$

In these formulae, r, r_1, r_2 are parameters in $\Bbbk$ with $r_2 \neq 0$.

Proposition 3.2 *Let A be a linear space, $\mu : A \otimes A \to A$ a linear multiplication on A, let $R : A \to A$ be a Rota-Baxter operator of weight λ for (A, μ) and $\alpha, \beta : A \to A$ two linear maps such that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$. Define a new multiplication on A by $x * y = \alpha(x)\beta(y)$, for all $x, y \in A$. Then R is also a Rota-Baxter operator of weight λ for $(A, *)$. In particular, if (A, μ) is associative and α, β are commuting algebra endomorphisms, then R is a Rota-Baxter operator of weight λ for the BiHom-associative algebra $A_{(\alpha,\beta)} = (A, \mu \circ (\alpha \otimes \beta), \alpha, \beta)$.*

Proof. We compute:

$$\begin{aligned}
R(x) * R(y) &= \alpha(R(x))\beta(R(y)) = R(\alpha(x))R(\beta(y)) \\
&= R(R(\alpha(x))\beta(y) + \alpha(x)R(\beta(y)) + \lambda\alpha(x)\beta(y)) \\
&= R(\alpha(R(x))\beta(y) + \alpha(x)\beta(R(y)) + \lambda\alpha(x)\beta(y)) \\
&= R(R(x) * y + x * R(y) + \lambda x * y),
\end{aligned}$$

finishing the proof. □

Proposition 3.3 *Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \to A$ a Rota-Baxter operator of weight λ commuting with α and β. Define the operations $\prec$, $\succ$ and $\cdot$ on A by $x \prec y = xR(y)$, $x \succ y = R(x)y$ and $x \cdot y = \lambda xy$, for all $x, y \in A$. Then $(A, \prec, \succ, \cdot, \alpha, \beta)$ is a BiHom-tridendriform algebra.*

Proof. We only prove (2.14)-(2.16) and leave the rest to the reader. We have:

$$(x \prec y) \prec \beta(z) - \alpha(x) \prec (y \prec z) - \alpha(x) \prec (y \succ z) - \alpha(x) \prec (y \cdot z)$$

$$\begin{aligned}
&= (xR(y)) \prec \beta(z) - \alpha(x) \prec (yR(z)) - \alpha(x) \prec (R(y)z) - \alpha(x) \prec (\lambda yz) \\
&= (xR(y))R(\beta(z)) - \alpha(x)R(yR(z)) - \alpha(x)R(R(y)z) - \alpha(x)R(\lambda yz) \\
&= (xR(y))\beta(R(z)) - \alpha(x)R(yR(z)) - \alpha(x)R(R(y)z) - \alpha(x)R(\lambda yz) \\
&= \alpha(x)(R(y)R(z)) - \alpha(x)R(yR(z) + R(y)z + \lambda yz) = 0,
\end{aligned}$$

$$(x \succ y) \prec \beta(z) - \alpha(x) \succ (y \prec z)$$

$$\begin{aligned}
&= (R(x)y) \prec \beta(z) - \alpha(x) \succ (yR(z)) = (R(x)y)R(\beta(z)) - R(\alpha(x))(yR(z)) \\
&= (R(x)y)\beta(R(z)) - \alpha(R(x))(yR(z)) = \alpha(R(x))(yR(z)) - \alpha(R(x))(yR(z)) = 0,
\end{aligned}$$

$$
\begin{aligned}
&\alpha(x) \succ (y \succ z) - (x \prec y) \succ \beta(z) - (x \succ y) \succ \beta(z) - (x \cdot y) \succ \beta(z) \\
&\qquad = \alpha(x) \succ (R(y)z) - (xR(y)) \succ \beta(z) - (R(x)y) \succ \beta(z) - (\lambda xy) \succ \beta(z) \\
&\qquad = R(\alpha(x))(R(y)z) - R(xR(y))\beta(z) - R(R(x)y)\beta(z) - R(\lambda xy)\beta(z) \\
&\qquad = \alpha(R(x))(R(y)z) - (R(x)R(y)\beta(z) = (R(x)R(y))\beta(z) - (R(x)R(y)\beta(z) = 0,
\end{aligned}
$$

finishing the proof. □

Corollary 3.4 *Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \to A$ a Rota-Baxter operator of weight 0 commuting with α and β. Define operations $\prec$ and $\succ$ on A by $x \prec y = xR(y)$ and $x \succ y = R(x)y$, for all $x, y \in A$. Then $(A, \prec, \succ, \alpha, \beta)$ is a BiHom-dendriform algebra.*

As a consequence of Propositions 2.11 and 3.3, we obtain:

Proposition 3.5 *Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \to A$ a Rota-Baxter operator of weight λ such that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$. Define operations $\prec'$ and $\succ'$ on A by $x \prec' y = xR(y) + \lambda xy$ and $x \succ' y = R(x)y$, for all $x, y \in A$. Then $(A, \prec', \succ', \alpha, \beta)$ is a BiHom-dendriform algebra.*

As a consequence of Propositions 2.14 and 3.3 we obtain:

Corollary 3.6 *Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \to A$ a Rota-Baxter operator of weight λ such that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$. If we define on A a new multiplication by $x * y = xR(y) + R(x)y + \lambda xy$, for all $x, y \in A$, then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra.*

Inspired by [2], we introduce the following concept:

Definition 3.7 *Let $(D, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra. A Rota-Baxter operator of weight 0 on D is a linear map $R : D \to D$ such that $R \circ \alpha = \alpha \circ R$, $R \circ \beta = \beta \circ R$ and the following conditions are satisfied, for all $x, y \in D$:*

$$R(x) \succ R(y) = R(x \succ R(y) + R(x) \succ y), \tag{3.1}$$
$$R(x) \prec R(y) = R(x \prec R(y) + R(x) \prec y). \tag{3.2}$$

From Corollary 2.15 we know that $(D, *, \alpha, \beta)$ is a BiHom-associative algebra. Adding the equations (3.1) and (3.2) we obtain that R is also a Rota-Baxter operator of weight 0 for $(D, *)$:

$$R(x) * R(y) = R(x * R(y) + R(x) * y).$$

Proposition 3.8 *Let $(D, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra and $R : D \to D$ a Rota-Baxter operator of weight 0 for D. Define new operations on D by*

$$x \searrow_R y = R(x) \succ y, \;\; x \nearrow_R y = x \succ R(y), \;\; x \swarrow_R y = R(x) \prec y \text{ and } x \nwarrow_R y = x \prec R(y).$$

Then $(D, \nwarrow_R, \swarrow_R, \nearrow_R, \searrow_R, \alpha, \beta)$ is a BiHom-quadri-algebra.

Proof. We only check (2.32) and leave the rest to the reader. We denote by $\succ_R, \prec_R, \vee_R, \wedge_R, *_R$ the operations defined on D by (2.21)-(2.25) corresponding to the operations $\nwarrow_R, \swarrow_R, \nearrow_R, \searrow_R$, which are then defined by

$$\begin{aligned} x \prec_R y &= x \nwarrow_R y + x \swarrow_R y = x \prec R(y) + R(x) \prec y, \\ x \succ_R y &= x \searrow_R y + x \nearrow_R y = R(x) \succ y + x \succ R(y), \\ x \wedge_R y &= x \nearrow_R y + x \nwarrow_R y = x \succ R(y) + x \prec R(y), \\ x \vee_R y &= x \searrow_R y + x \swarrow_R y = R(x) \succ y + R(x) \prec y, \\ x *_R y &= x \nwarrow_R y + x \swarrow_R y + x \searrow_R y + x \nearrow_R y \\ &= x \prec R(y) + R(x) \prec y + R(x) \succ y + x \succ R(y). \end{aligned}$$

For all $x, y, z \in D$ we have:

$$\begin{aligned} (x \wedge_R y) \nearrow_R \beta(z) &= (x \nwarrow_R y + x \nearrow_R y) \succ R(\beta(z)) \\ &= (x \prec R(y) + x \succ R(y)) \succ \beta(R(z)) \\ &\overset{(2.6)}{=} \alpha(x) \succ (R(y) \succ R(z)) \\ &\overset{(3.1)}{=} \alpha(x) \succ R(y \succ R(z) + R(y) \succ z) \\ &= \alpha(x) \succ R(y \nearrow_R z + y \searrow_R z) = \alpha(x) \nearrow_R (y \succ_R z), \end{aligned}$$

$$\begin{aligned} (x \swarrow_R y) \nwarrow_R \beta(z) &= (R(x) \prec y) \prec R(\beta(z)) = (R(x) \prec y) \prec \beta(R(z)) \\ &\overset{(2.4)}{=} \alpha(R(x)) \prec (y \prec R(z) + y \succ R(z)) \\ &= R(\alpha(x)) \prec (y \nwarrow_R z + y \nearrow_R z) = \alpha(x) \swarrow_R (y \wedge_R z), \end{aligned}$$

as needed. □

Remark 3.9 *In the setting of Proposition 3.8, the axioms (3.1) and (3.2) can be rewritten as* $R(x) \succ R(y) = R(x \succ_R y)$ *and* $R(x) \prec R(y) = R(x \prec_R y)$. *Thus, R is a morphism of BiHom-dendriform algebras from* $D_h = (D, \prec_R, \succ_R, \alpha, \beta)$ *to* $(D, \prec, \succ, \alpha, \beta)$.

On the other hand, if we denote by $(D, *, \alpha, \beta)$ the BiHom-associative algebra obtained from D as in Corollary 2.15, it is obvious that we have $x \wedge_R y = x * R(y)$ *and* $x \vee_R y = R(x) * y$, *for all $x, y \in D$. Thus, the BiHom-dendriform algebra structure obtained on D by applying Corollary 3.4 for the Rota-Baxter operator R on the BiHom-associative algebra $(D, *, \alpha, \beta)$ is exactly the vertical BiHom-dendriform algebra* $D_v = (D, \wedge_R, \vee_R, \alpha, \beta)$ *obtained from the BiHom-quadri-algebra* $(D, \nwarrow_R, \swarrow_R, \nearrow_R, \searrow_R, \alpha, \beta)$.

Similarly to the classical case in [2], examples of BiHom-quadri-algebras are provided by pairs of commuting Rota-Baxter operators on BiHom-associative algebras.

Proposition 3.10 *Let R and P be a pair of commuting Rota-Baxter operators of weight 0 on a BiHom-associative algebra (A, μ, α, β) such that* $R \circ \alpha = \alpha \circ R$, $R \circ \beta = \beta \circ R$, $P \circ \alpha = \alpha \circ P$ *and* $P \circ \beta = \beta \circ P$. *Then P is a Rota-Baxter operator of weight 0 on the BiHom-dendriform algebra* $(A, \prec_R, \succ_R, \alpha, \beta)$ *corresponding to R as in Corollary 3.4.*

Proof. We check the axioms (3.1) and (3.2). For all $x, y \in A$, we have:

$$P(x) \succ_R P(y) = R(P(x))P(y) = P(R(x))P(y)$$

$$
\begin{aligned}
&\overset{(1.1)}{=} \quad P(R(x)P(y) + P(R(x))y) = P(R(x)P(y) + R(P(x))y) \\
&= \quad P(x \succ_R P(y) + P(x) \succ_R y),
\end{aligned}
$$

$$
\begin{aligned}
P(x) \prec_R P(y) \quad &= \quad P(x)R(P(y)) = P(x)P(R(y)) \\
&\overset{(1.1)}{=} \quad P(xP(R(y)) + P(x)R(y)) = P(xR(P(y)) + P(x)R(y)) \\
&= \quad P(x \prec_R P(y) + P(x) \prec_R y),
\end{aligned}
$$

as needed. □

Corollary 3.11 *In the setting of Proposition 3.10, there exists a BiHom-quadri-algebra structure on the underlying linear space (A, α, β), with operations defined by*

$$
\begin{aligned}
&x \searrow y = P(x) \succ_R y = P(R(x))y = R(P(x))y, \\
&x \nearrow y = x \succ_R P(y) = R(x)P(y), \\
&x \swarrow y = P(x) \prec_R y = P(x)R(y), \\
&x \nwarrow y = x \prec_R P(y) = xR(P(y)) = xP(R(y)).
\end{aligned}
$$

In particular, $(A, *, \alpha, \beta)$ is a BiHom-associative algebra, where $a * b = R(P(a))b + R(a)P(b) + P(a)R(b) + aR(P(b))$, for all $a, b \in A$.

Proof. Apply Proposition 3.8 to the Rota-Baxter operator P of weight 0 on the BiHom-dendriform algebra $(A, \prec_R, \succ_R, \alpha, \beta)$. □

4 Free Rota-Baxter BiHom-associative algebra, categories and functors

We define the free BiHom-nonassociative algebra and free Rota-Baxter BiHom-associative algebra, generalizing the construction provided first by D. Yau for Hom-nonassociative algebras in [37] and then extended to Rota-Baxter Hom-associative algebras in [31]. A variation, in the multiplicative case, was presented in [23]. See also [32] for the free BiHom-nonassociative algebra construction.

4.1 BiHom-modules and BiHom-nonassociative algebras

We denote by **BiHomMod** the category of BiHom-modules. An object in this category is a triple (M, α_M, β_M) consisting of a $\Bbbk$-linear space M and linear maps $\alpha_M : M \to M$ and $\beta_M : M \to M$ satisfying $\beta_M \circ \alpha_M = \alpha_M \circ \beta_M$. A morphism $f : (M, \alpha_M, \beta_M) \longrightarrow (N, \alpha_N, \beta_M)$ of BiHom-modules is a linear map $f : M \to N$ such that $f \circ \alpha_M = \alpha_N \circ f$ and $f \circ \beta_M = \beta_N \circ f$.

There is a forgetful functor $U : \mathbf{BiHomMod} \to {}_{\Bbbk}\mathcal{M}$ that sends a BiHom-module (M, α, β) to the linear space M, forgetting about the maps α and β. Conversely, using a similar construction as in [37], one may construct a free BiHom-module associated to a linear space. Hence, we have an adjunction.

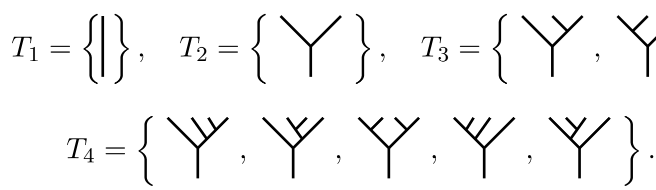
We call BiHom-nonassociative algebra a quadruple (A, μ, α, β), where (A, α, β) is a BiHom-module and $\mu : A \otimes A \to A$ is a linear map, called the multiplication of A, for which α and β are algebra maps.

Let (A, μ, α, β) and $(A', \mu', \alpha', \beta')$ be two BiHom-nonassociative algebras. A morphism of BiHom-nonassociative algebras from (A, μ, α, β) to $(A', \mu', \alpha', \beta')$ is a BiHom-module morphism $f : A \to A'$ such that $f \circ \mu = \mu \circ (f \otimes f)$. We denote the category of BiHom-nonassociative algebras by **BiHomNonAs**.

4.2 B-augmented and RB-augmented planar trees

The construction of the free BiHom-nonassociative algebra and respectively the free Rota-Baxter BiHom-nonassociative algebra involve planar trees and B-augmented trees (respectively RB-augmented trees) in order to freely generate products (for μ) and images of R.

For any natural number $n \geq 1$, let T_n denote the set of planar binary trees with n leaves and one root. Below are the sets T_n for $n = 1, 2, 3, 4$:

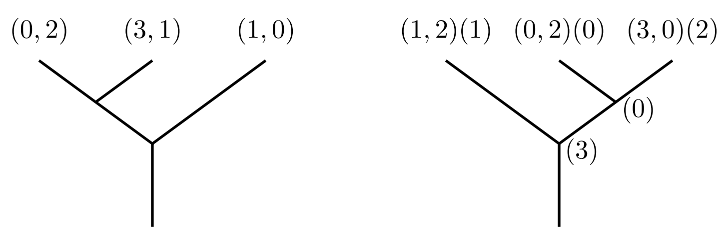


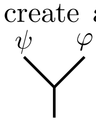
A **B-augmented** n**-tree** is a pair (φ, a), where $\varphi \in T_n$ is an n-tree and a is an n-tuple $(a_1, a_2, \ldots, a_n)$ consisting of n pairs of nonnegative integers.

A **RB-augmented** n**-tree** is a triple (φ, a, f) where (φ, a) is a B-augmented n-tree and f is a function that assigns a nonnegative integer to each element v in $N(\varphi)$, the set of nodes of the tree φ.

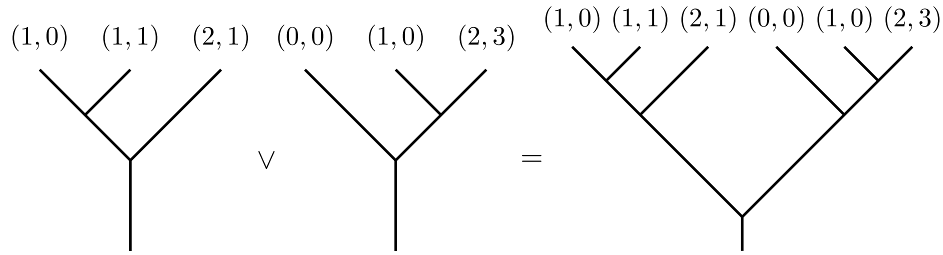
We call the tree φ the underlying tree of the B-augmented n-tree (φ, a) (respectively RB-augmented n-tree (φ, a, f)) while, for all $i = 1, \ldots, n$, the pair of integers a_i is called the (α, β)**-power** of the leaf i and $f(v)$ is the R**-power** of the node v. As we have labelled the n leaves of a tree $\varphi \in T_n$, we can consider that a is a function from the set $\{1, 2, \ldots, n\}$ to $\mathbb{N}^2$.

We denote by BT_n (respectively $\overline{BT}_n$) the set of B-augmented (respectively RB-augmented) planar binary trees with n leaves. The unions over $n \in \mathbb{N}$ of the sets T_n, BT_n, $\overline{BT}_n$ are denoted respectively T, BT and $\overline{BT}$. We depict the following examples of B-augmented (respectively RB-augmented) 3-trees:

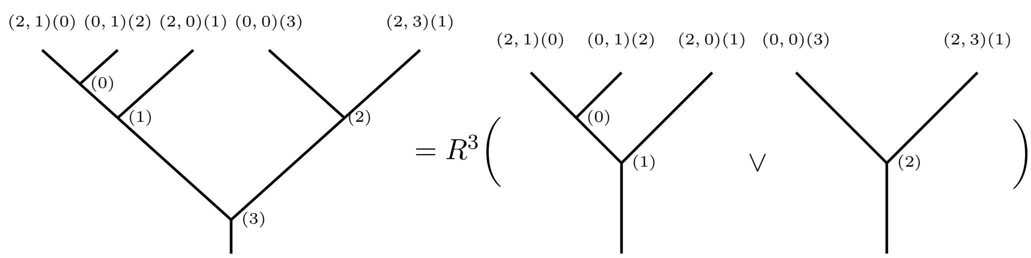


Given an n-tree (respectively B-augmented or RB-augmented n-tree) ψ and an m-tree (respectively B-augmented or RB-augmented m-tree) φ, their **grafting** $\psi \vee \varphi \in T_{n+m}$ (respectively in BT_{n+m} or $\overline{BT}_{n+m}$) is the tree obtained by joining the roots of ψ and φ to create a new root (in the RB-augmented case, one puts label 0 on the new node). Pictorially, 

. For example:



Note that grafting is neither an associative nor a commutative operation. For any tree $\varphi \in T_n$ (respectively B-augmented), there are unique integers p and q with $p + q = n$ and trees $\varphi_1 \in T_p$ (respectively in BT_p) and $\varphi_2 \in T_q$ (respectively in BT_q) such that $\varphi = \varphi_1 \vee \varphi_2$. It is clear that any tree in T_n (respectively in BT_n) can be obtained from $|$, the 1-tree (respectively B-augmented 1-tree), by successive grafting. For the B-augmented trees (φ, a), one has $(\varphi, a) = (\varphi_1, a_{(1)}) \vee (\varphi_2, a_{(2)})$, where $a_{(1)}, a_{(2)}$ are the corresponding leaf weights. For the RB-augmented trees one needs in addition to use the map R described in (4.5), that is there exists moreover an integer s such that $\varphi = R^s(\varphi_1 \vee \varphi_2)$ with $\varphi_1 \in \overline{BT}_p$ and $\varphi_2 \in \overline{BT}_q$. For example:



We consider the linear spaces $\mathbb{BT}$ freely generated by the set BT and $\overline{\mathbb{BT}}$ freely generated by the set $\overline{BT}$. The grafting is extended linearly to $\mathbb{BT}$ and $\overline{\mathbb{BT}}$, on which we define moreover two linear maps α and β and one more extra linear map R on $\overline{\mathbb{BT}}$, in the following way:

- The map $\alpha : \mathbb{BT} \to \mathbb{BT}$ (respectively $\alpha : \overline{\mathbb{BT}} \to \overline{\mathbb{BT}}$) sends a B-augmented (respectively RB-augmented) n-tree to a B-augmented (respectively RB-augmented) n-tree obtained by adding $+1$ to all first components of a_i in a, i.e.

$$\alpha((\varphi, ((a_{11}, a_{12}), (a_{21}, a_{22}), \cdots, (a_{n1}, a_{n2})))) = (\varphi, ((a_{11}+1, a_{12}), (a_{21}+1, a_{22}), \cdots, (a_{n1}+1, a_{n2}))) \quad (4.1)$$

respectively

$$\alpha((\varphi, ((a_{11}, a_{12}), (a_{21}, a_{22}), \cdots, (a_{n1}, a_{n2}))), f) = (\varphi, ((a_{11}+1, a_{12}), (a_{21}+1, a_{22}), \cdots, (a_{n1}+1, a_{n2})), f) \quad (4.2)$$

- The map $\beta : \mathbb{BT} \to \mathbb{BT}$ (respectively $\beta : \overline{\mathbb{BT}} \to \overline{\mathbb{BT}}$) sends a B-augmented (respectively RB-augmented) n-tree to a B-augmented (respectively RB-augmented) n-tree obtained by adding $+1$ to all second components of a_i in a, i.e.

$$\beta((\varphi, ((a_{11}, a_{12}), (a_{21}, a_{22}), \cdots, (a_{n1}, a_{n2})))) = (\varphi, ((a_{11}, a_{12}+1), (a_{21}, a_{22}+1), \cdots, (a_{n1}, a_{n2}+1))) \quad (4.3)$$

respectively

$$\beta((\varphi, ((a_{11}, a_{12}), (a_{21}, a_{22}), \cdots, (a_{n1}, a_{n2})), f)) = (\varphi, ((a_{11}, a_{12}+1), (a_{21}, a_{22}+1), \cdots, (a_{n1}, a_{n2}+1)), f) \quad (4.4)$$

- The map $R : \overline{\mathbb{BT}} \to \overline{\mathbb{BT}}$ sends a RB-augmented n-tree (φ, a, f) to a RB-augmented n-tree (φ, a, Rf) such that Rf is equal to f, except that for the lowest vertex v_{low} (or the only leaf if $\varphi \in T_1$) we have

$$Rf(v_{low}) = f(v_{low}) + 1. \quad (4.5)$$

Observe that the maps $\alpha,\ \beta,\ R$ defined above commute and $\alpha,\ \beta$ are algebra morphisms for the grafting of trees, that is $\alpha((\varphi,a)\vee(\psi,b))=\alpha((\varphi,a))\vee\alpha((\psi,b))$ and $\beta((\varphi,a)\vee(\psi,b))=\beta((\varphi,a))\vee\beta((\psi,b))$, for $(\varphi,a),(\psi,b)$ in $\mathbb{BT}$ (same for $\overline{\mathbb{BT}}$).

4.3 Free BiHom-nonassociative algebra

Let (A,μ,α,β) be a BiHom-nonassociative algebra. Given elements $x_1,\cdots,x_n\in A$, there are $\#T_n=C_{n-1}$ (Catalan's number) ways to parenthesize the monomial $x_1\cdots x_n$ to obtain an element in A. Hence, every parenthesized monomial $x_1\cdots x_n$ corresponds to an n-tree. Each B-augmented n-tree provides a way to multiply n elements in a BiHom-nonassociative algebra (A,μ,α,β). We define maps

$$\mathbb{BT}\otimes A^{\otimes n}\to A,\quad ((\varphi,a);x_1\otimes\cdots\otimes x_n)\to(\alpha^{a_{11}}\beta^{a_{12}}(x_1)\cdots\alpha^{a_{n1}}\beta^{a_{n2}}(x_n))_\varphi \tag{4.6}$$

inductively via the rules:
1) $(x)_I=x$ for $x\in A$, where I denotes the 1-tree.
2) If $\varphi=\varphi_1\vee\varphi_2$, where $\varphi_1\in T_p$ and $\varphi_2\in T_q$, then $(x_1\cdots x_n)_\varphi=(x_1\cdots x_p)_{\varphi_1}(x_{p+1}\cdots x_{p+q})_{\varphi_2}$.

Recall that $(x_1\cdots x_n)_\varphi$ means the element obtained by putting $x_1,\cdots,x_n$ on the leaves of the n-tree φ and applying the multiplication with respect to the tree.

There is a forgetful functor E : **BiHomNonAs** $\to$ **BiHomMod** that sends a BiHom-nonassociative algebra (A,μ,α,β) to the BiHom-module (A,α,β), forgetting about the multiplication μ. Conversely, using a similar construction as in [37], one may prove, by using the previous construction, for the functor E, the existence of a left adjoint F_{BHNAs} : **BiHomMod** $\to$ **BiHomNonAs** defined as

$$F_{BHNAs}(M)=\bigoplus_{n\geq 1}\bigoplus_{(\varphi,a)\in BT_n}M^{\otimes n}_{(\varphi,a)},$$

for $(M,\alpha,\beta)\in$ **BiHomMod**, where $M^{\otimes n}_{(\varphi,a)}$ is a copy of $M^{\otimes n}$ (its generators will be denoted by $(x_1\otimes\cdots\otimes x_n)_{(\varphi,a)}$).

Hence, $F_{BHNAs}(M)$ is called the free BiHom-nonassociative algebra of the BiHom-module (M,α,β), where the multiplication μ_F is defined by grafting and the structure linear maps α_F and β_F are defined in (4.1) and (4.3).

Remark 4.1 *One may get the free BiHom-associative algebra and the enveloping algebra of a BiHom-Lie algebra by considering quotients over ideals constructed in a similar way as in [37] and by adding the images of the second structure map. The free BiHom-associative algebra is obtained by a quotient with a two-sided ideal I^∞ defined as*

$$I^1=\langle im(\mu_F\circ(\mu_F\otimes\beta_F-\alpha_F\otimes\mu_F))\rangle,\quad I^{n+1}=\langle I^n\cup\alpha_F(I^n)\cup\beta_F(I^n)\rangle,\quad I^\infty=\cup_{n\geq 1}I^n.$$

4.4 The category of Rota-Baxter BiHom-associative algebras

We denote by **BiHomRB**$_\lambda$ the category of Rota-Baxter BiHom-associative algebras of weight λ. A Rota-Baxter BiHom-associative algebra of weight λ is a tuple (A,μ,α,β,R) in which (A,μ,α,β) is a BiHom-associative algebra, the linear map $R\colon A\to A$ satisfies the Rota-Baxter identity (1.1) and $\alpha\circ R=R\circ\alpha$ and $\beta\circ R=R\circ\beta$.

A morphism $f\colon(A,\mu_A,\alpha_A,\beta_A,R_A)\to(B,\mu_B,\alpha_B,\beta_B,R_B)$ of Rota-Baxter BiHom-associative algebras of weight λ is a morphism $f\colon(A,\alpha_A,\beta_A)\to(B,\alpha_B,\beta_B)$ of the underlying BiHom-modules such that $f\circ\mu_A=\mu_B\circ f^{\otimes 2}$ and $f\circ R_A=R_B\circ f$.

A Rota-Baxter algebra (A,μ,R) of weight λ gives rise to a Rota-Baxter BiHom-associative algebra of weight λ with $\alpha=\beta=id_A$. A morphism of Rota-Baxter algebras of weight λ is an algebra morphism that commutes with the Rota-Baxter operators R. In particular, the functor $(A,\mu,R)\mapsto(A,\mu,id_A,id_A,R)$ from the category of Rota-Baxter algebras of weight λ to the category $\mathbf{BiHomRB}_\lambda$ is a full and faithful embedding. So we can regard the category of Rota-Baxter algebras and Rota-Baxter Hom-associative algebras (for which $\alpha=\beta$) as subcategories of $\mathbf{BiHomRB}_\lambda$.

Similarly to [31] and using B-augmented and RB-augmented n-trees, we may get:

Theorem 4.2 *The forgetful functor $\mathcal{O}\colon \mathbf{BiHomRB}_\lambda\to\mathbf{BiHomMod}$ admits a right adjoint, where $\mathcal{O}(A,\mu,\alpha,\beta,R)=(A,\alpha,\beta)$.*

The proof uses an intermediate category $\mathbf{D}$, whose objects are tuples (A,μ,α,β,R), where A is a $\Bbbk$-linear space, $\mu: A\otimes A\to A$ is a multiplication and α,β,R are commuting two by two linear self-maps $A\to A$. A morphism $f\colon(A,\mu_A,\alpha_A,\beta_A,R_A)\to(B,\mu_B,\alpha_B,\beta_B,R_B)$ in $\mathbf{D}$ consists of a linear map $f\colon A\to B$ such that $f\circ\mu_A=\mu_B\circ f^{\otimes 2}$, $f\circ\alpha_A=\alpha_B\circ f$, $f\circ\beta_A=\beta_B\circ f$ and $f\circ R_A=R_B\circ f$.

There are forgetful functors $\mathbf{BiHomRB}_\lambda\xrightarrow{\mathcal{O}_2}\mathbf{D}\xrightarrow{\mathcal{O}_1}\mathbf{BiHomMod}$, whose composition is $\mathcal{O}$. One shows that each of these two forgetful functors $\mathcal{O}_i$ admits a left adjoint $\mathcal{F}_i$. The composition $\mathcal{F}=\mathcal{F}_2\circ\mathcal{F}_1$ is then the desired left adjoint.

First, consider the forgetful functor $\mathcal{O}_1\colon\mathbf{D}\to\mathbf{BiHomMod}$ defined as $\mathcal{O}_1(A,\mu,\alpha,\beta,R)=(A,\alpha,\beta)$. The functor $\mathcal{O}_1$ admits a left adjoint $\mathcal{F}_1\colon\mathbf{BiHomMod}\to\mathbf{D}$, defined as follows. If $(M,\alpha,\beta)\in\mathbf{BiHomMod}$, we set $\mathcal{F}_1(M)=\bigoplus_{n\geq 1}\bigoplus_{(\varphi,a,f)\in\overline{BT}_n}M^{\otimes n}_{(\varphi,a,f)}$, where $M^{\otimes n}_{(\varphi,a,f)}$ is a copy of $M^{\otimes n}$ (its generators will be denoted by $(x_1\otimes\ldots\otimes x_n)_{(\varphi,a,f)}$). We need to define on $\mathcal{F}_1(M)$ the four operations μ,α,β,R. The multiplication μ on $\mathcal{F}_1(M)$ is defined by

$$\mu((x_1\otimes\cdots\otimes x_n)_{(\varphi,a,f)},(x_{n+1}\otimes\cdots\otimes x_{n+m})_{(\psi,b,g)})=(x_1\otimes\cdots\otimes x_{n+m})_{(\varphi,a,f)\vee(\psi,b,g)}.$$

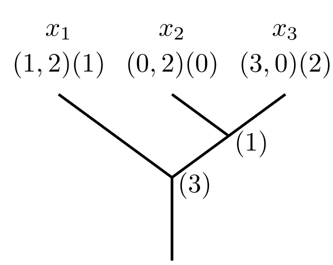
The maps α,β and R are defined, using (4.2), (4.4), (4.5), by

$$\begin{aligned}
\alpha((x_1\otimes\cdots\otimes x_n)_{(\varphi,a,f)})&=(x_1\otimes\cdots\otimes x_n)_{\alpha(\varphi,a,f)},\\
\beta((x_1\otimes\cdots\otimes x_n)_{(\varphi,a,f)})&=(x_1\otimes\cdots\otimes x_n)_{\beta(\varphi,a,f)},\\
R((x_1\otimes\cdots\otimes x_n)_{(\varphi,a,f)})&=(x_1\otimes\cdots\otimes x_n)_{R(\varphi,a,f)}.
\end{aligned}$$

In order to prove that $\mathcal{F}_1$ is the left adjoint of the forgetful functor $\mathcal{O}_1$, one has to define and use a certain action of RB-augmented trees. Namely, if $(A,\mu,\alpha,\beta,R)\in\mathbf{D}$, define $\overline{\mathbb{BT}}\otimes A^{\otimes n}\to A$,

$$((\varphi,a,f);x_1\otimes\cdots\otimes x_n)\to(\alpha^{a_{11}}\beta^{a_{12}}R^{f(v_{top,1})}(x_1)\cdots\alpha^{a_{n1}}\beta^{a_{n2}}R^{f(v_{top,n})}(x_n))_\varphi,\quad(4.7)$$

where $v_{top,i}$ are the highest nodes (leaves). Moreover, at each internal node v the map R is applied with power $f(v)$. For example, the following tree applied to $x_1\otimes x_2\otimes x_3$



translates to $R^3(\alpha\beta^2R(x_1)R(\beta^2(x_2)\alpha^3R^2(x_3)))$.

To construct $\mathcal{F}_2$, a left adjoint of $\mathcal{O}_2$, one picks an object $(A,\mu,\alpha,\beta,R)\in\mathbf{D}$ and take S a subset of A consisting of the generating relations in a Rota-Baxter BiHom-associative algebra, i.e. the elements $\alpha(x)(yz)-(xy)\beta(z)$ and $R(x)R(y)-R(R(x)y+xR(y)+\lambda xy)$, for all $x,y,z\in A$. Then we set $\mathcal{F}_2(A)=A/\langle S\rangle$.

4.5 The categories of BiHom-dendriform algebras and BiHom-tridendriform algebras

Naturally and similarly to what we did above, we consider **BiHomDend**, the category of BiHom-dendriform algebras and **BiHomTridend**, the category of BiHom-tridendriform algebras. One shows that the forgetful functors from **BiHomDend** (respectively **BiHomTridend**) to **BiHomMod**, the category of BiHom-modules, admit left adjoints. Moreover, we have:

Theorem 4.3 *(i) There is an adjoint pair of functors*

$$U_{\mathcal{BD}}\colon \mathbf{BiHomDend}\rightleftarrows \mathbf{BiHomRB}_0\colon \mathcal{BD}, \tag{4.8}$$

in which the right adjoint is given by $\mathcal{BD}(A,\mu,\alpha,\beta,R)=(A,\prec,\succ,\alpha,\beta)\in\mathbf{BiHomDend}$ *with* $x\prec y=xR(y)$ *and* $x\succ y=R(x)y$*, for* $x,y\in A$*.*

(ii) There is an adjoint pair of functors

$$U_{\mathcal{BT}}\colon \mathbf{BiHomTridend}\rightleftarrows \mathbf{BiHomRB}_\lambda\colon \mathcal{BT},$$

in which $U_{\mathcal{BT}}$ *is the left adjoint. For* $(A,\mu,\alpha,\beta,R)\in\mathbf{BiHomRB}_\lambda$*, the binary operations in the object* $\mathcal{BT}(A)=(A,\prec,\succ,\cdot,\alpha,\beta)\in\mathbf{BiHomTridend}$ *are defined as* $x\prec y=xR(y),\ x\succ y=R(x)y,\ x\cdot y=\lambda xy$*, for* $x,y\in A$*.*

The proof is similar to the Hom-type case in [31].

5 Weak BiHom-pseudotwistors

Let (A,μ) be an associative algebra. A weak pseudotwistor for A, as defined in [33] (extending the previous proposal from [26] called pseudotwistor) is a linear map $T: A\otimes A\rightarrow A\otimes A$ for which there exists a linear map $\mathcal{T}: A\otimes A\otimes A\rightarrow A\otimes A\otimes A$ such that

$$T\circ(id_A\otimes(\mu\circ T))=(id_A\otimes\mu)\circ\mathcal{T},$$
$$T\circ((\mu\circ T)\otimes id_A)=(\mu\otimes id_A)\circ\mathcal{T}.$$

If this is the case, then $(A,\mu\circ T)$ is also an associative algebra.

If (A,μ) is an associative algebra and $R: A\rightarrow A$ is a Rota-Baxter operator of weight λ, then the linear map

$$T: A\otimes A\rightarrow A\otimes A,\quad T(a\otimes b)=R(a)\otimes b+a\otimes R(b)+\lambda a\otimes b,\quad \forall\, a,b\in A,$$

is a weak pseudotwistor, and consequently we recover the fact that the new multiplication defined on A by $a*b=R(a)b+aR(b)+\lambda ab$, for all $a,b\in A$, is associative, see [33].

We want to obtain a BiHom-analogue of this fact. We begin by recalling the following concept and result from [18]:

Proposition 5.1 *Let (D, μ, α, β) be a BiHom-associative algebra and $\tilde{\alpha}, \tilde{\beta} : D \to D$ two multiplicative linear maps such that any two of the maps $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ commute. Let $T : D \otimes D \to D \otimes D$ be a linear map and assume that there exist two linear maps $\tilde{T}_1, \tilde{T}_2 : D \otimes D \otimes D \to D \otimes D \otimes D$ such that T commutes with $\alpha \otimes \alpha$, $\beta \otimes \beta$, $\tilde{\alpha} \otimes \tilde{\alpha}$, $\tilde{\beta} \otimes \tilde{\beta}$ and the following relations hold:*

$$T \circ (\alpha \otimes \mu) = (\alpha \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_D),$$
$$T \circ (\mu \otimes \beta) = (\mu \otimes \beta) \circ \tilde{T}_2 \circ (id_D \otimes T),$$
$$\tilde{T}_1 \circ (T \otimes id_D) \circ (\tilde{\alpha} \otimes T) = \tilde{T}_2 \circ (id_D \otimes T) \circ (T \otimes \tilde{\beta}).$$

Then $D^T_{\tilde{\alpha},\tilde{\beta}} := (D, \mu \circ T, \tilde{\alpha} \circ \alpha, \tilde{\beta} \circ \beta)$ is also a BiHom-associative algebra. The map T is called an $(\tilde{\alpha}, \tilde{\beta})$-BiHom-pseudotwistor and the two maps $\tilde{T}_1$, $\tilde{T}_2$ are called the companions of T. In the particular case $\tilde{\alpha} = \tilde{\beta} = id_D$, we call T a BiHom-pseudotwistor and we denote $D^T_{\tilde{\alpha},\tilde{\beta}}$ by D^T.

We introduce now a common generalization of this concept and of the one of weak pseudotwistor for an associative algebra:

Proposition 5.2 *Let (D, μ, α, β) be a BiHom-associative algebra and $\tilde{\alpha}, \tilde{\beta} : D \to D$ two multiplicative linear maps such that any two of the maps $\tilde{\alpha}, \tilde{\beta}, \alpha, \beta$ commute. Let $T : D \otimes D \to D \otimes D$ be a linear map and assume that there exist a linear map $\mathcal{T} : D \otimes D \otimes D \to D \otimes D \otimes D$ such that T commutes with $\alpha \otimes \alpha$, $\beta \otimes \beta$, $\tilde{\alpha} \otimes \tilde{\alpha}$, $\tilde{\beta} \otimes \tilde{\beta}$ and the following relations hold:*

$$T \circ ((\tilde{\alpha} \circ \alpha) \otimes (\mu \circ T)) = (\alpha \otimes \mu) \circ \mathcal{T}, \tag{5.1}$$
$$T \circ ((\mu \circ T) \otimes (\tilde{\beta} \circ \beta)) = (\mu \otimes \beta) \circ \mathcal{T}. \tag{5.2}$$

Then $D^T_{\tilde{\alpha},\tilde{\beta}} := (D, \mu \circ T, \tilde{\alpha} \circ \alpha, \tilde{\beta} \circ \beta)$ is also a BiHom-associative algebra. The map T is called a weak $(\tilde{\alpha}, \tilde{\beta})$-BiHom-pseudotwistor and $\mathcal{T}$ is called the weak companion of T. In the particular case $\tilde{\alpha} = \tilde{\beta} = id_D$, we call T a weak BiHom-pseudotwistor and we denote $D^T_{\tilde{\alpha},\tilde{\beta}}$ by D^T.

Proof. We only prove the BiHom-associativity condition and leave the rest to the reader:

$$\begin{aligned}(\mu \circ T) \circ ((\mu \circ T) \otimes (\tilde{\beta} \circ \beta)) &= \mu \circ T \circ ((\mu \circ T) \otimes (\tilde{\beta} \circ \beta)) \\ &\overset{(5.2)}{=} \mu \circ (\mu \otimes \beta) \circ \mathcal{T} = \mu \circ (\alpha \otimes \mu) \circ \mathcal{T} \\ &\overset{(5.1)}{=} (\mu \circ T) \circ ((\tilde{\alpha} \circ \alpha) \otimes (\mu \circ T)),\end{aligned}$$

finishing the proof. □

Remark 5.3 *If T is an $(\tilde{\alpha}, \tilde{\beta})$-BiHom-pseudotwistor with companions $\tilde{T}_1$, $\tilde{T}_2$ on a BiHom-associative algebra (D, μ, α, β), then T is also a weak $(\tilde{\alpha}, \tilde{\beta})$-BiHom-pseudotwistor for D, with weak companion $\mathcal{T} = \tilde{T}_1 \circ (T \otimes id_D) \circ (\tilde{\alpha} \otimes T) = \tilde{T}_2 \circ (id_D \otimes T) \circ (T \otimes \tilde{\beta})$.*

We can give now another proof for Corollary 3.6.

Proposition 5.4 *Let (A, μ, α, β) be a BiHom-associative algebra and $R : A \to A$ a Rota-Baxter operator of weight λ such that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$. Then the linear map*

$$T : A \otimes A \to A \otimes A, \quad T(a \otimes b) = R(a) \otimes b + a \otimes R(b) + \lambda a \otimes b, \quad \forall\, a, b \in A,$$

is a weak BiHom-pseudotwistor with weak companion $\mathcal{T} : A \otimes A \otimes A \to A \otimes A \otimes A$,

$$\begin{aligned}\mathcal{T}(a \otimes b \otimes c) = R(a) \otimes R(b) \otimes c + R(a) \otimes b \otimes R(c) + a \otimes R(b) \otimes R(c) + \lambda R(a) \otimes b \otimes c \\ + \lambda a \otimes R(b) \otimes c + \lambda a \otimes b \otimes R(c) + \lambda^2 a \otimes b \otimes c,\end{aligned}$$

and the new BiHom-associative multiplication $\mu \circ T$ on A is given by $a * b = R(a)b + aR(b) + \lambda ab$.

Proof. We only prove that $T\circ(\alpha\otimes(\mu\circ T))=(\alpha\otimes\mu)\circ\mathcal{T}$ and leave the rest to the reader:

$$
\begin{aligned}
&T\circ(\alpha\otimes(\mu\circ T))(a\otimes b\otimes c)\\
&\quad = R(\alpha(a))\otimes R(b)c+\alpha(a)\otimes R(R(b)c)+\lambda\alpha(a)\otimes R(b)c\\
&\qquad +R(\alpha(a))\otimes bR(c)+\alpha(a)\otimes R(bR(c))+\lambda\alpha(a)\otimes bR(c)\\
&\qquad +\lambda R(\alpha(a))\otimes bc+\lambda\alpha(a)\otimes R(bc)+\lambda^2\alpha(a)\otimes bc\\
&\quad = R(\alpha(a))\otimes R(b)c+\lambda\alpha(a)\otimes R(b)c+R(\alpha(a))\otimes bR(c)+\lambda\alpha(a)\otimes bR(c)\\
&\qquad +\lambda R(\alpha(a))\otimes bc+\lambda^2\alpha(a)\otimes bc+\alpha(a)\otimes R(R(b)c+bR(c)+\lambda bc)\\
&\quad \overset{(1.1)}{=} R(\alpha(a))\otimes R(b)c+\lambda\alpha(a)\otimes R(b)c+R(\alpha(a))\otimes bR(c)+\lambda\alpha(a)\otimes bR(c)\\
&\qquad +\lambda R(\alpha(a))\otimes bc+\lambda^2\alpha(a)\otimes bc+\alpha(a)\otimes R(b)R(c)\\
&\quad = \alpha(R(a))\otimes R(b)c+\lambda\alpha(a)\otimes R(b)c+\alpha(R(a))\otimes bR(c)+\lambda\alpha(a)\otimes bR(c)\\
&\qquad +\lambda\alpha(R(a))\otimes bc+\lambda^2\alpha(a)\otimes bc+\alpha(a)\otimes R(b)R(c)\\
&\quad = (\alpha\otimes\mu)\circ\mathcal{T}(a\otimes b\otimes c),
\end{aligned}
$$

finishing the proof. □

The next two results are BiHom-analogues of some results in [33].

Proposition 5.5 *Let (A,μ,α,β) be a BiHom-associative algebra and $T,D:A\otimes A\rightarrow A\otimes A$ two weak BiHom-pseudotwistors for A, with weak companions $\mathcal{T}$ and respectively $\mathcal{D}$, such that the following conditions are satisfied:*

$$D\circ(\alpha\otimes(\mu\circ T\circ D))=(\alpha\otimes(\mu\circ T))\circ\mathcal{D}, \tag{5.3}$$
$$D\circ((\mu\circ T\circ D)\otimes\beta)=((\mu\circ T)\otimes\beta)\circ\mathcal{D}. \tag{5.4}$$

Then $T\circ D$ is a weak BiHom-pseudotwistor for A, with weak companion $\mathcal{T}\circ\mathcal{D}$.

Proof. Obviously $T\circ D$ commutes with $\alpha\otimes\alpha$ and $\beta\otimes\beta$. Now we compute:

$$
\begin{aligned}
T\circ D\circ(\alpha\otimes(\mu\circ T\circ D)) &\overset{(5.3)}{=} T\circ(\alpha\otimes(\mu\circ T))\circ\mathcal{D}\\
&\overset{(5.1)}{=} (\alpha\otimes\mu)\circ\mathcal{T}\circ\mathcal{D},
\end{aligned}
$$

$$
\begin{aligned}
T\circ D\circ((\mu\circ T\circ D)\otimes\beta) &\overset{(5.4)}{=} T\circ((\mu\circ T)\otimes\beta)\circ\mathcal{D}\\
&\overset{(5.2)}{=} (\mu\otimes\beta)\circ\mathcal{T}\circ\mathcal{D},
\end{aligned}
$$

finishing the proof. □

Corollary 5.6 *Let (A,μ,α,β) be a BiHom-associative algebra and $T,D:A\otimes A\rightarrow A\otimes A$ two weak BiHom-pseudotwistors for A, with weak companions $\mathcal{T}$ and respectively $\mathcal{D}$, such that the following conditions are satisfied:*

$$\mu\circ T\circ D=\mu\circ D\circ T, \tag{5.5}$$
$$\mathcal{D}\circ(id_A\otimes T)=(id_A\otimes T)\circ\mathcal{D}, \tag{5.6}$$
$$\mathcal{D}\circ(T\otimes id_A)=(T\otimes id_A)\circ\mathcal{D}. \tag{5.7}$$

Then $T\circ D$ is a weak BiHom-pseudotwistor for A, with weak companion $\mathcal{T}\circ\mathcal{D}$.

Proof. We check (5.3), while (5.4) is similar and left to the reader:

$$\begin{aligned}
D \circ (\alpha \otimes (\mu \circ T \circ D)) &\overset{(5.5)}{=} D \circ (\alpha \otimes (\mu \circ D \circ T)) \\
&= D \circ (\alpha \otimes (\mu \circ D)) \circ (id_A \otimes T) \\
&\overset{(5.1)}{=} (\alpha \otimes \mu) \circ \mathcal{D} \circ (id_A \otimes T) \\
&\overset{(5.6)}{=} (\alpha \otimes \mu) \circ (id_A \otimes T) \circ \mathcal{D} \\
&= (\alpha \otimes (\mu \circ T)) \circ \mathcal{D},
\end{aligned}$$

finishing the proof. □

We can give now another proof for the second statement in Corollary 3.11.

Corollary 5.7 *Let (A, μ, α, β) be a BiHom-associative algebra and $R, P : A \to A$ two commuting Rota-Baxter operators of weight 0 such that $R \circ \alpha = \alpha \circ R$, $R \circ \beta = \beta \circ R$, $P \circ \alpha = \alpha \circ P$ and $P \circ \beta = \beta \circ P$. Define a new multiplication on A by*

$$a * b = R(P(a))b + R(a)P(b) + P(a)R(b) + aR(P(b)), \quad \forall\, a, b \in A. \tag{5.8}$$

Then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra.

Proof. We consider the weak BiHom-pseudotwistors $T, D : A \otimes A \to A \otimes A$,

$$T(a \otimes b) = R(a) \otimes b + a \otimes R(b), \quad D(a \otimes b) = P(a) \otimes b + a \otimes P(b),$$

with weak companions $\mathcal{T}$ and respectively $\mathcal{D}$ defined by

$$\begin{aligned}
\mathcal{T}(a \otimes b \otimes c) &= R(a) \otimes R(b) \otimes c + R(a) \otimes b \otimes R(c) + a \otimes R(b) \otimes R(c), \\
\mathcal{D}(a \otimes b \otimes c) &= P(a) \otimes P(b) \otimes c + P(a) \otimes b \otimes P(c) + a \otimes P(b) \otimes P(c).
\end{aligned}$$

Since R and P commute, it is clear that $T \circ D = D \circ T$, so (5.5) is satisfied. One can easily check that (5.6) and (5.7) are also satisfied, so $T \circ D$ is a weak BiHom-pseudotwistor and clearly $\mu \circ T \circ D$ is exactly the multiplication (5.8). □

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