

ABOUT THE NUCLEARITY OF $\mathcal{S}_{(M_p)}$ AND \mathcal{S}_ω

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Dedicated to Prof. Luigi Rodino on the occasion of his 70th birthday.

ABSTRACT. We use an isomorphism established by Langenbruch between some sequence spaces and weighted spaces of generalized functions to give sufficient conditions for the (Beurling type) space $\mathcal{S}_{(M_p)}$ to be nuclear. As a consequence, we obtain that for a weight function ω satisfying the mild condition: $2\omega(t) \leq \omega(Ht) + H$ for some $H > 1$ and for all $t \geq 0$, the space \mathcal{S}_ω in the sense of Björck is also nuclear.

1. Introduction and preliminaries

For a sequence $(M_p)_{p \in \mathbb{N}_0}$ which satisfies Komatsu's standard condition $(M2)'$ (stability under differential operators) and, moreover, the condition that there is $H > 0$ such that for any $C > 0$ there is $B > 0$ with

$$(1.1) \quad s^{s/2} M_p \leq BC^s H^{s+p} M_{s+p}, \quad \text{for any } s, p \in \mathbb{N}_0,$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, Langenbruch [8] proves that the Hermite functions are a Schauder basis in the spaces of ultradifferentiable functions of (Beurling type):

$$\mathcal{S}_{(M_p)}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \text{for any } j \in \mathbb{N}, \right. \\ \left. \sup_{\alpha, \beta \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| |j|^{\alpha+\beta} / M_{|\alpha+\beta|} < +\infty \right\}.$$

Moreover, in [8] it is also established an isomorphism between $\mathcal{S}_{(M_p)}$ and the Köthe sequence space:

$$\Lambda_{(M_p)} := \left\{ (c_k)_{k \in \mathbb{N}_0} : \text{for any } j \in \mathbb{N}_0, \sup_{k \in \mathbb{N}_0} |c_k| e^{M(jk^{1/2})} < +\infty \right\},$$

where

$$(1.2) \quad M(t) = \sup_p \log \frac{t^p M_0}{M_p}, \quad t > 0,$$

is the *associated function* of (M_p) .

In this paper we use Grothendieck-Pietsch criterion to characterize when the space $\Lambda_{(M_p)}$ is nuclear under the assumption that $(M_p/M_0)^{1/p}$ is bounded below by a positive constant and, hence, $M(t)$ is increasing and convex in $\log t$ (see [7, p. 49]). Indeed, we prove in Theorem 2.2 that $\Lambda_{(M_p)}$ is nuclear if and only if there is $H > 1$ such that for any $t > 0$ we have

$$(1.3) \quad M(t) + \log t \leq M(Ht) + H.$$

Key words and phrases. nuclear spaces; weighted spaces of ultradifferentiable functions of Beurling type.

As it is observed in [8, (2.2)], condition $(M2)'$ implies (1.3). Therefore, conditions $(M2)'$ and (1.1) imply that $\mathcal{S}_{(M_p)}$ is nuclear (see Corollary 2.3). This should be compared with [11], where the authors prove that $\mathcal{S}_{(M_p)}$ is nuclear under Komatsu's conditions $(M1)$ and $(M2)$.

As a consequence of Theorem 2.2 we give a simple proof of the nuclearity of the space \mathcal{S}_ω in the sense of Björck [1] given in Definition 3.1 under the following condition of Bonet, Meise and Melikhov [5] on the weight function ω :

$$(BMM) \quad \exists H > 1 \text{ s.t. } 2\omega(t) \leq \omega(Ht) + H, \quad t \geq 0.$$

In fact, in this case the space \mathcal{S}_ω is isomorphic to the space $\mathcal{S}_{(M_p)}$ for some suitable sequence (M_p) .

2. Results for the space $\mathcal{S}_{(M_p)}$

In this section we characterize the nuclearity of $\Lambda_{(M_p)}$ and give sufficient conditions for the nuclearity $\mathcal{S}_{(M_p)}$.

We consider a sequence $(M_p)_p$ satisfying the condition that $(M_p/M_0)^{1/p}$ is bounded from below by a positive constant, so that the associated function defined by (1.2) is increasing and convex in $\log t$.

From Grothendieck-Pietsch criterion it is easy to obtain the following

Lemma 2.1. *The Köthe sequence space $\Lambda_{(M_p)}$ is nuclear if and only if for every $j \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq j$ such that*

$$(2.1) \quad \sum_{k=0}^{+\infty} e^{M(jk^{1/2}) - M(mk^{1/2})} < +\infty.$$

Proof. It follows from Proposition 28.16 of [9]. □

Theorem 2.2. *The space $\Lambda_{(M_p)}$ is nuclear if and only if (1.3) holds.*

Proof. Let us first remark that (1.3) implies

$$\begin{aligned} M(t) + 2 \log t &= M(t) + \log t + \log t \\ &\leq M(Ht) + H + \log(Ht) - \log H \\ &\leq M(H^2t) + 2H - \log H \end{aligned}$$

and, more in general,

$$(2.2) \quad M(t) + N \log t \leq M(H^N t) + C_{N,H}, \quad \forall N \in \mathbb{N},$$

for some constant $C_{N,H} > 0$ depending on N and H .

Let us now assume that (1.3) is satisfied and prove the nuclearity of $\Lambda_{(M_p)}$, using (2.2) for a fixed $N > 2$. By Lemma 2.1, it's enough to prove the convergence of the series (2.1). Indeed,

for every fixed $j \in \mathbb{N}$, choosing $m \geq H^N j$,

$$\begin{aligned} e^{M(jk^{1/2})-M(mk^{1/2})} &\leq e^{M(jk^{1/2})-M(H^N jk^{1/2})} \\ &\leq e^{M(jk^{1/2})-M(jk^{1/2})-N \log(jk^{1/2})+C_{N,H}} \\ &= e^{C_{N,H}} j^{-N} \frac{1}{k^{N/2}} \end{aligned}$$

and the series $\sum_{k=1}^{+\infty} \frac{1}{k^{N/2}}$ converges since $N > 2$.

Let us now assume that the series (2.1) converges and prove (1.3). To this aim, let us first remark that, for $m > j$,

$$k \mapsto M(jk^{1/2}) - M(mk^{1/2})$$

is decreasing, because $M(e^t)$ is convex by our assumptions (see [7, p. 49]), and therefore its difference quotient $\frac{M(e^t)-M(e^s)}{t-s}$ is increasing with respect to both variables t and s ; this implies that

$$M(mk^{1/2}) - M(jk^{1/2}) = \frac{M\left(e^{\log mk^{1/2}}\right) - M\left(e^{\log jk^{1/2}}\right)}{\log mk^{1/2} - \log jk^{1/2}} \log \frac{m}{j}$$

is increasing with respect to k .

Then the convergence of (2.1) implies that

$$\lim_{k \rightarrow +\infty} k e^{M(jk^{1/2})-M(mk^{1/2})} = 0$$

and hence

$$\sup_{k \in \mathbb{N}} k e^{M(jk^{1/2})-M(mk^{1/2})} \leq A,$$

for some $A \in \mathbb{R}^+$. Then

$$\log k + M(jk^{1/2}) - M(mk^{1/2}) \leq \log A, \quad \forall k \in \mathbb{N},$$

and hence

$$\begin{aligned} (2.3) \quad M(jk^{1/2}) - M(mk^{1/2}) &\leq -\log k + \log A = -2 \log(jk^{1/2}) + \log(j^2 A) \\ &\leq -\log(jk^{1/2}) + \log(j^2 A). \end{aligned}$$

To prove that (2.3) implies (1.3) let us first consider $t \geq 1$ and choose the smallest $k \in \mathbb{N}$ such that $t \leq jk^{1/2}$. Since

$$j(k+1)^{1/2} - jk^{1/2} = \frac{j}{\sqrt{k+1} + \sqrt{k}} < j, \quad \forall k \in \mathbb{N},$$

we have that $jk^{1/2} \in [t, (j+1)t]$ and therefore, from (2.3),

$$\begin{aligned} M(t) + \log t &\leq M(jk^{1/2}) + \log(jk^{1/2}) \leq M(mk^{1/2}) + \log(j^2 A) \\ &= M\left(\frac{m}{j} jk^{1/2}\right) + \log(j^2 A) \\ &\leq M\left(\frac{m}{j}(j+1)t\right) + \log(j^2 A), \quad \forall t \geq 1, \end{aligned}$$

and hence, for $H = \max \left\{ \frac{m}{j}(j+1), \log(j^2 A) + M(1) \right\}$, we have that (1.3) is satisfied for all $t > 0$. \square

So, we automatically obtain

Corollary 2.3. *If (M_p) satisfies $(M2)'$ and (1.1), the space $\mathcal{S}_{(M_p)}$ is nuclear.*

Proof. The spaces $\mathcal{S}_{(M_p)}$ and $\Lambda_{(M_p)}$ are isomorphic because (M_p) satisfies $(M2)'$ and (1.1) by Theorem 3.4 of [8]. Since $(M2)'$ implies (1.3) (see for instance [8, (2.2)]), the result follows from Theorem 2.2. \square

Remark 2.4. Looking inside the proof of Theorem 3.4 of [8] we can see that in fact Langenbruch needs only (1.1) and (1.3), so that in the above corollary we could substitute the assumption $(M2)'$ with the condition that the associated function $M(t)$ satisfies (1.3).

3. Results for the space \mathcal{S}_ω . Examples

In this section we give a sufficient condition for the space \mathcal{S}_ω in the sense of Björck [1] to be nuclear.

We consider continuous increasing *weight functions* $\omega : [0, +\infty) \rightarrow [0, +\infty)$ satisfying:

- (α) $\exists L > 0$ s.t. $\omega(2t) \leq L(\omega(t) + 1)$, $\forall t \geq 0$;
- (β) $\omega(t) = o(t)$, as $t \rightarrow +\infty$;
- (γ) $\exists a \in \mathbb{R}, b > 0$ s.t. $\omega(t) \geq a + b \log(1 + t)$, $\forall t \geq 0$;
- (δ) $\varphi : t \mapsto \omega(e^t)$ is convex.

Then we define $\omega(\zeta) := \omega(|\zeta|)$ for $\zeta \in \mathbb{C}^d$.

We denote by φ^* the *Young conjugate* of φ , defined by

$$\varphi^*(s) := \sup_{t \geq 0} (ts - \varphi(t)).$$

We recall that φ^* is increasing and convex, $\varphi^{**} = \varphi$ and $\varphi^*(s)/s$ is increasing. Moreover, it will be not restrictive, in the following, to assume $\omega|_{[0,1]} \equiv 0$ and hence $\varphi^*(0) = 0$.

The space $\mathcal{S}_\omega(\mathbb{R}^d)$ of weighted rapidly decreasing functions is then defined by (see [1]):

Definition 3.1. $\mathcal{S}_\omega(\mathbb{R}^d)$ is the set of all $u \in L^1(\mathbb{R}^d)$ such that $u, \hat{u} \in C^\infty(\mathbb{R}^d)$ and

- (i) $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} e^{\lambda \omega(x)} |D^\alpha u(x)| < +\infty$,
- (ii) $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \sup_{\xi \in \mathbb{R}^d} e^{\lambda \omega(\xi)} |D^\alpha \hat{u}(\xi)| < +\infty$,

where $D^\alpha = (-i)^{|\alpha|} \partial^\alpha$.

Note that

$$(3.1) \quad \omega_0(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ \log t, & t > 1 \end{cases}$$

is a weight function for which $\mathcal{S}_{\omega_0}(\mathbb{R}^d)$ coincides with the classical Schwartz class $\mathcal{S}(\mathbb{R}^d)$.

The space $\mathcal{S}_\omega(\mathbb{R}^d)$ is a Fréchet space with different equivalent systems of seminorms (cf. [3], [4], [2]). In particular, we shall use in what follows the family of seminorms

$$(3.2) \quad p_\lambda(u) = \sup_{\alpha, \beta \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha u(x)| e^{-\lambda \varphi^*\left(\frac{|\alpha+\beta|}{\lambda}\right)}.$$

Given a weight function ω we construct the sequence (M_p) by

$$(3.3) \quad M_p = e^{\varphi^*(p)}, \quad \forall p \in \mathbb{N}_0.$$

Then the associated function of M_p is equivalent to the given weight ω . Indeed, on one side, since $M_0 = 1$, we have, for $t > 0$:

$$\begin{aligned} M(t) &= \sup_{p \in \mathbb{N}_0} \log \frac{t^p}{M_p} = \sup_{p \in \mathbb{N}_0} (\log t^p - \log e^{\varphi^*(p)}) \\ &\leq \sup_{s \geq 0} (s \log t - \varphi^*(s)) = \varphi(\log t) = \omega(t). \end{aligned}$$

On the other side, for $t > 0$:

$$\begin{aligned} \omega(t) &= \sup_{s \geq 0} (s \log t - \varphi^*(s)) = \sup_{p \in \mathbb{N}_0} \sup_{p \leq s < p+1} (s \log t - \varphi^*(s)) \\ &\leq \sup_{p \in \mathbb{N}_0} ((p+1) \log t - \varphi^*(p)) = \log t + M(t) \leq 2M(t) + \log M_1 \end{aligned}$$

since $M(t) \geq \log t - \log M_1$ by definition.

Therefore

$$(3.4) \quad M(t) \leq \omega(t) \leq M(t) + \log t \leq 2M(t) + A, \quad \forall t > 0,$$

and for some $A > 0$.

Moreover,

$$(3.5) \quad \begin{aligned} M_p &= e^{\varphi^*(p)} = \exp\left\{\sup_{t \geq 0} (pt - \omega(e^t))\right\} = \sup_{t \geq 0} \left(e^{pt} e^{-\omega(e^t)}\right) \\ &= \sup_{s \geq 1} (s^p e^{-\omega(s)}) = \sup_{s \geq 0} (s^p e^{-\omega(s)}), \end{aligned}$$

since $\omega|_{[0,1]} \equiv 0$.

Let us remark that the sequence (M_p) satisfies $(M_p/M_0)^{1/p} \geq 1$ and the condition of *logarithmic convexity*

$$(M1) \quad M_p^2 \leq M_{p-1} M_{p+1}, \quad p \in \mathbb{N},$$

since

$$\begin{aligned} 2\varphi^*(p) &= 2 \sup_{t \geq 0} (tp - \varphi(t)) \leq \sup_{t \geq 0} (t(p-1) - \varphi(t)) + \sup_{t \geq 0} (t(p+1) - \varphi(t)) \\ &= \varphi^*(p-1) + \varphi^*(p+1). \end{aligned}$$

If ω satisfies condition (BMM), then also $M(t)$ satisfies condition (BMM) because, by (3.4),

$$(3.6) \quad \begin{aligned} 2M(t) &\leq \frac{1}{2}(4\omega(t)) \leq \frac{1}{2}(2\omega(Ht) + 2H) \\ &\leq \frac{1}{2}\omega(H^2t) + \frac{3}{2}H \leq M(H^2t) + \frac{A}{2} + \frac{3}{2}H. \end{aligned}$$

Then, by [7, Prop. 3.6], we obtain that (M_p) satisfies also the condition of *stability under ultradifferential operators*:

$$(M2) \quad \exists A, H > 0 \text{ s.t. } M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}.$$

Moreover, the sequence $(M_p)_p$ satisfies (1.1). Indeed, since $\omega(t) = o(t)$ as $t \rightarrow +\infty$, we have that for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that $\omega(t) \leq \varepsilon t + R_\varepsilon$ for all $t \geq 0$. Therefore, for $s \geq \varepsilon$,

$$\varphi^*(s) = \sup_{t \geq 0} (ts - \omega(e^t)) \geq \sup_{t \geq 0} (ts - \varepsilon e^t) - R_\varepsilon = s \log \frac{s}{\varepsilon} - s - R_\varepsilon,$$

and hence

$$\left(\frac{s}{\varepsilon}\right)^s \leq e^{s+\varphi^*(s)+R_\varepsilon}, \quad \forall s \geq \varepsilon.$$

Since, for $s \leq \varepsilon$ we have that $s^s \leq (\varepsilon e)^s$, we finally have that for every $s > 0$:

$$s^{s/2} M_p \leq s^s e^{\varphi^*(p)} \leq e^{R_\varepsilon} (\varepsilon e)^s e^{\varphi^*(s)+\varphi^*(p)} \leq e^{R_\varepsilon} (\varepsilon e)^s e^{\varphi^*(p+s)} = e^{R_\varepsilon} (\varepsilon e)^s M_{p+s}.$$

If ω satisfies (BMM), then $\Lambda_{(M_p)}$ coincides with the sequence space

$$(3.7) \quad \Lambda_\omega := \left\{ (c_k)_{k \in \mathbb{N}_0} : \sup_{k \in \mathbb{N}_0} |c_k| e^{\omega(jk^{1/2})} < +\infty \forall j \in \mathbb{N}_0 \right\},$$

by (3.4) and (3.6).

Theorem 3.2. *Let ω be a weight function. Then Λ_ω is nuclear if and only if ω satisfies condition (1.3).*

Proof. As in Theorem 2.2, we use [9, Prop. 28.16] for the sequence space Λ_ω . \square

Example 3.3. Condition (1.3) for a weight function ω is weaker than (BMM). For instance

$$\omega(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ \log^2 t, & t > 1 \end{cases}$$

satisfies (1.3) but not (BMM).

Corollary 3.4. *Let ω be a weight function satisfying (BMM). Then Λ_ω is nuclear.*

Proposition 3.5. *Let ω be a weight function satisfying (BMM) and (M_p) the sequence defined by (3.3). Then $\mathcal{S}_\omega(\mathbb{R}^d)$ is equal (as vector space and as locally convex space) to $\mathcal{S}_{(M_p)}$ and isomorphic to $\Lambda_{(M_p)} = \Lambda_\omega$.*

Proof. We endow $\mathcal{S}_\omega(\mathbb{R}^d)$ with the family of seminorms (3.2). It is isomorphic (and hence equal) to $\mathcal{S}_{(M_p)}$ because, by [5, formulas (5), (6)], the following two conditions hold:

$$\forall j \in \mathbb{N} \exists \lambda, c > 0 \text{ s.t. } e^{\lambda \varphi^*(\frac{p}{j})} \leq c j^{-p} M_p, \quad \forall p \in \mathbb{N}_0,$$

and

$$\forall \lambda > 0 \exists j \in \mathbb{N}, C > 0 \text{ s.t. } j^{-p} M_p \leq C e^{\lambda \varphi^*(\frac{p}{j})}, \quad \forall p \in \mathbb{N}_0.$$

Finally, $\mathcal{S}_{(M_p)}$ is isomorphic to $\Lambda_{(M_p)}$ by Theorem 3.4 of [8], since (M_p) satisfies (M2) (stronger than (M2)') and (1.1). Moreover $\Lambda_{(M_p)}$ coincides with Λ_ω , as we already remarked in the comment for formula (3.7). \square

Condition (1.3), written in terms of the weight function ω , is equivalent to the nuclearity of Λ_ω by Theorem 3.2, but it is not necessary for the nuclearity of \mathcal{S}_ω . For example, the weight $\omega_0(t)$ defined by (3.1) does not satisfy (1.3) and hence Λ_{ω_0} is not nuclear, while \mathcal{S} is well known to be nuclear. In particular, Λ_{ω_0} and \mathcal{S} are not isomorphic. On the other hand, from the results that we have we do not know if condition (1.3) is sufficient for the nuclearity of \mathcal{S}_ω , but we need the stronger condition (BMM), as we state in the following

Theorem 3.6. *Let ω be a weight function satisfying (BMM). Then \mathcal{S}_ω is a nuclear space.*

Proof. It follows from Proposition 3.5 and Corollary 3.4. \square

Example 3.7. There exist sequences $(M_p)_p$ satisfying (1.1) and (1.3) (for the associated function), but not (M2).

Let us consider, for example, a weight function ω satisfying (1.3) but not (BMM) (see Example 3.3) and construct the sequence (M_p) as in (3.3). Then M_p satisfies (M1), (1.1) and its associated function satisfies (1.3) because, by (3.4) and (2.2):

$$\begin{aligned} M(t) + \log t &\leq \omega(t) + 2 \log t - \log t \leq \omega(H^2 t) + C_{2,H} - \log t \\ &\leq M(H^2 t) + \log(H^2 t) + C_{2,H} - \log t = M(H^2 t) + 2 \log H + C_{2,H}. \end{aligned}$$

However, $M(t)$ does not satisfy (BMM) because $\omega(t)$ does not satisfy (BMM) (see (3.4)), therefore M_p does not satisfy (M2) by [7, Prop. 3.6].

The example above furnishes a sequence (M_p) satisfying (M1), but not (M2), for which the space $\mathcal{S}_{(M_p)}$ is nuclear, by Corollary 2.3 and Remark 2.4. Comparing with [11] it is then interesting the following

Corollary 3.8. *Condition (M2) is not necessary for the nuclearity of $\mathcal{S}_{(M_p)}$.*

Acknowledgments. The authors were partially supported by the Projects FAR 2017, FAR 2018 and FIR 2018 (University of Ferrara), FFABR 2017 (MIUR). The research of the second author was partially supported by the project MTM2016-76647-P. The first and third authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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