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Contents

Introduction

An important goal of algebraic geometry is to give a classification of algebraic varieties. The relation induced by isomorphism is too strict and thus not very interesting. After the second half of the XIXth century many studies have focused their attention on birational equivalence. Namely, two varieties X and Y are birationally equivalent if there exist $U \subset X$ and $V \subset Y$ Zariski open subsets and an isomorphism $\phi : U \to V$. This means that X and Y are isomorphic outside a subvariety of positive codimension.

In this setting, a large number of birational invariants have been defined, for example the Kodaira dimension. The italian school (Enriques and Castelnuovo) classified rational complex surfaces at the beginning of the XXth century. They also proved that if a surface is rational, then it is also unirational, i.e. it is dominated by a finite morphism from a projective space. Only in the second half of the XXth century were provided examples of unirational but not rational varieties, such as the Zariski surfaces in positive characteristic. An other example is given by cubic threefolds: Clemens and Griffiths proved that they are not rational using a new technique, the intermediate Jacobian [**13**]. In dimension 4, the rationality of a general cubic hypersurface is still an open problem in algebraic geometry.

Nowadays, in the study of birational geometry we can proceed in many directions. Among them we follow the classical school inherited by Enriques and Castelnuovo, who focuses its attention on the Cremona group and a more algebraic approach which uses derived categories and semiorthogonal decompositions, defined by Bondal, Orlov and Kuznetsov.

Chapter 2

The Cremona group Cr_n is the group of birational self-transformations of $\mathbb{P}^n$. Its elements are called Cremona tranformations. Some examples of Cremona transformations, for example the inversion at a circle, had already been used in antiquity, but the true development of the study of this group begins in the XIXth century with the work of Cremona. In [**16**] he shows, for example, that for any integer m there exist a Cremona transformation which sends a line to a curve of degree m. In this discussion emerges his interest for de Jonquières transformations which provide the first example of transformation of arbitrary degree. Successively, transformations of order 8 and 17 are constructed by Geiser and Bertini.

In the second paper [**17**], Cremona introduces the idea of base points of a Cremona transformation and shows that the numbers of base points of the transformation and its inverse coincide. Cremona also proves that the number of fixed points of a general transformation of degree d is equal to $d+2$. In the case of de Jonquières transformations this was also done by de Jonquières in 1885.

A considerable number of results have been produced by classical algebraic geometers but the first major result after the works of Cremona has been the Noether-Castelnuovo theorem.

Theorem 1 (Noether-Castelnuovo). *The Cremona group of the projective plane over an algebraically closed field k is generated by the group of self-morphisms of $\mathbb{P}^2$ and the standard quadratic transformation σ*

$$\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0 : x_1 : x_2] \mapsto [x_1x_2 : x_0x_2 : x_0x_1].$$

The original proof of M. Noether, based on Noethers inequalities, contains a gap. In order to correct the mistake Noether introduces the idea of infinitely near points and uses them as base points of Cremona transformations. We have to wait until 1901 to have the first complete proof, given by Castelnuovo, where he uses de Jonquières transformations. For a more detailed historical account, see [**18**] and [**11**].
We remark that Noether-Castelnuovo theorem holds if the field k is algebraically closed.

Another important classical problem is the classification of finite subgroups of Cr_2. Dolgachev and Iskovskikh [**19**] provided an almost complete classification in the complex case. In positive characteristic the classification is still an open problem. On the other hand, classifying finite subgroups of Cr_3 is more difficult. Prokhorov described all finite simple groups that can be embedded into Cr_3. In dimensions strictly larger than 3 very little is known about finite subgroups of Cremona groups.

In the study of the Cremona group we can also focus on the orbit of the action of Cr_n on linear systems on $\mathbb{P}^n$. In particular we reduce to the case $n = 2$, where we can, for example, apply the Noether-Castelnuovo theorem. As we will see in Chapter 1, there are many results in this direction. For example, in Theorem 43, Castelnuovo and Enriques give a characterization of the contractibility of an irreducible curve in terms of adjoint linear systems. Later, in [**10**], [**12**] and [**9**], Calabri and Ciliberto show the relation between contractibility and Kodaira dimension.

In this thesis, in Chapter 2 we want to study completely reduced curves that we call configurations of lines. In particular we want to show when a configuration is contractible, that is when it is sent to a finite set of points by a Cremona transformation. For this reason we introduce the notion of marked divisor, which helps us to keep track of the lines already contracted during the contraction process. Let D be a configuration of d lines and let q be one of its points of maximal multiplicity. We denote with k the number of lines not containing the point q. Using this notation we prove that if $k \leq 2$, the configuration is contractible for any d; if $k \geq 7$, the configuration is not contractible for any d. For $3 \leq k \leq 6$ we recover a result from [**11**] about non contractibility of configurations with $d \geq 12$, we show that there is a unique contractible configuration in degree 11, and produce a partial classification in lower degrees.

Chapter 3

Another approach to birational geometry arises from the work of Bondal and Orlov [**7**], where they introduced the notion of semiorthogonal decomposition. Successively Orlov [**39**] described the semiorthogonal decomposition of the blow-up of a smooth variety. More precisely, the blow-up of a smooth subvariety induces a fully faithful functor which embeds the derived category of the base variety into that of the blow-up. Moreover, the orthogonal complement of its image can be decomposed into a number of subcategories which depends on the codimension of the exceptional divisor.

If the canonical divisor of a variety X is ample (or antiample), the derived category $D^b(X)$ characterizes the variety: two such varieties are isomorphic if and only if their derived categories are equivalent, [**8**]. Thus, in general it is not possible to make correspond the equivalence between derived categories to the birational equivalence.

In this context, M. Bernardara and M. Bolognesi [**6**] followed an idea of Bondal-Orlov and studied also by Kuznetsov and introduced the concept of categorical representability and formulated the following question: *is a rational variety always categorically representable in codimension 2?* That is, if X is a variety of dimension n, does there exist a semiorthogonal decomposition of the derived category which components are equivalent to admissible subcategories of the derived category of a variety of dimension at most $n-2$? Analogously, is it possible to characterize obstructions to rationality via natural components of some semiorthogonal decomposition which cannot be realized in codimension 2?

On the complex field, for example, if we consider a V_{14} Fano threefold X, its derived category admits a semiorthogonal decomposition with only one non trivial component $\mathcal{A}_X$. For a smooth cubic threefold we can find a similar decomposition and Kuznetsov showed that the non trivial component $\mathcal{A}_X$ is equivalent to $\mathcal{A}_Y$ where Y is the unique cubic threefold birational to X [**31**]. In this way he suggested that it is possible to look at $\mathcal{A}_X$ as a birational invariant. The derived category of a smooth cubic fourfold admits a similar decomposition and Kuznetsov conjectured that the non-trivial components should detect if the variety is rational.

If X is a smooth rational projective threefold, a necessary condition for rationality is that the intermediate Jacobian $J(X)$, as principally polarized abelian variety, is the direct sum of Jacobians of smooth projective curves. It follows that smooth cubic threefolds are not rational [**13**].
In the case of complex conic bundles $\pi : X \to S$ there are other results. From a categorical point of view it is possible to characterize the rationality of such conic bundle from a semiorthogonal decomposition of its derived category. By Kuznetsov we have

$$D^b(X) = \left\langle \Phi D^b(S, \mathcal{B}), \pi^* D^b(S) \right\rangle, \tag{1}$$

where $\mathcal{B}$ is the sheaf of even parts of Clifford algebras associated to the quadratic form defining the conic fibration and $\Phi : D^b(S, \mathcal{B}) \to D^b(X)$ is a fully faithful functor from the derived category of $\mathcal{B}$-algebras over S to $D^b(X)$. Bernardara and Bolognesi proved [**6**] that in the case where S is minimal, X is rational if and only if $D^b(S, \mathcal{B})$ has a decomposition whose components are equivalent to the derived categories of smooth rational curves or generated by exceptional objects.

All those results hold for varieties on the complex field $\mathbb{C}$, but the categorical-theoretical setting allows us to work on the problem over an arbitrary field k, not necessary separable nor algebraically closed.

In [**1**], Auel and Bernardara discuss the case of del Pezzo surfaces over an arbitrary field k. Let us consider a minimal del Pezzo surface S over a field k such that S has degree $d \geq 5$ and Picard rank 1, and let $\mathcal{A}_S$ be the complement of $\mathcal{O}_S$ in $D^b(X)$. Then we can define the Griffiths-Kuznetsov component GK to be zero if $\mathcal{A}_S$ is representable in dimension 0 or to be the product of all indecomposable elements of $\mathcal{A}_S$ otherwise. This means that it is the product of all components of $\mathcal{A}_S$ which are not representable in dimension 0.

Then the Griffiths-Kuznetsov component, where it is defined, it is the suitable birational invariant to detect the rationality of the given surface.

Our aim is to apply this approach to minimal surfaces with the structure of conic bundles: we want to define the Griffiths-Kuznetsov component and to show that in this case it is also a birational invariant.

As we will see in Definition (123), the Griffiths-Kuznetsov component is defined as follows.

Definition 2. Let $X \to C$ be conic bundle, where X is a smooth geometrically rational projective surface and C a geometrically rational smooth curve. Let us denote with V the vector bundle such that in k^s we have $\langle V\rangle = \langle \mathcal{O}(F)\rangle$ where F is the fiber of π. If X is minimal, the *Griffiths-Kuznetsov component* GK_X is defined as follows.

(1) if X is rational, $\mathrm{GK}_X = 0$;
(2) if $C \cong \mathbb{P}^1$ and X is birational to a quadric, $\mathrm{GK}_X = D^b(k, \mathcal{C}_0)$;
(3) if $C \cong \mathbb{P}^1$, X is not rational and not birational to a quadric, $\mathrm{GK}_X = D^b(\mathbb{P}^1, \mathcal{C}_0)$;
(4) if $C \ncong \mathbb{P}^1$ and X is birational to an involution surface, $\mathrm{GK}_X = \langle V\rangle \amalg D^b(k, \mathcal{C}_0)$;
(5) if $C \ncong \mathbb{P}^1$ and X is not birational to an involution surface, $\mathrm{GK}_X = \langle V\rangle \amalg D^b(C, \mathcal{C}_0)$.

where $\mathcal{C}_0$ is the Clifford class of X. If X is not minimal, the Griffiths-Kuznetsov component is $\mathrm{GK}_X = \mathrm{GK}_{X'}$ for a minimal model $X \to X'$.

The Griffiths-Kuznetsov component, as defined, is then the collection of subcategory generated by components which are not representable in dimension 0 in the decomposition (1).

THEOREM 3. *Let X be a surface and $X \to C$ be a conic bundle over an arbitrary field k. The Griffiths-Kuznetsov component as in Definition (123) is well defined and is a birational invariant.*

Moreover, in [**22**] we find a survey of the classification of surfaces over an arbitrary field, originally given by Iskovkikh and Fano. We can have the projective plane, the smooth quadric and other Del Pezzo surfaces and conic bundles. In [**1**] we can find the definition of the Griffiths-Kuznetsov component in the cases of Del Pezzo surfaces. Here, we discuss the problem for conic bundles and quadrics of Picard rank 2 and conclude this work.

Thus we can state the following result.

THEOREM 4. *Let X be a geometrically rational surface over an arbitrary field k. Then the Griffiths-Kuznetsov component* GK_X *is well defined and is a birational invariant. Moreover, if X is rational then* $\mathrm{GK}_X = 0$.

Notations

Let $\mathcal{L}$ be a linear system of degree d with base points $P_1, \dots, P_s$ of multiplicity respectively $m_1, \dots, m_s$. We will denote $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{L}_d(m_1 P_1, \dots, m_s P_s).$$

Moreover, when defined, the birational map associated to the linear system $\mathcal{L}$ will be denoted as $\phi_{\mathcal{L}}$.

Let X be a variety on an arbitrary field k and let k^s be a separable closure of k. We will denote by $X^s = X \times_k k^s$ and each object related to X but considered over k^s, will be munited of the superscript ^{s}.

CHAPTER 1

Background

1. Birational Geometry

We recall that a rational map f between two algebraic varieties X_1 and X_2, denoted as $f\colon X_1 \dashrightarrow X_2$, is a regular application defined on a non-empty Zariski open subset of X_1. The dashed arrow stresses the fact that there may exist some points of X_1 where f is not well defined. If f is well defined on every point of X_1, then it is a morphism and we denote it with a full arrow. A birational map is a rational map with a rational inverse. In other words, it is an isomorphism between two non-empty Zariski open subsets $U_1 \subset X_1$ and $U_2 \subset X_2$. We also refer to rational varieties, i.e. varieties that are birational to a projective space $\mathbb{P}^n$ for some n.

1.1. Minimal surfaces. In this section we will start introducing the main topic of this work, geometrically rational surfaces.

Let k be an arbitrary field. We denote with $\bar{k}$ an algebraic closure of k, that is an algebraic extension of k such that every polynomial $p(x)$ in $k[x]$ splits completely over $\bar{k}$. Moreover, with k^s we indicate a separable closure of k that is an extension field whose elements have a separable minimal polinomial (over k). We recall that if k is perfect, those two notions coincides. Examples of perfect fields are the finite ones, or those with characteristic 0.

In our discussion we will usually refer to a smooth projective geometrically integral variety X over k. By geometrically integral variety over k we mean a variety X such that for each field extension $k \subset k'$ we have that $X \otimes_k k'$ is integral. In particular, we are interested in the study of their rationality. We say that a variety X is *geometrically rational* if $\bar{X} = X \otimes_k \bar{k}$ is $\bar{k}$-rational.

We say that a field extension l of k is a splitting field for X if $X \otimes_k l$ is birational to $\mathbb{P}^n_l$ through a sequence of blow-ups centered at closed l-points. We observe that in the case of geometrically rational varieties, the algebraic closure $\bar{k}$ is a splitting field by definition. The following theorem shows that also separable closures k^s are splitting fields for surfaces, thus in our discussion we will usually refer to k^s.

THEOREM 5 ([**14**]). *If X is a geometrically rational surface over a field k, then it is split over k^s.*

Thanks to the Castelnuovo contraction theorem, we can reduce to the case of minimal geometrically rational surfaces.

THEOREM 6 (Castelnuovo). *Let X be a smooth surface over k and $E \subset X$ a k-rational smooth curve such that $E^2 = -1$. Then there exist a morphism $\phi : X \to X'$ to a non-singular projective surface X', such that ϕ contracts E and acts as the blow-up of X' at a closed point.*

This theorem suggests us to introduce the following definition.

DEFINITION 7. A smooth projective surface X is *minimal* over k if for any birational morphism $\phi : X \to Y$ defined over k, where Y is a smooth surface, ϕ is an isomorphism.

REMARK 8. This definition follows the one given by Castelnuovo and the Italian school of the beginning of the XXth century.

It means that no minimal surface contain a k-rational (-1)-curve. Thus, if k is separable, the only minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_n$ for $n \neq 1$.
For an arbitrary field k we have the following characterization.

THEOREM 9 ([**22**]). *A smooth projective surface X over k is minimal if and only if $X^s = X \otimes_k k^s$ admits no collections of pairwise disjoint (-1)-curves which are invariant for the action of the Galois group.*

In our discussion, a central role is played by the canonical divisor. We denote it with K_X and recall that if X is a surface it is defined as the divisor class associated to the invertible sheaf of differential two-forms $\Omega^2_X = \Lambda^2 \Omega^1_X$.
Over a general field, we have the following classification (for a complete survey, see [**22**]).

PROPOSITION 10. *Let X be a smooth projective minimal geometrically rational surface. Then X is one of the following:*
- *$X = \mathbb{P}^2$, so that $Pic(X) = \mathbb{Z}$;*
- *$X \subset \mathbb{P}^3$ a smooth quadric with $Pic(X) = \mathbb{Z}$;*
- *X is a del Pezzo surface with $Pic(X) = \mathbb{Z}K_X$;*
- *X is a conic bundle $\pi : X \to C$ over a geometrically rational curve C, with $Pic(X) = \mathbb{Z} \oplus \mathbb{Z}$.*

1.1.1. *The del Pezzo surfaces.* We want to give a brief recall about del Pezzo surfaces.

DEFINITION 11. Let X be a smooth projective surface with canonical class K_X. X is a *del Pezzo surface* if $-K_X$ is ample.

This definition means that there exist an embedding $X \subset \mathbb{P}^N$ such that

$$\mathcal{O}_{\mathbb{P}^N}(1)_{|X} = \mathcal{O}_X(-rK_X) \text{ for some } r > 0.$$

In particular, if X is a del Pezzo, then $K_X^2 > 0$.
We give now two examples of del Pezzo surfaces, the first one is rational, the second one is not.

EXAMPLE 12. Let k be an arbitrary field and k^s a separable closure. Let us consider $P_1, P_1', P_2, P_2', P_3, P_3'$ be six points in $\mathbb{P}^2_{k^s}$ such that P_i and P_i' are conjugate under the action of the Galois group of the extension k^s over k. The cubic plane curves passing through the six points define an injective map

$$\pi : \mathbb{P}^2_k \smallsetminus \{P_1, P_1', P_2, P_2', P_3, P_3'\} \longrightarrow \mathbb{P}^3_k$$

The image of π is a smooth cubic surface X in $\mathbb{P}^3$ with exactly three lines defined over k: they are the lines joining two of the three couples of conjugate points.
As we will see in Paragraph 1.1.3, a smooth cubic surface X can be geometrically realized as the blow-up of $\mathbb{P}^2_{k^s}$ in 6 points. In this case since the set of six points coincide with three pairs of conjugate points, X is described as the blow-up of $\mathbb{P}^2_k$ at those couples. We observe that it is a del Pezzo surface of degree

$$K_X^2 = K_{\mathbb{P}^2}^2 - 6 = 3.$$

We remark that since the blow-up is well defined over k, the surface is rational.

EXAMPLE 13. An example of non rational Del Pezzo surface is given by the following theorem due to Segre, for a reference see [**3**]

Proposition 14. *Let S be a smooth cubic surface over the arbitrary field k. Then the following are equivalent:*

- *S can be defined by the equation $\det(M) = 0$ for some 3×3 linear matrix M;*
- *S is rational;*
- *S contains a rational point and a set (defined over k) of six disjoint lines.*

1.1.2. *Conic bundles.* Let X be a smooth projective variety and $f : X \to B$ a dominant morphism. We say that f is *relatively minimal* if there exist no smooth variety Y and morphism $\phi : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

We observe that if k is separable, X is minimal if and only if there are no (-1)-curves in the fibers of f.

Definition 15. Let X be a smooth projective surface and C a smooth curve. A *conic bundle* is a relatively minimal dominant morphism $f : X \to C$ such that the generic fibre is a smooth curve of genus zero. If each fiber of f is isomorphic to $\mathbb{P}^1$, we say that $f : X \to B$ is a $\mathbb{P}^1$-bundle.

We observe that the minimality condition is necessary to ensure that also the degenerate fibers of $X \to C$ are conics. Moreover, in general we do not have a section of f defined over k.

Each fiber of f is a plane conic and it splits over a quadratic extension. The following theorem describes the geometry of these surfaces.

Theorem 16 ([**22**]). *Let $f : X \to C$ be a conic bundle. Every reducible fiber of $f^s : X^s \to C^s$ consists of two (-1)-curves intersecting in one point, conjugate under the Galois action.*

We denote with $\omega_{X|C}$ the *relative dualizing sheaf*:

$$\omega_{X|C} = \Omega^2_X \otimes (f^*\Omega^1_C)^{-1}.$$

Theorem 17. *Let $f : X \to C$ be a conic bundle with relative dualizing sheaf $\omega_{X|C}$. Then there exist an embedding over C*

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathbb{P}(f_*\omega^{-1}_{X|C}) \arrow[dl] \\ & C & \end{tikzcd}$$

which realizes each fiber of X as a plane conic.

We want now to describe the Picard group of X.

Remark 18. Let $\pi : X \to C$ be a conic bundle as above. We recall ([**42**]) that the Picard group of X is of the form

$$\operatorname{Pic} X \cong \pi^* \operatorname{Pic} C \oplus \mathbb{Z}D$$

where if X is rational $D \cdot F = 1$, otherwise $D \cdot F = 2$. In both cases, the Picard number of X is 2.

1.1.3. *Conic bundles and del Pezzo surfaces on a separable field.* On a separable field $k = k^s$, del Pezzo surfaces have been fully classified. In fact we have the following proposition.

Proposition 19. *Let X be a smooth del Pezzo surface. If $\mathrm{Pic}(X) = \mathbb{Z}$ then $X \cong \mathbb{P}^2$. Otherwise, X is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ in at most eight points in general position.*

Here general position has a precise meaning, namely:

- the points are distinct,
- no three points are collinear,
- no six points lie on a smooth conic
- there is not a cubic curve singular in one point and passing through the other 7.

A very interesting example is the cubic surface.

Example 20 (Cubic surface). Let us consider a smooth cubic hypersurface $X \subset \mathbb{P}^3$, we will refer to it as *cubic surface*. We recall a known construction.
Let $p_1, \ldots, p_6 \in \mathbb{P}^2$ be points in general position. The linear system of homogeneous cubics vanishing at these points has dimension four:

$$\mathcal{L} = \mathcal{L}_3(p_1, \ldots, p_6) = \langle F_0, F_1, F_2, F_3 \rangle$$

and has no other base points. The associated rational map is given by

$$\begin{aligned} \phi_{\mathcal{L}} : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ [x_0 : x_1 : x_2] &\mapsto [F_0 : F_1 : F_2 : F_3]. \end{aligned}$$

and is not defined at $p_1, \ldots, p_6$.

Let us consider the blow-up of $\mathbb{P}^2$ at those six points

$$\beta : X := Bl_{p_1,\ldots,p_6}\mathbb{P}^2 \to \mathbb{P}^2$$

and denote with $E_1, \ldots, E_6$ the respective exceptional divisors. We observe that the composition $j := \phi_{\mathcal{L}} \circ \beta : X \to \mathbb{P}^3$ is a morphism since β resolves the indeterminacy locus of $\phi_{\mathcal{L}}$. Moreover, it is a closed embedding of X in $\mathbb{P}^3$.

One interesting fact is that $j(X) \subset \mathbb{P}^3$ contains 27 lines:

- the exceptional curves E_i;
- the proper transform of the lines through two of the points;
- the proper transform of the conics through five of the basepoints.

Moreover, every cubic surface contains a pair of disjoint lines, say L_1 and L_2. Denote with $l_1, \ldots, l_5$ the lines in X which meet L_1 and L_2. There exist a birational morphism

$$\begin{aligned} \gamma : X &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ x &\mapsto (p_{L_1}(x), p_{L_2}(x)) \end{aligned}$$

where $p_{L_i} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ is the projection from L_i.
The lines $l_1, \ldots, l_5$ are contracted to distinct points $q_1 \ldots, q_5 \in \mathbb{P}^1 \times \mathbb{P}^1$ such that

- no pair of them lie on a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$;
- no four of them lie on a curve of bidegree (1, 1).

The inverse γ^{-1} of the birational map is associated to linear system of forms of bidegree $(2, 2)$ through $q_1, \ldots, q_5$.

This realizes $\mathbb{P}^1 \times \mathbb{P}^1$ birationally as a smooth cubic surface in $\mathbb{P}^3$.

We want to focus our attention on conic bundles.

Let $f : X \to C$ be a conic bundle and denote with F the class of a fiber of f. Let us consider a degenerate fiber of f, that is a fiber with irreducible components E_1, E_2. Then, $E_i^2 < 0$ and $E_i \cdot F = 0$ for each i. Thus, each reducible fiber is given by the union of two (-1)-curve. Under these hypothesis we have the following proposition.

PROPOSITION 21. *Let $f : X \to C$ be a conic bundle over a separable field. Then X is a $\mathbb{P}^1$-bundle over $C = \mathbb{P}^1$.*

We remark that if k is separable, the conic bundle cannot have degerate fibers since they would contain two (-1)-curves, against the minimality in the definition of conic bundles.

This proposition allows to proceed directly to the classification of conic bundles over separable fields.

THEOREM 22. *Let $k = k^s$ be a separable field. Every minimal ruled surface $f : X \to B$ over k is isomorphic to $\mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ a rank 2 vector bundle on B.*

Moreover, if $B \cong \mathbb{P}^1$ and X is minimal relative to $f : X \to \mathbb{P}^1$, then X is isomorphic to a Hirzebruch surface

$$\mathbb{F}_r := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-r)), \quad r \geq 0.$$

In particular, ruled surfaces over $\mathbb{P}^1$ are rational.

1.2. The Sarkisov program. In birational geometry we apply the Minimal Model Program to classify algebraic varieties. In particular, it associates to each projective variety X, a projective variety X' in the same birational class such that either X' is a Mori fiber space or it has a nef canonical divisor $K_{X'}$.

The Sarkisov program aims to factorize birational maps between Mori fiber Spaces in elementary modifications called links. We are not going to state it in full but only for the case of surfaces over an arbitrary field, see [**27**] for a reference.

For Sarkisov theory in arbitrary dimension, over an algebraically closed field of characteristic zero, see [**21**] and [**15**].

Let X be a minimal geometrically rational surface and $\overline{NE}(X)$ be its closed cone of curves, that is the closure of the cone of classes of effective 1-cycles of X. We recall that given a cone V, we say that a subcone W of V is extremal if it is closed and convex and if any two elements of V, whose sum is in W, are in W. An extremal subcone of dimension 1 is called extremal ray.

Let us denote with E an irreducible curve of X with $E \cdot K_X < 0$ and generating an extremal ray of $\overline{NE}(X)$. There exists a morphism $\pi : X \to S$ which contracts E. We observe that the structure of that morphism strictly depends on E.
If $E^2 < 0$, then E is a (-1)-curve, π is the blow-down of E and X is not minimal.
If $E^2 = 0$, then $\pi : X \to S$ is a conic bundle whose degenerate fibers are reducible singular conics.
If $E^2 > 0$, then E is big and π is constant (since it contracts E and all of its deformations). Then either S is a point and X has Picard number 1, or S is a curve and X is a conic bundle with Picard number 2.

We will follow the notation of Iskovskikh [**27**] both to describe the Sarkisov factorization and to give its classification.

NOTATION 23. We will denote with $\{\mathcal{D}\}$ the family of the smooth minimal geometrically rational surfaces with Picard number 1. It contains:

- $\mathbb{P}^2$;
- $Q \subset \mathbb{P}^3$ a non-singular quadric with Picard number 1;

- the del Pezzo surfaces X, with Picard group generated by the anticanonical class $-K_X$ and such that $\deg X = (-K_X)^2 = 9 - r$ with $0 \leq r \leq 8$.

On the other hand we denote with $\{\mathcal{C}\}$ the family of the relatively minimal geometrically rational surfaces with Picard number 2. Its elements are the conic bundles $X \to C$ (over a smooth geometrically rational curve C).

Let $\pi : X \longrightarrow S$ and $\pi' : X' \longrightarrow S'$ be two extremal contractions as defined above. Iskovskikh [**27**] proves the following theorem.

THEOREM 24. *Any birational map $\phi : X \dashrightarrow X'$ splits into a composition of finitely many elementary links of type I-IV.*

Links of type I. They are commutative diagrams of the form

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X' \arrow[l, "\phi"'] \arrow[d, "\pi'"] \\ S & S' \arrow[l, "\sigma"] \end{tikzcd}$$

where $\phi : X' \to X$ is a Mori divisorial elementary contraction, that is there exists a (-1)-curve E in X' which is contracted by ϕ. These links change the family of the variety, in fact $\pi : X \to S = Spec(k)$ is the contraction to a point thus X is in $\{\mathcal{D}\}$ and $\pi' : X' \to S'$ is a conic bundle in $\{\mathcal{C}\}$. Moreover $\sigma : S' \to S$ is the constant morphism.

EXAMPLE 25. An example of link of type I is given by the following. Let $X = \mathbb{P}^2$, $\phi : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in a point, with exceptional divisor the (-1)-curve of $\mathbb{F}_1$. Thus $S = x$ is a point in $\mathbb{P}^2$ and S' is the projective line $\mathbb{P}^1$. Moreover $\pi' : \mathbb{F}_1 \to \mathbb{P}^1$ is the natural $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and $S = \mathbb{P}^1 \to S' = x$ is the structure morphism.

$$\begin{tikzcd} \mathbb{P}^2 \arrow[d] & \mathbb{F}_1 \arrow[l, "\phi"'] \arrow[d] \\ x & \mathbb{P}^1 \arrow[l, "\sigma"] \end{tikzcd}$$

Links of type II. They are commutative diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[d, "\pi"'] & & X' \arrow[d, "\pi'"] \\ S \arrow[rr, "\simeq"'] & & S' \end{tikzcd}$$

where $p : Z \to X$ and $p' : Z \to X'$ are Mori divisorial elementary contractions. We observe that in this case X and X' have the same Picard number, thus ϕ transforms a surface in $\{\mathcal{D}\}$ into another surface in $\{\mathcal{D}\}$ and a conic bundle in $\{\mathcal{C}\}$ into another conic bundle in $\{\mathcal{C}\}$.

In this last case we consider $X \to S = C$ and $X' \to S' = C$ conic bundles. The morphism

$p : Z \to X$ is the blow-up of a closed point $x \in X$ not lying on a degenerate fiber of X and such that the geometric points of $x \otimes k^s$ lie in different geometric fibers. Thus $p' : Z \to X'$ is the contraction to a non-singular point $x' \in X'$ of the inverse image on Z of the fiber of X containing the point x.

Example 26. For example, we consider the elementary transformation from $\mathbb{F}_n$ to $\mathbb{F}_{n-1}$ with $n > 1$.That is, in our notation $X = \mathbb{F}_n$, $X' = \mathbb{F}_{n-1}$ and π, π' are the natural projection onto $\mathbb{P}^1$. The morphism $p : Z \to \mathbb{F}_n$ is the blow-up of a point x not lying on the $(-n)$-curve of $\mathbb{F}_n$ and $p' : Z \to \mathbb{F}_{n-1}$ is the contraction of the inverse image on Z of the fiber of $\mathbb{F}_n$ containing the point x.

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ \mathbb{F}_n \arrow[rr, dashed, "\phi"] \arrow[d, "\pi"'] & & \mathbb{F}_{n-1} \arrow[d, "\pi'"] \\ \mathbb{P}^1 \arrow[rr, "\simeq"'] & & \mathbb{P}^1 \end{tikzcd}$$

The inverse of this birational map is still a link of type II and is still an elementary transformation. Let us consider $X = \mathbb{F}_n$ and $X' = \mathbb{F}_{n+1}$. To obtain this transformation it is enough to take the point $x \in \mathbb{F}_n$ in the $(-n)$-curve.

Example 27. Now, we want to show an example of link of type II with X and X' in $\{\mathcal{D}\}$. Let X be the projective plane $X = \mathbb{P}^2$, then $S = \operatorname{Spec} k$. Let us consider a 0-cycle $x \in X$ of degree 2. There exists a unique line $L_x \subset \mathbb{P}^2$ containing x, since it has degree 2. Let $p\colon Z \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in x and $p'\colon Z \to X'$ be the contraction of the inverse image of the line L_x to a non-singular point $x' \in X'$ of degree 1. Then X' is a quadric $Q \subset \mathbb{P}^3$ with Picard number 1 and $S' = \operatorname{Spec} k$.

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi"] \arrow[d, "\pi"'] & & Q \subset \mathbb{P}^3 \arrow[d, "\pi'"] \\ S \arrow[rr, "\simeq"'] & & S' \end{tikzcd}$$

Links of type III. They are commutative diagrams of the form:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, "\sigma"'] & S' \end{tikzcd}$$

where $\phi : X \to X'$ is a Mori divisorial elementary contraction and $\sigma : S \to S'$ is a morphism. They are the inverse of the links of type I. In particular, $X \in \{\mathcal{C}\}$, $X' \in \{\mathcal{D}\}$ and $\phi\colon X \to X'$ is the contraction of an irreducible exceptional divisor of X to a non-singular point in X'.

EXAMPLE 28. Let us consider a smooth cubic surface $X \subset \mathbb{P}^3$ with a line $E \subset X$ defined over k. A plane in $\mathbb{P}^3$ containing E intersect X in a plane conic. Thus, the pencil of planes of $\mathbb{P}^3$ containing E determines on X a structure of conic bundle $X \to S = \mathbb{P}^1$. The morphism $\phi : X \to X'$ is the contraction of E to a non-singular 0-cycle $x' \in X'$ of degree 1. The surface $X' \in \{\mathcal{D}\}$ is a del Pezzo surface of degree 4 and the morphism $S = \mathbb{P}^1 \to S' = s$ is constant and contracts $\mathbb{P}^1$ to one point.

$$\begin{tikzcd}
\mathbb{P}^3 \supset X \arrow[r, "\phi"] \arrow[d] & X' \arrow[d] \\
\mathbb{P}^1 \arrow[r, "\sigma"'] & s
\end{tikzcd}$$

Links of type IV. They are commutative diagrams of the form

$$\begin{tikzcd}
X \arrow[rr, dashed, "\simeq"] \arrow[d, "\pi"'] & & X' \arrow[d, "\pi'"] \\
S \arrow[dr, "\psi"'] & & S' \arrow[dl, "\psi'"] \\
& Spec(k) &
\end{tikzcd}$$

where $\pi : X \to S$ and $\pi' : X' \to S'$ are two different conic bundle of the same variety $X = X'$. This means that in order to realize a link of type IV there must exist two extremal rays on X and $X \in \{\mathcal{C}\}$.

EXAMPLE 29. The simplest example is given by $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\pi, \pi' : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ the canonical projections onto the two factors. In this case there exists an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which exchanges the two factors but in general, such an automorphism does not exist.
For example, let us consider a minimal del Pezzo surface $X \subset \mathbb{P}^4$ of degree 4 with a structure of conic bundle $X \to \mathbb{P}^1$ with fiber F. The Picard number of X is 2 and it presents another structure of conic bundle whose fiber F' is equivalent to $-K_X - F$. For some choices of the field k there exists a biregular involution which exchanges the two factors, but in general it is false.

Iskovskikh [**27**] gives the complete classification of those links over an arbitrary field k.

Links on a separable field. We resume the above argument in the case of a separable field. We recall that the smooth minimal surfaces are $\mathbb{P}^2$ and $\mathbb{F}_n$ for $n \neq 1$.

Thus, links of type I consist in the blow-up of a closed point in one of those surfaces and analogously for links of type III. For links of type II, we just have the elementary transformations from $\mathbb{F}_n$ to $\mathbb{F}_{n\pm1}$. Lastly, links of type IV are realized only in $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and consists in the exchange between the two factors.

2. Complex Birational Geometry and Cremona transformations

In this section we will work over the complex field.

We recall the main properties and results we are going to use about the group of birational modifications of the projective plane.

Definition 30. A *Cremona transformation* is a birational map from $\mathbb{P}^n$ to $\mathbb{P}^n$ for some n. The set of all the Cremona transformations of $\mathbb{P}^n$ forms a group, denoted by $Cr(n)$ and called the *Cremona group.*

A fundamental result in the study of birational modifications of the plane is the Noether-Castelnuovo theorem: it presents a minimal set of generators for Cr(2).

Theorem 31 (Noether-Castelnuovo). *The Cremona group Cr_2 of the projective plane is generated by the self-morphisms of $\mathbb{P}^2$ and the standard quadratic transformation σ, where*

$$\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0 : x_1 : x_2] \mapsto [x_1x_2 : x_0x_2 : x_0x_1].$$

An interesting class of Cremona modifications is given by the de Jonquières transformations.

Definition 32. Let $\{x, y_1, \dots, y_{2d-2}\}$ be a set of distinct points such that no two points y_i and y_j lie on the same lime through x. Let $\phi_{\mathcal{L}}$ be the birational transformation associated to the linear system

$$\mathcal{L} = \mathcal{L}_d(d-1, 1^{2d-2}) = \mathcal{L}_d\big((d-1)x, 1y_1, \dots, 1y_{2d-2}\big)$$

with the point x of muliplicity $d-1$ and $y_1, \dots y_{2d-2}$ simple base points. We say that $\phi_{\mathcal{L}}$ is a *de Jonquières transformation* of degree d. Let us stress that we always assume that $\phi_{\mathcal{L}}$ is a birational map and it is defined by a linear system $\mathcal{L}$ without fixed components. If no confusion is likely to arise, we will identify birational modifications equivalent up to a linear automorphism of $\mathbb{P}^2$. A de Jonquières transformation of degree 2 is the standard Cremona transformation.

The map $\phi_{\mathcal{L}}$ contracts the lines spanned by x and the points y_i and the unique curve C of degree $d-1$ containing $\{x, y_1, \dots, y_{2d-2}\}$ with $\operatorname{mult}_x C = d-2$ and $\operatorname{mult}_{y_i} C = 1$ for $i = 1, \dots, 2d-2$.

Let l be a line and $\phi_{\mathcal{L}}$ a de Jonquières transformation. If $x \in l$, the line l is contracted if and only if $y_i \in l$ for some index i, otherwise $\phi_{\mathcal{L}}(l)$ is a line. If $x \notin l$ then $\phi_{\mathcal{L}}(l)$ is either contracted or it is a rational curve of degree $\delta := d - \sharp(l \cap \{y_1, \dots, y_{2d-2}\})$ with a point of multiplicity $\delta - 1$.

Remark 33. The Noether-Castelnuovo theorem holds only in dimension 2. In fact, it seems clear that for $n > 2$ and for any positive integer d, the set of Cremona transformations of degree d generates a proper subgroup of $\mathrm{Cr}(n)$, [**24**].

2.1. The action of the Cremona group. In the study of the Cremona transformations, we may focus our attention on the action of $\mathrm{Cr}(n)$ on subvarieties of $\mathbb{P}^n$. In particular, for $n = 2$ it reduces to the study of the properties of curves or linear systems of curves which are Cremona invariant, i.e. the properties preserved by Cremona transformations.

Let $\mathcal{L}$ be a linear system of plane curves. It is well known that the dimension of $\mathcal{L}$ is Cremona invariant. On the other hand, the degree of the linear system, that is the degree of the curves in $\mathcal{L}$, is not preserved.

Proposition 34. *Let $\mathcal{L}$ be a linear system of degree d. Then d is not preserved by the action of the Cremona group.*

Proof. Let $\mathcal{L} = \mathcal{L}_d(m_1, \dots, m_s)$ be the linear system of degree d with base points $p_1, \dots p_s$ of multiplicity respectively $m_1, \dots m_s$. We can assume that $d > 3$ and that three of the base points are distinct and not collinear, say p_1, p_2, p_3. Applying the standard Cremona transformation centered in those points, the image of $\mathcal{L}$ is:

$$\mathcal{L}_{2d-m_1-m_2-m_3}(d-m_2-m_3, d-m_1-m_3, d-m_1-m_2, m_4, \dots, m_s)$$

This means that in general the degree is not preserved. $\square$

However, we can define another invariant: the Cremona degree.

DEFINITION 35. The *Cremona degree* of a linear system $\mathcal{L}$ is the minimal degree of a linear system in the orbit of $\mathcal{L}$ through the action of Cr(2). The linear systems realizing the Cremona degree are called *Cremona minimal model* of $\mathcal{L}$.

Let $\mathcal{L}$ be a linear system of dimension 0, that is $\mathcal{L}$ consists in a unique curve C. Then the Cremona degree of C is 0 if and only if C can be contracted to a finite set of points using Cremona transformations. In this case we say that C is Cremona contractible.

DEFINITION 36. Let S be a rational surface and $D \subset S$ an effective divisor. The divisor D is *(Cremona) contractible* if there exists a birational map $\omega\colon S \dashrightarrow \mathbb{P}^2$ such that $\omega_* D = 0$ i.e. D is contracted by ω.

We remark that the definition of contractibility holds only for curves. In fact if the linear system $\mathcal{L}$ has positive dimension then the Cremona degree of $\mathcal{L}$ is strictly positive. The Cremona action on pencils of irreducible rational curves is particularly simple.

PROPOSITION 37. *Let $\mathcal{L}$ be a pencil of irreducible rational curves. Then $\mathcal{L}$ is Cremona equivalent to the pencil of lines through one point.*

PROOF. Firstly we observe that $\mathcal{L}$ has no fixed components since its generic element is irreducible. Let us denote with B the set of base points of $\mathcal{L}$. We blow-up the 0-dimensional scheme supported on B and obtain a base-point-free pencil $\mathcal{L}'$ on a smooth surface S. It follows that there is a structure of $\mathbb{P}^1$-bundle

$$\pi\colon S \longrightarrow \mathbb{P}^1$$

whose fibers are the element of the pencil $\mathcal{L}'$. Thus S is equivalent to $\mathbb{F}_1$ through a birational transformation $\phi : S \dashrightarrow \mathbb{F}_1$. Contracting the (-1)-curve of $\mathbb{F}_1$, we send $\phi_*(\mathcal{L}')$ to a pencil of lines in $\mathbb{P}^2$. $\square$

In 1889 G. Jung proved the following theorem

THEOREM 38 (Jung). *Let $\mathcal{L} = \mathcal{L}_d(m_1, m_2, m_3 \dots)$ be a linear system without multiple fixed components with $m_1 \geq m_2 \geq m_3 \geq \dots$ and $d \geq m_1 + m_2 + m_3$. Then it realizes its Cremona degree.*

Later Marletta [**36**] gave a sufficient condition that yelds Cremona minimality for an irreducible curve. The complete classification of irreducible Cremona minimal curves has been recently obtained by Mella and Polastri [**37**] and by Calabri and Ciliberto [**9**].

2.2. Adjoint linear systems and Kodaira dimension. The study of minimal model of irreducible curves and linear systems has highlighted the importance of adjoint linear systems.

In our discussion we reduce to the complex case, but in general they can be defined over an arbitrary field.

DEFINITION 39. Let $C \subset \mathbb{P}^2$ be a reduced curve and $f : S \to \mathbb{P}^2$ a log resolution, i.e. a resolution of the singularities of C with a normal crossing divisor. Let us denote with $\bar{C}$ the strict transform of C on S. The *Adjoint linear system* $\mathrm{adj}_{n,m}(C)$ of C is defined as

$$\mathrm{adj}_{n,m}(C) = f_*(|n\bar{C} + mK_S|)$$

for any pair of integers n, m with $m \geq n$ and $n \geq 1$. Moreover, the sequence

$$dim(\mathrm{adj}_{n,m})_{m \geq n}$$

is called the *n-adjoint sequence* of C.

Let (S, D) be a *pair*, with S a smooth, projective surface and D a reduced effective divisor on S. For any non-negative integer m, the *m-log plurigenus* of the pair is

$$P_m(S, D) = h^0(S, \mathcal{O}_S(mD + mK_s))$$

where K_S is the canonical divisor of S.
The *log Kodaira dimension* of (S, D) is defined as

$$\begin{aligned} &kod(S, D) = -\infty && \text{if } P_m(S, D) = 0 \text{ for all } m \geq 1 \\ &kod(S, D) = max(dim(Im(\phi_{|mD+mK_S|}))) && \text{otherwise} \end{aligned}$$

Remark 40. It is easy to see that for each n, the n-adjoint sequence stabilizes to -1 since the adjoint linear system $\mathrm{adj}_{n,m}(C) = \emptyset$ for a sufficiently big m.

Le us observe that if $\gamma : S \dashrightarrow S'$ contracts D, then D is contained in the exceptional locus of γ, thus $kod(S, D) = -\infty$ (see also [**11**]]).

Remark 41. Let us consider the pair $(\mathbb{P}^2, D)$ and let $\pi : S \to \mathbb{P}^2$ be the resolution of the singularities of D. Let us denote by $\bar{D}$ be the strict transform of D in S. If D is contractible, then so it is $\bar{D}$ and moreover $kod(S, \bar{D}) = -\infty$.

Since $\bar{D}$ is effective, we have also

$$\mathrm{adj}_{n,m} = \emptyset \text{ for each } m \geq n \geq 1$$

We will often use the non emptiness of the adjoint linear systems to prove the non-contractibility of a divisor.

An important aspect of these objects lies in the following lemma, due to Kantor.

Proposition 42. *If C is a reduced plane curve, the n-adjoint sequence of C is Cremona invariant.*

We observe that in this Proposition we consider the adjoint sequence instead of the adjoint linear systems. This is due to the fact that each adjoint linear system is not a birational invariant: it may contain an exceptional fixed component (which can be contracted by a Cremona transformation).

2.2.1. *Contractibility for irreducible curves.* Coming back to the classification of minimal models of irreducible curves, we can understand the important role of the adjoint linear systems. In particular, we will state some results which highlight the correlation between adjoint linear systems and contractibility. The first one is due to Castelnuovo and Enriques (1900) but proved later by Ferretti.

Theorem 43 (Castelnuovo-Enriques). *An irreducible curve C is Cremona contractible if and only if all adjoint linear systems to C vanish.*

Successively, this theorem has been improved by Kumar and Murthy (1982) [**30**].

Theorem 44 (Kumar-Murthy). *An irreducible plane curve C is Cremona contractible if and only if* $\mathrm{adj}_{1,1}(C)$ *and* $\mathrm{adj}_{1,2}(C)$ *vanish.*

A straightforward consequence is in the following corollary.

Corollary 45. *Let C be an irreducible plane curve and (S, C_S) be a pair where $S \to \mathbb{P}^2$ is a resolution of the singularities of C and C_S is the strict transform of C. Then the following are equivalent:*

- *C is Cremona contractible;*
- $\mathrm{kod}(S, C_S) = -\infty$*;*
- *the adjoint linear systems* $\mathrm{adj}_{1,m}(C)$ *are empty for any m;*
- *the adjoint linear systems* $\mathrm{adj}_{1,1}(C)$ *and* $\mathrm{adj}_{1,2}(C)$ *are empty.*

2.2.2. *Contractibility for reducible curves.* We may ask if we can generalize the theorems to the reducible case. Here the situation is more complicated: the only known result is due to Itaka, which extends the theorem of Kumar-Murthy.

THEOREM 46 (Iitaka,[**26**]). *Let C be a reduced plane curve with two irreducible components. Then C is Cremona contractible if and only if the adjoint linear systems* $\mathrm{adj}_{1,1}(C)$ *and* $\mathrm{adj}_{1,2}(C)$ *are empty.*

We observe that we cannot generalize further Corollary 45 to any reduced curve. In fact, we have the following example, for a reference see [**38**] or, more recent, [**10**].

EXAMPLE 47 (Pompilj). Let C be a reducible curve of with three irreducible components $C = C_1 + C_2 + C_3$. Let us suppose that C is of degree 9 and has 10 triple points $p_0, \dots p_9$. Let C_1 and C_2 be plane quartic curves and C_3 a line with the following intersection:

	deg	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
C_1	4	2	2	2	1	1	1	1	1	1	1
C_2	4	1	1	1	2	2	2	1	1	1	1
C_3	1	0	0	0	0	0	0	1	1	1	1
C	9	3	3	3	3	3	3	3	3	3	3

For this curve we have that $\mathrm{adj}_{2,3}(C) \neq \emptyset$ thus C is not contractible. But we can see that $\mathrm{adj}_{1,1}(C) = \mathrm{adj}_{1,2}(C) = \emptyset$.

Analogously, the following example shows that it is not possible to generalize Theorem (43) of Castelnuovo-Enriques to reducible curves.

EXAMPLE 48. Let C be a union of $d \geq 9$ distinct lines $l_1 \dots l_d$, and suppose that C has a point p_0 of multiplicity $d-3$ and $3(d-2)$ nodes $p_1, \dots p_{3(d-2)}$. This means that the lines $l_1, \dots l_{d-3}$ meet in p_0 and the other three lines are in general positions and meet the $l_1, \dots l_{d-3}$ in double points. It is easy to see that $\mathrm{adj}_{1,m}(C) = \emptyset$ for all $m \geq 1$. On the other hand, the adjoint linear system $\mathrm{adj}_{2,3}(C)$ is not empty since it is the system:

$$\mathrm{adj}_{2,3}(C) = \mathcal{L}_{2d-9}(2d-9) = l_1 + \dots l_{d-3} + l'_1 + l'_2 + l'_3$$

where l'_1, l'_2, l'_3 are the lines through p_0 and the intersection points of l_{d-2}, l_{d-1}, l_d.

It follows that the theorems that we have for irreducible curves cannot be generalized. We may also recall another result which involves curves with at most four irreducible components.

THEOREM 49 (Kojima-Takahashi). *Let (S, C) be a pair with S a smooth rational surface and C a reduced curve on S with at most four irreducible components. Then,* $\mathrm{kod}(S, C) = -\infty$ *if and only if* $P_6(S, C) = 0$.

This theorem does not relate Kodaira dimension with the contractibility.
In order to discuss this correlation we need the following definition.

DEFINITION 50. Let S be a rational surface, $D \subset S$ an effective divisor and $Z \subset S$ a 0-dimensional subscheme. We say that (D, Z) is a *marked divisor.* Moreover the triple (S, D, Z) is called *marked triple.*

The notion of marked triple must be compared with the notion of cluster, see [**12**].

Let $\phi : S \dashrightarrow S'$ be a birational map and (D, Z) a marked divisor in S. The image (D', Z') of (D, Z) through ϕ is defined as follows.
Let us consider a resolution of ϕ

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, dashed, "\phi"] & & S' \end{tikzcd}$$

such that all the valuations associated to Z are divisors on W, say $E_1, \ldots, E_s$. Assume that there exists an index h such that for $i \le h$, E_i is q-exceptional and for $j \ge h$ E_j is not q-exceptional. Then

$$(D', Z') = \big(q_* p_*^{-1} D + q_* \sum_{j=h+1}^{s} E_j, \quad q_* \sum_{i=1}^{h} E_i\big).$$

It is easy to see that it is still a marked divisor.

We can extend the notion of contractibility for marked divisors.

DEFINITION 51. Let (D, Z) be a marked divisor on a rational surface S and let $p : W \to S$ be a resolution of Z. Let us denote with $E_1, \ldots, E_r$ the divisors associated to the valuations of Z and with D_W the strict transform of D. We say that (D, Z) is *contractible* if $D_W + \sum_i E_i$ is contractible in the sense of definition 36

The following theorem has been recently proved by Calabri and Ciliberto [**12**].

THEOREM 52. *Let (S, D, Z) be a marked triple with S rational, D connected and simple normal crossing. If* $\operatorname{kod}(S, D) = -\infty$ *then (S, D, Z) is contractible.*

We recall that a divisor D is simple normal crossing if each component is smooth and D has at most nodes.

In particular, this theorem will be interesting in order to study a completely reduced divisor, that is a union of lines. In this case, there is a result of Calabri-Ciliberto [**11**] which lists the Cremona contractible unions of $d \ge 12$ lines in the projective plane $\mathbb{P}^2$.

THEOREM 53. *Let C be a reduced union of $d \ge 12$ lines. The adjoint linear systems* $\operatorname{adj}_{1,n}(C)$ *vanish for each $n \ge 1$ if and only if C has a point of maximal multiplicity $m \ge d - 3$. Moreover,* $\operatorname{kod}(C) = -\infty$ *if and only if $m \ge d - 2$. C is contractible if and only if* $\operatorname{kod}(C) = -\infty$.

They stated also the following conjecture:

CONJECTURE 54. *Let D be a reduced union of lines in $\mathbb{P}^2$. Then D is Cremona contractible if and only if* $\operatorname{kod}(\mathbb{P}^2, D) = -\infty$.

3. Derived categories

In this section we want to define the categorical theoretic setting for the geometric work in Chapter 3. In particular we introduce exceptional collections, mutations and semiorthogonal decompositions of triangulated categories. To have a complete introduction to these topics, see [**25**] and [**34**].

Let X be a smooth projective variety over a field k and $\mathrm{Coh}(X)$ the abelian category of coherent sheaves on X. We will denote its bounded derived category by $D^b(X) := D^b(\mathrm{Coh}(X))$. We recall that it is a triangulated k-linear category.

We can also define derived functors. In general, let us consider $\mathcal{A}$ and $\mathcal{B}$ abelian categories with enough injective objects and $F : \mathcal{A} \to \mathcal{B}$ a left exact functor. Using injective resolutions, we define the right derived functor $RF : D^b(\mathcal{A}) \to D^b(\mathcal{B})$. Analogously, we define the left derived functor using projective objects and right exact functors.

We can apply these constructions to functors arising from algebraic geometry. If $f : X \to Y$ is a morphism of smooth projective varieties, we can define Rf_*, Lf^*, $R\mathrm{Hom}$ and $\otimes^L$. From now on, we will omit the derived notation and, for example, we will denote by f_* the derived functor between derived categories Rf_*. (See notations in Chapter 3). For details, see [**25**].

Derived categories are an important tool since they can express some geometric aspects of the variety. For example, if two smooth projective varieties have equivalent derived categories, then they have the same dimension. Moreover, Bondal and Orlov [**7**] proved that two varieties with ample canonical (or anticanonical) class and equivalent derived categories, are isomorphic.

3.1. Exceptional objects and mutations.

Definition 55. Let $\mathcal{T}$ be a k-linear triangulated category and let A be a division k-algebra. An object $E \in \mathcal{T}$ is called *A-exceptional* if

$$\mathrm{Hom}_{\mathcal{T}}(E, E) = A \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{T}}(E, E[r]) = 0 \text{ for all } r \neq 0.$$

A totally ordered set $(E_1, \dots E_n)$ of exceptional objects is called an *exceptional collection* if $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[r]) = 0 \ \forall r$ and $\forall i > j$.

Given an exceptional sequence $(E_1, \dots, E_n)$ we denote as $\langle E_1, \dots, E_n \rangle$ the triangulated subcategory of T generated (as a category) by that sequence.

In the following example we investigate the correlation between exceptional divisors and exceptional objects in the derived category.

Example 56. Let X be a smooth projective surface and let $\sigma : \tilde{X} \to X$ be the blow-up of a rational point $x \in X$ with exceptional divisor E.

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d] & \tilde{X} \arrow[d, "\sigma"] \\ x \arrow[r, hook, "i"'] & X \end{tikzcd}$$

We want to show that $\mathcal{O}_E$ is an exceptional object. This means that we have to prove that the following complex is concentrated in degree 0.

$$\mathrm{Hom}^r(j_*\mathcal{O}_E, j_*\mathcal{O}_E) = \begin{cases} k & \text{if } r = 0 \\ 0 & \text{if } r \neq 0. \end{cases}$$

Since we are in the case of surfaces, it is enough to prove the claim for $r = 0, 1, 2$.
By [**25**] Ch.11, we have that

$$\mathrm{H}^p(E, \Lambda^q \mathcal{O}_E(-1)) = \mathrm{Ext}^{p+q}_{\tilde{X}}(\mathcal{O}_E, \mathcal{O}_E)$$

for any integers p, q. Thus, we have

$$\begin{aligned}
\mathrm{Hom}^0(\mathcal{O}_E, \mathcal{O}_E) &= \mathrm{H}^0(E, \mathcal{O}_E) \cong k \\
\mathrm{Hom}^1(\mathcal{O}_E, \mathcal{O}_E) &= \mathrm{H}^1(E, \mathcal{O}_E) \oplus \mathrm{H}^0(E, \mathcal{O}_E(-1)) = 0 \\
\mathrm{Hom}^2(\mathcal{O}_E, \mathcal{O}_E) &= \mathrm{H}^2(E, \mathcal{O}_E) \oplus \mathrm{H}^1(E, \mathcal{O}_E(-1)) \oplus \mathrm{H}^0(E, \mathcal{O}_E(-2)) = 0
\end{aligned}$$

Example 57. Let X be a del Pezzo surface. Kuleshov and Orlov [**29**] have characterized the exceptional objects of the derived category $D^b(X)$ in term of exceptional divisors. In particular they show that an object A of $D^b(X)$ is exceptional if and only if either it is a line bundle or it is isomorphic to $\mathscr{E}[i]$ for some exceptional sheaf $\mathscr{E}$ on X. This means that on a del Pezzo surface, exceptional objects are supported on (-1)-curves or on the whole surface.

We are ready to give the definition of mutations.

Definition 58. Let $\mathcal{T}$ be a triangulated category and let E be an exceptional object in $\mathcal{T}$. There exists a canonical morphism

$$l_{can} : \mathcal{T} \longrightarrow \mathcal{T} \tag{2}$$

$$R\,\mathrm{Hom}(E, A) \otimes A \longmapsto A \tag{3}$$

and its dual is

$$r_{can} : A \longmapsto R\,\mathrm{Hom}(A, E)^* \otimes E.$$

Those morphisms determine the natural transformations:

$$\begin{aligned}
l_{can} :& R\,\mathrm{Hom}(E,_)\otimes E \to \mathrm{id}_{\mathcal{T}} \\
r_{can} :& \mathrm{id}_{\mathcal{T}} \to R\,\mathrm{Hom}(-, E)^* \otimes E.
\end{aligned}$$

They define two objects: the *left mutation* $L_E(A)$ and the *right mutation* $R_E(A)$ of A with respect to E. In particular the triangles

$$\begin{aligned}
&(R\,\mathrm{Hom}(E, A) \otimes E, A, L_E(A)) \\
&(R_E(A), A, R\,\mathrm{Hom}(A, E)^* \otimes E)
\end{aligned}$$

are distinguished.

We observe that for any exceptional object E in $\mathcal{T}$, the functors L_E and R_E are exact additive functors.

Proposition 59. *Let (A, B) be an exceptional collection of objects of $\mathcal{T}$. There exists a morphism $\alpha : \mathrm{Hom}(A, B) \otimes A \to B$. Then the pair $(Cone(\alpha), A)$ is exceptional.*

Proof. We have to show that $\mathrm{Hom}^\bullet(A, Cone(\alpha)) = 0$ as a complex. Let us consider the sequence

$$\mathrm{Hom}(A, B) \otimes A \xrightarrow{\alpha} B \longrightarrow Cone(\alpha)$$

and apply the contravariant functor $\mathrm{Hom}(A, -)$:

$$\mathrm{Hom}(A, Cone(\alpha)) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}\,(A, \mathrm{Hom}(A, B) \otimes A) \cong \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(A, A)$$

Thus we obtain

$$\mathrm{Hom}(A, Cone(\alpha)) \longrightarrow \mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}(A, B).$$

Since the last arrow is an isomorphism, it follows that $\mathrm{Hom}(A, Cone(\alpha))$ is the zero complex. $\square$

3.2. Admissible subcategories and semiorthogonal decompositions. Let $\mathcal{T}$ be a k-linear triangulated category. A full triangulated subcategory $\mathcal{A}$ of $\mathcal{T}$ is admissible if the embedding functor admits left and right adjoints. The right orthogonal complement of an admissible subcategory $\mathcal{A}$ of $\mathcal{T}$ is the full subcategory $\mathcal{A}^{\perp} \subset \mathcal{T}$ of all objects $T \in \mathcal{T}$ such that $\mathrm{Hom}(A, T) = 0$ for all $A \in \mathcal{A}$. Analogously we define the left orthogonal ${}^{\perp}\mathcal{A}$ as the full subcategory of $\mathcal{T}$ of objects of $T \in \mathcal{T}$ such that $\mathrm{Hom}(T, A) = 0$ for all $A \in \mathcal{A}$ (see [**25**]).
Let $(\mathcal{A}_1, \mathcal{A}_2)$ be an exceptional collection of 2 subcategories of $\mathcal{T}$. It is *completely orthogonal* if $(\mathcal{A}_2, \mathcal{A}_1)$ is also exceptional. In this case we have that both $\mathrm{Hom}(A_1, A_2) = 0$ and $\mathrm{Hom}(A_2, A_1) = 0$ for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Definition 60. A *semiorthogonal decomposition* of $\mathcal{T}$ is a sequence of admissible subcategories $\mathcal{A}_1, \dots \mathcal{A}_n$ of $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(A_i, A_j) = 0$ for all $i > j$, A_i in $\mathcal{A}_i$ and A_j in $\mathcal{A}_j$ and for every object T in $\mathcal{T}$ there is a chain of morphisms $0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$ such that the cone of $T_r \to T_{r-1}$ is an object of $\mathcal{A}_r$ for all $r = 1, \dots, n$. We denote it by:

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

Example 61. Let $\mathcal{A} \subset \mathcal{T}$ be an admissible subcategory and let $\mathcal{A}^{\perp}$ and ${}^{\perp}\mathcal{A}$ be respectively the left and the right orthogonal of $\mathcal{A}$ in $\mathcal{T}$. Then there exist the following semiorthogonal decompositions:

$$\mathcal{T} = \left\langle \mathcal{A}^{\perp}, \mathcal{A} \right\rangle = \left\langle \mathcal{A}, {}^{\perp}\mathcal{A} \right\rangle .$$

Moreover if $(\mathcal{A}_1, \mathcal{A}_2)$ is a completely orthogonal collection, then

$$L_{\mathcal{A}_1}(A_2) = \mathcal{A}_2 \text{ and } R_{\mathcal{A}_2}(\mathcal{A}_1) = \mathcal{A}_1.$$

Some interesting examples of admissible subcategories are given by exceptional objects.

Proposition 62 ([**25**], Ch.1). *Let $\mathcal{T}$ be a k-linear triangulated category such that the vector space $\bigoplus_i \mathrm{Hom}(A, B[i])$ has a finite dimension for any $A, B \in \mathcal{T}$. Then, if E is an exceptional object of $\mathcal{T}$, the subcategory $\langle E \rangle$ generated by E is admissible.*

Remark 63. Given an exceptional collection $(E_1, \dots, E_n)$ in the derived category $D^b(X)$ of a smooth projective variety, then there exists a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{A}, E_1, \dots, E_n \rangle ,$$

where $\mathcal{A}$ is the full triangulated subcategory left orthogonal to $\langle E_1, \dots, E_n \rangle$.

Example 64 ([**4**]). Let us consider $X = \mathbb{P}^n$ the projective space. Then its derived category can be decomposed with exceptional objects:

$$D^b(X) = \langle \mathcal{O}_{\mathbb{P}^n}(i), \dots, \mathcal{O}_{\mathbb{P}^n}(n+i) \rangle$$

for any integer i.

We can apply mutations to semiorthogonal decompositions.

Proposition 65. *Suppose that $\mathcal{T}$ admits a semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

Then for each $1 \leq k \leq n-1$, there exists a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, L_{\mathcal{A}_k}(\mathcal{A}_{k+1}), \mathcal{A}_k, \mathcal{A}_{k+2}, ..., \mathcal{A}_n \rangle$$

where the functor $L_{\mathcal{A}_k} : \mathcal{A}_{k+1} \to L_{\mathcal{A}_k}(\mathcal{A}_{k+1})$ is the left mutation through $\mathcal{A}_k$ and it is an equivalence of categories.

In the same way, for each $2 \leq k \leq n$, there is a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}_1, ..., \mathcal{A}_{k-2}, \mathcal{A}_k, R_{\mathcal{A}_k}(\mathcal{A}_{k-1}), \mathcal{A}_{k+1}, ..., \mathcal{A}_n \rangle \,,$$

where the functor $R_{\mathcal{A}_k} : \mathcal{A}_{k-1} \to R_{\mathcal{A}_k}(\mathcal{A}_{k-1})$ is the right mutation through $\mathcal{A}_k$ and it is an equivalence.

The first property we will describe is in the case $n = 2$.

PROPOSITION 66. *Let X be a smooth projective variety and $D^b(X) = \langle A, B \rangle$ a semiorthogonal decomposition. Then $L_A(B) = B \otimes \omega_X$ and $R_B(A) = A \otimes \omega_X^*$*

In general, there is no way to compare two semiortogonal decompositions of the same triangulated category since the Jordan-Hölder property does not hold.

DEFINITION 67. Let $\mathcal{T}$ be a triangulated category. The *Jordan-Hölder property* states that for any pair of semiorthogonal decompositions $\mathcal{T} = \langle \mathcal{A}_1, \dots \mathcal{A}_r \rangle$ and $\mathcal{T} = \langle \mathcal{B}_1, \dots \mathcal{B}_s \rangle$ with indecomposable components, then $r = s$ and there exists a permutation $\sigma \in S_r$ such that $\mathcal{B}_i \cong \mathcal{A}_{\sigma(i)}$ for any $1 \leq i \leq r$.

In general it does not hold and there are very few examples of derived category of a variety where it is satisfied, for example $D^b(\mathbb{P}^1)$. On the other hand a counterexample to Jordan-Hölder property has been constructed by Bondal and Kuznetsov, [**33**].

EXAMPLE 68. Let $A \subset V$ be a pair of vector spaces of dimension 2 and 4 respectively. We observe that $\mathbb{P}(A) \subset \mathbb{P}(V)$ is a line in $\mathbb{P}^3$. Let us consider the blow-up of $\mathbb{P}(V)$ along $\mathbb{P}(A)$:

$$X := Bl_{\mathbb{P}(A)}\mathbb{P}(V) \to \mathbb{P}(V)$$

and let us denote with E the exceptional divisor. Then

$$E \cong \mathbb{P}(A) \times \mathbb{P}(V/A) = \mathbb{P}^1 \times \mathbb{P}^1$$

and the Picard group of X is generated by $\mathcal{O}_E(1, 0)$.
Let C be a smooth rational curve on X which intersects E in two distinct points. Let $\pi : Y \to X$ be the blow-up of X along C. Then the derived category $D^b(Y)$ of (Y) does not have the Jordan-Hölder property, see [**33**].

CHAPTER 2

Cremona contractibility

If D is irreducible then D is contractible if and only if it is Cremona equivalent to a line. These rational curves are classified, [**37**] and [**9**]. For reducible D the situation is much more complicated and the only results we are aware of are in [**11**], where the adjoint linear systems are used to study the contractibility of configurations of lines under some numerical hypothesis. Here we plug into this picture also Sarkisov theory. First we state some general result for any configuration of rational curves and then we use those to study the configuration of lines.

1. Notation

In this chapter we work over the complex field $\mathbb{C}$. We are interested in studying birational modification of $\mathbb{P}^2$ that contracts rational curves. In particular we are interested in completely reducible divisors D in $\mathbb{P}^2$, namely configurations of lines. For these we introduce a special notation.

Definition 69. Let $q \in \mathbb{P}^2$ be a point, $\{l_1, \dots, l_{d-k}\}$ a set of distinct lines through q, and $\{r_1, \dots, r_k\}$ a set of distinct lines not containing q. We always assume that $d-k \geq \operatorname{mult}_x D$ for any $x \in D$ and $k > 0$ and call the divisor

$$D = \sum_1^{d-k} l_i + \sum_1^k r_j$$

a configuration of lines.

To study the contractibility of a configuration of lines D it is useful to have a way to label its singular points

Definition 70. Let $D = \sum_1^{d-k} l_i + \sum_1^k r_j$ be a configuration of lines. We define

$$p_{\mathbf{i}_1 j_1 \dots j_s} := l_{i_1} \cap r_{j_1} \cap \dots \cap r_{j_s}, \quad p_{j_1, \dots j_t} := r_{j_1} \cap \dots \cap r_{j_t}$$

and we always assume that, $s \geq 1$, $t \geq 2$ and

$$\operatorname{mult}_{p_{\mathbf{i}_1 j_1 \dots j_s}} D = s+1, \quad \operatorname{mult}_{p_{j_1 \dots j_t}} D = t.$$

Let $D = \sum_1^{d-k} l_i + \sum_1^k r_j$ be a configuration of lines, then for any point x we have $\operatorname{mult}_x D \leq \min\{d-k, k+1\}$. Our technique forces us to keep track of all points in $D \setminus \{q\}$ with multiplicity strictly greater than $k/2$. In fact, those points are the singularities of D which are not canonical.

We have seen that in order to control the components of a configuration that are contracted by a birational map, we use the notion of marked divisor. Moreover, to distinguish between two different marked divisors we introduce the following definition.

Definition 71. When $D=\sum_1^{d-k} l_i+\sum_1^k r_j$ is a configuration of lines we attach to the marked divisor (D,Z) two multi-indexes

$$A:=(a_{\min\{d-k,k+1\}},\dots,a_{\lceil\frac{k+1}{2}\rceil}),\quad F:=(f_{\min\{d-k,k+1\}},\dots,f_{\lceil\frac{k+1}{2}\rceil}).$$

We say that (D,Z) is of type:

$(d,k;\mathbf{A}|F)$ if there are a_i points of multiplicity i in $\operatorname{Sing}D\setminus(Z\cap\operatorname{Sing}D\cup\{q\})$, $q\notin Z$, and f_i points of multiplicity i in Z,

$(d,k;A|\mathbf{F})$ if there are a_i points of multiplicity i in $\operatorname{Sing}D\setminus((Z\cap\operatorname{Sing}D)\cup\{q\})$, $q\in Z$, and f_i points of multiplicity i in Z.

We let $|\mathbf{A}|=\sum_{\lceil\frac{k+1}{2}\rceil}^{\min\{k+1,d-k\}}a_i$. To simplify the notation, when $Z=\emptyset$ we let $D:=(D,\emptyset)$ and denote its type by

$$(d,k;\mathbf{A}):=(d,k;\mathbf{A}|(0,\dots,0)).$$

Remark 72. We summarize here some observations that will be useful throughout the proofs. Any irreducible component of a contractible divisor is a rational curve. If $D=D_1+D_2$, D contractible implies that also the D_i are contractible. Viceversa, if one D_i is not contractible, so is D.

In particular we have the following.

Lemma 73. *Let us fix d and k and assume that any configuration of type $(d,k;\mathbf{A})$ is not contractible. Then any configuration of type $(d',k';\mathbf{A}')$ is not contractible for $d'=d+c$, $k\le k'\le k+c$.*

Proof. Let D be a configuration of type $(d',k';\mathbf{A}')$. Let $D_2\subset D$ be a configuration of $d'-d$ lines. We may choose D_2 in such a way that $D_1=D-D_2$, is of type $(d,k;\mathbf{A})$. Since D_1 is not contractible the configuration D cannot be contractible. □

We found useful, along our proofs, to use the classical dictionary of de Jonquières transformations that we recalled in Definition 32. For this we introduce the following notation.

Notation 74. Let $\{x,y_1,\dots,y_{2d-2}\}$ be a set of distinct points such that the y_i's are in linear general position with respect to x. We denote with

$$\omega:=\Omega(x,y_1,\dots,y_{2d-2})$$

the de Jonquières transformation of degree d centered at $x,y_1,\dots,y_{2d-2}$. Analogously, the map $\Omega(x,y_1,y_2)$ is the standard Cremona transformation of centers x,y_1,y_2.

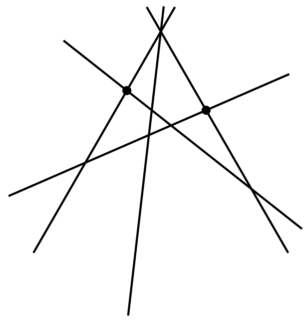
To help the reader to digest all these definitions we study in details some examples of contractible marked divisor. We will describe all birational modifications needed to contract them as a tutorial for future computation.

Example 75. Let (D,Z) be a marked configuration of type $(5,3;(\mathbf{9-a})|(a))$, with Z reduced. If $a\le 3$ then (D,Z) is contractible. We may assume $a=3$. We want to study the case with $Z=\bigcup_{i\neq j}(r_i\cap r_j)$. We stress the following convention we adopt here and in the following

(†) we keep the same name for lines and their images via ω.

This produces a marked divisor (D',Z'). Next we apply $\Omega(p_{\mathbf{2}1},p_{\mathbf{2}2},x_3)$ with $x_3\in r_3$ a general point. This map contracts l_2 and produces the marked divisor (D'',Z'') with $D''=r_1+r_2+r_3$. To contract r_3 we apply the transformation $\Omega(x_3',x_3'',p_{12})$ where $x_3',x_3''\in r_3$ are general points and $p_{12} \in Z''$. We obtain a divisor of the type $(3,1;(2),(\mathbf{1}))$ which is easily contractible, see also Lemma 77.

EXAMPLE 76. Let (D,Z) be a marked divisor of the type $(5,2;(\mathbf{0},\mathbf{5})|(0,2))$, as in the picture.



We can assume that the marked points are $p_{\mathbf{12}}$ and $p_{\mathbf{31}}$. Let us consider the transformation

$$\omega = \Omega(p_{\mathbf{21}}, p_{\mathbf{2}2}, p_{12}) \circ \Omega(q, p_{\mathbf{11}}, p_{\mathbf{3}2}).$$

The image of (D,Z) is a set of three points in $\mathbb{P}^2$. Thus it is contractible.

We have a quite general statement involving a pair of irreducible curves with points of the highest multiplicity, almost regardless of the marking.

LEMMA 77. *Assume that each $D_i \subset \mathbb{P}^2$ is irreducible and there is a point x with* $\operatorname{mult}_x D_i = \deg D_i - 1$. *Assume that one of the following is satisfied:*

a) *$D = D_1$ and Z any marking,*
b) *$D = D_1 + D_2$ and $x \notin Z$.*
c) *$D = D_1 + D_2 + l$, l is a line through x, and $x \notin Z$.*

Then the marked divisor (D,Z) is contractible.

PROOF.

CASE (a). Assume first that D is irreducible if $x \notin Z$ it is enough to consider the map $\omega := \Omega(x, y_1, \ldots, y_{2\deg D})$, with y_i general point in D. Then $\omega(D)$ is contracted by that de Jonquières transformation.
Assume that $x \in Z$. Let $\mu : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow up of x with exceptional divisor C_0. Then $D_1 := \mu_*^{-1} D$ is a section of the scroll structure and we are interested in the marked divisor $(C_0 + D_1, Z_1)$. For $N = 2m \gg 0$ we may perform N elementary transformations, say $\Phi_N : \mathbb{F}_1 \dashrightarrow \mathbb{F}_1$ in such a way that:

- $\Phi_N(D_1) \sim C_0 + f$,
- $\Phi_N(C_0) = C_0$,
- $\Phi_N(Z_1) \subset C_0 \cup D_1$.

where f is the fiber of $\mathbb{F}_1$. Then we blow down C_0, $\mu : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$, and are left with a marked divisor (l, Z'), with l a line, and conclude easily. Note that the map $\mu \circ \Phi_N \circ \mu^{-1}$ is a de Jonquières transformation of degree $m+1$.

CASE (b). Assume that $D = D_1 + D_2$ and that $x \notin Z$, then as above there is a map $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that:

- $\omega(D_1)$ is a line

- $\omega(D_2)$ is a curve with a point q with $\mathrm{mult}_q\,\omega(D_2) = \deg \omega(D_2) - 1$,
- $Z' := \omega(Z)$ is a zero dimensional subscheme and $q \notin Z'$.

Next we consider $\Omega(q, x_1, x_2)$ with x_i general points on $\omega(D_1)$ and we are back to an irreducible curve as in case a).

CASE (c). To conclude assume $D = D_1 + D_2 + l$. Let $\omega := \Omega(x, y_1, y_2)$, with $y_1 \in l$ general and $y_2 \in \mathbb{P}^2$ general. Then $\omega(D)$ is as in case b).

□

COROLLARY 78. *Let D be a configuration of lines of type $(d, k; \mathbf{A})$. If $k \leq 2$ then D is contractible.*

PROOF. Let $D = \sum_1^{d-k} l_i + \sum_1^k r_j$ be the configuration, $x_i \in l_i$ general, and $y_i \in \mathbb{P}^2$ general.

If $k = 0$ the map $\Omega(q, x_1, \ldots, x_d, y_1, \ldots, y_d)$ contracts D.

If $k = 1$ let $\omega := \Omega(q, x_1, \ldots, x_d, y_1, \ldots, y_d)$, then $\omega(D)$ is a marked divisor as in case a) of Lemma 77.

If $k = 2$ assume first that $p_{12} \in D$, in our notation this means that no lines l_i contains $r_1 \cap r_2$. Let $\omega = \Omega(q, x_1, \ldots, x_{d-2}, y_1, \ldots, y_{d-2})$, then $\omega(D) = r_1 + r_2$ is a marked divisor as in case b) of Lemma 77 and we conclude.

Assume that $p_{\mathbf{1}12} \in D$. Let $\omega = \Omega(q, x_2, \ldots, x_{d-2}, y_2, \ldots, y_{d-2})$, then $\omega(D) = l_1 + r_1 + r_2$, is a marked divisor as in case c) of Lemma 77 and we conclude. □

REMARK 79. In case $k \leq 1$ we may strenghten the result and extend it to any marked configuration of lines.

We give now another application of the Lemma 77, which will be very useful in the further calculations.

EXAMPLE 80. Let (D, Z) be a configuration of the type $(4, 2; (\mathbf{5} - \mathbf{a})|(a))$ with $a \leq 4$ and denote with q and x two double points for D not in Z. First, if the line passing through q and x does not belong to D, we observe that in this case, up to reordering the index, the point x is exactly p_{12}. We apply the transformation $\omega = \Omega(q, p_{12}, x_1)$ where x_1 is a general point on r_1. Then $\omega(D) = D'$ is the configuration (D', Z') where $D' = l_1 \cup l_2 \cup r_1$ and Z' is the set of double points of D'. We observe that we are using the convention †. Next, we consider the transformation $\Omega(y_1, y_2, s)$ where $y_1, y_2 \in l_1$, $s \in l_2$ are general points. Its image is a marked divisor, union of a conic and a line. Contracting the line we get a cubic curve with a double point, which is contractible by Lemma 77.

Otherwise, let us suppose that the line through q and x is contained in D, we denote it with l_1 and x with $p_{\mathbf{1}1}$. We consider the transformation $\Omega(q, p_{\mathbf{1}1}, x_1)$ where x_1 is a general point on r_2. The image is a set of three lines with three marked point as before, thus it is contractible.

1.1. Sarkisov theory and contractibility. We start extending the usual Noether–Fano inequalities to a wider context.

DEFINITION 81. Fix a rational smooth surface S, an effective reduced Cartier divisor $D \subset S$ and a non negative rational number α. Let (D, Z) in S be a marked divisor. Let $\mu : S_Z \to S$ be a resolution of Z and let us denote by $E_1, \ldots, E_r$ the divisors associated to the valuations of Z. We say that $(S, (\alpha D, Z))$ has canonical singularities if $(S_Z, \alpha(\mu_*^{-1}(D) + \sum_i E_i))$ has canonical singularities.

Proposition 82. *Let (D,Z) be a contractible marked divisor in S. Let $\chi : S \dashrightarrow \mathbb{P}^2$ be a birational map that contracts (D,Z). Then either there is an irreducible curve $C \subset S$ through the general point of S such that $(K_S+\alpha D)\cdot C<0$ and $K_S\cdot C<0$ or $(S,(\alpha D,Z))$ has singularities which are not canonical.*

Proof. Let

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, dashed, "\chi"] & & \mathbb{P}^2 \end{tikzcd}$$

be a resolution of the map χ and of the ideal of Z, with $E_i \subset T$ either p or q exceptional divisors and E_Z the divisor associated to the marking Z. Let $D_T := p_*^{-1}D$ be the strict transform. Then by hypothesis we may assume that $D_T+E_Z=\sum_1^l E_i$,

$$K_T+\alpha(D_T+E_Z)=q^*(K_S+\alpha D)+\sum_i a_iE_i,$$

and $K_T=q^*K_{\mathbb{P}^2}+\sum b_iE_i$, for non negative integers b_i. Let $r\subset \mathbb{P}^2$ be a general line. In particular q is an isomorphism in a neighborhood of r. Then

$$\begin{aligned}-3 =&(q^*K_{\mathbb{P}^2}+\sum b_iE_i)\cdot q^{-1}r=K_T\cdot q^{-1}r\\ =&(q^*(K_S+\alpha D)+\sum_i a_iE_i-\alpha(D_T+E_Z))\cdot q^{-1}r.\end{aligned}$$

By hypothesis D_T+E_Z is q exceptional. Therefore $(D_T+E_Z)\cdot q^{-1}r=0$ and we conclude that either there is a coefficient $a_i<0$ or $(K_S+\alpha D)\cdot p_*q^{-1}r<0$. Since $q^{-1}r$ is irreducible and contains a general point of T we have $D\cdot p_*q^{-1}r\geq 0$ and hence $K_S\cdot p_*q^{-1}r<0$. $\square$

Following Sarkisov theory we plan to use Proposition 82 to provide necessary condition for contractibility of divisors on $\mathbb{P}^2$.

Proposition 83. *Let $D=\sum D_i\subset\mathbb{P}^2$ be a divisor. Assume that the D_i are curves of degree d_i. If one of the following holds:*

- $\mathrm{mult}_q D\leq \sum d_i/3$ *for each* $q\in\mathbb{P}^2$;
- *there is a point* $p\in\mathbb{P}^2$ *such that for any* $q\in\mathbb{P}^2\setminus\{p\}$ *we have*

$$\mathrm{mult}_q D\leq \frac{\sum d_i-\mathrm{mult}_p D}{2}\ \textit{and}\ \mathrm{mult}_p D\geq\frac{\sum d_i}{3}$$

Then D is not contractible.

Proof. The first assumption is just saying that the pair $(\mathbb{P}^2,\frac{d}{3}K_{\mathbb{P}^2}+D)$ has canonical singularities and we apply Proposition 82 to conclude.

In the second case let $\nu:\mathbb{F}_1\to\mathbb{P}^2$ be the blow up in p with exceptional divisor C_0 and D_1 the strict transform. Let

$$\alpha=\frac{2}{\sum d_i-\mathrm{mult}_p D},$$

then by hypothesis $K_{\mathbb{F}_1}+\alpha D_1$ is nef. If there is a birational map $\chi:\mathbb{F}_1\dashrightarrow\mathbb{P}^2$ such that $\chi_*(D_1)=0$, by Proposition 82, $K_{\mathbb{F}_1}+\alpha D_1$ has not canonical singularities. By hypothesis it has canonical singularities outside C_0, therefore after finitely many elementary transformations centered on the exceptional section we land on $\mathbb{F}_e$, with $e\geq 2$, and with the unmarked divisor $(\mathbb{F}_e,\alpha D_1)$ canonical. Hence $K_{\mathbb{F}_e}+\alpha D_1$ is canonical and nef and D_1 is not contractible by Proposition 82. $\square$

As we will deduce from Proposition (88) and Section (2), the next example give us the non-contractible configuration with the smallest degree.

EXAMPLE 84. Let D be a configuration of 6 lines in general position, that is of the type $(6,4;(\mathbf{14}))$. Then D is not contractible by Lemma 83. In fact for any $x \in \mathbb{P}^2$

$$\operatorname{mult}_x D \leq 2 = \frac{d}{3}.$$

REMARK 85. We have shown that the configuration D of type $(6,4;(\mathbf{15}))$ is not contractible. On the other hand, we have that $D-l_1$ is a configuration of the type $(5,3;(\mathbf{9}))$ which is contractible by Example 75. We can therefore contract $D-l_1$ through a transformation $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Then $\chi_*(l_1)$ is a marked irreducible divisor that is not contractible. Thus we have constructed an example of marked irreducible divisors that is not contractible, even if the unmarked one is contractible.

The following is a general statement about configurations of lines, we can find it also in [**11**].

LEMMA 86. *Let D be a configuration of lines. Assume that D is of type $(d,k;\mathbf{A})$ and $|\mathbf{A}| \leq 1$. Then D is not contractible.*

PROOF. By Proposition 83 we may assume that $|\mathbf{A}| = 1$. According to our notation, let q be a point of multiplicity $d-k$. Then $(\mathbb{P}^2, D)$ is birational to $(\mathbb{F}_0, D)$ with $D \sim (d-k-\epsilon)F_0 + (m-\epsilon)F_1 + (k-m+\epsilon)(F_0+F_1)$, with $\epsilon \in \{0,1\}$. By our assumption the unmarked divisor $(\mathbb{F}_0, 2/kD)$ has canonical singularities and it is not nef only if

$$d-m < k.$$

We are assuming that $m \leq d-k$ therefore $(\mathbb{F}_0, 2/kD)$ is nef and it has canonical singularities. Hence D is not contractible by Proposition 82. ☐

1.2. Adjoint divisors and contractibility. Let $D \subset \mathbb{P}^2$ be a divisor and $\mu : S \to \mathbb{P}^2$ a log resolution of the singularities of D. We recall that for $a \leq b$ positive integers the adjoint linear system of type (a,b) is defined as

$$\operatorname{adj}_{a,b}(D) := \mu_*(aD_S + bK_S)$$

where D_S is the strict transform of D.

They provide us a tool for the non-contractibility of a configuration, as seen in Remark 41. This is a sample of our approach using adjoint linear systems.

LEMMA 87. *Let D be a configuration of type $(11,7;\mathbf{A})$, then D is not contractible.*

PROOF. Let $D = \sum_1^4 l_i + \sum_1^7 r_j$ be of type $(11,7;(\mathbf{a}))$, and let b be the number of its triple points. Our first aim is to bound a and b. Recall that in our notation D has $a+1$ 4-tuple points. If $a \leq 1$ D is not contractible by Proposition 86. In the remaining cases we will prove that the linear system $\operatorname{adj}_{2,4}(D)$ is effective and therefore conclude the non contractibility.

A direct computation shows that $a \leq 5$. Moreover a case by case analysis shows that the number of triple points b is bounded with respect to a. If D has $a = 5$ quartuple points, then the number of triple points is $b \leq 2$. Analogously we have:

$$a = 5 \ b \leq 2, \quad a = 4 \ b \leq 5, \quad a = 3 \ b \leq 7, \quad a = 2 \ b \leq 9$$

The adjoint linear system $\operatorname{adj}_{2,4}(D)$ is $\mathcal{L}(10;4^{a+1},2^b)$. The expected dimension of $\mathcal{L}(10;4^{a+1},2^b)$ is $66-10a-3b$. Hence $\operatorname{adj}_{2,4}(D)$ is effective if either $a \leq 4$ or $a = 5$ and $b = 1$. Note that if $a = 5$, up to reordering the indexes, we may assume that the line r_1 contains the three 4-uple points $p_{\mathbf{1}123}, p_{\mathbf{2}145}, p_{\mathbf{3}167}$. Therefore $\operatorname{adj}_{2,4}(D) - r_1$ is $\mathcal{L}(9;4^3,3^3,2^2)$ and it is effective. ☐

We are ready to state a first result about configuration of lines combining Sarkisov and adjoint linear system approach. The idea is to use Sarkisov to leave a finite number of cases and then apply adjoint divisors to study them.

PROPOSITION 88. *Let $D = \sum_1^{d-k} l_i + \sum_1^k r_j$ be a configuration of lines of type $(d, k; \mathbf{A})$. If $k \leq 2$ then D is contractible. If $k \geq 7$, D is not contractible.*

PROOF. For $k \leq 2$ see Corollary 78.
Let us assume $k \geq 7$. By Proposition 82 if D is contractible we have $d - k > d/3$, that is

$$d > \frac{3}{2}k. \tag{4}$$

Therefore if $k \geq 7$ and D is contractible we have $d \geq 11$. Thus a configuration D of the type $(d, k; \mathbf{A})$ with $k \geq 7$ and $d \geq 11$ must contain a configuration of the type $(11, 7; \mathbf{B})$ which is not contractible by Lemma 87. Using Remark 72 we conclude that D is not contractible.

□

REMARK 89. We will see that this result is almost optimal. For any $3 \leq k \leq 5$ there are both contractible and non contractible configurations, for $k = 6$ we have only non-contractible configurations up to now. One should compare it with Theorem 53 of Calabri and Ciliberto[**11**], where it is proven that any configuration with $d \geq 12, k \geq 3$ is not contractible.

In the next sections we proceed to a case by case analysis of the remaining possibilities, namely with $3 \leq k \leq 6$.

2. Configurations of lines with $k = 3$

PROPOSITION 90. *Let $D = \sum_1^{d-3} l_i + \sum_1^3 r_j$ be a configuration with one point of multiplicity $d - 3$ and $3(d - 2)$ double points. Then D is contractible if and only if $d \leq 8$.*

PROOF. Assume first that $d \geq 9$. Then we can see that the adjoint linear system $\mathrm{adj}_{2,3}(D)$ is not empty, see also 48. Then by Remark 72 we have only to prove the claim for $d = 8$.

Let us consider a configuration $D = \sum_1^5 l_i + \sum_1^3 r_j$ of type $(8, 3; (\mathbf{0}, \mathbf{0}, \mathbf{18}))$. Set

$$\omega := \Omega(q, p_{12}, p_{13}, p_{23}, p_{\mathbf{1}1}, p_{\mathbf{2}2}, p_{\mathbf{3}3}),$$

then ω is induced by quartic curves. Counting the indexes of points, we see that the lines l_1, l_2, l_3 are contracted and the lines l_4, l_5, r_1, r_2, and r_3 are mapped to lines. Then, recall convention (†), we let $\tilde{D} := \omega(D) = r_1 + r_2 + r_3 + l_4 + l_5$. Then $\tilde{D}$ is a general configuration of 5 lines. To study the contractibility of D we need to consider also the image of the contracted lines. This lead us to the marked divisor $(\tilde{D}, Z)$, where $Z = \omega(l_1 + l_2 + l_3)$. Let $x_i := \omega(l_i)$, then

$$x_i = r_h \cap r_k,$$

where $\{i, h, k\} = \{1, 2, 3\}$. Let $q := l_4 \cap l_5$, then $(\tilde{D}, Z)$ is of type $(5, 3; (\mathbf{7})|(3))$ and it is contractible as showed in Example 75. □

Next we study the special configurations with $k = 3$.

PROPOSITION 91. *Let D be a configuration of type $(d, 3; \mathbf{A})$. If $d \leq 8$, then D is contractible.*

Proof. By Example 75 we may assume that $d \geq 6$. Moreover, if there is no other triple point different from q, we can reduce to Proposition 90. For these reasons we assume that D is a configuration of degree $d \geq 6$ with at least two points of multiplicity strictly greater than 2, (one of them is q).

Our strategy is to apply a de Jonquières transformations to reduce the given configuration to a general one of lower degree. Let $D = \sum_1^{8-a-3} l_i + \sum_1^3 r_i$ with $a \in \{0, 1, 2\}$.

Firstly we analyse the case in which the r_i are in general position. The maximal multiplicity outside q is 3 and D is of type $(8 - a, 3; (\mathbf{0}, \mathbf{b}, \mathbf{18} - \mathbf{3}(\mathbf{a} + \mathbf{b})))$, with $0 < b \leq 3$.

If $b = 1$, up to reordering the indexes, we may define the map

$$\omega := \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{2}3}, p_{\mathbf{3}1}, p_{23}).$$

Then we have to consider the marked divisor $(\omega(D), Z)$, with $Z = \omega(l_1 + l_2 + l_3)$, of type $(5 - a, 3; (\mathbf{8})|(2))$ and we conclude as in Example 75.

If $b = 2$ we consider the map $\omega := \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{2}13}, p_{\mathbf{3}2}, x)$, with x a general point on r_3. Then we have to consider the marked divisor $(\omega(D), Z)$, with $Z = \omega(l_1 + l_2 + l_3)$, of type $(5 - a, 3; (\mathbf{9})|(1))$ and we conclude again as in Example 75.

If $b = 3$ we consider the map $\omega := \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{2}13}, p_{\mathbf{3}23}, x)$, with x a general point. Then we have to consider the marked divisor $(\omega(D), Z)$, with $Z = \omega(l_1 + l_2 + l_3)$, of type $(5 - a, 3; (\mathbf{10}))$ and we conclude by Proposition 90.

To conclude assume that the lines r_i are not in general position. Then the three lines are concurrent and, up to reordering the indexes we may assume that the configuration contains either the point $p_{\mathbf{1}123}$ or the point p_{123}.

In the former case we apply the transformation $\omega = \Omega(q, p_{\mathbf{1}123}, p_{\mathbf{2}1}, p_{\mathbf{3}2}, x_2)$ with $x_2 \in r_2$, while in the latter we use the modification $\omega = \Omega(q, p_{123}, p_{\mathbf{1}1}, p_{\mathbf{2}3}, p_{\mathbf{3}2})$. In both the cases this reduces the original configuration to the one of Example 75 and it concludes the proof. □

More interesting are the cases with $d \geq 9$. In fact some configurations with $d \geq 9$ are contractible.

Example 92. Let D be a configuration of type $(9, 3; (\mathbf{1}, \mathbf{0}, \mathbf{15}))$. Let $\omega = \Omega(q, p_{\mathbf{1}123}, x)$, with x a general point in $\mathbb{P}^2$. Then $\omega(D)$ is of type $(8, 3; \mathbf{A})$ and therefore it is contractible by Proposition 91.

For future reference we state the following special case

Lemma 93. *Let (D, Z) be a marked configuration of type $(6, 3; (\mathbf{0}, \mathbf{9})|(0, 3))$ such that $Z = \bigcup_{i,j=1}^3 r_i \cap r_j$. Then D is not contractible.*

Proof. Let D' be a configuration of type $(9, 3; (\mathbf{0}, \mathbf{0}, \mathbf{18}))$. Let

$$\omega = \Omega(q, p_{12}, p_{\mathbf{1}1}, p_{23}, p_{\mathbf{2}2}, p_{\mathbf{3}3}, p_{13}).$$

Then $\omega(D')$ is Cremona equivalent to a marked configuration of type $(6, 3; (\mathbf{0}, \mathbf{9})|(0, 3))$. Since D' is not contractible by Proposition 90 we conclude that D is not contractible. □

Proposition 94. *For $d \in \{9, 10, 11\}$ and $k = 3$ the only contractible configurations are the following:*

- *when the three lines r_i are in general position:*

$$(9, 3; (\mathbf{0}, \mathbf{1}, \mathbf{18}))\,, (9, 3; (\mathbf{0}, \mathbf{2}, \mathbf{15}))\,, (9, 3; (\mathbf{0}, \mathbf{3}, \mathbf{12}))\,,$$
$$(10, 3; (\mathbf{0}, \mathbf{2}, \mathbf{18}))\,, (10, 3; (\mathbf{0}, \mathbf{3}, \mathbf{15}))\,, (11, 3; (\mathbf{0}, \mathbf{3}, \mathbf{18}))\,;$$

- *when the r_i's are concurrent:*

$$(9, 3; (\mathbf{1}, \mathbf{0}, \mathbf{15}))\,.$$

Proof. Let $D = \sum_1^{11-a-3} l_i + \sum_1^3 r_i$ be a configuration of type $(d, 3; \mathbf{A})$. Assume first that the r_i's are in general position. Then the maximal multiplicity is 3 and these are at most 3 triple points. The type of the configuration is $(11 - a, 3; (\mathbf{0}, \mathbf{b}, \mathbf{27} - \mathbf{3}(\mathbf{a} + \mathbf{b})))$ and by Proposition 90 we may assume that $b > 0$.

If $b = 1$, up to reordering the indexes, we may define the map

$$\omega := \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{2}3}, p_{\mathbf{3}1}, p_{\mathbf{4}2}, p_{23}, p_{13}).$$

Then we have to consider the marked divisor $(\omega(D), Z)$, where $Z = \omega(l_1 + l_2 + l_3 + l_4)$. Note that $\omega(l_1)$ is a smooth point of $\omega(D)$, then possible types of $(\omega(D), Z)$ are:

$$(5, 3; (\mathbf{6})|(3)))\,,\ (6, 3; (\mathbf{0}, \mathbf{9})|(0, 3))\,,\ (7, 4; (\mathbf{0}, \mathbf{0}, \mathbf{12})|(0, 0, 3))\,.$$

The first is contractible by Example 75, while the latters are not by Lemma 93.

If $b = 2$ we consider the map

$$\omega := \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{2}13}, p_{\mathbf{3}2}, p_{\mathbf{4}3}, p_{\mathbf{5}1}, p_{23}).$$

Then we have to consider the marked divisor $(\omega(D), Z)$, with $Z = \omega(l_1 + l_2 + l_3 + l_4 + l_5)$ and $\omega(l_1 + l_2)$ are smooth points of $\omega(Z)$. Then the possible types are:

$$(4, 2; (\mathbf{2})|(3))\,,\ (5, 3; (\mathbf{6})|(3))\,,\ (6, 3; (\mathbf{0}, \mathbf{9})|(0, 3))\,.$$

The first two are contractible by Example 75, while the latter is not by Lemma 93.

If $b = 3$ we consider the map

$$\omega := \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{2}13}, p_{\mathbf{3}23}, x),$$

with x a general point. Then we have to consider the marked divisor $(\omega(D), Z)$, with $Z = \omega(l_1 + l_2 + l_3)$, of type $(8 - a, 3; (\mathbf{10}))$ and it is contractible by Proposition 90.

We resume the argumantation in the following scheme:

3 lines in general position		
$(9, 3; (\mathbf{0}, \mathbf{1}, \mathbf{18}))$	$(10, 3; (\mathbf{0}, \mathbf{1}, \mathbf{21}))$	$(11, 3; (\mathbf{0}, \mathbf{1}, \mathbf{24}))$
$\downarrow$	$\downarrow$	$\downarrow$
$(5, 3; (\mathbf{6})|(3)))$	$(6, 3; (\mathbf{0}, \mathbf{9})|(0, 3))$	$(7, 3; (\mathbf{0}, \mathbf{0}, \mathbf{12})|(0, 0, 3))$
contractible	*non* contractible	*non* contractible
$(9, 3; (\mathbf{0}, \mathbf{2}, \mathbf{15}))$	$(10, 3; (\mathbf{0}, \mathbf{2}, \mathbf{18}))$	$(11, 3; (\mathbf{0}, \mathbf{2}, \mathbf{21}))$
$\downarrow$	$\downarrow$	$\downarrow$
$(4, 2; (\mathbf{2})|(3)))$	$(5, 3; (\mathbf{6})|(3)))$	$(6, 3; (\mathbf{0}, \mathbf{9})|(0, 3))$
contractible	contractible	*non* contractible
$(9, 3; (\mathbf{0}, \mathbf{3}, \mathbf{12}))$	$(10, 3; (\mathbf{0}, \mathbf{3}, \mathbf{15}))$	$(11, 3; (\mathbf{0}, \mathbf{3}, \mathbf{18}))$
$\downarrow$	$\downarrow$	$\downarrow$
$(3, 1; (0)|(\mathbf{2}))$	$(4, 2; (\mathbf{2})|(3)))$	$(5, 3; (\mathbf{6})|(3)))$
contractible	contractible	contractible

We may argue similarly when the lines r_1, r_2, and r_3 are concurrent. This time there is either a 4-ple point or a 3-ple point. The possible types are

$$(11-a,3;(\mathbf{b},\mathbf{c},\mathbf{27}-\mathbf{3a}-\mathbf{6b}-\mathbf{3c}))$$

and using a similar construction we produce the following Cremona equivalences that conclude the proof.

3 lines in a pencil

$$\begin{array}{ccc}
(9,3;(\mathbf{0},\mathbf{1},\mathbf{18})) & (10,3;(\mathbf{0},\mathbf{1},\mathbf{21}) & (11,3;(\mathbf{0},\mathbf{1},\mathbf{24})) \\
\downarrow & \downarrow & \downarrow \\
(6,3;(\mathbf{0},\mathbf{9})|(0,3)) & (7,3;(\mathbf{0},\mathbf{0},\mathbf{12})|(0,0,3)) & (8,3;(\mathbf{0},\mathbf{0},\mathbf{15});(0,0,3)) \\
\textit{non}\text{ contractible} & \textit{non}\text{ contractible} & \textit{non}\text{ contractible} \\
 & & \\
(9,3;(\mathbf{1},\mathbf{0},\mathbf{15})) & (10,3;(\mathbf{1},\mathbf{0},\mathbf{18})) & (11,3;(\mathbf{1},\mathbf{0},\mathbf{21})) \\
\downarrow & \downarrow & \downarrow \\
(5,3;(\mathbf{6})|(3)) & (6,3;(\mathbf{0},\mathbf{9})|(0,3)) & (7,3;(\mathbf{0},\mathbf{0},\mathbf{12});(0,0,3)) \\
\text{contractible} & \textit{non}\text{ contractible} & \textit{non}\text{ contractible}
\end{array}$$

□

Remark 95. Note that in the Proposition 94 the distinction based on the position of the lines r_i is necessary. The configuration $(9,3;(\mathbf{0},\mathbf{1},\mathbf{18}))$ is contractible if the r_i's are in general position, it is not if they meet in a point.

The above list shows a particular feature of configurations of lines with $d \leq 11$ and $k = 3$. The configuration is non contractible if and only if it is Cremona equivalent to a marked divisor containing the one described in Lemma 93. We have not a theoretical argument to prove this fact.

We are ready to give the full picture of the $k = 3$ case.

Theorem 96. *Let D be a configuration of type $(d,3;\mathbf{A})$. Then it is contractible if and only if either $d \leq 8$ or its type is in the list of Proposition 94.*

Proof. By Proposition 91 and Proposition 94 we are left to consider the case $d \geq 12$. Let D be a configuration of type $(d,3;\mathbf{A})$. Then D contains a configuration of degree $d-3 \geq 9$ with one point of multiplicity $d-6$ and $3(d-4)$ double points. Hence it is not contractible by Proposition 90, see also 53. □

3. Configurations of lines with $k = 4$

To study configurations with higher k we have to introduce the following definition.

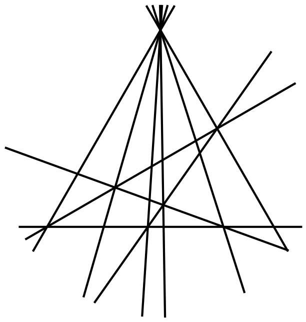
Definition 97. Let $D = \sum_1^{d-k} l_i + \sum_1^k r_j$ be a configuration of lines. Then

$$D_i = D - l_i,\ D^j = D - r_j,\ R_D = \sum_1^k r_j$$

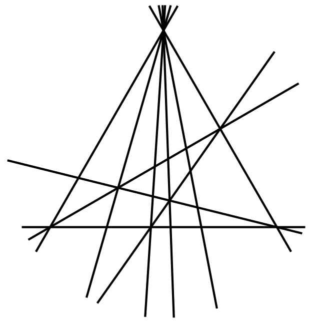
Lemma 98. *The following configurations are contractible:*

(1) $(10,4;(\mathbf{0},\mathbf{0},\mathbf{6}))$ *with* D_i *of type* $(9,4;(\mathbf{0},\mathbf{0},\mathbf{5}))$ *or* $(9,4;(\mathbf{0},\mathbf{0},\mathbf{4}))$
(2) $(10,4;(\mathbf{0},\mathbf{1},\mathbf{3}))$
(3) $(9,4;(\mathbf{0},\mathbf{0},\mathbf{6}))$ *with* D_1 *and* D_2 *of type* $(8,4;(\mathbf{0},\mathbf{4}))$

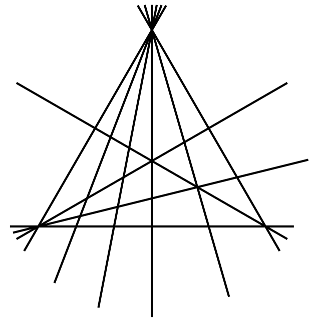
(4) $(7, 4; (\mathbf{2}))$.



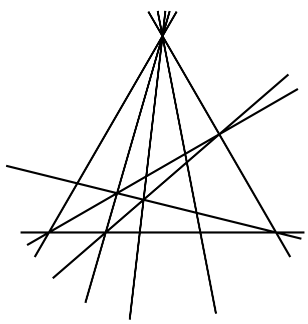
$(10, 4; (\mathbf{0}, \mathbf{0}, \mathbf{6}))$



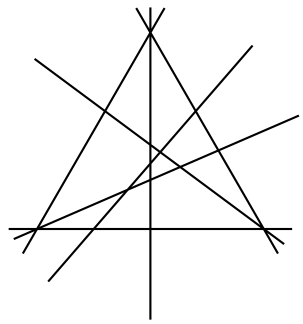
$(10, 4; (\mathbf{0}, \mathbf{0}, \mathbf{6}))$



$(10, 4; (\mathbf{0}, \mathbf{1}, \mathbf{3}))$



$(9, 4; (\mathbf{0}, \mathbf{0}, \mathbf{4}))$



$(7, 4; (\mathbf{2}))$

Proof. (1) Let D be a configuration of the type $(10, 4; (\mathbf{0}, \mathbf{0}, \mathbf{6}))$ as in the statement. We may assume that the map

$$\omega = \Omega(x_1, x_2, p_{\mathbf{6}2}) \circ \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{4}34}, p_{\mathbf{2}24}, p_{\mathbf{3}13})$$

is well defined, where $\{p_{\mathbf{1}12}, p_{\mathbf{4}34}, p_{\mathbf{2}24}, p_{\mathbf{3}13}, x_1, x_2\}$ is the set of all the triple points. We remark that we denote by x_1 and x_2 the last two triple points in order to consider both the possibilities in this configuration (see the picture above). Then $D' := \omega(\tilde{D})$ is a configuration of type $(4, 2; (\mathbf{a}), (5 - a))$ with $a \leq 2$ and it is easy to see that D' is contractible.

(2) Let D be a configuration of type $(10, 4; (0, 1, 3))$. We may assume that

$$\omega = \Omega(q, p_{\mathbf{6}123}, p_{\mathbf{1}14}, p_{\mathbf{2}24}, p_{\mathbf{3}34}),$$

is well defined. Then $D' := \omega(D)$ is of type $(5, 3; (6), (\mathbf{3}))$ where the p_{ij}'s are marked. Next consider $\omega' = \Omega(p_{\mathbf{5}2}, p_{\mathbf{5}3}, p_{23}) \circ \Omega(p_{\mathbf{4}1}, p_{\mathbf{5}1}, p_{\mathbf{4}2})$. Let $\overline{D} = \omega'(D')$ and $\overline{Z}$ the corresponding marking. Then $\overline{D} = C \cup l$, where C is a conic and l is a line and $\overline{Z} \not\supseteq C \cap l$. Therefore $(\overline{D}, \overline{Z})$ is contractible by Lemma 77.

(3) Let D be a configuration of type $(9, 4; (\mathbf{0}, \mathbf{0}, \mathbf{6}))$ as in the statement. We may assume that the map

$$\omega = \Omega(p_{\mathbf{2}13}, p_{\mathbf{1}23}, p_{\mathbf{1}14}) \circ \Omega(q, p_{\mathbf{3}12}, p_{\mathbf{4}34})$$

is well defined. Then $D' = \omega(D) = C \cup l_1 \cup l_2 \cup l_3$, with C a conic which is tangent to the line l_3. We need to contract the marked divisor (D', Z') and also this time $Z' \not\supseteq C \cap (\bigcup_i l_i)$.

We denote with x_i the point of $C \cap l_i$ not belonging to Z' for $i = 1, 2$ and with $y, y' \in l_3$ $c \in C$ general points. It suffices to apply the transformation $\omega = \Omega(y, y', c) \circ \Omega(q, x_1, x_2)$ to obtain a cubic curve with a cusp, which is contractible by Lemma 77.

(4) Let us consider a configuration D of type $(7, 4; (\mathbf{2}))$. If D is contained in one of the previous configurations, it is contractible. The only remaining case is when the D contains the triangle of vertices the triple points. We apply the Cremona transformation $\omega = \Omega(q, p_{\mathbf{1}12}, p_{\mathbf{2}13})$ is the marked divisor (D', Z') such that $D' = C \cup l_1 \cup l_2 \cup l_3$ and $C \cap (\bigcup l_i) \not\subseteq Z'$. Let us denote with x_i the point of $C \cap l_i$ but not in Z' for $i = 1, 2, 3$. After the transformation $\Omega(x_1, x_2, x_3)$, we obtain a configuration of the type $(4, 2; (\mathbf{2}), (3))$ which is contractible by lemma 80.

□

We are ready to characterize contractible configurations with $k = 4$.

THEOREM 99. *Let $D = \sum_1^{d-4} l_i + \sum_1^4 r_j$ be a configuration of lines of the type $(d, 4; \mathbf{A})$. Then D is contractible if and only if it is contained in one of the configurations of Lemma 98.*

PROOF. In Lemma 98 we have proved that the configurations in the statement are contractible. If $d \leq 6$ the configuration is not contractible by Lemma 83. If $d \geq 12$ it is easy to see that D always contains a configuration of type $(6, 4; \mathbf{A})$ (see Example 84) and we conclude that it is not contractible.

CASE ($d = 7$). In this case the maximal multiplicity is 3. Assume that all the 3-ple points are contained in a line m. If $m \subset D$ then $D - m$ is of type $(6, 4; (\mathbf{15}))$ and it is not contractible. If $m \not\subset D$ then there are at most 2 3-ple points in D and D is not contractible by Lemma 86. Assume that there are three non alligned 3-ple points in D then a direct calculation shows that D is contained in a configuration of type either (1) or (3).

CASE ($d = 8$). Let D be a contractible configuration. Then D_i is contractible therefore either $|A| \geq 3$ or there is a 4-uple point other than q and at least one 3-ple point. This forces type of R_D to be either $(4, 2; (6))$ or $(4, 1; (3))$. In the first case D is contained in configuration (1) or (3), while in the second D is contained in configuration (2).

CASE ($d = 9$). Let D be a contractible configuration. Then D_i is contractible and we argue exactly as for $d = 8$ introducing also configuration (3).

CASE ($d = 10$). Assume that D is contractible. Then the D_i's are contractible and this leads to $|A| \geq 4$ and if there are not 4-uple points $|A| = 6$. If $|A| = 6$, the configuration D is (1). If there is a 4-tuple point then D is configuration (2).

CASE ($d \geq 11$). Let D be a contractible configuration of the type $(11, 4; \mathbf{A})$, then R is of type $(4, k; \mathbf{B})$. By Lemma 86 and our previous analysis $k = 1, 2$ and if $k = 1$ the configuration D has a 4-uple point. If D has a 4-uple point we may assume it is $p_{\mathbf{11}23}$ and then D_1 is a configuration of degree 10 not in the list of the theorem hence it is not contractible. If D has only triple points then $|\mathbf{A}| \leq 6$ and we may assume $p_{\mathbf{11}2}$ is in the configuration. Then again D_1 is not contractible. This together with Lemma 73 concludes the proof.

□

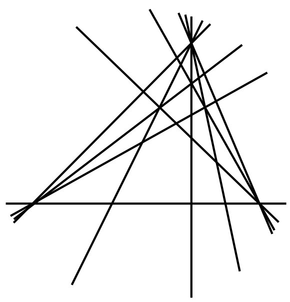
REMARK 100. We want to stress the fact that contractibility is neither opened nor closed as a property of deformation. Let us consider a flat family of deformations of a configuration of lines

D_t in $\mathbb{P}^2$ and let D_0 be a degenerate element of the family. We have examples of non-contractible families with D_0 contractible and examples of the viceversa.
If for $t \neq 0$, D_t is a configuration of the type $(6,4;(15))$, which is not contractible, then D_0 is of the type $(6,3;(0,12))$ and is contractible.
On the other hand, let us consider D_t a family of configurations of degree 7 with exactly three triple points x_1, x_2, x_3 such that the x_i are collinear only for $t = 0$. We have seen in the proof of Theorem 99 that D_t is contractible for $t \neq 0$ and it is not for $t = 0$.

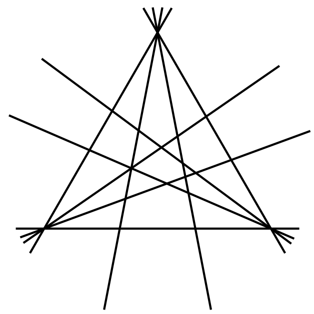
4. Configurations of lines with k=5

We proceed, as in the previous section, to provide a set of contractible configurations.

Lemma 101. *The following configurations are contractible:*
1) $(10,5;(\mathbf{0},\mathbf{2},\mathbf{a}))$ *with* $a \geq 3$
2) $(9,5;(\mathbf{2},\mathbf{b}))$ *with* $b \geq 1$



$(10,5;(\mathbf{0},\mathbf{2},\mathbf{3}))$



$(9,5;\mathbf{2},\mathbf{1})$

Proof. 1) Let us consider the configurations 1. First assume that $a = 3$ and each line l_i contains either a triple or a quartuple point. We apply the transformation:

$$\omega = \Omega(x_1, x_2, x_3) \circ \Omega(q, p_{\mathbf{1}123}, p_{\mathbf{5}345}).$$

where x_1, x_2, x_3 are the triple points. The image is a marked divisor of the type $(5,2;(\mathbf{0},\mathbf{5})|(\mathbf{0},\mathbf{2}))$, which is contractible by Example 76.
On the other hand, if $D - l_5$ is of the type $(9,5;(\mathbf{2},\mathbf{3}))$, we apply the transformation

$$\omega = \Omega(p_{\mathbf{4}35}, p_{\mathbf{2}4}, p_{\mathbf{5}4}) \circ \Omega(p_{\mathbf{2}15}, p_{\mathbf{4}14}, p_{\mathbf{5}3}) \circ \Omega(q, p_{\mathbf{1}123}, p_{\mathbf{3}245})$$

and conclude by Example 75.
For $a = 4$, we proceed analogously, applying the following transformation:

$$\omega = \Omega(p_{\mathbf{3}14}, p_{\mathbf{2}4}, p_{\mathbf{4}1}) \circ \Omega(p_{\mathbf{2}15} p_{\mathbf{3}25} p_{\mathbf{4}24}) \circ \Omega(q, p_{\mathbf{1}1234}, p_{\mathbf{5}345}).$$

2) Let D be the configuration in 2 with $b = 1$. We apply the transformation

$$\omega = \Omega(p_{\mathbf{4}14}, p_{\mathbf{2}5}, p_{\mathbf{4}3}) \circ \Omega(q, p_{\mathbf{1}123}, p_{\mathbf{3}245}).$$

Its image $\omega(D)$ is a configuration of the type $(5,3;(\mathbf{6})|(3))$ which is contractible by Example 75. For $b > 1$ we argue in the same way.

□

Lemma 102. *Let D be a configuration of the type $(d,5;\mathbf{A})$ with $d \geq 11$. Then D is not contractible.*

Proof. Let $D = \sum_1^6 l_i + \sum_1^5 r_j$ be of type $(11, 5; (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}))$. If $a > 0$ then the type is $(1, 0, 0, 0)$, while if $b > 0$ the type is $(0, 1, 0, d)$ where, up to reordering, the eventual triple points are alligned on the line r_1. In all these cases the configuration $D' := D - r_1$ is of type $(10, 4; \mathbf{A})$ with $|\mathbf{A}| = 1$ and it is not contractible by Proposition 86. Hence D is not contractible.

Hence we may assume that D is of type $(11, 5; (\mathbf{0}, \mathbf{0}, \mathbf{a}, \mathbf{b})$. Let us consider the linear system $\mathrm{adj}_{1,2}(D)$, it is not empty for $a = 0, b \leq 10$ $a = 1, b \leq 7$ and for $a = 2, b \leq 4$. It is easy to see all the configurations belong to those ranges, thus D is not contractible.

Since any configuration of type $(11, 5; \mathbf{A})$ is not contractible then the same is true for any degree $d \geq 12$, by Lemma 73. □

Theorem 103. *Let $D = \sum_1^{d-5} l_i + \sum_1^5 r_j$ be a configuration of lines not containing a subconfiguration $\tilde{D}$ of the type $(8, 5; \mathbf{A})$ with $|\mathbf{A}| \geq 5$. The configuration D is contractible if and only if it is contained in $(9, 5; (\mathbf{2}, \mathbf{a}))$ with $a \geq 1$.*

Proof. In Lemma 101 we have shown that the given configuration is contractible.

Now, let us consider D of the type $(d, 5; \mathbf{A})$. By Proposition 82 if D is contractible we have $d - 5 > d/3$, that is $d > 7$. Moreover, by Lemma 102 D is not contractible for $d \geq 11$.

Case $(d = 8)$. Assume that D is the configuration $(8, 5; (a))$ with $a \leq 4$. The adjoint linear system $\mathrm{adj}_{2,4}(D)$ is a quartic singular along the 3-ple points. Since they are at most five, it is not empty.

Case $(d = 9)$. Assume that D is contractible and of type $(9, 5; (\mathbf{a}, \mathbf{b}))$. We can see that $a \leq 2$. If $a = 0$, D contains a configuration $(8, 5; \mathbf{A})$, thus either it is not contractible or it contains $\tilde{D}$. If $a = 1$, then the line joining the two quartuple points is contained in D (otherwise D is not contractible) and we proceed as for $a = 0$. Then we can suppose that $a = 2$; if $b = 0$ the adjoint linear system $\mathrm{adj}_{2,4}(D)$ is not empty and D is not contractible. In order to show that for $b \geq 1$ D is contractible, we apply two Cremona transformations: the first centered in the 4-tuple points and the second in the triple points (and eventually a double point). We obtain a configuration of degree either 4 or 3, which is contractible.

Case $(d = 10)$. Assume that D is contractible and of type $(10, 5; (a, b, c))$. It is easy to see that if R is of type $(5, 0; 0)$, $(5, 1; (4))$, then D is not contractible. Moreover, if R is of the type $(5, 3; (10))$, then the adjoint linear system of coefficient $(2, 5)$ shows that D is not contractible. Therefore R is of type $(5, 2; \mathbf{A})$ and $a = 0$. We observe that the adjoint linear system $\mathrm{adj}_{1,2}(D)$ is not empty for $b = 0$ $c \leq 8$, for $b = 1$ $c \leq 5$ and for $b = 2$ $c \leq 2$. If $b = 0$, the configuration D contains at most 7 triple point, thus it is not contractible. This shows that $b \geq 1$. If $b = 1$, D is not contractible for $d \leq 5$ and it contains $\tilde{D}$ for $c = 6, 7$. Analogously, if $b = 2$ D is not contractible for $c \leq 2$, it contains $\tilde{D}$ for $c = 3, 4$.

□

4.1. The cases $(8, 5; \mathbf{A})$ with $|A| \geq 5$. The cases $(8, 5; \mathbf{A})$ with $|A| \geq 5$ need a further discussion since they include some contractible configurations and others not contractible. Thus we need to give a complete classification and a deeper notation to distiguish the different configurations.

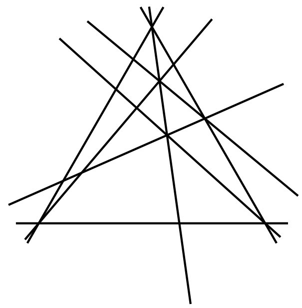
Definition 104. Let $D = \sum_1^d t_i$ be a configuration. A singular point $P = t_{a_1} \cap \cdots \cap t_{a_r}$ of D will be denoted by either $(a_1, \ldots a_r)$ or $P_{a_1, \ldots a_r}$.

To a configuration $D = \sum_1^d t_i$ we associate its sequence of singular points of multiplicity greater that $\frac{k}{2}$ where k denotes the difference between the degree d of the configuration and the maximal multiplicity of a point in D.

Moreover, we denote by $\mathcal{L}_d(m_1P_1,\dots m_sP_s)$ the linear system in $\mathbb{P}^2$ of degree d containing the points $P_1,\dots P_s$ with multiplicity respectively $m_1,\dots m_s$. The associated map will be written as $\phi_{\mathcal{L}_d(m_1P_1\dots m_sP_s)}$.

EXAMPLE 105. As a first example check that the configuration in the following figure is denoted by

$$(123)(145)(167)(257)(248)(368).$$



It is contracted by the transformation

$$\omega = \Omega(P_{37}, P_{78}, P_{46}) \circ \Omega(P_{123}, P_{167}, P_{248}) \circ \Omega(P_{145}, P_{257}, P_{368})$$

which can be denoted as $\phi_{\mathcal{L}}$ where $\mathcal{L}$ is the linear system

$$\mathcal{L} = \mathcal{L}_8(4P_{145}, 4P_{257}, 4P_{368}, 2P_{123}, 2P_{167}, 2P_{248}, P_{37}, P_{78}, P_{46})$$

.

REMARK 106. In some cases it seems not reasonable to consider standard Cremona transformations to contract a configuration. Thus we will use other kind of birational transformations, in particular the ones we use belong to the classification of plane Cremona transformation of degree smaller than 16, given by Hudson [**24**].
The linear system associated to a map in that classification has base points in general position. In our argumentation it is no more true since they may be collinear following the configuration, so the dimension of the linear system may be greater that 3.
As we will see we have adopted these transformations in case a) of Proposition 107 and in case c) of Proposition 108. We have used computational methods to show that the dimension does not change and thus that they are birational modifications.

THEOREM 107. *Let D be a configuration of the type $(8,5;(\mathbf{5}))$. Then D is one of the following:*

(a) (123)(148)(258)(246)(356)(678)
(b) (123)(145)(167)(257)(246)(348)
(c) (123)(145)(167)(257)(248)(368)
(d) (123)(145)(167)(257)(246)(347)
(e) (123)(145)(167)(257)(246)(478)
(f) (123)(145)(167)(258)(478)(368)

Moreover, the only contractible configurations are those in a,b,c.

PROOF.

CASE (a). Let us consider the linear system

$$\mathcal{L} = \mathcal{L}_{16}(8P_{148}, 7P_{123}, 6P_{246}, 6P_{57}, 5P_{678}, 5P_{356}, 3P_{258}, 3P_{37}, P_{45}, P_{47}).$$

Applying the transformation $\phi_{\mathcal{L}}$ to the configuration, we obtain a set of four lines with four marked points, which is contractible by Example 80.

Case (b). Let us consider the configuration in b and the linear system

$$\mathcal{L} = \mathcal{L}_8(4P_{167}, 4P_{257}, 4P_{348}, 2P_{145}, 2P_{123}, 2P_{246}, P_{36}, P_{58}, P_{68}).$$

Applying $\phi_{\mathcal{L}}$ we obtain a conic and two lines, which are contractible by Lemma 77.

Case (c). As concern the configuration in c, we use the transformation $\phi_{\mathcal{L}}$ where

$$\mathcal{L} = \mathcal{L}_8(4P_{145}, 4P_{257}, 4P_{368}, 2P_{123}, 2P_{167}, 2P_{248}, P_{37}, P_{78}, P_{46}).$$

The image is a set of four lines with four marked points, which is contractible, see Example 80.

Case (d). The configuration in case d is not contractible since the adjoint linear system $\mathrm{adj}_{(2,3)}$ is not empty because it contains $D - t_8$.

Case (e). Let D be a configuration as in e. It is not contractible by the adjoint linear system $\mathrm{adj}_{(2,4)}$.

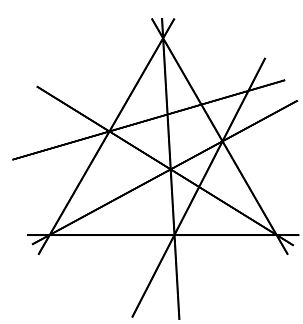
Case (f). If D is the configuration in f, then it is not contractible by the adjoint linear system $\mathrm{adj}_{(2,4)}$.

□

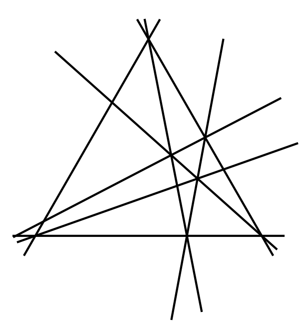
Theorem 108. *Let D be a configuration of the type $(8, 5; (\mathbf{6}))$. We have the following possible configurations:*

(a) (123)(145)(167)(246)(257)(348)(356)
(b) (123)(145)(167)(246)(257)(348)(568)
(c) (123)(145)(167)(246)(357)(348)(278)

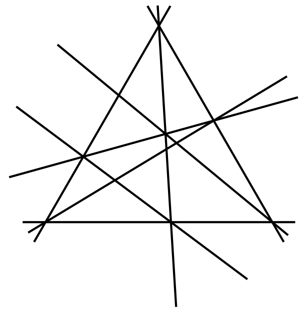
and they are contractible.



(123)(145)(167)(246)(257)(348)(356)



(123)(145)(167)(246)(257)(348)(568)



(123)(145)(167)(246)(357)(348)(278)

PROOF.

CASE (a). Applying to the configuration the transformation $\phi_{\mathcal{L}}$ where

$$\mathcal{L} = \mathcal{L}_8(4P_{167}, 4P_{257}, 4P_{348}, 2P_{145}, 2P_{246}, 2P_{356}, P_{123}, P_{18}, P_{280})$$

we have the a conic with a tangent line. We contract the line and apply Lemma 77.

CASE (b). Let us consider the linear system

$$\mathcal{L} = \mathcal{L}_8(4P_{167}, 4P_{257}, 4P_{348}, 2P_{123}, 2P_{145}, 2P_{568}, P_{246}, P_{18}, P_{28}).$$

The image $\phi_L(D)$ through the associated morphism, is a set of four lines with 4 marked points, which is contractible by Example 80.

CASE (c). The configuration D is mapped to the set of two lines by the transformation $\phi_{\mathcal{L}}$ where $\mathcal{L}$ is the linear system:

$$\mathcal{L} = \mathcal{L}_{16}(9P_{167}, 7P_{348}, 6P_{246}, 5P_{357}, 4P_{123}, 3P_{145}, 2P_{278}, 5P_{58}, 3P_{25}, P_{68}).$$

Thus it is contractible.

□

5. Configurations of lines with $k = 6$

We proceed with the study of the contractibility of configuration of lines. We observe that as k grows, the contractible configurations become fewer.

LEMMA 109. *Let D be a configuration of lines of the type $(d, 6; \mathbf{A})$ with $d \geq 11$. Then D is not contractible.*

PROOF. Assume that D is contractible and of type $(11, 6; (\mathbf{a}, \mathbf{b}))$. Then R_D is of the type $(6, k; \mathbf{B})$. If $k = 0, 1, 4$ then $a + b \leq 1$ and D is not contractible by Lemma 86. Assume that $k = 2$. Then, up to reordering, we may assume that $D - l_1 - r_1$ is of type $(9, 6; \mathbf{B})$ and it is not contractible. Therefore R is of type $(6, 3; \mathbf{B})$, $a = 0$, and $b \leq 4$. The adjoint linear system $\mathrm{adj}_{2,5}(D)$ has positive expected dimension for $b \leq 3$. If $b = 4$ then there are no triple points and the adjoint linear system $\mathrm{adj}_{2,4}(D)$ is not empty. This proves that D is not contractible and by Lemma 73 we conclude the same for $d \geq 12$. □

THEOREM 110. *Let $D = \sum_1^{d-6} l_i + \sum_1^6 r_j$ be a configuration of lines of type $(d, 6; \mathbf{A})$ not containing $\tilde{D}$ of the type $(8, 5; (a))$ with $a \geq 5$. Then D is not contractible.*

Proof. By Proposition 82 if D is contractible we have $d-6>d/3$, that is $d>9$. Moreover, by Lemma 109 if $d\geq 11$ then the configuration is not contractible.

Thus we can assume that D is of type $(10,6;(\mathbf{a}))$. We may assume also that $a\geq 2$ by Lemma 86.

Suppose firstly that $a=2$. Then if the 4-uple points are alligned, then D is not contractible. Assume that the 4-tuple points are not alligned and are the points q, $p_{\mathbf{1}123}$, and $p_{\mathbf{4}145}$. Let $\Delta := l_1+l_4+r_1$ the triangle determined by the 4-uple points. We are now interested in the 3-ple points. Note that: any line joining two 4-uple points can contain at most 1 triple point, any line through a 4-ple point can contain at most 3 triple points. Let b be the number of triple points, then a direct check shows that $b\leq 7$.

The linear system $\mathrm{adj}_{1,2}(D)$ has expected dimension

$$exp\dim\mathrm{adj}_{1,2}(D)=15-9-b=6-b.$$

Therefore D is not contractible if $b<6$.

If $b\geq 6$ then D contains a configuration of type $(8,5;(\mathbf{5}))$.

Assume that $a=3$ and let b be the number of triple points. Then by a direct calculation we see that we have $b\leq 6$. If $b\leq 2$ the linear system $\mathrm{adj}_{1,2}(D)$ is not empty. If $b\geq 3$, D contains a configuration of type $(8,5;(\mathbf{5}))$.

If $a=4$ then D has not triple points and the linear system $\mathrm{adj}_{2,4}(D)=(8;4^5)$ is effective. Thus D is not contractible. □

5.1. Classification. We can now state the main theorem on the classification of configurations of lines in $\mathbb{P}^2$.

Theorem 111. *Let D be a configuration of lines of the type $(d,k;\mathbf{A})$.*
If $d\geq 11$, D is contractible if and only if either $k\leq 2$ or D is of the type $(11,3;(\mathbf{0},\mathbf{3},\mathbf{18}))$.
If $d\leq 10$ and D does not contain a subconfiguration $(8,5;(\mathbf{a}))$ with $a\geq 5$, then D is contractible if and only if it is contained in one of the following configurations:

- $(d,2;\mathbf{A})$ *for any d;*
- $(11,3;(\mathbf{0},\mathbf{3},\mathbf{18}))$*;*
- $(10,4;(\mathbf{0},\mathbf{0},\mathbf{6}))$ *where at most one of the D_is is of the type $(9,4;(\mathbf{0},\mathbf{0},\mathbf{6}))$;*
- $(10,4;(\mathbf{0},\mathbf{1},\mathbf{3}))$*;*
- $(9,4;(\mathbf{0},\mathbf{0},\mathbf{6}))$ *where the D_is are of the type $(8,4;(\mathbf{0},\mathbf{b}))$ with $b\geq 4$;*
- $(9,5;(\mathbf{2},\mathbf{c}))$ *with $c\geq 1$.*

Moreover, if D is contained in a configuration of the type $(10,5;(\mathbf{0},\mathbf{2},\mathbf{a}))$ with $a\geq 3$, then D is contractible.

CHAPTER 3

Derived categories and rationality

1. Motivations

We are interested in characterizing the rationality of a surface X in terms of its derived category. We recall that a variety X is rational if there exists a birational transformation $\phi : X \dashrightarrow \mathbb{P}^n$ for some n. In characteristic 0, the weak factorization of a birational map ϕ suggest that we can decompose ϕ into a sequence of blow-ups and blow-downs with smooth exceptional loci, in particular these loci have codimension > 1.
Moreover, as we will see in the blow-up formula in Theorem (115), if $\tilde{X} \to X$ is a blow-up of X, a semiorthogonal decomposition of $D^b(\tilde{X})$ can be obtained by $D^b(X)$ and a finite number of derived categories of varieties of codimension > 1. For these reasons we expect that the obstruction to the rationality of a variety lies in codimension at most 1.

On the complex field we also have the following necessary condition for the rationality of a threefold X. That is, the intermediate Jacobian $J(X)$ has to be isomorphic, as principally polarized Abelian variety, to the direct sum of Jacobians of smooth projective curves. This result is due to Clemens and Griffiths [**13**]. They proved that if $X \subset \mathbb{P}^4$ is a smooth cubic, then it is not possible to split $J(X)$ as $J(X) = \bigoplus_i J(C_i)$ with C_i curves. Thus a cubic threefold X is not rational.

In this setting arise the notion of categorical representability.

Definition 112. A triangulated category $\mathcal{T}$ is *representable* in dimension m if it admits a semiorthogonal decomposition
$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle$$
where $\mathcal{A}_i$ is equivalent to an admissible subcategory of $D^b(Y_i)$ for some smooth variety Y_i of dimension at most m.

Moreover, if X is a smooth projective variety of dimension n, we say that X is *categorically representable* in dimension m (or in codimension $n - m$) if $D^b(X)$ is representable in dimension m.

This definition suggests the following conjectures.

Conjecture 113. *Let X be a geometrically rational variety of dimension n, if X is rational then it is representable in codimension $m \geq 2$.*

If X is a geometrically rational surface, X is rational if and only if X is categorically representable in dimension 0.

We usually refer to the second one as the Orlov conjecture.

In our work we are interested in the case of surfaces and thus of representability in dimension 0. By example 64, any rational surface is representable in dimension 0.
In order to study this aspect, in Section 4 we will generalize the definition of *Griffiths-Kuznetsov* component GK_X of a smooth geometrically rational surface X, which has been described by Auel and Bernardara [**1**] in the case of del Pezzo surfaces. It will appear clear that the Griffiths-Kuznetsov

component is the biggest subcategory of $D^b(X)$ which is not representable in dimension 0. In fact, it is generated by the terms in the semiorthogonal decomposition which are not representable in dimension 0. It will be a case by case definition in which it will appear clearly that the Griffiths-Kuznetsov component of the projective plane $\mathbb{P}^2$ is trivial.

2. An explicit mutation on derived categories

In Section 3.1 we have defined mutations and described some properties. Now we want to focus on mutations in $D^b(X)$ with X is surface.

Example 114. In the case of surfaces with ample anticanonical class, we can describe the mutations of exceptional objects as follows. (See [**40**]).

Let (A, B) be an exceptional collection in the triangulated category $\mathcal{T}$. We observe that in this case the complex $\mathrm{Ext}^\bullet_{\mathcal{T}}(A, B)$ is concentrated in the degrees 0 or 1. The left mutation $L_A(B)$ exists if one of the following possibilities occur:

(1) $\mathrm{Hom}^0_{\mathcal{T}}(A, B) \neq 0$ and the canonical map $\mathrm{Hom}^0_{\mathcal{T}}(A, B) \otimes A \to B$ is an epimorphism. Then $L_A(B)$ is defined by
$$0 \to L_A(B) \to \mathrm{Hom}^0_{\mathcal{T}}(A, B) \otimes A \to B \to 0$$
and it is called *division*.

(2) $\mathrm{Hom}^0_{\mathcal{T}}(A, B) \neq 0$ and the canonical map $\mathrm{Hom}^0_{\mathcal{T}}(A, B) \otimes A \to B$ is a monomorphism. Then we have
$$0 \to \mathrm{Hom}^0_{\mathcal{T}}(A, B) \otimes A \to B \to L_A(B) \to 0$$
and it is called *recoil*.

(3) $\mathrm{Hom}^1_{\mathcal{T}}(A, B) \neq 0$. Then $L_A(B)$ is given by the *extension*
$$0 \to B \to L_A(B) \to \mathrm{Hom}^1_{\mathcal{T}}(A, B) \otimes A \to 0.$$

Analogously we define the right mutations from the sequences:
$$\begin{aligned}
0 \to &A \to \mathrm{Hom}^0_{\mathcal{T}}(A, B)^* \otimes B \to R_B(A) \to 0 && \textit{division};\\
0 \to &R_B(A) \to A \to \mathrm{Hom}^0_{\mathcal{T}}(A, B)^* \otimes B \to 0 && \textit{recoil};\\
0 \to &\mathrm{Hom}^1_{\mathcal{T}}(A, B)^* \otimes B \to R_B(A) \to A \to 0 && \textit{extension}.
\end{aligned}$$

2.0.1. *An explicit example.* We want to describe explicitely an easy example of mutation on the projective plane $\mathbb{P}^2$. Let us consider the line bundles $\mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{O}_{\mathbb{P}^2}(1)$, we will omit the subscript since the situation is clear. They form an exceptional collection
$$(\mathcal{O}, \mathcal{O}(1)).$$
We want to the calculate the left mutation $L_{\mathcal{O}}(\mathcal{O}(1))$.

Firstly, let us denote with $[x_0 : x_1 : x_2]$ the coordinates of a point in $\mathbb{P}^2$. We observe that $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$ is generated as a group by the morphisms which acts on the unity as: $1 \mapsto x_0, 1 \mapsto x_1, 1 \mapsto x_2$. Thus it has dimension 3:
$$\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) \cong \mathcal{O}^{\oplus 3}.$$
There exists a natural morphism of complexes concentrated in degree 0:
$$\mathrm{Hom}^\bullet(\mathcal{O}, \mathcal{O}(1)) \otimes \mathcal{O} = \mathcal{O}^{\oplus 3} \xrightarrow{\alpha} \mathcal{O}(1)$$

and we observe that we are in Case (1) of Example 114. In order to complete the sequence into a triangle, we have to calculate the cone $Cone(\alpha)$ of the map α, object which coincides with the left mutation we are looking for. Thus we obtain

$$\mathcal{O}^{\oplus 3} \xrightarrow{\alpha} \mathcal{O}(1) \longrightarrow Cone(\alpha) = L_{\mathcal{O}}(\mathcal{O}(1)). \tag{5}$$

We recall the Euler sequence (see [**23**], II.8)

$$0 \to \Omega_{X|Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \to 0.$$

It is exact for $Y = \operatorname{Spec} R$, $X = \mathbb{P}^n_R$ for any ring R.
We tensorize the sequence with $\mathcal{O}(1)$:

$$0 \to \Omega_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_X^{\oplus 3} \xrightarrow{\alpha} \mathcal{O}(1) \to 0$$

where α is the same morphism as (5) since there we had a triangle.
Let us consider the objects of the sequence as complexes concentrated in degree 0. Thus, we want to find $Cone^{-1}(\alpha)$ and $Cone^0(\alpha)$ since $Cone^i(\alpha) = 0$ for $i \neq 0, 1$. We remark that α is surjective, thus $Cone^0(\alpha) = 0$. On the other side, $Cone^{-1}(\alpha) = \Omega_{\mathbb{P}^2}(1) = \Omega_{\mathbb{P}^2} \otimes \mathcal{O}(1)$. We have the following distinguished triangle

$$(\Omega_{\mathbb{P}^2}(1), \mathcal{O}^{\oplus 3}, \mathcal{O}(1)) = (\mathcal{O}^{\oplus 3}, \mathcal{O}(1), \Omega_{\mathbb{P}^2}(1)[1]).$$

It follows that $L_{\mathcal{O}}\mathcal{O}(1) = \Omega_{\mathbb{P}^2}(1)[1]$.

With analogous calculations we can find $R_{\mathcal{O}(1)}\mathcal{O}$. We consider the sequence

$$A \xrightarrow{\beta} \operatorname{Hom}(A, B)^* \otimes B \longrightarrow Cone(\beta).$$

Using the dual of the Euler sequence:

$$0 \to \mathcal{O} \xrightarrow{\beta} \mathcal{O}(1)^{\oplus 3} \longrightarrow T_{\mathbb{P}^2} \to 0$$

it is straightforward to see that $R_{\mathcal{O}(1)}(\mathcal{O}) = T_{\mathbb{P}^2}$. We observe that for the right mutation, we apply Case (1) of Example 114.

3. Semiorthogonal decompositions

In this section we describe semiorthogonal decompositions of the derived category of varieties relevant to our study. They are well known results and they will be useful to our proofs in the last section.

The first one is known as the Orlov formula for the blow-up and it holds for any dimension. It expresses the semiorthogonal decomposition of the derived category of the blow-up $\tilde{X} \to X$ of a variety X in terms of the semiorthogonal decomposition on X and the blown-up locus. In particular, if $D^b(X)$ is representable in dimension m, then $D^b(\tilde{X})$ is representable in dimension $\tilde{m}$ with $\tilde{m} \leq \max(n-2, m)$ where n is the dimension of X.

Theorem 115. *Let X be a smooth projective variety and $Y \subset X$ a smooth projective subvariety of codimension $d > 1$. Let us denote by $\epsilon : \tilde{X} \to X$ the blow-up of X along Y and let $E \overset{i}{\hookrightarrow} \tilde{X}$ be the exceptional divisor and $p : E \to Y$ the restriction of ϵ.*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\epsilon"] & X \\ E \arrow[u, hook, "i"] \arrow[r, "\sigma"'] & Y \arrow[u, hook] \end{tikzcd}$$

Then $\epsilon^* : D^b(X) \to D^b(\tilde{X})$ is fully faithful and for all j, there are fully faithful functors $\Psi^j : \epsilon^* D^b(X) \to D^b(\tilde{X})$ giving the following semiorthogonal decomposition:

$$D^b(\tilde{X}) = \left\langle \Psi^{-d+1} D^b(Y), \dots, \Psi^{-1} D^b(Y), \epsilon^* D^b(X) \right\rangle .$$

The functors Ψ^j are defined via $\Psi^j(-) = i_*(p^*(-) \otimes \mathcal{O}_{E/Y}(j))$.

It follows that for surfaces, this proposition shows how we can construct the semiorthogonal decomposition of the derived category of a surface from that of a minimal surface using the point of view of the Minimal Model Program.

COROLLARY 116. *Let X be a smooth projective surface over an arbitrary field k. Then there exist a smooth projective minimal surface X' and a fully faithful functor $\Phi : D^b(X') \to D^b(X)$ such that the orthogonal complement of $\Phi D^b(X')$ in $D^b(X)$ is representable in dimension 0.*

PROOF. Since X is not minimal, there exists a k-birational morphism $\pi : X \to X'$ to a minimal surface and it is exactly the blow-up of a closed subvariety Z of dimension 0. Then it is enough to use Proposition 115. □

3.1. A semiorthogonal decomposition for conic bundles. The next proposition explains how to construct a semiorthogonal decomposition for a conic bundle involving the derived category of the base. We observe that it holds also in higher dimension as we can see in [**32**].

THEOREM 117. *Let $\pi : X \to C$ be a conic bundle. Then $\pi^* : D^b(C) \to D^b(X)$ is fully faithful and there exists a fully faithful functor $\Phi : D^b(C, \mathcal{C}) \to D^b(X)$ such that*

$$D^b(X) = \left\langle \pi^* D^b(C), \Phi D^b(C, \mathcal{C}) \right\rangle .$$

where $\mathcal{C}$ is the sheaf of even parts of the Clifford algebra associated to $\pi : X \to C$, a locally free sheaf over C.

In order to fully express the semiorthogonal decomposition above, we reduce to the case we are interested in. Let $\pi : X \to C$ be a conic bundle where X is a smooth geometrically rational surface and C a genus 0 curve over an arbitrary field k. Since k is an arbitary field, we can write the derived category of the curve as

$$D^b(C) = \langle \mathcal{O}_C, V \rangle \text{ or equivalently } D^b(C) = \langle V^*, \mathcal{O}_C \rangle , \tag{6}$$

where V is a rank 2 vector bundle wih the following properties. If we consider the separable closure k^s of the field k, we have $C^s \cong \mathbb{P}^1$ and $V = \mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$. Moreover the subcategory generated by V in $D^b(C^s)$ is $\langle V \rangle = \langle \mathcal{O}(1) \rangle$. We observe that the equivalence 6 holds because V^* is the left mutation of V with respect to $\mathcal{O}$. Moreover in k^s the vector bundle V^* is of the form $V = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$

Then we can write the semiorthogonal decomposition of $D^b(X)$ as

$$D^b(X) = \langle \pi^* \mathcal{O}_C, \pi^* V, \mathcal{A}_X \rangle = \langle \mathcal{O}_X, \pi^* V, \mathcal{A}_X \rangle , \tag{7}$$

where $\mathcal{A}_X =^{\perp} \langle \mathcal{O}_X, \pi^* V \rangle$ and on k^s we have $\pi^* V = \mathcal{O}(F) \oplus \mathcal{O}(F)$. We may denote $\mathcal{A}_X$ as $\mathcal{A}_{X/C}$ if we want to stress the conic bundle structure. We observe that $\mathcal{A}$ is equivalent to $D^b(C, \mathcal{C})$ through the functor Φ from Theorem [**?**].

REMARK 118. By abuse of notation we will omit the pull-back functor and denote, for example, V instead of $\pi^* V$ as an element of $D^b(X)$.

In our discussion we will need to reduce to the separable closure k^s and to use the semiorthogonal decomposition of $D^b(X^s)$. We recall that $\pi : X^s \to \mathbb{P}^1$ can be obtained by the blow-up of $\mathbb{F}_n$ in r points on different fibers for some $n \neq 1$, where $r = 8 - K_X^2$. Let us denote by $F_1, \dots F_r$ the degenerate fibers. Thus by Theorem 115 and Theorem 117 we have

$$D^b(X^s) = \left\langle \pi^* D^b(\mathbb{P}^1), \mathcal{A}_{X^s} \right\rangle.$$

The subcategory $\mathcal{A}_{X^s}$can be written as

$$\mathcal{A}_{X^s} = \left\langle \mathcal{A}_{\mathbb{F}_n/\mathbb{P}^1}, \mathcal{O}_{F_1}, \dots \mathcal{O}_{F_r} \right\rangle = \langle \mathcal{O}(\Sigma), \mathcal{O}(\Sigma + F), \mathcal{O}_{F_1}, \dots \mathcal{O}_{F_r} \rangle$$

where Σ is the section of the conic bundle that is the strict transform of the curve of $\mathbb{F}_n$ with negative self-intersection $-n$.
Thus, from $D^b(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(F) \rangle$ it follows the following corollary.

Corollary 119. *On the separable closure of the field k, we have the semiorthogonal decompositon:*

$$D^b(X^s) = \langle \mathcal{O}_{X^s}, \mathcal{O}_{X^s}(F), \mathcal{O}_{X^s}(\Sigma), \mathcal{O}_{X^s}(\Sigma + F), \mathcal{O}_{F_1}, \dots, \mathcal{O}_{F_r} \rangle,$$

where F is the generic fiber, Σ the section and $F_1, \dots F_r$ the degenerate fibers of the conic bundle.

3.2. A semiorthogonal decomposition for del Pezzo surfaces. We want to study now the semiorthogonal decompositions of the derived category of a del Pezzo surface.

Let S be a del Pezzo surface on the arbitrary field k. First of all we notice that the line bundle $\mathcal{O}_S$ is an exceptional object of $D^b(S)$, thus there exist a non-trivial decomposition.

Theorem 120. *Let S be a del Pezzo surface, thus there exist the semiorthogonal decomposition:*

$$D^b(S) = \langle \mathcal{O}_S, \mathcal{A}_S \rangle$$

where $\mathcal{A}_S$ is the right orthogonal of $\mathcal{O}_S$ in $D^b(S)$.

In particular, the following examples will be useful in our discussion.

Example 121. Let S be a Del Pezzo surface of degree 4 with a rational point s and let $X \to S$ be the blow-up of S at s. It is a known fact that S can be realized as the complete intersection of two quadrics and X has a conic bundle structure $\pi : X \to \mathbb{P}^1$.
The derived category of X can be decomposed using the conic bundle structure on $\mathbb{P}^1$ to obtain

$$D^b(X) = \left\langle \mathcal{O}_X, \mathcal{O}_X(F), D^b(\mathbb{P}^1, \mathcal{C}) \right\rangle,$$

where F is the fiber of π. On the other hand it is possible to construct a decomposition using Theorem 120 and the Orlov formula for the blow-ups (Theorem 115).

$$D^b(X) = \langle \mathcal{O}_E(-E), \mathcal{O}_X, \mathcal{A}_S \rangle$$

Auel and Bernardara proved in [**1**] that $\mathcal{A}_S \cong D^b(\mathbb{P}^1, \mathcal{C})$.

Example 122. Let now consider S the Del Pezzo surface of degree 8 with a 0-cycle of degree 2. S is a quadric in $\mathbb{P}^3$ and its blow-up $\sigma : X \to S$ in s gives us a conic bundle structure $\pi : X \to C$ on a surface of degree 6.
As we have seen, the derived category of S is

$$D^b(S) = \langle \mathcal{O}_S, \mathcal{A}_S \rangle$$

and the component $\mathcal{A}_S$ is decomposable into $\mathcal{A}_S = \left\langle D^b(k, A), D^b(k, \mathcal{C}_0) \right\rangle$ where A is the algebra associated to S and $\mathcal{C}_0$ is the Clifford class of S, for a reference see [**1**]. In particular, $D^b(k, \mathcal{C}_0)$ can

be seen in k^s as $\langle \mathcal{O}_{S^s}(1,0) \oplus \mathcal{O}_{S^s}(0,1)\rangle$, and $D^b(k,A)$ as $\langle \mathcal{O}_{S^s}(1,1)\rangle$.
In order to compare $\sigma^*\mathcal{A}_S$ with $\mathcal{A}_{X/C}$, we write the derived category of X as a conic bundle

$$D^b(X) = \big\langle \mathcal{O}_X, V, \mathcal{A}_{X/C} \big\rangle$$

and as the blow-up of S

$$\begin{aligned} D^b(X) &= \big\langle \mathcal{O}_X, \sigma^*\mathcal{A}_S, D^b\big(k(s)\big)\big\rangle \\ &= \big\langle \mathcal{O}_X, \sigma^* D^b\big(k, A\big), \sigma^* D^b\big(k, \mathcal{C}_0\big), D^b\big(k(s)\big)\big\rangle \end{aligned}$$

where the last component is given by the exceptional divisor, thus is representable in dimension 0. Auel and Bernardara [**1**] show that

$$\langle V\rangle \cong \sigma^* D^b(k,A) \text{ and } \mathcal{A}_{X/C} \cong \big\langle D^b(k,\mathcal{C}_0), D^b(k(s))\big\rangle\,.$$

In particular, the components that are not representable in dimension 0 are:

- none if S is rational;
- $\sigma^* D^b(k,\mathcal{C}_0)$ if S is not rational and $C \cong \mathbb{P}^1$;
- $\sigma^* D^b\big(k,A\big), \sigma^* D^b\big(k,\mathcal{C}_0\big)$ if neither S nor C are rational.

4. The Griffiths-Kuznetsov component

In this section we give the definition of the Griffiths-Kuznetsov component. We want it to be a birational invariant which vanishes whenever the surface is rational. Firstly we construct the definition, then we show that it is a birational invariant.

We proceed with a case by case analysis based on the classification of minimal geometrically rational surfaces given in Proposition 10 adopting the notations of the previous Section 3.

Let X be a smooth projective minimal geometrically rational surface over an arbitrary field k. We want to determine the components of $D^b(X)$ which are not representable in dimension 0.

Case ($\mathbb{P}^2$). The semiorthogonal decomposition of the projective plane is representable in dimension 0. In fact we have

$$D^b(\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle\,.$$

Case (del Pezzo surface). Let X be a del Pezzo surface of degree d. We have seen that it admits the semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{O}, \mathcal{A}_X\rangle\,.$$

If $\mathcal{A}_X$ is not decomposable in two smaller subcategories, then $\mathcal{A}_X$ is the component we are looking for, otherwise, we take the subcategory not representable in dimension 0. See [**1**] for a detailed description.

Case (conic bundle). Let $\pi : X \to C$ be a conic bundle, by Theorem 117 it admits the semiortogonal decomposition

$$D^b(X) = \big\langle \mathcal{O}, V, \Phi D^b(C,\mathcal{C})\big\rangle\,.$$

Thus, if $C \cong \mathbb{P}^1$, V is representable in dimension 0 and $\Phi D^b(\mathbb{P}^1,\mathcal{C})$ may be not. If C is not rational, the components which are not representable in dimension 0 are V and possibly $\Phi D^b(C,\mathcal{C})$.

Among the del Pezzo surfaces, if X is a quadric in $\mathbb{P}^3$ or an involution surface, then the subcategory $\mathcal{A}_X$ can be decomposed into two smaller subcategories.

Case (quadric). Let $X \subset \mathbb{P}^3$ be a smooth non-rational quadric. There exists a conic bundle $X' \to \mathbb{P}^1$ such that $X' \to X$ is the blow-up of X. Combining the semiorthogonal decompositions of a del Pezzo surface with the blow-up formula, we see that

$$D^b(X) = \left\langle \mathcal{O}, \mathcal{O}(F), D^b(k, \mathcal{C}_0) \right\rangle$$

where $D^b(k, \mathcal{C}_0)$ is the orthogonal complement of $\langle \mathcal{O}, \mathcal{O}(F) \rangle$. The component non-representable in dimension 0 is $D^b(k, \mathcal{C}_0)$.

Case (involution surface). If X is an involution surface, then there exists a conic bundle $X' \to C$ with C a non-rational curve, such that $X' \to X$ is the blow-up of X. Using the blow-up formula, we obtain the following semiorthogonal decomposition:

$$D^b(X) = \left\langle \mathcal{O}, V, D^b(k, \mathcal{C}_0) \right\rangle$$

where V is a rank 4 vector bundle which can be seen in k^s as

$$V = \mathcal{O}_{X^s}(1, 0) \oplus \mathcal{O}_{X^s}(0, 1)$$

and $D^b(k, \mathcal{C}_0)$ is the orthogonal complement of $\langle \mathcal{O}, V \rangle$. The components non-representable in dimension 0 are V and $D^b(k, \mathcal{C}_0)$.

Resuming this discussion and using the classes $\{\mathcal{C}\}$ and $\{\mathcal{D}\}$ introduced in Notation 23, we can define the Griffiths-Kuznetsov component. For a reference about the definition on the class $\{\mathcal{D}\}$, see [**1**].

Definition 123. Let X be a smooth projective geometrically rational surface over an arbitrary field k.
If X belongs to the class $\{\mathcal{D}\}$, that is X is minimal and has Picard number 1, then the *Griffiths-Kuznetsov component* GK_X of the derived category $D^b(X)$ is defined as follows: if A_X is representable in dimension 0, set $\mathrm{GK}_X = 0$. If not, GK_X is either the product of all indecomposable components of A_X that are not representable in dimension 0 or else $\mathrm{GK}_X = A_X$.
If X belongs to the class $\{\mathcal{C}\}$, that is if it is a conic bundle $\pi : X \to C$ as in Definition 15 and has Picard number 2, then the *Griffiths-Kuznetsov component* GK_X of the derived category $D^b(X)$ is defined as follows:

(1) if X is rational, $\mathrm{GK}_X = 0$;
(2) if $C \cong \mathbb{P}^1$ and X is the blow-up of a quadric, $\mathrm{GK}_X = D^b(k, \mathcal{C}_0)$;
(3) if $C \cong \mathbb{P}^1$, X is not rational and not birational to a quadric, $\mathrm{GK}_X = D^b(\mathbb{P}^1, \mathcal{C})$;
(4) if $C \ncong \mathbb{P}^1$ and X is the blow-up of an involution surface, $\mathrm{GK}_X = \langle V \rangle \oplus D^b(k, \mathcal{C}_0)$;
(5) if $C \ncong \mathbb{P}^1$ and X is not birational to an involution surface, $\mathrm{GK}_X = \langle V \rangle \oplus D^b(C, \mathcal{C})$.

If X is not minimal, the Griffiths-Kuznetsov component is $\mathrm{GK}_X = \mathrm{GK}_{X'}$ for a minimal model $X \to X'$.

We want to show that is a good definition.

Theorem 124. *Let X be a geometrically rational surface over an arbitrary field k. Then the Griffiths-Kuznetsov component* GK_X *is well defined and is a birational invariant. Moreover, X is rational if and only if* $\mathrm{GK}_X = 0$.

Proof. Let $\phi : X \dashrightarrow Y$ be a birational map between two smooth projective geometrically rational surfaces X and Y over the arbitrary field k. By Theorem 24 ϕ can be decomposed into a finite number of links of Sarkisov. Thus we can assume that ϕ is a Sarkisov link.

Case (I). Firstly, let us assume that ϕ is a link of type I. Thus $Y \to X$ is the blow-up of X at a closed point. Then the semiorthogonal decomposition of $D^b(Y)$ differs from the semiorthogonal decomposition of $D^b(X)$ only for the component related to the exceptional divisor, which is representable in dimension 0 by Corollary 116. Thus the Griffiths-Kuznetsov components coincide: $\mathrm{GK}_X = \mathrm{GK}_Y$.

Case (II). Let us assume that ϕ is a link of type II. Thus X, Y belong to the same class: either $\{\mathcal{D}\}$ or $\{\mathcal{C}\}$.
If $X, Y \in \{\mathcal{D}\}$ the proof has been shown by Auel and Bernardara in [**1**]. If $X, Y \in \{\mathcal{C}\}$, the link has been described in Section 5. The derived categories admit the following semiorthogonal decompositions:

$$D^b(X) = \langle \mathcal{O}_X, V, \mathcal{A}_X \rangle, \quad D^b(Y) = \langle \mathcal{O}_Y, V', \mathcal{A}_Y \rangle.$$

The components which may not be representable in dimension 0 are V, V' and $\mathcal{A}_X, \mathcal{A}_Y$. It is straightforward that $\langle V \rangle \cong \langle V' \rangle$ and in Theorem 125 we show that $\mathcal{A}_X \cong \mathcal{A}_Y$. Thus $\mathrm{GK}_X = \mathrm{GK}_Y$.

Case (III). If ϕ is a link of type III, we proceed analogously to Case I.

Case (IV). Let us suppose that ϕ is a link of type IV, then $X = Y \in \{\mathcal{C}\}$. Iskovskikh has provided a classification of this link [**27**], in particular it can be realized by surfaces of degree 8,4,2 or 1.
We have the following semiorthogonal decompositions:

$$D^b(X) = \langle \mathcal{O}_X, V_1, \mathcal{A}_1 \rangle = \langle \mathcal{O}_X, V_2, \mathcal{A}_2 \rangle.$$

As above, the components which may not be representable in dimension 0 are V_i and $\mathcal{A}_i$ for $i = 1, 2$. In section 6, for each possible degree we show that $\langle V_1 \rangle \cong \langle V_2 \rangle$ and that $\mathcal{A}_1 \cong \mathcal{A}_2$. Thus the Griffiths-Kuznetsov components as defined by the two different conic bundle structures, are equivalent.

□

5. Links of type II

Let $X \xrightarrow{\pi} C$ and $Y \xrightarrow{\sigma} C'$ be conic bundles and F, F' the respective fibers. Let us consider $\phi : X \dashrightarrow Y$ a link of type II. Thus there exists a closed point $x \in X$ not lying in the degenerate geometric fibers of the structure morphism π such that ϕ is the blow-up of the point x with the subsequent contraction of the fiber X_x of π containing x. Let us denote with $p : Z \to X$ the blow-up of x in X. The following diagram commutes.

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[d, "\pi"'] & & Y \arrow[d, "\sigma"] \\ C \arrow[rr, "\simeq"'] & & C' \end{tikzcd}$$

Let E be the exceptional divisor of p and E' the exceptional divisor of q. By abuse of notations, we will denote by F and F' the preimages of the fibers of π and σ in Z. We have that

$$\begin{aligned} -K_Z =& p^*(-K_X) - E \\ =& q^*(-K_Y) - E' \end{aligned}$$

Moreover, since by the transformation $E = dF' - E'$ where d is the degree of the point x, it follows that

$$\begin{aligned} p^*(-K_X) =& q^*(-K_Y) + dF' - 2E' \\ F =& F' \\ E =& dF' - E' = p^*(-K_X) - q^*(-K_Y) + E'. \end{aligned}$$

Compare this result with Theorem 2.6 in Iskovskikh [**27**].

We recall that we have the following semiorthogonal decompositions for X and Y:

$$D^b(X) = \langle \mathcal{O}_X, V, \mathcal{A}_X \rangle, \quad D^b(Y) = \langle \mathcal{O}_Y, V', \mathcal{A}_X \rangle,$$

where V and V' are vector bundles of rank 2. We can show that $\langle V \rangle \cong \langle V' \rangle$. In fact, in k^s we can express those vector bundles as

$$V = \mathcal{O}(f) \oplus \mathcal{O}(f) \text{ and } V' = \mathcal{O}(f') \oplus \mathcal{O}(f')$$

where f and f' are respectively the fiber of $\pi^s : X^s \to \mathbb{P}^1$ and of $\sigma^s : Y^s \to \mathbb{P}^1$. In particular, $F = F'$ implies that F and F' consist of the same number of geometrical fibers. Thus $f = f'$ and $\langle \mathcal{O}(f) \rangle = \langle \mathcal{O}(f') \rangle$.

Theorem 125. *Let $\phi : X \dashrightarrow Y$ be a link of type II as above. Then $\mathcal{A}_X \cong \mathcal{A}_Y$.*

Proof. Let k^s be the separable closure of the field k, we use the notation seen previously. Then the derived category $D^b(X^s)$ admits the following semiorthogonal decomposition:

$$D^b(X^s) = \langle \mathcal{O}, \mathcal{O}(f), \mathcal{O}(\Sigma), \mathcal{O}(f + \Sigma), \mathcal{O}_{F_1}, \dots, \mathcal{O}_{F_r} \rangle$$

where f is the fiber of π^s, Σ is the section, $-n$ is its self-intersection, and $F_1, \dots, F_r$ are the degenerate fibers of π^s. Analogously for Y, we have

$$D^b(Y^s) = \langle \mathcal{O}, \mathcal{O}(f'), \mathcal{O}(\Sigma'), \mathcal{O}(f' + \Sigma'), \mathcal{O}_{F_1}, \dots, \mathcal{O}_{F_r} \rangle$$

We observe that $r = 8 - K_X^2$ and that for any $i = 1, \dots r$ the degenerate fiber F_i is preserved by link because the point x does not lie on it. Moreover, we have

$$-K_{X^s} = 2\Sigma + (n+2)F \text{ and } -K_{Y^s} = 2\Sigma' + (n-d+2)F' - 2E'$$

for some integer n. Using the substitutions in **??**, we have that in k^s

$$p^*(-K_X) - E = q^*(-K_Y) - E'$$

and by abuse of notations:

$$\begin{aligned} 2\Sigma + (n+2)F - E =& 2\Sigma' + (n-d+2)F - 2E' - E' \\ 2\Sigma = 2\Sigma' + (E - dF - E') - 2E'. \end{aligned}$$

We know that $E = dF - E'$, thus $2\Sigma = 2\Sigma' - 2E'$. In k^s we have that

$$\Sigma = \Sigma' - E'.$$

Let us consider the component $\mathcal{A}_{Y^s}$ of $D^b(Y^s)$ as defined in 7. We tensor it with $\mathcal{O}(-E')$ and obtain

$$\mathcal{A}_Y \otimes \mathcal{O}(-E') = \langle \mathcal{O}(\Sigma' - E'), \mathcal{O}(F' + \Sigma' - E'), \mathcal{O}_{F_1}, \dots \mathcal{O}_{F_r} \rangle .$$

It is straightforward to see that it is exactly the component

$$\mathcal{A}_{X^S} = \langle \mathcal{O}(\Sigma), \mathcal{O}(F + \Sigma), \mathcal{O}_{F_1}, \dots \mathcal{O}_{F_r} \rangle .$$

Thus,

$$\otimes \mathcal{O}(-E') : D^b(Z^s) \longrightarrow D^b(Z^s)$$

is an auto-equivalence of categories which sends

$$p^* \mathcal{A}_{X^s} \longmapsto q^* \mathcal{A}_{Y^s} .$$

We observe that $\otimes\mathcal{O}(-E')$ is well defined also in k, so there it gives an equivalence of categories

$$\begin{aligned} \otimes \mathcal{O}(-E') : D^b(Z) &\xrightarrow{\cong} D^b(Z) \\ \mathcal{A}_X &\longmapsto \mathcal{A}_Y \end{aligned}$$

. □

6. Links of type IV

As described in Chapter 1 a link of type IV is a birational transformation which sends one structure of conic bundle into another on the same variety. This means that if we consider a variety X, it must admit two conic bundle structures $\pi_1 : X \to C_1$ and $\pi_2 : X \to C_2$. In [**27**, Thm 2.6], it is proven that this is possible only for varieties with ample anticanonical class of degree 8,4,2 or 1. In this context, let us denote with $\mathcal{A}_i = \mathcal{A}_{X/C_i}$ the subcategory of $D^b(X)$ for $i = 1, 2$ and analogously each object related with the conic bundle structure π_i will be denoted with the subscript i.

In this section we will prove that the Griffiths-Kuznetsov components of X defined through the different conic bundle structures are equivalent. For this reason we introduce the following notation.

Notation 126. Let $\pi : X \to C$ be a conic bundle. The surface X admits the semiorthogonal decompositions

$$D^b(X) = \langle \mathcal{O}, V_1, \mathcal{A}_1 \rangle = \langle \mathcal{O}, V_2, \mathcal{A}_2 \rangle$$

given by the conic bundle structures. We have defined the Griffiths-Kuznetsov component GK_X of X starting from the semiorthogonal decomposition of $D^b(X) = \langle \mathcal{O}, V, \mathcal{A}_X \rangle$. We denote it as $\mathrm{GK}_X(\pi)$, that is, for $i = 1, 2$

(1) $\mathrm{GK}_X(\pi_i) = 0$ if X is rational;
(2) $\mathrm{GK}_X(\pi_i) = \mathcal{A}_i$ if C_i is rational and X is not;
(3) $\mathrm{GK}_X(\pi_i) = V_i \amalg \mathcal{A}_i$ if neither C_i nor X is rational.

In this way, we have to prove that a link of type IV acts as an equivalence between $\mathrm{GK}_X(\pi_1)$ and $\mathrm{GK}_X(\pi_2)$.

6.1. Conic bundles of degree 8. This case has been discussed in [**1**]. Let X be a geometrically rational conic bundle of degree 8. Firstly we notice that X have degenerate fibers neither for π_1 nor for π_2. Then by [**27**, Thm 2.6] we have that $X = C_1 \times C_2$ with C_1, C_2 smooth curves of genus 0 and π_1, π_2 are the canonical projections. The link acts as

$$F_1 = -\frac{1}{2}K_X - F_2$$

where F_i is the fiber of $\pi_i : X \to C_i$ for $i = 1, 2$. If $C_1 = C_2$ then the link consist in the involution of X which exchanges the two factors.

Theorem 127. *The Griffiths-Kuznetsov component* $\mathrm{GK}_X(\pi_1)$ *described through the conic bundle structure* $\pi_1 : X \to C_1$ *is equivalent to* $\mathrm{GK}_X(\pi_2)$*, described by* $\pi_2 : X \to C_2$*.*

Proof. Let us consider the separable closure of the field and the corresponding surface X^s. We recall that X^s is isomorphic to $\mathbb{F}_n$ for some n and is a $\mathbb{P}^1$-bundle. Thus X^s is isomorphic to $\mathbb{F}_0$. The derived category admits the following semiorthogonal decomposition:

$$D^b(X^s) = \langle \mathcal{O}, \mathcal{O}(F_1), \mathcal{A}_1 \rangle$$

and $\mathcal{A}_1 = \langle \mathcal{O}, \mathcal{O}(F_1) \rangle = \langle \mathcal{O}(\Sigma_1), \mathcal{O}(\Sigma_1 + F_1) \rangle$ where Σ is the section. We observe that in this case the section Σ_1 coincide with the fiber F_2. Thus we have

$$D^b(X^s) = \langle \mathcal{O}, \mathcal{O}(F_1), \mathcal{O}(F_2), \mathcal{O}(F_1 + F_2) \rangle .$$

Since $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$ are completely orthogonal, the left mutation of $\mathcal{O}(F_2)$ with respect to $\mathcal{O}(F_1)$ is $L_{\mathcal{O}(F_1)}(\mathcal{O}(F_2)) = \mathcal{O}(F_2)$. Thus we obtain the following

$$D^b(X^s) = \langle \mathcal{O}, \mathcal{O}(F_2), \mathcal{O}(F_1), \mathcal{O}(F_1 + F_2) \rangle .$$

which is exactly the semiorthogonal decomposition given by the conic bundle structure on the second factor.

Coming back to the field k, we have constructed an equivalence

$$\langle V_1, \mathcal{A}_1 \rangle = \langle V_2, \mathcal{A}_2 \rangle .$$

□

6.2. Conic bundles of degree 4. If X has degree 4, a link of type IV induces the following transformation on $\mathrm{Pic}(X)$:

$$F_1 = -K_{X_1} - F_2.$$

See [**27**, Thm 2.6] for a reference.
An example of this case is given over $\mathbb{R}$ by the canonical involution which exchange the two structures.

Theorem 128. *Let us consider X of degree 4. Then*

$$\mathrm{GK}_X(\pi_1) \cong \mathrm{GK}_X(\pi_1).$$

Proof. The derived categories of the two conics at the base admit the following semiorthogonal decompositions

$$D^b(C_1) = \langle \mathcal{O}_{C_1}, V_1 \rangle \qquad D^b(C_2) = \langle \mathcal{O}_{C_2}, V_2 \rangle$$

By abuse of notation we will denote $\pi_1^* V_1$ and $\pi_2^* V_2$ on X as V_1 and V_2.

We recall that if we consider V_1 and V_2 in k^s, the separable closure of k, they can be written as

$$V_1 = \mathcal{O}(F_1) \oplus \mathcal{O}(F_1), \qquad V_2 = \mathcal{O}(F_2) \oplus \mathcal{O}(F_2).$$

Using the transformation of $\mathrm{Pic}(X)$ induced by the link, we have that $\mathcal{O}(F_1) = \mathcal{O}(-K_X - F_2)$ and thus

$$\begin{aligned} V_1 =& \mathcal{O}(-K_X - F_2) \oplus \mathcal{O}(-K_X - F_2) = \\ =& \omega_X^* \otimes (\mathcal{O}(-F_2) \oplus \mathcal{O}(-F_2)) = \omega_X^* \otimes V_2^*, \end{aligned}$$

where we have used the fact that $V_2^* = \mathcal{O}(-F_2) \oplus \mathcal{O}(-F_2)$. Thus $\langle V_1 \rangle \cong \langle V_2 \rangle$ through the autoequivalence $\otimes \omega_X^*$ of $D^b(X)$.

Then the derived category of X has the following semiorthogonal decompositions

$$D^b(X) = \left\langle \pi_1^* D^b(C_1), \mathcal{A}_1 \right\rangle = \left\langle \mathcal{O}_X, V_1, \mathcal{A}_1 \right\rangle, \tag{8}$$

$$D^b(X) = \left\langle \pi_2^* D^b(C_2), \mathcal{A}_2 \right\rangle = \left\langle V_2^*, \mathcal{O}_X, \mathcal{A}_2 \right\rangle. \tag{9}$$

Let us consider (9) and tensor it with the anticanonical sheaf, then apply the mutation of $\langle \omega_X^*, \mathcal{A}_2 \otimes \omega_X^* \rangle$ to the left with respect to its orthogonal complement and use Proposition 66 to obtain

$$D^b(X) = \left\langle \mathcal{O}_X, \mathcal{A}_2, V_2^* \otimes \omega_X^* \right\rangle.$$

Since we have seen that $V_2^* \otimes \omega_X^* = V_1$, we can write:

$$D^b(X) = \left\langle \mathcal{O}_X, \mathcal{A}_2, V_1 \right\rangle.$$

Now we mutate $\mathcal{A}_2$ to the right with respect to V_1 and we obtain

$$D^b(X) = \left\langle \mathcal{O}_X, V_1, R_{V_1}\mathcal{A}_2 \right\rangle.$$

In this way we have shown that both $\mathcal{A}_1$ and $R_{V_1}\mathcal{A}_2 \cong \mathcal{A}_2$ are semiorthogonal to $\langle \mathcal{O}_X, V_1 \rangle$. Thus they are equivalent as subcategories of $D^b(X)$. $\square$

6.3. Conic bundles of degree 2. In this case we have that C_1 and C_2 are isomorphic and the link behaves as a Geiser involution. It acts on $\mathrm{Pic}(X)$ as

$$F_1 = -2K_X - F_2.$$

THEOREM 129. *Let X be of degree 2. Then $\mathrm{GK}_X(\pi_1)$ is equivalent to $\mathrm{GK}_X(\pi_2)$.*

PROOF. We proceed analogously to the proof of Proposition 128.
In k^s, the separable closure of k, we have that

$$V_1 = \mathcal{O}(F_1) \oplus \mathcal{O}(F_1) \quad \text{and} \quad V_2 = \mathcal{O}(F_2) \oplus \mathcal{O}(F_2).$$

Using the transformation induced on $\mathrm{Pic}(X)$ by the link described by Iskovskikh [**27**, Thm 2.6] we find the relation between V_1 and V_2.

$$\begin{aligned} V_1 =& \mathcal{O}(-2K_X - F_2) \oplus \mathcal{O}(-2K_X - F_2) \\ =& (\mathcal{O}(-F_2) \oplus \mathcal{O}(-F_2)) \otimes \mathcal{O}(-2K_X) \\ =& V_2^* \otimes (\omega_X^*)^{\otimes 2} \end{aligned} \tag{10}$$

Then $\langle V_1 \rangle \cong \langle V_2 \rangle$ through the autoequivalence $\otimes(\omega_X^*)^{\otimes 2}$ of $D^b(X)$.

The derived category of X admits the semiorthogonal decompositions:

$$D^b(X) = \langle \mathcal{O}_X, V_1, \mathcal{A}_1 \rangle \tag{11}$$

$$D^b(X) = \langle V_2^*, \mathcal{O}_X, \mathcal{A}_2 \rangle \tag{12}$$

Let us consider (11), we apply a right mutation of $\mathcal{O}_X$ with respect to its orthogonal complement as in Proposition 66 and obtain

$$D^b(X) = \langle V_1, \mathcal{A}_1, \omega_X^* \rangle = \langle \mathcal{A}_1', V_1, \omega_X^* \rangle \,, \tag{13}$$

where $\mathcal{A}_1' := L_{V_1}\mathcal{A}_1$ is equivalent to $\mathcal{A}_1$.
By definition it follows that $\mathrm{Hom}(\omega_X^*, V_1) = 0$.

Now, let us consider the semiorthogonal decomposition (12). Tensoring with $(\omega_X^*)^{\otimes 2}$ and applying a left mutation of $\left\langle (\omega_X^*)^{\otimes 2}, \mathcal{A}_2 \otimes (\omega_X^*)^{\otimes 2} \right\rangle$ with respect to $V_2^* \otimes (\omega_X^*)^{\otimes 2}$ as in Proposition 66 we have:

$$D^b(X) = \left\langle V_2^* \otimes (\omega_X^*)^{\otimes 2}, (\omega_X^*)^{\otimes 2}, \mathcal{A}_2 \otimes (\omega_X^*)^{\otimes 2} \right\rangle = \langle \omega_X^*, \mathcal{A}_2 \otimes \omega_X^*, V_1 \rangle$$

where we used the relation (10). Let us call $\mathcal{A}_2'$ the left mutation $L_{\omega_X^*}(\mathcal{A}_2 \otimes \omega_X^*)$. Then we can write the derived category as

$$D^b(X) = \langle \mathcal{A}_2', \omega_X^*, V_1 \rangle$$

It follows that $\mathrm{Hom}(V_1, \omega_X^*) = 0$, so that V_1 and ω_X^* are completely orthogonal. Thus we have:

$$D^b(X) = \langle \mathcal{A}_2', V_1, \omega_X^* \rangle \,.$$

Comparing this decomposition to (13) we have that $\mathcal{A}_2'$ and $\mathcal{A}_1'$ are orthogonal to the same subcategory, then they are equivalent. □

6.4. Conic bundles of degree 1. In this case C_1 and C_2 are smooth rational and the link is represented by a Bertini involution, that is the birational transformation $\phi_{\mathcal{L}}$ associated to the linear system of degree 17 with 8 points of multiplicity 6, see [**18**, Ch 7.3 and 8.8] and [**28**]. As described in [**27**, Thm 2.6], it acts on $\mathrm{Pic}(X)$ as

$$F_1 = -4K_X - F_2. \tag{14}$$

This last case requires some preliminary lemma.

Lemma 130 ([**20**]). *An exceptional bundle E on a surface with ample anticanonical sheaf is uniquely determined up to isomorphism by $\frac{c_1(E)}{r}$ where $c_1(E)$ and r are respectively the first Chern class and the rank of E.*

Lemma 131. *Let us consider the divisor $3K_X + F_2$. Then*

$$\begin{aligned} &dim\, \mathrm{H}^i(X, \mathcal{O}(3K_X + F_2)) = 0 \ \text{if} && i \neq 0 \\ &dim\, \mathrm{H}^i(X, \mathcal{O}(3K_X + F_2)) = 1 \ \text{if} && i = 1. \end{aligned}$$

Proof. Since $\dim X = 2$, we have that

$$\mathrm{H}^i(X, \mathcal{O}(3K_X + F_2)) = 0 \text{ for } i > 2 \text{ and } i < 0$$

by [**23**, Ch III.2]. Thus it remains to calculate it for $i = 0, 1, 2$. We observe that

$$\mathrm{H}^0(X, \mathcal{O}(3K_X + F_2)) = 0 \text{ since } (3K_X + F_2) \cdot F_2 < 0.$$

We apply Serre duality to the case $i = 2$, [**23**, Ch III.7] and (14).

$$\mathrm{H}^2(X, \mathcal{O}(3K_X + F_2)) = \mathrm{H}^0\big(X, (\mathcal{O}(3K_X + F_2))^* \otimes \omega_X\big) = \mathrm{H}^0(X, \mathcal{O}(2K_X + F_1)) = 0$$

since $(2K_X + F_1) \cdot F_1 < 0$.
To calculate $\mathrm{H}^1(X, \mathcal{O}(3K_X + F_2))$ we use the Riemann-Roch theorem [**23**, Ch IV.1]. We have that

$$\Sigma_l (-1)^l h^l\big(X, \mathcal{O}(3K_X + F_2)\big) = \frac{1}{2}(3K_X + F_2)(3K_X + F_2 - K_X) + 1 + p_a \tag{15}$$

where $h^*\big(X, \mathcal{O}(3K_X + F_2)\big) = \dim \mathrm{H}^*\big(X, \mathcal{O}(3K_X + F_2)\big)$ and the arithmetic genus $p_a = 0$ for X. In the left hand side, the only non-vanishing term is

$$-\dim \mathrm{H}^1\big(X, \mathcal{O}(3K_X + F_2)\big).$$

Thus we have

$$-\dim \mathrm{H}^1\big(X, \mathcal{O}(3K_X + F_2)\big) = \frac{1}{2}(6K_X^2 + 5K_X \cdot F_2 + F_2^2) + 1 = -1.$$

□

THEOREM 132. *Let X be a surface of degree 1. Then the Griffiths-Kuznetsov component $\mathrm{GK}_X(\pi_1)$ defined by $\pi_1 : X \to C_1$ is equivalent to $\mathrm{GK}_X(\pi_2)$ defined by the conic bundle structure $\pi_2 : X \to C_2$.*

PROOF. Since the curves are rational, we can show that $\langle V_1 \rangle \cong \langle V_2 \rangle$. We know that

$$V_i = \mathcal{O}(F_i) \oplus \mathcal{O}(F_i) \text{ and } V_i^* = \mathcal{O}(-F_i) \oplus \mathcal{O}(-F_i) \text{ for } i = 1, 2.$$

Thus, using the substitutions in 14, we obtain

$$\begin{aligned} V_1 =& \mathcal{O}(F_1) \oplus \mathcal{O}(F_1) \\ =& \mathcal{O}(-4K_X - F_2) \oplus \mathcal{O}(-4K_X - F_2) \\ =& (\omega_X^*)^{\oplus 4} \otimes V_2^*. \end{aligned}$$

The derived category of X admits the semiorthogonal decompositions

$$D^b(X) = \langle \mathcal{O}(-F_1), \mathcal{O}_X, \mathcal{A}_1 \rangle\,, \tag{16}$$

$$D^b(X) = \langle \mathcal{O}_X, \mathcal{O}(F_2), \mathcal{A}_2 \rangle\,. \tag{17}$$

Let us consider (16) and tensor it with $(\omega_X^*)^{\otimes 4}$. Using (14) we obtain:

$$\begin{aligned} D^b(X) &= \big\langle \mathcal{O}(-F_1 - 4K_X), (\omega_X^*)^{\otimes 4}, \mathcal{A}_1 \otimes (\omega_X^*)^{\otimes 4} \big\rangle \\ &= \big\langle \mathcal{O}(F_2), (\omega_X^*)^{\otimes 4}, \mathcal{A}_1 \otimes (\omega_X^*)^{\otimes 4} \big\rangle \end{aligned}$$

Applying the left mutation to $\big\langle (\omega_X^*)^{\otimes 4}, \mathcal{A}_1 \otimes (\omega_X^*)^{\otimes 4} \big\rangle$ with respect to $\mathcal{O}(F_2)$ as in Proposition 66, we have

$$D^b(X) = \big\langle (\omega_X^*)^{\otimes 3}, \mathcal{A}_1 \otimes (\omega_X^*)^{\otimes 3}, \mathcal{O}(F_2) \big\rangle = \big\langle (\omega_X^*)^{\otimes 3}, \mathcal{O}(F_2), \mathcal{A}_1' \big\rangle$$

where $\mathcal{A}_1'$ is the result of the right mutation of $\mathcal{A}_1 \otimes (\omega_X^*)^{\otimes 3}$ with respect to $\mathcal{O}(F_2)$, in particular $\mathcal{A}_1'$ is equivalent to $\mathcal{A}_1$.

Now we want to mutate $\mathcal{O}(F_2)$ to the left of $(\omega_X^*)^{\otimes 3}$. They are not completely orthogonal, so by Proposition 131 we apply the extension as defined in Example 114

$$0 \to \mathcal{O}(F_2) \longrightarrow \mathscr{F} \longrightarrow (\omega_X^*)^{\otimes 3} \to 0$$

where $\mathscr{F}$ is a rank 2 exceptional sheaf and its first Chern class is given by the formula

$$c_1(\mathscr{F}) = c_1\big(\mathcal{O}(F_2)\big) + c_1\big((\omega_X^*)^{\otimes 3}\big) = F_2 - 3K_X$$

Thus we have

$$D^b(X) = \big\langle \mathscr{F}, (\omega_X^*)^{\otimes 3}, \mathcal{A}_1' \big\rangle \tag{18}$$

On the other side, we apply proposition (66) to (17).

$$D^b(X) = \langle \mathcal{O}_X, \mathcal{O}(F_2), \mathcal{A}_2 \rangle = \langle \mathcal{O}(F_2), \mathcal{A}_2, \omega_X^* \rangle$$

and mutate $\mathcal{A}_2$ to the right of ω_X^* obtaining the subcategory $\mathcal{A}_2' \cong \mathcal{A}_2$.

$$D^b(X) = \langle \mathcal{O}(F_2), \omega_X^*, \mathcal{A}_2' \rangle$$

As above, we want to mutate $\mathcal{O}(F_2)$ to the left of the anticanonical sheaf but we have that

$$\mathrm{Hom}^i(\mathcal{O}(F_2), \omega_X^*) = \mathrm{Hom}^i\big((\omega_X^*)^{\otimes 3}, \mathcal{O}(F_1))\big) \otimes \omega_X^*.$$

Thus they are not completely orthogonal and the non-vanishing term is for $i = 1$ by Proposition 131. We apply Example 114 and consider the short exact sequence

$$0 \to \omega_X^* \to \mathscr{G} \to \mathcal{O}(F_2) \to 0$$

where $\mathscr{G}$ is a sheaf of rank 2. Its first Chern class is

$$c_1(\mathscr{G}) = c_1\big(\mathcal{O}(F_2)\big) + c_1\big(\omega_X^*\big) = F_2 - K.$$

Then the decomposition is written as

$$D^b(X) = \langle \omega_X^*, \mathscr{G}, \mathcal{A}_2' \rangle\,. \tag{19}$$

We observe that the rank 2 sheaves have the first Chern class differing by the divisor $2K_X$

$$c_1(\mathscr{G}) = c_1(\mathscr{F}) + 2K_X = c_1\big(\mathscr{F} \otimes \omega_X\big)$$

Then the exceptional bundles $\mathscr{G}$ and $\mathscr{F} \otimes \omega_X$ have the same rank and the same first Chern class. By Lemma 130 we have $\mathscr{G} \cong \mathscr{F} \otimes \omega_X$.

Then the derived category (19) can be decomposed as

$$D^b(X) = \langle \omega_X^*, \mathscr{F} \otimes \omega_X, \mathcal{A}_2' \rangle = \big\langle (\omega_X^*)^{\otimes 2}, \mathscr{F}, \mathcal{A}_2' \otimes \omega_X^* \big\rangle$$

where we obtain the last decompostion tensoring by ω_X^*. We mutate to the right $(\omega_X^*)^{\otimes 2}$ with respect to its orthogonal complement as in Proposition 66:

$$D^b(X) = \big\langle \mathscr{F}, \mathcal{A}_2' \otimes \omega_X^*, (\omega_X^*)^{\otimes 3} \big\rangle = \big\langle \mathscr{F}, (\omega_X^*)^{\otimes 3}, \mathcal{A}_2'' \big\rangle \tag{20}$$

where $\mathcal{A}_2'' = R_{(\omega_X^*)^{\otimes 3}} \mathcal{A}_2' \otimes \omega_X^*$.

Comparing (18) and (20) we obtain that the first two terms of the semiorthogonal decomposition are the same. Thus $\mathcal{A}_1'$ and $\mathcal{A}_2''$ are right orthogonal to $\big\langle \mathscr{F}, (\omega_X^*)^{\otimes 3} \big\rangle$ in $D^b(X)$ and then they are equivalent. $\square$

COROLLARY 133. *The Griffiths-Kuznetsov component is invariant for links of type IV.*

Bibliography

[1] Auel, Asher and Bernardara, Marcello. (2015). *Semiorthogonal decompositions and birational geometry of del Pezzo surfaces over arbitrary fields.* Proceedings of the London Mathematical Society. 117. 10.1112/plms.12119.

[2] A. Auel, M. Bernardara and M. Bolognesi, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, Journal de Math. Pures et Appliques 102, 249-291 (2014)

[3] A. Beauville, *Determinantal Hypersurfaces*, Michigan Math. J. 48 (2000), 3964.

[4] A.A. Beilinson, *The derived category of coherent sheaves on* $\mathbb{P}^n$, Selecta Math. Soviet. 3 (1983/84), 233237.

[5] M. Bernardara and M. Bolognesi, *Derived categories and rationality of conic bundles*, Compositio Math. 149 (2013), no. 11, 1789-1817.

[6] M. Bernardara, M. Bolognesi *Categorical representability and intermediate Jacobians of Fano threefolds* EMS Ser. Congr. Rep., Eur. Math. Soc., 2013.

[7] A.I. Bondal and D.O. Orlov, *Semiorthogonal decomposition for algebraic varieties*, MPIM preprint 1995-15, 1995.

[8] A.I. Bondal and D.O. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Comp. Math. 125 (2001), 327344.

[9] Calabri, Alberto, and Ciliberto, Ciro: *Birational classification of curves on rational surfaces*, Nagoya Math. J. 199 (2010), 4393.

[10] Calabri, Alberto, and Ciliberto, Ciro: *On Cremona contractibility*, to appear on Rend. Sem. Mat. Univ. Pol. Torino 71 (2013), 11 pages

[11] Calabri, Alberto and Ciliberto, Ciro. (2015). *On Cremona contractibility of unions of lines in the plane.* Kyoto Journal of Mathematics. 57. 10.1215/21562261-3759513.

[12] Calabri, Alberto and Ciliberto, Ciro. (2016). *Contractible curves on a rational surface*, arXiv:1611.02949.

[13] C.H. Clemens, P.A. Griffiths *The intermedite Jacobian of the cubic threefold* Ann. Math. (2) 95, 281356 (1972).

[14] K.R. Coombes, *Every rational surface is separably split*, Comment. Math. Helv. 63 (1988), no. 2, 305311.

[15] Corti Alessio, Kaloghiros Anne-Sophie. (2015). *The Sarkisov program for Mori fibred Calabi-Yau pairs.* Algebraic Geometry. 3. 10.14231/AG-2016-016.

[16] Luigi Cremona, *Sulle transformazioni geometriche delle figure piane*, Mem. Acad. Bologna 2 (1863), no. 2, 62130.

[17] LuigiCremona, *Sulle transformazioni geometriche delle figure piane*, Mem. Acad. Bologna 5 (1965), no. 2, 335.

[18] Dolgachev, I. (2012). *Classical Algebraic Geometry: A Modern View.* Cambridge: Cambridge University Press. doi:10.1017/CBO9781139084437

[19] Igor V. Dolgachev and Vasily A. Iskovskikh, *Finite subgroups of the plane Cremona group, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin.* Vol. I, Progr. Math., vol. 269, Birkhauser Boston, Inc., Boston, MA, 2009, pp. 443548. MR 2641179

[20] A.L. Gorodentsev, *Exceptional bundles on surfaces with a moving anticanonical class*, Izv. Akad. Nauk SSSR Ser. Mat. 52 (1988), no. 4, 740757, 895; translation in Math. USSR-Izv. 33 (1989), no. 1, 6783.

[21] Hacon, C. D. and McKernan, J., *The Sarkisov program*, J. Algebraic Geom. 22 (2013), 389405.

[22] B. Hassett *Rational surfaces over nonclosed fields* Arithmetic Geometry, proceedings of the 2006 Clay Mathematics Institute Summer School, Clay Mathematics Proceedings 8 155-210, American Mathematics Society, Providence, 2009

[23] Robin Hartshorne (1977): *Algebraic geometry*, Springer, ISBN 0-387-90244-9

[24] Hudson, Hilda P.: *Cremona Transformations in Plane and Space*, Cambridge University Press, 1927

[25] D. Huybrechts *Fourier-Mukai transforms in algebraic geometry* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006

[26] Iitaka, Shigeru. *Classification of Reducible Plane Curves.* Tokyo J. Math. 11 (1988), no. 2, 363–379.

[27] V.A. Iskovskikh, *Factorization of birational mappings of rational surfaces from the point of view of Mori theory* (Russian) Uspekhi Mat. Nauk 51 (1996), no. 4(310), 372; translation in Russian Math. Surveys 51 (1996), no. 4, 585652

[28] Kollár, János and Mella, Massimiliano. (2014). *Quadratic families of elliptic curves and unirationality of degree 1 conic bundles.* American Journal of Mathematics. 139. 10.1353/ajm.2017.0024.

[29] S. A. Kuleshov and D. O. Orlov, *Exceptional sheaves on del Pezzo surfaces*, Izv. Ross. Akad. Nauk Ser. Mat. 58 (1994), no. 3, 5387. MR 1286839 (95g:14048)

[30] Kumar, N. Mohan, and Murthy, M. Pavaman: *Curves with negative self-intersection on rational surfaces*, J. Math. Kyoto Univ., 22 (1982/83), no. 4, 767777.

[31] A. Kuznetsov *Derived categories of cubic and V14 threefolds* Proc. Steklov Inst. Math. 246, 171194 (2004); translation from Tr. Mat. Inst. Steklova 246, 183207 (2004)

[32] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. 218 (2008), no. 5, 13401369.

[33] A. Kuznetsov, *A simple counterexample to the Jordan-Hölder property for derived categories* preprint math.AG/1304.0903 (2013).

[34] A. Kuznetsov, *Semiorthogonal decompositions in algebraic geometry,* (2014) preprint arXiv:1404.3143.

[35] Kuznetsov A. (2016) *Derived Categories View on Rationality Problems.* In: Pardini R., Pirola G. (eds) Rationality Problems in Algebraic Geometry. Lecture Notes in Mathematics, vol 2172. Springer, Cham

[36] Marletta, Giuseppe: *Sulla identità cremoniana di due curve piane*, Rend. Circolo Mat. Palermo, 24 (1907), no. 1, 223242.

[37] M. Mella, E. Polastri *Equivlent birational embedding II: divisors* Math. Z. (2012) 270:11411161 DOI 10.1007/s00209-011-0845-3

[38] Pompilj, Giuseppe: *Sulle trasformazioni cremoniane del piano che posseggono una curva di punti uniti*, Rend. del Sem. Mat. dellUniv. di Roma (4) 2 (1937), 343

[39] D.O. Orlov, *Projective bundles, monoidal transformations and derived categories of coherent sheaves*, Russian Math. Izv. 41 (1993), 133141. 10, 15, 18, 30

[40] *Helices and vector bundles. Seminaire Rudakov*, London Mathematical Society Lecture Note Series, 148. A.N. Rudakov, ed. Cambridge University Press, Cambridge, 1990.

[41] A. N. Rudakov *Exceptional Vector bundles on a del Pezzo surface* In: Tikhomirov A., Tyurin A. (eds) Algebraic Geometry and its Applications. Aspects of Mathematics, vol 25. Vieweg+Teubner Verlag, Wiesbaden.

[42] V. G. Sarkisov *On conic bundle structure* Izv. Akad. Nauk SSSR Ser. Mat., 1982, Volume 46, Issue 2, Pages 371408 (Mi izv1620)

Abstracts

Geometria birazionale: classica e derivata

Nell'ambito della geometria algebrica, lo studio delle trasformazioni birazionali e delle loro proprietà riveste un ruolo di importanza primaria. In questo, si affiancano l'approccio classico della scuola italiana che si concentra sul gruppo di Cremona e quello più moderno che utilizza strumenti come categorie derivate e decomposizioni semiortogonali.

Del gruppo di Cremona Cr_n, cioè il gruppo degli automorfismi birazionali di $\mathbb{P}^n$, in generale non si conosce molto e ci si concentra sul caso complesso. Si conosce un insieme di generatori solo nel caso di dimensione 2. Inoltre non è ancora nota una classificazione tramite trasformazioni di Cremona delle curve e dei sistemi lineari di $\mathbb{P}^2$. Tra i casi noti ci sono: le curve irriducibili e quelle formate da due componenti irriducibili. In questa tesi ci si approccia al caso di una configurazione di d rette nel piano proiettivo. Il teorema finale fornisce condizioni necessarie o sufficienti alla contraibilità.

Da un punto di vista categoriale invece, le decomposizioni semiortogonali della categoria derivata di una varietà ci forniscono degli invarianti utili nello studio della varietà. Seguendo l'approccio di Clemens-Griffiths riguardante la cubica complessa di dimensione 3, si vuole caratterizzare le ostruzioni alla razionalità di una varietà X di dimensione n. L'idea è di raccogliere le componenti di una decomposizione ortogonale che non sono equivalenti a categorie derivate di varietà di dimensione almeno $n-1$ e in questo modo definire quella che chiamiamo componente di Griffiths-Kuznetsov di X. In questa tesi si studia il caso delle superfici geometricamante razionali su un campo arbitrario, si definisce tale componente e si mostra che essa è un invariante birazionale. Si vede anche che la componente di Griffiths-Kuznetsov è nulla solo se la superficie è razionale.

Classical and Derived Birational Geometry

In the field of algebraic geometry, the study of birational transformations and their properties plays a primary role. In this, there are two different approach: the classical one due to the Italian school who focuses on the Cremona group and a modern one which utilizes instruments like derived categories and semiorthogonal decompositions.

About the Cremona group, that is the group of birational self-morphisms of $\mathbb{P}^n$, we do not know much in general and we focus on the complex case. We know a set of generators only in dimension $n = 2$. Moreover, we do not have a classification of curves and linear systems in $\mathbb{P}^2$ up to Cremona transformations. Among the known results there are: irreducible curves and curves with two irreducible components. In this thesis we approach tha case of a configuration of lines in the projective plane. The last theorem lists the known contractible configurations.

From a categorical point of view, the semiorthogonal decompositions of the derived category of a variety provide some useful invariants in the study of the variety. Following the work of Clemens-Griffiths about the complex cubic threefold, we want to characterize the obstructions to the rationality of a variety X of dimension n. The idea is to collect the component of a semiorthogonal decomposition which are not equivalent to the derived category of a variety of dimension at least $n-1$. In this way we defined the so called Griffiths-Kuznetsov component of X. In this thesis we study the case of surfaces on an arbitrary field, we define that component and show that it is a birational invariant. It appears clearly that the Griffiths-Kuznetsov component vanishes only if the surface is rational.

Géométrie birationnelle: classique et dérivée

Dans le cadre de la géométrie algébrique, l'étude des transformations birationnelle et des leurs proprietés joue un rôle déterminant. Ou bien par l'approche classique de l'école italienne qui met l'accent sur le groupe de Cremona, ou bien par une approche plus moderne qui utilise des objets comme les catégories dérivées et leurs décompositions semiorthogonales.

Le groupe de Cremona Cr_n, notamment le groupe des automorphismes birationnels du $\mathbb{P}^n$, est en general peu connu notamment on travaille principalement sur le corp complexe. On connait un ensemble des générateurs seulement pour $n = 2$. On ne connait pas une classification des courbes et systemes linéaires de $\mathbb{P}^2$ pour transformations de Cremona . Un example des risultat qu'il y a est la charactérization de la contractibilité des courbes irreductibles et des courbes obtenues par union des deux component irreductibles. Le but de cette thèse est de s'approcher au cas d'une configuration de droites dans $\mathbb{P}^2$. Le théorème final fournit conditions necessaires ou suffisantes á la contractibilité.

En termes catégoriels, les décompositions semiorthogonales de la catégorie dérivée d'une variété fournissent des invariants pour étudier la variété. En s'inspirant de l'approche de Clemens-Griffiths sur la cubique complexe en diménsion 3, on veut characterizer les obstructions à la rationalité d'une variété de diménsion n. L'idée est de pouvoir isoler les composantes qui ne sont pas equivalentes à la catégorie dérivée d'une variété de dimension au plus $n-2$ et, de cette façon, définir ce que on appelle la composante de Griffiths-Kuznetsov. Dans cette thèse on étude le cas des surfaces sur un corps arbitraire, on définit de telles composantes et on démonstre que elles donnen un invariant birationnel.

On peut voir aussi que la composante de Griffiths-Kuznetsov est nulle si et seulement si la surface est rationnelle.