

# Explicit Resolutions of Double Point Singularities of Surfaces

ALBERTO CALABRI\*      RITA FERRARO†

## Abstract

Locally analytically, any isolated double point occurs as a double cover of a smooth surface. It can be desingularized explicitly via the canonical resolution, as it is very well-known. In this paper we explicitly compute the fundamental cycle of both the canonical and minimal resolution of a double point singularity and we classify those for which the fundamental cycle differs from the fiber cycle. Moreover we compute the conditions that a double point singularity imposes to pluricanonical systems.

**Mathematics Subject Classification (2000):** 14J17, 32S25.

**Keywords:** double points, surface singularities, canonical resolution, fundamental cycle, adjunction conditions.

## 1 Introduction

In this article, we give a detailed analysis of isolated surface singularities of multiplicity two, i.e., *double points*. Our goal is to be as explicit as possible.

A neighbourhood of an isolated double point  $p$  on a complex (normal) surface is analytically isomorphic to a double cover  $\pi_0 : X_0 \rightarrow Y_0$  branched along a reduced curve  $B_0$  with an isolated singularity at a point  $q_1$ , where  $Y_0$  is a smooth surface. If  $x, y$  are local coordinates of  $Y_0$  near  $q_1$ , then we may assume that  $X_0$  is defined locally by an equation  $z^2 = f(x, y)$ , where  $B_0$  is  $f = 0$  and  $f$  is a square-free polynomial in  $x, y$ .

It is very well-known that one may desingularize  $p = \pi_0^{-1}(q_1) \in X$  following the *canonical* resolution, which consists in desingularizing the branch curve and normalizing. This fact was implicitly used by the so-called Italian school (as one finds out for example by reading Castelnuovo and Enriques' papers on double covers of the projective plane), but it has been first written in 1946 by Franchetta [13], who computed the properties of the exceptional curves of the canonical resolution and the *fiber cycle*, that is the maximal effective divisor contained in the scheme-theoretic fiber over  $p$ .

\* Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma. E-mail: calabri@mat.uniroma2.it. Partially supported by E.C. project EAGER, contract n. HPRN-CT-2000-00099.

† Dipartimento di Matematica, Università di Roma Tre, Largo S. Leonardo Murialdo 1, 00146 Roma. E-mail: ferraro@mat.uniroma3.it.

Later, in 1978, Dixon [10] proved again Franchetta's results in a more modern language and he found sufficient conditions for the *fundamental cycle* of a double point singularity to be equal to the fiber cycle.

At the same time, Horikawa [16] and Laufer [19] explained the canonical resolution process too (see also [4]). In particular Laufer proved that the minimal resolution of a double point singularity is obtained from the canonical one by contracting simultaneously finitely many disjoint  $(-1)$ -curves and he described the relation between the topological types of the canonical resolution and of the minimal one, by using essentially the properties of the fundamental cycle.

We recall that the fundamental cycle of a resolution may well be computed inductively from the knowledge of the intersection matrix of the exceptional curves. However no explicit formula was known in the general case. In Theorem 11.2, we will give and prove such a formula, that turns out to be very simple, for the canonical resolution and then in (11.9) for the minimal one.

Moreover, this formula allows us also to classify those double points for which the fiber cycle strictly contains the fundamental cycle.

Finally in the last section we compute what are the conditions that a double point singularity imposes to canonical and pluricanonical systems of a surface. For this purpose, it is convenient to consider projective surfaces: we will assume that  $X_0$  is a double plane, i.e.,  $Y_0 = \mathbb{P}^2$ .

Although the general techniques used in this paper are very well-understood, we found no reference about our results. Thus we hope that this paper may serve as a natural complement to Dixon's and Laufer's papers and as an adequate reference for algebraic geometers that may use these results.

Moreover our approach is slightly different from the previous ones because we use, as discrete invariant of a plane curve singularity, the *Enriques* digraph, namely the directed graph involving the proximity relations among the infinitely near points to  $q_1$ , and we believe that this approach may be of independent interest as well (cf. Enriques diagrams in [15], [7] and [22]).

Thus we show also some examples of double point singularities which may help to understand the features of the combinatorial machinery we introduced. Please do not hesitate to contact the authors if you want to see the implementation of this approach on a computer.

The interested reader may also consult [3] for the analysis of an embedded resolution of  $\{z^2 = f(x, y)\} \subset \mathbb{C}^3$ .

**Acknowledgements.** We warmly thank prof. Ciro Ciliberto for addressing us to this subject and prof. Rick Miranda for several useful discussions and for joining us in preparing a preliminary version of this paper. We are grateful to prof. J. Lipman for some relevant bibliographical reference and to the referee for many suggestions that really improved the exposition of this paper.

## 2 Notation

To help the reader with the notation, which will soon become very heavy, we list here the main symbols used in this paper, together with the meaning and the reference formula or page where they are defined.

<i>Symbol</i>	<i>Meaning</i>	<i>Ref.</i>
$X_0$	normal complex surface	p. 4
$p$	isolated double point singularity of $X_0$	p. 4
$Y_0$	smooth surface	p. 4
$\pi_0 : X_0 \rightarrow Y_0$	double cover	p. 4
$B_0$	branch curve of $\pi_0$	p. 4
$q_1$	$= \pi_0(p)$ , isolated singular point of $B_0$	p. 5
$\sigma_i : Y_i \rightarrow Y_{i-1}$	blowing-up at a point $q_i \in Y_{i-1}$	p. 6
$q_i$	center of the blowing-up $\sigma_i$	p. 6
$(\cdot)_i, (\cdot)$	intersection pairing in $Y_i$ (resp. in $Y_n$ )	p. 6
$\sigma_{ij} : Y_j \rightarrow Y_i$	$= \sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_{i+1}$	(5.1)
$\sigma$	$= \sigma_{0n}$ , sequence of blowing-ups	p. 6
$E_i$	(proper transform of) the exceptional curve $\sigma_i^{-1}(q_i)$	p. 6
$E_i^*$	total transform of $E_i$ in $Y = Y_n$ via $\sigma = \sigma_{0n}$	p. 6
$N = (n_{ij})$	$n \times n$ matrix, $E_i = \sum_{j=1}^n n_{ij} E_j^*$	(5.3)
$M = (m_{ij})$	$n \times n$ matrix, $E_j^* = \sum_{k=1}^n m_{jk} E_k$	(5.3)
$q_j \rightarrow q_i$	$q_j$ is proximate to $q_i$	p. 6
$q_j >^s q_i$	$q_j$ is infinitely near of order $s$ to $q_i$	p. 7
$Q = (q_{ij})$	$n \times n$ matrix, $q_{ij} = 1$ if and only if $q_j \rightarrow q_i$	p. 6
$S = (s_{ij})$	$n \times n$ matrix of intersection numbers $s_{ij} = (E_i \cdot E_j)$	p. 7
$\tilde{B}_i$	proper transform of $B_0$ in $Y_i$	p. 7
$\tilde{\alpha}_i$	multiplicity of $\tilde{B}_{i-1}$ at $q_i$	(6.1)
$\tilde{\beta}_i$	$= \sum_{j=1}^n \tilde{\alpha}_i m_{ji}$	(6.2)
$\tilde{\Gamma}_i$	$= \tilde{B}_n _{E_i}$ , divisor on $E_i$	p. 8
$\tilde{\gamma}_i$	$= \deg(\tilde{\Gamma}_i) = \tilde{B}_n \cdot E_i$	p. 8
$\pi_i : X_i \rightarrow Y_i$	normal double cover induced by $\pi_0$ and $\sigma_{0i}$	p. 10
$B_i$	branch curve of $\pi_i$	p. 10
$\mu_i$	multiplicity of $B_{i-1}$ at $q_i$	p. 10
$\varepsilon_i$	$= \mu_i \bmod 2 \in \{0, 1\}$ , branchedness of $E_i$	(7.2)
$\alpha_i$	$= \mu_i - \varepsilon_i$ , even integer number	(8.1)
$\beta_i$	$= \sum_{j=1}^n \alpha_i m_{ji}$ , even integer number	(8.1)
$\tau : X \rightarrow X_0$	canonical resolution of $p \in X_0$	p. 11
$\bar{\tau} : \bar{X} \rightarrow X_0$	minimal resolution of $p \in X_0$	p. 15
$Y$	$= Y_n$ , smooth surface	p. 11
$\pi : X \rightarrow Y$	smooth double cover induced by the canonical resol.	p. 11
$B$	$= B_n$ , smooth branch curve of $\pi$	p. 11
$\Gamma_i$	$= B _{E_i}$ , divisor on $E_i$	(8.4)
$\gamma_i$	$= \deg(\Gamma_i) = B \cdot E_i$	p. 11
$F_i$	$= \pi^*(E_i)$ , exceptional curves of $\tau$	p. 13
$D$	$= \sum_{i=1}^n (\alpha_i/2 - 1) E_i^*$	p. 14
$D^*$	$= \pi^* D = \tau^* K_{X_0} - K_X$ , adjunction condition div.	(9.6)
$F$	fiber cycle of the canonical resolution	p. 14
$\bar{F}_i$	exceptional curves of $\bar{\tau}$	p. 16
$\bar{F}$	fiber cycle of the minimal resolution	(10.6)
$Z$	fundamental cycle of the canonical resolution	(11.1)
$\bar{Z}$	fundamental cycle of the minimal resolution	(11.9)

Symbol	Meaning	Ref.
defective	$q_i$ such that there exists $q_j >^1 q_i$ with $\alpha_j > \alpha_i$ .....	p. 27
Def	index set of defective points .....	p. 28

### 3 Double covers of surfaces

Throughout this paper a *double cover* of a smooth irreducible complex surface  $Y$  is a finite, surjective, proper holomorphic map  $\pi : X \rightarrow Y$  of degree 2 branched along a reduced curve, where  $X$  is a (normal) irreducible complex surface. In the algebraic setting  $\pi$  is a finite morphism of degree 2. For more details on covers and double covers the reader may consult [4].

Let  $p$  be an isolated double point singularity of a surface. Since any isolated double point occurs, locally analytically, as a double cover of a smooth surface, let us consider as our initial data a double cover  $\pi_0 : X_0 \rightarrow Y_0$  branched along the reduced curve  $B_0$  defined in local coordinates by  $f = 0$ , where  $f$  is a square-free polynomial. If  $x, y$  are local analytic coordinates at a point  $q \in Y_0$ , then  $X_0$  is defined by an equation  $z^2 = f(x, y)$  and  $X_0$  is normal.

If  $q \notin B_0$ , then  $f(q) \neq 0$  and there are two pre-images of  $q$  in  $X_0$ ; at each of these points  $X_0$  is smooth and  $\pi_0 : X_0 \rightarrow Y_0$  is unramified. If  $q \in B_0$ , then there is a single point of  $X_0$  lying above  $q$ ;  $X_0$  is smooth at this point if and only if  $B_0$  is smooth at  $q$ . The geometry of a smooth double cover is well-known:

**Lemma 3.1** *Let  $Y$  be a smooth surface. Let  $\pi : X \rightarrow Y$  be a double cover, branched along the smooth and reduced curve  $B$  in  $Y$ .*

1. *If  $C$  is an irreducible component of  $B$ , then  $\pi|_D : D = \pi^{-1}(C) \rightarrow C$  is an isomorphism,  $\pi^*(C) = 2D$  and  $D^2 = C^2/2$ .*
2. *If  $C$  is an irreducible curve in  $Y$  which meets  $B$  transversally in  $2k$  points, where  $k \geq 1$ , then  $D = \pi^{-1}(C)$  is a smooth irreducible curve on  $X$  and  $\pi|_D : D \rightarrow C$  is a double cover branched along the  $2k$  points of intersection of  $C$  with  $B$ . Moreover  $\pi^*(C) = D$  and  $D^2 = 2C^2$ .*

*Let  $E$  be a smooth rational curve in  $Y$  which is not part of the branch locus  $B$ . Let  $\Gamma = B|_E$  be the divisor of intersection of  $B$  with  $E$ .*

3. *If  $\Gamma$  is an even divisor, say  $\Gamma = 2Q$  (in particular if  $\Gamma = Q = 0$ ), then  $\pi^{-1}(E) = D_1 + D_2$ , where  $\pi|_{D_i} : D_i \rightarrow E$  is an isomorphism and  $D_i^2 = E^2 - \deg(Q)$  for  $i = 1, 2$ , and  $D_1 \cdot D_2 = \deg(Q)$ .*
4. *If  $\Gamma$  is not even, then  $D = \pi^{-1}(E)$  is an irreducible curve,  $\pi|_D : D \rightarrow E$  is a double cover and  $D^2 = 2E^2$ .  $D$  is singular at those points  $q \in E$  where  $\Gamma(q) \geq 2$ ; locally near  $p = \pi^{-1}(q)$  the curve  $D$  has the analytic equation  $z^2 = x^n$ , where  $n = \Gamma(q)$ .*

**Proof.** One can check locally the properties of  $\pi$ . If  $C$  is not contained in  $B$ , then  $\pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \rightarrow C$  is a double cover branched along  $B|_C$  and it is surely reducible if  $B|_C = 2Q$  for some divisor  $Q \neq 0$  on  $C$ . Since  $\pi$  has degree 2, we see that intersections double after applying  $\pi^*$ , i.e., for any divisors  $C_1$  and  $C_2$  in  $Y$ ,  $(\pi^*C_1 \cdot \pi^*C_2)_X = 2(C_1 \cdot C_2)_Y$ , where  $(\cdot)_X$  (resp.  $(\cdot)_Y$ ) is the intersection form

in  $X$  (resp.  $Y$ ). So all the claims about  $D^2$  are trivial. Regarding again point 3, note that an unramified double cover of  $\mathbb{P}^1$  is reducible, by Hurwitz formula (or the simply-connectedness of  $\mathbb{P}^1$ ).  $\square$

In the assumption of Lemma 3.1,  $B$  is an even divisor and

$$\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-B/2), \quad (3.2)$$

while Riemann-Hurwitz Formula is:

$$K_X = \pi^*(K_Y \otimes \mathcal{O}_Y(B/2)) \quad (3.3)$$

(see [4, Lemmas 17.1 and 17.2]). If  $\pi : X \rightarrow Y$  is a double cover with  $X$  normal but not smooth, one can still define the canonical divisor  $K_X$  and the same formula (3.3) holds. For a normal surface  $X$  the canonical divisor  $K_X$  is defined as the Weil divisor class  $\text{div}(s)$ , where  $s$  is a rational canonical differential. Since Weil divisors on a normal scheme do not depend on closed subsets of  $\text{codim} \geq 2$ , one can easily verify that (3.3) holds by considering the smooth double cover  $\pi|_{X_{sm}} : X_{sm} \rightarrow Y \setminus \text{Sing}(B)$ , where  $X_{sm} = X \setminus \pi^{-1}(\text{Sing}(B))$ .

## 4 Resolving a singular double cover

Suppose that the branch curve  $B_0$  of  $\pi_0 : X_0 \rightarrow Y_0$  is not smooth at the point  $q_1$ , thus  $p = \pi_0^{-1}(q_1)$  is a double point singularity of  $X_0$ .

Let us define  $\mu_1 = \text{mult}_{q_1}(B) = 2k + \varepsilon_1$ , with  $\varepsilon_1 \in \{0, 1\}$ . In order to get a resolution of the singularity  $p \in X_0$ , we begin with resolving the branch locus  $B_0$  of  $\pi_0$ . Let  $\sigma : Y_1 \rightarrow Y_0$  be the blowing-up at  $q_1 \in Y_0$ , with exceptional curve  $E_1$ , and let  $\tilde{B}_1$  be the proper transform of  $B_0$  in  $Y_1$ .

**Lemma 4.1** *Let  $X_1$  be the double cover of  $Y_1$  branched along  $\tilde{B}_1 + \varepsilon_1 E_1$ . Then  $X_1$  is the normalization of the pullback  $X_0 \times_{Y_0} Y_1$ , and as such is both a double cover of  $Y_0$  and dominates the singular surface  $X_0$ .*

**Proof.** The pullback  $X_0 \times_{Y_0} Y_1$ , is a double cover of  $Y_1$  (using the second projection) branched along the pullback of  $B_0$ , which is  $\pi^*(B_0) = \mu_1 E_1 + \tilde{B}_1$  (in fact it is defined by  $z^2 = f(x, y)$  also). However in the local coordinates  $(u, v)$  of  $Y_1$ , where  $x = u$  and  $y = uv$ , we have that  $f(x, y) = f(u, uv) = u^{\mu_1} g(u, v)$  where  $g(u, v)$  is a function whose series expansion is not divisible by  $u$ , and defines the proper transform  $\tilde{B}_1$  of  $B_0$ . Then the pullback is defined by  $z^2 = u^{\mu_1} g(u, v)$ , and is not normal if  $\mu_1 \geq 2$ ; if  $\mu_1 = 2k + \varepsilon_1$  with  $\varepsilon \in \{0, 1\}$ , then  $w = z/u^k$  satisfies the monic equation  $w^2 = u^{\varepsilon_1} g(u, v)$ . The normalization  $X_1$  of the pullback is defined by this monic equation, and is clearly a double cover of  $Y_1$  branched along  $\tilde{B}_1 + \varepsilon_1 E_1$ .  $\square$

This normal surface  $X_1$  dominates  $X_0$ , via the first projection, and gives a partial resolution of the double point singularity. We have passed to the double cover  $X_1 \rightarrow Y_1$ , and may iterate the procedure, continuing to blow up the branch curve at each of its singular points, then normalizing the double cover equation. It is already known that this process eventually terminates in a smooth double

cover  $X_n \rightarrow Y_n$  that is called the *canonical* resolution of  $X_0$  (see Theorem 7.4). Lemma 4.1 says that the series of double covers are determined by the parities of the multiplicities of the singular points involved. If the multiplicity is even, then the exceptional curve is not part of the new branch locus, only the proper transform; if the multiplicity is odd, then the exceptional curve is part of the new branch locus. This is all well-known, see for example [4]. We note that the multiplicity  $\mu_1$  may be determined on  $Y_1$  by  $\mu_1 = \tilde{B}_1 \cdot E$ .

## 5 Blowing up a smooth surface

Let us consider a sequence of blowing-ups of a smooth surface  $Y_0$ , each one at a single point. We fix a particular order for the blowing-ups, and let  $Y_i$  be the surface obtained after the  $i$ -th blowing-up  $\sigma_i : Y_i \rightarrow Y_{i-1}$  at a point  $q_i \in Y_{i-1}$ . Let  $\sigma_{ij}$  be the composition of the blowing-up maps from  $Y_j$  to  $Y_i$ :

$$\sigma_{ij} = \sigma_j \circ \cdots \circ \sigma_{i+2} \circ \sigma_{i+1} : Y_j \rightarrow Y_i, \quad (5.1)$$

for  $0 \leq i < j \leq n$ , where  $n$  is the total number of blowing-ups. Set  $\sigma = \sigma_{0n} : Y_n \rightarrow Y_0$ ,  $Y = Y_n$  and  $(\cdot)_i$ , resp.  $(\cdot)$ , the intersection form in  $Y_i$ , resp. in  $Y_n$ . The exceptional curve  $E_i = \sigma_i^{-1}(q_i)$  in  $Y_i$  satisfies  $(E_i \cdot E_i)_i = -1$  and  $(E_i \cdot \sigma_i^*(C))_i = 0$  for any divisor  $C$  of  $Y_{i-1}$ . We will abuse notation and refer to the proper transform of  $E_i$  on  $Y_{j \geq i}$  also as  $E_i$ , so  $E_1, \dots, E_n$  are the *exceptional curves* for  $\sigma$ . Let  $E_i^* = \sigma_{in}^*(E_i)$  be the total transform of  $E_i$  in  $Y_n$  via  $\sigma_{in}$ .

It is well-known that the relative Picard group  $\text{Pic}(Y_n)/\sigma^* \text{Pic}(Y)$  is freely generated by the classes  $\{E_i\}_{1 \leq i \leq n}$ , as well as by  $\{E_j^*\}_{1 \leq j \leq n}$ , and the latter ones are an *orthonormal* basis in the sense that:

$$(E_i^* \cdot E_j^*) = -\delta_{ij}, \quad (5.2)$$

where  $\delta$  is the Kronecker delta. Therefore we may write:

$$E_i = \sum_{j=1}^n n_{ij} E_j^*, \quad E_j^* = \sum_{k=1}^n m_{jk} E_k \quad (5.3)$$

where the matrix  $M = (m_{jk})$  is the inverse of  $N = (n_{ij})$ . Let  $Q = (q_{ij})$  be the strictly upper triangular matrix defined by  $q_{ij} = 1$  if  $q_j$  lies on  $E_i$  and  $q_{ij} = 0$  otherwise. Following the classical terminology, we say that the point  $q_j$  is *proximate* to  $q_i$ , and we write  $q_j \rightarrow q_i$ , if and only if  $q_{ij} = 1$ .

**Lemma 5.4** *Let  $I$  be the identity matrix. Then  $M = I + Q + \cdots + Q^{n-1}$  and*

$$N = I - Q. \quad (5.5)$$

**Proof.** The first formula follows from (5.5), because  $Q^m = 0$  for  $m \geq n$ . We prove (5.5) by induction on  $n$ . For  $n = 1$ , it is clear. If (5.5) holds for  $n - 1$ , then the proper transform of  $E_i$  in  $Y_{n-1}$  is  $\sigma_{i,n-1}^*(E_i) - \sum_{j=i+1}^{n-1} q_{ij} \sigma_{j,n-1}^*(E_j)$ , so the proper transform of  $E_i$  in  $Y_n$  is  $\sigma_n^*(E_i) - q_{in} E_n = \sigma_{in}^*(E_i) - \sum_{j=i+1}^{n-1} q_{ij} \sigma_{jn}^*(E_j) - q_{in} E_n^*$  and we conclude comparing with the first formula in (5.3).  $\square$

More explicitly,  $M$  can be computed inductively from  $Q$  as follows. Suppose that the first  $n - 1$  columns of  $M$  are known. If  $q_{in}$  is the unique non-zero entry of the last column of  $Q$ , then the last column of  $M$  is equal to the  $i$ -th column of  $M$  (apart  $m_{nn} = 1$ ), otherwise there is also  $q_{jn} = 1$  and the last column of  $M$  is the sum of the  $i$ -th and the  $j$ -th column of  $M$  (except  $m_{nn} = 1$ ). Let us consider the matrix  $Q$  as the adjacency matrix of a directed graph  $G$ , that we call the *Enriques digraph* of  $\sigma$ : the vertices of  $G$  are the points  $q_i$ , for  $i = 1, \dots, n$ , and there is an arrow from  $q_j$  to  $q_i$  if and only if  $q_{ij} = 1$  (i.e.  $q_j$  is proximate to  $q_i$ ).

**Remark 5.6** *The properties of  $Q$  imply that an Enriques digraph is characterized by the following four properties (see [5, §5], [15, pp. 213–214]):*

- i) *there is no directed cycle;*
- ii) *every vertex has out-degree at most 2;*
- iii) *if  $q_i \rightarrow q_j$  and  $q_i \rightarrow q_k$ , with  $j \neq k$ , then either  $q_j \rightarrow q_k$  or  $q_k \rightarrow q_j$ ;*
- iv) *there is at most one  $q_i$  with  $q_i \rightarrow q_j$  and  $q_i \rightarrow q_k$ , if  $j \neq k$ .*

Note that the *in-degree* of  $q_i$  (the number of arrows ending in  $q_i$ ) is  $-E_i^2 - 1$ .

Since we need only to resolve an isolated singularity at  $q_1$ , we assume to blow up only points lying on the total exceptional divisor, i.e., we assume that  $q_i \in \sigma_{1,i-1}^{-1}(q_1)$ , for every  $i > 1$ . This means that only the first column  $Q_1$  of  $Q$  is everywhere zero, so the Enriques digraph is connected. Recall that a point  $q_j$  is called *infinitely near* to  $q_i$ , and we write  $q_j > q_i$ , if  $q_j \in \sigma_{i,j-1}^{-1}(q_i)$ . Thus each  $q_i$  is infinitely near to  $q_1$ . Let us define the *infinitesimal order* inductively. If  $q_j > q_i$  and there is no  $q_k$  such that  $q_j > q_k > q_i$ , then  $q_j$  is infinitely near of *order one* to  $q_i$  and we write  $q_j >^1 q_i$ . If  $q_j >^1 q_k > q_i$ , then by induction  $q_k >^m q_i$  for some  $m$  and we set  $q_j >^{m+1} q_i$ .

Usually, the main combinatorial tool used for blowing-ups is the dual graph of the exceptional curves and their self-intersection numbers, that are the entries of the intersection matrix  $S = (s_{ij})$ , where  $s_{ij} = (E_i \cdot E_j)$ .

**Lemma 5.7** *The configuration of the exceptional curves  $E_i$  of  $\sigma$  may be given by only one of the following matrices:  $M$ ,  $N$ ,  $Q$ , or  $S$ . Indeed any one of them determines canonically all the others.*

**Proof.** Recall that  $N = M^{-1} = I - Q$ . Formulas (5.3) and (5.2) imply that:

$$s_{ij} = \left( \sum_{k=1}^n n_{ik} E_k^* \cdot \sum_{h=1}^n n_{jh} E_h^* \right) = \sum_{k=1}^n \sum_{h=1}^n n_{ik} n_{jk} (E_k^* \cdot E_h^*) = - \sum_{k=1}^n n_{ik} n_{jk},$$

so  $S = -NN^T = N(-I)N^T$ , that is the decomposition of  $S$  in an unipotent upper triangular, a diagonal and an unipotent lower triangular matrix. Such a decomposition is known to be unique by linear algebra.  $\square$

## 6 The proper transform of the singular curve

We now consider a reduced curve  $B_0$  on  $Y_0$  with a singular point at the point  $q_1 \in Y_0$  which is being blown up. Let us denote with  $\tilde{B}_i$  the proper transform of

$B_0$  in  $Y_i$  via the sequence of blowing-ups  $\sigma_{0i}$ , for every  $i = 1, \dots, n$ . Recall that the following formula holds in  $\text{Pic } Y_n$ :

$$\sigma^*(B_0) = \tilde{B}_n + \sum_{i=1}^n \tilde{\alpha}_i E_i^*, \quad \text{where } \tilde{\alpha}_i = \text{mult}_{q_i}(\tilde{B}_{i-1}). \quad (6.1)$$

Abusing language a little, usually  $\tilde{\alpha}_i$  is called the *multiplicity* of  $B_0$  at  $q_i$ . Note that  $\tilde{\alpha}_i$  may be determined also in  $Y_i$  by  $\tilde{\alpha}_i = (\tilde{B}_i \cdot E_i)_i$ .

On the other hand, in  $\text{Pic } Y_n$  we may write also:

$$\sigma^*(B_0) = \tilde{B}_n + \sum_{i=1}^n \tilde{\beta}_i E_i. \quad (6.2)$$

for some non-negative integers  $\tilde{\beta}_i$ . Putting the second formula of (5.3) in (6.1) we find that the  $\tilde{\beta}_i$ 's can be computed from the  $\tilde{\alpha}_i$ 's as follows:

$$\tilde{\beta}_i = \sum_{j=1}^n \tilde{\alpha}_j m_{ji}, \quad \text{or shortly} \quad \tilde{\beta} = \tilde{\alpha} M, \quad (6.3)$$

where  $\tilde{\alpha}$  and  $\tilde{\beta}$  are row vectors with the obvious entries.

The quantities  $\tilde{\alpha}$  and  $\tilde{\beta}$  may also be determined on  $Y_n$  knowing how  $\tilde{B}_n$  intersects the exceptional curves  $E_i$ . Indeed, intersecting (6.2) with  $E_i$  gives  $0 = \tilde{B}_n \cdot E_i + \sum_{j=1}^n \tilde{\beta}_j E_i \cdot E_j$  that is, setting  $\tilde{\gamma}_j = (\tilde{B}_n \cdot E_j)$  and  $\tilde{\gamma}$  the corresponding row vector:

$$\tilde{\gamma} = -\tilde{\beta} S = \tilde{\beta} N N^T = \tilde{\alpha} N^T, \quad \text{so} \quad \tilde{\alpha} = \tilde{\gamma} M^T. \quad (6.4)$$

Note that  $\tilde{B}_n$  satisfies  $(\tilde{B}_n \cdot E_i) \geq 0$  for every  $i$ , which is equivalent by (6.1), (5.3), (5.5) and (5.2) to the so-called *proximity inequality* at  $q_i$ :

$$\tilde{\alpha}_i \geq \sum_{j=1}^n q_{ij} \tilde{\alpha}_j = \sum_{j: q_j \rightarrow q_i} \tilde{\alpha}_j. \quad (6.5)$$

Suppose that  $B_0$  has only an isolated singularity in  $q_1$  and that  $\sigma : Y_n \rightarrow Y_0$  resolves the singularities of  $B_0$ , i.e., the proper transform  $\tilde{B}_n$  is smooth. Then the topological type of the singularity of  $B_0$  at  $q_1$  is completely determined by the matrix  $M$ , which carries the configuration of the exceptional curves  $E_i$  for  $\sigma$ , and the intersection divisor  $\tilde{\Gamma}_i = \tilde{B}_n|_{E_i}$ , which says how  $\tilde{B}_n$  meets  $E_i$ , for every  $i = 1, \dots, n$ . Each  $\tilde{\Gamma}_i$  is a non-negative divisor on  $E_i$  and if  $q$  is a point of intersection of two exceptional curves  $E_i$  and  $E_j$ , then:

$$\tilde{\Gamma}_i(q) = 0 \iff \tilde{\Gamma}_j(q) = 0. \quad (6.6)$$

Moreover if these numbers are non-zero, then at least one number is equal to one. Condition (6.6) says that either  $\tilde{B}_n$  passes through  $q$  or not, while the latter statement that  $\tilde{B}_n$  cannot be tangent in  $q$  to both exceptional curves simultaneously. These divisors  $\tilde{\Gamma}_i$  express the combinatorial information of the singularity of the

branch curve completely. Given the configuration of the exceptional curves, they can be arbitrary, subject to the above condition.

We remark that the knowledge of  $M$  and of the degrees  $\tilde{\gamma}_i = \deg(\tilde{\Gamma}_i)$  is equivalent to the knowledge of  $Q$  and the multiplicities  $\tilde{\alpha}_i$  of  $B_0$  at  $q_i$ , by (6.4).

Therefore we will define the *weighted Enriques digraph* of  $q_1 \in B_0$  by attaching to each vertex  $q_i$  of the Enriques digraph the weight  $\tilde{\alpha}_i = \text{mult}_{q_i}(\tilde{B}_{i-1})$ .

**Example 6.7** Let  $B_0 \subset Y_0$  be defined locally near the origin  $q_1 = (0, 0)$  by:

$$x(y^2 - x)(y^2 + x)(y^2 - x^3)(y^2 + x^3) = 0.$$

Clearly  $B_0$  has multiplicity  $\tilde{\alpha}_1 = 7$  at  $q_1$ . Blow up  $q_1$ :  $\tilde{B}_1 \subset Y_1$  has two singular points  $q_2$  and  $q_3$  on  $E_1$  of multiplicity  $\tilde{\alpha}_2 = 3$  and  $\tilde{\alpha}_3 = 2$  ( $q_2$  and  $q_3$  are infinitely near points of order one to  $q_1$ ). Then blow up  $q_2$  and  $q_3$ .  $\tilde{B}_2 \subset Y_2$  meets transversally  $E_2$  and it is smooth at those points, but  $\tilde{B}_3 \subset Y_3$  has in  $q_4 = E_1 \cap E_3$  a point of multiplicity  $\tilde{\alpha}_4 = 2$ , so  $q_4$  is proximate to both  $q_3$  and  $q_1$ , but  $q_4$  is infinitely near of order 2 to  $q_1$ . Classically,  $q_4$  is called a *satellite* point to  $q_1$ . Finally  $\tilde{B}_4 \subset Y_4$  is smooth.

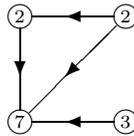
The configuration of the exceptional curves is determined by anyone of the following matrices:

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

and the combinatorial data of  $\tilde{B}_4$  by anyone of the following vectors:

$$\tilde{\alpha} = (7, 3, 2, 2), \quad \tilde{\beta} = (7, 10, 9, 18), \quad \tilde{\gamma} = (0, 3, 0, 2),$$

that we encode in the following weighted Enriques digraph:



## 7 Resolving the branch locus of a double cover

We return to consider a normal double cover  $\pi_0 : X_0 \rightarrow Y_0$  branched along the singular reduced curve  $B_0$ . We desingularize  $B_0$  by successive blowing-ups as in the previous sections. We have seen that  $\sigma$  induces a normal double cover  $\pi_n : X_n \rightarrow Y_n$ , which is the normalization of the pullback  $X_0 \times_{Y_0} Y_n$ . As in Lemma 4.1,  $\pi_n$  is branched along a reduced curve  $B_n$ , obtained from  $\sigma^*(B_0)$  by removing the multiple components an even number of times. Hence  $B_n$  is made of the proper transform  $\tilde{B}_n$  of  $B_0$  and possibly of some exceptional curves  $E_i$ . We set  $\varepsilon_i$  equal to one or zero, depending on whether  $E_i$  is part of the branch

locus of  $Y_n$  or not (and we say that  $E_i$  is *branched*, resp. *unbranched*). Setting  $\varepsilon$  the corresponding row vector, by (6.2) and (6.3) we have that:

$$\varepsilon = \tilde{\beta} \bmod 2 \quad \text{and} \quad \varepsilon = \tilde{\alpha}M \bmod 2. \quad (7.1)$$

Note that the branchedness of  $E_i$  is determined at the moment  $E_i$  is created on  $Y_i$  by blowing up the point  $q_i \in Y_{i-1}$ . Indeed, Lemma 4.1 says that  $\varepsilon_i = 1$  (resp.  $\varepsilon_i = 0$ ) if the multiplicity  $\mu_i$  at  $q_i$  of the branch locus  $B_{i-1}$  of  $\pi_{i-1} : X_{i-1} \rightarrow Y_{i-1}$  is odd (resp. even). Shortly:

$$\varepsilon = \mu \bmod 2. \quad (7.2)$$

Assuming inductively to know  $\varepsilon_j$  for  $j < i$ , the multiplicity of  $B_{i-1}$  at  $q_i$  is

$$\mu_i = \tilde{\alpha}_i + \sum_{j=1}^{i-1} \varepsilon_j q_{ji} = \tilde{\alpha}_i + \sum_{j:q_i \rightarrow q_j} \varepsilon_j, \quad \text{or shortly} \quad \mu = \tilde{\alpha} + \varepsilon Q. \quad (7.3)$$

If we have blown up to make the proper transform  $\tilde{B}_n$  smooth, we still may not have the total branch locus  $B_n$  smooth. Singularities of the total branch locus now come from two sources: intersections of  $\tilde{B}_n$  with branched  $E_i$ 's, and intersections between two different branched  $E_i$ 's. We first take up the former case. Suppose that  $\tilde{B}_n$  meets the exceptional configuration  $\bigcup_i E_i$  at a point  $q_{n+1}$ . We blow up  $q_{n+1}$  to create a new surface  $Y_{n+1}$ , a new proper transform  $\tilde{B}_{n+1}$ , and a new exceptional curve  $E_{n+1}$ . Since  $\tilde{B}_n$  is smooth at  $q_{n+1}$ , we have  $(\tilde{B}_{n+1} \cdot E_{n+1})_{n+1} = 1$ . We now have new intersection divisors  $\tilde{\Gamma}'_i$  for each  $i = 1, \dots, n+1$ . These are related to the previous intersection divisors  $\tilde{\Gamma}_i$ 's for  $i = 1, \dots, n+1$  as follows:

$$\tilde{\Gamma}'_i = \begin{cases} \tilde{\Gamma}_i & \text{if } q_{n+1} \notin E_i \\ \tilde{\Gamma}_i - q_{n+1} & \text{if } q_{n+1} \in E_i \end{cases}$$

for every  $i = 1, \dots, n$  and  $\tilde{\Gamma}'_{n+1} = q$ , where  $q$  is the point of intersection of  $\tilde{B}_{n+1}$  with  $E_{n+1}$ . We may now iterate this construction, and arrive at the situation (increasing the number of blowing-ups  $n$ ) that the proper transform  $\tilde{B}_n$  does not meet any branched exceptional curves, i.e., the new exceptional divisors  $\tilde{\Gamma}'_i$  are zero for each  $i$  such that  $\varepsilon_i = 1$ . Finally if any two exceptional curves now meet, we simply blow up the point of intersection once, and obtain an unbranched exceptional curve which now separates the two branched exceptional curves.

At this point we have a nonsingular total branch locus, hence a smooth double cover. We note that we still have the matrices  $M, N, Q, S$  for the current configuration of exceptional curves and the numbers  $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i$  and  $\varepsilon_i$ , defined for each  $i$ , as before. All the above process gives a proof of the following theorem (cf. [4, III.§6], [19, Theorem 3.1]):

**Theorem 7.4 (The canonical resolution)** *Let  $\pi_0 : X_0 \rightarrow Y_0$  be a double cover with  $X_0$  normal and  $Y_0$  smooth. Then there exists a birational morphism*

$\sigma : Y \rightarrow Y_0$  such that the normalization  $X$  of the pullback  $X_0 \times_{Y_0} Y$  is smooth. Moreover  $\pi : X \rightarrow Y$  is a double cover and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\tau} & X_0 \\ \downarrow \pi & & \downarrow \pi_0 \\ Y & \xrightarrow{\sigma} & Y_0 \end{array} \quad (7.5)$$

commutes. So  $\tau : X \rightarrow X_0$  is a resolution of singularities of  $X_0$ . □

We say that  $\tau : X \rightarrow X_0$  (together with the double cover map  $\pi : X \rightarrow Y$ ) is the *canonical* resolution of the double cover  $\pi_0 : X_0 \rightarrow Y_0$ , because  $X$  and  $Y$  are unique, up to isomorphism, assuming that the centers of the blowing-ups  $\sigma_i : Y_i \rightarrow Y_{i-1}$ , which factorizes  $\sigma$ , are always singular points of the branch curve of  $X_{i-1} \rightarrow Y_{i-1}$ . We will see in section 10 that the canonical resolution might not be minimal. However, we may think the canonical resolution as the “minimal” resolution in the category of double covers over smooth surfaces.

## 8 The branch curve of the canonical resolution

As in the previous section, let  $\tau : X \rightarrow X_0$  be the canonical resolution of  $X_0$  and  $\pi : X \rightarrow Y$  be the smooth double cover. Let us write  $B$  for the branch curve of  $\pi : X \rightarrow Y$  and suppose that  $\sigma : Y \rightarrow Y_0$  factorizes in  $n$  blowing-ups  $\sigma_i : Y_i \rightarrow Y_{i-1}$ , with  $Y_n = Y$ , so the exceptional curves for  $\sigma$  are  $E_1, \dots, E_n$  and all the formulas in the previous sections hold for  $B_n = B$  and  $\tilde{B}_n = \tilde{B}$ . The branch curve  $B$  can be written in  $\text{Pic } Y$  as:

$$B = \tilde{B} + \sum_{i=1}^n \varepsilon_i E_i = \sigma^*(B_0) - \sum_{i=1}^n \beta_i E_i = \sigma^*(B_0) - \sum_{i=1}^n \alpha_i E_i^* \quad (8.1)$$

for some non negative integers  $\beta_i$  and  $\alpha_i$ . Comparing (8.1) with (6.2), (6.1) and (5.3) we see that:

$$\beta = \tilde{\beta} - \varepsilon \quad \text{and} \quad \alpha = \tilde{\alpha} + \varepsilon N = \tilde{\alpha} + \varepsilon(Q - I). \quad (8.2)$$

Moreover, from formulas (8.2), (6.3) and (7.3) we find that:

$$\alpha = \mu - \varepsilon \quad \text{and} \quad \alpha = \beta N. \quad (8.3)$$

By (8.2) and (7.1), (8.3) and (7.2) it follows that the  $\beta_i$ 's and the  $\alpha_i$ 's are all even. From (7.3) and (8.2) it is clear that  $\alpha = \mu = \tilde{\alpha}$  if and only if  $\varepsilon = 0$ , that happens if  $\tilde{\alpha}_i$  is even for every  $i = 1, \dots, n$ , i.e. if  $B_0$  has even multiplicity at each singular point, including the infinitely near ones.

In order to measure how the branch curve  $B$  of the canonical resolution meets the exceptional curves, let us introduce the following intersection divisors:

$$\Gamma_i = B|_{E_i} = \tilde{B}|_{E_i} + \sum_{j:\varepsilon_j=1} E_j|_{E_i} \quad (8.4)$$

and set  $\gamma_i = \deg(\Gamma_i)$ . By definition  $\Gamma_i = 0$  if  $E_i$  is branched. However,  $\Gamma_i$  could be zero even if  $\varepsilon_i = 0$ . We claim that the  $\Gamma_i$ 's have the following properties:

1.  $\Gamma_i$  is a non-negative divisor and  $\gamma_i$  is even;
2. if  $q = E_i \cap E_j$  and  $\varepsilon_i = \varepsilon_j = 0$ , then  $\Gamma_i(q) = 0$  if and only if  $\Gamma_j(q) = 0$ . If these numbers are non-zero, then at least one of them is equal to 1.
3. if  $q = E_i \cap E_j$ ,  $\varepsilon_i = 1$  and  $\varepsilon_j = 0$ , then  $\Gamma_j(q) = 1$ .

It suffices to show that  $\gamma_i$  is even, since all the other properties of  $\Gamma_i$  are induced by those of  $\tilde{\Gamma}_i$ , which we have already seen in section 6. If  $E_i$  is branched, then  $\gamma_i = 0$  and the thesis is trivial. Otherwise, if  $E_i$  is unbranched, then:

$$\gamma_i = \tilde{\gamma}_i + \sum_{j \neq i} \varepsilon_j (E_i \cdot E_j) \equiv \tilde{\gamma}_i + \sum_{j=1}^n \tilde{\beta}_j s_{ij} \pmod{2}$$

hence  $\gamma \equiv \tilde{\gamma} + \tilde{\beta}S \pmod{2}$  and the claim follows from  $\tilde{\gamma} = -\tilde{\beta}S$  (see (6.4)).

In order to encode the combinatorial data of the double point singularity  $p = \pi_0^{-1}(q_1) \in X_0$ , we define the *weighted Enriques digraph* of  $p$  by attaching to each vertex  $q_i$  of the Enriques digraph of  $q_1 \in B_0$  the weight  $\mu_i = \text{mult}_{q_i}(B_{i-1})$ .

In the next example we will illustrate in detail how the canonical resolution process goes on.

**Example 8.5** Let  $B_0$  be a curve defined locally near the origin  $q_1 = (0, 0)$  by:

$$y(y^2 - x^3) = 0.$$

Clearly, we need just two blowing-ups to smooth the proper transform  $\tilde{B}_2$  of  $B_0$ . However, we have to blow up 5 more times in order to get a smooth double cover. In fact  $E_1, E_2$  are branched and  $\tilde{B}_2$  passes through  $q_3 = E_1 \cap E_2$  and meets  $E_2$  also in another point  $q_4$ . Thus  $B_2 = \tilde{B}_2 + E_1 + E_2$  has multiplicity  $\mu_3 = 3$  at  $q_3$ , hence  $\varepsilon_3 = 1$  and  $\tilde{B}_3$  meets  $E_3$  in a point  $q_5$ . Now  $B_3$  has only nodes in  $q_4, q_5, q_6 = E_1 \cap E_3$  and  $q_7 = E_2 \cap E_3$ , therefore  $E_4, \dots, E_7$  are unbranched and  $B_7$  is smooth. Our combinatorial data are:

$$M = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} \tilde{\alpha} &= (3, 2, 1, 0, 0, 1, 1) \\ \tilde{\gamma} &= (0, 0, 0, 1, 1, 0, 0), \\ \mu &= (3, 3, 3, 2, 2, 2, 2), \\ \varepsilon &= (1, 1, 1, 0, 0, 0, 0), \\ \alpha &= (2, 2, 2, 2, 2, 2, 2), \\ \gamma &= (0, 0, 0, 2, 2, 2, 2), \end{aligned}$$

that we encode in the Enriques digraph, weighted with the  $\mu_i$ 's (see the right-hand side graph of Figure 1). For the readers' convenience, we inserted in Figure 1 (on the left-hand side) also the Enriques digraph weighted with the  $\tilde{\alpha}_i$ 's and labelled with the  $q_i$ 's.

Figure 1 may help to understand formula (7.3), namely how to compute inductively the  $\mu_i$ 's (thus the  $\varepsilon_i$ 's) from the  $\tilde{\alpha}_i$ 's. Start from  $q_1$ : the weight of  $q_1$  is 3, thus  $\mu_1 = 3$ ,  $\varepsilon_1 = 1$  and add 1 to all the weights attached to vertices with arrows ending in  $q_1$  (namely  $q_2, q_3$  and  $q_6$ ). Now consider  $q_2$ : the actual weight of  $q_2$  is 3, hence  $\mu_2 = 3$ ,  $\varepsilon_2 = 1$  and add 1 to the weights of  $q_3, q_4$  and  $q_7$ , which are the vertices with arrows ending in  $q_2$ . Then go on inductively, for all the  $q_i$ 's. Clearly, no change of weights is made at step  $i$  if  $\varepsilon_i = 0$ .

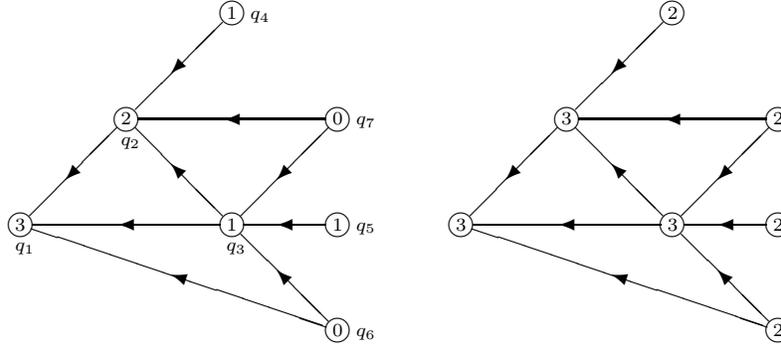


Figure 1: The Enriques digraph weighted resp. with the  $\tilde{\alpha}_i$ 's and the  $\mu_i$ 's

### 9 The description of the canonical resolution

The description of the canonical resolution of the singularity on  $X_0$  is now a combinatorial problem, using the information of the configuration of the exceptional curves  $E_i$  (described by the matrix  $M$  or the Enriques digraph) and the divisors  $\tilde{\Gamma}_i$  (subject to the conditions stated in section 6).

We then can compute the quantities  $\tilde{\gamma}, \tilde{\alpha}, \mu, \varepsilon$ , and determine (from  $\varepsilon$ ) which of the  $E_i$ 's are branched curves, and finally determine the divisors  $\Gamma_i$ .

We now apply Lemma 3.1 for each exceptional curve. If  $\pi : X \rightarrow Y$  is the double cover map, let us define  $F_i = \pi^{-1}(E_i)$  for each  $i$ , thus  $F_1, \dots, F_n$  are all the exceptional curves for the canonical resolution  $\tau : X \rightarrow X_0$ .

**Remark 9.1** *A curve  $F_i$  is reducible if and only if  $\varepsilon_i = 0$  and  $\Gamma_i = B_{|E_i}$  is an even divisor. In that case,  $F_i$  splits in two smooth rational curves  $F'_i$  and  $F''_i$ , with  $F'_i \cdot F''_i = \gamma_i/2$  and  $F_i'^2 = F_i''^2 = E_i^2 - \gamma_i/2$ .*

If  $\varepsilon_i = 0$ , then  $\pi^*(E_i) = F_i$ , otherwise, if  $\varepsilon_i = 1$ , then  $\pi^*(E_i) = 2F_i$ , i.e.

$$\pi^*(E_i) = (1 + \varepsilon_i)F_i. \tag{9.2}$$

Moreover since intersections double after applying  $\pi^*$  we have that:

$$F_i^2 = \frac{2}{(1 + \varepsilon_i)^2} E_i^2 \quad \text{and} \quad (F_i \cdot F_j) = (2 - \varepsilon_i - \varepsilon_j)(E_i \cdot E_j). \tag{9.3}$$

We claim that the arithmetic genus of  $F_i$  is, for each  $i$ :

$$p_a(F_i) = \frac{\gamma_i}{2} + \varepsilon_i - 1. \tag{9.4}$$

If  $\varepsilon_i = 1$ , then  $F_i$  is a smooth rational curve,  $\Gamma_i = 0$  and the claim is trivial. If  $F_i$  splits, (9.4) follows from Remark 9.1. Otherwise,  $F_i$  is a double cover of  $E_i$  branched along  $\Gamma_i$  and (9.4) is just Hurwitz formula.

Moreover  $F_i$  is singular at a point  $P$  if and only if  $\Gamma_i(\pi(P)) > 1$ , thus in particular  $F_i$  is smooth at the intersection points with  $F_j$ , for each  $j \neq i$ .

Now we want to find an explicit formula for the canonical divisor  $K_X$ . By Riemann-Hurwitz formula (3.3) we know that  $K_X = \pi^*(K_Y + B/2)$ , where  $K_Y = \sigma^*(K_{Y_0}) + \sum_i E_i^*$ . Therefore by (8.1):

$$K_Y + \frac{B}{2} = \sigma^* \left( K_{Y_0} + \frac{B_0}{2} \right) - \sum_{i=1}^n \left( \frac{\alpha_i}{2} - 1 \right) E_i^*. \quad (9.5)$$

We define  $D = \sum_{i=1}^n (\alpha_i/2 - 1) E_i^*$ , so:

$$K_X = (\sigma \circ \pi)^*(K_{Y_0} + B_0/2) - \pi^* D = \tau^* K_{X_0} - D^*, \quad (9.6)$$

where  $D^* = \pi^* D$  is called the *adjunction condition* divisor (cf. section 14). We remark that  $D^* \geq 0$ , because  $\alpha_i \geq 2$  for each  $i$  (since in the canonical resolution process we blow up only singular points of the branch curve).

Let us show now the explicit formula for the *fiber cycle*  $F$  of the canonical resolution, already written without proof by Franchetta and Dixon. The *fiber cycle* of  $\tau$  is the maximal effective divisor  $F$  contained in the *scheme theoretic fiber* of  $\tau$ , i.e., the subscheme of  $X$  defined by the inverse image ideal sheaf  $\tau^{-1}m_{p,X_0}$  of the maximal ideal  $m_{p,X_0}$  of  $p$  in  $X_0$ . Therefore:

$$F = \gcd \{ \operatorname{div}(g) \mid \text{for all } g \in \tau^{-1}m_{p,X_0} \}.$$

For example, the fiber cycle of the sequence  $\sigma : Y \rightarrow Y_0$  of blowing-ups is  $E_1^*$ .

**Theorem 9.7 (Franchetta)** *The fiber cycle of the canonical resolution is:*

$$F = \pi^*(E_1^*) = \sum_{i=1}^n m_{1i}(1 + \varepsilon_i) F_i. \quad (9.8)$$

**Proof.** The second equality in (9.8) follows from (9.2) and from the definition of  $M$  (see (5.3)), so it suffices to show the first equality. Recall that locally  $x, y$  are the coordinates of  $Y_0$  near  $q_1 = (0, 0)$ ,  $X_0$  is defined by  $z^2 = f(x, y)$  and  $\pi_0$  is the projection  $(x, y, z) \mapsto (x, y)$ . Hence  $x, y$  and  $z$  are generators of the ideal sheaf  $\tau^{-1}m_{p,X_0}$  as  $\mathcal{O}_{X_0,p}$ -module. Therefore  $F$  is the greatest effective divisor contained in the pullback divisors  $\tau^* \operatorname{div} x$ ,  $\tau^* \operatorname{div} y$  and  $\tau^* \operatorname{div} z$ . From the commutativity of the diagram (7.5) we see that  $\tau^* \operatorname{div} x = (\pi_0 \circ \tau)^* \operatorname{div} x = (\sigma \circ \pi)^* \operatorname{div} x$  and  $\tau^* \operatorname{div} y = (\sigma \circ \pi)^* \operatorname{div} y$ , thus the gcd of the divisors  $\tau^* \operatorname{div} x$  and  $\tau^* \operatorname{div} y$  is the pullback  $\pi^*(E_1^*)$  of the fiber cycle  $E_1^*$  of  $\sigma$ . Moreover  $\tau^* \operatorname{div} z^2 = (\pi_0 \circ \tau)^* \operatorname{div} f = (\sigma \circ \pi)^* \operatorname{div} f$ , thus the divisor  $2\tau^* \operatorname{div} z = \tau^* \operatorname{div} z^2$  is equal to the pullback  $\pi^*(\sigma^* B_0)$  of the total transform  $\sigma^* B_0$  which contains  $E_1^*$  with multiplicity  $\tilde{\alpha}_1 \geq 2$ . So  $\tau^* \operatorname{div} z \supseteq \pi^*(E_1^*)$ , hence  $F = \pi^*(E_1^*)$ .  $\square$

The self-intersection of the fiber cycle is:

$$F^2 = \pi^*(E_1^*) \cdot \pi^*(E_1^*) = 2E_1^{*2} = -2 = -\operatorname{mult}_p(X_0). \quad (9.9)$$

By (9.8) and (9.2), the fiber cycle  $F$  has the following properties:

$$F \cdot F_1 = (2 - \varepsilon_1) E_1^* \cdot E_1 = -2 + \varepsilon_1, \quad F \cdot F_i = (2 - \varepsilon_i) E_1^* \cdot E_i = 0 \quad (9.10)$$

for every  $i > 1$ . Thus the normal sheaf  $\mathcal{O}_{F_i}(F_i)$  of  $F_i$ , for  $i > 1$  (that can be useful if  $p_a(F_i) > 0$ ), is given by how  $F_i$  meets the other components of  $F$ :

$$m_{1i}(1 + \varepsilon_i)F_i|_{F_i} = -F^{\hat{i}}|_{F_i} \quad (9.11)$$

where  $F^{\hat{i}} = F - m_{1i}(1 + \varepsilon_i)F_i$ . Finally, by (9.6), (9.8) and (9.9), the arithmetic genus of the fiber cycle is:

$$p_a(F) = (F \cdot K_X + F^2)/2 + 1 = E_1^* \cdot (K_Y + B/2) = \alpha_1/2 - 1. \quad (9.12)$$

## 10 The minimal resolution

It may happen that the canonical resolution  $\tau : X \rightarrow X_0$  of  $p \in X_0$  is not minimal. However the following theorem (cf. [19, Th. 5.4]) shows that the canonical resolution is not too far to be minimal.

**Theorem 10.1** *Let  $\tau : X \rightarrow X_0$  be the canonical resolution and  $\bar{\tau} : \bar{X} \rightarrow X_0$  the minimal one. Then  $\tau = \tau' \circ \bar{\tau}$ , where  $\tau' : X \rightarrow \bar{X}$  is the blowing-up at finitely many distinct points. Moreover none of these points is singular for the exceptional curves of  $\bar{\tau}$  and neither lies on the intersection (necessarily transverse) of more than three of them.*

The following lemma characterizes the  $(-1)$ -curves in  $\tau^{-1}(p)$  and allows us to give an elementary proof of Theorem 10.1, easier than Laufer's original one.

**Lemma 10.2** *An exceptional curve  $F_j$  for  $\tau$  is a  $(-1)$ -curve if and only if  $F_j = \pi^{-1}(E_j)$ , where  $E_j$  is branched and  $E_j^2 = -2$ . Moreover the  $(-1)$ -curves in  $\tau^{-1}(p)$  are disjoint.*

**Proof.** Let  $E_j$  be unbranched. By Lemma 3.1, if  $\Gamma_j = B|_{E_j}$  is not even, then  $F_j$  cannot be a  $(-1)$ -curve, because  $F_j^2 = 2E_j^2$  would be even. If  $\Gamma_j$  is even, then  $\pi^{-1}(E_j) = F_j' + F_j''$  and  $F_j' = F_j'' = E_j^2 - \deg(\Gamma_j/2)$ , thus  $F_j'$  (and  $F_j''$ ) could be a  $(-1)$ -curve only if  $E_j^2 = -1$  and  $\Gamma_j = 0$ , that means that  $q_j$  was unnecessarily blown up. Hence we are left only with the possibility that  $E_j$  is branched and  $E_j^2 = 2F_j^2 = -2$ . Since no branched exceptional curves meet in  $Y$ , neither do two  $(-1)$ -curves in  $X$ .  $\square$

**Remark 10.3** *If  $F_j = \pi^{-1}(E_j)$  is a  $(-1)$ -curve in  $X$ , then  $\mu_j$  is odd and there is exactly one  $q_i$  such that  $q_i >^1 q_j$  and  $\mu_i = \mu_j + 1$ . Usually one says that  $B_{j-1}$  has two infinitely near points of the same, odd, multiplicity  $\mu_j$  at  $q_j$ .*

**Proof.** Clearly  $\mu_j$  is odd (and  $> 2$ ), because  $\varepsilon_j = 1$ . Since  $E_j$  is a  $(-2)$ -curve, we blew up only one point  $q_i$  on  $E_j$ : this means that all the intersections of  $\bar{B}_j$  with  $E_j$  are supported on  $q_i$ , i.e.  $\tilde{\alpha}_i = \tilde{\alpha}_j$ , and the thesis follows from (7.3).  $\square$

Let us denote by  $\bar{\tau} : \bar{X} \rightarrow X_0$  the minimal resolution of  $p \in X_0$ .

**Proof of Theorem 10.1.** Let  $\tau' : X \rightarrow \bar{X}$  be the contraction of all the  $(-1)$ -curves in  $\tau^{-1}(p)$ . We claim that  $\bar{X}$  is isomorphic to  $\bar{X}$  and  $\tau = \tau' \circ \bar{\tau}$ , namely

there is no  $(-1)$ -curve in  $\tau'(\tau^{-1}(p))$ . The only way for a  $(-1)$ -curve to be created after blowing down a  $(-1)$ -curve  $F_j = \pi^{-1}(E_j)$  is for a smooth rational curve  $F_k$  of self-intersection  $-2$  to meet the given  $(-1)$ -curve  $F_j$ . Since no two branched curve meet on  $Y$  and  $E_j$  is branched by Lemma 10.2, then  $F_k$  must lie over an unbranched curve  $E_k$  which meets  $E_j$  at one point. Therefore  $\pi^*(E_k)$  cannot split (its divisor  $\Gamma_k$  is not even) so that  $F_k = \pi^*(E_k)$  and  $F_k^2 = 2E_k^2$ . So  $E_k$  is a  $(-1)$ -curve on  $Y$  and, since  $F_k$  is smooth and rational, the divisor  $\Gamma_k$  must consist of two simple points (one of which is the intersection point with  $E_j$ ). In this case we see that  $E_k$  and  $E_j$  should be blown down on  $Y$ , so that both of these curves were unnecessarily blown up. This proves our claim. Let  $F_j = \pi^{-1}(E_j)$  be a  $(-1)$ -curve of  $X$ . Let  $E_k$  be an exceptional curve that meets  $E_j$ . Since  $E_j$  is branched,  $E_k$  is unbranched and their intersection is transversal, as any intersection of the  $E_i$ 's. Hence  $F_j$  contracts to a smooth point of  $F_k$  (cf. Lemma 3.1). Finally,  $E_j^2 = -2$  implies that  $E_j$  meets at most three exceptional curves of  $\sigma$ , namely one curve that corresponds to a blown up point on  $E_j$  and the exceptional curves on which  $q_j$  lies, that are at most two.  $\square$

The previous analysis offers also an alternative route to obtaining the minimal resolution of  $p \in X_0$ : we may first contract the branched  $(-2)$ -curves among the  $E_i$ 's and then take the double cover, namely the diagram

$$\begin{array}{ccc} X & \xrightarrow{\tau'} & \bar{X} \\ \downarrow \pi & & \downarrow \bar{\pi} \\ Y & \xrightarrow{\sigma'} & \bar{Y} \end{array}$$

commutes, where  $\sigma'$  is the contraction of the branched  $(-2)$ -curves in  $\sigma^{-1}(q_1) \subset Y$  and  $X$  is the fiber product  $\bar{X} \times_{\bar{Y}} Y$ .

Note that contracting a  $(-2)$ -curve on a smooth surface produces a singularity, namely an ordinary double point (of type  $A_1$ , see p. 24). Since the  $(-2)$ -curves are branched for  $\pi : X \rightarrow Y$ , the singular points of  $\bar{Y}$  must be considered as *branched points* for  $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ .

Let us denote by  $\bar{F}_i$  (resp.  $\bar{E}_i$ ) the image of  $F_i$  in  $\bar{X}$  (resp. of  $E_i$  in  $\bar{Y}$ ). For simplicity, suppose that we blow down only a  $(-1)$ -curve  $F_k$  (the general case can be computed inductively). If  $E_k$  meets two unbranched divisors  $E_i$  and  $E_j$  in  $X$ , then  $\bar{E}_i$  and  $\bar{E}_j$  meet in a branched point, hence:

$$\bar{F}_i \cdot \bar{F}_j = \begin{cases} 1 & \text{if } E_i \cdot E_k = E_j \cdot E_k = 1, \\ F_i \cdot F_j & \text{if } E_i \cdot E_k = 0 \text{ or } E_j \cdot E_k = 0. \end{cases} \quad (10.4)$$

Moreover the self-intersection numbers change as follows:

$$\bar{F}_i^2 = \begin{cases} F_i^2 + 1 & \text{if } E_i \cdot E_k = 1, \\ F_i^2 & \text{otherwise,} \end{cases} \quad (10.5)$$

while the arithmetic genera stay unchanged. Formula (9.8) and the fact that  $\tau'^{-1}(\bar{\tau}^{-1}(m_{p, X_0})) = \tau^{-1}(m_{p, X_0})$  imply that the fiber cycle of the minimal resolution  $\bar{\tau} : \bar{X} \rightarrow X$  is:

$$\bar{F} = \sum m_{1i}(1 + \varepsilon_i)\bar{F}_i. \quad (10.6)$$

## 11 The fundamental cycle

The fundamental cycle of the (canonical) resolution is the unique smallest positive cycle:

$$Z = \sum_{i=1}^n z_i F_i \tag{11.1}$$

with  $z_i > 0$  such that  $Z \cdot F_k \leq 0$  for every  $k = 1, \dots, n$ . If  $F_i$  splits in  $F'_i$  and  $F''_i$  as in Remark 9.1, *a priori* we should consider in (11.1) two distinct coefficients  $z'_i$  and  $z''_i$ . But if  $z'_i$  were different from  $z''_i$  in (11.1), then we could exchange them and take the g.c.d. of the cycles, so  $z_i = \min\{z'_i, z''_i\}$  would fulfill the properties of the fundamental cycle. Therefore we may and will assume  $z'_i = z''_i = z_i$ .

In general the fundamental cycle of any resolution can be computed inductively as follows. Let  $F_1, \dots, F_n$  be the exceptional curves.

- (1) Set  $Z = \sum_{i=1}^n F_i$ .
- (2) Check if  $Z \cdot F_j \leq 0$  for every  $j$ .
- (3) If (2) is false, there exists  $j$  such that  $Z \cdot F_j > 0$ . Replace  $Z$  with  $Z + F_j$  and go back to (2).
- (4) Otherwise, if (2) is true,  $Z$  is the fundamental cycle.

In [17] Laufer used essentially the properties of the fundamental cycle in order to describe precisely the relation between the topological types of the canonical and the minimal resolution. However Laufer showed only an implicit formula for  $Z$  (see Lemma 5.5 in [17]). In the next theorem we give an explicit formula for  $Z$ , that turns out to be very simple and that may help to understand better the algorithms described by Laufer in [17, Theorems 5.7 and 5.10]. It is natural to compare the fiber cycle with the fundamental one. The definitions implies that  $F \geq Z$ . It is known that the equality holds for every resolution for special types of singularities, for example rational [2] and minimally elliptic [18, Theorem 3.13] ones. Regarding double points, Dixon showed that  $Z = F$  for every resolution if  $\mu_1 = \text{mult}_{q_1}(B_0)$  is even [10, Theorem 1], and that  $Z = F$  for the minimal resolution if  $B_0$  is analytically irreducible at  $q_1$  [10, Theorem 2]. In section 12 we will classify all the double point singularities for which  $F > Z$ .

**Theorem 11.2** *The fundamental cycle  $Z$  of the canonical resolution  $\tau : X \rightarrow X_0$  of  $p \in X_0$  differs from the fiber cycle  $F = \pi^*(E_1^*)$  if and only if there exists  $j > 1$  such that  $\varepsilon_j = 0$ ,  $q_j$  is infinitely near of order one to  $q_1$  and:*

$$m_{1i} + m_{ji} \text{ is even for every } i \text{ such that } \varepsilon_i = 0. \tag{11.3}$$

*In that case the fundamental cycle is:*

$$Z = \frac{1}{2} \pi^*(E_1^* + E_j^*) = \sum_{i=1}^n \frac{1}{2} (1 + \varepsilon_i) (m_{1i} + m_{ji}) F_i. \tag{11.4}$$

We remark that condition (11.3) implies that  $\varepsilon_1 = 1$ , because  $m_{11} = 1$  and  $m_{j1} = 0$ , thus the multiplicity  $\tilde{\alpha}_1 = \mu_1$  of  $B_0$  at  $q_1$  is odd. Furthermore  $j$  is

uniquely determined. Indeed, if  $q_i >^1 q_1$  (and  $i \neq j$ ), then  $m_{1i} = 1$  and  $m_{ji} = 0$ , so (11.3) would imply that  $\varepsilon_i = 1$ .

**Proof.** By (9.10),  $Z \leq F$ . If  $Z \neq F$ , we may write:

$$F = Z + P \quad (11.5)$$

where  $P = \sum_i t_i F_i$  is a positive (non-zero) divisor. It follows from (9.9) that:

$$-2 = F^2 = Z^2 + P^2 + 2Z \cdot P.$$

Since  $Z > 0$  and  $P > 0$ , then  $Z^2 < 0$  and  $P^2 < 0$ . Moreover  $Z \cdot P \leq 0$  because  $Z$  is the fundamental cycle, so the only possibility is:

$$Z^2 = P^2 = -1, \quad Z \cdot P = 0.$$

Hence  $F \cdot P = -1$  and by formulas (9.10), one finds that:  $-1 = F \cdot P = F \cdot t_1 F_1 = -(2 - \varepsilon_1)t_1$ , which forces  $t_1 = \varepsilon_1 = 1$ , so  $E_1$  must be branched and  $F_1$  is forced to belong to  $P$  with multiplicity one. Since  $Z \cdot F_k \leq 0$  for each  $k$ ,  $Z^2 = -1$  implies that there exists a unique  $j$  such that:

$$Z \cdot F_j = -1, \quad Z \cdot F_k = 0 \quad \text{for every } k \neq j,$$

and  $z_j = 1$ . Therefore  $t_j = -Z \cdot P = 0$ , i.e.,  $F_j$  is not a component of  $P$ , and the coefficient of  $F_j$  in  $F$  is  $(1 + \varepsilon_j)m_{1j} = z_j + t_j = 1$ , so

$$\varepsilon_j = 0 \quad \text{and} \quad m_{1j} = 1,$$

in particular  $j \neq 1$ . It follows from (11.5) that:  $P \cdot F_1 = -1$ ,  $P \cdot F_j = 1$ ,  $P \cdot F_k = 0$  for  $k \neq 1, j$ . The previous three equations are equivalent to:

$$\sum_{i=1}^n (2 - \varepsilon_i) t_i s_{ik} = \begin{cases} -2 & \text{if } k = 1 \\ 1 & \text{if } k = j \\ 0 & \text{if } k \neq 1, j. \end{cases}$$

Recalling that:

$$\sum_{i=1}^n 2m_{1i} s_{ik} = 2(E_1^* \cdot E_k) = \begin{cases} -2 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$$

and setting  $m_1$  and  $t$  row vectors with the obvious entries, one finds that:

$$((2 - \varepsilon)t - 2m_1)S = e_k$$

where  $e_k$  is the row vector with the  $k$ -th entry equal to 1 and 0 everywhere else. Multiplying both sides with  $S^{-1}$ , the vector  $((2 - \varepsilon)t - 2m_1)$  is the  $k$ -th row of the matrix  $S^{-1}$ . In particular:  $2t_j - 2m_{1j} = 0 - 2 = -2$  is the  $(j, j)$ -entry in  $S^{-1} = -M^t M$ . Therefore:

$$-2 = - \sum_{1 \leq i \leq j} m_{ij}^2 = -m_{1j}^2 - m_{jj}^2 - \sum_{1 < i < j} m_{ij}^2 = -2 - \sum_{1 < i < j} m_{ij}^2$$

that is possible if and only if  $m_{ij} = 0$  for every  $1 < i < j$ . This means that  $q_j$  is proximate to  $q_1$ , but  $q_j \not\prec q_i$  for  $i \neq 1$ , i.e.  $q_j >^1 q_1$ . Furthermore, the  $(k, j)$ -entry in  $S^{-1}$  is:

$$(2 - \varepsilon_k)t_k - 2m_{1k} = - \sum_{1 \leq i \leq j} m_{ik}m_{ij} = -m_{1k} - m_{jk},$$

that we may rewrite as follows:  $t_k = \frac{1}{2}(1 + \varepsilon_k)(m_{1k} - m_{jk})$ . Since the coefficient of  $F_k$  in  $F$  is  $(1 + \varepsilon_k)m_{1k} = t_k + z_k$ , then

$$z_k = \frac{1}{2}(1 + \varepsilon_k)(m_{1k} + m_{jk}) \quad (11.6)$$

must be an integer, that proves (11.3) and (11.4).

Vice versa, if there exists  $j$  as in the statement, we may order the blowing-ups  $\sigma_i$  in such a way that  $j = 2$ . It suffices to show that  $Z' = \pi^*(E_1^* + E_2^*)/2$  has the property that  $Z' \cdot F_k \leq 0$  for every  $k = 1, \dots, n$ . Indeed  $Z' < F$ , so the first part of the proof implies that  $Z'$  has to be the fundamental cycle. If  $k > 2$ , then:  $Z' \cdot F_k = (1 - \varepsilon_k/2)(E_1^* \cdot E_k + E_2^* \cdot E_k) = 0$ . Moreover  $Z' \cdot F_2 = E_2^{*2} = -1$  and  $Z' \cdot F_1 = E_1^* \cdot E_1/2 + E_2^* \cdot E_1/2 = 0$ .  $\square$

If  $F > Z$ , the arithmetic genus of  $Z$  is, by  $Z^2 = -1$ , (11.4) and (9.5):

$$\begin{aligned} p_a(Z) &= \frac{E_1^* + E_j^*}{2} \cdot \left( \sigma^* \left( K_{Y_0} + \frac{B_0}{2} \right) - \sum_{k=1}^n \left( \frac{\alpha_k}{2} - 1 \right) E_k^* \right) + \frac{1}{2} = \\ &= \frac{\alpha_1 + \alpha_j - 2}{4} = \frac{\tilde{\alpha}_1 + \tilde{\alpha}_j - 2}{4}. \end{aligned} \quad (11.7)$$

where the last equality follows from  $\alpha_1 = \tilde{\alpha}_1 - 1$  and  $\alpha_j = \tilde{\alpha}_j + 1$ .

Now we compute the fundamental cycle of the minimal resolution.

**Lemma 11.8** *Let  $\bar{\tau} : \bar{X} \rightarrow X_0$  be the minimal resolution of  $p \in X_0$ . Then the fundamental cycle  $\bar{Z}$  of  $\bar{\tau}$  is:*

$$\bar{Z} = \sum_i z_i \bar{F}_i \quad (11.9)$$

where  $\bar{F}_i$  are the exceptional curves of  $\bar{\tau}$  and  $Z = \sum_i z_i F_i$  is the fundamental cycle of the canonical resolution.

**Proof.** Without any loss of generality, we may assume to blow down only a  $(-1)$ -curve  $F_k = \pi^{-1}(E_k)$ , where  $E_k^2 = -2$  and  $\varepsilon_k = 1$ . Recall that  $E_k$  meets at least one and at most three unbranched divisors. First, we shall prove that:

$$\sum_{i \neq k} z_i \bar{F}_i \cdot \bar{F}_j \leq 0. \quad (11.10)$$

for every  $j \neq k$ . For this purpose, we claim that:

$$z_k = \sum_{i: E_i \cdot E_k = 1} z_i. \quad (11.11)$$

Suppose that  $E_k$  meets three unbranched divisors, i.e.,  $q_k = E_{k_1} \cap E_{k_2}$  and we blew up a point  $q_{k_3}$  lying on  $E_k$ , with  $\varepsilon_{k_i} = 0$  for  $i = 1, 2, 3$ . Then  $m_{1k} = m_{1k_3} = m_{1k_1} + m_{1k_2}$ , so:

$$2m_{1k} = (1 + \varepsilon_k)m_{1k} = \sum_{i=1}^3 (1 + \varepsilon_{k_i})m_{1k_i} = m_{1k_1} + m_{1k_2} + m_{1k_3}$$

which proves (11.11) if  $Z = F$ , by (9.8). Similarly,  $2m_{2k} = m_{2k_1} + m_{2k_2} + m_{2k_3}$  and (11.11) holds even if  $Z < F$ , by (11.6). If  $E_k$  meets only one or two unbranched divisors, there are four possible configurations and the proof of (11.11) is analogous.

Clearly (11.10) holds if  $E_j \cdot E_k = 0$ . Otherwise if  $E_j \cdot E_k = 1$ , by (10.4), (10.5) and the fact that if  $E_j \cdot E_k = E_i \cdot E_k = 1$  then  $F_i \cdot F_j = 0$ , formula (11.11) implies that:

$$\begin{aligned} \sum_{i \neq k} z_i \bar{F}_i \cdot \bar{F}_j &= \sum_{i: E_i \cdot E_k = 1} z_i \bar{F}_i \cdot \bar{F}_j + \sum_{i: E_i \cdot E_k = 0} z_i F_i \cdot F_j = \\ &= \sum_{i: E_i \cdot E_k = 1} z_i + \sum_{i \neq k} z_i F_i \cdot F_j = z_k + \sum_{i \neq k} z_i F_i \cdot F_j = Z \cdot F_j \leq 0 \end{aligned}$$

which proves (11.10). Let  $\bar{Z} = \sum_{i \neq k} s_i \bar{F}_i$  be the fundamental cycle of  $\bar{\tau}$ . If we show that for every  $j$ :

$$\left( \sum_{i \neq k} s_i F_i + \sum_{i: E_i \cdot E_k = 1} s_i F_k \right) \cdot F_j \leq 0, \quad (11.12)$$

then  $s_i = z_i$ , for  $i \neq k$ , and (11.9) holds. Indeed if  $E_j \cdot E_k = 0$  then (11.12) is trivial. If  $E_j \cdot E_k = 1$ , then the left hand side of (11.12) becomes:

$$\sum_{i: E_i \cdot E_k = 0} s_i F_i \cdot F_j + \sum_{i: E_i \cdot E_k = 1} s_i F_i \cdot F_j + \sum_{i: E_i \cdot E_k = 1} s_i F_k \cdot F_j = \sum_{i \neq k} s_i \bar{F}_i \cdot \bar{F}_j = \bar{F} \cdot \bar{F}_j \leq 0.$$

Finally, for  $j = k$ , the left hand side of (11.12) is:

$$\sum_{i \neq k} s_i F_i \cdot F_k + \sum_{i: E_i \cdot E_k = 1} s_i F_k^2 = \sum_{i: E_i \cdot E_k = 1} s_i F_i \cdot F_k - \sum_{i: E_i \cdot E_k = 1} s_i = 0. \quad \square$$

**Corollary 11.13** *Let  $F, Z$  (resp.  $\bar{F}, \bar{Z}$ ) be the fiber and the fundamental cycle of the canonical (resp. minimal) resolution. Then  $F > Z$  and  $\bar{F} = \bar{Z}$  if and only if  $q_2$  is the unique proximate point to  $q_1$  and  $\tilde{\alpha}_1 = \tilde{\alpha}_2$  is odd. Furthermore, this happens if and only if  $\bar{F}^2 = \bar{Z}^2 = -1$ .*

**Proof.** Suppose that  $F > Z$  and  $\bar{F} = \bar{Z}$ . This means that  $F_1$  is a  $(-1)$ -curve that we blow down, hence  $E_1^2 = -2$  and there is only one proximate point to  $q_1$ , that is  $q_2$ , so  $\tilde{\alpha}_1 = \tilde{\alpha}_2$ . Moreover  $\tilde{\alpha}_1$  is odd by Theorem 11.2. Conversely, if  $E_1$  is branched and  $E_1^2 = -2$ , then  $F_1$  is a  $(-1)$ -curve that we blow down. Hence



**Theorem 12.2** *The fundamental cycle is strictly contained in the fiber cycle of the canonical resolution of  $p \in X_0$  if and only if the weighted Enriques digraph of  $q_1 = \pi(p)$  has the following properties:*

1.  $q_1$  is a very odd vertex (in particular its weight  $\mu_1$  is odd);
2. a proximate point  $q_i$  to  $q_1$ , belonging to a petal of even (resp. odd) length of the proximity subgraph of  $q_1$ , is a very odd vertex if and only if  $q_i$  is infinitely near of odd (resp. even) order to  $q_1$ ;
3. inductively, 1 and 2 hold replacing  $q_1$  with any very odd vertex.

**Proof.** Suppose that the fundamental cycle  $Z$  is strictly contained in the fiber cycle  $F$ . By Theorem 11.2,  $\mu_1$  is odd and there exists  $j$  such that  $\varepsilon_j = 0$ ,  $q_j >^1 q_1$  and condition (11.3) holds. Moreover we may and will assume that  $j = 2$ , so  $\varepsilon_2 = 0$  and  $\mu_2$  is even.

Consider the proximity subgraph of  $q_1$  as above (cf. formula (12.1) and Figure 2). We claim that  $i_j$  is even for every  $j = 1, \dots, h$ , thus  $q_1$  is a very odd vertex.

Suppose that  $i_1 > 2$ , namely in the canonical resolution process we blow up  $q_3 = E_2 \cap E_1$ . Then  $m_{13} = 2$  and  $m_{23} = 1$ , so condition (11.3) implies that  $\varepsilon_3 = 1$  and  $\mu_3$  is odd. Since  $\varepsilon_1 = \varepsilon_3 = 1$ , the intersection  $q_4 = E_3 \cap E_1$  is a singular point of the branch curve  $B_3$  of  $\pi_3 : X_3 \rightarrow Y_3$ , so we must blow up also  $q_4$ . Similarly, if we blow up  $q_5 = E_4 \cap E_1$ , then  $m_{15} + m_{25} = 5$ , thus condition (11.3) forces  $\varepsilon_5 = 1$  and we must blow up also  $q_6 = E_5 \cap E_1$ . Repeating this argument, it follows that the first petal has odd length and  $i_1$  is even.

Look at the second petal. Now  $q_{i_1+1} >^1 q_1$ , so  $m_{1,i_1+1} = 1$  and  $m_{2,i_1+1} = 0$ . Hence (11.3) implies that  $\varepsilon_{i_1+1} = 1$  and  $\mu_{i_1+1}$  is odd. Therefore we must blow up  $q_{i_1+2} = E_{i_1+1} \cap E_1$ . If  $i_2 > i_1 + 2$ , it means that we blow up also  $q_{i_1+3} = E_{i_1+2} \cap E_1$ , then  $m_{1,i_1+3} + m_{2,i_1+3} = 3$ , so (11.3) forces that  $\varepsilon_{i_1+3} = 1$  and we must blow up  $q_{i_1+4} = E_{i_1+3} \cap E_1$  too. Proceeding in this way, this shows that the second petal has even length and  $i_2$  is even. The same argument works for the  $j$ -th petal, with  $j > 2$ , just by replacing  $i_1$  with  $i_{j-1}$ . This proves our claim that  $q_1$  is a very odd vertex.

Now we want to show that  $\varepsilon_i \equiv i \pmod{2}$  for every  $i = 1, \dots, i_h$ , and  $q_{2l-1}$  is a very odd vertex for every  $l = 2, \dots, i_h/2$ .

We already know that  $\varepsilon_2 = 0$  and  $\varepsilon_{2i-1} = 1$  for every  $i = 1, \dots, i_h/2$ . Suppose by contradiction that  $\varepsilon_4 = 1$  (and  $i_h > 2$ ). Since  $\varepsilon_3 = 1$ , we must blow up also  $q_k = E_4 \cap E_3$  and  $m_{1k} + m_{2k}$  is odd, so condition (11.3) implies that  $\varepsilon_k = 1$ . Hence we must blow up  $q_{k+1} = E_k \cap E_4$  too, and  $m_{1,k+1} + m_{2,k+1}$  is again odd, thus  $\varepsilon_{k+1} = 1$  by (11.3). Going on in this way, we produce each time another branched exceptional curve, so we should never stop blowing up, contradicting Theorem 7.4. This shows that  $\varepsilon_4 = 0$  and  $\mu_4$  is even. The proof that  $\varepsilon_{2l} = 0$  for every  $l = 3, \dots, i_h/2$  is similar.

Consider the proximity subgraph of  $q_{2l-1}$ , for  $l = 2, \dots, i_h/2$ . Repeating exactly the same arguments as for  $q_1$  and  $q_2$ , one finds out that  $q_{2l}$  (which is proximate to  $q_{2l-1}$ ) belongs to a petal of odd length, while all other petals of the proximity subgraph of  $q_{2l-1}$  have even length, thus  $q_{2l-1}$  is a very odd vertex.

It remains to prove that the proximity subgraph of a non-very-odd vertex can be arbitrary. For every  $l = 1, \dots, i_h/2$ , we proved that  $\varepsilon_{2l} = 0$ , so  $q_{2l}$  cannot be very odd. Moreover  $m_{1,2l} + m_{2,2l}$  is even. If  $q_k$  is proximate to  $q_{2l}$  (and

$k \neq 2l + 1$ ), then  $m_{1k} + m_{2k}$  is a multiple of  $m_{1,2l} + m_{2,2l}$ , hence it is even and (11.3) imposes no condition on  $q_k$ . This means that the proximity subgraph of a non-very-odd vertex, as  $q_{2l}$ , can be arbitrary and concludes the proof that the Enriques digraph has properties 1, 2 and 3.

Conversely, suppose that the three properties hold. One may easily check that the  $m_{ij}$ 's satisfy condition (11.3), where the wanted  $q_j$  is the infinitely near point of order one to  $q_1$  belonging to the petal of odd length, therefore one concludes by Theorem 11.2.  $\square$

Note that, with the notation of the proof,  $\tilde{\alpha}_1$  and  $\tilde{\alpha}_2$  are odd, while  $\tilde{\alpha}_i$  is even for every  $i = 3, \dots, i_h$ , by (7.3). Moreover  $\varepsilon_1 = 1$  forces  $\tilde{B} \cdot E_1 = 0$ , or equivalently  $\tilde{\alpha}_1 = \sum_{j=1}^n \tilde{\alpha}_j q_{1j} = \sum_{j=2}^{i_h} \tilde{\alpha}_j$ . By induction on the number  $i_h$  of proximate points to  $q_1$ , it is easy to check that  $\tilde{\alpha}_1 = \tilde{\alpha}_2 + \sum_{j=3}^{i_h} \tilde{\alpha}_j \equiv \tilde{\alpha}_2 \pmod{4}$ , thus  $\tilde{\alpha}_1 + \tilde{\alpha}_2 \equiv 2 \pmod{4}$  (cf. the genus formula (11.7)).

### 13 Some examples

**Example 13.1** Let  $B_0$  be defined by:  $y(y - x^2)(y + x^2) = 0$ . One usually says that  $B_0$  has two infinitely near triple points at  $q_1$ . Our combinatorial data are:

$$\textcircled{3} \longleftarrow \textcircled{4} \quad \text{or equivalently} \quad M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{matrix} \mu = (3, 4), \\ \varepsilon = (1, 0). \end{matrix}$$

The exceptional curves for  $\tau : X \rightarrow X_0$  are a smooth rational curve  $F_1$  with  $F_1^2 = -1$  and a smooth elliptic curve  $F_2$  with  $F_2^2 = -2$  that meet in a point  $P$ . By Theorems 9.7 and 11.2, the fiber cycle of the canonical resolution is  $F = 2F_1 + F_2$ , while the fundamental cycle is  $Z = F_1 + F_2 < F$ , as one may also check directly. Moreover  $F_{2|F_2} = -2P$  by (9.11).

The minimal resolution  $\bar{\tau} : \bar{X} \rightarrow X_0$  is obtained by contracting the  $(-1)$ -curve  $F_1$ . Therefore  $\bar{F}_2$  is the only exceptional curve for  $\bar{\tau}$  and  $\bar{F}_2$  is a smooth elliptic curve with  $\bar{F}_2^2 = -1$ . Clearly the fiber cycle and the fundamental cycle of the minimal resolution  $\bar{\tau}$  are  $\bar{Z} = \bar{F} = \bar{F}_2$ .

The previous example can be generalized as follows.

**Example 13.2** Let  $B_0$  be a curve with  $2k$  infinitely near points  $q_1, \dots, q_{2k}$  of the same odd multiplicity  $\tilde{\alpha}_1 = \dots = \tilde{\alpha}_{2k} = 2g + 1$ , for some  $g \geq 1$ . More precisely,  $q_i >^1 q_{i-1}$  for  $1 < i \leq 2k$  and the weighted Enriques digraph is:



The exceptional curves for  $\tau : X \rightarrow X_0$  are the following:  $(-1)$ -curves  $F_{2i-1}$ , for every  $i = 1, \dots, k$ ; smooth rational curves  $F_{2i}$  with self-intersection  $-4$ , for  $i = 1, \dots, k-1$ , and a smooth curve  $F_{2k}$  of genus  $g$  with  $F_{2k}^2 = -2$ . By Theorems 9.7 and 11.2, the fiber cycle and the fundamental cycle of the canonical resolution are respectively:

$$F = \sum_{i=1}^k (2F_{2i-1} + F_{2i}), \quad Z = F_1 + F_2 + \sum_{i=2}^k (2F_{2i-1} + F_{2i}).$$

Blow down the  $F_{2i-1}$ 's, for  $i = 1, \dots, k$ , thus the exceptional curves for the minimal resolution  $\bar{\tau} : \bar{X} \rightarrow X_0$  are the  $\bar{F}_{2i}$ 's, for  $i = 1, \dots, k$ , which are smooth rational curves with self-intersection  $-2$ , except  $\bar{F}_{2k}$  which is smooth of genus  $g$  with  $\bar{F}_{2k}^2 = -1$ . The fundamental cycle equals the fiber cycle of the minimal resolution  $\bar{Z} = \bar{F} = \sum_{i=1}^k \bar{F}_{2i}$ .

**Example 13.3** (cf. [19, p. 322]) Let  $B_0$  be defined by:  $y(x^4 + y^6) = 0$ . In this case our combinatorial data are:

$$M = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} \mu &= (5, 2, 3, 4), \\ \varepsilon &= (1, 0, 1, 0). \end{aligned}$$

The fiber cycle of the canonical resolution is  $F = 2F_1 + F_2 + 2F_3 + 2F_4$ , while the fundamental cycle is  $Z = F_1 + F_2 + F_3 + F_4 < F$ , as it should be by Theorem 12.2, because  $q_1$  is a very odd vertex, its proximity digraph has just two petals (one of length 1 and the other of length 2) and  $q_3$  is also very odd. The minimal resolution is obtained by blowing down  $F_3$ , therefore:

$$\bar{F} = 2\bar{F}_1 + \bar{F}_2 + 2\bar{F}_4 > \bar{F}_1 + \bar{F}_2 + \bar{F}_4 = \bar{Z}.$$

*Rational double points* (see [2], [11] and [4]). It is very well-known that the rational double points are given by the following equations:

$$\begin{aligned} A_n &: z^2 = x^2 + y^{n+1}, & (n \geq 1), \\ D_n &: z^2 = y(x^2 + y^{n-2}), & (n \geq 3), \\ E_6 &: z^2 = x^3 + y^4, \\ E_7 &: z^2 = x(x^2 + y^3), \\ E_8 &: z^2 = x^3 + y^5 \end{aligned}$$

and the minimal resolution consists in smooth rational curves of self-intersection  $-2$  whose dual graph is the corresponding Dynkin diagram.

Note that, for a rational double point, the fundamental cycle  $Z$  of the canonical resolution equals the fiber cycle  $F$ . Indeed if  $Z < F$ , then formula (11.7) says that  $p_a(Z) = (\tilde{\alpha}_1 + \tilde{\alpha}_2 - 2)/4$ , where  $\tilde{\alpha}_1$  is odd and  $\tilde{\alpha}_1 \geq 2$ , so  $p_a(Z) > 0$ , contradicting Artin's criterion (which says that  $p_a(Z) = 0$  if and only if the singularity is rational, cf. [2, Theorem 3]).

Moreover, starting from the well-known formula for the arithmetic genus of a sum of two curves, that is  $p_a(C + D) = p_a(C) + p_a(D) + (C \cdot D) - 1$ , and using (9.3) and (9.4) we find out that (cf. (9.12)):

$$p_a(Z) = p_a(F) = \frac{1}{2} \sum_{i=1}^n m_{1i}(\gamma_i - (\varepsilon_i - 2)E_i^2 - 4).$$

Hence we can see directly that  $p_a(Z) = 0$  if and only if every branched exceptional divisor has self-intersection  $-4$ , every unbranched exceptional divisor has self-intersection  $-1$  and  $\gamma_i = 2$ , or self-intersection  $-2$  and  $\gamma_i = 0$ . Thus every  $F_i$  (if

$F_i$  splits, every irreducible component of  $F_i$ ) is rational with self-intersection  $-2$  and the canonical resolution is minimal.

## 14 Adjunction conditions

We want to study the conditions that a double point singularity  $p \in X_0$  imposes to canonical and pluricanonical systems of a surface. Recall that locally  $p$  is  $\pi_0^{-1}(q_1)$ , where  $\pi_0 : X_0 \rightarrow Y_0$  is a double cover,  $Y_0$  is a smooth surface,  $X_0$  is normal and  $q_1$  is an isolated singular point of the branch curve  $B_0$  of  $\pi_0$ . Then we consider the canonical resolution  $\pi : X \rightarrow Y$ , that is a double cover branched along the smooth curve  $B$ .

In (9.6) we defined the adjunction condition divisor  $D^*$  as the pullback of

$$D = \sum_i (\alpha_i/2 - 1)E_i^* = \sigma^*(K_{Y_0} + B_0/2) - (K_Y + B/2). \quad (14.1)$$

So it suffices to understand what are the conditions that  $D$  imposes to the adjoint linear system  $|K_Y + B/2|$ . It is well known that  $D = 0$ , or equivalently  $D^* = 0$ , if and only if  $p \in X_0$  is a rational double point (cf. previous section).

By applying  $\sigma_*$  to the exact sequence  $0 \rightarrow \mathcal{O}_Y(-D) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_D \rightarrow 0$ , one sees that  $\mathcal{I}_\Gamma := \sigma_*\mathcal{O}_Y(-D)$  is the ideal sheaf of a zero-dimensional scheme  $\Gamma$  supported at  $q_1 \in Y_0$ . Let us call  $\mathcal{I}_\Gamma$  the *adjoint ideal* of the singularity.

For our convenience, let us assume that  $X_0$  is a double plane, i.e.,  $Y_0 = \mathbb{P}^2$ . Indeed the double point singularity is locally given as a double cover of an open disc, thus we may always find an irreducible plane curve  $B_0$  of arbitrarily high (and even) degree whose germ at  $q_1$  is analitically isomorphic to the germ of the branch curve of the double cover at  $q_1$ .

By (3.2), (3.3) and the projection formula we have that  $\pi_*K_X \cong K_Y \oplus (K_Y + B/2)$ , so  $p_g(X) = h^0(X, K_X) = h^0(Y, K_Y + B/2)$  and  $q(X) = h^1(X, K_X) = h^1(Y, K_Y + B/2)$ . Riemann–Roch Theorem for  $K_Y + B/2$  on  $Y$  and for  $K_{\mathbb{P}^2} + B_0/2$  on  $\mathbb{P}^2$  implies that:

$$\begin{aligned} h^0(K_X) - h^1(K_X) &= B \cdot (K_Y + B/2)/4 + 1, \\ h^0(K_{X_0}) &= h^0(K_{\mathbb{P}^2} + B_0/2) = B_0 \cdot (K_{\mathbb{P}^2} + B_0/2)/4 + 1. \end{aligned}$$

It follows from (9.5) that

$$h^0(K_{X_0}) - h^0(K_X) + h^1(K_X) = \sum_{i=1}^n \frac{\alpha_i(\alpha_i - 2)}{8} =: c, \quad (14.2)$$

where  $c$  is defined by (14.2). Let us recall a well-known theorem of De Franchis:

**Theorem 14.3 (De Franchis)** *Let  $\pi_0 : X_0 \rightarrow \mathbb{P}^2$  be a double plane and  $\pi : X \rightarrow Y$  its canonical resolution. Then  $q(X) > 0$  if and only if there is a plane curve  $B'$  (possibly  $B' = 0$ ) such that:*

$$B_0 + 2B' = C_1 + C_2 + \cdots + C_m \quad (14.4)$$

where  $C_1, \dots, C_m$  are curves belonging to one and the same pencil and  $m = 2q(X) + 2$  (resp. possibly  $m = 2q(X) + 1$  if the pencil contains a double curve).

**Proof.** See [9] (or [8] for a modern proof).  $\square$

**Corollary 14.5** *With the above notation, the number  $h^0(K_{X_0}) - h^0(K_X)$  of conditions that the singularity  $p \in X_0$  imposes to the canonical system is:*

$$c = \sum_{i=1}^n \frac{\alpha_i(\alpha_i - 2)}{8} = \sum_{i=1}^n \frac{\alpha_i/2(\alpha_i/2 - 1)}{2}. \quad (14.6)$$

**Proof.** Since we assumed  $B_0$  to be irreducible, then  $q(X) = h^1(K_X) = 0$  by De Franchis' Theorem and (14.6) follows from (14.2).  $\square$

We remark that De Franchis' Theorem allows us to compute the adjunction conditions even if  $B_0$  were a given reducible curve and its degree were not assumed to be arbitrarily high.

We want to determine which singularity the general element  $C$  in  $|\mathcal{I}_\Gamma(h)| = |\sigma_* \mathcal{O}_Y(K_Y + B/2)| = |\sigma_*(\sigma^*(\mathcal{O}_{\mathbb{P}^2}(hL)) \otimes \mathcal{O}_Y(-D))|$  has at  $q_1$ , where  $L$  is a general line in  $\mathbb{P}^2$  and  $h = \deg(B_0)/2 - 3$ . According to formulas (14.1) and (14.6), one might expect that  $C$  has exactly multiplicity  $\alpha_i/2 - 1$  at  $q_i$ , for every  $i = 1, \dots, n$ . The next example shows that this is not always the case.

Suppose that  $q_1 \in B_0$  is the same singularity of Example 13.1. Since  $\alpha = \mu - \varepsilon$ , we have  $\alpha_1 = 2$  and  $\alpha_2 = 4$ , thus one expects the general element  $C$  in  $|\mathcal{I}_\Gamma(h)|$  to pass simply through  $q_2$  and not to pass through  $q_1$ . But this is not possible, because  $q_2$  is infinitely near to  $q_1$ .

Actually, we see that  $(K_Y + B/2) \cdot E_1 < 0$  and  $E_1$  is a fixed component of  $|K_Y + B/2| = |\sigma^*(hL) - E_2|$ . Moreover

$$K_Y + B/2 - E_1 = \sigma^*(hL) - 1 \cdot E_1^* - 0 \cdot E_2^* \quad (14.7)$$

meets non negatively  $E_1$  and  $E_2$ , so  $|K_Y + B/2 - E_1|$  has no fixed components by the next Lemma 14.8. Formula (14.7) means that  $C$  passes simply through  $q_1$  and does not pass through  $q_2$  (which is 1 adjunction condition as well).

**Lemma 14.8** *Let  $\mathcal{L}$  be a linear system on  $Y$  which we write as:*

$$\mathcal{L} = \left| \sigma^*(hL) - \sum_{i=1}^n m_i E_i^* \right|,$$

where  $L$  is a general line in  $\mathbb{P}^2$ ,  $m_i$  are non-negative integers and  $h$  is arbitrarily high. Suppose that  $\deg \mathcal{L}|_{E_i} \geq 0$ , for every  $i = 1, \dots, n$ . Then  $\mathcal{L}$  has no fixed component. In particular the general member of  $\sigma_* \mathcal{L}$  is a plane curve with multiplicity exactly  $m_i$  at  $q_i$ , for  $i = 1, \dots, n$ .

**Proof.** Since  $h \gg 0$ , we may assume that the only possible fixed components of  $\mathcal{L}$  are among the  $E_i$ 's. For every  $i = 1, \dots, n$ , consider the exact sequence  $0 \rightarrow \mathcal{L}(-E_i) \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_{E_i} \rightarrow 0$ . We need to show that  $h^0(\mathcal{L}(-E_i)) < h^0(\mathcal{L})$ . This will follow from  $H^1(\mathcal{L}(-E_i)) = 0$ , because  $H^0(\mathcal{L}|_{E_i}) \neq 0$  by assumption. We claim that  $R^1 \sigma_* \mathcal{L}(-E_i) = 0$ . This will imply that  $H^1(\mathcal{L}(-E_i)) = H^1(\sigma_* \mathcal{L}(-E_i)) = 0$ ,

where the last equality follows from Serre's Theorem, because  $h \gg 0$ , and we will be done. Indeed  $H^1(\mathcal{L}(-E_i)|_{E_1^*}) = 0$ , since  $E_1^*$  is 1-connected,  $p_a(E_1^*) = 0$  and  $\deg \mathcal{L}(-E_i)|_{E_1^*} \geq m_1 \geq 0$ .  $\square$

The above discussion suggested us to introduce the following notion: we say that a point  $q_i$  is *defective* if there exists a point  $q_j$  such that  $\alpha_j > \alpha_i$  and  $q_j$  is infinitely near of order one to  $q_i$ . Hence, if  $q_i$  is defective, then  $B \cdot E_i < 0$ , while we know that  $\tilde{B} \cdot E_i \geq 0$  for every  $i = 0, \dots, n$ , because  $\tilde{B}$  is the proper transform of a plane curve. In Example 13.1 (recalled before the previous lemma),  $q_1$  is defective, because  $q_2 >^1 q_1$  and  $4 = \alpha_2 > \alpha_1 = 2$ .

**Lemma 14.9** *A point  $q_i$  is defective if and only if  $D \cdot E_i > 0$ . More precisely, a point  $q_i$  is defective if and only if  $\varepsilon_i = 1$  and there exists a (necessarily unique) point  $q_j >^1 q_i$  with  $\tilde{\alpha}_j = \tilde{\alpha}_i$  and  $\varepsilon_j = 0$ . Furthermore, either:*

- (i)  $\alpha_i = \tilde{\alpha}_i - 1$ , or
- (ii)  $\alpha_i = \tilde{\alpha}_i$  and both  $q_i, q_j$  are proximate to a point  $q_k$  with  $\varepsilon_k = 1$ .

Finally a point  $q_i$  is defective if and only if  $F_i$  is a  $(-1)$ -curve.

**Proof.** The last statement follows easily from the other ones and Remark 10.3. By definition, if  $q_i$  is defective, then  $D \cdot E_i \geq -(B \cdot E_i)/2 > 0$ . Conversely,  $D \cdot E_i \leq 0$  is equivalent to  $(B + 2K_Y) \cdot E_i \geq 0$ , that holds, if  $q_i$  is not defective, by Lemma 4.3 in [6]. This proves the first statement.

Note that if there is a point  $q_j >^1 q_i$  with  $\tilde{\alpha}_j = \tilde{\alpha}_i$ , then  $q_j$  is the unique proximate point to  $q_i$ .

Suppose that  $q_i >^1 q_k$  and  $q_k$  is the only point which  $q_i$  is proximate to. Then  $\alpha_i = \tilde{\alpha}_i + \varepsilon_k - \varepsilon_i$  by (7.3). Let  $q_j$  be an infinitely near point of order one to  $q_i$ . If  $q_j$  is proximate only to  $q_i$ , then  $\alpha_j = \tilde{\alpha}_j + \varepsilon_i - \varepsilon_j$ , thus  $\alpha_j \geq \alpha_i + 2$  if and only if

$$(\tilde{\alpha}_i - \tilde{\alpha}_j) + \varepsilon_k + \varepsilon_j + 2 \leq 2\varepsilon_i,$$

which (recalling that  $\tilde{\alpha}_i \geq \tilde{\alpha}_j$  because  $\tilde{B} \cdot E_i \geq 0$ ) holds only if  $\tilde{\alpha}_i - \tilde{\alpha}_j = \varepsilon_k = \varepsilon_j = 0$  and  $\varepsilon_i = 1$ , that is case (i).

If  $q_j$  is proximate also to  $q_k$ , then  $\alpha_i = \tilde{\alpha}_j + \varepsilon_i + \varepsilon_k - \varepsilon_j$ , hence  $\alpha_j \geq \alpha_i + 2$  if and only if  $(\tilde{\alpha}_i - \tilde{\alpha}_j) + \varepsilon_j + 2 \leq 2\varepsilon_i$ , that is either case (i) or (ii) depending on the value of  $\varepsilon_k$ .

This concludes the proof in case  $q_i$  is proximate to only one point. One may proceed similarly for the other configurations of  $q_j >^1 q_i$ , namely if  $q_i$  is not infinitely near to any point or if  $q_i$  is proximate to more than one point.  $\square$

Both of the cases (i) and (ii) of Lemma 14.9 may occur, as the point  $q_1$  in Example 13.1 and the point  $q_3$  in Example 13.3 respectively show.

We remark that if  $q_i$  is defective and  $q_j$  is as above, namely  $q_j >^1 q_i$  and  $\alpha_j > \alpha_i$ , then  $q_j$  cannot be defective. However there may exist a defective point  $q_l$  with  $q_l >^1 q_j$  and  $\alpha_l = \alpha_i$ , as  $q_3, q_5, \dots, q_{2k-1}$  in Example 13.2.

We say that a point  $q_i$  is *1-defective*, and we write  $\text{def}(q_i) = 1$ , if  $q_i$  is defective and there is no defective point  $q_j > q_i$  with  $\alpha_j = \alpha_i$ . Inductively, we say that  $q_i$

is  $k$ -defective, and we write  $\text{def}(q_i) = k$ , if there exists a  $(k-1)$ -defective point  $q_j >^2 q_i$  with  $\alpha_j = \alpha_i$ .

In Example 13.2, the point  $q_1$  is  $k$ -defective. We set  $\text{def}(q_i) = 0$  if  $q_i$  is not defective and  $\text{Def} = \{i \mid \text{def}(q_i) > 0\}$ . Thus  $i \in \text{Def}$  if and only if  $q_i$  is defective.

Now we are ready to show what exactly happens to an element in  $|\mathcal{I}_\Gamma(h)|$  at a defective point. To simplify the notation, by ordering conveniently the blowing-ups, we may and will assume that if  $q_j, q_k$  are defective, with  $\alpha_j = \alpha_k$  and  $q_k >^2 q_j$ , then  $k = j + 2$  and  $q_{j+2} >^1 q_{j+1} >^1 q_j$ .

**Theorem 14.10** *The fixed part of  $|K_Y + B/2|$  is exactly:*

$$\bar{E} = \sum_{j \in \text{Def}} \sum_{r=0}^{\text{def}(q_j)-1} E_{j+r}. \quad (14.11)$$

**Proof.** By Lemma 14.9, the non-defective points do not mind, so we may focus only on what happens at a defective point. Any  $k$ -defective point looks like the point  $q_1$  of Example 13.2, thus we will assume that  $q_1$  is  $k$ -defective and we will follow the notation of that example. Recall that, for every  $i = 1, \dots, k$ , the point  $q_{2i-1}$  is  $(k-i+1)$ -defective,  $\alpha_{2i}/2 - 1 = g$  and  $\alpha_{2i-1}/2 - 1 = g - 1$ .

We claim that the fixed part of  $|K_Y + B/2|$  is exactly:

$$\bar{E} = \sum_{l=0}^{k-1} \sum_{i=1}^{k-l} E_{2i+l-1} = \sum_{j \in \text{Def}} \sum_{r=0}^{\text{def}(q_j)-1} E_{j+r}, \quad (14.12)$$

and formula (14.11) clearly follows. Note that (14.12) means that the general element of  $|\mathcal{I}_\Gamma(h)|$  has multiplicity  $g$  at  $q_1, \dots, q_k$  and  $g-1$  at  $q_{k+1}, \dots, q_{2k}$ , giving  $kg^2$  adjunction conditions as expected.

Now we prove our claim. By Lemma 14.9, the exceptional curve  $E_{2i-1}$  is a fixed component of  $|K_Y + B/2|$ , for  $i = 1, \dots, k$ . Then

$$D + \sum_{i=1}^k E_{2i-1} = \sum_{j=1}^k (gE_{2j-1}^* + (g-1)E_{2j}^*)$$

meets positively  $E_{2j}$  for  $j = 1, \dots, k-1$ , which therefore are fixed components of  $|K_Y + B/2|$  too. Now

$$D + \sum_{i=1}^k E_{2i-1} + \sum_{j=1}^{k-1} E_{2j} = gE_1^* + \sum_{l=1}^{k-1} (gE_{2l}^* + (g-1)E_{2l+1}^*) + (g-1)E_{2k}^*$$

meets again positively  $E_{2i-1}$ , for  $i = 2, \dots, k$  (but not  $E_1$  and  $E_{2k}$ ).

Going on in this way, by induction on  $k$ , it follows that  $|K_Y + B/2|$  contains (14.12). On the other hand,  $D + \bar{E} = \sum_{i=1}^k (gE_i^* + (g-1)E_{k+i}^*)$  does not meet positively anyone of the  $E_i$ 's, thus the fixed part of  $|K_Y + B/2|$  is exactly  $\bar{E}$ , by Lemma 14.8, and our claim is proved.  $\square$

We remark that the previous theorem gives an alternative proof of Corollary 14.5, independent from De Franchis' Theorem.

Finally we want to compute the number of conditions that the singularity  $p \in X_0$  imposes to pluricanonical systems. The plurigenera of  $X$  are:

$$P_m(X) = h^0(X, mK_X) = h^0(Y, mK_Y + mB/2) + h^0(Y, mK_Y + (m-1)B/2).$$

Riemann–Roch Theorem and (9.5) imply that:

$$h^0(mK_{X_0}) - h^0(mK_X) + h^1(mK_X) = \sum_{i=1}^n \frac{2(m^2 - m)(\alpha_i - 2)^2 + \alpha_i^2 - 2\alpha_i}{8}.$$

**Theorem 14.13** *The number of conditions  $h^0(mK_{X_0}) - h^0(mK_X)$  that the singularity  $p \in X_0$  imposes to the  $m$ -canonical system are:*

$$\sum_{i=1}^n \frac{2(m^2 - m)(\alpha_i - 2)^2 + \alpha_i^2 - 2\alpha_i}{8} - \frac{dm(m-1)}{2} \quad (14.14)$$

where  $d := \sharp \text{Def}$  is the number of defective points.

**Proof.** We shall show that  $h^1(mK_X) = dm(m-1)/2$ . Since we are dealing with local questions, we may assume that the  $(-1)$ -curves of  $X$  are contained in  $\tau^{-1}(p)$ . Recall that these  $(-1)$ -curves are disjoint and there are exactly  $d$  of them. Let  $\tau' : X \rightarrow \bar{X}$  be their contraction (see section 10), then  $\bar{X}$  is a minimal surface of general type and we may assume that  $h^0(K_{\bar{X}}) \gg 0$  (because  $h$  is arbitrarily high). With no loss of generality, we may also assume that  $\tau'$  is the blowing-down of just a  $(-1)$ -curve  $F_i$ . Thus it suffices to show that, under these assumptions,  $h^1(mK_X) = m(m-1)/2$ . By Serre duality,  $h^1(mK_X) = h^1(-(m-1)K_X)$ . Let  $C$  be a curve in  $|(m-1)K_X|$ . Clearly  $C = \tau'^*(C_0) + (m-1)F_i$ , where  $C_0 \in |(m-1)K_{\bar{X}}|$ . It is well-known that  $C_0$  and  $\tau'^*(C_0)$  are 1-connected (see [4, Proposition 6.1] and [14, §1]), therefore  $h^0(\mathcal{O}_{\tau'^*(C_0)}) = 1$ . Since  $q(X) = h^1(\mathcal{O}_X) = 0$ , the exact sequence of sheaves  $0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$  implies that  $h^1(\mathcal{O}_X(-C)) = h^0(\mathcal{O}_{(m-1)F_i})$ . Finally one easily checks that  $h^0(\mathcal{O}_{(m-1)F_i}) = m(m-1)/2$ .  $\square$

We remark that, if  $h$  is not assumed to be arbitrarily high,  $\bar{X}$  may not be of general type and one should compute, or estimate,  $h^1(mK_X)$ .

As we did for  $|K_Y + B/2|$ , we want to determine the fixed components of  $|mK_Y + mB/2|$  and  $|mK_Y + (m-1)B/2|$ . After having ordered the blowing-ups as explained just before Theorem 14.10, we are ready to prove the following:

**Theorem 14.15** *The fixed part of  $|mK_Y + \bar{m}B/2|$ , for  $\bar{m} = m$  or  $\bar{m} = m-1$ , is exactly:*

$$\left[ \frac{\bar{m}}{2} \right] \sum_{j \in \text{Def}} E_j + (\bar{m} \bmod 2) \bar{E}, \quad (14.16)$$

where  $\bar{E}$  is (14.11),  $[\bar{m}/2]$  is the largest integer smaller than or equal to  $\bar{m}/2$  and  $\bar{m} \bmod 2 = \bar{m} - 2[\bar{m}/2] \in \{0, 1\}$ .  $\square$

**Proof.** As in the proof of Theorem 14.10, it suffices to understand what happens at a defective point, thus we assume that  $q_1$  is a  $k$ -defective point as in Example 13.2. Let us set

$$\tilde{E} = \sum_{i=1}^k E_{2i-1} = \sum_{j \in \text{Def}} E_j.$$

Suppose that  $\bar{m} = m$ . If  $m$  is even, then

$$mD = \sum_{i=1}^k (m(g-1)E_{2i-1}^* + mgE_{2i}^*),$$

hence  $mD$  meets positively  $E_{2i-1}$  for every  $i = 1, \dots, k$ . Moreover  $(mD + j\tilde{E}) \cdot E_{2i-1} > 0$  for every  $i = 1, \dots, k$  and  $j = 1, \dots, m/2 - 1$ . This means that  $m\tilde{E}/2$  is a fixed component of  $|mK_Y + mB/2|$ . Then

$$mD + \frac{m}{2}\tilde{E} = \sum_{i=1}^{2k} \left( mg - \frac{m}{2} \right) E_i$$

does not meet positively anyone of the  $E_i$ 's, therefore the fixed part of  $|mK_Y + mB/2|$  is exactly  $m\tilde{E}/2$ , by Lemma 14.8.

If  $m$  is odd, following the same argument, one easily shows that the fixed part of  $|mK_Y + mB/2|$  is  $(m-1)\tilde{E}/2 + \bar{E}$ .

One proceeds similarly in the case that  $\bar{m} = m - 1$ . Indeed, one can prove that the fixed part of  $|mK_Y + (m-1)B/2|$  is  $(m-1)\tilde{E}/2$  (resp.  $m\tilde{E}/2 + \bar{E}$ ) if  $m$  is odd (resp. if  $m$  is even).  $\square$

Note that Theorem 14.13 can be proved also as corollary of Theorem 14.15.

## References

- [1] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces, *Amer. J. Math.* **84** (1962), 485–496.
- [2] M. Artin, On isolated rational singularities of surfaces, *Amer. J. Math.* **88** (1966), 129–136.
- [3] C. Ban, L.J. McEwan and A. Nemethi, The Embedded Resolution of  $f(x, y) + z^2 : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ , preprint, math.AG n. 9911187.
- [4] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, *Ergebnisse der Mathematik*, 3. Folge, Band 4, Springer, Berlin, 1984.
- [5] A. Calabri, *Sulla razionalità dei piani doppi e tripli ciclici*, Tesi di dottorato (X ciclo), Università di Roma “La Sapienza”, 1999.
- [6] A. Calabri, On rational and ruled double planes, to appear on *Annali di Matematica Pura ed Applicata*.
- [7] E. Casas-Alvero, *Singularities of plane curves*, London Math. Soc. Lecture Note Series 276, Cambridge University Press, 2000.
- [8] F. Catanese and C. Ciliberto, On the irregularity of cyclic covers of algebraic surfaces, in *Geometry of Complex Projective Varieties*, Seminars and Conferences 9, Mediterranean Press, 1993, 89–115.

- [9] M. De Franchis, I piani doppi dotati di due o più differenziali totali di prima specie, *Rend. Accad. Lincei* **13** (1904), 688–695.
- [10] D. Dixon, The fundamental divisor of normal double points of surfaces, *Pacific J. Math.* **8** (1979), 105–115.
- [11] A.H. Durfee, Fifteen characterizations of rational double points and simple critical points, *Enseign. Math.*, ser. 2, **25** (1979), 131–163.
- [12] R. Ferraro, *Curve di genere massimo in  $\mathbb{P}^5$* , and *Explicit Resolutions of Double Point Singularities of Surfaces*, Tesi di dottorato (IX ciclo), Università di Roma “Tor Vergata”, 1998.
- [13] A. Franchetta, Sui punti doppi isolati delle superficie algebriche. Nota I, and Nota II, *Atti. Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur.*, ser. VIII, **1** (1946), 49–57 and 162–168.
- [14] A. Franchetta, Sulle curve riducibili appartenenti ad una superficie algebrica, *Rend. Mat. Appl., Univ. Roma*, ser. 5, **8** (1949), 378–398.
- [15] S. Kleiman and R. Piene, Enumerating singular curves on surfaces, in *Algebraic Geometry: Hirzebruch 70 (Warsaw, 1998)*, Contemporary Mathematics 241, American Mathematical Society, Providence, RI, 1999, 209–238.
- [16] E. Horikawa, On deformations of quintic surfaces, *Invent. Math.* **31** (1975), 43–85.
- [17] H. Laufer, *Normal two-dimensional singularities*, Annals of Mathematical Studies 71, Princeton University Press, Princeton, 1971.
- [18] H. Laufer, On minimally elliptic singularities, *Amer J. Math.* **99** (1977), 1257–1295.
- [19] H. Laufer, On normal two-dimensional double point singularities, *Israel J. Math.* **31** (1978), 315–334.
- [20] U. Persson, Double covers and surfaces of general type, in *Algebraic Geometry. Proceedings, Tromsø, Norway, 1977*, ed. L.D. Olson, Lecture Notes in Mathematics 687, Springer, Berlin, 1978, 168–195.
- [21] M. Reid, Chapters on Algebraic Surfaces, in *Complex Algebraic Geometry. Lectures of a summer program, Park City, UT, 1993*, ed. Kollár J., IAS/Pak City Mathematics Series 3, American Mathematical Society, Providence, RI, 1997, 5–159.
- [22] J. Roè, Varieties of clusters and Enriques diagrams, preprint, math.AG n. 0108023.