# QUANTUM LINES FOR DUAL QUASI-BIALGEBRAS 

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#### Abstract

In this paper, the theory to construct quantum lines for general dual quasi-bialgebras is developed followed by some specific examples where the dual quasi-bialgebras are pointed with cyclic group of points.


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## 1. Introduction

For $H$ a bialgebra over a field $\mathbb{k}$ and $R$ a bialgebra in the category of Yetter-Drinfeld modules ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$, the Radford biproduct or bosonization $R \# H$ is a well-known construction giving a new bialgebra. Similarly, if $H$ is a dual quasi-bialgebra and $R$ is a bialgebra in a suitably defined category of Yetter-Drinfeld modules over $H$, then by [AP], a new dual quasi-bialgebra $R \# H$ can be defined, called the bosonization of $R$ and $H$.

Given a bialgebra $H$, however, finding $R$ is nontrivial. For $H=\mathbb{k} G$, a group algebra, finding $R$ is the key to constructing pointed bialgebras with $G$ as the group of points and finding finite dimensional $R$ is crucial to the classification of pointed Hopf algebras of finite dimension. If $R$ is finite dimensional and is generated as an algebra by a one-dimensional vector space in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$, then $R$ is called a quantum line. For various $H$ semisimple of even dimension, not necessarily group algebras, the question of the existence of a quantum line for $H$ was completely settled in CDMM, as well as the question of liftings of the bosonizations.

In this paper, we adapt the methods and language of CDMM to dual quasi-bialgebras. We find necessary and sufficient conditions for quantum lines to exist for a given dual quasi-bialgebra $H$ and we compute several examples for the dual quasi-bialgebra $(H, \omega)$ where $H$ is the group algebra

[^0]of a cyclic group of any order and $\omega$ is a 3 -cocycle. Finally we give an example of the existence of a quantum line for a bosonization $R \# \mathbb{k} C_{n}$ where $n$ is an even integer.

The duals of our examples will be quasi-bialgebras as studied in the papers of Angiono [An], Gelaki Ge, and Etingof and Gelaki EGe1 EGe2 EGe3.

The first section of this paper is used for notation and some preliminary material. In the second section, we define quasi-Yetter-Drinfeld data for dual quasi-bialgebras, and then in the next section we construct quantum lines. Section 5 discusses conditions to construct a quantum line for a bosonization and then the last section gives examples of these constructions. The examples are based on knowledge of the dual quasi-bialgebra $\mathbb{k} C_{n}$ with Drinfeld reassociator given by a nontrivial 3-cocycle.

## 2. Preliminaries

Throughout we work over $\mathbb{k}$, an algebraically closed field of characteristic zero. The tensor product over $\mathbb{k}$ will be denoted by $\otimes$. Vector spaces, algebras and coalgebras are all understood to be over $\mathbb{k}$ and all maps are understood to be $\mathbb{k}$-linear. The usual twist map from the tensor space $V \otimes W$ to $W \otimes V$ will be denoted $\tau$, i.e., $\tau(v \otimes w)=w \otimes v$. The multiplicative group of nonzero elements of $\mathbb{k}$ is denoted by $\mathbb{k}^{\times}$.

For any coalgebra $C$ and algebra $A, *$ will denote the convolution product in $\operatorname{Hom}(C, A)$. Composition of functions may be written as concatenation if the emphasis of the symbol $\circ$ is not required. The tensor product of a map with itself will often be written exponentially, i.e., we will write $\phi^{\otimes 3}$ to denote $\phi \otimes \phi \otimes \phi$. Similarly $H \otimes H$ is denoted $H^{\otimes 2}$, etc.

The group algebra over a group $G$ will be written $\mathbb{k} G$. The set of grouplike elements of a coalgebra $C$ will be denoted $G(C)$ and the subcoalgebra generated by $G(C)$ will be denoted $\mathbb{k} G(C)$.

We make the convention that an empty product, for example, a product of the form $\prod_{1 \leq j \leq a}$ with $a<1$, is defined to be 1 .
2.1. Definitions. A coalgebra with multiplication and unit is a datum $(H, \Delta, \varepsilon, m, u)$ where $(H, \Delta, \varepsilon)$ is a coalgebra, $m: H \otimes H \rightarrow H$ is a coalgebra homomorphism called multiplication and $u: \mathbb{k} \rightarrow H$ is a coalgebra homomorphism called unit Ka, dual to page 368]. Denote $u\left(1_{\mathbb{k}}\right)$ by $1_{H}$.

For $H$ a coalgebra with multiplication and unit, a convolution invertible map $\omega: H^{\otimes 3} \rightarrow \mathbb{k}$ is called a 3-cocycle if and only if

$$
\begin{equation*}
(\varepsilon \otimes \omega) * \omega(H \otimes m \otimes H) *(\omega \otimes \varepsilon)=\omega(H \otimes H \otimes m) * \omega(m \otimes H \otimes H) \tag{1}
\end{equation*}
$$

and we say that a cocycle $\omega$ is unitary or normalized if for all $h, h^{\prime} \in H$,

$$
\begin{equation*}
\omega\left(h \otimes 1_{H} \otimes h^{\prime}\right), \text { or equivalently either } \omega\left(1 \otimes h \otimes h^{\prime}\right) \text { or } \omega\left(h \otimes h^{\prime} \otimes 1\right), \text { is } \varepsilon(h) \varepsilon\left(h^{\prime}\right) \tag{2}
\end{equation*}
$$

If $H, L$ are coalgebras with multiplication and unit, a coalgebra map $\phi: L \rightarrow H$ is a morphism of coalgebras with multiplication and unit, if

$$
m_{H}(\phi \otimes \phi)=\phi m_{L}, \quad \phi u_{L}=u_{H}
$$

If $\phi: L \rightarrow H$ is a morphism of coalgebras with multiplication and unit and $\omega$ is a (normalized) 3-cocycle for $H$, then $\omega \circ \phi^{\otimes 3}$ is a (normalized) 3-cocycle for $L$.

For $H$ a coalgebra with unit, a convolution invertible map $v: H^{\otimes 2} \rightarrow \mathbb{k}$ such that $v(1 \otimes h)=$ $v(h \otimes 1)=\varepsilon(h)$ for all $h \in H$, i.e., $v$ is unitary or normalized, is called a gauge transformation.

Note that for $H$ a cocommutative bialgebra and $v: H \otimes H \rightarrow \mathbb{k}$ a (normalized) convolution invertible map then the map $\partial^{2} v: H \otimes H \otimes H \rightarrow \mathbb{k}$, defined by

$$
\partial^{2} v:=(\varepsilon \otimes v) * v^{-1}(m \otimes H) * v(H \otimes m) *\left(v^{-1} \otimes \varepsilon\right)
$$

is a (normalized) cocycle called a (normalized) coboundary. Conversely, if $v$ is convolution invertible and $\partial^{2} v$ is normalized then $v(1 \otimes h)=v(h \otimes 1)=\varepsilon(h) v(1 \otimes 1)$ and so $v(1 \otimes 1)^{-1} v$ is normalized. (See also Lemma 2.3.)

Now we define the objects of interest in this paper.
Definition 2.1. A dual quasi-bialgebra is a datum $(H, \Delta, \varepsilon, m, u, \omega)$ where $(H, \Delta, \varepsilon, m, u)$ is a coalgebra with multiplication and unit and $\omega: H \otimes H \otimes H \rightarrow \mathbb{k}$ is a normalized 3-cocycle such that

$$
\begin{align*}
{[m(H \otimes m)] *(u \omega) } & =(u \omega) *[m(m \otimes H)]  \tag{3}\\
m\left(1_{H} \otimes h\right) & =h=m\left(h \otimes 1_{H}\right), \text { for all } h \in H \tag{4}
\end{align*}
$$

Unless it is needed to emphasis the structure of the coalgebra $H$ with multiplication and unit, we will write $(H, \omega)$ for a dual quasi-bialgebra. The map $\omega$ is called the (Drinfeld) reassociator of the dual quasi-bialgebra. Note that if $H$ is cocommutative, then $(H, \omega)$ has associative multiplication for every reassociator $\omega$.

Following [Sd, Section 2], we say that $\Phi:(H, \omega) \rightarrow\left(H^{\prime}, \omega^{\prime}\right)$ is a morphism of dual quasi-bialgebras if $\Phi: H \rightarrow H^{\prime}$ is a morphism of coalgebras with multiplication and unit and $\omega^{\prime} \circ \Phi^{\otimes 3}=\omega$.

A bijective morphism of dual quasi-bialgebras is an isomorphism.

A dual quasi-subbialgebra of a dual quasi-bialgebra $\left(H^{\prime}, \omega^{\prime}\right)$ is a dual quasi-bialgebra $(H, \omega)$ such that $H$ is a subcoalgebra of $H^{\prime}$ and the canonical inclusion $\sigma: H \rightarrow H^{\prime}$ is a morphism of dual quasi-bialgebras.

We note that by (3) multiplication in the dual quasi-subbialgebra $\mathbb{k} G(H)$ of $H$ is associative.
Let $(H, \omega)$ be a dual quasi-bialgebra. It is well-known that the category ${ }^{H} \mathfrak{M}$ of left $H$-comodules becomes a monoidal category as follows. Given a left $H$-comodule $V$, we denote the left coaction of $V$ by $\rho=\rho_{V}^{l}: V \rightarrow H \otimes V, \rho(v)=v_{-1} \otimes v_{0}$. The tensor product of two left $H$-comodules $V$ and $W$ is a comodule via diagonal coaction i.e. $\rho(v \otimes w)=v_{-1} w_{-1} \otimes v_{0} \otimes w_{0}$. The unit is the trivial left $H$-comodule $\mathbb{k}$, i.e. $\rho(k)=1_{H} \otimes k$. The associativity and unit constraints are defined, for all $U, V, W \in{ }^{H} \mathfrak{M}$ and $u \in U, v \in V, w \in W, k \in \mathbb{k}$, by

$$
\begin{array}{ccc}
{ }^{H} a_{U, V, W}(u \otimes v \otimes w) & := & \omega^{-1}\left(u_{-1} \otimes v_{-1} \otimes w_{-1}\right) u_{0} \otimes\left(v_{0} \otimes w_{0}\right),  \tag{5}\\
l_{U}(k \otimes u):=k u & \text { and } & r_{U}(u \otimes k):=u k .
\end{array}
$$

The monoidal category we have just described will be denoted by ( $\left.{ }^{H} \mathfrak{M}, \otimes, \mathbb{k},{ }^{H} a, l, r\right)$.
The monoidal categories $\left(\mathfrak{M}^{H}, \otimes, \mathbb{k}, a^{H}, l, r\right)$ and $\left({ }^{H} \mathfrak{M}^{H}, \otimes, \mathbb{k},{ }^{H} a^{H}, l, r\right)$ are defined similarly. We just point out that

$$
\begin{gathered}
a_{U, V, W}^{H}(u \otimes v \otimes w):=u_{0} \otimes\left(v_{0} \otimes w_{0}\right) \omega\left(u_{1} \otimes v_{1} \otimes w_{1}\right), \\
H_{a_{U, V, W}^{H}(u \otimes v \otimes w):=\omega^{-1}\left(u_{-1} \otimes v_{-1} \otimes w_{-1}\right) u_{0} \otimes\left(v_{0} \otimes w_{0}\right) \omega\left(u_{1} \otimes v_{1} \otimes w_{1}\right) .} .
\end{gathered}
$$

For $(H, \omega)$ a dual quasi-bialgebra and $v: H^{\otimes 2} \rightarrow \mathbb{k}$ a convolution invertible map, define maps $m^{v}: H^{\otimes 2} \rightarrow H$ and $\omega^{v}: H^{\otimes 3} \rightarrow \mathbb{k}$ by setting

$$
\begin{align*}
m^{v} & :  \tag{6}\\
\omega^{v} & :=v * m * v^{-1}  \tag{7}\\
& =(\varepsilon \otimes v) * v(H \otimes m) * \omega * v^{-1}(m \otimes H) *\left(v^{-1} \otimes \varepsilon\right)
\end{align*}
$$

If $v$ is a gauge transformation, then the datum

$$
(H, \omega)^{v}=\left(H^{v}, \omega^{v}\right)=\left(H, m^{v}, u, \Delta, \varepsilon, \omega^{v}\right)
$$

is a dual quasi-bialgebra with reassociator $\omega^{v}$ called the twisted dual quasi-bialgebra of $H$ by $v$.
REMARK 2.2. (i) If $a \in \mathbb{K}^{\times}$and $v$ is convolution invertible, then so is $a v$, the composition of $v$ with multiplication by $a$, and $a v$ has inverse $a^{-1} v^{-1}$. Note that $m^{v}=m^{a v}$ and $\omega^{v}=\omega^{a v}$.
(ii) Note that $\left(m^{v}\right)^{v^{-1}}=m$. It is straightforward to verify that $\left(\omega^{v}\right)^{v^{-1}}=\omega$, remembering that the multiplication in $\left(H^{v}, \omega^{v}\right)$ is $m^{v}$. Thus, since $v^{-1}$ is a gauge transformation for $\left(H^{v}, \omega^{v}\right)$, we have that $\left(H^{v}, \omega^{v}\right)^{v^{-1}} \cong(H, \omega)$.
(iii) Note that if $H$ is cocommutative then $\omega^{v}=\partial^{2} v * \omega$ so that $\left(H^{v}, \varepsilon^{v}\right)=\left(H, \partial^{2} v\right)$.

Lemma 2.3. For $(H, \omega)$ a dual quasi-bialgebra, suppose $v: H \otimes H \rightarrow \mathbb{k}$ is a convolution invertible map such that $\omega^{v}$ as defined in (7) is normalized, i.e., satisfies (夷). Then av is a gauge transformation for $a=v(1 \otimes 1)^{-1} \in \mathbb{k}^{\times}$.

Proof. For all $h, h^{\prime} \in H$,

$$
\varepsilon(h) \varepsilon\left(h^{\prime}\right) \stackrel{\text { (2) }}{\underline{2}} \omega^{v}\left(h \otimes 1 \otimes h^{\prime}\right) \stackrel{\text { (8) }}{\underline{\text { B }}} v\left(1 \otimes h^{\prime}\right) v^{-1}(h \otimes 1) .
$$

Setting $h$ and $h^{\prime}$ equal to 1 in turn, we obtain for all $h \in H$,

$$
\varepsilon(h)=v(1 \otimes 1) v^{-1}(h \otimes 1)=a^{-1} v^{-1}(h \otimes 1) \quad \text { and } \quad \varepsilon(h)=v(1 \otimes h) v^{-1}(1 \otimes 1)=a v(1 \otimes h) .
$$

Since $a^{-1} v^{-1}(h \otimes 1)=\varepsilon(h)$ and $a v$ is the inverse of $a^{-1} v^{-1}$ then $a v(h \otimes 1)=\varepsilon(h)$ also.
Corollary 2.4. Let $H$ be a cocommutative bialgebra. If $v: H \otimes H \rightarrow \mathbb{k}$ is a convolution invertible map such that $\partial^{2} v$ is a normalized cocycle, then av is normalized for $a=v(1 \otimes 1)^{-1} \in \mathbb{k}^{\times}$.

Proof. Take $\omega=\varepsilon$ in Lemma 2.3. Since $H$ is cocommutative, $\omega^{v}=\partial^{2} v$.
Proposition 2.5. For $\sigma:\left(H, \omega_{H}\right) \rightarrow\left(A, \omega_{A}\right)$ a morphism of dual quasi-bialgebras and $v: A^{\otimes 2} \rightarrow \mathbb{k}$ a gauge transformation for $A$, then $v \circ(\sigma \otimes \sigma)$ is a gauge transformation for $H$. Also

$$
\omega_{A}^{v} \circ \sigma^{\otimes 3}=\omega_{H}^{v \circ \sigma^{\otimes 2}} \quad \text { and } \quad m_{A}^{v} \circ \sigma^{\otimes 2}=\sigma m_{H}^{v \circ \sigma^{\otimes 2}}
$$

Thus $\sigma:\left(H^{v \circ \sigma^{\otimes 2}}, \omega_{H}^{v \circ \sigma^{\otimes 2}}\right) \rightarrow\left(A^{v}, \omega_{A}^{v}\right)$ is also a morphism of dual quasi-bialgebras between the twisted dual quasi-bialgebras obtained from $\left(H, \omega_{H}\right)$ and $\left(A, \omega_{A}\right)$.

Proof. We have

$$
\begin{aligned}
& \omega_{A}^{v} \circ \sigma^{\otimes 3} \quad \stackrel{\text { (I) }}{ }\left[\left(\varepsilon_{A} \otimes v\right) * v\left(A \otimes m_{A}\right) * \omega_{A} * v^{-1}\left(m_{A} \otimes A\right) *\left(v^{-1} \otimes \varepsilon_{A}\right)\right] \circ \sigma^{\otimes 3} \\
&=\left(\varepsilon_{H} \otimes v \sigma^{\otimes 2}\right) * v \sigma^{\otimes 2}\left(H \otimes m_{H}\right) * \omega_{H} * v^{-1} \sigma^{\otimes 2}\left(m_{H} \otimes H\right) *\left(v^{-1} \sigma^{\otimes 2} \otimes \varepsilon_{H}\right) \\
& \underline{\text { (I) }} \omega_{H}^{v \sigma^{\otimes 2} .}
\end{aligned}
$$

Also

$$
\begin{aligned}
m_{A}^{v} \sigma^{\otimes 2} & \stackrel{(6)}{ }\left[v * m_{A} * v^{-1}\right]\left(\sigma^{\otimes 2}\right)=v \sigma^{\otimes 2} * m_{A} \sigma^{\otimes 2} * v^{-1} \sigma^{\otimes 2} \\
& =v \sigma^{\otimes 2} * \sigma m_{H} *\left[v \sigma^{\otimes 2}\right]^{-1}=\sigma m_{H}^{v \sigma^{\otimes 2}}
\end{aligned}
$$

Definition 2.6. Dual quasi-bialgebras $A$ and $B$ are called quasi-isomorphic (or equivalent) whenever $\left(A, \omega_{A}\right) \cong\left(B^{v}, \omega_{B}^{v}\right)$ as dual quasi-bialgebras for some gauge transformation $v \in(B \otimes B)^{*}$.

By Remark 2.2 ii., if $\left(A, \omega_{A}\right) \cong\left(B^{v}, \omega_{B}^{v}\right)$, then $\left(B, \omega_{B}\right) \cong\left(A, \omega_{A}\right)^{v^{-1}}$.
Corollary 2.7. If $\sigma:\left(H, \omega_{H}\right) \rightarrow\left(A, \omega_{A}\right)$ is a morphism of dual quasi-bialgebras and $\left(A, \omega_{A}\right)$ is quasi-isomorphic to an ordinary bialgebra so is $\left(H, \omega_{H}\right)$.

Proof. Suppose $\gamma_{A}: A \otimes A \rightarrow \mathbb{k}$ is a gauge transformation such that $A^{\gamma_{A}}$ has trivial reassociator. Then $\gamma_{H}:=\gamma_{A}(\sigma \otimes \sigma): H \otimes H \rightarrow \mathbb{k}$ is a gauge transformation, and, by Proposition 2.5, the map $\sigma:\left(H^{\gamma_{H}}, \omega_{H}^{\gamma_{H}}\right) \rightarrow\left(A^{\gamma_{A}}, \omega_{A}^{\gamma_{A}}=\varepsilon_{A^{\otimes 3}}\right)$ is a morphism of dual quasi-bialgebras. Hence

$$
\omega_{H^{\gamma_{H}}}=\omega_{H}^{\gamma_{H}}=\omega_{A}^{\gamma_{A}} \circ \sigma^{\otimes 3}=\varepsilon_{A^{\otimes 3}} \circ \sigma^{\otimes 3}=\varepsilon_{H^{\otimes 3}}
$$

so that $H^{\gamma_{H}}$ has trivial reassociator.

## 3. Quasi-Yetter-Drinfeld data for dual quasi-bialgebras

3.1. Yetter-Drinfeld modules. In this subsection, we first recall some facts from AP about the category of Yetter-Drinfeld modules for a dual quasi-bialgebra.

Definition 3.1 ( $\widehat{\mathrm{AP}}$, Definition 3.1]). For $(H, \omega)$ a dual quasi-bialgebra, the category ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ of Yetter-Drinfeld modules over $H$ is defined as follows. An object is a tern $\left(V, \rho_{V}, \triangleright\right)$, where
$(V, \rho)$ is an object in ${ }^{H} \mathfrak{M}$ and $\mu: H \otimes V \rightarrow V$ is a $\mathbb{k}$-linear map written $h \otimes v \mapsto h \triangleright v$ such that, for all $h, l \in H$ and $v \in V$

$$
\begin{align*}
& (h l) \triangleright v=\left[\begin{array}{c}
\omega^{-1}\left(h_{1} \otimes l_{1} \otimes v_{-1}\right) \omega\left(h_{2} \otimes\left(l_{2} \triangleright v_{0}\right)_{-1} \otimes l_{3}\right) \\
\omega^{-1}\left(\left(h_{3} \triangleright\left(l_{2} \triangleright v_{0}\right)_{0}\right)_{-1} \otimes h_{4} \otimes l_{4}\right)\left(h_{3} \triangleright\left(l_{2} \triangleright v_{0}\right)_{0}\right)_{0}
\end{array}\right],  \tag{8}\\
& 1_{H} \triangleright v=v \text { and }  \tag{9}\\
& \left(h_{1} \triangleright v\right)_{-1} h_{2} \otimes\left(h_{1} \triangleright v\right)_{0}=h_{1} v_{-1} \otimes\left(h_{2} \triangleright v_{0}\right) . \tag{10}
\end{align*}
$$

A morphism $f:(V, \rho, \triangleright) \rightarrow\left(V^{\prime}, \rho^{\prime}, \triangleright^{\prime}\right)$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ is a morphism $f:(V, \rho) \rightarrow\left(V^{\prime}, \rho^{\prime}\right)$ in ${ }^{H} \mathfrak{M}$ such that

$$
f(h \triangleright v)=h \triangleright^{\prime} f(v)
$$

REmARK 3.2. The category ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ is isomorphic to the weak right center of ${ }^{H} \mathfrak{M}$, see AP, Theorem A.2.]. As a consequence ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ has a prebraided monoidal structure given as follows. The unit is $\mathbb{k}$ regarded as an object in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ via the trivial structures $\rho_{\mathrm{k}}(k)=1_{H} \otimes k$ and $h \triangleright k=\varepsilon_{H}(h) k$. The tensor product is defined by

$$
\left(V, \rho_{V}, \triangleright\right) \otimes\left(W, \rho_{W}, \triangleright\right)=\left(V \otimes W, \rho_{V \otimes W}, \triangleright\right)
$$

where $\rho_{V \otimes W}(v \otimes w)=v_{-1} w_{-1} \otimes v_{0} \otimes w_{0}$ and

$$
h \triangleright(v \otimes w)=\left[\begin{array}{c}
\omega\left(h_{1} \otimes v_{-1} \otimes w_{-2}\right) \omega^{-1}\left(\left(h_{2} \triangleright v_{0}\right)_{-2} \otimes h_{3} \otimes w_{-1}\right)  \tag{11}\\
\omega\left(\left(h_{2} \triangleright v_{0}\right)_{-1} \otimes\left(h_{4} \triangleright w_{0}\right)_{-1} \otimes h_{5}\right)\left(h_{2} \triangleright v_{0}\right)_{0} \otimes\left(h_{4} \triangleright w_{0}\right)_{0}
\end{array}\right] .
$$

The constraints are the same as ${ }^{H} \mathfrak{M}$ i.e.

$$
\begin{aligned}
& { }^{H} a_{U, V, W}(u \otimes v \otimes w): \quad=\omega^{-1}\left(u_{-1} \otimes v_{-1} \otimes w_{-1}\right) u_{0} \otimes\left(v_{0} \otimes w_{0}\right), \\
& l_{U}(k \otimes u) \quad:=k u \quad \text { and } \quad r_{U}(u \otimes k):=u k .
\end{aligned}
$$

viewed as morphisms in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. The prebraiding $c_{V, W}: V \otimes W \rightarrow W \otimes V$ is given by

$$
\begin{equation*}
c_{V, W}(v \otimes w)=\left(v_{-1} \triangleright w\right) \otimes v_{0} \tag{12}
\end{equation*}
$$

REmARK 3.3. The coproduct of a family $\left(V^{i}\right)_{i \in I}$ of objects in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ is the vector space $\oplus_{i \in I} V^{i}$ regarded as an object in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ via the action and the coaction defined by

$$
h \triangleright\left(v^{i}\right)_{i \in I}=\left(h \triangleright v^{i}\right)_{i \in I} \quad \text { and } \quad \rho\left(\left(v^{i}\right)_{i \in I}\right)=\sum_{i \in I} v_{-1}^{i} \otimes u_{i}\left(v_{0}^{i}\right)
$$

respectively, where $u_{i}: V^{i} \rightarrow \oplus_{i \in I} V^{i}$ is the canonical injection. Let $W \in{ }_{H}^{H} \mathcal{Y} \mathcal{D}$. By the universal property of the coproduct, the canonical morphisms $W \otimes u_{i}: W \otimes V^{i} \rightarrow W \otimes\left(\oplus_{i \in I} V^{i}\right)$ yield a morphism in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$

$$
\oplus_{i \in I}\left(W \otimes V^{i}\right) \rightarrow W \otimes\left(\oplus_{i \in I} V^{i}\right)
$$

This is bijective because it is bijective at the level of vector spaces. This proves that the functor $W \otimes(-):{ }_{H}^{H} \mathcal{Y} \mathcal{D} \rightarrow{ }_{H}^{H} \mathcal{Y} \mathcal{D}$ commutes with coproducts. Similarly $(-) \otimes W:{ }_{H}^{H} \mathcal{Y} \mathcal{D} \rightarrow{ }_{H}^{H} \mathcal{Y} \mathcal{D}$ commutes with coproducts.

Therefore we can apply Mad , Theorem 2, page 172] to construct a left adjoint $T:{ }_{H}^{H} \mathcal{Y} \mathcal{D} \rightarrow$ Mon $\left({ }_{H}^{H} \mathcal{Y} \mathcal{D}\right)$ of the forgetful functor. For every $V \in{ }_{H}^{H} \mathcal{Y} \mathcal{D}$, the algebra $T(V)$ will be called the tensor algebra of $V$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. By standard arguments one can endow $T:=T(V)$ with a bialgebra structure in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ where the comultiplication $\Delta_{T}$ and the counit $\varepsilon_{T}$ are uniquely defined by setting $\Delta_{T}(v)=v \otimes 1_{T}+1_{T} \otimes v$ and $\varepsilon_{T}(v)=0$.

The next theorem from AP gives the structure of a bosonization in this setting.
Theorem 3.4 (AP, Theorem 5.2]). Let $\left(H, \omega_{H}\right)$ be a dual quasi-bialgebra.
Let $\left(R, \mu_{R}, \rho_{R}, \Delta_{R}, \varepsilon_{R}, m_{R}, u_{R}\right)$ be a bialgebra in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ and use the following notations

$$
\begin{array}{rlll}
h \triangleright r & : & =\mu_{R}(h \otimes r), & r_{-1} \otimes r_{0}:=\rho_{R}(r), \\
r \cdot{ }_{R} s & : & =m_{R}(r \otimes s), & 1_{R}:=u_{R}\left(1_{\mathbb{k}}\right), \\
r^{1} \otimes r^{2} & : & =\Delta_{R}(r) . &
\end{array}
$$

Consider on $B:=R \otimes H$ the following structures:

$$
\begin{aligned}
m_{B}[(r \otimes h) \otimes(s \otimes k)] & =\left[\begin{array}{c}
\omega_{H}^{-1}\left(r_{-2} \otimes h_{1} \otimes s_{-2} k_{1}\right) \omega_{H}\left(h_{2} \otimes s_{-1} \otimes k_{2}\right) \\
\omega_{H}^{-1}\left[\left(h_{3} \triangleright s_{0}\right)_{-2} \otimes h_{4} \otimes k_{3}\right] \omega_{H}\left(r_{-1} \otimes\left(h_{3} \triangleright s_{0}\right)_{-1} \otimes h_{5} k_{4}\right) \\
r_{0} \cdot_{R}\left(h_{3} \triangleright s_{0}\right)_{0} \otimes h_{6} k_{5}
\end{array}\right] \\
u_{B}(k) & =k 1_{R} \otimes 1_{H} \\
\Delta_{B}(r \otimes h) & =\omega_{H}^{-1}\left(r_{-1}^{1} \otimes r_{-2}^{2} \otimes h_{1}\right) r_{0}^{1} \otimes r_{-1}^{2} h_{2} \otimes r_{0}^{2} \otimes h_{3} \\
\varepsilon_{B}(r \otimes h) & =\varepsilon_{R}(r) \varepsilon_{H}(h) \\
\omega_{B}((r \otimes h) \otimes(s \otimes k) \otimes(t \otimes l)) & =\varepsilon_{R}(r) \varepsilon_{R}(s) \varepsilon_{R}(t) \omega_{H}(h \otimes k \otimes l) .
\end{aligned}
$$

Then $\left(B, \Delta_{B}, \varepsilon_{B}, m_{B}, u_{B}, \omega_{B}\right)$ is a dual quasi-bialgebra.
Definition 3.5 ( $\boxed{A P}$, Definition 5.4]). For $H, R, B$ as in Theorem 3.4, the dual quasi-bialgebra $B$ will be called the bosonization of $R$ by $H$ and denoted by $R \# H$. Elements of $B$ may be written $r \# h$ instead of $r \otimes h$ to emphasize that we are working in the bosonization.

Remark 3.6. Let $A:=R \# H, B:=S \# L$ where $H, L$ are cosemisimple dual quasi-bialgebras and $R, S$ are bialgebras in the categories of Yetter-Drinfeld modules over $H$ and $L$ respectively such that $A_{0}=\mathbb{k} \# H$ and $B_{0}=\mathbb{k} \# L$. Then if $A$ and $B$ are quasi-isomorphic, so are $H$ and $L$.

For suppose that there is an isomorphism $\varphi: A \rightarrow B^{v}$ of dual quasi-bialgebras. Since $\varphi$ is a coalgebra isomorphism, $\varphi\left(A_{0}\right)=\left(B^{v}\right)_{0}=B_{0}$.

Write $\varphi(1 \otimes h)$ as $1 \otimes \varphi^{\prime}(h)$ for some $\varphi^{\prime}(h) \in L$. In this way we get the following commutative diagram.


By the same argument using $\varphi^{-1}$ we get an inverse for $\varphi^{\prime}$. By Proposition 2.5, the right-hand side vertical map is an injective morphism of dual quasi-bialgebras. Since $\sigma_{H}$ and $\varphi$ are also morphisms of dual quasi-bialgebras we get that $\varphi^{\prime}$ also is. Thus $\varphi^{\prime}$ is an isomorphism of dual quasi-bialgebras as required.

The proof of the next lemma is straightforward and so is left to the reader.
Lemma 3.7. Take the hypothesis and notations of Theorem 3.4. Let $\pi: R \# H \rightarrow H$ be defined by $\pi(r \# h)=\varepsilon_{R}(r) h$. Then $\pi$ is a morphism of dual quasi-bialgebras and

$$
\begin{equation*}
\pi\left((r \# h)_{1}\right) \otimes(r \# h)_{2} \# \pi\left((r \# h)_{3}\right)=r_{-1} h_{1} \otimes\left(r_{0} \# h_{2}\right) \otimes h_{3} \tag{13}
\end{equation*}
$$

REmARK 3.8. Note that for $R \# H$ as above, the map $\sigma: H \hookrightarrow R \# H$ defined by $\sigma(h)=1_{R} \# h$ is also a morphism of dual quasi-bialgebras and, for $\pi$ as defined in Lemma 3.7, $\pi \sigma=I d_{H}$. Corollary 2.7 then implies that $H$ is quasi-isomorphic to an ordinary bialgebra if and only if $R \# H$ is.
3.2. Quasi-Yetter-Drinfeld data. Let $(H, \omega)$ be a dual quasi-bialgebra. In this subsection we study one-dimensional vector spaces in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ and the bialgebras in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ generated by these.

Proposition 3.9. Let $(H, \omega)$ be a dual quasi-bialgebra and let $V$ be a one-dimensional vector space. Then $V$ is an object in ${ }_{H}^{H} \mathcal{Y D}$ if and only if for all $v \in V, h, l \in H$,
(i) $V \in{ }^{H} \mathfrak{M}$ and $\rho(v)=g \otimes v$ for some $g \in G(H)$;
(ii) There is a unitary map $\chi \in H^{*}$ such that for $g$ the grouplike in $(i)$,

$$
\begin{equation*}
\chi(h l)=\omega^{-1}\left(h_{1} \otimes l_{1} \otimes g\right) \chi\left(l_{2}\right) \omega\left(h_{2} \otimes g \otimes l_{3}\right) \chi\left(h_{3}\right) \omega^{-1}\left(g \otimes h_{4} \otimes l_{4}\right) \tag{14}
\end{equation*}
$$

(iii) For $g$ the grouplike from part (i),

$$
\begin{equation*}
g \chi\left(h_{1}\right) h_{2}=h_{1} \chi\left(h_{2}\right) g \tag{15}
\end{equation*}
$$

Proof. First let $V \in{ }_{H}^{H} \mathcal{Y} \mathcal{D}$. Since $V$ is one-dimensional, there is a grouplike element $g \in H$ such that $\rho(v)=g \otimes v$ for all $v \in V$, and also there is a map $\chi: H \rightarrow \mathbb{k}$ such that $h \triangleright v=\chi(h) v$. Equation (8) of the definition of Yetter-Drinfeld modules now translates to (14) and equation (9) implies that $\chi$ is unitary. Finally here equation (10) of Definition 3.1 is equivalent to (15) above.

Now suppose that $V$ is a one-dimensional vector space satisfying $(i)$ to (iii) above. Then it is easy to see that $V$ with coaction given by $\rho(v)=g \otimes v$ and action given by $h \triangleright v=\chi(h) v$ for all $v \in V$ is an object in ${ }_{H}^{H} \mathcal{Y D}$.

Definitions 3.10. Let $(H, \omega)$ be a dual quasi-bialgebra. For $g \in G(H)$ and $\chi \in H^{*}, \chi$ unitary, the triple $((H, \omega), g, \chi)$ is called a quasi-Yetter-Drinfeld datum, abbreviated to quasi- $Y D$ datum, whenever equations (14) and (15) above hold. If $q:=\chi(g)$, we also say that $((H, \omega), g, \chi)$ is a quasi-Yetter-Drinfeld datum for $q$.
Remark 3.11. (i) When $H$ is a Hopf algebra, $\omega$ is trivial and $q \neq 1$, then the previous definition reduces to CDMM, Definition 2.1].
(ii) Note that Proposition 3.9, roughly speaking, says that a one-dimensional vector space $V$ with action and coaction defined by $\chi$ and $g$, is an object in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ if and only if $((H, \omega), g, \chi)$ is a quasi- $Y D$ datum.
(iii) Equation (15) implies that if $((H, \omega), g, \chi)$ is a quasi- $Y D$ datum and $\ell \in G(H)$ with $\chi_{H}(\ell) \neq$ 0 , then $g \ell=\ell g$.

Lemma 3.12. Let $G$ be a group and let $\omega$ be a normalized 3 -cocycle on $G$. Let $g \in G$ and $\chi: \mathbb{k} G \rightarrow \mathbb{k}$. The following are equivalent.
(i) $((\mathbb{k} G, \omega), g, \chi)$ is a quasi-YD datum.
(ii) $g \in Z(G), \chi$ is unitary and (14) holds for all $h, \ell \in G$.

Proof. It suffices to prove that $(i)$ implies that $g \in Z(G)$. Let $h \in G$. Then

$$
1=\chi(1)=\chi\left(h^{-1} h\right) \stackrel{14}{=} \omega^{-1}\left(h^{-1} \otimes h \otimes g\right) \chi(h) \omega\left(h^{-1} \otimes g \otimes h\right) \chi\left(h^{-1}\right) \omega^{-1}\left(g \otimes h^{-1} \otimes h\right)
$$

so that $\chi(h)$ is invertible, and $g h=h g$ by Remark 3.11.

Definition 3.13. For $((H, \omega), g, \chi)$ and $((L, \alpha), l, \xi)$ quasi- $Y D$ data, a dual quasi-bialgebra homomorphism $\varphi:(H, \omega) \rightarrow(L, \alpha)$ such that $\varphi(g)=l$ and $\xi \varphi=\chi$ is called a morphism of quasi- $Y D$ data.

Lemma 3.14. Let $\pi:\left(A, \omega_{A}\right) \rightarrow\left(H, \omega_{H}\right)$ be a morphism of dual quasi-bialgebras and $((H, \omega), g, \chi)$ a quasi-YD datum. If there exists $a \in G(A)$ such that $\pi(a)=g$ and $a \chi \pi\left(b_{1}\right) b_{2}=b_{1} \chi \pi\left(b_{2}\right) a$, for every $b \in A$, then $\left(\left(A, \omega_{A}\right), a, \chi_{A}:=\chi \pi\right)$ is also a quasi-YD datum and $\pi$ is a morphism of quasi-YD data.

Proof. We need only verify (14) for $\left((A, \omega), a, \chi_{A}\right)$. For $h, l \in A$, since (14) holds for $\chi$, we have:

$$
\begin{aligned}
& \omega_{A}^{-1}\left(h_{1} \otimes l_{1} \otimes a\right) \chi_{A}\left(l_{2}\right) \omega_{A}\left(h_{2} \otimes a \otimes l_{3}\right) \chi_{A}\left(h_{3}\right) \omega_{A}^{-1}\left(a \otimes h_{4} \otimes l_{4}\right) \\
= & \omega_{H}^{-1}\left(\pi(h)_{1} \otimes \pi(l)_{1} \otimes g\right) \chi\left[\pi(l)_{2}\right] \omega_{H}\left(\pi(h)_{2} \otimes g \otimes \pi(l)_{3}\right) \chi\left[\pi(h)_{3}\right] \omega_{H}^{-1}\left(g \otimes \pi(h)_{4} \otimes \pi(l)_{4}\right)
\end{aligned}
$$

(14)

$$
\chi(\pi(h) \pi(l))=\chi \pi(h l)=\chi_{A}(h l) .
$$

Lemma 3.15. Suppose $(H, \omega)$ is a dual quasi-bialgebra and $((H, \omega), g, \chi)$ is a quasi-YD datum. Then for $c \in G(H)$ and $1 \leq t$,

$$
\begin{equation*}
\chi\left(c^{t}\right)=\chi(c)^{t} \prod_{0 \leq i \leq t-1}\left[\omega^{-1}\left(c^{i} \otimes c \otimes g\right) \omega\left(c^{i} \otimes g \otimes c\right) \omega^{-1}\left(g \otimes c^{i} \otimes c\right)\right], \tag{16}
\end{equation*}
$$

and, in particular,

$$
\begin{equation*}
\chi\left(g^{t}\right)=\chi(g)^{t} \prod_{0 \leq i \leq t-1} \omega^{-1}\left(g \otimes g^{i} \otimes g\right) . \tag{17}
\end{equation*}
$$

Proof. Let $s>1$ and then by (14) we have

$$
\begin{equation*}
\chi\left(c^{s-1} c\right)=\omega^{-1}\left(c^{s-1} \otimes c \otimes g\right) \chi(c) \omega\left(c^{s-1} \otimes g \otimes c\right) \chi\left(c^{s-1}\right) \omega^{-1}\left(g \otimes c^{s-1} \otimes c\right) . \tag{18}
\end{equation*}
$$

Equation (16) now follows by induction on $t \geq 1$. For $t=1$, there is nothing to prove. Let $t>1$ and assume that the statement holds for $t-1$. Then by (18),

$$
\chi\left(c^{t}\right)=\chi\left(c^{t-1}\right)\left[\chi(c) \omega^{-1}\left(c^{t-1} \otimes c \otimes g\right) \omega\left(c^{t-1} \otimes g \otimes c\right) \omega^{-1}\left(g \otimes c^{t-1} \otimes c\right)\right]
$$

and if we then expand $\chi\left(c^{t-1}\right)$ using (16), the result is immediate.
Remark 3.16. If $\omega=\omega(H \otimes \tau)$, then equation (16) simplifies to:

$$
\begin{equation*}
\chi\left(c^{t}\right)=\chi(c)^{t} \prod_{0 \leq i \leq t-1} \omega^{-1}\left(g \otimes c^{i} \otimes c\right) . \tag{19}
\end{equation*}
$$

## 4. Quantum lines

Our first lemma will be useful in the computations to follow.
Lemma 4.1. Let $(H, \omega)$ be a dual quasi bialgebra and let $g \in G(H)$. For all $0 \leq a, b, c$,

$$
\begin{equation*}
\omega^{-1}\left(g^{a} \otimes g^{b} \otimes g^{c}\right)=\prod_{0 \leq j \leq a-1} \omega^{-1}\left(g \otimes g^{j+b} \otimes g^{c}\right) \omega^{-1}\left(g \otimes g^{j} \otimes g^{b}\right) \omega\left(g \otimes g^{j} \otimes g^{b+c}\right) . \tag{20}
\end{equation*}
$$

Proof. Let $\Phi(j):=\omega^{-1}\left(g \otimes g^{j+b} \otimes g^{c}\right) \omega^{-1}\left(g \otimes g^{j} \otimes g^{b}\right) \omega\left(g \otimes g^{j} \otimes g^{b+c}\right)$. The proof is by induction on $a \geq 1$. For $a=0,1$ there is nothing to prove. Let $a>1$ and assume the formula holds for $a-1$. By (1) evaluated on $g \otimes g^{a-1} \otimes g^{b} \otimes g^{c}$, we have

$$
\omega\left(g^{a-1} \otimes g^{b} \otimes g^{c}\right) \omega\left(g \otimes g^{a+b-1} \otimes g^{c}\right) \omega\left(g \otimes g^{a-1} \otimes g^{b}\right)=\omega\left(g \otimes g^{a-1} \otimes g^{b+c}\right) \omega\left(g^{a} \otimes g^{b} \otimes g^{c}\right)
$$

so that

$$
\omega\left(g^{a-1} \otimes g^{b} \otimes g^{c}\right) \omega^{-1}\left(g^{a} \otimes g^{b} \otimes g^{c}\right)=\Phi(a-1)
$$

and the statement then follows from the induction assumption.
Next we introduce some useful notation.
Notation 4.2. Let $(H, \omega)$ be a dual quasi-bialgebra. For $U, V, W, Z$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$, we define $\Omega_{U, V, W, Z}$ : $(U \otimes V) \otimes(W \otimes Z) \rightarrow(U \otimes W) \otimes(V \otimes Z)$ by

$$
\begin{equation*}
\Omega_{U, V, W, Z}:=a_{U, W, V \otimes Z}^{-1}\left(U \otimes a_{W, V, Z}\right)\left(U \otimes c_{V, W} \otimes Z\right)\left(U \otimes a_{V, W, Z}^{-1}\right) a_{U, V, W \otimes Z}, \tag{21}
\end{equation*}
$$

where $a:={ }^{H} a$ is the associativity constraint (国) in ${ }^{H} \mathfrak{M}$. If $U=V=W=Z$, we write $\Omega_{U}$ in place of $\Omega_{U, U, U, U}$.

As observed in Remark 3.3, we can consider the tensor algebra $T(V)$ of $V$ in ${ }_{H}^{H} \mathcal{Y D}$ for any object $V$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. Explicitly

$$
T(V):=\oplus_{n \in \mathbb{N}} T^{n}(V),
$$

where $T^{0}(V)=\mathbb{k}, T^{1}(V)=V$ and, for $n>1, T^{n}(V):=V \otimes T^{n-1}(V)$. Thus, for instance, $T^{2}(V)=V \otimes V$ and $T^{3}(V)=V \otimes(V \otimes V)$. Note that the order of the brackets is important here.

Let $((H, \omega), g, \chi)$ be a quasi- $Y D$ datum for $q$. Let $V=\mathbb{k} v$ be a one-dimensional vector space and then $(V, \rho, \mu)$ with $\rho(v)=g \otimes v$ and $\mu(h \otimes v)=\chi(h) v$ is an object in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ by Remark 3.11. Set $v^{[0]}:=1, v^{[1]}:=v$ and, for $n>1, v^{[n]}:=v \otimes v^{[n-1]}$. As a vector space $T(V)$ may be identified with the polynomial ring $\mathbb{k}[X]$ via the correspondence $v^{[n]} \leftrightarrow X^{n}$. However the multiplication is different.

Proposition 4.3. With hypothesis and notations as above, the tensor algebra $T(V)$ has basis $\left(v^{[n]}\right)_{n \in \mathbb{N}}$ and has the following bialgebra structure in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$.
(i) The left coaction of $H$ on $T(V)$ is given by

$$
\begin{equation*}
\rho\left(v^{[n]}\right)=v_{-1}^{[n]} \otimes v_{0}^{[n]}=g^{n} \otimes v^{[n]} \tag{22}
\end{equation*}
$$

The left action of $H$ on $T(V)$ is given by

$$
\begin{equation*}
h \triangleright v^{[n]}=\chi_{[n]}(h) v^{[n]}, \tag{23}
\end{equation*}
$$

where $\chi_{[n]} \in H^{*}$ is defined iteratively by setting $\chi_{[0]}:=\varepsilon$, and for $n \geq 1$,

$$
\begin{equation*}
\chi_{[n]}:=\omega\left(-\otimes g \otimes g^{n-1}\right) * \chi * \omega^{-1}\left(g \otimes-\otimes g^{n-1}\right) * \chi_{[n-1]} * \omega\left(g \otimes g^{n-1} \otimes-\right) . \tag{24}
\end{equation*}
$$

Furthermore, for $n \geq 1$

$$
\begin{equation*}
\chi_{[n]}=\left[\prod_{0 \leq i \leq n-1}^{*} \omega\left(-\otimes g \otimes g^{n-1-i}\right) * \chi * \omega^{-1}\left(g \otimes-\otimes g^{n-1-i}\right)\right] *\left[\prod_{0 \leq i \leq n-1}^{*} \omega\left(g \otimes g^{i} \otimes-\right)\right] \tag{25}
\end{equation*}
$$

and in particular

$$
\begin{equation*}
\chi_{[n]}(g)=\left[\prod_{0 \leq i \leq n-1} \omega\left(g \otimes g^{i} \otimes g\right)\right] \chi(g)^{n}=q^{n} \prod_{0 \leq i \leq n-1} \omega\left(g \otimes g^{i} \otimes g\right) . \tag{26}
\end{equation*}
$$

(ii) The algebra structure on $T(V)$ is given by $1_{T(V)}=1_{\mathbb{k}} \in T^{0}(V)$ and

$$
\begin{equation*}
v^{[a]} v^{[b]}=\left[\prod_{0 \leq i \leq a-1} \omega^{-1}\left(g \otimes g^{i} \otimes g^{b}\right)\right] v^{[a+b]}, \text { for } a \geq 1, b \in \mathbb{N} \tag{27}
\end{equation*}
$$

(iii) The coalgebra structure is given by $\varepsilon_{T}\left(v^{[n]}\right)=\delta_{n, 0}$, and

$$
\begin{equation*}
\Delta_{T}\left(v^{[n]}\right)=\sum_{0 \leq i \leq n} \beta(i, n) v^{[i]} \otimes v^{[n-i]} \tag{28}
\end{equation*}
$$

where we define

$$
\begin{equation*}
\beta(i, n)=\binom{n}{i}_{q} \prod_{0 \leq j \leq i-1} \omega\left(g \otimes g^{j} \otimes g^{n-i}\right) \text { for all } 0 \leq i \leq n \tag{29}
\end{equation*}
$$

Note that $\beta(0, n)=1=\beta(n, n)$ for all $n \geq 0$.
Proof. Equation (22) follows from the definition of the comodule structure on the tensor product in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ in Remark 3.2 and the fact that $\rho(v)=g \otimes v$.

Next we compute $h \triangleright v^{[n]}$ for $h \in H$. If $n=0$, then $\chi_{[0]}=\varepsilon$ satisfies (23). We prove, by induction on $n \geq 1$, that (23) holds for $\chi_{[n]} \in H^{*}$ defined inductively by equation (24). Equation (24) gives $\chi_{[1]}=\chi$, which satisfies (23). Let $n>1$ and assume that the statement holds for $n-1$. Then using the induction assumption and (22), we have:

$$
\begin{aligned}
h \triangleright v^{[n]} & =h \triangleright\left(v \otimes v^{[n-1]}\right) \\
& \left.\stackrel{\omega\left(h_{1} \otimes v_{-1} \otimes v_{-2}^{[n-1]}\right) \omega^{-1}\left(\left(h_{2} \triangleright v_{0}\right)_{-2} \otimes h_{3} \otimes v_{-1}^{[n-1]}\right)}{\omega\left(\left(h_{2} \triangleright v_{0}\right)_{-1} \otimes\left(h_{4} \triangleright v_{0}^{[n-1]}\right)_{-1} \otimes h_{5}\right)\left(h_{2} \triangleright v_{0}\right)_{0} \otimes\left(h_{4} \triangleright v_{0}^{[n-1]}\right)_{0}}\right]
\end{aligned}
$$

$$
\begin{aligned}
& =\left[\begin{array}{c}
\omega\left(h_{1} \otimes g \otimes g^{n-1}\right) \omega^{-1}\left(g \otimes h_{3} \otimes g^{n-1}\right) \\
\omega\left(g \otimes\left(h_{4} \triangleright v^{[n-1]}\right)_{-1} \otimes h_{5}\right) \chi\left(h_{2}\right) v \otimes\left(h_{4} \triangleright v^{[n-1]}\right)_{0}
\end{array}\right] \\
& =\left[\begin{array}{c}
\omega\left(h_{1} \otimes g \otimes g^{n-1}\right) \omega^{-1}\left(g \otimes h_{3} \otimes g^{n-1}\right) \\
\omega\left(g \otimes g^{n-1} \otimes h_{5}\right) \chi\left(h_{2}\right) v \otimes \chi_{[n-1]}\left(h_{4}\right) v^{[n-1]}
\end{array}\right] \\
& =\left[\omega\left(-\otimes g \otimes g^{n-1}\right) * \chi * \omega^{-1}\left(g \otimes-\otimes g^{n-1}\right) * \chi_{[n-1]} * \omega\left(g \otimes g^{n-1} \otimes-\right)\right](h) v^{[n]} .
\end{aligned}
$$

Now, using this formula, we prove by induction on $n \geq 1$ that (25) holds. For $n=1$, since $\omega$ is a normalized cocycle and $g^{0}=1$, then the right hand side of (25) is just $\chi=\chi_{[1]}$.

Let $n>1$ and assume the formula holds for $n-1$. Then

$$
\begin{aligned}
& \chi_{[n]}=\omega\left(-\otimes g \otimes g^{n-1}\right) * \chi * \omega^{-1}\left(g \otimes-\otimes g^{n-1}\right) * \chi_{[n-1]} * \omega\left(g \otimes g^{n-1} \otimes-\right) \\
&=\left[\left[\prod_{0 \leq i \leq n-2}^{*} \omega\left(-\otimes g \otimes g^{n-2-i}\right) * \chi * \omega^{-1}\left(g \otimes-\otimes g^{n-2-i}\right)\right] *\left[\prod_{0 \leq i \leq n-2}^{*} \omega\left(g \otimes g^{i} \otimes-\right)\right] *\right] \\
& \omega\left(g \otimes g^{n-1} \otimes-\right) \\
&=\left[\prod_{0 \leq i \leq n-1}^{*} \omega\left(-\otimes g \otimes g^{n-1-i}\right) * \chi * \omega^{-1}\left(g \otimes-\otimes g^{n-1-i}\right)\right] *\left[\prod_{0 \leq i \leq n-1}^{*} \omega\left(g \otimes g^{i} \otimes-\right)\right]
\end{aligned}
$$

and so (25) holds for all $n \geq 1$. We note that if (25) is applied to a cocommutative element, then since the product for $0 \leq i \leq n-1$ is the same as taking the product over $0 \leq n-1-i \leq n-1$,

$$
\begin{equation*}
\chi_{[n]}=\chi^{n} *\left[\prod_{0 \leq i \leq n-1}^{*} \omega\left(-\otimes g \otimes g^{i}\right) * \omega^{-1}\left(g \otimes-\otimes g^{i}\right) * \omega\left(g \otimes g^{i} \otimes-\right)\right] \tag{30}
\end{equation*}
$$

Equation (26) follows immediately.
(ii) Let $b \in \mathbb{N}$ and we prove by induction on $a \geq 1$ that (27) holds. For $a=1$, we have by definition $v^{[a]} v^{[b]}=v v^{[b]}=v \otimes v^{[b]}=v^{[1+b]}=v^{[a+b]}$. Let $a>1$ and assume (27) for $a-1$. Then

$$
\begin{aligned}
v^{[a]} v^{[b]} & =\left(v \otimes v^{[a-1]}\right) v^{[b]}=\left(v v^{[a-1]}\right) v^{[b]} \underline{\text { 雨 }} \omega^{-1}\left(v_{-1} \otimes v_{-1}^{[a-1]} \otimes v_{-1}^{[b]}\right) v_{0}\left(v_{0}^{[a-1]} v_{0}^{[b]}\right) \\
& \stackrel{\text { 22] }}{ } \omega^{-1}\left(g \otimes g^{a-1} \otimes g^{b}\right) v\left(v^{[a-1]} v^{[b]}\right) \\
& =\omega^{-1}\left(g \otimes g^{a-1} \otimes g^{b}\right) v\left[\prod_{0 \leq i \leq a-2} \omega^{-1}\left(g \otimes g^{i} \otimes g^{b}\right)\right] v^{[a-1+b]} \\
& =\left[\prod_{0 \leq i \leq a-1} \omega^{-1}\left(g \otimes g^{i} \otimes g^{b}\right)\right] v v^{[a-1+b]}=\left[\prod_{0 \leq i \leq a-1} \omega^{-1}\left(g \otimes g^{i} \otimes g^{b}\right)\right] v^{[a+b]},
\end{aligned}
$$

and so we have proved that (27) holds for $a \geq 1$.
(iii) We wish to show that

$$
\Delta_{T}\left(v^{[n]}\right)=\sum_{0 \leq i \leq n}\binom{n}{i}_{q} \prod_{0 \leq j \leq i-1} \omega\left(g \otimes g^{j} \otimes g^{n-i}\right) v^{[i]} \otimes v^{[n-i]}
$$

where if $i=0$, the empty product is defined to be 1 . If $n=0$ the formula holds since $\Delta_{T}\left(v^{[0]}\right)=$ $1 \otimes 1$. If $n=1$, the formula holds since $\Delta_{T}\left(v^{[1]}\right)=v \otimes 1+1 \otimes v,\binom{1}{0}_{q}=\binom{1}{1}_{q}=1$, and $\omega(-\otimes 1 \otimes-)=1$. Let $n>1$ and suppose that the formula holds for $n-1$. Then

$$
\begin{aligned}
& \Delta_{T}\left(v^{[n]}\right)=\Delta_{T} m_{T}\left(v \otimes v^{[n-1]}\right)=\left(m_{T} \otimes m_{T}\right) \Omega_{T}\left(\Delta_{T} \otimes \Delta_{T}\right)\left(v \otimes v^{[n-1]}\right) \\
= & \left(m_{T} \otimes m_{T}\right) \Omega_{T}\left[(v \otimes 1+1 \otimes v) \otimes\left(\sum_{0 \leq i \leq n-1} \beta(i, n-1) v^{[i]} \otimes v^{[n-1-i]}\right)\right] .
\end{aligned}
$$

From the definition of $\Omega_{T}$, it is easily seen that:

$$
\Omega_{T}\left((v \otimes 1) \otimes\left(v^{[i]} \otimes v^{[n-1-i]}\right)\right)=\omega\left(g \otimes g^{i} \otimes g^{n-1-i}\right)\left(\left(v \otimes v^{[i]}\right) \otimes\left(1 \otimes v^{[n-1-i]}\right)\right)
$$

and
$\Omega_{T}\left((1 \otimes v) \otimes\left(v^{[i]} \otimes v^{[n-1-i]}\right)=\chi_{[i]}(g) \omega\left(g \otimes g^{i} \otimes g^{n-1-i}\right) \omega^{-1}\left(g^{i} \otimes g \otimes g^{n-1-i}\right)\left(1 \otimes v^{[i]}\right) \otimes\left(v \otimes v^{[n-1-i]}\right)\right.$.
Then

$$
\begin{aligned}
& \Delta_{T}\left(v^{[n]}\right) \\
& =\sum_{0 \leq i \leq n-1} \beta(i, n-1)\left[\begin{array}{c}
\omega\left(g \otimes g^{i} \otimes g^{n-1-i}\right) v^{[i+1]} \otimes v^{[n-1-i]} \\
\left.+\chi_{[i]}(g) \omega\left(g \otimes g^{i} \otimes g^{n-1-i}\right) \omega^{-1}\left(g^{i} \otimes g \otimes g^{n-1-i}\right) v^{[i]} \otimes v^{[n-i]}\right]
\end{array}\right]
\end{aligned}
$$

The coefficient of $v^{[0]} \otimes v^{[n]}$ in the expression above is $\chi_{[0]}(g) \omega\left(g \otimes 1 \otimes g^{n-1}\right) \omega^{-1}\left(1 \otimes g \otimes g^{n-1}\right)=$ $1=\beta(0, n)$ and similarly the coefficient of $v^{[n]} \otimes v^{[0]}$ is $\beta(n-1, n-1)=1$. For $1 \leq j \leq n-1$, we compute the coefficient of $v^{[j]} \otimes v^{[n-j]}$ in this expression to be:

$$
\begin{aligned}
& \beta(j-1, n-1) \omega\left(g \otimes g^{j-1} \otimes g^{n-j}\right)+\beta(j, n-1) \omega\left(g \otimes g^{j} \otimes g^{n-1-j}\right) \chi_{[j]}(g) \omega^{-1}\left(g^{j} \otimes g \otimes g^{n-1-j}\right) \\
= & {\left[\binom{n-1}{j-1}_{q} \prod_{0 \leq k \leq j-2} \omega\left(g \otimes g^{k} \otimes g^{n-j}\right)\right] \omega\left(g \otimes g^{j-1} \otimes g^{n-j}\right) } \\
& +\left[\binom{n-1}{j}_{q} \prod_{0 \leq i \leq j-1} \omega\left(g \otimes g^{i} \otimes g^{n-1-j}\right)\right] \omega\left(g \otimes g^{j} \otimes g^{n-1-j}\right) \chi_{[j]}(g) \omega^{-1}\left(g^{j} \otimes g \otimes g^{n-1-j}\right) \\
= & \binom{n-1}{j-1} \prod_{q} \prod_{0 \leq k \leq j-1} \omega\left(g \otimes g^{k} \otimes g^{n-j}\right) \\
& +\binom{n-1}{j}_{q}\left[\prod_{0 \leq i \leq j} \omega\left(g \otimes g^{i} \otimes g^{n-1-j}\right)\right] \chi_{[j]}(g) \omega^{-1}\left(g^{j} \otimes g \otimes g^{n-1-j}\right) \\
= & \binom{n-1}{j-1}_{q} \prod_{0 \leq k \leq j-1} \omega\left(g \otimes g^{k} \otimes g^{n-j}\right) \\
& +\binom{n-1}{j}_{q}\left[\prod_{0 \leq s \leq j-1} \omega\left(g \otimes g^{s+1} \otimes g^{n-1-j}\right)\right] \chi_{[j]}(g) \omega^{-1}\left(g^{j} \otimes g \otimes g^{n-1-j}\right) .
\end{aligned}
$$

By Lemma 4.1

$$
\omega^{-1}\left(g^{j} \otimes g \otimes g^{n-1-j}\right)=\prod_{0 \leq s \leq j-1} \omega^{-1}\left(g \otimes g^{s+1} \otimes g^{n-1-j}\right) \omega^{-1}\left(g \otimes g^{s} \otimes g\right) \omega\left(g \otimes g^{s} \otimes g^{n-j}\right)
$$

so that the last summand in the expression above becomes:

$$
\begin{aligned}
& \quad\binom{n-1}{j}_{q} \chi_{[j]}(g) \prod_{0 \leq s \leq j-1} \omega^{-1}\left(g \otimes g^{s} \otimes g\right) \prod_{0 \leq t \leq j-1} \omega\left(g \otimes g^{t} \otimes g^{n-j}\right) \\
& =\binom{n-1}{j}_{q} q^{j} \prod_{0 \leq s \leq j-1}\left(\omega\left(g \otimes g^{s} \otimes g\right) \omega^{-1}\left(g \otimes g^{s} \otimes g\right)\right) \prod_{0 \leq t \leq j-1} \omega\left(g \otimes g^{t} \otimes g^{n-j}\right) \\
& = \\
& \binom{n-1}{j}_{q} q^{j} \prod_{0 \leq t \leq j-1} \omega\left(g \otimes g^{t} \otimes g^{n-j}\right)
\end{aligned}
$$

Thus the coefficient of $v^{[j]} \otimes v^{[n-j]}$ in $\Delta_{T}\left(v^{[n]}\right)$ is

$$
\left[\binom{n-1}{j-1}_{q}+\binom{n-1}{j}_{q} q^{j}\right] \prod_{0 \leq t \leq j-1} \omega\left(g \otimes g^{t} \otimes g^{n-j}\right)=\binom{n}{j}_{q} \prod_{0 \leq t \leq j-1} \omega\left(g \otimes g^{t} \otimes g^{n-j}\right)
$$

and this is indeed $\beta(j, n)$ as required. It is then straightforward to see that $\varepsilon_{T}\left(v^{[n]}\right)=\delta_{n, 0}$.
The next technical results allow us to construct a bialgebra quotient of the tensor algebra.

Proposition 4.4. Let $\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}\right)$ be a bialgebra in an abelian prebraided monoidal category $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r, c)$ where the tensor functors are additive and right exact. Let $\left(I, i_{I}: I \rightarrow A\right)$ be a subobject of $A$ in $\mathcal{M}$ such that

$$
\begin{gather*}
\left(p_{R} \otimes p_{R}\right) \circ \Delta_{A} \circ i_{I}=0,  \tag{31}\\
\varepsilon_{A} \circ i_{I}=0,  \tag{32}\\
p_{R} \circ m_{A} \circ i_{K}=0 . \tag{33}
\end{gather*}
$$

where $R:=A / I, p_{R}: A \rightarrow R$ denotes the canonical projection and $\left(K, i_{K}: K \rightarrow A \otimes A\right):=$ $\operatorname{Ker}\left(p_{R} \otimes p_{R}\right)$. Then there are maps $m_{R}, u_{R}, \Delta_{R}, \varepsilon_{R}$ such that $\left(R, m_{R}, u_{R}, \Delta_{R}, \varepsilon_{R}\right)$ is a bialgebra in $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r, c)$ and $p_{R}$ is a bialgebra morphism.

Proof. In this proof we omit the constraints as in view of the coherence theorem they take care of themselves. By (31) and (32), there are morphisms

$$
\Delta_{R}: R \rightarrow R \otimes R \quad \text { and } \quad \varepsilon_{R}: R \rightarrow \mathbb{k}
$$

defined by $\Delta_{R} p_{R}=\left(p_{R} \otimes p_{R}\right) \Delta_{A}$ and $\varepsilon_{R}\left(p_{R}\right)=\varepsilon_{A}$. The first equality yields

$$
\begin{aligned}
\left(\Delta_{R} \otimes R\right) \Delta_{R} p_{R} & =\left(\left(p_{R} \otimes p_{R}\right) \otimes p_{R}\right)\left(\Delta_{A} \otimes A\right) \Delta_{A} \\
& =\left(p_{R} \otimes\left(p_{R} \otimes p_{R}\right)\right)\left(A \otimes \Delta_{A}\right) \Delta_{A}=\left(R \otimes \Delta_{R}\right) \Delta_{R} p_{R}
\end{aligned}
$$

so that $\left(\Delta_{R} \otimes R\right) \Delta_{R}=\left(R \otimes \Delta_{R}\right) \Delta_{R}$. The other equality leads to counitarity of $\Delta_{R}$. Since the tensor functors are right exact, we have that $p_{R} \otimes p_{R}$ is an epimorphism and hence $\left(R \otimes R, p_{R} \otimes p_{R}\right)=$ Coker $\left(i_{K}\right)$. Thus, by (33), we have that there is a unique map $m_{R}: R \otimes R \rightarrow R$ such that $m_{R}\left(p_{R}\right)=\left(p_{R} \otimes p_{R}\right) m_{A}$. Set $u_{R}:=p_{R}\left(u_{A}\right)$. The first equality yields

$$
m_{R}\left(m_{R} \otimes R\right) p_{R}^{\otimes 3}=m_{R}\left(R \otimes m_{R}\right) p_{R}^{\otimes 3}
$$

so that, by right exactness of tensor functors, we get $m_{R}\left(m_{R} \otimes R\right)=m_{R}\left(R \otimes m_{R}\right)$. Similarly one gets $m_{R}\left(u_{R} \otimes R\right)=l_{R}$ and $m_{R}\left(R \otimes u_{R}\right)=r_{R}$. Finally, we have

$$
\begin{aligned}
& \left(m_{R} \otimes m_{R}\right)\left(R \otimes c_{R, R} \otimes R\right)\left(\Delta_{R} \otimes \Delta_{R}\right)\left(p_{R} \otimes p_{R}\right) \\
= & \left(p_{R} \otimes p_{R}\right)\left(m_{A} \otimes m_{A}\right)\left(A \otimes c_{A, A} \otimes A\right)\left(\Delta_{A} \otimes \Delta_{A}\right) \\
= & \left(p_{R} \otimes p_{R}\right) \Delta_{A} m_{A}=\Delta_{R} m_{R}\left(p_{R} \otimes p_{R}\right) .
\end{aligned}
$$

Since $p_{R} \otimes p_{R}$ is an epimorphism, we get $\left(m_{R} \otimes m_{R}\right)\left(R \otimes c_{R, R} \otimes R\right)\left(\Delta_{R} \otimes \Delta_{R}\right)=\Delta_{R} m_{R}$. Thus $\left(R, m_{R}, u_{R}, \Delta_{R}, \varepsilon_{R}\right)$ is a bialgebra in $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r, c)$. Clearly $p_{R}$ is a bialgebra morphism.
Lemma 4.5. Let $(H, \omega)$ be a dual quasi-bialgebra and let $I$ be an ideal of a bialgebra $A$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. Let $z, u \in A$ and assume $\Delta_{A}(u) \in A \otimes I+I \otimes A$. Then $\Delta_{A}(z u) \in A \otimes I+I \otimes A$.

Proof. Since $\Delta_{A}(u) \in A \otimes I+I \otimes A$, then

$$
\begin{aligned}
& \Delta_{A}(z u)=\Delta_{A} m_{A}(z \otimes u)=\left(m_{A} \otimes m_{A}\right) \Omega_{A}\left(\Delta_{A} \otimes \Delta_{A}\right)(z \otimes u) \\
\in & \left(m_{A} \otimes m_{A}\right) \Omega_{A}[(A \otimes A) \otimes(A \otimes I)+(A \otimes A) \otimes(I \otimes A)] \\
\subseteq & \left(m_{A} \otimes m_{A}\right)[(A \otimes A) \otimes(A \otimes I)+(A \otimes I) \otimes(A \otimes A)] \subseteq A \otimes I+I \otimes A
\end{aligned}
$$

Let $(H, g, \chi)$ be a quasi- $Y D$ datum for $q$ with $q$ a primitive $N$-th root of unity with $N>0$. Let $V=\mathbb{k} v \in{ }_{H}^{H} \mathcal{Y} \mathcal{D}$ with coaction and action defined by $g$ and $\chi$ as in Remark 3.11. Let $\left(v^{[n]}\right)_{n \in \mathbb{N}}$ be the basis of $T:=T(V)$ considered at the beginning of this section.

Let $I$ be the two-sided ideal of $T$ generated by $v^{[N]}$, i.e, $I=: T(I T)$. Since $v v^{[n]}=v^{[n+1]}$, by (27), $I$ is the vector space with basis $\left(v^{[n]}\right)_{n \geq N}$. Thus, $T / I$ identifies with $K[X] /\left(X^{N}\right)$. By formulas (23) and (22), we deduce that $I$ is a subobject of $T$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. Hence $I$ is a two-sided ideal of $T$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. Moreover $R:=T / I$ with the induced structures is in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ so that the canonical projection $p_{R}: T \rightarrow R$ is in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$.

To check (31) for $I$ we must show that

$$
\Delta_{T}\left(v^{[n]}\right) \in T \otimes I+I \otimes T, \text { for every } n \geq N
$$

For $n=N$, this follows from (28) and the fact that, since $q$ has order $N$, then $\binom{N}{i}_{q}=0$ for $i \neq 0, N$. For $n \geq N+1$, in view of (27) we have

$$
v^{[n]}=\prod_{0 \leq i \leq n-N-1} \omega\left(g \otimes g^{i} \otimes g^{N}\right) v^{[n-N]} v^{[N]}
$$

Hence, Lemma 4.5 implies that $\Delta_{T}\left(v^{[n]}\right) \in T \otimes I+I \otimes T$.
Since by Proposition 4.3, $\varepsilon_{T}\left(v^{[n]}\right)=\delta_{n, 0}$ and $N \neq 0$, it is clear that $\varepsilon_{T}(I)=0$. Since $I$ is a twosided ideal of $T=T(V)$, we have that $m_{T}(T \otimes I+I \otimes T) \subseteq I$. Since $\operatorname{Ker}\left(p_{R} \otimes p_{R}\right)=T \otimes I+I \otimes T$, we deduce that $m_{T}\left(\operatorname{Ker}\left(p_{R} \otimes p_{R}\right)\right) \subseteq I$. By Proposition 4.4, there are maps $m_{R}, u_{R}, \Delta_{R}, \varepsilon_{R}$ such that $\left(R, m_{R}, u_{R}, \Delta_{R}, \varepsilon_{R}\right)$ is a bialgebra in $\left({ }_{H}^{H} \mathcal{Y} \mathcal{D}, \otimes, \mathbb{k},{ }^{H} a, l, r, c\right)$ and $p_{R}$ is a bialgebra morphism.

Recall that the Iverson bracket $[[P]]$ is a notation that denotes a number that is 1 if the condition $P$ in double square brackets is satisfied, and 0 otherwise.

By the above we have the following result.
Theorem 4.6. Let $((H, \omega), g, \chi)$ be a quasi-YD datum for $q>1$, a primitive $N^{t h}$ root of unity.
(i) There is a bialgebra $R=R((H, \omega), g, \chi)$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ with basis $\left(x^{[n]}\right)_{0 \leq n \leq N-1}$ and structure given as follows:

$$
\begin{aligned}
\rho\left(x^{[n]}\right) & : \quad=g^{n} \otimes x^{[n]}, \\
h \triangleright x^{[n]} & :=\chi_{[n]}(h) x^{[n]}, \text { where } \chi_{[n]} \in H^{*} \text { is defined in (2S), } \\
1_{R} & :=x^{[0]}, \\
m_{R}\left(x^{[a]} \otimes x^{[b]}\right) & =[[a+b \leq N-1]]\left[\prod_{0 \leq i \leq a-1} \omega^{-1}\left(g \otimes g^{i} \otimes g^{b}\right)\right] x^{[a+b]} \text { when } a, b \geq 0, \\
\Delta_{R}\left(x^{[n]}\right) & =\sum_{0 \leq i \leq n} \beta(i, n) x^{[i]} \otimes x^{[n-i]}, \text { where } \beta(i, n) \text { is defined in (29), } \\
\varepsilon_{R}\left(x^{[n]}\right) & :=\delta_{n, 0 .}
\end{aligned}
$$

(ii) For $R$ the bialgebra in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ from (i), let $B:=R \# H$, the bosonization of $R$ by $H$. Then

$$
B_{0} \subseteq \mathbb{k} 1_{R} \otimes H
$$

Proof. (i) Take $R:=T(V) / I$ as above and set $x^{[n]}:=v^{[n]}+I$.
 subobject of $R$ in ${ }_{H}^{H} \mathcal{Y D}$ such that

$$
\Delta_{R}\left(R_{[n]}\right) \subseteq \sum_{0 \leq i \leq n} R_{[i]} \otimes R_{[n-i]}
$$

Set $B_{[n]}:=R_{[n]} \otimes H$. By the structure maps for $B$ in Theorem 3.4, we see

$$
\Delta_{B}\left(B_{[n]}\right) \subseteq \sum_{0 \leq i \leq n} B_{[i]} \otimes B_{[n-i]}
$$

Since $B=\cup_{n \in \mathbb{N}} B_{[n]}$, we have proved that $B$ is a filtered coalgebra so that, by Sw2, Proposition 11.1.1, page 226],

$$
B_{0} \subseteq B_{[0]}=R_{[0]} \otimes H=\mathbb{k} x^{[0]} \otimes H=\mathbb{k} 1_{R} \otimes H
$$

Note that the result in the previous theorem still holds formally if $q=1$ but is not so interesting, since, in this case $R$ collapses to the base field $\mathbb{k}$.

Definition 4.7. Let $((H, \omega), g, \chi)$ be a quasi- $Y D$ datum for $q \neq 1$, a primitive $N$-th root of unity. The bialgebra $R=R((H, \omega), g, \chi)$ of the previous theorem will be called a quantum line for the given datum.

Proposition 4.8. The bialgebra $R$ from Theorem 4.6 is a Hopf algebra in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ with bijective antipode $S_{R}: R \rightarrow R$ defined by

$$
S_{R}\left(x^{[n]}\right)=(-1)^{n} \chi(g)^{\frac{n(n-1)}{2}} x^{[n]} \text { for } 0 \leq n \leq N-1 .
$$

Proof. Consider the basis $\left(x^{[n]}\right)_{0 \leq n \leq N-1}$ of the bialgebra $R=R((H, \omega), g, \chi)$ in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. We want to define a linear map $S_{R}: R \rightarrow R$ on the basis which a posteriori is expected to be antimultiplicative in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$. Set $S_{R}\left(x^{[1]}\right)=-x^{[1]}$. Then, for $1<n \leq N-1$, we have

$$
\begin{aligned}
S_{R}\left(x^{[n]}\right) & =S_{R} m_{R}\left(x^{[1]} \otimes x^{[n-1]}\right)=m_{R}\left(S_{R} \otimes S_{R}\right) c_{R, R}\left(x^{[1]} \otimes x^{[n-1]}\right) \\
& =m_{R}\left(S_{R} \otimes S_{R}\right)\left(x_{-1}^{[1]} \triangleright x^{[n-1]} \otimes x_{0}^{[1]}\right) \\
& =m_{R}\left(S_{R} \otimes S_{R}\right)\left(g \triangleright x^{[n-1]} \otimes x^{[1]}\right)=\chi_{[n-1]}(g) m_{R}\left(S_{R} \otimes S_{R}\right)\left(x^{[n-1]} \otimes x^{[1]}\right) \\
& =\chi_{[n-1]}(g) S_{R}\left(x^{[n-1]}\right) S_{R}\left(x^{[1]}\right)=-\chi_{[n-1]}(g) S_{R}\left(x^{[n-1]}\right) x^{[1]},
\end{aligned}
$$

where $c_{R, R}: R \otimes R \rightarrow R \otimes R$ denotes the braiding of ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ evaluated in $R$. Let us check that this forces

$$
S_{R}\left(x^{[n]}\right)=(-1)^{n} \chi(g)^{\frac{n(n-1)}{2}} x^{[n]} \text { for } 0 \leq n \leq N-1
$$

For $n=0,1$ the formula trivially holds. Let $n$ with $1<n \leq N-1$ such that the formula holds for $n-1$. Then

$$
\begin{aligned}
S_{R}\left(x^{[n]}\right) & =-\chi_{[n-1]}(g) S_{R}\left(x^{[n-1]}\right) x^{[1]} \\
& =-\left[\prod_{0 \leq i \leq n-2} \omega\left(g \otimes g^{i} \otimes g\right)\right] \chi(g)^{n-1}(-1)^{n-1} \chi(g)^{\frac{(n-1)(n-2)}{2}} x^{[n-1]} x^{[1]} \\
& =\left[\prod_{0 \leq i \leq n-2} \omega\left(g \otimes g^{i} \otimes g\right)\right](-1)^{n} \chi(g)^{\frac{n(n-1)}{2}} x^{[n-1]} x^{[1]} \\
& =\left[\prod_{0 \leq i \leq n-2} \omega\left(g \otimes g^{i} \otimes g\right)\right](-1)^{n} \chi(g)^{\frac{n(n-1)}{2}} \prod_{0 \leq i \leq n-2} \omega^{-1}\left(g \otimes g^{i} \otimes g\right) x^{[n]} \\
& =(-1)^{n} \chi(g)^{\frac{n(n-1)}{2}} x^{[n]} .
\end{aligned}
$$

We have

$$
\begin{aligned}
S_{R}\left(\left(x^{[n]}\right)^{1}\right)\left(x^{[n]}\right)^{2} & =\sum_{0 \leq i \leq n} \beta(i, n) S_{R}\left(x^{[i]}\right) x^{[n-i]} \\
& =\sum_{0 \leq i \leq n} \beta(i, n)(-1)^{i} \chi(g)^{\frac{i(i-1)}{2}} x^{[i]} x^{[n-i]} \\
& =\sum_{0 \leq i \leq n} \beta(i, n)(-1)^{i} \chi(g)^{\frac{i(i-1)}{2}} \prod_{0 \leq j \leq i-1} \omega^{-1}\left(g \otimes g^{j} \otimes g^{n-i}\right) x^{[n]}
\end{aligned}
$$

But

$$
\begin{equation*}
\beta(i, n) \prod_{0 \leq j \leq i-1} \omega^{-1}\left(g \otimes g^{j} \otimes g^{n-i}\right)=\binom{n}{i}_{q} \tag{34}
\end{equation*}
$$

so that

$$
S_{R}\left(\left(x^{[n]}\right)^{1}\right)\left(x^{[n]}\right)^{2}=\left[\sum_{0 \leq i \leq n}\binom{n}{i}_{q}(-1)^{i} q^{\frac{i(i-1)}{2}}\right] x^{[n]}
$$

By Ka, Proposition IV.2.7] we have that

$$
\sum_{0 \leq i \leq n}\binom{n}{i}_{q}(-1)^{i} q^{\frac{i(i-1)}{2}} a^{n-i} X^{i}=\prod_{0 \leq i \leq n-1}\left(a-q^{i} X\right)
$$

for any scalar $a$ and variable $X$. If we take $a=1$ and evaluate this polynomial in $X=1$, we get $\sum_{0 \leq i \leq n}\binom{n}{i}_{q}(-1)^{i} q^{\frac{i(i-1)}{2}}=\delta_{n, 0}$. Hence

$$
S_{R}\left(\left(x^{[n]}\right)^{1}\right)\left(x^{[n]}\right)^{2}=\delta_{n, 0} x^{[n]}=\delta_{n, 0} x^{[0]}=\varepsilon_{R}\left(x^{[n]}\right) 1_{R}
$$

On the other hand we have

$$
\begin{aligned}
& \left(x^{[n]}\right)^{1} S_{R}\left(\left(x^{[n]}\right)^{2}\right) \\
= & \sum_{0 \leq i \leq n} \beta(i, n) x^{[i]} S_{R}\left(x^{[n-i]}\right) \\
= & \sum_{0 \leq w \leq n} \beta(n-w, n) x^{[n-w]} S_{R}\left(x^{[w]}\right) \\
= & \sum_{0 \leq w \leq n} \beta(n-w, n)(-1)^{w} \chi(g)^{\frac{w(w-1)}{2}} x^{[n-w]} x^{[w]} \\
\text { (34) } & \sum_{0 \leq w \leq n}\binom{n}{w}_{q}(-1)^{w} q^{\frac{w(w-1)}{2}} x^{[n]}=\delta_{n, 0} x^{[n]}=\delta_{n, 0} x^{[0]}=\varepsilon_{R}\left(x^{[n]}\right) 1_{R} .
\end{aligned}
$$

We note that $S_{R}: R \rightarrow R$ is trivially bijective.
Recall the definition of a morphism of quasi- $Y D$ data from Definition 3.13. Note that if $\varphi$ : $((H, \omega), g, \chi) \rightarrow((L, \alpha), \ell, \xi)$ is a morphism of quasi- $Y D$ data, with $((H, \omega), g, \chi)$ a quasi- $Y D$ datum for $q$ then $((L, \alpha), \ell, \xi)$ is also a quasi- $Y D$ datum for $q$ since $\xi(\ell)=\xi \varphi(g)=\chi(g)$. It follows easily from equation (25) that $\xi_{[n]} \varphi=\chi_{[n]}$ for all $n \geq 1$. The proof of the next proposition is straightforward and so the details are left to the reader.
Proposition 4.9. Let $\varphi:((H, \omega), g, \chi) \rightarrow((L, \alpha), l, \xi)$ be a morphism of quasi-YD data with $q:=\chi(g)$ a primitive $N$-th root of unity,$N>0$. Let $\left(x^{[n]}\right)_{0 \leq n \leq N-1}$ be the canonical basis for $R_{H}:=R((H, \omega), g, \chi)$ and $\left(y^{[n]}\right)_{0 \leq n \leq N-1}$ the canonical basis for $R_{L}:=R((L, \alpha), l, \xi)$. Consider the $\mathbb{k}$-linear isomorphism $f: R_{H} \rightarrow R_{L}$ mapping $x^{[n]}$ to $y^{[n]}$ for all $n \in\{0, \ldots, N-1\}$. Then

$$
\begin{gathered}
\rho_{R_{L}} f=(\varphi \otimes f) \rho_{R_{H}}, \quad \mu_{R_{L}}(\varphi \otimes f)=f \mu_{R_{H}}, \\
1_{R_{L}}=f\left(1_{R_{H}}\right), \quad m_{R_{L}}(f \otimes f)=f m_{R_{H}}, \\
\Delta_{R_{L}} f=(f \otimes f) \Delta_{R}, \quad \varepsilon_{R_{L}} f=\varepsilon_{R_{H}} .
\end{gathered}
$$

Moreover $f \otimes \varphi: R_{H} \# H \rightarrow R_{L} \# L$ is a dual quasi-bialgebra homomorphism.

## 5. Quasi-Yetter-Drinfeld data for bosonizations

In this section we consider quasi- $Y D$ data for bosonizations $R \# H$. In the next lemma, we assume that we have a bosonization $B=R \# H$ with a quasi- $Y D$ datum and we find that this yields a quasi-YD datum for $H$.

Lemma 5.1. For $(H, \omega)$ a dual quasi-bialgebra and $R$ a bialgebra in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$, consider the dual quasibialgebra $B:=R \# H$, the bosonization of $R$ by $H$, defined in Theorem 3.4. Assume that $B_{0} \subseteq$ $\mathbb{k} 1_{R} \otimes H$ and let $\left(\left(B, \omega_{B}\right), g, \chi_{B}\right)$ be a quasi-YD datum. Then there exists $c \in G(H)$ such that $g=1_{R} \# c$, and $\left((H, \omega), c, \chi_{B} \sigma\right)$ is a quasi-YD datum where $\sigma: H \hookrightarrow B$ is the inclusion. Moreover, for every $r \in R, h \in H$ we have

$$
\begin{align*}
\chi_{B}(r \# h) & =\omega_{H}^{-1}\left(r_{-2} \otimes h_{1} \otimes c\right) \chi_{B}\left(1 \# h_{2}\right) \omega_{H}\left(r_{-1} \otimes c \otimes h_{3}\right) \chi_{B}\left(r_{0} \# 1\right),  \tag{35}\\
\chi_{B}\left(r r_{R} s \# 1\right) & =\omega_{H}^{-1}\left(r_{-1} \otimes s_{-1} \otimes c\right) \chi_{B}\left(s_{0} \# 1_{H}\right) \chi_{B}\left(r_{0} \# 1\right)  \tag{36}\\
\chi_{B}\left(r \# h_{1}\right) c h_{2} & =\left(r_{-1} h_{1}\right) \chi_{B}\left(r_{0} \# h_{2}\right) c  \tag{37}\\
\chi_{B}\left(r^{1} \# r_{-1}^{2}\right) c \triangleright r_{0}^{2} & =\omega_{H}^{-1}\left(r_{-1}^{1} \otimes r_{-1}^{2} \otimes c\right) r_{0}^{1} \chi_{B}\left(r_{0}^{2} \# 1_{H}\right) \tag{38}
\end{align*}
$$

Proof. Since $g \in G(B) \subseteq B_{0} \subseteq \mathbb{k} \otimes H$, then $g=1_{R} \# c=\sigma(c)$ for some $c \in H$. Thus $c=(\pi \sigma)(c)=$ $\pi(g)$ and since the maps $\pi, \sigma$ from Section 3.1 are coalgebra maps and $g$ is grouplike, then $c$ is grouplike.

In order to apply Lemma 3.14 to conclude that $\left((H, \omega), c, \chi_{B} \sigma\right)$ is a quasi- $Y D$ datum, we must show that $c \chi_{B} \sigma\left(h_{1}\right) h_{2}=h_{1} \chi_{B} \sigma\left(h_{2}\right) c$ for all $h \in H$. Since $\left(\left(B, \omega_{B}\right), g, \chi_{B}\right)$ is a quasi- $Y D$ datum, and so satisfies (15), then for every $r \in R, h \in H$,

$$
g \chi_{B}\left((r \# h)_{1}\right)(r \# h)_{2}=(r \# h)_{1} \chi_{B}\left((r \# h)_{2}\right) g
$$

If we let $r=1_{R}$ in the equation above, and apply $\pi$ to both sides, we obtain

$$
c \chi_{B}\left(\sigma(h)_{1}\right) \pi\left(\sigma(h)_{2}\right)=\pi\left(\sigma(h)_{1}\right) \chi_{B}\left(\sigma(h)_{2}\right) c
$$

and since $\sigma, \pi$ are coalgebra maps with $\pi \sigma$ the identity, then (15) holds for $\left((H, \omega), c, \chi_{B} \sigma\right)$ and by Lemma 3.14, $\left((H, \omega), c, \chi_{B} \sigma\right)$ is a quasi- $Y D$ datum.

Since $\omega_{B}=\omega_{H} \circ \pi^{\otimes 3}$ then for all $x, y \in B$, by (14) for the quasi- $Y D$ datum for $B$, we have that $\chi_{B}(x y)$ is:

$$
\begin{align*}
& \omega_{B}^{-1}\left(x_{1} \otimes y_{1} \otimes g\right) \chi_{B}\left(y_{2}\right) \omega_{B}\left(x_{2} \otimes g \otimes y_{3}\right) \chi_{B}\left(x_{3}\right) \omega_{B}^{-1}\left(g \otimes x_{4} \otimes y_{4}\right)  \tag{39}\\
= & \omega_{H}^{-1}\left(\pi\left(x_{1}\right) \otimes \pi\left(y_{1}\right) \otimes c\right) \chi_{B}\left(y_{2}\right) \omega_{H}\left(\pi\left(x_{2}\right) \otimes c \otimes \pi\left(y_{3}\right)\right) \chi_{B}\left(x_{3}\right) \omega_{H}^{-1}\left(c \otimes \pi\left(x_{4}\right) \otimes \pi\left(y_{4}\right)\right)
\end{align*}
$$

By (13),

$$
\begin{align*}
(\pi \otimes \pi \otimes B \otimes \pi) \Delta_{B}^{3}(r \# 1) & =r_{-2} \otimes r_{-1} \otimes\left(r_{0} \# 1\right) \otimes 1_{H}  \tag{40}\\
(\pi \otimes B \otimes \pi \otimes \pi) \Delta_{B}^{3}(1 \# h) & =h_{1} \otimes\left(1 \# h_{2}\right) \otimes h_{3} \otimes h_{4} \tag{41}
\end{align*}
$$

and so, letting $x=r \# 1$ and $y=1 \# h$, we have that $\chi_{B}(r \# h)$ is:

$$
\begin{equation*}
\omega_{H}^{-1}\left(r_{-2} \otimes h_{1} \otimes c\right) \chi_{B}\left(1 \# h_{2}\right) \omega_{H}\left(r_{-1} \otimes c \otimes h_{3}\right) \chi_{B}\left(r_{0} \# 1\right) \omega_{H}^{-1}\left(c \otimes 1 \otimes h_{4}\right), \tag{42}
\end{equation*}
$$

and since $\omega_{H}$ is normalized, (35) holds.
Similarly $\chi_{B}\left(r \cdot_{R} s \# 1\right)=\chi_{B}((r \# 1)(s \# 1))$ and then, using (40) and (39), along with

$$
(\pi \otimes B \otimes \pi \otimes \pi) \Delta_{B}^{3}\left(s \# 1_{H}\right)=s_{-1} \otimes\left(s_{0} \# 1\right) \otimes 1_{H} \otimes 1_{H}
$$

it is straightforward to verify (36).
Now we prove (37). Since $\left(\left(B, \omega_{B}\right), g, \chi_{B}\right)$ satisfies (15), we have,

$$
g \chi_{B}\left((r \# h)_{1}\right)(r \# h)_{2}=(r \# h)_{1} \chi_{B}\left((r \# h)_{2}\right) g, \text { for every } r \in R, h \in H
$$

Recall from Theorem 3.4 that

$$
\Delta_{B}(r \# h)=\omega_{H}^{-1}\left(r_{-1}^{1} \otimes r_{-2}^{2} \otimes h_{1}\right) r_{0}^{1} \# r_{-1}^{2} h_{2} \otimes r_{0}^{2} \# h_{3}
$$

so that applying $\pi$ to the left hand side of (15) for $B$ we obtain:

$$
c \chi_{B}\left(r_{0}^{1} \# r_{-1}^{2} h_{2}\right) \omega_{H}^{-1}\left(r_{-1}^{1} \otimes r_{-2}^{2} \otimes h_{1}\right) \varepsilon\left(r_{0}^{2}\right) h_{3}=\chi_{B}\left(r \# h_{1}\right) c h_{2}
$$

Applying $\pi$ to the right hand side yields

$$
\omega_{H}^{-1}\left(r_{-1}^{1} \otimes r_{-2}^{2} \otimes h_{1}\right) \varepsilon\left(r_{0}^{1}\right) r_{-1}^{2} h_{2} \chi_{B}\left(r_{0}^{2} \# h_{3}\right) c=\chi_{B}\left(r_{0} \# h_{2}\right) r_{-1} h_{1} c,
$$

and thus (37) holds.
Equation (38) is verified in a similar fashion. Let $h=1$ in the left hand side of equation (15) for $B$ and then apply $R \otimes \varepsilon_{H}$ to obtain

$$
\chi_{B}\left((r \# 1)_{1}\right)\left(R \otimes \varepsilon_{H}\right)\left[(1 \# c)(r \# 1)_{2}\right]=\chi_{B}\left(r_{0}^{1} \# r_{-1}^{2}\right)\left(R \otimes \varepsilon_{H}\right)\left[c \triangleright r_{0}^{2} \# c\right]=\chi_{B}\left(r^{1} \otimes r_{-1}^{2}\right) c \triangleright r_{0}^{2}
$$

Now let $h=1$ in the right hand side of (15) for $B$ and apply $R \otimes \varepsilon_{H}$ to obtain

$$
\begin{aligned}
& \chi_{B}\left((r \# 1)_{2}\right)\left(R \otimes \varepsilon_{H}\right)\left[(r \# 1)_{1}(1 \# c)\right]=\chi_{B}\left(r_{0}^{2} \# 1\right)\left(R \otimes \varepsilon_{H}\right)\left[\left(r_{0}^{1} \# r_{-1}^{2}\right)(1 \# c)\right] \\
& =\chi_{B}\left(r_{0}^{2} \# 1\right) \omega_{H}^{-1}\left(\left(r_{0}^{1}\right)_{-1} \otimes\left(r_{-1}^{2}\right)_{1} \otimes c\right)\left(r_{0}^{1}\right)_{0} \varepsilon_{H}\left(\left(r_{-1}^{2}\right)_{2} c\right) \\
& =\chi_{B}\left(r_{0}^{2} \# 1\right) \omega_{H}^{-1}\left(r_{-1}^{1} \otimes r_{-1}^{2} \otimes c\right) r_{0}^{1}
\end{aligned}
$$

and this finishes the proof of (38).
In the next proposition we show how an arbitrary quasi- $Y D$ datum on a bosonization $R \# H$ where $R:=R\left(\left(H, \omega_{H}\right), g_{H}, \chi_{H}\right)$ is related to $g_{H}$ and $\chi_{H}$.

QUANTUM LINES FOR DUAL QUASI-BIALGEBRAS
Proposition 5.2. Let $\left(\left(H, \omega_{H}\right), g_{H}, \chi_{H}\right)$ be a quasi-YD datum for a primitive $N$-th root of unity $q$ and let $R=R\left(\left(H, \omega_{H}\right), g_{H}, \chi_{H}\right)$ be the bialgebra in ${ }_{H}^{H} \mathcal{Y D}$ introduced in Theorem 4.6. Let $B=R \# H$, the bosonization of $R$ by $H$ and suppose that $\left(\left(B, \omega_{B}\right), g_{B}, \chi_{B}\right)$ is a quasi- $Y D$ datum. Then there exists $d \in G(H)$ such that $g_{B}=1_{R} \# d$. If $d \neq g_{H} d$, then
(i) $\chi_{B}(r \# h)=\varepsilon_{R}(r) \chi_{B}\left(1_{R} \# h\right)$, for every $r \in R, h \in H$,
(ii) $\chi_{B}\left(1_{R} \# g_{H}\right) \chi_{H}(d)=1$,
(iii) $d g_{H}=g_{H} d$.

Proof. Theorem 4.6 implies that $B_{0} \subseteq \mathbb{k} 1_{R} \otimes H$. Then Lemma 5.1 implies that there exists $d \in G(H)$ such that $g_{B}=1_{R} \# d$. By (37) with $r=x^{[1]}$ and $h=1_{H}$,

$$
\chi_{B}\left(x^{[1]} \# 1_{H}\right) d=\chi_{B}\left(x^{[1]} \# 1_{H}\right) g_{H} d
$$

If $\chi_{B}\left(x^{[1]} \# 1_{H}\right) \neq 0$, then $d=g_{H} d$, contrary to our assumption and so $\chi_{B}\left(x^{[1]} \# 1_{H}\right)=0$.
Now, let $2 \leq n \leq N-1$ and assume $\chi_{B}\left(x^{[n-1]} \# 1_{H}\right)=0$. Then

$$
\begin{aligned}
\chi_{B}\left(x^{[n]} \# 1_{H}\right) & =\chi_{B}\left(x^{[1]} \cdot R x^{[n-1]} \# 1_{H}\right) \\
& \stackrel{(36)}{=} \omega_{H}^{-1}\left(g_{H} \otimes g_{H}^{n-1} \otimes d\right) \chi_{B}\left(x^{[n-1]} \# 1_{H}\right) \chi_{B}\left(x^{[1]} \# 1_{H}\right)=0
\end{aligned}
$$

so that

$$
\chi_{B}\left(x^{[n]} \# 1_{H}\right)=\delta_{n, 0}, \text { for } 0 \leq n \leq N-1
$$

Now

$$
\begin{aligned}
\chi_{B}\left(x^{[n]} \# h\right) & \stackrel{ }{=} \omega_{H}^{-1}\left(g_{H}^{n} \otimes h_{1} \otimes d\right) \chi_{B}\left(1_{R} \# h_{2}\right) \omega_{H}\left(g_{H}^{n} \otimes d \otimes h_{3}\right) \chi_{B}\left(x^{[n]} \# 1_{H}\right) \\
& =\delta_{n, 0} \chi_{B}\left(1_{R} \# h\right)
\end{aligned}
$$

and so

$$
\chi_{B}(r \# h)=\varepsilon_{R}(r) \chi_{B}\left(1_{R} \# h\right), \text { for every } r \in R, h \in H
$$

Next we consider equation (38) with $r=x^{[1]}$.
The left hand side is

$$
\begin{aligned}
& \chi_{B}\left(\left(x^{[1]}\right)^{1} \#\left(x^{[1]}\right)_{-1}^{2}\right) d \triangleright\left(x^{[1]}\right)_{0}^{2} \\
= & \chi_{B}\left(x^{[1]} \# 1_{H}\right) d \triangleright 1_{R}+\chi_{B}\left(1_{R} \# g_{H}\right) d \triangleright x^{[1]} \\
= & \chi_{B}\left(1_{R} \# g_{H}\right) d \triangleright x^{[1]}=\chi_{B}\left(1_{R} \# g_{H}\right) \chi_{H}(d) x^{[1]}
\end{aligned}
$$

and the right hand side is

$$
\begin{aligned}
& \omega_{H}^{-1}\left(\left(x^{[1]}\right)_{-1}^{1} \otimes\left(x^{[1]}\right)_{-1}^{2} \otimes d\right)\left(x^{[1]}\right)_{0}^{1} \chi_{B}\left(\left(x^{[1]}\right)_{0}^{2} \# 1_{H}\right) \\
= & \omega_{H}^{-1}\left(1_{H} \otimes g_{H} \otimes d\right) \chi_{B}\left(x^{[1]} \# 1_{H}\right)+\omega_{H}^{-1}\left(g_{H} \otimes 1_{H} \otimes d\right) x^{[1]} \chi_{B}\left(1_{R} \# 1_{H}\right) \\
= & x^{[1]}+1_{R} \chi_{B}\left(x^{[1]} \# 1_{H}\right)=x^{[1]} .
\end{aligned}
$$

and we can conclude that

$$
\chi_{B}\left(1_{R} \# g_{H}\right) \chi_{H}(d)=1
$$

Now we apply (15) for the quasi- $Y D$ datum $\left(H, d, \chi_{B}\left(1_{R} \otimes-\right)\right)$ from Lemma 5.1 with $h=g_{H}$ to obtain

$$
\chi_{B}\left(1_{R} \# g_{H}\right) d g_{H}=g_{H} d \chi_{B}\left(1_{R} \# g_{H}\right)
$$

and since $\chi_{B}\left(1_{R} \# g_{H}\right)$ is invertible, we obtain

$$
d g_{H}=g_{H} d
$$

## 6. Examples

In this section, we present examples illustrating the theory in the previous sections. The problem of course is to find the reassociator explicitly. Our examples are based on the coalgebra $\mathbb{k} C_{n}$ where the cocycles are well-known.
6.1. Group cohomology. First we set some notation. Our examples will involve cyclic groups of order $n$ and $n^{2}, n>1$. We will denote $C_{n}=\langle c\rangle$ and $C_{n^{2}}=\langle\mathfrak{c}\rangle$. We will always denote by $q$ a primitive $n^{2}$-rd root of unity and set $\zeta:=q^{n}$. Let $\phi: C_{n^{2}} \rightarrow C_{n}$ be the canonical projection with $\phi(\mathfrak{c})=c$ and denote by the same symbol the corresponding map $\mathbb{k} C_{n^{2}} \rightarrow \mathbb{k} C_{n}$. For every $a \in \mathbb{Z}$, let $a^{\prime} \in\{0, \ldots, n-1\}$ be congruent to $a$ modulo $n$.

Since $\mathbb{k}$ is an algebraically closed field of characteristic zero, by Sw1, Theorem 3.1], the Sweedler cohomology can be computed through an isomorphism

$$
H_{s w}^{t}\left(\mathbb{k} C_{n}, \mathbb{k}\right) \cong H^{t}\left(C_{n}, \mathbb{k}^{\times}\right),
$$

where the latter is the group cohomology computed as in WE, page 167].
For $0 \leq i \leq n-1$ and $0 \leq a, b, d$, define $\omega_{\zeta^{i}}:\left(\mathbb{k} C_{n}\right)^{\otimes 3} \rightarrow \mathbb{k}$ by

$$
\begin{equation*}
\omega_{\zeta^{i}}\left(c^{a} \otimes c^{b} \otimes c^{d}\right)=\omega_{\zeta^{i}}\left(c^{a} \otimes c^{d} \otimes c^{b}\right)=\zeta^{\left.i a\left[b^{\prime}+d^{\prime}>n-1\right]\right]} \tag{43}
\end{equation*}
$$

Since $\zeta=q^{n}$, it is easy to check that

$$
\begin{equation*}
\omega_{\zeta^{i}}\left(c^{a} \otimes c^{b} \otimes c^{d}\right)=\zeta^{i a\left[\left[b^{\prime}+d^{\prime}>n-1\right]\right]}=q^{i n a\left[\left[b^{\prime}+d^{\prime}>n-1\right]\right]}=q^{i a\left(b^{\prime}+d^{\prime}-(b+d)^{\prime}\right)} \tag{44}
\end{equation*}
$$

One can prove that the set of Sweedler 3-cocycles is given by
$Z_{s w}^{3}\left(\mathbb{k} C_{n}, \mathbb{k}\right)=\left\{\left(\omega_{\zeta^{i}}\right)^{v}=\omega_{\zeta^{i}} * \partial^{2} v \mid 0 \leq i \leq n-1, v: \mathbb{k} C_{n}^{\otimes 2} \rightarrow \mathbb{k}\right.$ is convolution invertible $\}$.
This follows from the fact (see e.g. MS, formulas (E.13) and (E.14)] over $\mathbb{C}$ ) that the map

$$
\begin{equation*}
\left\{k \in \mathbb{k}^{\times} \mid k^{n}=1\right\} \rightarrow H_{s w}^{3}\left(\mathbb{k} C_{n}, \mathbb{k}\right): k \mapsto\left[\omega_{k}\right] \tag{46}
\end{equation*}
$$

is a group isomorphism.
Proposition 6.1. Let $\mathbb{k} C_{n}$ be the group algebra with its standard bialgebra structure and $\omega$ a normalized 3-cocycle. Then $\left(\mathbb{k} C_{n}, \omega\right)$ is a dual quasi-bialgebra and there is a gauge transformation $\alpha:\left(\mathbb{k} C_{n}\right)^{\otimes 2} \rightarrow \mathbb{k}$, and $0 \leq i \leq n-1$ such that $\left(\mathbb{k} C_{n}, \omega\right)=\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right)^{\alpha}=\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}} * \partial^{2} \alpha\right)$.
Proof. The first statement follows from the fact that $\mathbb{k} C_{n}$ is cocommutative. Since $\omega$ is a normalized Sweedler 3-cocycle, by (45) there exists a convolution invertible map $v: \mathbb{k} C_{n}^{\otimes 2} \rightarrow \mathbb{k}$ and $i \in$ $\{0, \ldots, n-1\}$ such that $\omega=\left(\omega_{\zeta^{i}}\right)^{v}=\partial^{2} v * \omega_{\zeta^{i}}$ and $\left(\mathbb{k} C_{n}, \omega\right)=\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}^{v}\right)=\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right)^{v}$. Since $\omega$ and $\omega_{\zeta^{i}}$ are normalized, so is $\partial^{2} v$. Thus, by Corollary 2.4, $a v$ is a gauge transformation for $a=v(1 \otimes 1)^{-1}$. Since $\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right)^{v}=\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right)^{a v}$, the statement is proved.

REmARK 6.2. In fact, $\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right)$ is a dual quasi-Hopf algebra, meaning that there exists an antipode $S$ and maps $\alpha, \beta$ from $\mathbb{k} C_{n}$ to $\mathbb{k}$ such that for all $h \in \mathbb{k} C_{n}$ :

$$
\begin{align*}
& S\left(h_{1}\right) \alpha\left(h_{2}\right) h_{3}=\alpha(h) 1 \text { and } h_{1} \beta\left(h_{2}\right) S\left(h_{3}\right)=\beta(h) 1  \tag{47}\\
& \omega_{\zeta^{i}}\left(h_{1} \beta\left(h_{2}\right) \otimes S\left(h_{3}\right) \otimes \alpha\left(h_{4}\right) h_{5}\right)=\omega_{\zeta^{i}}^{-1}\left(S\left(h_{1}\right) \otimes \alpha\left(h_{2}\right) h_{3} \otimes \beta\left(h_{4}\right) S\left(h_{5}\right)\right)=\varepsilon(h) \tag{48}
\end{align*}
$$

In this case, $S$ is the usual antipode for $\mathbb{k} C_{n}, \alpha$ and $\beta$ are both equal to the counit $\varepsilon$ and then since $\omega_{\zeta^{i}}\left(c^{j} \otimes c^{-j} \otimes c^{j}\right)=1$, the statement follows.

Since by the above discussion the maps $\omega_{\zeta^{i}}$ are not coboundaries, we have the following:
Corollary 6.3. The dual quasi-bialgebra $\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right)$ is not quasi-isomorphic to an ordinary bialgebra, i.e., one with reassociator $\varepsilon_{\mathrm{k} C_{n}^{\otimes 3}}$.

On the other hand $\mathbb{k} C_{n^{2}}$ with dual quasi-bialgebra structure via the bialgebra epimorphism $\phi: \mathbb{k} C_{n^{2}} \rightarrow \mathbb{k} C_{n}, \phi(\mathfrak{c})=c$, is quasi-isomorphic to an ordinary bialgebra since if $\omega$ is a normalized 3 -cocycle for $\mathbb{k} C_{n}$, then $\omega \phi^{\otimes 3}$ is a coboundary. In fact, one can see by direct computation that $\omega_{\zeta^{i}} \phi^{\otimes 3}=\partial^{2} v_{i}$ where $v_{i}:\left(\mathbb{k} C_{n^{2}}\right)^{\otimes 2} \rightarrow \mathbb{k}$ is defined by $v_{i}\left(\mathfrak{c}^{a} \otimes \mathfrak{c}^{b}\right)=q^{i a\left(b-b^{\prime}\right)}$.
6.2. Quasi- $Y D$ data for $\mathbb{k} C_{n}$. To find quasi- $Y D$ data for $\left(\mathbb{k} C_{n}, \omega_{\zeta^{w}}\right)$, we apply the results of Section 3.2. For $0 \leq z \leq n-1$, from (19) we will be able to show that if $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta^{w}}\right), g:=c^{z}, \chi\right)$ is a quasi- $Y D$ datum then

$$
\begin{equation*}
\chi\left(c^{t}\right)=\chi(c)^{t} \text { for } 0 \leq t \leq n-1 \text { and } \chi(c)^{n}=\zeta^{w z} \tag{49}
\end{equation*}
$$

and thus, unless $\chi(c)^{n}=1$, i.e., $\zeta^{w z}=1$, then $\chi$ is not a character.
We show (49) as follows. From (19) and the definition of $\omega_{\zeta^{w}}$, we deduce that $\chi\left(c^{t}\right)=\chi(c)^{t}$ for $1 \leq t \leq n-1$. By unitarity of $\chi$, this equality also holds for $t=0$. By unitarity of $\chi$ and the fact that $c^{n}=1$, we get $1=\chi(1)=\chi\left(c^{n}\right)$. On the other hand a direct computation of $\chi\left(c^{n}\right)$ using (19) and the definition of $\omega_{\zeta^{w}}$ yields $\chi\left(c^{n}\right)=\chi(c)^{n} \zeta^{-w z}$ and so (49) is proved.

Take $t \in \mathbb{N}$. Then, since $c^{n}=1$, we have $\chi\left(c^{t}\right)=\chi\left(c^{t^{\prime}}\right) \stackrel{(49)}{=} \chi(c)^{t^{\prime}}$. Thus (49) is equivalent to

$$
\begin{equation*}
\chi\left(c^{t}\right)=\chi(c)^{t^{\prime}} \text { for } t \in \mathbb{N} \text { and } \chi(c)^{n}=\zeta^{w z} \tag{50}
\end{equation*}
$$

Proposition 6.4. Consider the dual quasi-bialgebra $\left(\mathbb{k} C_{n}, \omega_{\zeta^{w}}\right)$. Let $c^{z} \in C_{n}, 0 \leq z \leq n-1$ and $\chi \in \mathbb{k} C_{n}^{*}$. If $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta^{w}}\right), c^{z}, \chi\right)$ is a quasi-YD datum, then (50), or equivalently (49), holds. Conversely if $\chi$ is a unitary map satisfying (50), or equivalently (49), for some $0 \leq z \leq n-1$ then $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta^{w}}\right), c^{z}, \chi\right)$ is a quasi-YD datum.

Proof. The first assertion follows immediately from Lemma 3.15 and Remark 3.16.
Since $c^{z} \in G\left(\mathbb{k} C_{n}\right)$, since $\chi \in \mathbb{k} C_{n}^{*}$ is unitary by assumption, and since (15) holds because $\mathbb{k} C_{n}$ is both commutative and cocommutative, it remains only to check (14). We check (14) on generators. The equality holds trivially for $h=1_{\mathbb{k} C_{n}}$ or $k=1_{\mathbb{k} C_{n}}$. Hence we can assume that $h=c^{a}$ and $k=c^{b}$ for $1 \leq a, b \leq n-1$. Then the left side of (14) is

$$
\chi\left(c^{a} c^{b}\right)=\chi\left(c^{a+b}\right) \chi(c)^{(a+b)^{\prime}}=\chi(c)^{a+b-[[a+b \geq n]] n} \chi(c)^{a+b} \zeta^{-[[a+b \geq n]] w z}
$$

Since $\omega_{\zeta^{w}}=\omega_{\zeta^{w}}\left(\mathbb{k} C_{n} \otimes \tau\right)$ and $\mathbb{k} C_{n}$ is cocommutative, the right hand side of (14) is:

$$
\omega_{\zeta^{w}}^{-1}\left(c^{z} \otimes c^{a} \otimes c^{b}\right) \chi\left(c^{a}\right) \chi\left(c^{b}\right) \stackrel{50}{=} \chi(c)^{a+b} \omega_{\zeta^{w}}^{-1}\left(c^{z} \otimes c^{a} \otimes c^{b}\right) \stackrel{\text { 43) }}{=} \chi(c)^{a+b} \zeta^{-[[a+b \geq n]] w z} .
$$

Thus (14) holds and the proof is complete.
Example 6.5. Consider the dual quasi-bialgebra ( $\mathbb{k} C_{n}, \omega_{\zeta^{i}}$ ) with $i>0$. Let $c^{z} \in C_{n}$ with $1 \leq z \leq$ $n-1$, and then for $\chi \in\left(\mathbb{k} C_{n}\right)^{*}$ to satisfy (49), we must have that $\chi(c)^{n}=\zeta^{i z}=q^{n i z}$. Thus if we define $\chi\left(c^{t}\right)=q^{i z t}$ for $0 \leq t \leq n-1$, then $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right), c^{z}, \chi\right)$ is a quasi $Y D$-datum for $\chi\left(c^{z}\right)=q^{i z^{2}}$. Note that $q^{i z^{2}}$ is a primitive $r$ th root of unity where $r=\frac{n^{2}}{\left(n^{2}, i z^{2}\right)}$.

More generally, for $0 \leq j \leq n-1$, let $\chi_{j}: \mathbb{k} C_{n} \rightarrow \mathbb{k}$ be defined by $\chi_{j}\left(c^{t}\right)=\zeta^{j t} q^{i z t}$ if $0 \leq t \leq n-1$. Since $\chi_{j}(c)^{n}=\left(\zeta^{j}\right)^{n} q^{i z n}=\zeta^{i z}$, so that (49) holds, $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta^{i}}\right), c^{z}, \chi_{j}\right)$ is a quasi- $Y D$ datum for $\chi_{j}\left(c^{z}\right)=\zeta^{j z} q^{i z^{2}}$ and the order of $\zeta^{j z} q^{i z^{2}}$ is $\frac{n^{2}}{\left(n j z+i z^{2}, n^{2}\right)}$.
Example 6.6. Let $n=p$, a prime. For the quasi- $Y D$ datum $\left(\left(\mathbb{k} C_{p}, \omega_{\zeta^{i}}\right), c^{z}, \chi_{j}\right)$ in Example 6.5 with $\chi_{j}\left(c^{t}\right)=\zeta^{j z} q^{i z t}$, there are $p-1$ choices for $i$ and also for $z$ that give quasi- $Y D$ data for a primitive $p^{2}$ rd root of unity and since $j=0, \ldots, p-1$, there are $p$ choices for $j$. Thus one may form $p(p-1)^{2}$ bosonizations $R \# \mathbb{k} C_{p}$ where $R$ has dimension $p^{2}$. Below we discuss which of these can be isomorphic or quasi-isomorphic.

Suppose that $H:=\left(\mathbb{k} C_{p}, \omega_{\zeta^{i}}\right)$ and $L:=\left(\mathbb{k} C_{p}, \omega_{\zeta^{i^{\prime}}}\right)$. Then by the discussion in Section 6.1, $H$ and $L$ are quasi-isomorphic if and only if $i=i^{\prime}$. If $R \# H$ is quasi-isomorphic to $S \# L$ for some $R, S$ as in Theorem 4.6, then, by Remark 3.6, $H$ is quasi-isomorphic to $L$ and thus $i=i^{\prime}$. Thus if two bosonizations as constructed above are quasi-isomorphic, then $i=i^{\prime}$, i.e., $H=L$.

Now fix $H:=\left(\mathbb{k} C_{p}, \omega_{\zeta^{i}}\right)$, and consider the quasi- $Y D$ data $\mathcal{D}:=\left(H, c^{z}, \chi_{j}\right)$ and $\mathcal{E}:=\left(H, c^{w}, \chi_{k}\right)$. Let $R$ (respectively $S$ ) be the Hopf algebra in ${ }_{H}^{H} \mathcal{Y} \mathcal{D}$ constructed from $\mathcal{D}$ ( $\mathcal{E}$ respectively) with basis $x^{[n]}$ (respectively $y^{[n]}$ ). Set $x=x^{[1]}, y=y^{[1]}$. Suppose that there is a dual quasi-bialgebra isomorphism $\Phi: R \# H \rightarrow S \# H$. By the formula for $\Delta_{R}$ in Theorem 4.6, the comultiplication formula from Theorem 3.4 and the fact that the coefficients $\beta(i, n)$ are nonzero, we have that
$P_{1 \# 1,1 \# c^{i}}(R \# H)=\mathbb{k} 1 \#\left(1-c^{i}\right)+\delta_{i, z} \mathbb{k}(x \# 1)$ (respectively $P_{1 \# 1,1 \# c^{i}}(S \# H)=\mathbb{k} 1 \#\left(1-c^{i}\right)+$ $\left.\delta_{i, w} \mathbb{k}(y \# 1)\right)$.

Since $\Phi$ is a morphism of dual quasi-bialgebras, by Remark 3.6, we get that $\Phi\left(1 \# c^{z}\right)=1 \# \Phi^{\prime}\left(c^{z}\right)$. Write $\Phi^{\prime}\left(c^{z}\right)=c^{a}$ with $0 \leq a \leq p-1$. Since $x \# 1 \in P_{1 \# 1,1 \# c^{z}}(R \# H)$, we get that $\Phi(x \# 1) \in$ $P_{1 \# 1,1 \# c^{a}}(S \# H)=\mathbb{k} 1 \#\left(1-c^{a}\right)+\delta_{a, w} \mathbb{k}(y \# 1)$. If $a \neq w$, then $\Phi(x \# 1) \in \mathbb{k} 1 \#\left(1-c^{a}\right)$ and hence $x \# 1 \in \mathbb{k} \Phi^{-1}\left(1 \#\left(1-c^{a}\right)\right) \subseteq \mathbb{k} \# H$, a contradiction. Thus $a=w$ and hence $\Phi\left(1 \# c^{z}\right)=1 \# c^{w}$, and $\Phi(x \# 1)=\alpha y \# 1+\beta 1 \#\left(1-c^{w}\right)$. Since $\Phi^{-1}(1 \# H)=1 \# H$ then $\alpha \neq 0$.

Then we have

$$
\begin{aligned}
\Phi\left[\left(1 \# c^{z}\right)(x \# 1)\right] & =\Phi\left[\chi_{j}\left(c^{z}\right) x \# c^{z}\right] \\
& =\Phi\left[\chi_{j}\left(c^{z}\right)(x \# 1)\left(1 \# c^{z}\right)\right] \\
& =\chi_{j}\left(c^{z}\right)\left[\alpha y \# 1+\beta 1 \#\left(1-c^{w}\right)\right]\left[1 \# c^{w}\right] \\
& =\chi_{j}\left(c^{z}\right)\left[\alpha y \# c^{w}+\beta 1 \#\left(1-c^{w}\right) c^{w}\right]
\end{aligned}
$$

However,

$$
\begin{aligned}
\Phi\left(1 \# c^{z}\right) \Phi(x \# 1) & =\left(1 \# c^{w}\right)\left(\alpha y \# 1+\beta 1 \#\left(1-c^{w}\right)\right) \\
& =\chi_{k}\left(c^{w}\right) \alpha y \# c^{w}+\beta 1 \#\left(1-c^{w}\right) c^{w}
\end{aligned}
$$

Thus $\beta=0$ and $\chi_{j}\left(c^{z}\right)=\chi_{k}\left(c^{w}\right)$, i.e., $\zeta^{j z} q^{i z^{2}}=\zeta^{k w} q^{i w^{2}}$. Thus $p^{2}$ divides $p(j z-k w)+i\left(z^{2}-w^{2}\right)$ so that $p$ divides $i(z-w)(z+w)$. Then either $z=w$ or $z+w=p$.

Suppose that $z=w$. Then $p$ divides $z(j-k)$. This is impossible unless $j=k$ and then the two quasi- $Y D$ data are the same.

Suppose that $z+w=p$. Then $p$ divides $j z-k w+i(z-w)=j z-k(p-z)+i(2 z-p)$ so that $p$ divides $z(j+k+2 i)$, i.e., $p \mid(j+k+2 i)$.

In any case, there are at least $p(p-1)$ nonisomorphic bosonizations. Fix $z=1$. Then there are $p$ choices for $j$ and $p-1$ choices for $i$ giving nonisomorphic bosonizations.

In the next example, for a change, we consider the group algebra of a nonabelian group and find a quasi- $Y D$ datum.
Example 6.7. Let $G:=D i c_{p}$, the dicyclic group of order $4 p$ for $p$ an odd prime. Then $D i c_{p}=$ $C_{p} \rtimes C_{4}=\left\langle x, y \mid x^{4}=1=y^{p}, x y x^{-1}=y^{-1}\right\rangle$ and $Z(G)=\left\{1, x^{2}\right\}$. Since $C_{p}$ is a normal subgroup of $G$ then there is a bialgebra projection $\pi$ from $\mathbb{k} G$ to $\mathbb{k} C_{4}=\mathbb{k}\langle c\rangle$ by $\pi\left(y^{i} x^{j}\right)=c^{j}$. Let $\omega:=\omega_{\zeta}$ be the cocycle defined in Subsection 6.1 for $\mathbb{k} C_{4}$ with $q$ a primitive 16 th root of unity and $\zeta=q^{4}$. Let $\omega_{G}: \mathbb{k} G^{\otimes 3} \rightarrow \mathbb{k}$ be defined by $\omega_{G}:=\omega \pi^{\otimes 3}$ and thus $\left(\mathbb{k} G, \omega_{G}\right)$ is a dual quasi-bialgebra and $\pi$ is a dual quasi-bialgebra morphism. By Corollary 2.7, since $\left(\mathbb{k} C_{4}, \omega_{\zeta}\right)$ is nontrivial and since there is an inclusion $\sigma: \mathbb{k} C_{4} \hookrightarrow \mathbb{k} G$ such that $\pi \sigma$ is the identity, then $\left(\mathbb{k} G, \omega_{G}\right)$ is also nontrivial.

By Example 6.5 with $n=4,\left(\left(\mathbb{k} C_{4}, \omega_{\zeta}\right), c^{2}, \chi\right)$ with $\chi\left(c^{t}\right)=q^{2 t}$ is a quasi- $Y D$ datum for $\chi\left(c^{2}\right)=q^{4}=\zeta$, a primitive 4th root of unity. By Lemma 3.14, since $\pi\left(x^{2}\right)=c^{2}$ and $x^{2} \in Z(G)$, then $\left(\left(G, \omega_{G}\right), x^{2}, \chi_{G}:=\chi \pi\right)$ is a quasi- $Y D$ datum for $\left(\mathbb{k} G, \omega_{G}\right)$.

Note that for a nonabelian group with trivial centre, the construction in the example above can only yield a trivial $Y D$ datum for $q=1$. On the other hand, the same construction is possible for any nonabelian group $G$ with a projection onto a cyclic group such that the kernel does not contain the centre of $G$.

The next example shows that (49) need not hold for a quasi- $Y D$ datum for $\mathbb{k} C_{N}$ if $\omega \neq \omega_{\zeta^{w}}$, in particular it can happen that $\chi\left(c^{t}\right) \neq \chi(c)^{t}$ for some $0<t<N$.
Example 6.8. Let $\phi: \mathbb{k} C_{n^{2}}=\mathbb{k}\langle\mathfrak{c}\rangle \rightarrow \mathbb{k} C_{n}=\mathbb{k}\langle c\rangle$ be the surjection of bialgebras from Section 6.1 given by $\phi(\mathfrak{c})=c$. Then $\phi$ induces a morphism of dual quasi-bialgebras from $\left(\mathbb{k} C_{n^{2}}, \omega_{\zeta} \phi^{\otimes 3}=\partial^{2} v\right)$ to $\left(\mathbb{k} C_{n}, \omega_{\zeta}\right)$ where, by Section 6.1, $v\left(\mathfrak{c}^{a} \otimes \mathfrak{c}^{b}\right)=q^{a\left(b-b^{\prime}\right)}$.

By Example 6.5, $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c, \chi\right)$ is a quasi $Y D$-datum, with $\chi\left(c^{t}\right)=q^{t}$ for $0 \leq t \leq n-1$, and so by Lemma 3.14, $\left(\left(\mathbb{k} C_{n^{2}}, \partial^{2} v\right), \mathfrak{c}, \chi \phi\right)$ is a quasi- $Y D$ datum also. However, taking $t=n<n^{2}-1$ and checking (49), we find that $\chi \phi\left(\mathfrak{c}^{n}\right)=\chi\left(c^{n}\right)=\chi(1)=1$ while $(\chi \phi(\mathfrak{c}))^{n}=\chi(c)^{n}=q^{n}=\zeta$. Thus in this case (49) is not satisfied. Note that $\omega_{\zeta} \phi^{\otimes 3}=\omega_{\zeta} \phi^{\otimes 3}\left(\mathbb{k} C_{n^{2}} \otimes \tau\right)$ so that (19) still holds.

Example 6.9. Let $((H, \omega), g, \chi)$ be a quasi- $Y D$ datum for some primitive $N$-th root of unity $q, N>0$. Let $L=\mathbb{k}\langle g\rangle$ and let $\varphi:\left(L, \omega_{L}\right) \hookrightarrow\left(H, \omega_{H}\right)$ be the canonical inclusion where $\omega_{L}=\omega_{\mid L \otimes 3}$. Note that $\varphi:\left(\left(L, \omega_{L}\right), g, \chi_{\mid L}\right) \rightarrow((H, \omega), g, \chi)$ is a morphism of quasi- $Y D$ data. By Proposition 4.9, we have a dual quasi-bialgebra homomorphism $f \otimes \varphi: R_{L} \# L \rightarrow R_{H} \# H$ where $R_{L}:=R\left(\left(L, \omega_{L}\right), g, \chi_{\mid L}\right), R_{H}:=R((H, \omega), g, \chi)$ and $f: R_{L} \rightarrow R_{H}$ is a $\mathbb{k}$-linear isomorphism. Note that, since $\varphi$ is injective, so is $f \otimes \varphi$ so that $R_{L} \# L$ identifies with a dual quasi-subbialgebra of $R_{H} \# H$.

We point out that the following example is dual to one given by Gelaki in Ge, subsection 3.1]. There a quasi-Hopf algebra is given which is quasi-isomorphic to an ordinary Hopf algebra but contains a sub-quasi-Hopf algebra which is not.
Example 6.10. Recall the setting of Example 6.8 where we have a morphism of quasi- $Y D$ data from $\left(\left(\mathbb{k} C_{n^{2}}, \omega_{n^{2}}:=\partial^{2} v\right), \mathfrak{c}, \chi \phi\right)$ to $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c, \chi\right)$ with $\chi\left(c^{t}\right)=q^{t}, 0 \leq t \leq n-1$, induced by the bialgebra surjection $\phi: \mathbb{k} C_{n^{2}} \rightarrow \mathbb{k} C_{n}$ with $\phi(\mathfrak{c})=c$. Note that both are quasi- $Y D$-data for $q$ where $q$ is a primitive $n^{2}$-rd root of unity.

The isomorphism $f: R_{n^{2}} \rightarrow R_{n}$ from Proposition 4.9 yields a dual quasi-bialgebra surjection $f \otimes \phi: R_{n^{2}} \# \mathbb{k} C_{n^{2}} \rightarrow R_{n} \# \mathbb{k} C_{n}$ where $R_{n^{2}}:=R\left(\left(\mathbb{k} C_{n^{2}}, \omega_{n^{2}}\right), \mathfrak{c}, \chi \phi\right), R_{n}:=R\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c, \chi\right)$. Set

$$
A:=R_{n^{2}} \# \mathbb{k} C_{n^{2}} \quad \text { and } \quad B:=R_{n} \# \mathbb{k} C_{n},
$$

so that $B$ is a quotient of the dual quasi-bialgebra $A$. Since $\left(\mathbb{k} C_{n^{2}}, \partial^{2} v\right)$ can be twisted by $v^{-1}$ to $\left(\mathbb{k} C_{n^{2}}, \varepsilon_{\left(\mathbb{k} C_{n^{2}}\right)^{\otimes 3}}\right)$, then by Remark 3.8, $A$ is also quasi-isomorphic to an ordinary bialgebra. In fact, it is easy to check that $A$ should be deformed by the gauge transformation $\mu:=v^{-1}(\pi \otimes \pi)$ to obtain an ordinary bialgebra. On the other hand, since $\omega_{\zeta}$ is not trivial in $H^{3}\left(\mathbb{k} C_{n}, \mathbb{k}\right),\left(\mathbb{k} C_{n}, \omega_{\zeta}\right)$ cannot be quasi-isomorphic to an ordinary bialgebra and thus by Remark 3.8, neither can $B$.

We can say more about $A^{\mu}$. Since $A^{\mu}$ is finite dimensional with coradical $\mathbb{k} C_{n^{2}}$, which is a Hopf algebra, then $A^{\mu}$ is also a Hopf algebra Ta, Remark 36]. Now, let $\left(x^{[n]}\right)_{0 \leq n \leq N-1}$ be the canonical basis for $R_{n^{2}}$. Then $X:=x^{[1]} \# 1_{\mathbb{k} C_{n^{2}}}$ is a nontrivial skew-primitive element since

$$
\begin{aligned}
\Delta_{A}\left(x^{[1]} \# 1_{\mathfrak{k} C_{n^{2}}}\right) & =\left(x^{[1]}\right)^{1} \#\left(x^{[1]}\right)_{-1}^{2} \otimes\left(x^{[1]}\right)_{0}^{2} \# 1_{\mathfrak{k} C_{n^{2}}} \\
& =\left(1_{R_{n^{2}}} \# \mathfrak{c}\right) \otimes\left(x^{[1]} \# 1_{\mathbb{k} C_{n^{2}}}\right)+\left(x^{[1]} \# 1_{\mathfrak{k} C_{n^{2}}}\right) \otimes\left(1_{R_{n^{2}}} \# 1_{\mathfrak{k} C_{n^{2}}}\right) \\
& =\left(1_{R_{n^{2}}} \# \mathfrak{c}\right) \otimes\left(x^{[1]} \# 1_{\mathbb{k} C_{n^{2}}}\right)+\left(x^{[1]} \# 1_{\mathfrak{k} C_{n^{2}}}\right) \otimes 1_{A}
\end{aligned}
$$

so that, if we set $\Gamma:=1_{R_{n^{2}}} \# \mathfrak{c}$, we get

$$
\Delta_{A}(X)=X \otimes 1_{A}+\Gamma \otimes X
$$

Since $A$ and $A^{\mu}$ have the same coalgebra structure, $X$ is a $\left(1_{A}, \Gamma\right)$-primitive element also in $A^{\mu}$. Consider the sub-Hopf algebra of $A^{\mu}$ generated by $X$ and $\Gamma$. This is a Taft algebra of dimension $o(\Gamma)^{2}=n^{4}$. Hence $A^{\mu}=T_{q}$.
6.3. Quasi- $Y D$ data for $R \# \mathbb{k} C_{n}$. Now we apply Proposition 5.2 to a quasi- $Y D$ datum used in our examples.

Proposition 6.11. Let $H:=\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c, \chi\right)$ with $\chi\left(c^{t}\right)=q^{t}$ for $0 \leq t \leq n-1$, be the quasi- $Y D$ datum from Example 6.5, let $R:=R\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c, \chi\right)$, and let $B:=R \# H$. Suppose that there is a quasi-YD datum for $B,\left(\left(B, \omega_{B}\right), g_{B}, \chi_{B}\right)$ as in Proposition 5.2. Then $g_{B}=1_{R} \# c^{w}$ for some $0 \leq w \leq n-1$, and for $0 \leq t \leq n-1$ and $r \in R$,

$$
\begin{equation*}
\chi_{B}\left(r \# c^{t}\right)=\varepsilon_{R}(r) q^{-w t} \prod_{0 \leq i \leq t-1} \omega_{\zeta}^{-1}\left(c^{w} \otimes c^{i} \otimes c\right)=\varepsilon_{R}(r) q^{-w t} \tag{51}
\end{equation*}
$$

In particular $g_{B}$ and $\chi_{B}$ are uniquely determined by $w$ and $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c, \chi\right)$.
Proof. By Proposition 5.2 there exists $d=c^{w}$ such that $g_{B}=1_{R} \# d$. Since $c^{w} \neq c c^{w}$, Proposition 5.2 (ii) may be applied to get that,

$$
\chi_{B}\left(1_{R} \# c\right)=\chi\left(c^{w}\right)^{-1}=q^{-w}
$$

Since by Lemma $5.1\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c^{w}, \chi_{B}\left(1_{R} \#-\right)\right)$ is a quasi- $Y D$ datum, then $\chi_{B}\left(1_{R} \#-\right)$ must satisfy (19), i.e.,

$$
\chi_{B}\left(1_{R} \# c^{t}\right)=\chi_{B}\left(1_{R} \# c\right)^{t} \prod_{0 \leq i \leq t-1} \omega_{\zeta}^{-1}\left(c^{w} \otimes c^{i} \otimes c\right)=q^{-w t} \prod_{0 \leq i \leq t-1} \omega_{\zeta}^{-1}\left(c^{w} \otimes c^{i} \otimes c\right)
$$

The statement now follows from (i) of Proposition 5.2.
Corollary 6.12. Let $H, R, B$ be as in the proposition with $n=2 m$. If $\left(\left(B, \omega_{B}\right), g_{B}=1 \# c^{w}, \chi_{B}\right)$ is a quasi-YD datum for $B$ with $c^{w} \neq 1$, then $w=m$.

Proof. By Lemma 5.1, if $\left(\left(B, \omega_{B}\right), g_{B}=1 \# c^{w}, \chi_{B}\right)$ is a quasi- $Y D$ datum, then $\left(\left(\mathbb{k} C_{2 m}, \omega_{\zeta}\right), c^{w}, \chi_{B} \sigma\right)$ is also a quasi- $Y D$ datum where $\sigma$ is the inclusion map. Then by (49), $\chi_{B} \sigma(c)^{n}=\zeta^{w}$. By (51),

$$
\chi_{B}(1 \# c)^{n}=q^{-w n}=\zeta^{-w}
$$

so that $\zeta^{2 w}=1$ and we must have that $w=m$.
We are now able to construct a quasi- $Y D$ datum on a dual quasi-bialgebra which is a bosonization of a group algebra. We begin with a useful lemma.
Lemma 6.13. Let $n, a \in \mathbb{N}$ with $0 \leq a \leq n^{2}-1$. Then

$$
\left|\left\{i \mid 0 \leq i \leq a-1, i^{\prime}=n-1\right\}\right|=\frac{a-a^{\prime}}{n}
$$

Proof. Note that the left hand side of the equation above is the number of nonnegative integers congruent to $n-1 \bmod n$ and strictly less than $a$. For $t \geq 1$, define an interval $I_{t}$ of $n$ integers by $I_{t}=\{j \in \mathbb{N} \mid(t-1) n \leq j \leq t n-1\}$. Then the left hand side is the number of intervals $I_{t}$ whose entries are less than $a$. If $a=a^{\prime}+s n$, then $a \in I_{s+1}$ and this number is clearly $s$.

In the next example, we find a quasi- $Y D$ datum for $B:=R \# \mathbb{k} C_{n}$ where $n$ is even. As always, $q$ denotes a primitive $n^{2}$-rd root of unity and $\zeta:=q^{n}$.

EXAMPLE 6.14. Let $n=2 m$ and let $\left(B:=R \# \mathbb{k} C_{n}, \omega_{B}=\omega_{\zeta} \pi\right.$ ) be the dual quasi-bialgebra of dimension $n^{3}$ constructed via the quasi- $Y D$ datum $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c, \chi\right)$ for $q$ with $\chi\left(c^{t}\right)=q^{t}$, $0 \leq t \leq n-1$, as in Example 6.5. We will construct a quasi- $Y D$ datum for $B$ for $\iota:=q^{-m^{2}}$, a primitive 4 -th root of unity.

First note that $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c^{m}, \widetilde{\chi}\right)$ with $\widetilde{\chi}\left(c^{t}\right)=q^{-m t}$ for $0 \leq t \leq n-1$ is a quasi- $Y D$ datum by Proposition 6.4 since $\widetilde{\chi}(c)^{n}=q^{-m n}=\zeta^{-m}=\zeta^{m}$ since $\zeta$ has order $2 m$. Also $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c^{m}, \widetilde{\chi}\right)$ is a quasi- $Y D$ datum for $\iota$ since $\widetilde{\chi}\left(c^{m}\right)=q^{-m^{2}}$.

We will now show that $\left(\left(B, \omega_{B}:=\omega_{\zeta} \pi^{\otimes 3}\right), g_{B}:=\sigma\left(c^{m}\right), \chi_{B}:=\widetilde{\chi} \pi\right)$, where $\pi, \sigma$ are the usual projection and inclusion maps from Remark 3.8, is a quasi-YD datum. Since $\pi$ is a surjection of dual quasi-bialgebras from $\left(B, \omega_{B}\right)$ to $\left(\mathbb{k} C_{n}, \omega_{\zeta}\right)$ with $\pi \sigma\left(c^{m}\right)=c^{m}$, it remains to show that for all $b \in B$,

$$
\sigma\left(c^{m}\right) \widetilde{\chi} \pi\left(b_{1}\right) b_{2}=b_{1} \widetilde{\chi} \pi\left(b_{2}\right) \sigma\left(c^{m}\right)
$$

in order to apply Lemma 3.14 and conclude that $\left(\left(B, \omega_{B}:=\omega_{\zeta} \pi\right), g_{B}:=\sigma\left(c^{m}\right), \chi_{B}:=\widetilde{\chi} \pi\right)$ is a quasi- $Y D$ datum for $\iota$.

Let $b=x^{[a]} \# c^{\ell}$ for $0 \leq a \leq n^{2}-1$ and $0 \leq \ell \leq n-1$. Since by Theorem 4.6, $\Delta_{R}\left(x^{[a]}\right)=$ $\sum_{0 \leq i \leq a} \beta(i, a) x^{[i]} \otimes x^{[a-i]}$ and since $\beta(0, a)=\beta(a, a)=1$, by applying $\varepsilon_{H}$ on the left and on the right of (13),

$$
\begin{equation*}
\pi\left(b_{1}\right) \otimes b_{2}=c^{a} c^{\ell} \otimes x^{[a]} \# c^{\ell}=c^{a} c^{\ell} \otimes b \text { and } b_{1} \otimes \pi\left(b_{2}\right)=x^{[a]} \# c^{\ell} \otimes c^{\ell}=b \otimes c^{\ell} \tag{52}
\end{equation*}
$$

By the formula for multiplication in $B=R \# \mathbb{k} C_{n}$ in Theorem 3.4, we have that

$$
\left(1 \# c^{m}\right)\left(x^{[a]} \# c^{\ell}\right)=\omega_{\zeta}\left(c^{m} \otimes c^{a} \otimes c^{\ell}\right) \omega_{\zeta}^{-1}\left(c^{a} \otimes c^{m} \otimes c^{\ell}\right) c^{m} \triangleright x^{[a]} \# c^{m} c^{\ell}
$$

and

$$
\left(x^{[a]} \# c^{\ell}\right)\left(1 \# c^{m}\right)=\omega_{\zeta}^{-1}\left(c^{a} \otimes c^{\ell} \otimes c^{m}\right) x^{[a]} \# c^{\ell} c^{m}=\omega_{\zeta}^{-1}\left(c^{a} \otimes c^{m} \otimes c^{\ell}\right) x^{[a]} \# c^{\ell} c^{m}
$$

Since $c^{m} \triangleright x^{[a]}=\chi_{[a]}\left(c^{m}\right) x^{[a]}$ where $\chi_{[a]}$ is defined in Proposition 4.3, it remains to show that

$$
\omega_{\zeta}\left(c^{m} \otimes c^{a} \otimes c^{\ell}\right) \chi_{[a]}\left(c^{m}\right) \widetilde{\chi}\left(c^{a} c^{\ell}\right)=\widetilde{\chi}\left(c^{\ell}\right)
$$

$\operatorname{By}(14)$ for the quasi- $Y D$ datum $\left(\left(\mathbb{k} C_{n}, \omega_{\zeta}\right), c^{m}, \widetilde{\chi}\right)$,

$$
\widetilde{\chi}\left(c^{a} c^{\ell}\right)=\omega_{\zeta}^{-1}\left(c^{m} \otimes c^{a} \otimes c^{\ell}\right) \widetilde{\chi}\left(c^{a}\right) \widetilde{\chi}\left(c^{\ell}\right)
$$

and thus it suffices to prove that $\widetilde{\chi}\left(c^{a}\right) \chi_{[a]}\left(c^{m}\right)=1$. Since $\widetilde{\chi}\left(c^{a}\right)=\widetilde{\chi}\left(c^{a^{\prime}}\right)=q^{-m a^{\prime}}$, this is equivalent to showing that

$$
\chi_{[a]}\left(c^{m}\right)=q^{m a^{\prime}}
$$

Since $c^{m}$ is a cocommutative element, by equation (30)

$$
\begin{aligned}
\chi_{[a]}\left(c^{m}\right) & =\chi\left(c^{m}\right)^{a} \prod_{0 \leq i \leq a-1} \omega_{\zeta}\left(c^{m} \otimes c \otimes c^{i}\right)=q^{m a} \prod_{0 \leq i \leq a-1} \omega_{\zeta}\left(c^{m} \otimes c \otimes c^{i^{\prime}}\right) \\
& =q^{m a} \prod_{0 \leq i \leq a-1} \zeta^{m\left[\left[1+i^{\prime} \geq n\right]\right]}=q^{m a} \prod_{0 \leq i \leq a-1} \zeta^{m \delta_{i^{\prime}, n-1}}
\end{aligned}
$$

Thus we have to prove that

$$
q^{m a} \prod_{0 \leq i \leq a-1} \zeta^{m \delta_{i^{\prime}, n-1}}=q^{m a^{\prime}}
$$

But $q^{m s}=q^{-m s}$ for every $s \in n \mathbb{Z}$ since, writing $s=n \widehat{s}, q^{2 m n \widehat{s}}=q^{n^{2} \widehat{s}}=1$. Thus it suffices to prove that

$$
\prod_{0 \leq i \leq a-1} \zeta^{m \delta_{i^{\prime}, n-1}}=q^{-m\left(a-a^{\prime}\right)}=q^{m\left(a-a^{\prime}\right)}
$$

By Lemma 6.13, we have

$$
\left|\left\{i \mid 0 \leq i \leq a-1, i^{\prime}=n-1\right\}\right|=\frac{a-a^{\prime}}{n}
$$

so that

$$
\prod_{0 \leq i \leq a-1} \zeta^{m \delta_{i^{\prime}, n-1}}=\prod_{0 \leq i \leq a-1} q^{n m \delta_{i^{\prime}, n-1}}=q^{m n \sum_{0 \leq i \leq a-1} \delta_{i^{\prime}, n-1}}=q^{m n \frac{a-a^{\prime}}{n}}=q^{m\left(a-a^{\prime}\right)} .
$$

This shows that $\left(\left(B, \omega_{B}\right), g_{B}:=\sigma\left(c^{m}\right), \chi_{B}:=\widetilde{\chi} \pi\right)$ is a quasi- $Y D$ datum for $q^{-m^{2}}$ and thus one can form the bosonization $S \# B$ of dimension $4 n^{3}=32 m^{3}$ where $S$ has basis $y^{[i]}$ for $0 \leq i \leq 3$.

## References

[An] I. E. Angiono, Basic quasi-Hopf algebras over cyclic groups. Adv. Math. 225 (2010), 3545-3575.
[AP] A. Ardizzoni, A. Pavarin, Bosonization for Dual Quasi-Bialgebras and Preantipode, J. Algebra, Vol. 390 (2013), 126-159.
[CDMM] C. Călinescu, S. Dăscălescu, A. Masuoka, C. Menini, Quantum lines over non-cocommutative cosemisimple Hopf algebras. J. Algebra 273 (2004), no. 2, 753-779.
[EGe1] P. Etingof, S. Gelaki, Finite-dimensional quasi-Hopf algebras with radical of codimension 2, Math. Res. Lett. 11 (2004) 685-696.
[EGe2] _, On radically graded finite-dimensional quasi-Hopf algebras, Mosc. Math. J. 5 (2005), no. 2, 371-378.
[EGe3] , Liftings of graded quasi-Hopf algebras with radical of prime codimension, J. Pure Appl. Algebra 205 (2006), no. 2, 310-322.
[Ge] S. Gelaki, Basic quasi-Hopf algebras of dimension $n^{3}$. J. Pure Appl. Algebra 198 (2005), no. 1-3, 165-174.
[Ka] C. Kassel, Quantum groups, Graduate Text in Mathematics 155, Springer, 1995.
[Mac] S. Mac Lane, Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998.
[MS] G. Moore, N. Seiberg, Classical and quantum conformal field theory. Comm. Math. Phys. 123 (1989), no. 2, 177-254.
[Sc] P. Schauenburg, Quotients of finite quasi-Hopf algebras. Hopf algebras in noncommutative geometry and physics, 281-290, Lecture Notes in Pure and Appl. Math., 239, Dekker, New York, 2005.
[Sw1] M. E. Sweedler, Cohomology of algebras over Hopf algebras. Trans. Amer. Math. Soc. 1331968 205-239. [Sw2] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[Ta] M. Takeuchi, Free Hopf algebras generated by coalgebras. J. Math. Soc. Japan 23 (1971), 561582.
[We] C. Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

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