Università degli Studi di Ferrara

DOTTORATO DI RICERCA
in
Matematica

In convenzione con
Università degli studi di Parma
Università degli studi di Modena e Reggio Emilia

CICLO XXIX

COORDINATORE Prof. Massimiliano Mella

Sharp Estimates for Fundamental Solutions of some degenerate Kolmogorov equations arising in Finance

Settore Scientifico Disciplinare MAT/05

Dottorando
Dott. Gennaro Cibelli

Tutore
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Contents

Introduction

This thesis is devoted to the study of the following parabolic degenerate operator

$$\mathscr{L}u = x\partial_x(a(x,y,t)x\partial_x u) + b(x,y,t)x\partial_x u + x\partial_y u - \partial_t u \tag{1}$$

where $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ and the coefficients $a(x,y,t), b(x,y,t)$ are bounded real functions, with a bounded by below, which satisfy suitable regularity and growth conditions that will be specified in the sequel.
The aim of this work is to prove sharp upper and lower bounds for the fundamental solution of $\mathscr{L}$ in (1): let $\Gamma(x,y,t;x_0,y_0,t_0)$ denote the fundamental solution of the operator $\mathscr{L}$ with pole at (x_0,y_0,t_0), then for sufficiently small $\varepsilon > 0$, there exists two functions $\Gamma_\varepsilon^-, \Gamma_\varepsilon^+ : \mathbb{R}^6 \mapsto \mathbb{R}$ and two positive constants $k_\varepsilon^+, k_\varepsilon^-$ such that

$$k_\varepsilon^- \Gamma_\varepsilon^-(x,y,t;x_0,y_0,t_0) \leq \Gamma(x,y,t;x_0,y_0,t_0) \leq k_\varepsilon^+ \Gamma_\varepsilon^+(x,y,t;x_0,y_0,t_0) \tag{2}$$

where $x,\ x_0 \in \mathbb{R}^+,\ y,\ y_0 \in \mathbb{R},\ t,\ t_0 \in [0,T]$ with $t > t_0$. The interest in this result is in that an expression of Γ is not available, whereas explicit information on the asymptotic behaviour of $\Gamma_\varepsilon^\pm$ are provided in this thesis. Estimates as in (2) are also called *two sided bounds.*

We recall that the operator in (1) appears in several physical and mathematical applications, in particular in Mathematical Finance since it plays a crucial role in the problem of the Pricing of Arithmetic Average Asian Options in the framework introduced by Black, Scholes and Merton in their celebrated papers [14] and [58]. For an exhaustive treatment of this subject we refer to the monographs by Pascucci [66], Björk [13], Shreve [78], Lamberton [46] and Hull [35].

The Mathematical Problem

The problem of proving upper and lower estimates for the fundamental solution of a *second order partial evolution operator* has been considered by many authors in the study of partial differential operators and the theory, nowadays, is rich of interesting results. It started with the seminal works by Nash [63], and by Moser [60, 61] for uniformly parabolic operators. A keystone result in the theory of uniformly parabolic partial differential equations reads as follows: if $\Gamma = \Gamma(x, t, \xi, \tau)$ is the fundamental solution of a parabolic PDE

$$\partial_t u(x,t) = \sum_{i,j=1}^{N} \partial_{x_i} \left(a_{ij}(x,t) \partial_{x_j} u(x,t) \right), \qquad (x,t) \in \mathbb{R}^N \times]0,T], \tag{3}$$

then there exist positive constants c^-, C^-, c^+, C^+ such that

$$\frac{c^-}{(t-\tau)^{N/2}} \exp\left(-C^- \frac{|x-\xi|^2}{t-\tau} \right) \le \Gamma(x,t,\xi,\tau) \le \frac{C^+}{(t-\tau)^{N/2}} \exp\left(-c^+ \frac{|x-\xi|^2}{t-\tau} \right), \tag{4}$$

for every $(x,t), (\xi,\tau) \in \mathbb{R}^N \times]0,T]$ with $\tau < t$. The upper bound has been proved by Aronson [2] for operators with bounded measurable coefficients a_{ij}'s, satisfying the uniform parabolicity condition

$$\exists \Lambda > 0 \; s.t. \; \forall \xi \in \mathbb{R}^N, \quad \Lambda^{-1} |\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x,t) \xi_i \xi_j \le \Lambda |\xi|^2, \quad \forall t > 0, \tag{5}$$

whereas the lower bound was proved by Moser [60], by Fabes and Strook [29] following the fundamental works of Nash [63] for divergence form parabolic operators (3). We also recall that Kusuoka and Stroock in [45] prove (4) by probabilistic methods. We eventually refer to the monograph of Bass [10] where elliptic-parabolic operators and the related results are introduced from a probabilistic point of view.

Many authors such as Davies [24], Jerison and Sánchez-Calle [37] and Varopoulos, Saloff-Coste, Coulhon [81] have been interested in extending the bounds (4) to fundamental solutions of non uniformly parabolic operators

$$\mathcal{L}_0 = \sum_{i=1}^{m} X_i^2(x) - \partial_t, \quad (x,t) \in \mathbb{R}^{N+1}, \tag{6}$$

satisfying the strong Hörmander condition and suitable Hypothesis on the Lie group stucture related to them (see section 2.2 below for the precise notions and definitions). Here $X_1, ..., X_m$ are smooth vector fields on $\mathbb{R}^{N+1}$ and $m \le N$. In this setting, the quantity $|x-\xi|$ appearing in (4) is replaced by the the *Carnot-Carathéodory* distance $d_{CC}(x,\xi)$, which is its natural counterpart in the subelliptic setting. We also remind that the quantity $\sum_{i=1}^m X_i^2$ is called *sub-Laplacian* and operators as in (6), satisfying Hörmander condition, are said *hypoelliptic* (see Definition 2.2.1 below).

The above results was also extended by Boscain-Polidoro [16], Cinti-Polidoro [23] to hypoelliptic operators with drift, that is

$$\mathcal{L}_0 = \sum_{i=1}^{m} X_i^2(x,t) + X_0(x,t) - \partial_t, \quad (x,t) \in \mathbb{R}^{N+1}, \tag{7}$$

satisfying the weak Hörmander condition. Here $X_0, X_1, ..., X_m$ are smooth vector fields on $\mathbb{R}^{N+1}$ and $m \leq N$. In this case, not even the Carnot-Carathéodory distance is appropriate to bound the fundamental solution.

A particular class of second order degenerate parabolic operator which belongs to the family described in (7) are *Kolmogorov Operators.* They are operators in the form

$$\mathcal{K}u \equiv \sum_{i,j=1}^{p_0} a_{i,j}(x,t)\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u, \quad (x,t) \in \mathbb{R}^{N+1} \tag{8}$$

where $(x,t) \in \mathbb{R}^{N+1}$, $p_0 \leq N$ and the coefficients $a_{i,j}(x,t)$ are bounded and smooth functions. Moreover, for this kind of operators, several results are available when the coeffcents a_{ij} are bounded Hölder continuous or measurable. Such operators are widely studied in literature both from the analytical and the probabilistic point of view as they play a crucial role in the description of the transition probability density of a N-dimensional stochastic process and they form a bridge between the theory of Stochastic Differential Equations and PDE's.
We summarize some of the most interesting results on Kolmogorov operators. In Di Francesco-Pascucci [26] and Polidoro [71], the authors prove the existence and upper bounds for the fundamental solution of degenerate operators of Kolmogorov type by using an adaptation of the Levi parametrix method for parabolic operators. The Levi parametrix method was previously successfully used for uniformly parabolic operators (see Friedman [31]). Also, from an analytical point of view Lanconelli-Polidoro [47], Pascucci-Polidoro [68], Cinti-Pascucci-Polidoro [22], Di Francesco-Polidoro [27], Polidoro [73] and Pascucci-Polidoro [67] extended several results on uniformly parabolic operators to the degenerate operators of Kolmogorov type. To reach this goal, very interesting geometric properties linked with the Lie group structure which underlies the operators have been pointed out. The main results are the invariant Harnack inequality for Kolmogorov operators, the Moser's iterative method and the two sided bounds for fundamental solutions under suitably assumptions on the coefficients a_{ij}'s and on the coefficients of the matrix $B = (b_{ij})_{i,j=1,...,N}$.

Other related results are due to Lunardi [52], Lorenzi [51] and Priola [75] who focused on the problem of the existence, uniqueness, optimal Schauder estimates and other regularity properties of the solutions u of Kolmogorov equations.

Besides the analytical aspects, Kolmogorov operators are also discussed from a probabilistic point of view. The most recent results are given by Delarue-Menozzi [25] who extended the two sided bounds to Kolmogorov Operators of the form

$$\mathcal{K}u \equiv \sum_{i,j=1}^{p_0} a_{i,j}(x,t)\partial_{x_i x_j} u + \sum_{j=1}^{N} F_i(x,t)\partial_{x_j} u - \partial_t u, \quad (x,t) \in \mathbb{R}^{N+1}, \tag{9}$$

in that the function F_i are assumed spatial Lipschitz-continuous (uniformly in t) and the coefficients $a_{i,j}(x,t)$ are bounded and spatial Hölder-continuous (uniformly in t). Their approach exploits the parametric expansion for fundamental solutions previously introduced by Singer-McKean [79] combined with elements of Stochastic Control Theory.

Concerning the operator $\mathscr{L}$ in (1) discussed in this thesis, it is clearly degenerate. Consider for instance the prototype of the operator $\mathscr{L}$:

$$\begin{aligned} &\mathscr{L}_0 = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t = X^2(x,y,t) + Y(x,y,t), \\ &X(x,y,t) = x\partial_x, \; Y(x,y,t) = x\partial_y - \partial_t. \end{aligned}$$

Note that, even if $\mathscr{L}_0$ can be written in the form (7) the estimates proven in [16], [23] do not apply to it for the following reason. This operator does not satisfy the Hörmander condition on the whole space $\mathbb{R}^3$, since the vector field $X(x,y,t)$ vanishes at $x = 0$. This fact yields some consequences on the underlying Lie group structure which satisfy weaker Hypotheses than ones made for hypoelliptic operators (7) in [39], [16], [23].

On the other hand, the operator $\mathscr{L}_0$ cannot be globally seen as a Kolmogorov operator with bounded variable variable coefficients, as those introduced in Delarue-Menozzi [25], Di Francesco-Polidoro [27], Pascucci-Polidoro [68, 67], Di Francesco-Pascucci [26]: indeed the variable coefficient x^2 is unbounded and its infimum equals zero. Hence, the results proven in the above references cannot be applied.

In this work we suitably adapt the methodology appearing in [27] and [23] in order to achieve the bounds (2), giving an explicit expression to the function $\Gamma^{\pm}_{\varepsilon}$ appearing in (2). The new difficulty concerning the operator $\mathscr{L}$ is thus in managing the unbounded and possibly vanishing coefficient x which appears in the vector field $X(x,y,t) = x\partial_x$. As we will see in the sequel, the operator $\mathscr{L}$ exhibits different features from a classical Kolmogorov operator with bounded Hölder continuous coefficients.

Our Approach and Main Results

Our main result extends the bounds (4) to the operator $\mathscr{L}$. Specifically, we prove that

$$\begin{aligned} \frac{c_\varepsilon^-}{\xi^2(t-t_0)^2} \exp\left(-C^-\Psi(x, y+x_0\varepsilon(t-t_0), t-\varepsilon(t-t_0), x_0, y_0, t_0)\right) \le \\ \Gamma(x,y,t,x_0,y_0,t_0) \le \qquad\qquad (10) \\ \frac{C_\varepsilon^+}{\xi^2(t-t_0)^2} \exp\left(-c^+\Psi(x, y-x_0\varepsilon, t+\varepsilon, x_0, y_0, t_0)\right), \end{aligned}$$

for every $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$, every arbitrary $\varepsilon \in]0, \frac{1}{4T}[$ and

$$(x,y,t) \in \mathbb{R}^+ \times]-\infty, y_0 - x_0\varepsilon(t-t_0)[\times]t_0, T].$$

Here the constants c_ε^-, C_ε^+ depend on ε and T, whereas the constants c^+, C^- depend only on $\mathscr{L}$. The function Ψ is the *value function* of a suitable optimal control problem with quadratic cost (see chapter 4).
The presence of ε is due to the method that we use to establish the estimates (10), which lead us to consider a smaller time range $(1-\varepsilon)(t-t_0)$ than $t-t_0$. At this level, we only precise that it is not possible to obtain estimates for $\varepsilon \to 0$ since the constants c_ε^- and C_ε^+ appearing in (10) may explode (see Remark 4.3.3 in chapter 4 for further details).

In order to emphasize the application of our main result to the existing literature for the operator $\mathscr{L}$ and to the corresponding stochastic theory, we note that (4) can be alternatively written as

$$k^-\Gamma^-(x,t,\xi,\tau) \le \Gamma(x,t,\xi,\tau) \le k^+\Gamma^+(x,t,\xi,\tau), \qquad (11)$$

where $\Gamma^\pm$ is the fundamental solution of the heat equation $\partial_t u = \lambda^\pm \Delta u$, and the constants $k^\pm, \lambda^\pm$ are strictly positive and depend on $c^\pm, C^\pm$. From this point of view, it would be natural write (10) in terms of the fundamental solution of a suitable *constant coefficients operator* analogous to $\mathscr{L}$. Actually, as we have noticed above, the simplest form of $\mathscr{L}$ appears as we choose $a = 1$ and $b = 0$:

$$\mathscr{L}_0 u = x^2\partial_{xx}u + x\partial_x u + x\partial_y u - \partial_t u, \quad (x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]. \qquad (12)$$

The fundamental solution Γ_0 of $\mathscr{L}_0$ can be written starting from the results given in Yor [84] on the joint distribution of the process $\left(W_t, A_t\right)_{t\geq 0}$, where $(W_t)_{t\geq 0}$ is a Wiener process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with the standard Brownian filtration and

$$A_t = \int_0^t \exp\left(2W_s\right) ds. \tag{13}$$

As we will see in Section 2 , it is possible to write almost explicitly the expression of the fundamental solution Γ_0 of $\mathscr{L}_0$. The authors Yor and Geman in their monograph [85] widely studied the stochastic process related to the operator $\mathscr{L}_0$ above by means of probabilistic instruments. However, to our knowledge, the results apply only to the specific case of $\mathscr{L}_0$.

Our approach is completely different. We adapt in this thesis the approach introduced in [27], [23], and we provide a well-structured methodology in order to prove upper and lower bound for the fundamental solution of a hypoelliptic operators with drift which satisfy Hörmander condition.

We emphasize that the methodology used in the proof of (10) involves several techniques belonging to the theory of Partial Differential Equations, Stochastic Processes and Optimal Control Theory, and can be applied to several different problems.

In particular, the proof of the lower bound relies on the repeated application of the Harnack inequality, combined with a suitable optimization procedure, for positive solution of $\mathscr{L}u = 0$. Such a procedure will lead us to naturally consider a relevant optimal control problem which will be explicitly solved.

Concerning the upper bound we combine analytical results with elementary tools belonging to Optimal Control Theory. We will use an analogous result of the Moser iteration and the *Hamilton-Jacobi-Bellman equation* for the value function Ψ related to a relevant optimal control problem.

The existence of a fundamental solution of $\mathscr{L}$ is guaranteed by Malliavin Calculus under some regularity conditions on the coefficients, since the operator $\mathscr{L}$ can be seen as the infinitesimal generator of a suitable stochastic process. We will clarify in the sequel the intrinsic link between the joint transition probability densities of a n-dimensional Stochastic Processes and the fundamental solution of its infinitesimal generator.

A further consequence of (10) is the following result, again in the spirit of (11).

By applying (10) to Γ and to the fundamental solutions $\Gamma^{\pm}$ of the operators

$$\mathscr{L}^{\pm}u = \lambda^{\pm}x^2\partial_{xx}u + x\partial_x u + x\partial_y u - \partial_t u, \quad (x,y,t) \in \mathbb{R}^+ \times \mathbb{R}\times]0,T],$$

for some strictly positive constants $\lambda^{\pm}$, we can state that there exist two constants $k^{\pm} > 0$ such that

$$\begin{aligned}
&k^-\Gamma^-(x, y + x_0\varepsilon(t - t_0 + 1), t - \varepsilon(t - t_0 + 1); x_0, y_0, t_0) \leq \\
&\quad \Gamma(x, y, t, x_0, y_0, t_0) \leq \\
&\quad k^+\Gamma^+\left(x, y - x_0\frac{\varepsilon}{1-\varepsilon}(t - t_0 + 1), t + \frac{\varepsilon}{1-\varepsilon}(t - t_0 + 1), x_0, y_0, t_0\right),
\end{aligned}$$

for every $(x, y, t), (x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}\times]0, T]$ with $y + x_0\varepsilon(t - t_0 + 1) < y_0$ and $t > t_0 + \varepsilon/(1 - \varepsilon)$. The constants $\lambda^{\pm}$, $k^{\pm}$ depend on c_ε^-, C_ε^+, C^-, c^+. This is an important theoretical result, as it allows us to extend to $\mathscr{L}$ any quantitative information we know on the fundamental solution of $\mathscr{L}^{\pm}$. Of course, the same result holds for the densities of the respective stochastic processes.

Outline of the Thesis

The thesis is organized as follows.

In *Chapter 1* we will preliminary recall some basic notion and facts about the Theory of Stochastic Processes. We will introduce the application to Mathematical Finance: we will provide a short description of financial markets and the Option Pricing Theory in the Black-Scholes setting. In particular we will briefly introduce some financial derivatives (*plain-vanilla options*, *path-dependent options* such as Arithmetic Asian Options and Geometric ones) and we will introduce the classical problem of *Pricing Financial Derivatives*. In particular, we will use some elements of the theory of the Martingale combined with some basic tools of Linear SDE (see [13, Chapter 10,11] and [66, Chapter 9,10] for a detailed presentation of the topic).

In *Chapter 2* we will discuss about Degenerate Evolution Operators in Hypoelliptic setting and we give the main ingredients of the Thesis: the invariant Harnack inequality and the Harnack chains which are strongly related to the Harnack inequality. A particular attention will be given to Degenerate Kolmogorov Operators with constant and variable coefficients. More specifically, we consider Kolmogorov Operators with variable coefficients

$$\mathcal{K}u \equiv \sum_{i,j=1}^{d_0} a_{i,j}(x,t)\partial_{x_i x_j}u + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u, \quad (x,t) \in \mathbb{R}^{N+1}$$

with $d_0 \leq N$ under the assumptions that the coefficients a_{ij}'s satisfy the following uniform parabolicity condition

$$\forall t \in \mathbb{R}, \quad \mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{d_0} a_{i,j}(x,t)\xi_i\xi_j \leq \mu|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^{d_0}, \tag{14}$$

for a suitable $\mu > 0$ and they are bounded Hölder continuous functions. We will then consider the related constant coefficents Kolmogorov operators

$$K^{\pm}u \equiv \mu^{\pm 1} \sum_{i=1}^{d_0} \partial_{x_i x_i} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u$$

and the main result of this chapter will concern the two sided bound of the fundamental solution $\Gamma(x,t;y,s)$ of $\mathcal{K}$ in terms of the fundamental solutions $\Gamma^{\pm}(x,t;y,s)$ of the constant operators $K^{\pm}$:

$$k^{-}\Gamma^{-}(x, t - \varepsilon(t-s); y, s) \leq \Gamma(x,t;y,s) \leq k^{+}\Gamma^{+}(x, t + \varepsilon(t-s); y, s)$$

for every (x,t), $(y,s) \in \mathbb{R}^{N+1}$ and arbitrary $\varepsilon \in]0,1[$, where the constants $k^{-} > 0$, $k^{+} > 0$ and only depend on the operator $\mathcal{K}$ and on T.

Chapter 3 will be devoted to the probabilistic part of this Thesis. In the first part, we will focus on Geman-Yor's results for Arithmetic Asian Options with constant drift and volatility which have been introduced in (12), then we will write the analytical expression of the transition probability density $p(x_0, y_0, t_0; x, y, t)$ of the stochastic process $(W_t, A_t)_{t \geq 0}$ in (13).

We will then introduce a short survey on Malliavin Calculus, exhibiting some key results of the Theory and its link with Hörmander Condition. We will emphasize the deep connection between joint density of a n-dimensional stochastic process and the fundamental solution of its infinitesimal generator and we compare our methodology with a similar probabilistic results existing in literature. In the end of this chapter we prove the existence of the fundamental solution Γ of (1), under further conditions on the coefficients.

In *Chapter 4* we will prove our main results. The main purpose is to prove formulas (10) and (11) and to give a description of the asymptotic behaviours of the functions $\Gamma_{\varepsilon}^{\pm}$ in (2). In this chapter, we will show in detail the method which we use: we will split the treatment into two part since lower and upper bound involve different techniques. Concerning the lower bound we will introduce Harnack type inequalities for positive solution u of the operator $\mathscr{L}$ in (1) and Harnack chains for the operator $\mathscr{L}$. By combining Harnack chains with a suitable optimization procedure we will show how the problem of deriving sharp lower bound is strongly related to the resolution of an optimal control problem. We will solve such optimal control problem and we will give some asymptotic behaviours of its Value Function. The final part of the chapter will be devoted to the research of the optimal upper bound for the fundamental solution $\Gamma(x, y, t; x_0, y_0, t_0)$ of $\mathscr{L}$. We will introduce several results in the spirit of the Moser iteration (see [23] and [22]) and by using Hamilton-Jacobi-Bellman equation we will show that the Value function related to the optimal control problem introduced for the lower bound can be also used to get upper bound. This justifies the therm *Optimal* or *Sharp* estimates.

The last Chapter contains final remarks about the main results and possible developments and extensions of the topic. We also give a brief plan of future researches.

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I wish to warmly thank my parents, my sister and my brother for their support and the great help they gave to me. Moreover, I would like to thank my dear Barbara for having been close every time during these years. In the end, special thanks go to colleagues and friends who encouraged me.

I hope I have given some confirmations and I have been up to the task, my diligence and my passion are not missed for sure.

However, in the future, I hope to continue to face with very interesting and challenging problems and I would like to meet people always open to discussions and the dialogue.

Chapter 1

Elements of Option Pricing in Mathematical Finance

In this Chapter we briefly recall some notions and details about the classic Option Pricing Theory. We start with the introduction of some basic probabilistic notions and tools, then passing to a briefly presentation of the Black-Scholes Theory [14] for the Option Pricing. We only introduce in this section notions that we need in this thesis and we refer to the monographs of Pascucci [66], Björk [13] and Hull [35] for a complete treatment of the topic.

1.1 Stochastic Processes Prerequisites

We start with preliminary notions of the theory of stochastic processes. We suggest skipping ahead if the reader already knows the basic definitions.
We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and we introduce the following objects.

Definition 1.1.1 (Continuous stochastic process)**.** *A $\mathbb{R}^N$-valued continuous time stochastic process is a family of random variables $X = (X_t)_{t\in I}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that the function:*

$$X \ : \ I \times \Omega \longrightarrow \mathbb{R}^N, \qquad X(t,\omega) = X_t(\omega)$$

is measurable with respect to the σ-algebra $\mathscr{F} \times \mathcal{B}(I)$, where $\mathcal{B}(I)$ is the Borel σ-algebra on the interval $I \subseteq \mathbb{R}$.

Definition 1.1.2 (Filtration)**.** *A **filtration** on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is an increasing family of σ-algebras $(\mathscr{F}_t)_{t\in I}$ (i.e. $\mathscr{F}_s \subseteq \mathscr{F}_t$ for every $s < t$) such that*

$\mathscr{F}_t \subseteq \mathscr{F}$. *Given a stochastic process* $X = (X_t)_{t\in I}$, *the family of* σ*-algebras* $(\mathscr{F}_t^X)_{t\in I}$ *defined by:*

$$\mathscr{F}_t^X = \sigma(X_s; 0 \le s \le t)$$

is said natural filtration related to the process $(X_t)_{t\in I}$.

We convene to set:

$$\mathscr{F}_\infty = \sigma\left(\bigcup_{t\in I} \mathscr{F}_t\right)$$

A filtration is said right continuous (respectively left continuous) if

$$\mathscr{F}_t = \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon}, \quad (\text{respectively} \mathscr{F}_t = \bigcap_{\varepsilon>0} \mathscr{F}_{t-\varepsilon})$$

for every $t \in I$.

We say that the process is **adapted** with respect to the filtration $(\mathscr{F}_t)_t$ if

$$\mathscr{F}_t^X \subseteq \mathscr{F}_t, \qquad \forall t \in I.$$

or, equivalently, the random variable X_t is $\mathscr{F}_t$ measurable for every $t \in I$. Obviously, a stochastic process $(X_t)_{t\in I}$ is always adapted with respect to the natural filtration related to it.

A stochastic process is said continuous if has continuous trajectories $\omega \mapsto X_t$, for almost every fixed ω and for every $t \in I$. A stochastic process is said right continuous if:

$$X_{t^+}(\omega) = \lim_{s\to t^+} X_s(\omega) = X_t(\omega)$$

and it is said integrable in I if

$$E\left[\int_I |X_t| dt\right] < +\infty.$$

Definition 1.1.3 (Progressively measurable process)**.** *A stochastic process* $X = (X_t)_{t\in I}$ *is said* ***progressively measurable****, with respect to* $(\mathscr{F}_t)_{t\in I}$, *if it holds:*

$$\{(s,\omega) \in [0;t] \times \Omega \ s.t \ X_s(\omega) \in H\} \in \mathcal{B}([0;t]) \times \mathscr{F}_t$$

for every fixed $t \in I$ *and every Borel set* H *in* $\mathbb{R}^N$.

Remark 1.1.4. *It holds that a right continuous process is also progressively measurable.*

Given a Probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we denote by:

$$\mathrm{N} = \{A \in \mathscr{F} \ t.c \ \mathbb{P}(A) = 0\}$$

the collection of the sets A with null measure.

Definition 1.1.5. *A filtration $(\mathscr{F}_t)_{t\in I}$ is said **standard** if:*

i) *the family $\mathscr{F}_t$ contains* N *for every $t \in I$;*

ii) *the filtration is right continuous* [1].

Definition 1.1.6. *We say that a stochastic process $X = (X_t)_{t\in I}$ belongs to the class $\mathbb{L}^p(I)$ if:*

- *X is progressively measurable with respect to the filtration $(\mathscr{F}_t)_{t\in I}$;*
- *there exists $\int_I E\left[|X_t|^p\right] dt$ and it is finite.*

Furthermore we say that the process $(X_t)_{t\in I}$ belongs to the class $\mathbb{L}^p$ if it belongs to $\mathbb{L}^p(I)$ for every compact $I \subset \mathbb{R}$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space endowed with the filtration $(\mathscr{F}_t)_{t\in I}$. We now recall the definition of Martingale, Submartingale and Supermartingale.

Definition 1.1.7. *An integrable stochastic process $(X_t)_{t\in I}$ is said **Martingale** adapted with respect to $(\mathscr{F}_t)_{t\in I}$ if:*

$$X_s = E[X_t|\mathscr{F}_s] \ a.s. \quad \forall\, 0 \leq s \leq t.$$

*Analogously, an integrable stochastic process $(X_t)_{t\in I}$ is said **supermartingale** (respectively **submartingale**) adapted with respect to $(F_t)_{t\in I}$ if:*

$$X_s \geq E[X_t|\mathscr{F}_s], \quad (X_s \leq E[X_t|\mathscr{F}_s]), \quad a.s. \quad \forall\, 0 \leq s \leq t$$

It holds the following properties

- $E[X_0] = E[X_s] = E[X_t], \quad \forall\, 0 \leq s \leq t$, for Martingale;

[1] We only consider filtrations that contain negligible events in order to avoid the unpleasant situation in which, given two random variables X, Y one has $X = Y$ a.e., but X is $\mathscr{F}_t$-measurable and Y not.

- $E[X_0] \geq E[X_s] \geq E[X_t], \quad \forall\, 0 \leq s \leq t$ for Supermartingale;
- $E[X_0] \leq E[X_s] \leq E[X_t], \quad 0 \leq s \leq t$ for Submartingale.

Definition 1.1.8 (Markov Property)**.** *An adapted stochastic process $(X_t)_{t \in I}$, satisfies the **Markov Property** if:*

$$\mathbb{P}(X_t \in H | \mathscr{F}_s) = \mathbb{P}(X_t \in H | X_s) \ a.s. \tag{1.1}$$

for all borel sets H of $\mathbb{R}^N$.

1.1.1 Brownian motion and Itô Calculus

Let $(\Omega, \mathscr{F}; \mathbb{P}; (\mathscr{F}_t)_{t>0})$ be a probability space with filtration.

Definition 1.1.9 (scalar Brownian motion)**.** *A scalar Brownian motion on the filtered probability space $(\Omega, \mathscr{F}; \mathbb{P}; (\mathscr{F}_t))$ is a stochastic process $(W_t)_{t \geq 0}$ in $\mathbb{R}$ satisfying the following properties:*

1) *W_t is adapted with respect to $(\mathscr{F}_t)_{t \geq 0}$;*

2) $\mathbb{P}(W_0 = 0) = 1, \qquad (W_0 = 0\ a.s.)$;

3) *the random variable $W_t - W_s$ is independent of $\mathscr{F}_s$ for $t > s$ (i.e for every $y \in \mathbb{R}$ and $A \in \mathscr{F}_s$ the events $\{W_t - W_s \leq y\}$ and A are independent);*

4) *The random variable $W_t - W_s$ is normally distributed $\mathcal{N}(0; t - s)$, that is for every borel set $A \in \mathbb{R}$*

$$\mathbb{P}\big((W_t - W_s) \in A\big) = \int_A \frac{1}{\sqrt{2\pi(t-s)}}\, e^{\left(-\frac{x^2}{2t}\right)} dx$$

We now list some properties of scalar Brownian motions:

Proposition 1.1.10. *A scalar Brownian motion satisfies the following properties:*

- *For all $t \geq 0$ the random variable $W_t = W_t - W_0$ has normal distribution $\mathcal{N}(0; t)$;*
- *$Cov(W_s, W_t) = \min(s, t)$;*
- *$(W_t)_{t \geq 0}$ is a martingale with respect to the filtration $(\mathscr{F}_t)_{t \geq 0}$:*

$$E\left[W_t | \mathscr{F}_s\right] = W_s, \qquad \forall\ s < t.$$

- *A scalar Brownian motion* $(W_t)_{t\geq 0}$ *satisfies the Markov property: for every borel set* $A \subseteq \mathbb{R}$

$$\mathbb{P}(W_t \in A|\mathscr{F}_s) = \mathbb{P}(W_t \in A|W_s).$$

Moreover it holds the following formula

$$\mathbb{P}(W_t \in A|W_s = x) = \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} dy$$

We refer to

$$f(y,t|x,s) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}}$$

as the transition function of the Brownian motion.

We remind that the family of trajectories with bounded variation of a Brownian motion W_t is negligible. In other words W_t has almost all paths that are irregular, non-rectifiable. If $\{t_1, ..., t_m\}$ is a partition of $[0,T]$ then

$$\lim_{\Delta t \to 0} \sum_{j=1}^{m} |W_{t_j} - W_{t_{j-1}}| = +\infty \tag{1.2}$$

Indeed, it holds the following crucial proposition

Theorem 1.1.11 (Quadratic Variation). *If* W_t *is a Brownian motion on* $[0,T]$*, then we have*

$$\langle W_t \rangle = \lim_{\Delta t \to 0} \sum_{j=1}^{m} |W_{t_j} - W_{t_{j-1}}|^2 = t, \quad \text{in } L^2[0,T].$$

We say that the Brownian motion has quadratic variation $\langle W_t \rangle$ *equal to* t*.*

With abuse of notation we will sometimes write $(dW_t)^2 = dt$.

We now introduce an n-dimensional Brownian motion.

Definition 1.1.12 (n-dimensional Brownian motion). *A* n*-dimensional stochastic process* $W_t = (W_t^1, ..., W_t^n)$ *is said* n*-dimensional Brownian motion on the filtered probability space* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t>0})$ *if:*

1) *for every* $i = 1, ..., n$ *the process* W_t^i, $t \geq 0$ *is a scalar Brownian motion on* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t>0})$*;*

2) *for every fixed* $t \geq 0$ *and* $s \geq 0$*, the random variables* W_t^i *and* W_s^j *are independent for every* $i \neq j$*.*

A n-dimensional Brownian motion satisfies analogous properties:

Lemma 1.1.13. *Let W_t be a n-dimensional Brownian motion. It holds the following properties*

- *cov* $\left[W_t^k W_s^k\right] = \min(t,s)$, *cov* $\left[W_t^k W_s^l\right] = 0$, *if* $k \neq l$;
- $E\left[(W_t^k - W_s^k)^2\right] = (t-s)$, $E\left[(W_t^k - W_s^k)(W_t^l - W_s^l)\right] = 0$, *if* $k \neq l$;
- *For every borel set* $A \in \mathbb{R}^n$ *the probability* $\mathbb{P}(W_t \in A | W_s = x)$, *where* $x \in \mathbb{R}^n$, *admits a gaussian density with respect to the Lebesgue measure:*

$$f(y) = f(y,t|x,s) = \frac{1}{(2\pi(t-s))^{n/2}} e^{-\frac{\|y-x\|^2}{2(t-s)}}.$$

Correlated Brownian motion. Sometimes it is convenient to build models based on Brownian motions which are correlated. In order to define such objects, we start by considering a d-dimensional Brownian motion $W_t = (W_t^1, ..., W_t^d)$ on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t>0})$, with $t \geq 0$. Let furthermore consider a constant $n \times d$ matrix $\delta = (\delta_{ij})$ with $i = 1, ..., n$ and $j = 1, ..., d$ whose rows have unit length, i.e. denoting by $|\cdot|$ the usual Euclidean norm in $\mathbb{R}^d$

$$|\delta_i| = 1, \quad i = 1, ..., n, \tag{1.3}$$

and construct the following n-dimensional process $\bar{W}_t = \delta W_t$, in other words

$$\bar{W}_t^i = \sum_{j=1}^{d} \delta_{ij} W_t^j, \quad i = 1, ..., n.$$

It is easy to check that each of the components $\bar{W}_t^i, ..., \bar{W}_t^n$ separately are standard Brownian motions on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t>0})$, but the process is correlated as it holds:

$$\rho_{ij}\, t = Cov[\bar{W}_t^i, \bar{W}_t^j] = \left(\sum_{k=1}^{d} \delta_{ik}\delta_{jk}\right) t.$$

Note that, denoting by $\rho = (\rho_{ij})$, we have $\rho = \delta\delta^*$ [2].

[2] By means of analogous arguments, it is also possible to built correlated Brownian motions whose correlation matrix is a deterministic function of t, that is $\rho(t) = \delta(t)\delta^*(t)$.

Definition 1.1.14. *The process* $\bar{W}_t = (\bar{W}_t^i, ..., \bar{W}_t^n)$ *is called a vector of correlated Brownian motion on the filtered probability space* $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t>0})$ *, with correlation matrix* ρ.

The random variable $\bar{W}_t$ has a multivariate normal distribution with zero mean an covariance matrix

$$\Sigma(t) = (\rho_{ij} t)_{ij}, \quad i, j = 1, ..., n.$$

Itô Formula

Let $(W_t^1, ..., W_t^d)$ be a d-dimensional Brownian motion on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_t)$. A n-valued stochastic process $X_t = (X_t^1, ..., X_t^n)$ is called an *Itô process* if it writes as follows:

$$X_t^i = X_0^i + \int_0^t \mu_i(s, X_s) ds + \sum_{j=1}^{d} \int_0^t \sigma_{ij}(s, X_s) dW_s^j, \quad i = 1, ..., n, \tag{1.4}$$

where $t \in [0, T]$ and:

- X_0^i is a $\mathscr{F}_0$ measurable random variable for all $i = 1, ..., n$;
- each $\mu_i = \mu_i(t, X_t)$ (also called *drift* term) is a stochastic process belonging to $\mathbb{L}^1[0, T]$;
- σ is a $n \times d$ matrix (also called *diffusion matrix*) and each $\sigma_{i,j} = \sigma_{i,j}(t, X_t)$ is a stochastic process belonging to $\mathbb{L}^2[0, T]$.

We explicitly say that the first integral $\int_0^t \mu_i(s, X_s) ds$ appearing in (1.4) can be interpreted in the usual sense of Riemann integral. But how can we interpret the stochastic integral $\int_0^t \sigma_{ij}(s, X_s) dW_s^j$ in (1.4)? The definition of the stochastic integral is not so obvious. Indeed, in view of property (1.2), it is not possible to define the stochastic integral in the sense of Riemann-Stieltjes.

Given a stochastic process u_t belonging to $\mathbb{L}^2[0, T]$ and a scalar Brownian motion W_t, the first definition of the stochastic integral $\int_0^t u_s dW_s$, is due to Itô. Its construction strongly exploits the result given in Theorem 1.1.11. We refer to Pascucci [66, Chap 4] or Karatzas-Shreve [38, Chap 3] for a rigorous definition of it.

From now on, we will rewrite (1.4) in its differential form

$$dX_t^i = \mu_i(t, X_t) dt + \sum_{j=1}^{d} \sigma_{ij}(t, X_t) dW_t^j, \quad i = 1, ..., n, \tag{1.5}$$

or in compact form:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \in [0, T].$$

Let $f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be a function and consider the process $f(t, X_t)$. It holds the following Itô formula:

Theorem 1.1.15. *Let X_t be a n-dimensional Itô process and let $f \in C^{1,2}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$. Then the process $Y_t = f(t, X_t)$ has the following Itô-differential:*

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} c_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_i dW_t,$$

Where $\sigma_i = [\sigma_{i1}, ..., \sigma_{id}]$ is the i^{th} row of the matrix σ and

$$C = (c_{ij}) = \sigma \cdot \sigma^*, \quad \text{with } \sigma^* \text{ transpose matrix of } \sigma.$$

We can also use the following formal rules:

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

where:

$$\begin{cases} (dt)^2 = 0 \\ dt \cdot dW_t^i = 0, & i = 1, ..., d, \\ (dW_t^i)^2 = dt, & i = 1, ..., d, \\ dW_t^i dW_t^j = 0, & i \neq j. \end{cases} \tag{1.6}$$

The operator

$$\mathcal{L}f = \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} c_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

is called **infinitesimal generator** of the stochastic process $(X_t)_{t\in[0,T]}$.

If, instead of a d-dimensional Brownian motion, we consider a correlated n-dimensional Brownian motion $W_t = (W_t^1, ..., W_t^n)$ on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t>0})$, with correlation matrix ρ, we have

Theorem 1.1.16 (Multidimensional Correlated Itô Formula)**.** *Let $X_t = (X_t^1, ..., X_t^n)$ a Itô process having the dynamic*

$$dX_t^i = \mu_i(t, X_t)dt + \sum_{j=1}^{n} \sigma_{ij}(t, X_t)dW_t^j, \quad i = 1, ..., n,$$

where $t \in [0,T]$ and $(W_t^1, ..., W_t^n)$ is a vector of correlated Brownian motion with correlation matrix ρ. Let $f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. It holds:

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} c_{ij} \rho_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_i dW_t.$$

One may use the following informal computation rules:

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

with:

$$\begin{cases} (dt)^2 = 0 \\ dt \cdot dW_t^i = 0, & i = 1, ..., d, \\ (dW_t^i)^2 = dt, & i = 1, ..., d, \\ dW_t^i dW_t^j = \rho_{i,j} dt, & i \neq j. \end{cases} \tag{1.7}$$

If the diffusion matrix $\sigma_{ij}(t, X_t)$ is a deterministic function, depending only on the t time variable, i.e

$$\sigma_{ij}(t, X_t) = \sigma_{ij}(t), \quad i = 1, ..., n, \quad j = 1, ..., d,$$

it is possible to simply shift between the two formalisms above. It suffices to repeat the construction introduced above for correlated Brownian motions by considering correlations which depend on t. Suppose, therefore, that the n-dimensional process X_t, $t \in [0,T]$ has a stochastic differential

$$dX_t^i = \mu_i(t, X_t) dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW_t^j \quad i = 1, ..., n.$$

Here W_t is a d-dimensional standard Brownian motion. Let denote by σ_i the i^{th} row of the matrix σ and define n new scalar Wiener processes $\bar{W}_t^1, ..., \bar{W}_t^n$ by

$$\bar{W}_t^i = \frac{1}{|\sigma_i(t)|} \sum_{j=1}^{d} \sigma_{ij}(t) W_t^j, \quad i = 1, ..., n.$$

We can thus rewrite the X dynamics as

$$dX_t^i = \mu_i(t, X_t) dt + |\sigma_i(t)| d\bar{W}_t^i, \quad i = 1, ..., n.$$

Each $\bar{W}_t^i$ is a standard scalar Wiener process, but $\bar{W}_t^1, ..., \bar{W}_t^n$ are of course a correlated Brownian motion. To see this, we first note that

$$d\bar{W}_t^i \, d\bar{W}_t^j = \frac{\sigma_i(t) \cdot \sigma_j^*(t)}{|\sigma_i(t)| \, |\sigma_j(t)|} dt = \rho_{ij}(t) dt,$$

where $\sigma_j^*(t)$ denotes the transpose of the row vector $\sigma_j(t)$.

By applying the Ito formula we have

$$d(\bar{W}_t^i \bar{W}_t^j) = \bar{W}_t^i d\bar{W}_t^j + d\bar{W}_t^i \bar{W}_t^j + d\bar{W}_t^i d\bar{W}_t^j$$

and integrating from zero to t we obtain

$$\bar{W}_t^i \bar{W}_t^j = \int_0^t \bar{W}_s^i d\bar{W}_s^j + \int_0^t \bar{W}_s^j d\bar{W}_s^i + \int_0^t \rho_{ij}(s) ds.$$

Taking the expected value we have

$$cov[\bar{W}_t^i, \bar{W}_t^j] = \int_0^t \rho_{ij}(s) ds.$$

1.1.2 From SDE's to PDE's: the Kolmogorov Equations

In this section we examine the deep connection between SDE's and PDE's. Let y be a point belonging to $\mathbb{R}^N$ and let consider the following N-dimensional Stochastic Differential Equation

$$\begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \\ X_s = y, \end{cases} \tag{1.8}$$

where W_t is an assigned d-dimensional Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, $t \in]0, T]$, $s \in [0, t[$, $\mu(t, X(t))$ is a $\mathbb{R}^N$-valued stochastic process belonging to $\mathbb{L}^1[0, T]$ and the $N \times d$ matrix $\sigma(t, X_t)$ belonging to $\mathbb{L}^2[0, T]$.
We say that the stochastic process X_t is a **strong solution** of (1.8) if X_t satisfies the integral equation

$$X_t = y + \int_s^t \mu(\tau, X_\tau)dt + \int_s^t \sigma(\tau, X_\tau)dW_\tau, \quad \text{for all } t > s. \tag{1.9}$$

We explicitly write $X_t^{s,y}$ to emphasize the fact that at time s the random variable is deterministic and equal to $X_s = y$.

Proposition 1.1.17. *Suppose that:*

i) μ, σ *are locally Lipschitz continuous functions in* x *uniformly with respect to* $t \in [0, T]$*, that is for every compact set* $K \subset \mathbb{R}^N$ *there exists a positive constant* L *such that for every* x, $y \in K$ *and for every* $t \in [0, T]$ *it holds*

$$|\mu_i(t, x) - \mu_i(t, y)| \leq L|x - y|, \quad i = 1, ..., n, \quad \forall t \in [0, T],$$
$$|\sigma_{i,j}(t, x) - \sigma_{ij}(t, y)| \leq L|x - y|, \quad i = 1, ..., n, \ j = 1, ..., d, \quad \forall t \in [0, T].$$

ii) *μ, σ have at most linear growth, i.e for every $t \in [0,T]$ there exists a positive constant $M = M(T)$ such that*

$$|\mu_i(t,x)| \le M(1+|x|), \quad i = 1, ..., n,$$
$$|\sigma_{i,j}(t,x)| \le M(1+|x|), \quad i = 1, ..., n, \; j = 1, ..., d,$$

for every $x \in \mathbb{R}^N$.

Then there exists a unique strong solution to the SDE (1.8). *Such a solution has the following properties:*

1. *X_t is $\mathscr{F}_t^W$-adapted, where $\mathscr{F}_t^W$ denotes the standard filtration generated by the Brownian motion W_t for every $t \in [s,T]$;*
2. *the stochastic process X_t has continuous trajectories and satisfies the Markov Property* (1.1)*;*
3. *there exists a constant C such that*
$$\int_s^T E[|X_t|^2]dt \le Ce^{C(T-s)}(1+|y|^2).$$

For every fixed $t \in [s,T]$ the application $W_t \mapsto (X_t^{s,y}(W_t))$ is a functional from $C([s,T])$ (space of continuous function in $[s,T]$) to $\mathbb{R}^N$ and $X_t(\cdot)$ is a random variable. For every Borel set $A \subseteq \mathbb{R}^N$ the induced measure $\nu_{s,y}(t,A) = \mathbb{P}(X_t \in A | X_s = y)$ is a probability measure on $\mathbb{R}^N$. Let $\mathcal{L}$ denotes the infinitesimal generator of the process X_t. It holds the following proposition

Proposition 1.1.18. *For every fixed $(s,y) \in [0,T] \times \mathbb{R}^N$, the measure $(t,A) \mapsto \nu_{s,y}(t,A)$ is a weak solution of the following equation*

$$\partial_t u(t,x) - \mathcal{L}^* u(t,x) = 0 \quad \text{for all } t \in [s,T]. \tag{1.10}$$

where $\mathcal{L}^*$ denotes the formal adjoint of the operator $\mathcal{L}$.

Proof. Let $X_t^{s,y}$ be the unique solution of (1.9), and let $f \in C_0^\infty([s,T] \times \mathbb{R}^N)$. By applying the Itô formula, reminding that f has compact support, we have

$$\begin{aligned} 0 &= f(T, X_T^{s,y}) - f(s,y) \\ &= \int_s^T (\partial_t f(t, X_t^{s,y}) + \mathcal{L}f(t, X_t^{s,y}))dt + \int_s^T \sum_{i=1}^N \partial_{x_i} f(t, X_t^{s,y}) \sigma_i dW_t. \end{aligned}$$

Taking the expectation value, since

$$E\Big[\int_s^T \sum_{i=1}^N \partial_{x_i} f(t, X_t^{s,y})\sigma_i dW_t\Big] = 0,$$

we obtain

$$\begin{aligned} 0 &= \int_\Omega d\mathbb{P} \int_s^T (\partial_t f(t, X_t^{s,y}) + \mathcal{L} f(t, X_t^{s,y}))dt \\ &= \int_{\mathbb{R}^N} \int_s^T (\partial_t f(t,x) + \mathcal{L} f(t,x))\nu_{s,y}(t,x)dt. \end{aligned}$$

Therefore, the induced measure $\nu_{s,y}$ solves weakly the PDE

$$\partial_t u(t,x) - \mathcal{L}^* u(t,x) = 0, \quad (t,x) \in [s,T] \times \mathbb{R}^N.$$

□

Provided that, for every fixed $t \in [0,T]$, we are able to show that the measure $\nu_{y,s}(t,\cdot)$ admits a density $p(x,t;y,s)$ with respect to the Lebesgue measure on $\mathbb{R}^N$, that is

$$\nu_{y,s}(t,dx) = p(x,t;y,s)dx$$

we can conclude that $p(x,t;y,s)$ is a weak solution of

$$\begin{cases} \partial_t p(t,x;y,s) - \mathcal{L}^* p(x,t;y,s)) = 0, & \text{for all } t \in]s,T]; \\ p(x,s;y,s) = \delta_y(x). \end{cases} \tag{1.11}$$

In this case we refer to (1.11) as the forward Kolmogorov equations since the operator $\partial_t - \mathcal{L}^*$ acts on the forward variables (x,t).

Feynman-Kac Formula. Let W_t, $t \in [0,T]$ be a d-dimensional Brownian motion and consider the N-dimensional SDE

$$\begin{cases} dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \\ X_s = y, \end{cases} \tag{1.12}$$

whose coefficients satisfy the Hypothesis of Proposition 1.1.17. Consider a continuous stochastic process $r_t = r(t,X_t)$, $t \in [0,T]$ and suppose that $r_t \in \mathbb{L}^1[0,T]$ and is bounded from below. The Feynman-Kac representation formula allows us to give a probabilistic representation of the function $f(s,y)$ solution of the following Cauchy problem

$$\begin{cases} \partial_s u(s,y) + \mathcal{L} u(s,y) - r(s,y)u(s,y) = 0, & (s,y) \in [0,T[\times \mathbb{R}^N, \\ u(T,y) = \varphi(y). \end{cases} \tag{1.13}$$

where $\varphi(y)$ is a continuous function $\varphi : \mathbb{R}^N \mapsto \mathbb{R}$ which satisfies suitable growth condition (for simplicity one can assume $\varphi \in C_0^\infty(\mathbb{R}^N)$).

Theorem 1.1.19 (Feynman-Kac)**.** *Let $u(s,y)$ a solution of the parabolic Cauchy problem* (1.13)*. Let X_t be a stochastic process solution of the SDE* (1.12)*. Assume that*

i) *the coefficients μ, σ are uniformly Lipschitz functions and have at most linear growth in x;*

ii) *for every $(t,x) \in [0,T] \times \mathbb{R}^N$, there exists a solution $X^{s,y}$ of the SDE* (1.12) *related to a d-dimensional Brownian motion W_t on the space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_t)$;*

iii) *there exist two positive constants M, p such that*

$$|u(s,y)| \leq M(1+|y|^p), \quad (s,y) \in [0;T] \times \mathbb{R}^N,$$

or the matrix σ is bounded and there exist two positive constants M and α, with α small enough, such that:

$$|u(t,x)| \leq M e^{\alpha|x|^2}, \quad (s,y) \in [0;T] \times \mathbb{R}^N.$$

Then $u(y,s)$ has the following representation

$$u(s,y) = E\left[\exp\Big(-\int_s^T r_\tau d\tau\Big)\varphi(X_T)|X_s = y\right]. \tag{1.14}$$

Proof. Let $X_t^{s,y}$ be the unique solution of (1.9). For the sake of simplicity we give the proof of the theorem only if the process r_t is a deterministic function $r(t)$ belonging to $L^1[0,T]$ and we refer to [66, Chap. 9] for the general proof of the statement. By applying the Itô formula we have

$$\begin{aligned} d\left[\exp\Big(-\int_s^t r(\tau)d\tau\Big)u(t,X_t^{s,y})\right] &= -r(t)\exp\Big(-\int_s^t r(\tau)d\tau\Big)u(t,X_t^{s,y}) \\ + \exp\Big(-\int_s^t r(\tau)d\tau\Big)\Big[\Big(\partial_t u(t,X_t^{s,y}) &+ \mathcal{L}u(t,X_t^{s,y})\Big)dt + \sum_{i=1}^N \partial_{x_i}u(t,X_t^{s,y})\sigma_i dW_t\Big] \\ &= \sum_{i=1}^N \partial_{x_i}u(t,X_t^{s,y})\sigma_i dW_t, \end{aligned}$$

in view of (1.13).

The result follows by integrating from s to T:

$$\exp\Big(-\int_s^T r(\tau)d\tau\Big)\varphi(X_T) - u(s,y) = \int_s^T \sum_{i=1}^N \partial_{x_i} u(t, X_t^{s,y})\sigma_i dW_t$$

and taking the expectation value, reminding that

$$E\Big[\int_s^T \sum_{i=1}^N \partial_{x_i} u(t, X_t^{s,y})\sigma_i dW_t\Big] = 0.$$

□

As said before, provided that the measure $\mathbb{P}(X_t \in dx | X_s = y)$ admits a density $p(t,x;y,s)$, we can rewrite the Faynman-Kac formula as follows

$$u(s,y) = \int_{\mathbb{R}^N} \exp\Big(-\int_s^T r(\tau)d\tau\Big)\varphi(x)p(t,x;y,s)dx \tag{1.15}$$

where the function $p(x,t;y,s)$ solves weakly the problem

$$\begin{cases} \partial_t p(x,t;y,s) + \mathcal{L}p(x,t;y,s) - r(s,y)p(x,t;y,s) = 0, & \text{for all } s \in [0,t[; \\ p(x,t;y,t) = \delta_x(y) \end{cases} \tag{1.16}$$

In this case we refer to the equation in (1.16) as the backward Kolmogorov equation.

Linear Equations

In this section we study the simplest and most important class of stochastic equations, namely those whose coefficients are linear functions of the solution, and we introduce the corresponding class of second-order differential operators, the Kolmogorov operators. Let us consider the following linear SDE in $\mathbb{R}^N$

$$dX_t = (B(t)X_t + b(t))dt + \sigma(t)dW_t \tag{1.17}$$

where b, B and σ are $L^\infty[0,T]$ functions with values in the space of $(N \times 1), (N \times N)$ and $(N \times d)$-dimensional matrices respectively, and W_t is a d-dimensional Brownian motion with $d \leq N$ and $t \in [0,T]$. For these equations a strong solution exists and is unique. Further, it is also possible to obtain the explicit expression of the solution.

Let us denote by $\Phi(t)$ the solution of the ordinary Cauchy problem

$$\begin{cases} \Phi'(t) = B(t)\Phi(t), \\ \Phi(t_0) = Id_N \end{cases}$$

where Id_N is the identity $N \times N$ matrix.

Proposition 1.1.20. *The solution of the SDE equation* (1.17), *with initial condition* $X_s = \xi$ *is given by*

$$X_t^{s,\xi} = \Phi(t-s)\Big(\xi + \int_s^t \Phi^{-1}(\tau)b(\tau)d\tau + \int_s^t \Phi^{-1}(\tau)\sigma(\tau)dW_\tau\Big), \quad t \geq s. \tag{1.18}$$

Furthermore, $X_t^{s,\xi}$ *has multivariate normal distribution with mean*

$$m_\xi(t-s) = \Phi(t-s)\Big(\xi + \int_s^t \Phi^{-1}(\tau)b(\tau)d\tau\Big) \tag{1.19}$$

and covariance matrix

$$C(t-s) = \Phi(t-s)\Big(\int_s^t \Phi^{-1}(\tau)\sigma(\tau)\cdot(\Phi^{-1}(\tau)\sigma(\tau))^* d\tau\Big)\Phi^*(t-s). \tag{1.20}$$

PROOF. To prove that $X_t^{s,\xi}$ in (1.18) is the solution, we just have to use the Itô formula: we set

$$Y_t = \xi + \int_s^t \Phi^{-1}(\tau)b(\tau)d\tau + \int_s^t \Phi^{-1}(\tau)\sigma(\tau)dW_\tau$$

and we note that

$$dX_t^{s,\xi} = d(\Phi(t)Y_t) = \Phi'(t)Y_t + \Phi(t)dY_t = (B(t)X_t^{s,\xi} + b(t))dt + \sigma(t)dW_t.$$

Therefore, (1.18) holds true. The distribution of $X_t^{s,\xi}$ in (1.18) is multivariate normal since it is the sum of integrals of deterministic functions.

In order to compute its mean and covariance matrix we start from the integral representation of (1.17) with initial data $X_s = \xi$

$$X_t^{s,\xi} = \xi + \int_s^t (B(t)X_t^{s,\xi} + b(t))dt + \int_s^t \sigma(t)dW_t,$$

and, taking the expectation value we have

$$m(t) = E[X_t^{s,\xi}] = \xi + \int_s^t \Big(B(t)E[X_t^{s,\xi}] + b(t)\Big)dt$$

therefore the mean $m(t)$ satisfies the following Cauchy problem

$$\begin{cases} \dot m(t) = B(t)m(t) + b(t), \quad \forall\, t > s; \\ m(s) = \xi. \end{cases}$$

Then, (1.19), plainly follows.

Concerning the computation of the covariance matrix: we have

$$C(t-s) = cov[X_t^{s,\xi}] = E[(X_t^{s,\xi} - m_\xi(t-s))(X_t^{s,\xi} - m_\xi(t-s))^*]$$
$$= \Phi(t-s)E\left[\int_s^t \Phi^{-1}(\tau)\sigma(\tau)dW_\tau \cdot \Big(\int_s^t (\Phi^{-1}(\tau)\sigma(\tau))dW_\tau\Big)^*\right]\Phi^*(t-s)$$

and (1.20) follows by Itô isometry. □

We explicitly note that, since $d \leq N$, in general, the matrix $C(t)$ is only positive semi-definite. The case $C(t) > 0$ is particularly significative: indeed in this case the transition density $(\xi, s) \to p(x,t;\xi,s)$ of $X_t^{\xi,s}$ is

$$p(x,t;\xi,s) = \frac{2\pi^{-\frac{N}{2}}}{\sqrt{C(t-s)}} \exp\left(-\tfrac{1}{2}\langle C^{-1}(t-s)(x-m_\xi(t-s)),(x-m_\xi(t-s))\rangle\right) \tag{1.21}$$

for $x, y \in \mathbb{R}^N$ and $t \in]s,T]$. Moreover, by the Faynman-Kac representation formula, $p(x,t,\xi,s)$ satisfies (1.16), where the differential operator in $\mathbb{R}^N \times [0,t[$ associated to the linear SDE writes:

$$\mathcal{L} + \partial_s = \frac{1}{2}\sum_{i,j=1}^N c_{i,j}(s)\partial_{\xi_i\xi_j} + \langle (b(s) + B(s)\xi); \nabla_\xi\rangle + \partial_s,$$

where $(c_{ij}) = \sigma\,\sigma^*$ and $\nabla_\xi = (\partial_{\xi_1}, \ldots, \partial_{\xi_N})$.

The case of constant coefficients, $b(t) = b, B(t) = B$ and $\sigma(t) = \sigma$, is quite simple. First of all, let us recall that in this case we have $\Phi(t) = e^{tB}$ where

$$e^{tB} = \sum_{n=0}^\infty \frac{(tB)^n}{n!}.$$

We observe that

$$\left(e^{tB}\right)^* = e^{tB^*}, \quad \left(e^{tB}\right)^{-1} = e^{-tB}, \quad e^{(t_1+t_2)B} = e^{t_1B}e^{t_2B}$$

Therefore the solution of the linear SDE

$$dX_t = (BX_t + b)dt + \sigma dW_t \tag{1.22}$$

with initial condition $X_s = \xi$ is given by

$$X_t^{\xi,s} = e^{(t-s)B}\Big(\xi + \int_s^t e^{-\tau B}bd\tau + \int_s^t e^{-\tau B}\sigma dW_\tau\Big), \quad t \in [s,T] \tag{1.23}$$

and we have

$$m_\xi(t-s) = E[X_t^{\xi,s}] = e^{(t-s)B}\xi + e^{(t-s)B}\int_s^t e^{-\tau B}bd\tau \tag{1.24}$$

and covariance matrix

$$C(t-s) = cov[X_t^{\xi,s}] = e^{(t-s)B}\Big(\int_s^t e^{-\tau B}\sigma \cdot (e^{-\tau B}\sigma)^* d\tau\Big)e^{(t-s)B^*} \tag{1.25}$$

Geometric Brownian motion. In the one dimensional case, given a scalar Brownian Motion $(W_t)_t$ on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_t)$, a linear differential equation having the following dynamic:

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x_0 \tag{1.26}$$

with constant $\mu,\ \sigma \in \mathbb{R}$, is called Geometric Brownian motion.
In this case a solution of (1.26) explicitly writes as

$$X_t = x_0 \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right) \tag{1.27}$$

and the process has respectively mean and variance

$$E[X_t] = x_0 e^{\mu t}, \quad Var(X_t) = x_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1). \tag{1.28}$$

We conclude this paragraph by recalling the Girsanov's Theorem. We write its statement in the general case of correlated Brownian motion $(W_t)_{t\in[0,T]}$ with correlation matrix ρ.

Theorem 1.1.21 (Girsanov's Theorem)**.** *Let $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F})_t)$ be a probability space, let λ be a stochastic process belonging to $\mathbb{L}^2[0,T]$, and let Z_t^λ the exponential martingale with respect to $\mathbb{P}$ related to it*

$$dZ_t = -Z_t(\rho^{-1}\lambda_t) \cdot dW_t, \quad t \in [0,T],$$

or equivalently

$$Z_t = \exp\left(-\int_0^T \rho^{-1} \cdot \lambda_t dW_t - \frac{1}{2}\int_0^T |\rho^{-1} \cdot \lambda_t|^2 dt\right).$$

Consider the measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^\lambda.$$

Then the process

$$W_t^\lambda = W_t + \int_0^t \lambda_s ds, \quad t \in [0,T]$$

is a Brownian motion on $((\Omega, \mathscr{F}, \mathbb{Q}, (\mathscr{F})_t)$ with correlation matrix ρ.

The main assumption of the Girsanov's theorem is the martingale property of the process Z_t. In financial applications we frequently assume that $\rho^{-1} \cdot \lambda_t$ is a bounded process: under this hypothesis Z_t is martingale. Nevertheless, in general, the fact that $\rho^{-1} \cdot \lambda_t$ is bounded may not be verified directly, so the following Novikov condition can be very useful:

Proposition 1.1.22 (Novikov Condition). *If $\rho^{-1} \cdot \lambda_t$ such that*

$$E\left[\exp\left(\frac{1}{2}\int_0^T |\rho^{-1} \cdot \lambda_t|^2 dt\right)\right] < \infty,$$

then the exponential martingale Z_t is a strict martingale.

1.2 Options, Bonds and Interest rates

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, where Ω is a set such that $\mathbb{P}(w) > 0$ for every $w \in \Omega$.

Definition 1.2.1. *A continuous market is a vector of $n+1$ stochastic process:*

$$S_t = (S_t^0, S_t^1, ..., S_t^n), \qquad t \in [0, T],$$

*where $S_t^i,\ i = 0, ..., n$ are commonly called **assets**. For every fixed $t \in [0, T]$, $S_t^i,\ i = 0, ..., n$ is a nonnegative random variable which denotes the price of the i^{th} asset at time t.*

In this paragraph we only deal with continuous stochastic processes.

Let W_t be a d-dimensional Brownian motion and $t \in [0, T]$. In the Financial application the d-dimensional Brownian motion W_t represents the d independent sources of the risk of the market. We also suppose that $d \leq n$.

Typically, in the Black-Scholes setting, the dynamic of $S_t^i,\ i = 1, ..., n$, with $t \in [0, T]$ is modeled as follows:

$$dS_t^i = \mu_i(t, S_t) S_t^i dt + \sum_{j=1}^{d} \sigma_{ij}(t, S_t) S_t^i dW_t^j, \tag{1.29}$$

where $S_t = (S_t^1, ..., S_t^n)$, the vector $\mu(t, S_t)$ is called *drift* and the matrix $\sigma(t, S_t)$ is the *volatility*.

We endow the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the usual filtration $(\mathscr{F}_t^W)$, $t \in [0, T]$ generated by the Brownian motion W_t for every $t \in [0, T]$. As a consequence,

the stochastic processes in (1.29) are adapted.
We also suppose that $F_0 = \{\emptyset, \Omega\}$, i.e at time zero the random variables are deterministic and observable (the price of the assets at time zero are observable), and we further assume that $\mathscr{F} = \mathscr{F}_T$. The drift $\mu_i(t, S_t),\ i = 1, ..., n$ and the volatility $\sigma_{ij}(t, S_t),\ i = 1, ..., n,\ j = 1, ..., d$ belong to $\mathbb{L}^1[0, T]$ and $\mathbb{L}^2[0, T]$ respectively.

We assume that the first asset S_t^0 is a risk-less asset which we will denote by B_t, having the following deterministic dynamic

$$dB_t = r_t B_t dt, \quad B_0 = 1, \qquad B_t = \exp\left(\int_0^T r_t dt\right). \tag{1.30}$$

The process $B_t,\ t \in [0, T]$ is called **Bond** and the stochastic process $r_t = r(t, S_t),\ t \in [0, T]$ is the **interest rate**. In the general case we assume that $r_t,\ t \in [0, T]$ belongs to $\mathbb{L}^1[0, T]$ and is a positive stochastic process for all $t \in [0, T]$[3].

A financial derivative is a contract whose value depends on one or more securities or assets, called underlying assets. Typically the underlying asset is a stock, a bond, a currency exchange rate or the quotation of commodities such as gold, oil or wheat.

An option is the simplest example of a derivative instrument. This is a contract that gives the right (but not the obligation) to its holder to buy or sell some amount of the underlying asset at a future date, for a pre-specified price. The parties that sign an option establish at time $t = 0$ (today) the amount of money which will be paid or cashed for the purchase or sale of the underlying asset.

Who sells an option has the obligation, at time T, to sell or buy the underlying asset if the holder decides to exercise. Therefore, to write the contract, we need to specify: an underlying asset, an exercise price K called *strike price*, a date T, *called maturity*. The function φ which describes the profits or losses at time T is said *pay-off function*.

Who subscribes on option is mainly concerned with two problems:

- **The Evaluation Problem**. This matter mainly concerns the buyer: one knows the exact value of an option at maturity T (actually, its value is given by the pay-off function φ). The goal is to establish a *fair price* c_0, which

[3] In the Black-Scholes setting the interest rate $(r_t)_{t\in[0,T]}$ plays a marginal role and it is modeled as a continuous positive stochastic process. Actually, nowadays, deriving an exact formula for stochastic interest rates is not an easy task to perform. Moreover, negative interest rates can be addressed in the modern class models.

represents the initial value of the option, to get into the contract. The money will be paid at time $t = 0$.

- **The Hedging Problem**. This matter mainly concerns the seller and, typically, who sells an option is a bank. Provided that the amount c_0 is cashed, the goal is to exploit the money to construct an investment strategy (commonly said *hedging portfolio*), which allows to cover from unexpected losses.

Actually, the two problems above are strictly connected. In particular, one may use a suitable approach to the problem which finds out the price of an option at time $t = 0$ by using hedging issues. The idea is to compute the value of an option by means of a strategy which time by time assumes exactly the same value of the option. This approach is called **pricing by replication**, and the portfolio is the **replication strategy**. This method has the advantage of using few and simple instruments of Stochastic Analysis even if the procedure requires additional notions and economic reasonings. We suggest the manuscripts [35], [66], [13] for an exhaustive presentation of the topic.

We follow in this thesis a bit sophisticated mathematical approach based on Stochastic Differential Equations.

Financial derivatives: Vanilla Options and Path dependent Options

We introduce some of the most common financial derivatives: the European (Vanilla) Call and Put Option and the path dependent option of Asian type.

An European Put Option is a contract that gives the owner the right to sell an asset at the expiry date T and at a prescribed price K. A Call Option gives instead to him the right to buy the same asset at the date T and at the price K. The value of the Option at its expiry date T is given by a function $\varphi(S_T)$, where S_t denotes the price of the asset at time t. For instance, the payoff of a call option is

$$\varphi_C(S_T) = \max(0, S_T - K),$$

while the payoff of a put option is

$$\varphi_P(S_T) = \max(0, K - S_T)$$

Path dependent Options are characterized by the fact that their value also depends on some average of the past price of the stock, that is $Z_t = Z(S_t, A_t, t)$ for

$0 \leq t \leq T$. For instance, in an *Arithmetic Average Floating Strike Option* the strike price of an option is computed as the average of the stock price, then its payoff is

$$\varphi_C(S_T, A_T) = \max\left(0, S_T - \frac{1}{T}\int_0^T S_t dt\right),$$
$$\varphi_P(S_T, A_T) = \max\left(0, \frac{1}{T}\int_0^T S_t dt - S_T\right),$$

while in the *Arithmetic Average Fixed Strike Option* the payoff is

$$\varphi_C(S_T, A_T) = \max\left(0, \frac{1}{T}\int_0^T S_t dt - K\right),$$
$$\varphi_P(S_T, A_T) = \max\left(0, K - \frac{1}{T}\int_0^T S_t dt\right).$$

When considering *Geometric Average Options*, the arithmetic average is replaced by the geometric one

$$\exp\left(\frac{1}{T}\int_0^T \log(S_t)\, dt\right).$$

1.2.1 Option Pricing Theory

In this section we introduce the evaluation problem in Option Pricing in the Black-Scholes framework. We discuss the topic in the general situation of financial derivatives which are written on n underlying assets.

We consider a market in which there are $n+1$ assets. The first n assets are risky and have the following dynamic

$$dS_t^i = \mu(t, S_t)S_t^i dt + \bar{\sigma}_i(t, S_t)S_t^i d\bar{W}_t^i, \quad i = 1, ..., n. \tag{1.31}$$

where $\bar{W}_t = (\bar{W}_t^1, ..., \bar{W}_t^n)$ is a vector of n-correlated Brownian motion with correlation matrix ρ. We assume that ρ is a deterministic matrix whose elements can be constant or depending on t.
We also assume the existence of a bond B_t, $t \in [0, T]$ whose dynamic is described in (1.30) with stochastic interest rate $r_t \in \mathbb{L}^1[0, T]$.

In the general case, we suppose that the asset S_t^i pays dividends $\delta_i(t, S_t)$, where δ_t is a stochastic process belonging to $\mathbb{L}^1[0, T]$. Such a process is also known as dividend-yield. We denote $\hat{S}_t^i = S_t^i \exp\left(\int_0^t \delta_i(t, S_t)dt\right)$. By applying Itô formula to the process $\hat{S}_t^i$, we have

$$d\hat{S}_t^i = (\mu(t, \hat{S}_t) + \delta(t, \hat{S}_t))\hat{S}_t^i dt + \bar{\sigma}_i(t, \hat{S}_t)\hat{S}_t^i d\bar{W}_t^i. \tag{1.32}$$

Discounting the (1.32) we have

$$d\hat{S}_t^i = (\mu_i(t,\hat{S}_t) - r_t + \delta_i(t,\hat{S}_t))\hat{S}_t^i dt + \bar{\sigma}_i(t,\hat{S}_t)\hat{S}_t^i d\bar{W}_t^i$$

and by applying the Girsanov Theorem (Theorem 1.1.21), the process defined by

$$d\widetilde{W}_t^i = d\bar{W}_t^i + \frac{(\mu_i(t,\hat{S}_t) - r_t + \delta_i(t,\hat{S}_t))}{\bar{\sigma}_i(t,\widetilde{S}_t)} dt \tag{1.33}$$

is a Brownian motion under the unique measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \rho^{-1}\cdot\lambda_t dW_t - \frac{1}{2}\int_0^T |\rho^{-1}\cdot\lambda_t|^2 dt\right),$$

where we have set

$$\lambda_t^i = \frac{\mu_i(t,\hat{S}_t) + \delta_i(t,\hat{S}_t) - r_t}{\bar{\sigma}_i(t,\hat{S}_t)}, \quad \lambda_t = (\lambda_t^1, ..., \lambda_t^n).$$

In order to write the new dynamic of the assets under the measure $\mathbb{Q}$, we substitute (1.33) in the expression (1.31). Hence, under the measure $\mathbb{Q}$, we have

$$dS_t^i = (r_t - \delta(t,S_t))S_t^i dt + \bar{\sigma}_i(t,S_t)S_t^i d\widetilde{W}_t^i.$$

which, basically says the fact that under $\mathbb{Q}$ we

$$E[S_T^i|S_t^i] = S_t^i \exp\left(\int_t^T (r_t - \delta_t^i)dt\right), \quad E[\hat{S}_T^i|\hat{S}_t^i] = \hat{S}_t^i \exp\left(\int_t^T r_t dt\right). \tag{1.34}$$

The formula (1.34) resumes two crucial fact:

- The discounted process $e^{-\int_0^t r_t dt}\hat{S}_t^i$ is a martingale under the measure $\mathbb{Q}$;
- At time t the conditional expectation $E[\hat{S}_T^i|\hat{S}_t^i]$ has the same dynamic of a Bond under $\mathbb{Q}$. This fact tell us that, under the measure $\mathbb{Q}$, we realize (in mean) a gain which is exactly equal to the profit arising from a risk-free investment (B_t is the risk-less asset): in this sense we are *risk-neutral.*

This explain the fact why the measure $\mathbb{Q}$ is also called **risk-neutral measure** or **Equivalent Martingale Measure (EMM)**:

Definition 1.2.2 (EMM)**.** *An equivalent martingale measure $\mathbb{Q}$ with numeraire $B_t,\ t \in [0,T]$ is a probability measure on $(\Omega, \mathscr{F})$ such that*

i) *$\mathbb{Q}$ is equivalent to $\mathbb{P}$;*

ii) *the process of the discounted prices*

$$\exp\left(-\int_0^t r_s ds\right)\hat{S}_t^i, \quad t\in[0,T],$$

is a strict $\mathbb{Q}$-martingale. In particular, it holds the following risk-neutral pricing formula

$$\hat{S}_t^i = E^{\mathbb{Q}}\left[\exp\Big(-\int_t^T r_s ds\Big)\hat{S}_T^i \,|\, \mathscr{F}_t\right], \quad t\in[0,T] \tag{1.35}$$

If an EMM exists we say that the ***Market is Arbitrage free****.*

Let $(\Omega, \mathscr{F}, \mathbb{Q}, (\mathscr{F}_t)_t)$ be a filtered probability space, where $\mathscr{F}_t$ is the filtration generated by a Brownian motion $\widetilde{W}_t = (\widetilde{W}_t^1, ..., \widetilde{W}_t^n)$.
We consider the processes whose dynamic,, under $\mathbb{Q}$, is

$$dS_t^i = (r_t - \delta(t,S_t))S_t^i dt + \bar{\sigma}_i(t,S_t)S_t^i d\widetilde{W}_t^i, \tag{1.36}$$

$$dB_t = r_t B_t dt, \quad t\in[0,T]. \tag{1.37}$$

We now introduce the the financial derivative $\Phi(S_T^1, ..., S_T^n)$, where $\Phi : \mathbb{R}^n \mapsto \mathbb{R}$ is a continuous function which satisfy the following growth condition

$$\int_{\mathbb{R}^n} e^{-c|(x_1,...,x_n)|^2}|\Phi(x_1,...,x_n)|dx_1\cdots dx_n < \infty. \tag{1.38}$$

for a suitable positive constant c. We assume that, in accordance with (1.35), for every $t\in[0,T]$ it holds the following **risk neutral valuation formula**, which reflects the fact that the Market is arbitrage free

$$V_t = V(t,\Phi) = E^{\mathbb{Q}}\left[\exp\Big(-\int_t^T r_s ds\Big)\Phi(S_T^1, ..., S_T^n)\,|\,\mathscr{F}_t\right]. \tag{1.39}$$

We refer to V_t, $t\in[0,T]$ as the pricing process.

We explicitly note that, since the process $S_t = (S_t^1, ..., S_t^n)$, is Markovian we have that the pricing process must be of the form $V(t,\Phi) = F(t,S_t)$ for some function $F(t,x_1,...,x_n)\in C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R})$.

Reminding the dynamic (1.36) and the relation (1.39), by using the Feynman-Kac theorem, we can conclude that the function $F(t,x_1,...,x_n)$ must solve the following backward Cauchy Problem (denote by $X=(x_1,...,x_n)$):

$$\begin{cases} \frac{\partial f}{\partial t} + \sum_{i=1}^n (r(t,x)-\delta_i(t,x))x_i\frac{\partial f}{\partial x_i} \\ \quad +\frac{1}{2}\sum_{i,j=1}^n \bar{\sigma}_i(t,x)\,\bar{\sigma}_j(t,x)\rho_{i,j}x_i x_j\frac{\partial^2 f}{\partial x_i\partial x_j} = r(t,x)f, \\ F(T,x_1,...,x_n) = \Phi(x_1,...,x_n). \end{cases} \tag{1.40}$$

The Cauchy problem (1.40) and the representation formula (1.39), fully solve the problem of Pricing Financial Derivatives instrument both from theoretical and numerical point of views.

The rank of the matrix $A = [\bar{\sigma} \cdot \bar{\sigma}^*](t, x_1, ..., x_n)$ plays a crucial role in our treatment. Indeed, if we have $rank(A) = n$ (or equivalently $d = n$), the problem (1.40) is a standard n-dimensional Cauchy problem of parabolic type and the matrix A is uniformly parabolic for every $t \in [0, T]$ (provided that the volatility is strictly positive and uniformly bounded from below and above).

On the other hand, if the $rank(A) < n$ (or equivalently $d < n$), the problem (1.40) become strongly degenerate and the matrix A is not uniformly parabolic for every $t \in [0, T]$. In this case we refer to (1.40) as a Degenerate Kolmogorov Problem.

Asian Options

Geometric Average. A Geometric Average Asian Option is a contract which depends on a stock S_t and its geometric mean

$$\exp\Big(\frac{1}{T}\int_0^T \log(S_\tau)d\tau\Big).$$

In the simplest case of an option written on one asset with constant drift and volatility, we introduce directly the process of the mean and we consider the following system

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t, \\ dG_t = \log(S_t)dt \end{cases} \tag{1.41}$$

The payoff is given by $\varphi = \varphi(S_T, G_T)$. We always assume the existence of a bond $B_t,\ t \in [0, T]$ whose dynamic is given by $B(t) = B_0 e^{rt}$ and the interest rate r is constant on $[0, T]$ and non-negative.

By applying the change of variables $X_t = \log(S_t)$ and the bi-dimensional Itô formula, we get:

$$\begin{cases} dX_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t, & X_0 = x_0; \\ dG_t = X_t dt, & G_0 = y_0. \end{cases} \tag{1.42}$$

We now observe that the system (1.42) is linear with constant coefficients, and according to the notations used in Proposition 1.1.20 we have

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} \left(\mu - \frac{\sigma^2}{2}\right) \\ 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore we have that the process (X_t, G_t) is Gaussian. In order to compute its mean and its covariance matrix we note that, since B is nilpotent we obtain

$$e^{tB} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

and the covariance matrix is given by

$$\Sigma(t) = \begin{pmatrix} \sigma^2 t & \sigma^2 \frac{t^2}{2} \\ \sigma^2 \frac{t^2}{2} & \sigma^2 \frac{t^3}{3} \end{pmatrix} \tag{1.43}$$

Therefore the random vector has distribution

$$(X_t, G_t) \sim \mathcal{N}\left(x_0 + (\mu - \frac{\sigma^2}{2})t, y_0 + x_0 t + \left(\mu - \frac{\sigma^2}{2}\right)\frac{t^2}{2};\ \Sigma(t)\right). \tag{1.44}$$

Moreover, since the matrix $\Sigma(t)$ in (1.43) is positive definite for every $t \in [0, T]$, the transition density $p(\bar{x}, \bar{y}, T; x, y, t)$ of the process (X_t, G_t) was given by (1.21).

According to the general Pricing Theory introduced before, we explicitly note that the dynamic of the stochastic process (S_t, G_t) in (1.41), with $t \in [0, T]$ under the equivalent Martingale Measure $\mathbb{Q}$ is obtained by replacing $\mu = r$ in the system (1.41).

In this case, reminding (1.44) and that $S_t = \exp(X_t)$, we able to explicitly compute the risk evaluation formula

$$V_t = e^{-r(T-t)} E^{\mathbb{Q}} \left[\varphi(\exp(X_T), G_T) \mid (X_t, G_t) = (x, y)\right],$$

Furthermore, the price $f(t, x, y) = V_t$ at time t of a geometric average asian option satisfies the following degenerate parabolic Cauchy problem (we denote by $x = S_t$ and by $y = G_t$):

$$\begin{cases} \partial_t f + rx\partial_x f + \log(x)\partial_y f(t, x, y) + \frac{\sigma^2 x^2}{2}\partial_{xx} f = rf, \\ f(T, x, y) = \varphi(x, y), \end{cases} \tag{1.45}$$

where $\varphi(S_T, G_T)$ is a prescribed payoff function.

The peculiarity of the Cauchy problem (1.45), is that by means of the change of variable

$$u(x, y, t) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x + \left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)t} f\left(e^{\frac{\sigma}{\sqrt{2}}x}, \frac{\sigma}{\sqrt{2}}y, T - t\right)$$

we are able to lead back the problem (1.45) to the following degenerate problem

$$\begin{cases} \partial_{xx} u(x, y, t) + x\partial_y u(x, y, t) - \partial_t u(x, y, t) = 0, & (x, y, t) \in \mathbb{R}^2 \times]0, T]; \\ u(x, y, 0) = e^{\frac{2r-\sigma^2}{2\sqrt{2}\sigma}x} \varphi\left(e^{\frac{\sigma}{\sqrt{2}}x}, \frac{\sigma}{\sqrt{2}}y\right), & (x, y) \in \mathbb{R}^2. \end{cases} \tag{1.46}$$

The equation in (1.46) is the simplest example of a degenerate equation given by Kolmogorov in 1934. Such problem has an explicit solution

$$u(x,y,t) = \int_{\mathbb{R}^2} \Gamma(x,y,t;\xi,\eta,\tau)\varphi\Big(\exp\big(\tfrac{\sigma}{\sqrt{2}}\xi\big), \eta\Big) d\xi d\eta, \quad t > \tau,$$

where

$$\Gamma_0(x,y,t;\xi,\eta,\tau) = \frac{\sqrt{3}}{2\pi(t-\tau)^2} \exp\left(-\tfrac{(x-\xi)^2}{4(t-\tau)} - \tfrac{3}{(t-\tau)^3}\left(y - \eta - (t-\tau)\tfrac{(x+\xi)}{2}\right)^2 \right).$$

We will go back on Kolmogorv equations in the next chapter.

Arithmetic Average. An Arithmetic Average Asian Option is a contract which depends on a stock S_t and on its arithmetic mean

$$\frac{1}{T}\int_0^T S_t dt.$$

In the simplest case of an option written on one assets we introduce directly the process of the mean and we consider the following system

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t, \\ dA_t = S_t dt \end{cases} \tag{1.47}$$

The payoff is a function $\varphi = \varphi(S_T, A_T)$. As done before, we assume the existence of a bond $B_t,\ t \in [0,T]$ whose dynamic is given by $B(t) = B_0 e^{rt}$ and the interest rate r is constant on $[0,T]$ and non negative. The case of arithmetic Asian Option is much more challenging then geometric average: the main difficulty lies in the fact that the system (1.47) cannot be led back to a linear system by means of change of variables. In order to compute explicitly the risk evaluation formula, we need the transition density of the random variable (S_t, A_t). Unfortunately, the distribution of the vector (S_t, A_t) is not easy to find. The distribution of A_t itself is not immediate. Many authors tried to bypass the problem by using various approach. Levy [50], for instance, in 1992, had a simplified approach, which was used to approximate the density of A_t by matching the first two moments. The first two moment of the random variable A_t can be computed directly by integration (see formulas in chapter 3).

Another interesting approach was proposed by Ingersoll, who used a suitable change of variable for the Arithmetic asian Option through which one can find the value of an Arithmetic asian option just by solving a parabolic PDE equation of

Black-Scholes type. The approach proposed by Ingersoll work only for suitable pay-off functions.

The problem was solved by Yor in [84] and Geman-Yor in [33]. In these works the authors were able to derive explicitly the joint law of the random vector (S_t, A_t) for every $t \in [0, T]$ and they wrote explicitly the Laplace transform at time t of the price of an Arithmetic Asian Option. Their approach is based on some advanced techniques in Probability Theory, and we list the most meaningful aspects of their methodology in chapter 3.

Ingersoll aprroach for Arithmetic Asian Option

Let consider an Arithmetic asian Option whose pay-off is a homogeneous function

$$\varphi(A_T, S_T) = S_T \varphi(A_T/S_T, 1). \tag{1.48}$$

The dynamic of the asset under the martingale measure $\mathbb{Q}$ is

$$\begin{cases} dS_t = rS_t dt + \sigma S_t dW_t, \\ dA_t = S_t dt. \end{cases} \tag{1.49}$$

were the interest rate r is constant and non negative, the volatility σ is constant and $t \in [0, T]$. By default one takes B_t as numeraire and, given a EMM $\mathbb{Q}$, the risk evaluation formula writes

$$V_t = B_t E^{\mathbb{Q}} \Big[\frac{\varphi(A_T, S_T)}{B_T} \,|\, \mathscr{F}_t \Big] = E^{\mathbb{Q}} \Big[e^{-\int_t^T r_\tau d\tau} \varphi(A_T, S_T) \,|\, \mathscr{F}_t \Big].$$

In general, for a generic numeraire S_t^0 with associated EMM $\mathbb{Q}_0$, one has the following risk evaluation formula

$$V_t = S_t^0 E^{\mathbb{Q}_0} \Big[\frac{\varphi(A_T, S_T)}{S_T^0} \,|\, \mathscr{F}_t \Big]. \tag{1.50}$$

Provided that (1.48) holds, with B_t as numeraire we have to calculate

$$V_t = B_t E^{\mathbb{Q}} \Big[\frac{S_T \varphi(A_T/S_T, 1)}{B_T} \,|\, \mathscr{F}_t \Big],$$

whereas, with S_t as numeraire, and $\mathbb{Q}_0$ the corresponding EMM, the same V_t can be obtained as

$$V_t = S_t E^{\mathbb{Q}_0} \Big[\varphi(A_T/S_T, 1) \,|\, \mathscr{F}_t \Big],$$

which appears much more simple.

Therefore, a change of numeraire induces a change of the EMM, which may result convenient in certain cases for the calculation of derivative prices. The question is, once we have computed the martingale measure $\mathbb{Q}$ with numeraire B_t, how can we find the martingale measure $\mathbb{Q}^0$ related to S_t as numeraire? In other terms, what is the Radon-Nikodym derivative $L = d\mathbb{Q}^0/d\mathbb{Q}$?

In this case, it is quite simple to compute L. Indeed, if we set $L = \frac{S_T B_0}{B_T S_0}$, since S_t/B_t is a $\mathbb{Q}$ martingale, we have

$$L_t = E^{\mathbb{Q}}\big[L \mid \mathscr{F}_t\big] = \frac{S_t B_0}{B_t S_0}.$$

By applying the Bayes formula we have

$$\begin{aligned} S_t E^{\mathbb{Q}_0}\Big[\frac{\varphi}{S_T} \mid \mathscr{F}_t\Big] &= \frac{S_t E^{\mathbb{Q}}\Big[L\frac{\varphi}{S_T} \mid \mathscr{F}_t\Big]}{E^{\mathbb{Q}}\Big[L \mid \mathscr{F}_t\Big]} = \\ S_t E^{\mathbb{Q}}\Big[\frac{L}{L_t}\frac{\varphi}{S_T} \mid \mathscr{F}_t\Big] &= B_t E^{\mathbb{Q}}\Big[\frac{\varphi}{B_T} \mid \mathscr{F}_t\Big]. \end{aligned}$$

The previous computation can be formalized in the following proposition

Proposition 1.2.3 (Change of numeraire)**.** *Let $\mathbb{Q}$ be an EMM with numeraire B_t and let U_t a process such that U_t/B_t is a $\mathbb{Q}$-martingale.*

Consider the probability measure $\mathbb{Q}^U$ on $(\Omega, \mathscr{F})$ defined by

$$\frac{dQ^U}{dQ} = \frac{U_T B_0}{B_T U_0},$$

Then for any $X \in L^1(\Omega, \mathbb{Q})$ we have

$$B_t E^{\mathbb{Q}}\Big[\frac{X}{B_T} \mid \mathscr{F}_t\Big] = U_t E^{\mathbb{Q}^U}\Big[\frac{X}{U_T} \mid \mathscr{F}_t\Big], \quad t \in [0, T].$$

A direct consequence of the previous result is the following

Corollary 1.2.4. *Let U_t, V_t be stochastic processes such that U_t/B_t, V_t/B_t are martingale under $\mathbb{Q}$. Le denote by $\mathbb{Q}^U$, $\mathbb{Q}^V$ the corresponding EMMs Then we have*

$$\frac{dQ^U}{dQ^V}\Big|_{\mathscr{F}_t} = \frac{U_t V_0}{V_t U_0}.$$

Note that if U_t, V_t are be two positive Itô processes of the form

$$\begin{aligned} dU_t &= (...)dt + \sigma_t^U dW_t \\ dV_t &= (...)dt + \sigma_t^V dW_t, \end{aligned}$$

where W_t is a correlated d-dimensional Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{Q}, (\mathscr{F}_t)_t)$ with correlation matrix ρ, by applying the Itô formula we have

$$d\frac{U_t}{V_t} = (...)dt + \frac{U_t}{V_t}\Big(\tfrac{\sigma^U}{U_t} - \tfrac{\sigma^V}{V_t}\Big)dW_t \tag{1.51}$$

By combining Corollary 1.2.4 with (1.51) and Girsanov Theorem 1.1.21, we obtain the following proposition which basically shows how the dynamic of the assets change when we make a change of numeraire.

Theorem 1.2.5. *Let U_t, V_t be two positive Itô processes of the form*

$$dU_t = (...)dt + \sigma_t^U dW_t,$$
$$dV_t = (...)dt + \sigma_t^V dW_t,$$

where W_t is a correlated d-dimensional Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{Q}, (\mathscr{F}_t)_t)$ with correlation matrix ρ. Let $\mathbb{Q}^U$, $\mathbb{Q}^V$ be the EMMs related to U, V respectively and W_t^U, W_t^V be the related Brownian motions. Then the following formula for the change of drift holds:

$$dW_t^V = dW_t^U + \rho\Big(\tfrac{\sigma^U}{U_t} - \tfrac{\sigma^V}{V_t}\Big)dt, \quad t \in [0, T]. \tag{1.52}$$

With Theorems 1.2.3 and 1.2.5 in hand, it is quite simple to solve the problem of Arithmetic Asian Option, when the pay-off function $\varphi(A_T, S_T)$ is homogeneous with respect to S_T. In this case, by picking the process S_t as numeraire, and $\mathbb{Q}^0$ the associated EMM, we have from (1.50)

$$V_t = S_t E^{\mathbb{Q}^0}\Big[\varphi(A_T/S_T, 1)\,|\,\mathscr{F}_t\Big] = e^{-r(T-t)} E^{\mathbb{Q}}\Big[\varphi(A_T, S_T)\,|\,\mathscr{F}_t\Big]$$

In order to compute the mean with respect to the measure $\mathbb{Q}^0$ we need to write the S_t dynamic under $\mathbb{Q}^0$. We use the Theorem 1.2.5, we denote by $W_t^{\mathbb{Q}}$ the Brownian motion which drives S_t under the measure $\mathbb{Q}$ having B_t as numeraire, and by $W_t^{\mathbb{Q}^0}$ the Brownian motion which drives S_t under the measure $\mathbb{Q}^0$ having S_t as numeraire. In view of Theorem 1.52 we have

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{Q}_0} + \sigma dt.$$

Therefore, under $\mathbb{Q}^0$ we obtain

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} = (r + \sigma^2)S_t dt + \sigma S_t dW_t^{\mathbb{Q}_0} \tag{1.53}$$

Defining the change of variable $Z_t = A_t/S_t$ where the dynamic of S_t is given in (1.53), we have that, by applying the Itô formula, the differential of Z_t under $\mathbb{Q}^0$ is given by

$$dZ_t = (1 - rZ_t)dt - \sigma Z_t dW_t^{\mathbb{Q}^0}, \quad t \in [0, T].$$

Therefore, by the Faynman-Kac Formula we have that the function

$$g(t, z) = E^{\mathbb{Q}^0}\Big[\varphi(Z_T, 1) \,|\, Z_t = z\Big], \quad (t, z) \in [0, T) \times \mathbb{R}^+$$

satisfies the following backward Cauchy problem

$$\begin{cases} \partial_t g(t, z) + (1 - rz)\partial_z g(t, z) + \frac{\sigma^2 z^2}{2}\partial_{zz} g(t, z) = 0, & (t, z) \in [0, T) \times \mathbb{R}^+; \\ g(T, z) = \varphi(z, 1), & z \in \mathbb{R}^+. \end{cases} \tag{1.54}$$

which in turns is a parabolic problem of Black-Scholes type (i.e. the partial differential equation is not degenerate).

We end this paragraph by pointing out that the Ingersoll approach for Arithmetic Asian Option works both in the case of arithmetic floating strike and for fixed strike Asian Options. Indeed, when $\varphi(A_T, S_T) = \max(S_T - A_T/T, 0)$ the function is homogeneous with respect to the variable S_T.
In case of fixed strike $\varphi(A_T, S_T) = \max(A_T/T - K, 0)$, we use the change of variable $Y_t = A_t - KT$. In this case we have $dY_t = dA_t$ and the pay-off function becomes $\varphi(Y_T, S_T) = \max(Y_T/T, 0)$ which in turns satisfies the condition $\varphi(Y_T, S_T) = S_T\varphi(Y_T/S_T, 1)$.

Chapter 2

Degenerate Evolution operators

In this chapter we introduce some evolution operators of parabolic type, with non-negative and degenerate characteristic form, appearing in literature. The aim is to give a brief description of them and, in particular, we mainly focus on the Harnack inequality and its applications. The investigation of the local regularity properties of the solutions of degenerate elliptic-parabolic operators have produced very interesting results. The keystone result is the celebrated **Hörmander Theorem** proved in 1967 [34]. As we will see in the sequel, the Hörmander theorem is strongly linked with underlying algebraic-geometric structures of sub-Riemannian type which will lead us to associate to the operators which will be considered in the sequel a Lie group structure on $\mathbb{R}^{N+1}$. In this frame, many geometrical aspects can be pointed out.

In this chapter we show the key ideas and results of the method to derive two sided bounds for fundamental solutions of degenerate operators of Kolmogorov type. Some results will be proven in detail, for other ones will give specific references.

Summarizing, the key ingredients which we will apply in this chapter rely on the following tools:

- concerning the lower bound we will exploit the invariant Harnack inequality. Such inequality will be applied recursively along a suitable chain of points and combined with an optimization procedure. We mainly refer to the works given by Boscain-Polidoro [16], Polidoro [73] and Di Francesco-Polidoro [27],

- concerning the upper bound, we just announce the main techniques involved. In particular we mainly refer to the work by Pascucci-Polidoro [67], to the work by Cinti-Polidoro [23] where the upper bounds are proven by using Moser's iteration, and Di Francesco-Pascucci [26], Polidoro [71] where the upper bounds are derived by using and adaptation of the Levi parametrix method.

We first discuss general Hypoelliptic Operators with smooth coefficients: we introduce the main definition and tools linked with them and we prove the invariant Harnack inequality.

In the second part of the Chapter we discuss Kolmogorov Operators with variable and bounded coefficients: here we introduce the methodology which we use to derive two sided bounds for their fundamental solutions. The same method will be suitably adapted and shown in detail in chapter 4 to the case of the Arithmetic Asian Option type Operators.

2.1 Overview of the Method

In the first part of this chapter, smooth linear degenerate evolution operators of second order on $\mathbb{R}^{N+1}$ of the form

$$\mathcal{L}_0 := \sum_{i=1}^{m} X_i^2 + X_0 - \partial_t. \tag{2.1}$$

will be discussed. In (2.7) X_i's are smooth vector fields on $\mathbb{R}^{N+1}$, i.e. denoting $z = (x, t)$ the point in $\mathbb{R}^{N+1}$

$$X_i(z) = \sum_{j=1}^{N} d_{i,j}(z)\partial_{x_j}, \qquad i = 0, \dots, m,$$

where $d_{i,j}(z) : \mathbb{R}^{N+1} \mapsto \mathbb{R}$ are smooth functions for every $i, j = 1, \dots, N$, and $m \leq N$.

To take confidence with the operators in (2.7), we list some of meaningful operators belonging to the class described by $\mathcal{L}_0$.

1. **Heat operator on the Heisenberg group**: whose prototype is an operator acting on the variable $(x, y, w, t) \in \mathbb{R}^4$ which writes:

$$\mathscr{L}_0 = \partial_{xx} + \partial_{yy} + \frac{1}{4}(x^2 + y^2)\partial_{ww} + x\partial_{yw} - y\partial_{xw} - \partial_t = X_1^2 + X_2^2 - \partial_t \tag{2.2}$$

Here we have $X_1 = \partial_x - \frac{1}{2}y\partial_w$, $X_2 = \partial_y + \frac{1}{2}x\partial_w$ and the vector field

$$\Delta_{\mathbb{H}} = X_1^2 + X_2^2,$$

denotes the canonical sublaplacian in sub-riemmanian setting. Note that in this case we have $X_0 = 0$;

2. **Kolmogorov Type Operators**: the simplest version of a Kolmogorov operator $\mathcal{L}_0$, acting on $(x, y, t) \in \mathbb{R}^3$, is a linear operator of the form

$$\mathcal{L}_0 = \partial_{xx} + x\partial_y - \partial_t. \tag{2.3}$$

   In this case, the vector fields are the following

$$X(x, y, t) = \partial_x, \quad X_0(x, y, t) = x\partial_y;$$

3. **More Degenerate Kolmogorov-type Operators**: they are operators $\mathcal{L}_0$, acting on $(x, y, t) \in \mathbb{R}^3$, whose prototype is the following

$$\mathcal{L}_0 = \partial_{xx} + x^2\partial_y - \partial_t. \tag{2.4}$$

   In this case, the vector fields are the following

$$X(x, y, t) = \partial_x, \quad X_0(x, y, t) = x^2\partial_y.$$

4. **Asian Option type Operators**: whose prototype is represented by the following operator:

$$\mathcal{L}_0 = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t. \tag{2.5}$$

   In this case, the vector fields are the following

$$X(x, y, t) = x\partial_x, \quad X_0(x, y, t) = x\partial_y.$$

General second order partial differential operators with nonnegative and degenerate characteristic form appeared in literature with the seminal works of M.Picone, who proved the celebrated Weak Maximum Principle for their solutions. The interest in this type of equations in application fields arose because of works given by Fokker, Planck and Kolmogorov, who showed that PDEs with nonnegative characteristic form played a crucial role in several research areas, such as in mathematical models in theoretical physics and in diffusion processes. Since then, this type of equations appeared in many other different research fields, including for instance Cauchy-Riemann geometry, Minimal Surfaces and convexity in sub- Riemannian settings; Kinetic Theory of gases; Mathematical models in Finance and in Human

vision.
A first systematic study of boundary value problems for the class of elliptic-parabolic operators described by (2.1) was performed by Fichera, who proved existence theorems on weak solutions of the *Dirichlet problem* related to operators belonging to the family (2.7). Several existence and regularity results for elliptic- parabolic operators were proved by O.A. Oleĭnik and E.V.Radkevič and by J.J. Kohn and L.Nirenberg (see the monograph [65]).

More recent results for operators belonging to the family $\mathcal{L}_0$ described in (2.1) were given by Garofalo and Lanconelli [32], Lanconelli and Polidoro [47], Pascucci and Polidoro [70, 69], Kogoj and Lanconelli [39], Boscain and Polidoro [16] who concerned with Harnack inequality for non negative solutions and provided upper and lower estimates for fundamental solutions. We will see them in the next section.

In the second part of the chapter particular attention will be given to the study of *Kolmogorov Operators* $\mathcal{K}$ with non-negative characteristic form

$$\mathcal{K}u \equiv \sum_{i,j=1}^{m} a_{ij}(x,t)\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u$$

where $m \leq N$ and whose coefficients a_{ij}'s can be constant or variable satisfying suitable conditions which will be specified in the sequel. We here describe the method to derive two-sided bounds for fundamental solutions of the operator $\mathcal{K}$.

We want to close this paragraph by giving a short overview on the results about Kolmogorov Operators in literature.

In Lunardi [52], the problem on existence, uniqueness, optimal Schauder estimates and optimal Hölder regularity results for solutions u of the Cauchy problem

$$\begin{cases} \sum_{i,j=1}^{m} a_{ij}(x)\partial_{x_1 x_j} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u = f(x,t), & (x,t) \in \mathbb{R}^N \times]0,T]; \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.6}$$

were discussed in the framework of the semigroup theory. The main assumption appearing in [52] are that the coefficients of the matrix $(a_{i,j})_{i,j=1,\dots,m}$ satisfy the condition

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \ \forall \xi, \, x \in \mathbb{R}^N,$$

where λ is a strictly positive constant, and that the functions φ, $\partial_{x_i}\varphi$, $\partial_{x_i x_j}\varphi$ and f are Hölder continuous. Note that, in this frame, the coefficients a_{ij}'s are allowed to be unbounded. We also quote the works given by Lorenzi [51] and Priola [75] who extended the previous results given by Lunardi in [52].

In the framework of sub-riemmanian geometry and Hörmander setting, several results on interior Schauder estimates for the solution u of the Cauchy problem related to the Kolmogorov operator $\mathcal{K}$ and on the Dirichlet problem for it were discussed by Manfredini [54], Di Francesco-Polidoro [27]. The Dirichlet problem was also studied by Lascialfari-Morbidelli in [48] for quasilinear ultraparabolic operator. We also remind that in Polidoro [72] uniqueness and representation for the solutions of the Cauchy problem associated to the equation $\mathcal{K}u = 0$ were proven.

We eventually quote the work of Delarue-Menozzi [25], where the authors considered a wider class of Kolmogorov including, for instance, operators with non linear drifts. In their paper the two sided bounds for the fundamental solution is derived by applying parametrix method linked with a Stochastic Control approach. We provide a comparison with their results in the next chapter.

2.2 Hypoelliptic Operators

Consider a linear second order operator on $\mathbb{R}^{N+1}$ of the form

$$\mathcal{L}_0 := \sum_{k=1}^{m} X_k^2 + X_0 - \partial_t. \tag{2.7}$$

In (2.7) X_k's are smooth vector fields on $\mathbb{R}^{N+1}$, i.e. denoting $z = (x, t)$ the point in $\mathbb{R}^{N+1}$

$$X_i(z) = \sum_{j=1}^{N} d_{i,j}(z)\partial_{x_j}, \qquad i = 0, \ldots, m, \tag{2.8}$$

where $d_{i,j}(z) : \mathbb{R}^{N+1} \mapsto \mathbb{R}$ are smooth functions for every $i, j = 1, \ldots, N$. Here $m \le N$ and we denote by $Y = X_0 - \partial_t$.

As usual in the PDEs theory, we identify the *directional derivatives* with their *vector fields*

$$X_i(z) \sim \begin{pmatrix} d_{i1}(z) \\ \vdots \\ d_{iN}(z) \end{pmatrix}$$

In general, we consider operators with $m < N$, in this specific case we say that $\mathcal{L}_0$ is strongly degenerate. We now introduce an important notion about the regularity of an operator $\mathcal{L}_0$ in (2.7).

Definition 2.2.1. *We say that an operator $\mathcal{L}_0$ is hypoelliptic in $\Omega \subseteq \mathbb{R}^N \times \mathbb{R}$ if every distributional solution of $\mathcal{L}_0 u = f$ is a $C^\infty(\Omega)$ solution whenever $f \in C^\infty(\Omega)$.*

The Hörmander condition provides us with a simple sufficient condition for the hypoellipticity of $\mathcal{L}_0$. It requires the definition of *commutator* or *Lie Bracket* of two vector fields W and Z, acting on $u \in C^\infty(\Omega)$ as $[W, Z]u := WZu - ZWu$.

Definition 2.2.2. *We say that the vector fields $X_1(z), \dots, X_m(z), Y(z)$ satisfy the* Hormander Condition *at step k if:*

- *the vector fields $X_i(z)$ $i = 1, \dots N$ and $Y(z)$;*
- *the Lie brackets $[X_i, X_j](z)$ and $[X_i, Y](z)$ for $i = 1, \dots N$;*
- *the Lie bracket obtained by commuting the vector fields $X_1(z), \dots, X_m(z), Y(z)$ with the commutators $[X_i, X_j](z)$, $[X_i, Y](z)$ for $i, j = 1, \dots N$,*

until the step k, form a basis for $\mathbb{R}^{N+1}$ for every $z \in \mathbb{R}^{N+1}$.

The Hörmander condition can be stated in terms of *Lie Algebra*

$$\text{Lie}\,\{X_1, \dots, X_m, Y\}\,(z),$$

which denotes the vector space generated by the vector fields $X_1(z), \dots, X_m(z), Y(z)$ and by their commutators. Whit this notation the Hörmander condition holds true if

$$\text{rank } (\text{Lie}\,\{X_1, \dots, X_m, Y\})(z) = N + 1, \tag{2.9}$$

at every $z \in \Omega$. The celebrated hypoellipticity result due to Hörmander reads as follows.

Theorem 2.2.3 (Hörmander [34])**.** *Let $\mathcal{L}_0$ be a smooth operator as in* (2.7) *and $X_1(z), \dots, X_m(z), Y(z)$ its vector fields. If* (2.9) *holds at every $z \in \Omega$, then $\mathcal{L}_0$ is hypoelliptic in Ω.*

The main assumption we make in this section are that it holds the Hörmander condition and it holds invariance properties for the operator $\mathcal{L}_0$ with respect to suitable left translations and dilations.

Hypothesis [H] *The vector fields $X_1, \dots, X_m, Y$ satisfy the* Hörmander condition

$$\text{rank } (\text{Lie}\{X_1, \dots, X_m, Y\}(z)) = N + 1, \qquad \forall z \in \mathbb{R}^{N+1}. \tag{2.10}$$

Hypothesis [H1] *There exists a homogeneous Lie group* $\mathbb{G} = \big(\mathbb{R}^{N+1}, \circ, \delta_\lambda\big)$ [1] *such that*

(i) $X_1, \dots, X_m, Y$ *are left translation invariant on* $\mathbb{G}$, i.e. given $\xi \in \mathbb{R}^{N+1}$, denoting by

$$l_\xi(z) = \xi \circ z$$

the left translation of $z \in \mathbb{R}^{N+1}$, we have

$$X_i(u(l_\xi(z))) = (X_i u)(l_\xi(z)).$$

We also assume that the composition law $\circ$ is Euclidean in the time variable. More explicitly

$$(x,t) \circ (\xi, \tau) = (f(x,t;\xi,\tau); t+\tau)$$

for a suitable smooth function f;

(ii) $X_1, \dots, X_m$ *are* δ_λ*-homogeneous of degree one and* Y *is* δ_λ*-homogeneous of degree two.* i.e.:

$$X_i(u(\delta_\lambda(z))) = \lambda (X_i u)(\delta_\lambda(z))$$
$$Y(u(\delta_\lambda(z))) = \lambda^2 (Yu)(\delta_\lambda(z))$$

Here, δ_λ is a matrix of the form $(\lambda^M)_{\lambda>0}$ where M is a diagonal and positive definite matrix on $\mathbb{R}^{N+1}$. Hypothesis [H1] allows us to state that if $u(z)$ is a solution of $\mathcal{L}_0 u(z) = 0$ in $\mathbb{R}^{N+1}$, then for every fixed point $z_0 \in \mathbb{R}^{N+1}$, the function $u(z_0 \circ \delta_\lambda(z))$ is still a solution of $\mathcal{L}_0 u(z_0 \circ \delta_\lambda(z)) = 0$. Indeed, one has

$$\mathcal{L}_0\big(u(z_0 \circ \delta_\lambda(z))\big) = \lambda^2 \big(\mathcal{L}_0 u\big)\big(z_0 \circ \delta_\lambda(z)\big) = 0 \tag{2.11}$$

In our setting, hypotheses [H1] and [H] imply that $\mathbb{R}^N$ has a direct sum decomposition

$$\mathbb{R}^N = V_1 \oplus \cdots \oplus V_n$$

such that, if $x = x^{(1)} + \cdots + x^{(n)}$ with $x^{(k)} \in V_k$, then the dilations are

$$\delta_\lambda(x^{(1)} + \cdots + x^{(n)}, t) = (\lambda x^{(1)} + \cdots + \lambda^n x^{(n)}, \lambda^2 t), \tag{2.12}$$

[1] A Lie group $\mathbb{G} = \big(\mathbb{R}^{N+1}, \circ\big)$ is called *homogeneous* if a family of dilations $(\delta_\lambda)_{\lambda>0}$ exists on $\mathbb{G}$ and $\delta_\lambda(z \circ \zeta) = (\delta_\lambda z) \circ (\delta_\lambda \zeta)$ for every $z, \zeta \in \mathbb{R}^{N+1}$ and for any $\lambda > 0$.

for any $(x,t) \in \mathbb{R}^{N+1}$ and $\lambda > 0$. We may assume that

$$x^{(1)} = (x_1, \dots, x_{m_1}, 0, \dots, 0) \in V_1,$$
$$x^{(k)} = (0, \dots, 0, x_1^{(k)}, \dots, x_{m_k}^{(k)}, 0, \dots, 0) \in V_k,$$

for some basis of $\mathbb{R}^N$, where

$$x_i^{(k)} = x_{m_1+\ldots+m_{k-1}+i}, \qquad i = 1, \dots, m_k \equiv \dim V_k.$$

The natural number

$$Q = \sum_{k=1}^{n} k\, m_k + 2$$

is usually called the *homogeneous dimension* of $\mathbb{G}$ with respect to $(\delta_\lambda)_{\lambda>0}$. We also introduce the following δ_λ-homogeneous norms on $\mathbb{R}^N$ and $\mathbb{R}^{N+1}$:

$$|x|_{\mathbb{G}} = \sum_{k=1}^{n} \sum_{j=1}^{m_k} |x_j|^{\frac{1}{k}}, \qquad \|(x,t)\|_{\mathbb{G}} = |x|_{\mathbb{G}} + |t|^{\frac{1}{2}}.$$

Since $X_1, \dots, X_m$ and Y are smooth vector fields which are δ_λ-homogeneous respectively of degree one and two, then

$$\begin{aligned} X_i &= \sum_{j=1}^{n} d_{i,j-1}(x^{(1)}, \dots, x^{(j-1)}) \cdot \nabla^{(j)}, \qquad i = 1, \dots, m, \\ Y := X_0 - \partial_t &= \sum_{j=2}^{n} d_{0,j-2}(x^{(1)}, \dots, x^{(j-2)}) \cdot \nabla^{(j)} - \partial_t, \end{aligned} \tag{2.13}$$

where

$$\nabla^{(j)} = \big(0, \dots, 0, \partial_{x_1^{(j)}}, \dots, \partial_{x_{m_j}^{(j)}}, 0, \dots, 0\big).$$

and $d_{i,j}$ and b_j are $\delta_\lambda-$homogeneous polynomial functions of degree j with values in V_{j+1} and V_{j+2} respectively.

Remark 2.2.4. *Let us explicitly note that formula* (2.13) *says that*

$$\text{span}\big\{X_1(0), \dots, X_m(0)\big\} = V_1,$$

then we may assume $m = m_1$ *and* $X_j(0) = \mathbf{e}_j$ *for* $j = 1, \dots, m$ *where* $\{\mathbf{e}_i\}_{1\le i\le N}$ *denotes the canonical basis of* $\mathbb{R}^N$. *Also note that from* (2.13) *it follows that*

$$\text{span}\big\{X_0(0), [X_j, X_i](0)\big\} = V_2, \quad i, j = 1, ..., m.$$

Moreover, up to a linear change of variable $(x,t) \to (x - t d_{0,0}, t)$, *we may assume that* $d_{0,0} = X_0(0) = 0$.

2.2.1 Admissible Paths and Controllability

Given an operator of the form (2.7), we define

Definition 2.2.5. *We say that an $\mathbb{R}^N$ valued function $\gamma : [0,T] \to \mathbb{R}^{N+1}$ is an* $\mathcal{L}_0$-admissible path *starting from $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ for the vector fields X_i, $i = 1 \dots m$, Y if it is an absolutely continuous solution of the following ODE*

$$\dot{\gamma}(\tau) = \sum_{k=1}^{m} \omega_k(\tau) X_k(\gamma(\tau)) + Y(\gamma(\tau)) \tag{2.14}$$
$$\gamma(0) = z_0.$$

with $\omega_1, \dots, \omega_m \in L^1([0,T])$.

Note that, from (2.14), we always have $\dot{t}(\tau) = -1$, then we can only steer points (x_0, t_0) and (x_1, t_1) with $t_1 < t_0$.

Definition 2.2.6 (Attainable set). *For every $z_0 \in \Omega \subseteq \mathbb{R}^{N+1}$ the* attainable set $\mathscr{A}_{z_0}$ *from z_0 in Ω is*

$$\mathscr{A}_{z_0} = \big\{ z \in \Omega \mid \text{there exists a time } \bar{t} \in \mathbb{R}^+ \text{ and an } \mathscr{L}\text{-admissible path}$$
$$\gamma : \ [0, \bar{t}] \to \Omega \ s.t \ z_0 = \gamma(0), \ z = \gamma(\bar{t}) \big\}. \tag{2.15}$$

Let fix a point $(x,t) \in \mathbb{R}^{N+1}$. We introduce the following *controllability condition*

Hypothesis [C]. *For every $(x,t), (y,s) \in \mathbb{R}^{N+1}$ with $t > s$, there exists an absolutely continuous path $\gamma : [0, t-s] \to \mathbb{R}^{N+1}$ solution of* (2.14) *such that $\gamma(0) = x, \quad \gamma(t-s) = y$.*

We explicitly remind that the controllability condition is a precious property that gives us a connectivity on $\mathbb{R}^{N+1}$ by combining the vector fields $X_i(z) \quad i = 1, \dots, m$ and Y. It is not always satisfied by an hypoelliptic operator: concerning the example introduced above, operators as in (2.2) and (2.3) satisfy such property, whereas the operators as in (2.4) and (2.5) do not satisfy it (see section 2.2.2 below). We also remind that it is not true that condition [C] is a consequence of (2.10), nevertheless it is well known that the strong Hörmander condition

$$\text{rank Lie}\big\{X_1, \dots, X_m\big\}(x) = N, \qquad \forall x \in \mathbb{R}^N, \tag{2.16}$$

(which is stronger than (2.10)) implies [C].

In the sequel we denote by $\gamma((x,t),(y,s),\omega)$ any admissible path joining (x,t) to (y,s).

For a fixed path $\gamma((x_0,t_0),(x_1,t_1),\omega)$, which starts from $z_0=(x_0,t_0)$ and ends to $z_1=(x_1,t_1)$, the following quantity

$$\Phi(\omega)=\int_0^{t_0-t_1}(\omega_1^2(s)+\cdots+\omega_m^2(s))ds \tag{2.17}$$

is said *cost related to the controls* $\omega_1,\ldots,\omega_m$.

Fundamental Solution

Operators of the form (2.7), verifying assumptions [C] and [H1], have been considered by Kogoj and Lanconelli in [39] and [40]. In particular, in [39], it is proved that $\mathcal{L}_0$ has a fundamental solution Γ_0, which can be defined as follows

Definition 2.2.7 (Fundamental Solution). *Let* $(x_0,t_0)\in\mathbb{R}^{N+1}$. *We say that the function*

$$(x,t)\in\mathbb{R}^{N+1}\mapsto\Gamma_0(x,t,x_0,t_0)\in\mathbb{R}$$

is a fundamental solution for the operator $\mathcal{L}_0$ *with pole in* $(x_0;t_0)$ *if satisfies the following property*

i) *For any fixed* $(x,t)\in\mathbb{R}^{N+1}$, $\Gamma_0(\cdot;x,t)$ *and* $\Gamma_0(x,t;\cdot)$ *belong to* $L^1_{loc}(\mathbb{R}^{N+1})$*;*

ii) *For every* $f\in C_0^\infty(\mathbb{R}^{N+1})$ *and* $(x,t)\in\mathbb{R}^{N+1}$ *it holds*

$$\mathcal{L}_0\int_{\mathbb{R}^{N+1}}\Gamma_0(x,t;x_0,t_0)f(x_0,t_0)dx_0dt_0=$$
$$\int_{\mathbb{R}^{N+1}}\Gamma_0(x,t;x_0,t_0)\mathcal{L}_0f(x_0,t_0)dx_0dt_0=-f(x,t);$$

as a consequence it holds $\mathcal{L}_0\Gamma_0(x,t;x_0,t_0)=-\delta_{(x_0,t_0)}$ *for every fixed* $(x_0,t_0)\in\mathbb{R}^{N+1}$ *in distributional sense;*

iii) *For any bounded function* $\varphi\in C(\mathbb{R}^N)$ *and* $x,y\in\mathbb{R}^N$, *we have*

$$\lim_{(x,t)\to(y,s)}u(x,t)=\varphi(y),\qquad \lim_{(y,s)\to(x,t)}v(y,s)=\varphi(x), \tag{2.18}$$

where

$$u(x,t):=\int_{\mathbb{R}^N}\Gamma_0(x,t;y,s)\varphi(y)dy,\qquad v(y,s):=\int_{\mathbb{R}^N}\Gamma_0(x,t;y,s)\varphi(x)dx. \tag{2.19}$$

Note that the functions in (2.19) *are weak solutions of the following backward and forward Cauchy problems:*

$$\begin{cases} \mathcal{L}_0 u(t,x) = 0, & (x,t) \in]s,+\infty[\times\mathbb{R}^N, \\ u(x,s) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \quad \begin{cases} \mathcal{L}_0^* v(y,s) = 0, & (y,s) \in]-\infty,t[\times\mathbb{R}^N, \\ v(y,t) = \varphi(y), & y \in \mathbb{R}^N. \end{cases}$$

iv) $\Gamma_0 \in C_0^\infty\big(\{(x,t;x_0,t_0) \in \mathbb{R}^{N+1}\times\mathbb{R}^{N+1} \mid (x,t) \neq (x_0,z_0)\}\big)$*;*

v) $\Gamma_0 \geq 0$ *and* $\Gamma_0(x,t;x_0,t_0) > 0$ *if and only if* $t > t_0$*;*

vi) *If we define* $\Gamma_0^*(x,t;x_0,t_0) := \Gamma_0(x_0,t_0;x,t)$ *then* Γ^* *is a fundamental solution for the adjoint operator* $\mathcal{L}_0^*$ *of* $\mathcal{L}_0$ *satisfying the dual properties of* **ii)** *and* **iv)**.

The crucial results appeared [39] and [40] regards the existence of a fundamental solution Γ for $\mathcal{L}_0$ and the following remarkable properties

i) $\Gamma_0(x,t;\xi,\tau)$ is δ_λ-homogeneous of degree $2-Q$:

$$\Gamma_0\left(\delta_\lambda(x,t);\delta_\lambda(\xi,\tau)\right) = \lambda^{2-Q}\Gamma_0(x,t;\xi,\tau), \qquad \lambda > 0, \tag{2.20}$$

ii) Γ_0 is invariant with respect to the left translation $\circ$ of $\mathbb{G}$

$$\Gamma_0\big((\xi,\tau)\circ(x,t);(\xi,\tau)\circ(y,s)\big) = \Gamma_0(x,t;y,s), \tag{2.21}$$

for every (x,t), (ξ,τ), $(y,s) \in \mathbb{R}^N \times [0,T]$ with $t > \tau$.

iii) It holds the following integral identity

$$\int_{\mathbb{R}^N} \Gamma_0(x,t;\xi,\tau)d\xi = 1, \quad \forall t > \tau,\ x \in \mathbb{R}^N. \tag{2.22}$$

2.2.2 Examples

In this section, we introduce some examples of operators belonging to the family described in $\mathcal{L}_0$. Some operators satisfy the assumptions [H] and [H1], whereas some not.

Heat operator on Carnot groups. Consider operators $\mathscr{L}_0$ in (2.7) with $X_0 = 0$ which satisfy Hypothesis [H1] and the srong Hörmander condition (2.16). In this setting we write

$$\mathcal{L}_0 = \Delta_{\mathbb{G}} - \partial_t \tag{2.23}$$

where $\Delta_{\mathbb{G}}$ denotes the canonical sublaplacian on $\mathbb{G}$

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i^2$$

with $X_i(x) : x \in \mathbb{R}^N \mapsto \mathbb{R}$ smooth vector fields for every $i = 1, ..., N$. The prototype of operators in the form (2.23) is the operator $\mathscr{L}_0 = X_1^2 + X_2^2 - \partial_t$, where

$$X_1 = \partial_x - \tfrac{1}{2} y \partial_w, \qquad X_2 = \partial_y + \tfrac{1}{2} x \partial_w. \tag{2.24}$$

Note that $\mathscr{L}_0$ acts on the variable $(x, y, w, t) \in \mathbb{R}^4$, and writes in the form (2.7) with $X_0 = 0$. The vector fields X_1, X_2 are invariant with respect to the left translation in $\mathbb{R}^4$ defined by

$$(x_0, y_0, w_0, t_0) \circ (x, y, w, t) = \left(x_0 + x, y_0 + y, w_0 + w + \tfrac{1}{2}(x_0 y - y_0 x), t_0 + t\right).$$

Moreover $\mathscr{L}_0$ is invariant with respect to the following dilation

$$\delta_r(x, y, w, t) = \left(rx, ry, r^2 w, r^2 t\right) = \mathrm{diag}(r, r, r^2, r^2) \cdot (x, y, w, t).$$

The group $\mathbb{H} = (\mathbb{R}^3, \circ, (\delta_r)_{r>0})$ is called *Heisenberg group*. The vector field $\Delta_{\mathbb{H}} = X_1^2 + X_2^2$ is said *sub-Laplacian* on the *Heisenberg group*. The operator defined by the vector fields (2.24) is the simplest example of degenerate operator on a homogeneous Lie group.

In general, operators as in (2.23) satisfies the *controllability condition* [C]. Moreover, for every $\Omega \subset \mathbb{R}^{N+1}$ and for every $(x_0, t_0) \in \Omega$ there exist a positive ε and a neighborhood U of x_0 such that $U \times]t_0, t_0 - \varepsilon[\subset \mathscr{A}_{(x_0,t_0)}(\Omega)$. This particular geometric property of the attainable set implies that an invariant Harnack inequality analogous to the standard parabolic one holds for these operators. The only difference is that the Euclidean translation and the parabolic dilations are replaced by the operations used to satisfy hypotheses [H1]. A keystone result for the fundamental solution Γ of operators $\mathcal{L}_0$ as in (2.23) is the following: there exist positive constants c^-, C^-, c^+, C^+ such that

$$\frac{1}{C\sqrt{|\mathcal{B}_{t-\tau}(x)|}} \exp\left(-\frac{C\, d_{CC}^2(x,\xi)}{t-\tau}\right) \leq \Gamma(x, t, \xi, \tau) \leq \frac{C}{\sqrt{|\mathcal{B}_{t-\tau}(x)|}} \exp\left(-\frac{d_{CC}^2(x,\xi)}{C\,(t-\tau)}\right), \tag{2.25}$$

for every $(x, t), (\xi, \tau) \in \mathbb{R}^N \times]T_0, T_1[$ with $t > \tau$, where where d_{CC} denotes the *Carnot-Caratheodory distance*

$$d_{CC}(x_0, x) = \inf\{\ell(\gamma) \mid \gamma \text{ is as in (2.14)}\}, \quad \ell(\gamma) := \int_0^T \|\omega(s)\| ds.$$

and $|\mathcal{B}_r(x)|$ is the volume of the metric ball with center at x and radius r. Since $|\mathcal{B}_r(x)|$ is proportional to the homogeneous spatial dimension

$$r^{Q+2} = \det\big(\delta_r\big),$$

we can easily rewrite

$$\frac{c^-}{(t-\tau)^{Q/2}} \exp\left(-C^- \tfrac{d_{CC}(x,\xi)^2}{t-\tau}\right) \le \Gamma(x,t,\xi,\tau) \le \frac{C^+}{(t-\tau)^{Q/2}} \exp\left(-c^+ \tfrac{d_{CC}(x,\xi)^2}{t-\tau}\right), \tag{2.26}$$

The upper and lower bounds are due to Jerison and Sánchez-Calle [37], and to Varopoulos, Saloff-Coste and Coulhon [81]. Note that it holds the identity

$$\Psi(x_0,t_0,x,t) = \frac{d_{CC}(x_0,x)^2}{t_0-t}.$$

Indeed, if we consider the path $\gamma(s) = (\gamma_0(s), t_0 - s)$ with $0 \le s \le t_0 - t$, then by the Hölder inequality, we obtain $\ell(\gamma_0) \le \sqrt{\Phi(\omega)}\sqrt{t_0-t}$. Moreover the equality occurs only if the norm of the control ω is constant, that is

$$\ell(\gamma_0) = \sqrt{\Phi(\omega)}\sqrt{t_0-t} \iff (\omega_1^2 + ... + \omega_m^2)(s) = \tfrac{\Phi(\omega)}{t_0-t} \quad \text{for every } s \in [0, t_0-t].$$

These results apply to Lie groups which are not necessarily homogeneous. We also quote the estimates by Saloff-Coste and Stroock [77].

Kolmogorov Type Operators. The following two examples appear in the analysis of the Kolmogorov-Fokker-Plank equation related to some SDEs equations

1. *Kolmogorov Operators.* The simplest version of a Kolmogorov operator $\mathscr{L}_0$, acting on $(x,t) \in \mathbb{R}^{2N+1}$, is a linear operator

$$\mathcal{L}_0 = \sum_{i=1}^{N} \partial_{x_i}^2 + \sum_{i=1}^{N} x_i \partial_{x_{N+i}} - \partial_t.$$

   In this case, the vector fields are the following ones

$$X_i(x,t) = \partial_{x_i}, \; i = 1,...,N, \quad X_0(x,t) = \sum_{i=1}^{N} x_i \partial_{x_{N+i}}. \tag{2.27}$$

   The introduction of this operator is due to A.N. Kolmogorov in order to describe the probability density of a system with $2N$ degree of freedom. The $2N$-dimensional space is the phase space, where $(x_1,...,x_N)$ is the velocity and $(x_{N+1},...,x_{2N})$ the position of the system. We deal with this kind of operator and its extension in the next section.

2. *More Degenerate Kolmogorov-type Operators.* An interesting presentation of some more degenerate Kolmogorov operators $\mathcal{L}_0$, acting on $(x,t) \in \mathbb{R}^{N+1}$, is given in [21]. In their work, they consider differential operators with a quadratic or super-quadratic vector field X_0 in (2.27). The prototype of such operator is the following one:

$$\mathcal{L}_0 = \sum_{i=1}^{N-1} \partial_{x_i}^2 + \sum_{i=1}^{N-1} x_i^2 \partial_{x_N} - \partial_t.$$

In this case, the vector fields are the following

$$X_i(x,t) = \partial_{x_i}, \;\; i = 1, ..., N-1, \quad X_0(x,t) = \sum_{i=1}^{N-1} x_i^2 \partial_{x_N}.$$

The name *More Degenerate* denotes the fact that the commutator

$$[X_i, X_0] = 2x_i \partial_{x_N} \sim \begin{pmatrix} 0 \\ \vdots \\ 2x_i \\ \vdots \\ 0 \end{pmatrix}$$

vanishes in the set $(x \in \mathbb{R}^N | x_i = 0)$, therefore, in a neighborhood of $x = 0$, we need to consider another commutator

$$[X_i, [X_1, X_0]] = 2\partial_{x_N} \sim \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2 \end{pmatrix}$$

to satisfy the Hörmander condition. As a consequence, a Lie group leaving invariant the equation $\mathcal{L}_0 u = 0$ cannot exist, so the Hypothesis [H1]-*i)* is not satisfied. This problem is overcome by a *lifting procedure* (see Rothshild and Stein [76]). Specifically, we consider the following operator

$$\widetilde{\mathcal{L}}_0 := \partial_x^2 + x\partial_w + x^2 \partial_y - \partial_t, \qquad (x, y, w, t) \in \mathbb{R}^3 \times (0, T),$$

and we consider any solution of $\mathcal{L}_0 u = 0$ as a function that does not depend on w, and that solves the equation $\widetilde{\mathcal{L}}_0 u = 0$. The lifting procedure allows us to rely on the Lie group invariance of $\widetilde{\mathcal{L}}_0$ in the study of the positive solutions of $\mathcal{L}_0 u = 0$. Indeed, we have *i)* The operator $\widetilde{\mathcal{L}}_0$ is invariant with respect to the following Lie group operation

$$(x_0, y_0, w_0, t_0) \circ (x, y, w, t) = (x + x_0, y + y_0 + 2x_0 w - t x_0^2, w + w_0 - t x_0, t + t_0),$$

defined for every $(x, y, w, t), (x_0, y_0, w_0, t_0) \in \mathbb{R}^4$. In particular, it holds

$$(\widetilde{\mathcal{L}}_0 u)(z_0 \circ z) = \widetilde{\mathcal{L}}_0(u(z_0 \circ z)),$$

for every $z_0 = (x_0, y_0, w_0, t_0)$ and $z = (x, y, w, t) \in \mathbb{R}^4$.

ii) The operator $\widetilde{\mathcal{L}}_0$ is invariant with respect to the following dilation

$$(\delta_\rho)_{\rho \geq 0} : (x, y, w, t) \mapsto (\rho x, \rho^4 y, \rho^3 w, \rho^2 t)$$

that is it holds:

$$\rho^2 \, (\widetilde{\mathcal{L}}_0 u)(\rho x, \rho^4 y, \rho^3 w, \rho^2 t) = \widetilde{\mathcal{L}}_0(u(\rho x, \rho^4 y, \rho^3 w, \rho^2 t)).$$

Another peculiarity of this operator is that its fundamental solution is supported in a subset of $\mathbb{R}^{N+1}$, hence the property [C] is not satisfied. Indeed, the attainable set of the is

$$\mathcal{A}_{(0,0,0,0)}(\Omega) = \Big\{(x, w, y, t) \in \mathbb{R}^4 \;\mid 0 \leq y \leq -t, w^2 \leq -ty\Big\}.$$

We refer to the work [21] for an exhaustive presentation of the problem.

3. *Asian Option type Operators.* The prototype operator is

$$\mathcal{L}_0 = x^2 \partial_{xx} + x\partial_x + x\partial_y - \partial_t, \quad (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T], \tag{2.28}$$

of the operator $\mathscr{L}$ in (1), which will be discussed in chapter 4. We let known in advance that only the Hypotheses [H] and [H1]-*i)* hold true for the operator $\mathcal{L}_0$ in (2.28), whereas [H1]-*ii)* and [C] do not hold.

2.2.3 Harnack inequality for non-negative solution of $\mathcal{L}_0 u = 0$

In this section we prove an invariant Harnack inequality for non-negative smooth solutions u of $\mathcal{L}_0 u = 0$. Let us introduce the following sets

$$H_r(z_0) = z_0 \circ \delta_r(H_1), \quad S_r(z_0) = z_0 \circ \delta_r(S_1), \tag{2.29}$$

where

$$H_1 = \{z = (x,t) \in \mathbb{R}^{N+1} \ \mid \ \|z\|_{\mathbb{G}} \le 1, \ t \le 0\}, \quad S_1 = \left\{(x,t) \ \in H_1 \mid \frac{1}{4} \le -t \le \frac{3}{4}\right\}.$$

The next result is an invariant Harnack inequality proved by Kogoj and Lanconelli in [39, Theorem 7.1]:

Theorem 2.2.8. *Let Ω be an open subset of $\mathbb{R}^{N+1}$ containing $H_r(z_0)$ for some $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$. Then, there exist two positive constants θ and M, only depending on the operator $\mathcal{L}_0$, such that*

$$\sup_{S_{\theta r}(z_0)} u \le M\, u(z_0), \tag{2.30}$$

for every non-negative solution u of $\mathcal{L}_0 u = 0$ in Ω.

The proof of this theorem is based on the *mean value property* of the solutions of $\mathcal{L}_0 u = 0$. Given $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$ we define the $\mathcal{L}_0$-ball of center z_0 and radius r as the following set

$$\Omega_r(z_0) := \left\{z \in \mathbb{R}^{N+1} \ : \ \Gamma(z^{-1} \circ z_0) > \tfrac{1}{r^{Q-2}}\right\}.$$

In virtue of formulas (2.20), (2.21) and (2.22), we have that the set $\Omega_r(z_0)$ is bounded, $\Omega_r(z_0) = z_0 \circ \Omega_r$ where $\Omega_r = \Omega_r(0)$ and $\Omega_r(0) = \delta_r\big(\Omega_1(0)\big)$. Moreover it holds the following proposition

Proposition 2.2.9. *Let $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$. There exists two positive constants θ, ρ_0 only depending on the operator $\mathcal{L}_0$ such that it holds the following inclusion*

$$S_{\theta r}(z_0) \subseteq \Omega_{\frac{r}{\rho_0}}(z_0) \subseteq H_r(z_0) \tag{2.31}$$

PROOF. Since $\Omega_r(z_0)$ is a bounded set, denoting by $\rho_0 = \max_{z \in \bar{\Omega}_1} \|z\|_{\mathbb{G}}$ we have $\bar{\Omega}_1 \subseteq H_{\rho_0}$. Therefore it holds that

$$\Omega_{\frac{r}{\rho_0}}(z_0) \subseteq H_r(z_0), \quad \forall z_0 \in \mathbb{R}^{N+1}, \forall r > 0. \tag{2.32}$$

Furthermore, it also holds that

$$\bigcup_{r>0} \Omega_r = \mathbb{R}^N \times]-\infty, 0[.$$

Indeed, if $z = (x,t) \in \Omega_r$ then $\Gamma((x,t)^{-1}) > 1/r^{Q-2}$. On the other hand since $(x,t)^{-1} = (y,-t)$ for a suitable $y \in \mathbb{R}^N$, from $\Gamma(y,-t) > 1/r^{Q-2}$ we deduce $-t > 0$

in view of **v)** of Definition (2.2.7). Therefore, $z \in \mathbb{R}^N \times]-\infty, 0[$. Viceversa, given $z = (x,t) \in \mathbb{R}^N \times]-\infty, 0[$ for a suitable $y \in \mathbb{R}^N$, we have $\Gamma((x,t)^{-1}) = \Gamma(y, -t) > 0$. This implies $\Gamma((x,t)^{-1}) > 1/r^{Q-2}$ for a suitable $r > 0$, hence $(x,t) \in \Omega_r$.
Therefore, since S_1 is a compact subset of $\mathbb{R}^N \times]-\infty, 0[$, there exists $\rho_1 > 0$ such that $S_1 \subseteq \Omega_{\rho_1}$. Then, we have

$$S_r \subseteq \Omega_{r\rho_1}, \qquad \text{for every } r > 0. \tag{2.33}$$

By setting $\theta = \rho_0 \cdot \rho_1$, merging (2.32) and (2.33), we obtain (2.31). □

Proposition 2.2.10 (Mean Value Formula)**.** *Let u be a solution to $\mathcal{L}_0 u = 0$ in the open set $O \in \mathbb{R}^{N+1}$. Then, for every $\mathcal{L}_0$-ball $\Omega_r(z_0)$ with closure contained in O, we have*

$$u(z_0) = \tfrac{1}{r^{Q-2}} \int_{\Omega_r(z_0)} u(x,t) K(x_0, t_0; x, t) dx dt$$

where

$$K(x_0, t_0; x, t) = \frac{|\nabla_{\mathcal{L}_0} \Gamma(x_0, t_0; x, t)|^2}{\Gamma^2(x_0, t_0; x, t)}, \quad \nabla_{\mathcal{L}_0} = (X_1, ..., X_m).$$

Here $\nabla_{\mathcal{L}_0}$ acts on the variable (x,t).

Lemma 2.2.11. *Let $z = (x,t) \in \mathbb{R}^{N+1}$ be fixed. Then the set*

$$\Sigma = \{\zeta = (\xi, \tau) \in \mathbb{R}^{N+1} \mid \tau < t, K(z, \zeta) = 0\},$$

does not contain interior points.

Proof. By contradiction, assume that $K(z,\zeta) = 0$ for every $\xi \in U$, with U an open subset of $\mathbb{R}^N \times]-\infty, t[$. Then, it holds $X_j \Gamma(z; \cdot) = 0$ in U for any $j = 1, ..., m$. Therefore,

$$\sum_{j=1}^{m} X_j^2 (\Gamma(z, \cdot)) = 0 \quad \text{in } U.$$

Since $\mathcal{L}_0^*(z; \cdot) = 0$ in $\mathbb{R}^N \times]-\infty, t[$, this implies

$$(X_0 - \partial_t) \Gamma(z, \cdot) = 0, \quad \text{in } U.$$

As a consequence, by the Remark 2.2.4, the euclidean gradient of $\Gamma(z; \cdot)$ is identically zero in U. Then $\Gamma(\xi^{-1} \circ z) = C_0$ for any $\xi \in U$. Note that the time component of $\xi^{-1} \circ z$ is $t - \tau > 0$, therefore, by the definition of fundamental solution, we have

$C_0 > 0$. This contradicts the fact that $\Gamma(x,t;\xi,\tau)$ is δ_λ-homogeneous of degree $2-Q$ and $Q \neq 2$. □

The main ingredient required for the proof of Theorem 2.2.8 is the following weak Harnack inequality, first proven by Bony [15, Theorem 7.1]

Proposition 2.2.12 (Weak Harnack inequality)**.** *Let O be an open set of $\mathbb{R}^{N+1}$ and let T be a dense subset of O. Then, for every compact set $K \subset O$ there exists $z_1, ..., z_p \in T$ and a constant $c > 0$ depending on $\mathcal{L}_0$ and K, such that*

$$\sup_K u \leq c(u(z_1) + ... + u(z_p)),$$

for every non-negative C^∞ solution u of $\mathcal{L}_0 u = 0$ in O.

We now state a convergence theorem, easy consequence of a weak Harnack inequality due to Bony.

Proposition 2.2.13. *Let u_n be a sequence of $\mathcal{L}_0$-harmonic functions in an open set $O \subseteq \mathbb{R}^{N+1}$. Suppose u_n monotone increasing and such that*

$$u = \sup_{n\in\mathbb{N}} u_n < \infty$$

in a dense subset T of O. Then $u < \infty$ everywhere, $u \in C^\infty(O)$ and satisfies $\mathcal{L}_0(u) = 0$ in O.

Proof. By the previous Proposition 2.2.12, for every fixed compact set $K \subseteq O$, there exist $x_1, ..., x_p \in T$ and a constant $C > 0$ such that

$$\sup_K (u_n - u_m) \leq C \sum_{j=1}^{p} (u_n(x_j) - u_m(x_j)), \quad \forall n \geq m$$

Then, since $(u_n - u_m)(x_j) \to 0$, as $n, m \to +\infty$, for any $j \in \{1, ..., p\}$, the sequence (u_n) is locally uniformly convergent in O, so that u is finite everywhere. Moreover, since $\mathcal{L}_0 u_n = 0$ for any $n \in N$, it follows that $\mathcal{L}_0 u = 0$ in O in the weak sense of distributions. The hypoellipticity of $\mathcal{L}_0$ implies that $u \in C^\infty(O)$ and satisfies the equation in the classical sense. □

Proof of Theorem 2.2.8. Since $\mathcal{L}_0$ is left invariant and δ_r-homogeneous (properties (2.20) and (2.21)), it suffices to prove the claim for $z_0 = 0$ and $r = 1$. By contradiction, assume that (2.31) is false with $z_0 = 0$ and $r = 1$. Then, for every $n \in N$ there exists a non-negative solution $u_n \in C^\infty(O)$ of $\mathcal{L}_0 u_n = 0$ such that

$$\sup_{S_\theta(z_0)} u_n > 4^n u_n(z_0). \tag{2.34}$$

By using the relation (2.32) and the mean value property stated in Proposition 2.2.10 we have

$$u_n(z_0) = \frac{1}{\rho^{Q-2}} \int_{\Omega_\rho(z_0)} K(z_0, \zeta) u_n(\zeta) d\zeta, \quad \rho = \frac{1}{\rho_0}$$

In view of Lemma 2.2.11, K is nonnegative and strictly positive in a dense open subset of $\Omega_\rho(z_0)$, this identity and inequality (2.34) imply that $u_n(z_0) > 0$.
Let define

$$w_m = \frac{u_n}{u_n(z_0)}, \quad \text{and} \quad w = \sum_{n=1}^{\infty} \frac{w_n}{2^n},$$

we obtain

$$1 = w(z_0) = \frac{1}{\rho^{Q-2}} \int_{\Omega_\rho(z_0)} K(z_0, \zeta) w(\zeta) d\zeta.$$

As a consequence, by the positivity property of K, we have $w < \infty$ in a dense subset of $\Omega_\rho(z_0)$. By Proposition 2.2.13, $w \in C^\infty(\Omega_\rho(z_0))$ and $\mathcal{L}_0 w = 0$ in $\Omega_\rho(z_0)$. In particular, since $S_\theta(z_0)$ is a compact subset of $\Omega_\rho(z_0)$, we have

$$\sup_{S_\theta(z_0)} w < \infty \tag{2.35}$$

But, on the other hand, by inequality (2.34), we obtain

$$\sup_{S_\theta(z_0)} w \geq \sup_{S_\theta(z_0)} \frac{w_n}{2^n} \geq 2^n, \quad \text{for any } n \in \mathbb{N}.$$

This contradicts (2.35) and completes the proof. □

Remark 2.2.14. *We refer to the inequality in* (2.30) *as an* ***invariant Harnack inequality*** *in the sense that the constant M appearing in* (2.30) *does not depend on the point z_0 and r, but only on the operator $\mathcal{L}_0$.*

By the invariance with respect to the dilations $(\delta_\lambda)_{\lambda>0}$, Pascucci and Polidoro [70] obtained a slight different version of Harnack inequality by replacing the cylinder $S_{\theta r}(z_0)$ with the cone

$$\mathcal{P}_r(z_0) = \left\{ z_0 \circ z \in \mathbb{R}^{N+1} \quad s.t. \quad z \in \mathcal{P}_r \right\},$$

where

$$\mathcal{P}_r = \left\{(x,-t) \in \mathbb{R}^{N+1} \;\mid\; |x|_{\mathbb{G}}^2 \leq 2t,\ 0 < t \leq 2\theta^2 r^2\right\}. \tag{2.36}$$

Such statement reads as follows and it is a corollary of Theorem 2.2.8

Proposition 2.2.15. *Let Ω be an open set in $\mathbb{R}^{N+1}$ containing $H_r(z_0)$ for some $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$. Then*

$$u(z_0 \circ z) \leq M\, u(z_0) \tag{2.37}$$

for every non-negative solution u of $\mathcal{L}_0 u = 0$ in Ω and for every z in $\mathcal{P}_r$

Proof. We only give the proof when $z_0 = 0$ since the general case follows by using the invariance properties (2.11) of the operator $\mathcal{L}_0$. For every positive $\rho > 0$ we denote

$$D_\rho = \left\{(x,-t) \in \mathbb{R}^{N+1} \;\mid\; |x|_{\mathbb{G}}^2 \leq \rho^2,\ t = \tfrac{\rho^2}{2}\right\}.$$

Then, for every $\rho \in [0, \theta r]$ we have that u is a non-negative solution of $\mathcal{L}_0 u = 0$ in the domain H_ρ. Since $D_\rho \subset S_\rho$, from Theorem 2.2.8 we obtain

$$\sup_{D_\rho} u \leq \sup_{S_\rho} u \leq C u(0),$$

and the conclusion follows by observing that $2t = \rho^2$ on D_ρ and from the fact that $P_r = \cup_{0<\rho\leq\theta r} D_\rho$. □

By iterating Proposition 2.2.15, Boscain and Polidoro proved the following *non-local* version of Harnack inequality:

Proposition 2.2.16. *Let $\mathcal{L}_0$ be as defined in (2.7), satisfying assumptions [C] and [H]. Then there exist three constants $\theta \in]0,1[, h > 0$ and M, only depending on the operator $\mathcal{L}_0$, such that the following statement is true.*
If $u : \mathbb{R}^N \times]T_0, T_1[\to \mathbb{R}$ is a non negative solution to $\mathcal{L}_0 u = 0$, $(x,t), (y,s) \in \mathbb{R}^N \times]T_0, T_1[$ are two points such that $T_1 - \theta^2(T_1 - T_0) \leq s < t < T_1$, and $\gamma((x,t),(y,s),\omega)$ is a solution to (2.14), then

$$u(y,s) \leq M^{1+\frac{\Phi(\omega)}{h}} u(x,t),$$

where $\Phi(\omega)$ is the function defined in (2.17).

We refer to Boscain and Polidoro [16] for the detailed proof of this proposition.

2.2.4 Elements of Control Theory

Theorem 2.2.16 provides us with an upper bound for non negative solutions u of $\mathcal{K}$ for every $(x_1, t_1) \in \mathscr{A}_{(x_0,t_0)} \cap \mathbb{R}^N \times]T_0, T_1[$ in terms of the value which the function u assumes in the vertex (x_0, t_0).

In order to find the best exponent appearing in formula (2.73), it is natural introducing the following Optimal Control Problem:

$$\begin{aligned}
&\dot{\gamma}(s) = \sum_{k=1}^{m} \omega_k(s) X_k(\gamma(s)) + Y(\gamma(s)) \qquad 0 \le s \le T, \\
&\Phi(\omega) = \int_0^T (\omega_1^2(s) + \cdots \omega_m^2(s))\, ds \to \min \quad \text{with } T \text{ fixed} \\
&\gamma(0) = (x_0, t_0), \quad \gamma(T) = (x_1, t_1), \quad \omega_i \in L^1[0, T].
\end{aligned} \tag{2.38}$$

The minimum $V(x_0, t_0; x_1, t_1)$ of this problem:

$$V(x_0, t_0; x_1, t_1) = \inf \left\{ \Phi(\omega) | \gamma(x_0, t_0; x_1, t_1; \omega) \text{ is a } \mathcal{L}_0 \text{ admissible path} \right\}$$

is called *Value funtion* of the problem (2.38). In virtue of Theorem 2.2.16, by the definition of value function V we get the following sharp inequality

$$u(x_1, t_1) \le M^{1 + \frac{V(x_0, t_0; x_1, t_1)}{h}} u(x_0, t_0), \tag{2.39}$$

for every $(x_0, t_0), (x_1, t_1) \in \mathbb{R}^N \times]T_0, T_1[$ with $T_1 - \theta^2(T_1 - T_0) < t_1 < t_0 < T_1$.

A key tool used to solve the optimal control problem (2.38) is the Pontryagin Maximum Principle. Since we always have $\dot{t}(s) = -1$, from now on, we drop the t time variable.

Let $\Omega \subset \mathbb{R}^N$ be an open set and let (x_0, t_0), (x_1, t_1) (with $t_1 < t_0$) be a fixed point of $\mathbb{R}^N \times]T_0, T_1[$. We denote by $T = t_0 - t_1$ and we consider the following *optimal control problem*:

$$\begin{aligned}
&\dot{q} = X_0 + \sum_{i=1}^{m} \omega_i X_i(q) \quad 0 \le s \le T, \\
&\int_0^T \sum_{i=1}^{m} \omega_i^2(s)\, ds \to \min, \quad T \text{ is fixed.} \\
&q(0) = q_0, \quad q(T) = q_1, \quad \omega_i \in L^1[0, T].
\end{aligned} \tag{2.40}$$

For such optimal control problem, the Pontryagin Maximum Principle provides a first-order condition for the minimizing controls $\omega(.)$ and the corresponding trajectories $q(.)$. We now recall its statement in the particular case in which variables and controls belong to the Euclidean spaces $\mathbb{R}^n, \mathbb{R}^m$, respectively. For a more general statement on manifolds, see e.g. [1].

Theorem 2.2.17 (PMP for the problem (2.40))**.** *Consider the minimization problem* (2.40)*, in the class of Lipschitz continuous curves, where X_i, $i = 0, \ldots, m$ are smooth vector fields on $\mathbb{R}^N$ and the final time T is fixed. Consider the map $H : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ defined by*

$$H(q, \lambda, p_0, \omega) \ := \ \big\langle \lambda, X_0 + \sum_{i=1}^{m} \omega_i X_i(q) \big\rangle + p_0 \sum_{i=1}^{m} \omega_i^2. \tag{2.41}$$

If the curve $q(.) : [0,T] \to \mathbb{R}^N$ corresponding to the control $\omega(.) : [0,T] \to \mathbb{R}^m$ is optimal, then there exist a Lipschitz continuous covector $\lambda(.) : s \in [0,T] \mapsto \lambda(s) \in \mathbb{R}^N$ and a constant $p_0 \leq 0$ such that:

- *the pair $(\lambda(s), p_0)$ is never vanishing;*
- *the optimal control $\omega(s)$ satisfies*

$$H(q(s), \lambda(s), p_0, \omega(s)) = \max_{\nu \in \mathbb{R}^m} H(q(s), \lambda(s), p_0, \nu);$$

- *for a.e. $s \in [0,T]$ it holds*

$$\begin{cases} \dot{q}(s) = \frac{\partial H}{\partial \lambda}(q(s), \lambda(s), p_0, \omega(s)), \\ \dot{\lambda}(s) = -\frac{\partial H}{\partial q}(q(s), \lambda(s), p_0, \omega(s)). \end{cases} \tag{2.42}$$

The Hamiltonian $H^*(q, \lambda, p_0) := \max_{\nu \in \mathbb{R}^m} H(q, \lambda, p_0, \nu)$ is called the **maximized Hamiltonian***.*

Solutions to the system (2.42) *are called* **extremals***. When $p_0 = 0$, they are called* **abnormal extremals***, while when $p_0 < 0$ they are called* **normal extremals***.*

Remark 2.2.18. *The original statement [74] of the Pontryagin Maximum Principle provides optimal controls in the space $L^\infty([0,T], \mathbb{R}^m)$. Instead, we are interested in optimal controls in the larger space $L^1([0,T], \mathbb{R}^m)$. For this reason, we aim to apply a generalized version of the Pontryagin Maximum Principle, such as the one stated in [82, chapter 6]. For our optimal control problem, such generalized version has a statement completely equivalent to Theorem 2.2.17.*

We further emphasizes that the function V is a viscosity solution of a specific Hamilton-Jacobi-Bellman equation (this is a general fact coming from the Optimal Control Theory, see for instance [9]). The Hamilton-Jacobi-Bellman equation related to the Optimal control problem (2.38) for the value function V is the following

$$YV + \frac{1}{4}\sum_{j=1}^{m}(X_j V)^2 = 0. \tag{2.43}$$

where $X_i \quad i = 0, ..., m$ are defined in (2.8). Moreover, a result by Cannarsa and Rifford [18] states that the solution of the the Hamilton-Jabobi-Bellman equation is a semiconcave function, provided that the coefficients of the vector fields $X_0, \ldots, X_m$ have sub-linear growth

$$|X_j(x)| \le M(1 + |x|) \quad \text{for every} \quad x \in \mathbb{R}^N, \quad j = 1, ..., m,$$

and that the optimal control problem related to the Cauchy problem (2.14) admits no singular minimizing controls. As a consequence, V satisfies the partial differential equation (2.43) in the distributional sense, since all semiconcave functions are locally Lipschitz continuous.

2.3 Degenerate Kolmogorv Operators

In this section we introduce the *Degenerate Parabolic Evolution Operators of Kolmogorov type* in $\mathbb{R}^N \times]0, T[$. A Kolmogorov Operator is an operator of the form

$$\mathcal{K}u \equiv \sum_{i,j=1}^{d_0} a_{ij}(x,t)\partial_{x_i x_j} + \sum_{i=1}^{d_0} a_i(x,t) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u. \tag{2.44}$$

where $d_0 \le N$ and whose coefficients $a_{i,j}$'s satisfy the following *uniform parabolicity condition*

$$\exists \lambda > 0 \ s.t \quad \lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{d_0} a_{i,j}(x,t)\xi_i\xi_j \le \lambda|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^{d_0},\ t > 0. \tag{2.45}$$

Other regularity assumptions on the coefficients $a_{i,j}(x,t)$ and on $a_i(x,t)$ will be specified in the sequel. The matrix $B = (b_{ij})_{i,j=1,\ldots,N}$ has constant elements. We denote by $\nabla = (\partial_{x_1}, \ldots, \partial_{x_N})^T$ and $\langle \cdot, \cdot \rangle$, respectively, the gradient and the inner product in $\mathbb{R}^N$.

A Kolmogorov operator can also appear in its divergence form

$$\mathcal{K}u \equiv \text{div } (\langle A(x,t); \nabla u\rangle) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j}u - \partial_t u, \tag{2.46}$$

where the matrix $A(x,t) = (a_{i,j}(x,t))$ is a $N \times N$ symmetric square matrix of rank $d_0 \leq N$ having the following representation

$$A(x,t) = \begin{pmatrix} A_0(x,t) & 0 \\ 0 & 0 \end{pmatrix}$$

with $A_0(x,t)$ symmetric $d_0 \times d_0$ square matrix, and the coefficients $a_{i,j}$'s, with $i,j = 1, ..., p_0$, satisfy the uniform parabolicity condition (2.45).

If the matrix A_0 is constant we set

$$Ku := \sum_{i,j=1}^{d_0} a_{i,j}\partial_{x_i x_j}u + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u, \tag{2.47}$$

Throughout this section the *constant coefficients operator* K in (2.47) plays the role of the smooth operator $\mathcal{L}_0$ as in (2.7) satisfying the hypotheses [H] and [C]. These assumptions make sense, since K can be written in the form (2.7) by choosing

$$X_i = \sum_{j=1}^{d_0} \bar{a}_{ij}\partial_{x_j}, i = 1, \dots, d_0, \qquad X_0 = \langle x, B\nabla\rangle, \tag{2.48}$$

where $A_0^{\frac{1}{2}} = (\bar{a}_{ij})_{i,j=1,\dots,d_0}$ is the unique positive $d_0 \times d_0$ matrix such that $A_0^{\frac{1}{2}} \cdot A_0^{\frac{1}{2}} = A_0$. In the sequel we will denote by A and $A^{\frac{1}{2}}$ the $N \times N$ matrices

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^{\frac{1}{2}} = \begin{pmatrix} A_0^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.49}$$

On the other hand we refer to the operator $\mathcal{K}$ in (2.44) and in (2.46) as a *variable coefficients operator*. We further denote by K_0 the operator as in (2.47) when $A = Id$:

$$K_0 = \sum_{i=1}^{d_0} \partial_{x_i x_i}u + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u \tag{2.50}$$

and we refer to K_0 as the *model operator* related to $\mathcal{K}$.
A key results available in literature are upper and lower estimates for the fundamental solution $\Gamma(x,t;\xi,\eta)$ of the operator $\mathcal{K}$ in (2.46) or in (2.44) in terms of the fundamental solution $\Gamma_0^{\pm}(x,t;\xi,\eta)$ of the operators

$$K_0^{\pm} = \mu^{\pm}\sum_{i=1}^{d_0} \partial_{x_i x_i}u + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u,$$

where the constants $\mu^\pm$ depend on the vector fields $\mathcal{K}$ and on λ in (2.45). The two sided bounds write as follows

$$k^-_{\varepsilon,T}\Gamma^-_0(x,t-\varepsilon(t-\tau);\xi,\tau) \le \Gamma(x,t;\xi,\tau) \le k^+_{\varepsilon,T}\Gamma^+_0(x,t+\varepsilon(t-\tau);\xi,\tau), \qquad (2.51)$$

where $(x,t),(x_0,t_0) \in \mathbb{R}^{N+1}$ with $t > t_0$, $\varepsilon > 0$ is arbitrary and the constants $k^\pm_{\varepsilon,T}$ depend on T and ε.

We remark that estimates (2.51) extend the results previously given for uniformly parabolic operators. Specifically, let $h(x,t;x_0,t_0)$ denote the fundamental solution of an uniformly parabolic operator (in non divergence form)

$$\mathscr{L}u = \sum_{i,j=1}^N a_{i,j}(x,t)\partial_{x_ix_j}u(x,t) + \sum_{i=1}^N b_i(x,t)\partial_{x_i}u(x,t) + c(x,t)u(x,t) - \partial_t u(x,t),$$

where $(x,t) \in \mathbb{R}^N \times]t_0, t_0+T[$ and whose coefficients are bounded Hölder continuous functions. The lower bound for h writes as follows: there exists $\lambda > 0$ and a positive constant M, depending on T such that

$$h(x-x_0,t-t_0) \ge \frac{1}{M}\frac{\lambda^{N/2}}{(4\pi(t-t_0))^{N/2}} e^{-\frac{\lambda|x-x_0|^2}{4(t-t_0)}}, \qquad (2.52)$$

for every $(x,t),(x_0,t_0) \in \mathbb{R}^N \times]t_0,t_0+T[$ such that $t > t_0$. We explicitly note that the term on the right side is the fundamental solution of the heat equation

$$\frac{1}{\lambda}\Delta u(x,t) - \partial_t u(x,t) = 0 \quad \text{for every } (x,t) \in \mathbb{R}^N \times]t_0,t_0+T[.$$

The results exploit the recursive use of the Harnack inequality given by Krylov-Safonov in [42].

Upper estimates are obtained by using the parametrix method (see Friedman [31]); this method provides also the existence of a fundamental solution. Such result reads as follows:

$$h(x-x_0,t-t_0) \le \frac{M}{(4\pi\lambda(t-t_0))^{N/2}} e^{-\frac{|x-x_0|^2}{4\lambda(t-t_0)}}, \qquad (2.53)$$

where λ and M are the constants introduced before. We explicitly note that the term on the right side is the fundamental solution of the heat equation

$$\lambda\Delta u(x,t) - \partial_t u(x,t) = 0 \quad \text{for every } (x,t) \in \mathbb{R}^N \times]t_0,t_0+T[.$$

For divergence form uniformly parabolic operators

$$\mathscr{L} = \sum_{i,j=1}^N \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}) - \partial_t, \quad (x,t) \in \mathbb{R}^N \times]t_0,t_0+T[\qquad (2.54)$$

the upper bound was proved by Aronson [2] for measurable coefficients α_{ij}'s. Moreover, estimates (2.52) and (2.53) were proved by Davies [24] and by Fabes-Stook [29] following the pioneer work of Nash [63]. The striking aspect is that for divergence form operator (2.54) with measurable coefficients, the constant M appearing in the bounds (2.52) and (2.53) does not depend on T.

Also, as Fabes and Stroock emphasized in [29], the upper bound is an important tool for using the ideas of Nash in order to directly obtain the lower bound and then to derive the Harnack inequality and the local Hölder continuity of weak solutions.

In this chapter, we collect the results given by Polidoro [73] and by Di Francesco-Polidoro [27] which lead to the prof of the lower bound in (2.51), by Pascucci-Polidoro [67] and by Cinti-Polidoro [23] for the upper bound. Concerning the lower bound, generally, the methodology is an adaptation of the method due to Moser [60, 62] in order to prove a Gaussian lower bound of Γ and is based on the repeated use of an invariant Harnack inequality combined with an optimization procedure.

On the other hand, the upper bound can be derived following the techniques and ideas appearing in Aronson [2] by using result analogous to the Moser iteration for uniformly parabolic operators. Furthermore, it can also be derived by using the parametrix expansion of the fundamental solution Γ of (2.46), given in [72] and [26].

For the sake of completeness we start by introducing the theory underlying the Kolmogorov operator with constant coefficients (2.47) and then we pass to the variable coefficients ones.

2.3.1 Kolmogorov operators with constant coefficients

In this section we consider Kolmogorov operators with constant coefficients

$$Ku := \sum_{i,j=1}^{d_0} a_{i,j}\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u, \tag{2.55}$$

and we recall some known results about it. We suggest the reference [47] for an exhaustive presentation of the topic. In this section the operator K in (2.55) is studied under the hypothesis [C].

It suffices only assuming [C] since it is equivalent to any of the following statements:

i) K satisfies the Hörmarder condition (2.10) (then it is hypoelliptic);

ii) $\mathrm{Ker}(A^{\frac{1}{2}})$ does not contain non-trivial subspaces which are invariant for B;

iii) there exists a basis of $\mathbb{R}^N$ such that B has the form

$$\begin{pmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_n \\ * & * & * & \dots & * \end{pmatrix} \tag{2.56}$$

where B_j is a matrix $d_{j-1} \times d_j$ of rank d_j, with

$$d_0 \geq d_1 \geq \ldots \geq d_n \geq 1, \quad d_0 + d_1 + \ldots + d_n = N,$$

while $*$ are constant and arbitrary blocks;

iv) if we set

$$E(s) = \exp(-sB^T), \qquad \mathcal{C}(t) = \int_0^t E(s)AE^T(s)ds, \tag{2.57}$$

then $\mathcal{C}(t)$ is positive, for every $t > 0$.

For the equivalence of the above conditions we refer to [47]. We explicitly remark that [C] is equivalent to the well known Kalman condition: *the* $N \times N^2$ *matrix*

$$\left[A^{\frac{1}{2}}, B^T A^{\frac{1}{2}}, \ldots, (B^T)^{N-1} A^{\frac{1}{2}}\right] \tag{2.58}$$

has rank N (see, for instance, [49], Theorem 5, p. 81).

We recall that the group law related to the Kolmogorov operator is

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \qquad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1}. \tag{2.59}$$

Moreover, K is invariant with respect to some group of dilations $(\delta_\lambda)_{\lambda>0}$ if, and only if, every $*$-block in (2.56) is null. In this case the dilations are defined as

$$\delta_\lambda = \mathrm{diag}(\lambda I_{d_0}, \lambda^3 I_{d_1}, \ldots, \lambda^{2n+1} I_{d_n}, \lambda^2), \qquad \lambda > 0, \tag{2.60}$$

where I_{d_i} is the $d_i \times d_i$ identity matrix, with $i = 0, ..., n$. According to (2.12), we decompose the space $\mathbb{R}^N$ as follows

$$\mathbb{R}^N = W_0 \oplus \cdots \oplus W_n, \qquad x = x^{(0)} + \cdots + x^{(n)}, \tag{2.61}$$

with $x^{(k)} \in W_k$, so that

$$\delta_\lambda(x^{(0)} + \cdots + x^{(n)}, t) = (\lambda\, x^{(0)} + \cdots + \lambda^{2n+1} x^{(n)}, \lambda^2 t),$$

for any $(x,t) \in \mathbb{R}^{N+1}$ and $\lambda > 0$. We also define the following norms

$$|x|_{\mathbb{K}} = \sum_{k=0}^{n} \sum_{i=0}^{d_k} |x_i^{(k)}|^{\frac{1}{2k+1}}, \quad \|(x,t)\|_{\mathbb{K}} = |x|_{\mathbb{K}} + |t|^{\frac{1}{2}}.$$

We precise that we do not require explicitly that a group of dilation exist for K. Therefore, the point *i)* in [H1] is always satisfied in view of (2.59), but *ii)* of [H1] could be not verified.

The fundamental solution of K, with singularity at point (ξ, τ), is explicitly known:

$$\Gamma(x,t,\xi,\tau) = \frac{e^{-Tr(B)(t-\tau)}}{(4\pi)^{\frac{N}{2}} \sqrt{\det \mathcal{C}(t-\tau)}} \exp\left(-\tfrac{1}{4}\langle \mathcal{C}^{-1}(t-\tau)(x - E(t-\tau)\xi), x - E(t-\tau)\xi\rangle\right), \tag{2.62}$$

for $t > \tau$, and $\Gamma(x,t,\xi,\tau) = 0$ for $t \le \tau$, in particular, when a group of dilations exists, we have $Tr(B) = 0$.

Remark 2.3.1. *We point out that, if K is dilation invariant, then it holds the following identity between K and its formal adjoint K^**

$$K^* = \sum_{ij=1}^{d_0} a_{ij}\partial_{x_i x_j} u - Y, \quad Y = \sum_{i,j=1}^{n} b_{ij} x_i \partial_{x_j} - \partial_t. \tag{2.63}$$

Indeed, by the representation of B given in (2.56)*, being the $*$-blocks null if and only if the Lie group is homogeneous, we can rewrite K as follows*

$$K_1 = \sum_{i,j=1}^{d_0} a_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{ij} x_i \partial_{x_j} u - \partial_t u \tag{2.64}$$

and for the operator K_1 in (2.64) *plainly holds* (2.63)*.*
The converse is not true. Consider for instance the operator in $\mathbb{R}^3$

$$\bar{K} = \partial_{xx} u(x,y,t) + x\partial_y u(x,y,t) + y\partial_x u(x,y,t) - \partial_t u, \quad (x,y,t) \in \mathbb{R}^3,$$

which satisfies the Hörmander condition and (2.63)*. In this case the matrix $\bar{B}$ assumes the following representation*

$$\bar{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

hence a group of dilations does not exist.

We note that the operator K in (2.47) is a particular smooth operator of the form (2.7), hence it holds true the Harnack inequality stated in Theorem 2.2.8.

2.3.2 Kolmogorov operators with variable coefficients

In this paragraph we consider Kolmogorov-Fokker-Planck operators with variable coefficients in the form (2.46) or (2.44). In both cases, the coefficients $a_{i,j}$ satisfy the uniform parabolicity condition (2.45) and are bounded Hölder continuous, in the sense that:

There exists $\alpha \in]0,1]$ *and a positive constant* $\overline{M}$ *such that*

$$|a_{i,j}(x,t) - a_{i,j}(\xi,\tau)| \leq \overline{M} \|(\xi,\tau)^{-1} \circ (x,t)\|_{\mathbb{K}} \tag{2.65}$$

for every (x,t), (ξ,τ) *belonging to* $\mathbb{R}^{N+1}$ *and for any* $i,j = 1,...,d_0$.

We suppose that the coefficients $a_{ij}(x,t)$ and $a_i(x,t)$ appearing in (2.44) are bounded and Hölder continuous in the sense of (2.65).

Concerning the divergence form operator (2.46), we also suppose that for every $i,j = 1,...,d_0$ there exists the derivatives $\partial_{x_i} a_{i,j}(z)$ and they are bounded and Hölder continuous in the sense of (2.65) (with this assumption the divergence form operator in (2.46) is a particular case of (2.44)).

We consider the model operator K_0 in (2.50), and we require that it satisfies the Hörmander condition [H]. Moreover, we introduce for K_0 the sets (2.29) according to the operation (2.59) and the dilation (2.60). We explicitly remind that the existence of a dilation group appearing in the Hypothesis [H1] is not required. In this frame Di Francesco-Pascucci [26] proves the existence of a fundamental solution Γ for the operator in (2.44) by using an adaptation of Levy's parametrix method.
The same result was actually first proved by Polidoro [71] by requiring that there exists a group of dilation with respect to whom the operator is invariant.

Let $(\xi,\tau) \in \mathbb{R}^N \times [0,T[$ and let $\Gamma(x,t;\xi,\tau)$ denotes the fundamental solution of $\mathcal{K}$ with pole in (ξ,τ). The main result of this section is to prove the following Theorem

Theorem 2.3.2. *Let* $\mathcal{K}$ *be the operator defined in* (2.46) *or in* (2.44) *whose coefficients satisfies the condition* (2.45). *Assume that the model operator* K_0 *related to* $\mathcal{K}$ *satisfies the Hörmander condition [H]. Then, for every* $\varepsilon > 0$, *there exist two* *positive constants* $k^{\pm}_{\varepsilon,T}$ *such that*

$$k^{-}_{\varepsilon,T}\Gamma^{-}_{0}(x,t-\varepsilon(t-\tau);\xi,\tau)\leq\Gamma(x,t,\xi,\tau)\leq k^{+}_{\varepsilon,T}\Gamma^{+}_{0}(x,t+\varepsilon(t-\tau);\xi,\tau) \qquad (2.66)$$

for every $(x,t)\in\mathbb{R}^N\times[0,T]$, *and* $k^{\pm}_{\varepsilon,T}$ *depends on* $\mathcal{K}$, T, λ *and* ε. *Here,* $\Gamma^{\pm}_{0}(x,t;\xi,\tau)$ *are the fundamental solutions respectively of the operators*

$$K^{\pm}_0 \;=\; \mu^{\pm}\sum_{i=1}^{d_0}\partial^2_{x_i}+\langle x,B\nabla\rangle-\partial_t. \qquad (2.67)$$

where the constant $\mu^{\pm}$ *are such that* $\mu^{+}>\lambda$ *and* $\mu^{-}<1/\lambda$ *and depend only on the operator* $\mathcal{K}$.

The proof of Theorem 2.3.2 will be given by means of two proposition: one for the lower bound and one for the upper bound.

2.3.3 Harnack Inequality and Lower Bound.

The first step is to give Harnack inequality Di Francesco and Polidoro prove in [27] the following invariant Harnack inequality analogous to Theorem 2.2.8. Let $H_r(z_0), S_r(z_0)$ the sets introduced in (2.29) according to the operation (2.59) and the dilation (2.60).

Theorem 2.3.3. *Let* Ω *be an open subset of* $\mathbb{R}^{N+1}$. *Consider the* $\mathcal{K}$ *as in* (2.44) *and assume that the coefficients* $a_{ij}(x,t)$ *and* $a_i(x,t)$ *are bounded, Hölder continuous in the sense of* (2.65) *and satisfy* (2.45) *for every* $(x,t)\in\mathbb{R}^{N+1}$. *Let* $z_0\in\Omega$, $r\in]0,1]$ *be such that* $H_r(z_0)\subseteq\Omega$. *Then, there exist two positive constants* $0<\theta<1$ *and* M, *only depending on the operator* $\mathcal{K}$, *such that*

$$v(z)\leq M\,v(z_0),\qquad \textit{for every } z\in S_{\theta r}(z_0), \qquad (2.68)$$

and for every non-negative solution v *of* $\mathcal{K}v=0$ *in* Ω.

It also holds the following Theorem analogous to Proposition 2.2.15.

Theorem 2.3.4. [Di Francesco-Polidoro] *Let* O *be an open set in* $\mathbb{R}^{N+1}$ *containing* $H_r(z_0)$ *for some* $z_0\in\mathbb{R}^{N+1}$ *and* $r\in]0,1]$. *Then, there exist two positive constants* θ *and* M, *only depending on the operator* $\mathcal{K}$ *in* (2.44), *such that*

$$u(z_0\circ z)\leq M\,u(z_0) \qquad (2.69)$$

for every non-negative solution u of $\mathcal{K}u = 0$ in O and for every z in the positive cone

$$\mathscr{P}_r = \left\{(x,-t) \in \mathbb{R}^{N+1} \;\mid\; |x|_{\mathbb{K}}^2 \le 2\,t,\; 0 < t \le 2\,\theta^2 r^2\right\}. \tag{2.70}$$

For fixed points (ξ,τ), (x,t) belonging to $\mathbb{R}^{N+1}$ with $t > \tau$, we define the admissible paths steering (ξ,τ) and (x,t) as in (2.14), then the admissible paths for $\mathcal{K}$ make use of the vector fields of K_0 in (2.50). In this case the problem assumes the following form

$$\begin{cases} \dot\gamma(s) = & B^T\gamma(s) + \widetilde{Id}\,\omega(s) \\ \gamma(0) = & (x,t), \quad \gamma(t-\tau) = (\xi,\tau), \end{cases} \tag{2.71}$$

where $\omega(s) = (\omega_1(s),\ldots,\omega_{d_0}(s),0,\ldots,0)^T \in \mathbb{R}^N$ and $s \in [0,t-\tau]$ and

$$\widetilde{Id} = \begin{pmatrix} Id & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \text{where } Id \text{ is the } d_0 \times d_0 \text{ identity matrix.} \tag{2.72}$$

We explicitly note that, if we write $\gamma(s) = (x_1(s),...,x_N(s),t(s))$ in its components, we always have $\dot t(s) = -1$, therefore $\tau < t$.

For positive solutions $\mathcal{K}u = 0$, it holds this version of Harnack inequality which involves the cost function Φ defined in (2.17)

Theorem 2.3.5. *Let $T_0, T_1 \in \mathbb{R}$ be fixed and let $u : \mathbb{R}^N\times]T_0,T_1[$ be a positive solution of $\mathcal{K}u = 0$. Fix $(\xi,\tau),(x,t) \in \mathbb{R}^N\times]T_0,T_1[$ with $t > \tau$ and let $\gamma(s)$ be of an admissible path such that $\gamma(0) = (x,t)$ and $\gamma(t-\tau) = (\xi,\tau)$ and $\omega \in L^1[0,t-\tau]$ the corresponding control. Then, there exist three positive constants $\theta \in]0,1[, h$ and $M > 1$, only depending on $\mathcal{K}$, such that*

$$u(\xi,\tau) \le M^{1+\frac{\Phi(\omega)}{h}+\frac{t-\tau}{\theta^2}+\frac{1}{|\log(1-\theta^2)|}\log\left(\frac{\tau-T_0}{t-T_0}\right)} u(x,t), \tag{2.73}$$

Here, the function $\Phi(\omega)$ is cost (2.17) of the path γ .

Proof. If $\omega \in L^1([\tau,t]) \setminus L^2([\tau,t])$, then the estimate reads as $u(x,y,t) \le +\infty$, that is clearly true. We now assume $\omega \in L^2([\tau,t])$.

Before starting we show the following fact: it is possible to pick a positive constant h and a time $\bar t$ only depending on the operator $\mathcal{K}$ such that as long as

$$\int_{\bar t}^{t} \omega^2(s)ds \le h$$

then

$$|\gamma(\bar t) - E(-\bar t)x|_{\mathbb{K}} \le 2\bar t. \tag{2.74}$$

Indeed, the solution to (2.71) is

$$\gamma(\bar t) = E(-\bar t)x + \int_0^{\bar t} E(s-\bar t)\,\widetilde{Id}\,\omega(s)ds. \tag{2.75}$$

moreover, if we decompose the matrix E defined in (2.57) as in (2.56):

$$E(s) = \begin{pmatrix} E_{0,0}(s) & E_{0,1}(s) & \dots & E_{0,r}(s) \\ E_{1,0}(s) & E_{1,1}(s) & \dots & E_{1,r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ E_{r,0}(s) & E_{r,1}(s) & \dots & E_{r,r}(s) \end{pmatrix} \tag{2.76}$$

then $E_{0,0}(s) = I_{p_0} + s\,O_0(s)$,

$$E_{j,0}(s) = \frac{(-s)^j}{j!}\big(I_{p_j} + s\,O_j(s)\big)B_j^T \dots B_1^T, \qquad j = 1, \dots, r,$$

where O_j denotes a $p_j \times p_j$ matrix whose coefficients continuously depend on $s \in [0, +\infty[$. As a consequence, in accordance with the decomposition (2.61), and using (2.75), we find

$$|(\gamma(\bar t) - E(-\bar t)x)^{(j)}| \le \int_0^{\bar t} (\bar t - s)^j\,(c_j + (\bar t - s)g_j(\bar t - s))\,|\omega(s)|ds, \qquad j = 0, \dots, n,$$

for some positive constants $c_0, \dots, c_n$, and positive functions $g_0, \dots, g_n \in C([0, +\infty[)$ only depending on A and B. Hence

$$|(\gamma(\bar t) - E(-\bar t)x)^{(j)}|_{\mathbb{K}}^2 \le c'_j\,\bar t\left(\int_0^{\bar t} |\omega(s)|^2 ds\right)^{\frac{1}{2j+1}} \qquad j = 0, \dots, n,$$

for every $\bar t \in [0, T]$, and for some positive constants $c'_0, \dots, c'_n$ only depending on T, A and B. The claim then follows by choosing h is sufficiently small.

The proof of the proposition is based on the construction of a Harnack chain which is made by applying several times Corollary 2.3.4. We perform the proof into two steps.

Step 1. We fix three restrictive assumptions:

- it holds $t - T_0 \le 1$;
- the path γ is defined on the time interval $[0, t-\tau]$ with $t - \tau \le \theta^2(t - T_0)$;
- the function $\Phi(\omega)$ satisfies $\Phi(\omega) \le h$, where the constant h is a positive constant, only depending on the operator $\mathcal{K}$ and T, and it is suitably chosen.

Under this assumption, we have $(\gamma(s), s) \in (x,t) \circ \mathscr{P}_r$ for every $s \in [0, t-\tau]$, with $r = \sqrt{t - T_0}$, therefore it holds (2.69).

Step 2. We now remove the three hypotheses of Step 1 and prove the main statement. Consider any control $\omega \in L^2([\tau, t])$ and the corresponding curve $\gamma(.)$. Define the sequence of times $\tau < t_k < t_{k-1} < \ldots < t_2 < t_1 < t$ recursively starting from t as follows

$$t_{j+1} = \max\left\{\tau,\ t_j - \theta^2, t_j - \theta^2(t_j - T_0),\ \inf\left\{s \ \text{ s.t. } \ \int_s^{t_j} |\omega(\sigma)|^2\, d\sigma \le h\right\}\right\}. \tag{2.77}$$

The recursive formula terminates when the lower boundary τ is reached. For simplicity of notation, we denote $t_{k+1} = \tau$.

We now define $r_j = \sqrt{t_j - t_{j+1}}/\theta$, then we note that $r_j \le 1$ and

$$H_{r_j}(x(t - t_j), t_j) \subset \mathbb{R}^N \times [T_0, T_1],$$

by (2.77). Moreover we clearly have $t_j - t_{j+1} \le \theta^2 r_j^2$ and by applying Step 1 on the $k+1$ intervals $[t_{j+1}, t_j]$, it holds

$$u(\xi, \tau) \le M^{1+k} u(x,t).$$

We point out that the points $(x(t-t_j), y(t-t_j), t_j)$, $j = 1, \ldots k+1$, selected on the path $\gamma(.)$, represent a Harnack chain. Since (2.77) implies

$$k \le \frac{\int_\tau^t |\omega(s)|^2 ds}{h} + 4(t-\tau)/\theta^2 + \frac{1}{|\log(1-\theta^2)|} \log\left(\tfrac{\tau - T_0}{t - T_0}\right),$$

this concludes the proof. □

Let fix $(x,t), (\xi, \tau) \in \mathbb{R}^{N+1}$, in view of the above proposition it is natural considering the following Optimal Control Problem:

$$\begin{aligned} &\dot\gamma(s) = \widetilde{Id}\omega(s) + B^T\gamma(s) \qquad 0 \le s \le T, \\ &\Phi(\omega) = \int_0^T \sum_{i=1}^m \omega_i^2(s)\, ds \to \min \quad \text{with } T \text{ fixed} \\ &\gamma(0) = (x,t), \qquad \gamma(t-\tau) = (\xi, \tau). \end{aligned} \tag{2.78}$$

We denote by $V_0(x,t;\xi,\tau)$ its value function.

A crucial result to prove the lower bound in (2.66) is the following

Proposition 2.3.6. *Let $\mathcal{K}$ the operator defined in* (2.44) *whose coefficients satisfies the condition* (2.45). *Assume that the model opertor K_0 related to $\mathcal{K}$ satisfies the Hörmander condition [H]. Let T_0, T_1 be fixed and let $\Gamma(x,t;\xi,\tau)$ be the fundamental solution of $\mathcal{K}u = 0$ where (x,t), (ξ,τ) belong to $\mathbb{R}^N \times]T_0,T_1[$ with $t > \tau$. Denote by $T = T_1 - T_0$ then there exist two positive constants $c^-_{\varepsilon,T}$ and C^- such that*

$$\Gamma(x,t;\xi,\tau) \geq \frac{c^-_{\varepsilon,T}}{(t-\tau)^{\frac{Q-2}{2}}} e^{-C^- V_0(x,t-\varepsilon(t-\tau);\xi,\tau)}. \tag{2.79}$$

PROOF. We first prove the claim in the case $(\xi,\tau) = (0,0)$ after we pass to the general one. With the same notation used in proposition 2.3.5, we pick $T_1 = t$, $T_0 = 0$, let $\varepsilon \in]0,1[$ arbitrary chosen and by using (2.39) for the fundamental solution $\Gamma_0(x,t,0,0)$ one gets

$$\begin{aligned}\Gamma_0(x,t,0,0) &\geq M^{-1-t-\frac{V((x,t);(0,\varepsilon t))}{h}} \Gamma_0(0,\varepsilon t;0,0)\\ &\geq M^{-1-T-\frac{V_0((x,t);(0,\varepsilon t))}{h}} \Gamma_0(0,\varepsilon t;0,0)\end{aligned}$$

In Di Francesco Polidoro [27, Theorem 1.5] it is shown that there exists a positive constant C_1 such that

$$\Gamma(0,\varepsilon t) \geq \frac{C_1}{(\varepsilon t)^{\frac{Q-2}{2}}}, \quad \forall t \in (0,T]. \tag{2.80}$$

Finally, by using simple algebraic manipulations we get

$$\Gamma_0(x,t,0,0) \geq \frac{c^-_{\varepsilon,T}}{t^{\frac{Q-2}{2}}} \exp\big(-C^- V_0((x,t);(0,\varepsilon t))\big),$$

with

$$c^-_{\varepsilon,T} = \frac{C_1}{M^{1+T}\varepsilon^{\frac{Q-2}{2}}},$$

so the thesis holds for $(y,s) = (0,0)$.
The general case follows from the fact that, in view of (2.21), we have

$$\Gamma_0(x,t;\xi,\tau) = \Gamma_0((\xi,\tau)^{-1} \circ (x,t);0,0)$$

so, since the time translation is Euclidean, we get

$$\Gamma_0((\xi,\tau)^{-1} \circ (x,t);0,0) \geq \frac{1}{C \cdot (\varepsilon(t-\tau))^{\frac{Q-2}{2}}} e^{-CV_0\left((\xi,\tau)^{-1}\circ(x,t);(0,\varepsilon(t-\tau))\right)},$$

The thesis follows from the invariance property of the optimal cost

$$
\begin{aligned}
V_0\left((\xi,\tau)^{-1}\circ(x,t);(0,\varepsilon(t-\tau))\right) &= V_0\left((x,t);(\xi,\tau)\circ(0,\varepsilon(t-\tau))\right)\\
&= V_0\left((x,t)\circ(0,-\varepsilon(t-\tau);(\xi,\tau)\right).
\end{aligned}
$$

□

Remark 2.3.7. *The inequality* (2.80) *is a strict equality under the Hypothesis that there exist a group of dilation with respect to whom the operator is invariant. Indeed, the proof of the equality*

$$\Gamma(0,\varepsilon t)=\frac{C_1}{(\varepsilon t)^{\frac{Q-2}{2}}}.$$

is a plain consequence of the property (2.20) *by observing that* $(0,\varepsilon t)=\delta_{\sqrt{\varepsilon t}}(0,1)$ *and by denoting* $C_1=\Gamma(0,1;0,0)$.

We explicitly note that the bounds stated in Proposition 2.3.6 agrees with the one stated in (2.26) for operators in the form (2.7) without the drift term X_0 since

$$V(x,t,\xi,\tau)=\frac{d^2(x,\xi)}{t-\tau}\qquad\text{when } X_0=0. \tag{2.81}$$

The identity (2.81) *fails when the drift term* X_0 *is needed to fulfill condition [C].* Aiming to emphasize this assertion, we announce the following proposition which states the remarkable fact that the Value function of the problem (2.38) has the same expression appearing at the exponent of the fundamental solution of K (2.62).

Proposition 2.3.8. *The solution* $\gamma:[0,t-\tau]\to\mathbb{R}^N$ *of* (2.78)*, which minimizes the problem* (2.38)*, is the solution related to the control*

$$\omega(s)=\widetilde{Id}\,E(s)^T\mathcal{C}^{-1}(t-\tau)(x-E(t-\tau)y),$$

the optimal cost is

$$\Psi(\omega)=\int_0^{t-\tau}|\omega(s)|^2d\tau=\langle\mathcal{C}^{-1}(t-\tau)(x-E(t-\tau)\xi),x-E(t-\tau)\xi\rangle.$$

where $\widetilde{Id}$ *is defined in* (2.72) *and*

$$C(t)=\int_0^t E(s)\,\widetilde{Id}\,E^T(s)ds$$

PROOF. Let denote by $q(s) = (x_1(s), \dots, x_N(s))$, we apply the Pontryagin Maximum Principle (proposition 2.2.17) to problem (2.78) whose Hamiltonian is

$$\mathcal{H}(x, \lambda, \omega) = \lambda \widetilde{Id}\, \omega + \lambda\, B^T x + p_0 |\omega|^2, \qquad \lambda = (\lambda_1, \dots, \lambda_N),\ x = (x_1, \dots, x_N)^T. \tag{2.82}$$

since the operator satisfies the Hörmander condition, according to the representation of B in (2.56), we can rewrite

$$\lambda = (\lambda^0, \dots, \lambda^n), \quad x = (x^{(0)}, \dots, x^{(n)})^T. \tag{2.83}$$

We first remark that the problem (2.78) admits no abnormal extremals. Indeed, assume by contradiction $p_0 = 0$ in (2.82), one has

$$\mathcal{H}(x, \lambda, \omega) = \lambda \widetilde{Id}\, \omega + \lambda\, B^T x \tag{2.84}$$

Hence, the maximization of the Hamiltonian $\frac{\partial \mathcal{H}}{\partial \omega}(x, \lambda, \omega) = 0$ is equivalent to

$$\lambda_i(s) = 0, \quad i = 1, ..., d_0, \quad \forall s \in [0, t_1]. \tag{2.85}$$

By plugging (2.85) into (2.84), in view of (2.56) and (2.83), we have:

$$\mathcal{H}(x, \lambda, \omega) = \lambda\, B^T x =$$
$$(\lambda^{(1)} B_1^T, \lambda^{(1)}(*)^T + \lambda^{(2)} B_2^T, ..., \lambda^{(1)}(*)^T + ... + \lambda^{(n)} B_n^T, \lambda^{(1)}(*)^T + ... + \lambda^{(n)}(*)^T) \cdot x,$$

where $(*)$'s are the blocks appearing in (2.56).
Hence, in view of (2.85), we have

$$0 = \dot{\lambda}^{(0)} = -\frac{\partial \mathcal{H}}{\partial x^{(0)}} = \lambda^{(1)} B_1^T,$$

which yields

$$\lambda^{(1)}(s) = 0, \quad \forall s \in [0, t_1], \tag{2.86}$$

since B_1^T has rank $p_1 > 0$.
By plugging (2.85) and (2.86) into (2.84), with analogous arguments we obtain

$$0 = \dot{\lambda}^{(1)} = -\frac{\partial \mathcal{H}}{\partial x^{(1)}} = \lambda^{(2)} B_2^T,$$

which yields

$$\lambda^{(2)}(s) = 0, \quad \forall s \in [0, t_1], \tag{2.87}$$

since B_2^T has rank $p_2 > 0$.
The arguments can be applied recursively up to the step n. As a consequence we obtain

$$(\lambda^{(0)}(s), \lambda^{(1)}(s), \lambda^{(2)}(s), ..., \lambda^{(n)}(s)) = (\lambda_1(s), ..., \lambda_N(s)) = (0, ..., 0),$$

Hence, we conclude that

$$(\lambda_1(s), ..., \lambda_N(s), p_0) = (0, 0, ..., 0) \quad \text{for every} \quad s \in [0, t_1]. \tag{2.88}$$

The (2.88) is in contrast with the fact that $(\lambda_1(s), ..., \lambda_N(s), p_0)$ is always non-vanishing.

Consider the Hamiltonian function $\mathcal{H}$ in (2.82) related to the control problem (2.78) in the interval $[0, t - \tau]$.
The Pontryagin Maximum Principle states that the optimal control is given by

$$\omega(s) = \widetilde{Id}\, p(s)^T, \qquad \text{for some solution to} \quad \dot{p} = -p\, B^T. \tag{2.89}$$

We get

$$p(s) = p(0) \exp(-B^T(s)) = p(0)E(s). \tag{2.90}$$

We use the identities (2.89) and (2.90) in (2.78), and we compute the explicit solution:

$$\gamma(s) = E(-s) \left(x + \mathcal{C}(s) p(0)^T \right)$$

for a specific constant vector $p(0)$ which is given by the final condition $\gamma(t - \tau) = \xi$. At last, we find

$$p(0)^T = \mathcal{C}^{-1}(t - \tau)(E(t - \tau)x - \xi).$$

This concludes the proof. □

As a byproduct, Proposition 2.3.6 and Proposition 2.3.8 yield the lower bound in (2.66).

Remark 2.3.9. *A result analogous to the lower bound in* (2.66) *was first proven by Polidoro [73] for divergence form Kolmogorov operators* $\mathcal{K}$ *in* (2.46) *under the Hypothesis that* K_0 *in* (2.50) *is invariant with respect to a class of dilation group. In his article the result was proved by using Harnack chain without passing through the formulation of an optimal control problem. Moreover, in case of divergence form Kolmogorov operators* $\mathcal{K}$*, provided that there exist an homogeneous Lie group structure, it is possible to show that the lower bound is* ***uniform****, in the sense that the constant* c_T^- *appearing on the lower bound in* (2.66) *does not depend on* T*.*

2.3.4 Upper Bounds

An upper bound analogous to the one stated in Theorem 2.3.2 was given in [26], by using the parametrix representation of the fundamental solution Γ of $\mathcal{K}$ in (2.44) under the hypothesis of boundedness and Hölder continuity of the coefficients. The parametrix method is a powerful instrument as it allows to construct a fundamental solution Γ of $\mathcal{K}$. In some particular cases, it was first introduced by Weber [83], Il'in [36] and Sonin [80] in order to construct a fundamental solution for Kolmogorov operators with variable coefficients. Besides an existence results, the parametrix method provides upper bounds for Γ and for its first and second derivatives. Such results writes as follows (see Di Francesco Pascucci [26, Theorem 1.4]):

Consider the $\mathcal{K}$ *as in* (2.44) *and assume that the coefficients* $a_{ij}(x,t)$ *and* $a_i(x,t)$ *are bounded, Hölder continuous in the sense of* (2.65) *and satisfy* (2.45) *for every* $(x,t) \in \mathbb{R}^N \times]0,T[$. *Assume that the model operator* K_0 *satisfies the Hörmander condition. Let* Γ *be the fundamental solution of the operator* $\mathcal{K}$ *and let* Γ_0^ε *denote the fundamental solution of the constant coefficient operator*

$$K_0^\varepsilon = (\mu+\varepsilon)\sum_{i=1}^{d_0} \partial_{x_i}^2 + \langle x, B\nabla\rangle - \partial_t.$$

Then, there exists a constant $C_{\varepsilon,T}$, *depending on* μ, B, T *and* ε, *such that*

$$\begin{aligned}
&\Gamma(x,t;x_0,t_0) \le C_{\varepsilon,T}\Gamma_0^\varepsilon(x,t;x_0,t_0),\\
&|\partial_{x_i}\Gamma(x,t;x_0,t_0)| \le \frac{C_{\varepsilon,T}}{\sqrt{t-t_0}}\Gamma_0^\varepsilon(x,t;x_0,t_0),\\
&|\partial_{x_ix_j}\Gamma(x,t;x_0,t_0)| \le \frac{C_{\varepsilon,T}}{t-t_0}\Gamma_0^\varepsilon(x,t;x_0,t_0),\\
&|Y\Gamma(x,t;x_0,t_0)| \le \frac{C_{\varepsilon,T}}{t-t_0}\Gamma_0^\varepsilon(x,t;x_0,t_0),
\end{aligned}$$

for any $i,j = 1,...,p_0$ and $(x,t),(x_0,t_0) \in \mathbb{R}^{N+1}$ with $0 \le t_0 < t \le T$.

Concerning divergence form operators $\mathcal{K}$ in (2.46), under the assumption that the related model operator satisfies [H] and [H1], Pascucci-Polidoro [67] proved upper bound for the fundamental solution Γ of $\mathcal{K}$ in the form

$$\Gamma(x,t;x_0,t_0) \le C\Gamma_0^+(x,t;x_0,t_0), \quad \forall\ (x,t),\ (x_0,t_0) \in \mathbb{R}^{N+1},\ t > t_0,$$

where Γ_0^+ is the fundamental solution of the operator K_0^+ appearing in (2.67), the constant C only depend on the operator $\mathcal{K}$ and on λ in (2.45) and is independent of the moduli of continuity of the coefficients (and therefore the results can be applied to operators in the form (2.46) with bounded and measurable coefficients) and independent of T.

We end this Chapter by recalling that an upper bound analogous to that one given in the Proposition 2.3.6 was proved by Cinti-Polidoro [23, Theorem 1.6] for divergence form operators with bounded and measurable coefficients. In their work, more general hypoelliptic operators were discussed, including $\mathcal{K}$ in (2.46) as particular case. Furthermore, the proof of the upper bound is made under the assumption that the related model operator satisfies [H1]-*ii)* and [C]. For the reader convenience we write here the result Theorem 1.6 in [23] in our specific case:

Let $\mathcal{K}$ be the operator in (2.46)*, whose model operator K_0 satisfies the Hypotheses [H1], [H], [C], and let Γ be its fundamental solution. Then, for every positive ε, there exists a positive constant $C_\varepsilon > 0$, only depending on the vector fields $X_1, ..., X_{p_0}, Y$, on the constant μ in* (2.67) *and on ε such that*

$$\Gamma(x,t;\xi,\tau) \leq \frac{C_\varepsilon}{(t-\tau)^{\frac{Q-2}{2}}} \exp\big(- \tfrac{1}{32\mu} V(x, t+\varepsilon(t-\tau), \xi, \tau)\big).$$

Chapter 3

Geman-Yor's results and elements of Malliavin Calculus

In this chapter we collect the probabilistic notions and results of the thesis.

In the first part of the chapter we provide a description of Geman-Yor's type results concerning the case model of Arithmetic Asian Option in the Black-Scholes setting. In this simplest case the problem is related to the following degenerate operator

$$\mathscr{L}_0 = x^2\partial_{xx} + x\partial_x + x\partial_y - \partial_t, \tag{3.1}$$

first introduced in (12). Many authors as Dufresne [28], Yor [55] were first interested in the probabilistic description of the process $(A_t)_{t\geq 0}$ in (13). Moreover, they also discussed exponential type functionals of Brownian motion as they play an important role in several domains, e.g., Mathematical Finance, Diffusion Processes, Stochastic Analysis related to Brownian motions on hyperbolic spaces. These facts motivated detailed studies about these functionals. In particular we focus on the works given by Yor and Geman about the problem of Pricing Arithmetic Average Asian Options [33]. We recommend the monograph [85] for further developments which is a collection of the main works written by Yor and Geman about this subject.

In the second part of the chapter we discuss some basic facts on Malliavin Calculus. This topic, also known as the *Stochastic Calculus of Variation* was first introduced by Malliavin in [53] and well developed by Kusuoka and Strook in the papers [43, 44, 45] and finally formalized in Nualart [64]. We also quote the monograph given by Bally [4] in which Malliavin Calculus is introduced in an elementary way.

Besides the basic facts, notions and results of the Malliavin Calculus, we want to focus on the link with the Hypoelliptic Operators and on the strict connection between the existence of a probability density and the Hörmander condition. Roughly speaking, we can say that the primitive goal of Malliavin Calculus is that to prove the Hörmander Theorem without passing through the Partial Differential Equations. A Particular attention will be paid for the the Malliavin reprensentation formula for the probability density of a stochastic process. This formula has large use in the applications.

In the last part of the chapter we briefly mention some modern applications of Malliavin Calculus and we compare our main result with a similar result given by Delarue and Menozzi in [25].

3.1 Yor type results

In this section we recall some results given by Yor, Matsumoto and Geman [55] [56] [33] and collected by Yor in [85]. These authors first formalized the problem of Pricing of Arithmetic Average Asian Option by using probabilistic techniques, specifically Yor and Geman exhibited in [33] a closed formula for the Laplace transform of the price of an Arithmetic Average Asian Option at time t. The results concern the simplest case of an Option written on an underlying asset with constant drift and volatility:

$$\begin{aligned} X_t &= \xi \exp\left(\sigma W_t + (\mu - \frac{\sigma^2}{2})t\right) \\ A_t &= \eta + \xi \int_0^t \exp\left(\sigma W_s + (\mu - \frac{\sigma^2}{2})s\right) ds \end{aligned} \tag{3.2}$$

where $(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+$, $t \in [0, T]$ and $(W_t)_{t\in[0,T]}$ is a scalar Brownian Motion on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The main difficulty emerging in this problem is understanding how to manage the integral of a geometric Brownian motion $(A_t)_{t\in[0,T]}$. Besides Yor, other authors such as Carr-Shröder [20]. [19], Dufresne [28] have been interested in studying this process in order to give an expression of the laws of $(A_t)_{t\in[0,T]}$ and $(X_t, A_t)_{t\in[0,T]}$ as explicit as possible.

Let fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We assume the existence of a *risk-neutral* probability measure $\mathbb{Q}$ (equivalent to $\mathbb{P}$) under which the underlying stock price dynamic is given by

$$dX_t = rX_t dt + \sigma X_t dW_t^{\mathbb{Q}}, \quad t \in [0,T],$$

where r denotes the *instantaneous risk free interest rate* of a riskless asset (bond), σ is the volatility of the asset in $[0,T]$ and $(W_t^{\mathbb{Q}})_t$, $t \in [0,T]$ is a scalar Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{Q})$. From now on, we only work with the probability space $(\Omega, \mathscr{F}, \mathbb{Q})$ and we endow it with the standard Brownian filtration $(\mathscr{F}_t^{W^{\mathbb{Q}}})_{t\in[0,T]}$. In order not to complicate the notation, we simply write W_t instead of $W_t^{\mathbb{Q}}$.
Working under the martingale measure $\mathbb{Q}$ an expression of the density of the random vector (X_t, A_t) allows us to obtain the pricing at zero of an Arithmetic Average Asian Option by the formula

$$C_{\mathcal{A}}(T,K) = e^{-rT} E\left[\varphi(X_T, A_T)\right]. \tag{3.3}$$

In order to get information about the stochastic price process

$$C_{\mathcal{A}}(t,K) = e^{-r(T-t)} E\left[\varphi(X_T, A_T)|\mathcal{F}_t\right], \tag{3.4}$$

we need the transition probability density $\mathbb{Q}((X_T, A_T) \in A_1 \times A_2 | X_t = x, A_t = y)$, where A_1, A_2 are borelian sets of $\mathbb{R}^+$ and $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+$ is fixed.

A particular approach was used in [33], where the authors rewrote the formula (3.4) in a suitable representation which does not involve conditional expectation. Following their work, consider for instance an Asian Call options with fixed strike

$$C_{\mathcal{A}}(t,K) = e^{-r(T-t)} E\left[\left(\frac{A_T}{T} - K\right)^+ |\mathcal{F}_t\right]. \tag{3.5}$$

We can split the price into two terms

$$C_{\mathcal{A}}(t,K) = \frac{e^{-r(T-t)}}{T} \cdot \frac{4X_t}{\sigma^2} C^{(\nu)}(\tau, q), \tag{3.6}$$

where

$$C^{(\nu)}(\tau, q) = E\left[(A_\tau^{(\nu)} - q)^+\right]. \tag{3.7}$$

Herein, $A_\tau^{(\nu)}$ is Yor's process

$$A_\tau^{(\nu)} = \int_0^\tau \exp\left(2(W_s + \nu s)\right) ds$$

and

$$\nu = \frac{2r}{\sigma^2} - 1, \quad \tau = \frac{\sigma^2}{4}(T-t), \quad q = \frac{\sigma^2}{4X_t}\left(KT - \int_0^t X_s ds\right). \tag{3.8}$$

In order to compute the quantity appearing in (3.7) we need the distribution of $A_T^{(\nu)}$. As we will see in the sequel the law of $A_t^{(\nu)}$ can be obtained by the joint distribution of (A_t, e^{W_t}).

In [85] Yor writes the moment of all order of the random variable $A_t^{(\nu)}$, we recall here only the first two for the sake of completeness, which can be easily computed by integration

$$E(A_t^{(\nu)}) = \frac{1}{2(\nu+1)}\left(e^{(2(\nu+1)t} - 1\right), \quad E((A_t^{(\nu)})^2) = \frac{1}{4}\left(\frac{1}{\nu^2+3\nu+2} - \frac{2e^{2(\nu+1)t}}{\nu^2+4\nu+3} + \frac{e^{4(\nu+2)t}}{\nu^2+5\nu+6}\right). \tag{3.9}$$

Formula (3.9) can be directly employed when q appearing in (3.7) is not positive. In this case, the computation of $C^{(\nu)}(\tau, q)$ in (3.7) reduces to

$$C^{(\nu)}(\tau, q) = E[A_\tau^{(\nu)}] - q = \frac{1}{2(\nu+1)}\left(e^{(2(\nu+1)t} - 1\right) - q \tag{3.10}$$

By plugging this formula into (3.6), reminding the expression given in (3.8), we obtain the following closed-form expression of the Asian option price, namely:

$$C_{t,T} = X_t\left(\frac{1 - e^{-r(T-t)}}{rT}\right) - e^{-r(T-t)}\left(K - \frac{1}{T}\int_0^t X_\tau d\tau\right) \tag{3.11}$$

This expression deserves some comments: it has an interesting resemblance to the Black and Scholes (1973) formula but, as shown in the following remarks, the comparison should not be carried too far:

- the volatility does not appear explicitly in the call price, but it is carried implicitly in X_t and in the integral $\int_0^t X_\tau d\tau$. This appears clearly in a direct and simple proof of (3.11), using the property that the discounted underlying asset price is a $\mathbb{Q}$-martingale; moreover, the result can be easily extends to stochastic interest rates, since it indeed involves the forward price of the underlying asset.

- It is interesting to observe that $C_{t,T}$ is simultaneously and separately increasing in the integral $\int_0^t X_\tau d\tau$ and in X_t.

- The *delta* and the *gamma* of the Asian call can be immediately derived from (3.11). We explicitly note that *gamma* is zero while *delta* is not constant, which is an uncommon situation. This property reflects the fact that the hedging strategy consists in selling every day (or continuously) the same fraction of the underlying asset.

For $q > 0$, it is not possible to have a simple computation of $C^{(\nu)}(\tau, q)$, but Geman-Yor in their celebrated paper [33] were able to give an explicit formula for the Laplace transform with respect to the variable τ of the quantity

$$C_a^{(\nu)}(\tau) = E\left[(A_\tau^{(\nu)} - a)^+\right], \tag{3.12}$$

where a is a fixed real parameter and the expected value is made with respect to the Martingale measure $\mathbb{Q}$. Aiming to introduce their main results, we follow the works [33], [84] and we present a summary of the involved methodologies.

The crucial quantity to compute is (3.7). By applying the Girsanov's theorem (Theorem 1.1.21) one has

$$\begin{aligned} E\left[f(A_\tau^{(\nu)})\right] &= E\left[f(A_\tau)\exp\left(\nu W_\tau - \frac{\nu^2}{2}\tau\right)\right] \\ &= \exp\left(-\frac{\nu^2}{2}\tau\right) E\left[f(A_\tau)\left(e^{W_\tau}\right)^\nu\right] \end{aligned} \tag{3.13}$$

so the distribution of $A_\tau^{(\nu)}$ can be obtained once the joint distribution of (e^{W_τ}, A_τ) is known.
With this aim we denote by $W_\tau^{(\nu)} = W_\tau + \nu\tau$ and we consider the process

$$A_\tau = \int_0^\tau \exp(aW_\tau + bs)ds, \quad a, b \in \mathbb{R}$$

and because of the time scaling property of Brownian motion we fix $a = 2, \nu = b/a$ and we focus on the process

$$A_\tau^{(\nu)} = \int_0^\tau \exp(2W_s^{(\nu)})ds$$

Theorem 3.1.1. *For any reals positive τ, ν, we have*

$$C^{(\nu)}(\tau, q) = $$
$$c_\nu(\tau)\int_0^{+\infty} x^\nu dx \int_0^{+\infty} \frac{1}{y^2}\exp\left(-\frac{1+x^2}{2y}\right)[y-q]^+\psi\left(\frac{x}{y}, \tau\right)dy \tag{3.14}$$

Here the function $\psi(\frac{x}{y},\tau)$, *is given by the following integral representation*

$$\psi\left(\frac{x}{y},\tau\right)=\int_0^\infty e^{-\frac{\xi^2}{2\tau}}e^{-\frac{x}{y}\cosh(\xi)}\sinh(\xi)\sin\left(\frac{\pi\xi}{\tau}\right)d\xi \tag{3.15}$$

and

$$c_\nu(\tau)=\frac{1}{\pi\sqrt{2\pi\tau}}e^{\frac{\pi^2}{2\tau}-\frac{\nu^2\tau}{2}}. \tag{3.16}$$

Before showing the theorem, we next recall a preliminary result about the law of $(A^{(\nu)},X^{(\nu)})$ taken at an exponential time independent on the underlying Brownian motion. We now consider S_μ, an exponentially distributed random variable with parameter μ

$$\mathbb{P}(S_\mu\in dt)=\frac{\mu^2}{2}\exp\left(-\frac{\mu^2}{2}t\right)dt$$

and we assume that the random variable S_μ is independent of W_t.

For $\mu>0$, we denote by $p_t^\mu(\xi,\eta)$ the semigroup (transition probability density) of the Bessel process $R^\mu_{t\geq 0}$ of index μ

$$p_t^\mu(\xi,\eta)=\frac{\eta}{\xi}\left(\frac{\eta}{t}\right)^\mu\exp\left(-\frac{\xi^2+\eta^2}{2t}\right)I_\mu\left(\frac{\xi\eta}{t}\right) \tag{3.17}$$

were $I_\mu(w)$ stands for the modified Bessel function of the first kind of order μ and complex argument $w\in\mathbb{C}$ with $\mathfrak{Re}(w)>0$

$$I_\mu(w)=\frac{1}{\pi}\int_0^\pi e^{(w\cos\theta)}\cos(\mu\theta)d\theta-\frac{\sin(\mu\pi)}{\pi}\int_0^\infty e^{-(w\cosh t+\mu t)}dt.$$

Looking at the distribution of $\left(e^{W_{S_\mu}},A_{S_\mu}\right)$ may be very helpful since it can be led back to the Laplace transform of the distribution of $\left(e^{W_t},A_t\right)$ by means of the following equality

$$E\left[f\left(\exp\left(W_{S_\mu}\right)\right)g\left(A_{S_\mu}\right)\right]=\frac{\mu^2}{2}E\left[\int_0^{+\infty}e^{-\frac{\mu^2}{2}t}f\left(\exp\left(W_t\right)\right)g\left(A_t\right)dt\right] \tag{3.18}$$

for any Borel functions $f,g\ :\mathbb{R}^+\mapsto\mathbb{R}^+$.

We state the following result

Theorem 3.1.2. *The joint law of* $\left(e^{W_{S_\mu}},A_{S_\mu}\right)$ *is given by*

$$\mathbb{P}\left(e^{W_{S_\mu}}\in dx,A_{S_\mu}\in dy\right)=\frac{\mu^2}{2x^{\mu+2}}p_y^\mu(1,x)dxdy. \tag{3.19}$$

More generally

$$\mathbb{P}\left(e^{W^{(\nu)}_{S_\mu}}\in dx,A^{(\nu)}_{S_\mu}\in dy\right)=\frac{\mu^2}{2x^{\sqrt{\mu^2+\nu^2}-\nu+2}}p_y^{\sqrt{\mu^2+\nu^2}}(1,x)dxdy. \tag{3.20}$$

PROOF. We only show (3.20) starting from (3.19). The proof of (3.19) is not so immediate since it requires some notions and results about the Theory of Bessel Processes. We refer to [84] and [33] for further details.

By using (3.18) and (3.13) we obtain

$$
\begin{aligned}
E\left[f(e^{W^{(\nu)}_{S_\mu}})g(A^{(\nu)}_{S_\mu})\right] &= \frac{\mu^2}{2}\int_0^{+\infty} e^{-\frac{\mu^2}{2}t} E\left[f(e^{W^{(\nu)}_t})g(A^{(\nu)}_t)\right] dt \\
&= \frac{\mu^2}{2}\int_0^{+\infty} e^{-\frac{\mu^2}{2}t} E\left[f(e^{W_t})g(A_t)e^{(\nu W_t - \frac{\nu^2 t}{2})}\right] dt \\
&= \frac{\mu^2}{\mu^2+\nu^2}\frac{\mu^2+\nu^2}{2}\int_0^{+\infty} e^{-\frac{\mu^2+\nu^2}{2}t} E\left[f(e^{W_t})g(A_t)e^{(\nu W_t)}\right] dt
\end{aligned}
$$

and going back

$$
\begin{aligned}
&\frac{\mu^2}{\mu^2+\nu^2}\frac{\mu^2+\nu^2}{2}\int_0^{+\infty} e^{-\frac{\mu^2+\nu^2}{2}t} E\big[f(e^{W_t})g(A_t)e^{(\nu W_t)}\big] dt = \\
&\frac{\mu^2}{\mu^2+\nu^2}E\left[f\big(e^{W_{S_{\sqrt{\mu^2+\nu^2}}}}\big)g\big(A_{S_{\sqrt{\mu^2+\nu^2}}}\big)e^{\left(\nu W_{S_{\sqrt{\mu^2+\nu^2}}}\right)}\right] = \\
&\mu^2\int_0^{+\infty}\int_0^{+\infty} f(x)g(u)x^{\nu}\frac{1}{2x^{\sqrt{\mu^2+\nu^2}+2}}p_u^{\sqrt{\mu^2+\nu^2}}(1,x)dudx,
\end{aligned}
$$

which implies formula (3.20). □

PROOF OF THEOREM 3.1.1. In order to show the result we denote by

$$
a_\tau(x,y) := \mathbb{P}(A_\tau \in dy | W_\tau = x), \quad p_\tau(x,y) := \mathbb{P}(A_\tau \in dy; W_\tau = x) \tag{3.21}
$$

We firstly show that the following identity holds

$$
\begin{aligned}
p_\tau(x,y) &= a_\tau(x,y)\cdot\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{x^2}{2\tau}} = \\
&= \frac{e^x}{y^2}\exp\left(-\frac{1+e^{2x}}{2y}\right)\frac{e^{\frac{\pi^2}{2\tau}}}{\pi\sqrt{2\pi\tau}}\cdot\psi\left(\frac{e^x}{y},\tau\right)
\end{aligned} \tag{3.22}
$$

The introduction of the function ψ plays a central role since, as shown in [84], it holds

$$
\int_0^{+\infty} e^{-\frac{\nu^2}{2}\tau}\frac{e^{\frac{\pi^2}{2\tau}}}{\pi\sqrt{2\pi\tau}}\cdot\frac{e^x}{y}\psi\left(\frac{e^x}{y},\tau\right)d\tau = I_{|\nu|}\left(\frac{e^x}{y}\right) \tag{3.23}
$$

By applying Fubini theorem we have

$$
\begin{aligned}
&E\left[\int_0^{+\infty} e^{-\frac{\mu^2}{2}t} f(\exp(W_t))\, g(A_t)\, dt\right] = \\
&\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty} e^{-\frac{\mu^2}{2}t} f(e^x)g(y)a_t(x,y)e^{-\frac{x^2}{2\tau}}dtdxdy = \\
&\int_0^{+\infty} e^{-\frac{\mu^2}{2}t}dt\int_{-\infty}^{+\infty} f(e^x)e^{-\frac{x^2}{2\tau}}dx\int_0^{+\infty} g(y)a_t(x,y)dy
\end{aligned} \tag{3.24}
$$

But on the other hand, reminding (3.18)

$$E\left[\int_0^{+\infty} e^{-\frac{\mu^2}{2}t} f\left(\exp\left(W_t\right)\right) g\left(A_t\right) dt\right] =$$
$$\int_0^{\infty}\int_0^{\infty} f(z)g(y)\frac{1}{z^{\mu+2}}\, p_y^{\mu}(1,z)dzdy =$$
$$\int_{\infty}^{\infty} e^{-(\mu+1)x} f(e^x)dx \int_0^{\infty} g(y)\, p_y^{\mu}(1,e^x)dy \tag{3.25}$$

Hence, comparing (3.24) with (3.25) we get

$$e^{-(\mu+1)x}p_y^{\mu}(1,e^x) = \int_0^{+\infty} e^{-\frac{\mu^2}{2}t}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2\tau}}a_t(x,y)dt$$

and finally, in virtue of (3.23) and the analytical expression of Bessel semigroup (3.17)

$$\int_0^{+\infty} e^{-\frac{\mu^2}{2}t}\exp\left(-\frac{1+e^{2x}}{2y}\right)\frac{e^{\frac{\pi^2}{2\tau}}}{\pi\sqrt{2\pi\tau}}\cdot\frac{e^x}{y}\psi\left(\frac{e^x}{y},\tau\right)dt =$$
$$\int_0^{+\infty} e^{-\frac{\mu^2}{2}t}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2\tau}}a_t(x,y)dt \tag{3.26}$$

and the identity (3.22) follows from the injectivity of Laplace transform.
The statement (3.14) follows from the fact that, reminding (3.22) we have

$$\mathbb{P}(A_\tau\in dy; e^{W_\tau}\in dx) = \frac{1}{x}p_\tau(y,\log x)$$
$$= \frac{1}{y^2}\exp\left(-\frac{1+x^2}{2y}\right)\frac{e^{\frac{\pi^2}{2\tau}}}{\pi\sqrt{2\pi\tau}}\cdot\psi\left(\frac{x}{y},\tau\right) \tag{3.27}$$

and so from (3.13)

$$\exp\left(-\frac{\nu^2}{2}\tau\right)E\left[f(A_\tau)\left(e^{W_\tau}\right)^{\nu}\right] =$$
$$= c_\nu(\tau)\int_0^{+\infty}\int_0^{+\infty} x^{\nu}\frac{1}{y^2}\exp\left(-\frac{1+x^2}{2y}\right)f(y)\psi\left(\frac{x}{y},\tau\right)dxdy \tag{3.28}$$

□

Remark 3.1.3. *By using the Girsanov theorem one may obtain the joint distribution of* $(W_\tau^{(\nu)}, A_\tau^{(\nu)})$ *for every* $\nu\in\mathbb{R}$*. As a consequence, the distribution of* $(e^{2(W_\tau^{(\nu)})}, A_\tau^{(\nu)})$ *plainly follows:*

$$\mathbb{P}(W_\tau^{(\nu)}\in dx, A_\tau^{(\nu)}\in dy) =$$
$$e^{\nu x-\frac{\nu^2}{2}t}\exp\left(-\frac{1+e^{2x}}{2y}\right)\frac{e^{\frac{\pi^2}{2\tau}}}{\pi\sqrt{2\pi\tau}}\cdot\frac{e^x}{y^2}\psi\left(\frac{e^x}{y},\tau\right) \tag{3.29}$$

and therefore

$$\mathbb{P}(e^{2(W_\tau^{(\nu)})} \in dx, A_\tau^{(\nu)} \in dy) =$$

$$e^{-\frac{\nu^2}{2}\tau} \exp\left(-\frac{1+x}{2y}\right) \frac{e^{\frac{\pi^2}{2\tau}}}{2\pi\sqrt{2\pi\tau}} \cdot \frac{x^{\frac{(\nu-1)}{2}}}{y^2} \psi\left(\frac{\sqrt{x}}{y}, \tau\right) \tag{3.30}$$

The following Theorem provides the main result given by Geman and Yor in [33]. A detailed proof of the result is out of the goal of the current Thesis. The proof makes use of a particular representation of exponential function of Brownian motion in terms of Bessel processes (see [33]). An alternative proof of the Geman-Yor formula was given by Yor and Matsumoto in [56] starting from a suitable representation in law of $A_{S_\mu}^{(\nu)}$ in terms of beta variables and gamma variables.

Denoting by $d = \frac{\sqrt{\mu^2+\nu^2}+\nu}{2}$ and $b = \frac{\sqrt{\mu^2+\nu^2}-\nu}{2}$, the Geman-Yor Laplace transform writes as follows [see Geman-Yor [33]).

Theorem 3.1.4 (Geman-Yor [33])**.** *For all $\nu \in \mathbb{R}$, $\frac{\mu^2}{2} > 2(1+\nu)$ and $a > 0$ one has*

$$\frac{\mu^2}{2} \int_0^{+\infty} e^{-\frac{\mu^2}{2}t} E\left[\left(A^{(\nu)} - a\right)^+\right] dt =$$

$$\frac{2}{(\mu^2 - 4(1+\nu))G(b-1)} \int_0^{1/2a} e^{-t} t^{b-2} (1-2at)^{d+1} dt \tag{3.31}$$

where $G(t)$ is the Gamma Function*.*

The inversion of this Laplace transform for a fixed τ would provide the quantity $C_a^{(\nu)}(\tau)$ and, finally, the Asian option price in view of (3.6). This inversion is not easy. There are some softwares for inverting Laplace transforms, which may be useful for a numerical solution of this problem.

Link with Fundamental Solution of $\mathscr{L}_0$. The operator $\mathscr{L}_0$ in (3.1) appears when we fix $\sigma = \sqrt{2}$ and $\mu = 1$ in (3.2). We first suppose that $\xi = 1$, $\eta = 0$; this assumptions are not restrictive as we will see in the sequel. Therefore we consider

$$X_t = e^{\sqrt{2}W_t}, \qquad A_t = \int_0^t e^{\sqrt{2}W_s} ds. \tag{3.32}$$

Because of the scaling property of Brownian motion $W_t = aW_{\frac{t}{a^2}}$, for all $a > 0$ (the equality holds in law), we can rewrite (3.32) as follows

$$X_t = e^{2W_{\frac{t}{2}}}, \quad A_t = \int_0^t e^{2W_{\frac{s}{2}}} ds. \tag{3.33}$$

We precise that the equalities in (3.33) hold in law as we have

$$E[X_t] = E[e^{\sqrt{2}W_t}] = e^t = E[e^{2W_{\frac{t}{2}}}], \quad t \in [0,T].$$

Starting from (3.30), by replacing $\nu = 0$ and τ with $\frac{t}{2}$ we get

$$\mathbb{P}(X_t \in dx, A_t \in dy) = \frac{e^{\frac{\pi^2}{t}}}{2\pi\sqrt{\pi t}} \cdot \frac{1}{\sqrt{x}y^2} \exp\left(-\frac{1+x}{2y}\right) \psi\left(\frac{\sqrt{x}}{y}, \frac{t}{2}\right) \tag{3.34}$$

In order to obtain the probability transition density

$$\mathbb{P}(X_t \in dx, A_t \in dy | X_s = \xi, A_s = \eta),$$

we consider the following left translation

$$(\xi, \eta, s) \circ (x, y, t) = (\xi x, \eta + \xi y, t + s),$$

which corresponds to the fact that we shift the system (3.33) into

$$X_t = \xi e^{2W_{\frac{t-s}{2}}}, \quad A_t = \eta + \xi \int_s^t X_u du. \tag{3.35}$$

We note that $(\xi, \eta, s)^{-1} = \left(\frac{1}{\xi}, \frac{-\eta}{\xi}, -s\right)$ and starting from

$$\int_{\mathbb{R}^+ \times \mathbb{R}} p(x, y, t; \xi, \eta, s) dx dy = 1,$$

by applying the change of variable $(x, y, t) \to (\xi, \eta, s)^{-1} \circ (x, y, t)$ we obtain

$$\begin{aligned} 1 &= \int_{\mathbb{R}^+ \times \mathbb{R}} p(x, y, t, \xi, \eta, s) dx dy \\ &= \int_{\mathbb{R}^+ \times \mathbb{R}} \frac{1}{\xi^2} p\left(\frac{x}{\xi}, \frac{y-\eta}{\xi}, t-s; 1, 0, 0\right) dx dy. \end{aligned}$$

Then, we deduce that

$$p(x, y, t, \xi, \eta, s) = \frac{1}{\xi^2} p\left(\frac{x}{\xi}, \frac{y-\eta}{\xi}, t-s; 1, 0, 0\right) \quad a.s. \tag{3.36}$$

As a consequence, the following result holds true

Theorem 3.1.5. *Consider the stochastic process*

$$dX_t = X_t dt + \sqrt{2} X_t dW_t, \qquad X_s = \xi \tag{3.37}$$

$$dA_t = X_t dt, \qquad A_s = \eta \tag{3.38}$$

which have

$$\mathscr{L}_0 = \xi^2 \partial_{\xi\xi} + \xi \partial_\xi + \xi \partial_\eta + \partial_s$$

as infinitesimal generator. Then the transition probability density $p(x, y, t; \xi, \eta, s)$ *of the random vector* (X_t, A_t) *exists and has got the following representation*

$$p(x, y, t; \xi, \eta, s) = \frac{1}{\xi^2} p((\xi, \eta, s)^{-1} \circ (x, y, t); 1, 0, 0) =$$

$$\frac{e^{\frac{\pi^2}{t-s}}}{2\pi\sqrt{\pi(t-s)}} \cdot \frac{\sqrt{\xi}}{\sqrt{x}(y-\eta)^2} \exp\left(-\frac{x+\xi}{2(y-\eta)}\right) \psi\left(\frac{\sqrt{x\xi}}{y-\eta}, \frac{t-s}{2}\right) \tag{3.39}$$

for every $x > 0$, $y > \eta$, $t > s$*; and* $p(x, y, t; \xi, \eta, s) = 0$ *otherwise.*

In the end, we note that Theorem 3.1.5 and the relationship between probability transition density and fundamental the solution Γ_0 of (3.1)

$$\Gamma(x, y, t, x_0, y_0, t_0) = p(x_0, y_0, T - t_0; x, y, T - t), \tag{3.40}$$

provide the existence and an explicit formula of Γ_0 analogous to (3.39). Formula (3.40) also allows us to obtain two crucial properties for Γ_0:

1. Since $p(x_0, y_0, T - t_0; x, y, T - t)$ is a density with respect to the variables (x_0, y_0), we have that

$$\int_{\mathbb{R}^+ \times (y, +\infty)} \Gamma_0(x, y, t; x_0, y_0, t_0) dx_0 dy_0 = 1, \quad t > t_0;$$

2. Since p is the transition probability density of a Markovian process (X_t, A_t) with $t \in [0, T]$, then it satisfies the reproduction property. The same property also holds for Γ_0

$$\Gamma_0(x, y, t; x_0, y_0, t_0) = \int_{\mathbb{R}^+ \times (y, +\infty)} \Gamma_0(x, y, t; \xi, \eta, \tau) \Gamma_0(\xi, \eta, \tau; x_0, y_0, t_0) d\xi d\eta,$$

for all $t > \tau > t_0$.

3.2 Elements of Malliavin Calculus

This section contains some known results about the theory of *Stochastic Calculus of Variations* we need in this work. The aim of this section is to prove the existence of a fundamental solution Γ of the hypoelliptic operator

$$\mathscr{L}u := x\partial_x\big(a(x, y)x\partial_x u\big) + x\, b(x, y)\partial_x u + x\partial_y u - \partial_t u, \tag{3.41}$$

with $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times]0, T]$, under the assumption that the coefficients $a(x, y)$, $b(x, y)$ are bounded and smooth functions, $x\partial_x a(x, y)$ is also bounded and

$$\inf_{\mathbb{R}^+ \times \mathbb{R}} a(x, y) > 0. \tag{3.42}$$

To keep the exposition in a general frame, we introduce the space $C^\infty_{l,b}(\mathbb{R}^N, \mathbb{R})$ of smooth functions with bounded derivatives of any order and the space $C^\infty_p(\mathbb{R}^N)$ of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all its partial derivatives have polynomial growth.
We then consider the n-dimensional Markovian diffusion process $(X_t)_{t\in[0,T]}$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, strong solution of the SDE:

$$dX^i_t = \sum_{j=1}^{d} \sigma^i_j(X_t) dW^j_t + F^i(X_t) dt, \qquad i = 1, \dots, N, \quad t \in [0, T], \tag{3.43}$$

where $W_t = (W^1_t, \dots, W^d_t)$ is an assigned d-dimensional Brownian motion and

$$F^i,\ \sigma^i_j\ \in C^\infty_{l,b}(\mathbb{R}^N, \mathbb{R}) \quad i = 1, ..., N \quad j = 1, ..., d.$$

We endow the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ generated by $(W_t)_{t\in[0,T]}$ and we remind that $(X_t)_{t\in[0,T]}$ belongs to the space $L^2([0, T] \times \Omega; \mathcal{B} \times \mathscr{F}; \lambda \times \mathbb{P})$, where λ stands for the Lebesgue measure in $\mathbb{R}^N$ and $\mathcal{B}$ is the Borel σ-algebra in $[0, T]$. We denote by X^x_t the solution of the SDE (3.43) with initial condition $X^x_0 = x \in \mathbb{R}^n$.

We further note that the functions belonging to $C^\infty_{l,b}(\mathbb{R}^N, \mathbb{R})$ have sub-linear growth since they are globally Lipschitz by definition.

Given a stochastic process as in (3.43), it is well known that under uniform ellipticity and boundedness assumptions for the diffusion coefficients matrix, its law is absolutely continuous with respect to the Lebesgue measure. Moreover, one may obtain Gaussian type lower and upper bounds for such density. This classical result has been extended by many authors in the case of degenerate processes with bounded and smooth coefficients, by using Malliavin Calculus. The first results about degenerate diffusion processes concerning the existence, positivity and estimates of their densities, was given by Kusuoka and Stroock in [43, 44, 45], Ben Arous and Leandre in [11, 12] and Sanchez in [30], which, instead of ellipticity, the authors assume Hörmander type hypotheses on the coefficients of the diffusion.

3.2.1 Malliavin Calculus for Random variables

We consider the Hilbert space of functions $\mathcal{H} = L^2([0,T],\mathbb{R}^d)$ and we denote by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ its scalar product. Let $(W_t)_{t\geq 0}$ a d-dimensional Brownian motion on the probability space $(\Omega,\mathscr{F},\mathbb{P})$. For each $h(t) = (h^1(t),...,h^d(t)) \in \mathcal{H}$ we introduce the Gaussian random variable:

$$W(h) = \sum_{j=1}^{d}\int_0^T h^j(t)dW_t^j$$

We denote by $\mathcal{S}$ the class of n-dimensional smooth functions of Brownian motion of the form:

$$F = f(W(h_1),...,W(h_n)), \quad f \in C_p^\infty(\mathbb{R}^n,\mathbb{R}), \quad h_1,...,h_n \in \mathcal{H}$$

where $C_p^\infty(\mathbb{R}^n,\mathbb{R})$ is the set of smooth functions with polynomial growth.
For every $F \in \mathcal{S}$ we define the *Malliavin derivative* $(D_tF)_{t\in[0,T]}$ of F as the $\mathbb{R}^d$-dimensional (non adapted) process:

$$D_tF = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(W(h_1),...,W(h_n)))h_i(t).$$

For example, $D_tW(h) = h(t)$. In order to interpret D_tF as a directional derivative, note that for any element $h \in \mathcal{H}$ we have

$$\langle DF,h\rangle_{\mathcal{H}} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big[f(W(h_1)+\varepsilon\langle h_1,h\rangle_{\mathcal{H}},...,W(h_n)+\varepsilon\langle h_n,h\rangle_{\mathcal{H}})-f(W(h_1),...,W(h_n))\big]$$

Each $h_i(t) = (h_i^1(t),...,h_i^d(t))$ has d components and we write D_t^jF for the j^{th} component of D_tF, $j = 1,...,d$. We introduce the Sobolev norm:

$$\|F\|_{1,p} = \Big[\mathbb{E}(|F|_{\mathcal{H}}^p) + \mathbb{E}\big(\big|DF\big|_{\mathcal{H}}^p\big)\Big]^{1/p} \tag{3.44}$$

where

$$\big|DF\big|_{\mathcal{H}} = \Big(\int_0^T |D_tF|^2dt\Big)^{1/2}$$

It is possible to show that the operator $D : \mathcal{S} \to L^p(\Omega,L^2[0,T])$ is closable with respect to the norm $\|\cdot\|_{1,p}$. We denote by $\mathbb{D}^{1,p} = Dom(D)$ the domain of D in $L^p[0,T]$, which is the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,p}$.

For the high order derivatives, let $\alpha = (j_1,...,j_k)$ be a multi-index of length k, we define the k^{th}-order derivative as the random vector on $[0,T]^k\times\Omega$ with coordinates:

$$D_{t_1,...,t_k}^\alpha F = D_{t_k}^{j_k}\cdots D_{t_1}^{j_1}F.$$

We introduce the Sobolev norm:

$$\|F\|_{k,p} = \Big[\mathbb{E}(|F|^p_{\mathcal{H}}) + \sum_{j=1}^{k} \mathbb{E}(|D^{(j)}F|^p_{\mathcal{H}})\Big]^{1/p} \tag{3.45}$$

where

$$|D^{(j)}F|^p_{\mathcal{H}} = \sum_{|\alpha|=j} \left(\int_{[0,T]^k} |D^{\alpha}_{t_1,\dots,t_k} F|^2 dt_1 \dots dt_k \right)^{1/2}$$

We denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$ and finally we denote by

$$\mathbb{D}^{\infty} = \bigcap_{k,p\geq 1} \mathbb{D}^{k,p}.$$

We now introduce the space $\mathcal{P}$ of simple processes, that is the subspace of

$$\mathcal{P} \subseteq L^2([0,T] \times \Omega; \mathcal{B} \times \mathscr{F}; \lambda \times \mathbb{P})$$

of $\mathbb{R}^n$-valued processes $(u_t)_{t\in[0,T]}$ which can be written

$$u_t = \sum_{i=1}^{m} F_i(W(h_1),...,W(h_m))h_i(t), \quad F_i(W(h_1),...,W(h_m)) \in \mathcal{S},\ i = 1,...,m$$

for some $m \in \mathbb{N}$. For $u \in \mathcal{P}$ we define the *Skorohod integral* (we denote by $W(h) = (W(h_1),...,W(h_m))$ for brevity)

$$\delta(u) := \sum_{i=1}^{m} \big(F_i(W(h))W(h_i) - \sum_{j=1}^{m} \partial_{x_j} F_i(W(h)) \langle h_i(s) h_j(s) \rangle_{L[0,T]} \big).$$

We further note that $(D_s F_i)_{s\in[0,T]} \in \mathcal{P}$ for $i = 1,...,m$ and $\delta(u) \in \mathcal{S}$. One can show that the Skorohod operator $\delta :\ \mathcal{P} \to L^2\big([0,T] \times \Omega\big)$ is also closable with respect to the norm

$$\|u\|_{L^2([0,T]\times\Omega)} = \int_0^T E[|u_t|^2|]dt, \quad u_t \in \mathcal{P},$$

so we can extend the definition of Skorohod integral on the whole domain

$$Dom(\delta) = \Big\{ u \in L^2([0,T] \times \Omega) \ :\ \exists u_n \in \mathcal{P}\ s.t.\ \|u_n - u\|_{L^2([0,T]\times\Omega)} \to 0 \Big\}$$

In this case we define

$$\delta(u_n) \to_{L^2(\Omega)} F := \delta(u).$$

We eventually define the *Ornstein-Uhlenbeck* operator

$$LF := \delta D(F) = \sum_{i=1}^{m} \partial_i f(W(h))W(h_i) - \sum_{i,j=1}^{m} \partial_{x_i x_j} f(W(h)) \int_0^T h_i(s) h_j(s) ds.$$

for $F \in \mathcal{S}$. This operator is also closable and L is defined on $\mathbb{D}^{\infty}$.

We now list some meaningful formulas:

Proposition 3.2.1 (Duality Formula)**.** *Let $F \in \mathbb{D}^{1,2}$ and $u \in Dom(\delta)$. It holds the following identity*

$$E\Big[\langle DF, u\rangle_{\mathcal{H}}\Big] = E[F\delta(u)] \tag{3.46}$$

hence, the Skorohod integral δ is the adjoint of the Malliavin derivative D. As a consequence, for $F, G \in Dom(L)$ we have

$$E[FLG] = E[F\delta(DG)] = E\Big[\langle DF, DG\rangle_{\mathcal{H}}\Big] = E[FLG]$$

i.e. L is self-adjoint.

Proposition 3.2.2. *It holds the following computation formulas:*

1) (Chain Rule). *Let $\varphi \in C^1(\mathbb{R}^N, \mathbb{R})$ and $F = (F^1, ..., F^N)$ a random vector derivable in Malliavin sense. Then*

$$D_s\varphi(F^1, ..., F^N) = \sum_{k=1}^{N} \partial_k\varphi(F^1, ..., F^N) D_s F^k. \tag{3.47}$$

2) (Differentiation of Itô Integral). *Let $W_t = (W_t^1, ..., W_t^d)$, with $t \in [0, T]$ a d-dimensional Brownian motion and let v_t an adapted process belonging to $Dom(D)$ such that $\int_0^T E[|u_t|^2]dt < \infty$. Therefore it holds*

1. $D_s^i \int_0^T v_t \, dW_t^i = v_s + \int_s^T D_s^i v_t \, dW_t^i$;
2. $D_s^i \int_0^T v_t \, dt = \int_s^T D_s^i v_t \, dt$;
3. $D_s^i \int_0^T v_t dW_t^j = \int_s^T D_s^i v_t \, dW_t^j$, *for* $i \neq j$.

3) (Clark-Ocone Formula). *Let $W_t = (W_t^1, ..., W_t^d)$, with $t \in [0, T]$ a d-dimensional Brownian motion and let $F \in Dom(D)$ and $\sigma(W_s^1, ..., W_s^d, s \leq T))$-measurable, then it holds the following representation:*

$$F = E[F] + \int_0^T \sum_{i=1}^{d} \varphi_s^i dW_s^i, \quad \varphi_s^i = E\left[D_s^i F | \mathscr{F}_s\right].$$

Density representation

Let $F = (F^1, ..., F^N)$ be a random variable differentiable in Malliavin sense. We now introduce the Malliavin covariance matrix of F.

Definition 3.2.3. *Let $F = (F^1, \dots, F^N)$ be a random vector which is derivable in Malliavin sense. We define the* ***Malliavin Covariance Matrix*** *of the random variable F as follows:*

$$\gamma_F^{ij} = \langle DF^i, DF^j \rangle_{\mathcal{H}} = \sum_{k=1}^{d} \int_0^T D_s^k F^i \, D_s^k F^j ds \quad i, j = 1, ..., N. \tag{3.48}$$

We say that F is non-degenerate *if its Malliavin covariance matrix is invertible a.s and satisfies*

$$\mathbb{E}(|\text{det}\gamma_F|^{-p}) < \infty, \quad \forall p \in \mathbb{N}. \tag{3.49}$$

In the sequel, we denote the inverse of the Malliavin matrix by

$$\Gamma_F = \gamma_F^{-1}.$$

The invertibility of the Malliavin Matrix γ_F yields a sufficient condition to ensure that the law of the random vector F is absolutely continuous with respect to the Lebesgue measure. Moreover, the *non-degeneracy* (3.49) condition is a sufficient condition to ensure that such density is regular. We refer to [64, Chapter 2], for the following proposition

Proposition 3.2.4 (Hirch-Bouleau). *Let $F = (F^1, ..., F^N)$ a random variable. If each $F^i \in \mathbb{D}^{1,p}$ with $p > 1$ and if γ_F is invertible almost surely, then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$, that is*

$$P_F(dx) = p_F(x)dx.$$

Moreover, if γ_F satisfies the non degeneracy condition *(3.49), then such a density is smooth.*

The following theorem gives us a second integration by parts formula in terms of some *weights* which depend on the coefficients of Malliavin covariance matrix.

Theorem 3.2.5 (Malliavin Representation Formula). *Let $F = (F^1, \dots, F^N) \in (\mathbb{D}^\infty)^N$. Then, for all smooth function $\varphi \in C_p^\infty(\mathbb{R}^N)$, $G \in \mathbb{D}^\infty$ and all multi-index $\alpha = (\alpha_1, ..., \alpha_N)$, it holds the following representation formula*

$$E\big[\partial_\alpha \varphi(F) G\big] = E\big[\varphi(F) H_\alpha(F, G)\big], \tag{3.50}$$

where

$$H_i(F, G) = -\sum_{j=1}^{N} \left(G \langle D\Gamma_F^{ij}, DF^j \rangle_{L^2[0,T]} + \Gamma_F^{ij} \langle DG, DF^j \rangle_{L^2[0,T]} - \Gamma_F^{ij} G L F^j \right),$$

for $i = 1, \ldots, N$ *and*

$$H_\alpha(F, G) = H_{(\alpha_1,\ldots,\alpha_m)}(F, G) = H_{\alpha_m}(F, H_{(\alpha_1,\ldots,\alpha_{m-1})}(F, G)).$$

An crucial consequence of The Malliavin formula is the following *Density Representation Formula*

Theorem 3.2.6 (Density representation)**.** *Let* $F = (F^1, \ldots, F^N) \in (\mathbb{D}^\infty)^N$ *satisfy the non degeneracy condition* (3.49)*. The random vector* F *admits a density on* $\mathbb{R}^N$*. Fix* $y \in \mathbb{R}^N$*, the density writes*

$$p_F(y) = E\left[\mathbf{1}_{I(y)}(F)H_\alpha(F, 1)\right], \quad \alpha = (1, ..., N), \tag{3.51}$$

where $I(y) = \prod_{i=1}^N [y_i, +\infty)$*.*
Moreover, $\partial_{y_i} p_F(y)$ *exists and is given by*

$$\partial_{y_i} p_F(y) = -E\left[\mathbf{1}_{I(y)}(F)H_{(\alpha,i)}(F, 1)\right], \quad i = 1, ..., N, \tag{3.52}$$

where $H_{(\alpha,i)}(F, 1) = H_i(F, H_\alpha(F, 1))$*.*

As a consequence of (3.51), by applying the Cauchy-Schwartz inequality we obtain:

Corollary 3.2.7. *In the same Hypothesis of Theorem 3.50. it holds the following upper bounds for the density:*

$$p_F(y) \le \sqrt{\mathbb{P}(F_1 > y_1, ..., F_N > y_N)}\|H_\alpha(F, 1)\|_2, \tag{3.53}$$
$$|\partial_{y_i} p_F(y)| \le \sqrt{\mathbb{P}(F_1 > y_1, ..., F_N > y_N)}\|H_{(\alpha,i)}(F, 1)\|_2, . \tag{3.54}$$

The great advantage of formulas (3.53) and (3.54) is that they allows us to factorize the estimates of the density (or of its derivatives), into the product of a diagonal decay, which is given by the L^2-norm of Malliavin weights, and an off-diagonal bound which in turns constitutes an estimates of the tales of the distribution. Formulas (3.53) and (3.54) have large uses in the applications.

We want to emphasise that, supposing that the non degeneracy condition holds true, formula (3.52) ensures that the density $p_F(y)$ is a $C(\mathbb{R}^N, \mathbb{R})$. Moreover, it holds

$$\lim_{|y| \to +\infty} p_F(y) = 0$$

and the rate of convergence is controlled by the tails of the distribution. From formula (3.54), if $H_{(\alpha,i)}(F,1) \in L^2(\Omega)$ and F has finite moments of all order $q \in \mathbb{N}$, by using Chebishev inequality we get

$$|\partial_{y_i} p_F(y)| \leq \sqrt{\mathbb{P}(F_1 > y_1, ..., F_N > y_N)} \|H_{(\alpha,i)}(F,1)\|_2 \leq \frac{C}{|y|^{\frac{q}{2}}}, \quad q \in \mathbb{N},$$

therefore the density p is a C^1 function. By iterating the previous argument for higher order derivatives we obtain that the density $p_F(y)$ belongs to the Schwartz space of infinitely differentiable functions which decrease rapidly to infinity, along with all its derivatives.

The following results concerns the conditional expectations:

Proposition 3.2.8 (Conditional Expectation)**.** *Let* F, G. *It holds that*

$$E(G \,|\, F = x) = \frac{E\left[\mathbf{1}_{I(x)}(F) H_\alpha(F,G)\right]}{E\left[\mathbf{1}_{I(x)}(F) H_\alpha(F,1)\right]},$$

with the convention that the the term on the right side is null when the denominator is null.

We end this paragraph with the following Remark

Remark 3.2.9. *Due to the fact that* $\mathbf{1}_{I(y)}(F) = \prod_{i=1}^N \mathbf{1}_{[y_i,+\infty)}(F_i)$, *then starting from* (3.53) *and* (3.54), *by applying repeatedly the Cauchy-Schwartz inequality we have*

$$\exists C > 0: \; p_F(y) \leq C \prod_{i=1}^N \left(\mathbb{P}(F_i > y_i)\right)^{\nu(i)} \|H_\alpha(F,1)\|_2, \quad \nu(i) = 2^{-(i+1)},$$

$$\exists C > 0: \; |\partial_{y_i} p_F(y)| \leq C \prod_{i=1}^N \left(\mathbb{P}(F_i > y_i)\right)^{\nu(i)} \|H_{(\alpha,i)}(F,1)\|_2, \quad \nu(i) = 2^{-(i+1)}.$$

3.2.2 Malliavin Calculus for Diffusion Processes

In this section we want to show that Malliavin Calculus is a powerful instrument in the Theory of Diffusion Process which allow us to prove results about the strictly positivity and smoothness for the density of a stochastic process $(X_t)_{t\geq 0}$.

In particular, Theorem 3.2.14 below provides the crucial results of Malliavin Calculus: the proof of the Hörmander Theorem, and accurate Gaussian-type estimate for degenerate process with bounded coefficients.

To start with, we introduce the following probabilistic meanings of *Lie Bracket* and *Hörmander condition.*

Definition 3.2.10 (Lie Bracket). *For each pair of functions $f, g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ we define the Lie Bracket*

$$[f, g] = f\nabla g - g\nabla f \tag{3.55}$$

We construct by recurrence the set of functions:

$$M_0 = \{\sigma^1, \ldots, \sigma^d\}, \quad M_k = \{[F, \varphi], [\sigma_1, \varphi], \ldots, [\sigma_d, \varphi], \quad \varphi \in M_{k-1},\ k \geq 1\},$$

where σ_j is the $j - th$ column of the matrix σ and $F = (F_1, F_2, ..., F_n)$.

Definition 3.2.11 (weak Hörmander Condition). *We say that Hörmander condition holds at the step k in x_0 if:*

$$span\left\{\varphi(x_0) : \varphi \in \bigcup_{i=0}^{k} M_i\right\} = \mathbb{R}^n \tag{3.56}$$

In order not to make confusion between the two definitions of Lie Bracket and Hörmander given in chapter 2, we clarify in the following Remark that the above definitions are exactly the same

Remark 3.2.12. *The above definitions* (3.55) *and* (3.56) *agree with the ones we used for directional derivatives in chapter 1. As usual happened in sub-riemannian framework, the vector fields are identified with their coefficients, so, indeed, it easy to check that, if $X = \sum f_j \partial_{x_j}$ and $Y = \sum g_j \partial_{x_j}$, then $[X, Y] = \sum h_j \partial_{x_j}$, where $h = [f, g]$.*

Let $(X_t(x))_{t\in[0,T]}$ a stochastic process having the dynamic as in (3.43) and starting point $X_0(x) = x \in \mathbb{R}^N$. We introduce the first variation of $(X_t(x))_{t\in[0,T]}$ which is the matrix

$$Y_t(x) = \nabla_x X_t(x), \quad \text{or in components} \quad Y_t^{ij}(x) = \partial_{x_j} X_t^i(x),\ i, j = 1, ..., n.$$

It is possible to show that the matrix $Y_t(x)$ is invertible and we denote by $Z_t(x)$ its inverse. The following proposition, also known as *variation of constants*, plays a crucial role in the proof of Theorem 3.2.14 below in order to check if the Malliavin covariance matrix of the stochastic process $X_t(x)$ satisfies or not the non degeneracy condition (3.49). We refer to [4] for its proof and to [3] for useful applications.

Proposition 3.2.13. *Let $(X_t(x))_{t\in[0,T]}$ a stochastic process having the dynamic as in* (3.43) *and starting point $X_0(x) = x \in \mathbb{R}^N$. Let $\sigma(X_t(x))$ denote the diffusion* *matrix* $\sigma^i_j(X_t(x)),\ i = 1, ..., n,\ j = 1, ..., d$ *of the stochastic process* $X_t(x)$ *in* (3.43) *for every* $t \in [0, T]$*. It holds the following formula*

$$D_s X_t(x) = Y_t(x) \cdot Z_s(x) \cdot \sigma(X_t(x)).$$

We now state the main result of the Stochastic Calculus of Variations Theory (see Kusuoka-Stroock [43, 44, 45]).

Theorem 3.2.14 (Malliavin)**.** *Consider the* n*-dimensional diffusion process (3.43) and suppose that* $F^i,\ \sigma^i_j\ \in C^\infty_{l,b}$.

i) *Then for every* $t \in [0, T]$*,* X_t *belongs to* $\mathbb{D}^\infty$ *and*

$$\|X^x_t\|_{k,p} \le c_{k,p}(t)(1 + |x|)^{\beta_{k,p}} \tag{3.57}$$

where $\beta_{k,p} \in \mathbb{N}$ *and* $c_{k,p}(t)$ *is a constant which depends on* k, p, t *and on the bounds of the derivatives of* b, σ *up to order* k.

ii) *Suppose that the Hörmander condition* (3.56) *holds true. Then there exist a function* $C_{k,p}(t)$ *and some constants* $n_k, m_k \in \mathbb{N}$ *such that it is satisfied the non-degeneracy condition, more precisely:*

$$\| (\gamma_{X^x_t})^{-1}\|_p \le \frac{C_{k,p}(t)(1 + |x|)^{m_k}}{t^{n_k/2}}. \tag{3.58}$$

The function $t \to C_{k,p}(t)$ *is increasing. In particular the right hand side in* (3.58) *blows up as* $t^{-n_k/2}$ *as* $t \to 0$.

iii) *Suppose that the Hörmander condition* (3.56) *and* $F^i,\ \sigma^i_j\ \in C^\infty_{l,b}$*. Then for every* $t \in [0, T]$ *the law of* X^x_t *is absolutely continuous with respect to the Lebesgue measure and the transition density* $y \mapsto p(y, t; x, 0)$ *is a* C^∞ *function. Moreover, if* b, σ *are bounded, one has*

$$p(y, t; x, 0) \le \frac{C_0(1 + |x|)^{m_0}}{t^{n_0/2}} \exp\left(-\frac{D_0(t)|y - x|^2}{t}\right), \tag{3.59}$$

$$|D^\alpha_y p(y, t; x, 0)| \le \frac{C_\alpha(1 + |x|)^{m_\alpha}}{t^{n_\alpha/2}} \exp\left(-\frac{D_\alpha(t)|y - x|^2}{t}\right), \tag{3.60}$$

where all above constants depend on the step for which Hörmander condition holds true and the functions $C_0, D_0, C_\alpha, D_\alpha$ *are increasing functions of* t.

3.2.3 Existence of a Fundamental Solution for $\mathscr{L}$ and further comments

We now consider the operator $\mathscr{L}$ in (3.41), assuming that the coefficients a, b are bounded $C^\infty(\mathbb{R}^2)$, $x\partial_x a(x,y)$ is also bounded and $a(x,y)$ satisfies the condition (3.42). We denote by

$$\mathcal{L} = a(x,y)x^2\partial_{xx} + \big(a_x(x,y)x + a(x,y) + b(x,y)\big)x\partial_x + x\partial_y. \tag{3.61}$$

and we have that $\mathcal{L} + \partial_t$ is the infinitesimal generator of the process

$$\begin{cases} dX_t = \mu(X_t, Y_t)X_t dt + \sigma(X_t, Y_t)X_t dW_t \\ dY_t = X_t dt. \end{cases} \tag{3.62}$$

with

$$a(x,y) = \frac{\sigma^2(x,y)}{2}, \qquad b(x,y) + \frac{\sigma^2(x,y)}{2} + \sigma(x,y)\sigma_x(x,y)x = \mu(x,y), \tag{3.63}$$

and $t \in [0,T]$. It is simple to show that the process $(X_t, Y_t)_{t\in[0,T]}$ belongs to the space $C^\infty_{l,b}(\mathbb{R}^+ \times \mathbb{R})$ provided that $\partial_x(xa(x,y))$ is bounded. We now show that the Hörmander condition (3.56) holds true for (3.62) We state the following result

Corollary 3.2.15. *The process $(X_t, Y_t)_{t\in[0,T]}$ in (3.62) satisfies the Hörmander condition* (3.56)*.*

Proof. Consider the process $(X_t, Y_t)_{t\geq 0}$. The functions

$$F^1(x,y) = \Big(b(x,y) + \frac{\sigma^2(x,y)}{2} + \sigma(x,y)\sigma_x(x,y)x\Big)x,$$
$$F^2(x,y) = x, \quad \sigma = \sigma(x,y)x$$

belong to the space $C^\infty_{l,b}(\mathbb{R}^+ \times \mathbb{R})$ and the Hörmander condition holds true for every $(x,y) \in \mathbb{R}^+ \times \mathbb{R}$.

Indeed we have:

$$F(x,y) = \ \big(F^1(x,y), \ \ F^2(x,y)\big) \qquad \sigma(x,y) = \ (\sigma(x,y)x, \ \ 0)$$

So, one has (we omit the dependence on x, y of the coefficients):

$$\begin{aligned} [F,\sigma] = & \ \big((b + \tfrac{\sigma^2}{2})x + \sigma\sigma_x x^2, \ \ x\big) \begin{pmatrix} \partial_x(\sigma x) & 0 \\ 0 & 0 \end{pmatrix} + \\ & - \ (\sigma x, \ \ 0) \begin{pmatrix} \partial_x\big((b + \tfrac{\sigma^2}{2})x + \sigma\sigma_x x^2\big) & 1 \\ 0 & 0 \end{pmatrix} \\ = & \ \big(x^2\big(\sigma_x b - \tfrac{3}{2}\sigma^2\sigma_x - b_x\sigma\big) - \sigma^2\sigma_{xx}x^3, \ \ -\sigma x\big) \end{aligned}$$

and Hörmander condition holds true at step 1 for every $(x, y) \in \mathbb{R}^+ \times \mathbb{R}$ since the volatility $\sigma(x, y)$ is never vanishing in view of (3.42).
This concludes the proof. □

As a consequence, the density p of the process $(X_t, Y_t)_{t\in[0,T]}$ exists and is smooth in view of **i)** and **ii)** of Theorem 3.2.14 and from Proposition 3.2.4.

By using the Feynman-Kac representation formula, one can state that the transition density $p(x_0, y_0, t_0; x, y, t)$, of the process (3.62) satisfies the Fokker-Planck equation $\mathcal{L}u + \partial_t u = 0$, with $\mathcal{L}$ in (3.61), with final condition $p(T, x_0, y_0; T, x, y) = \delta_{(x_0,y_0)}(x, y)$. Specifically, the function

$$u(x, y, t) = \mathbb{E}[\varphi(X_T, A_T)|(X_t, Y_t) = (x, y)] = \int_{\mathbb{R}^+\times\mathbb{R}} \varphi(\xi, \eta)p(\xi, \eta, T; x, y, t)d\xi d\eta \tag{3.64}$$

is a solution of the Cauchy problem related to the equation $\mathcal{L}u + \partial_t u = 0$ with prescribed bounded continuous final condition φ.

We would remind that, since the stochastic process in (3.62) is Markovian, p satisfies the *Reproduction Property* (also said *Chapman-Kolmogorov identity*)

$$p(x_0, y_0, t_0; x, y, t) = \int_{\mathbb{R}^+\times\mathbb{R}} p(x_0, y_0, t_0; \xi, \eta, \tau)p(\xi, \eta, \tau; x, y, t)d\xi d\eta, \qquad t < \tau < t_0. \tag{3.65}$$

The following proposition summarizes the result about the fundamental solution of $\mathscr{L}$ we have obtained in this Section.

Proposition 3.2.16. *Let $a = a(x, y), b = b(x, y) \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$, with a, b and $\partial_x(xa(x, y))$ bounded. Suppose that $\inf a > 0$. Then, there exists a smooth fundamental solution of $\mathscr{L}$. Moreover it holds the following properties:*

$$1)\ \ \Gamma(x, y, t; \xi, \eta, \tau) = 0 \qquad \textit{whenever} \qquad t \le \tau \ \ \textit{or} \ \ y \ge \eta, \tag{3.66}$$

$$2)\ \ \int_{\mathbb{R}^+\times(y,+\infty)} \Gamma(x, y, t; \xi, \eta, \tau)d\xi d\eta = 1, \quad (x, y) \in \mathbb{R}^+ \times \mathbb{R}, \tag{3.67}$$

and the reproduction property holds true

$$3)\ \ \Gamma(x, y, t; x_0, y_0, t_0) = \int_{\mathbb{R}^+\times\mathbb{R}} \Gamma(x, y, t; \xi, \eta, \tau)\Gamma(\xi, \eta, \tau; x_0, y_0, t_0)d\xi d\eta \tag{3.68}$$

for every $(x, y, t), (x_0, y_0, t_0), (\xi, \eta, \tau)$ belonging to $\mathbb{R}^+ \times \mathbb{R}^2$ with $t > \tau > t_0$.

Proof. Malliavin Calculus provides us with the existence of a smooth probability density $p(x_0, y_0, t_0; x, y, t)$ for the process (3.62). By setting

$$\Gamma(x, y, t; \xi, \eta, \tau) = p(\xi, \eta, T - \tau; x, y, T - t). \tag{3.69}$$

it easy to check that (3.69) defines a smooth fundamental solution for $\mathscr{L}$ in the sense of the Definition 2.2.7. The relation (3.66) simply follows from (3.62).
The relation (3.67) follows from the fact that $p(\cdot, \cdot, T - \tau; x, y, T - t)$ is a density. Moreover, since p is the transition probability density of a Markovian process, the reproduction property (3.68) follows from (3.65). □

Further comments

As seen above, Mallavin Calculus provides a lots of instruments to prove the existence, positivity, and upper bound for densities of Stochastic Processes. Unfortunately, the involved techniques do not allow us to infer on lower bounds. Such problem is much more challenging and several authors focused on lower estimates for diffusion process (3.43) by using probabilistic techniques linked to Malliavin Calculus. We mainly quote the works due to Kohatsu-Higa [41], Bally [5] and Kohatsu-Higa-Bally [8] which are the seminal works where new methodologies and ideas appear in order to get lower buound for densities via Malliavin Calculus. In these work less restrictive assumption and new objects are introduced: specifically, the authors avoid boundedness assumptions on the coefficients of the underlying dynamics (3.43) of X_t, and they relax the uniform parabolicity condition (2.45).

A very technical use of Malliavin Calculus was given by Caramellino-Bally [6, 7] where the authors discussed about the positivity and lower bounds for densities of stochastic processes and the tube estimates for degenerate processes.[1]

We want to quote the paper given by Cinti-Menozzi-Polidoro where Malliavin Calculus was used to get upper estimates. In this paper, the authors consider $n+1$-dimensional stochastic processes $X_t = (X_t^1, ..., X_t^n, X_t^{n+1})$ of two type:

$$\begin{cases} X_t^i = x_i + W_t^i, \quad i = 1, ..., n, \\ X_t^{n+1} = x_{n+1} + \int_0^t |X_s^{1,n}|^k ds \ (k \text{ even}). \end{cases} \qquad \begin{cases} X_t^i = x_i + W_t^i, \quad i = 1, ..., n, \\ X_t^{N+1} = x_{N+1} + \int_0^t \sum_{i=1}^n (X_s^i)^k ds. \end{cases}$$

[1] A tube kind estimate is a lower bound of the probability that a stochastic process X_t remains close (in a suitable metric) a deterministic curve in a fixed interval $[0, T]$.

where $W_t = (W_t^1, ..., W_t^n)$ is a n-dimensional Brownian motion. This problem led the authors to consider particular degenerate hypoelliptic operators

$$\mathcal{L} = \tfrac{1}{2}\Delta_{x_{1,n}} + |x_{1,n}|^k \partial_{x_{n+1}} + \partial_t, \tag{3.70}$$

$$\mathcal{L} = \tfrac{1}{2}\Delta_{x_{1,n}} + \sum_{i=1}^{n} x_i^k \partial_{x_{n+1}} + \partial_t \tag{3.71}$$

The task was performed by using Malliavin Calculus for diffusion processes for the upper bound and Harnack chains for the lower bound.

Concerning the upper bound of the density the main ideas used by the authors is that to exploit the non degeneracy of the first n components of the stochastic process X_t and decompose the density $p(t, x, \xi)$ of X_t into the product of two factors

$$p(t,x,\xi) = p_{X^{1,n}}(t, x_{1,n}, \xi_{1,n}) p_{X^{n+1}}\big(t, x_{n+1}, \xi_{n+1} \,|\, X_0^{1,n} = X_{1,n},\, X_t^{1,n} = \xi_{1,n}\big)$$

where

$$p_{X^{1,n}}(t, x_{1,n}, \xi_{1,n}) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|\xi_{1,n} - x_{1,n}|^2}{2t}\right),$$

is the usual n-dimensional gaussian function.

They used Malliavin Calculus in order to prove upper bound for the density

$$p_{X^{n+1}}\big(t, x_{n+1}, \xi_{n+1} \,|\, X_0^{1,n} = X_{1,n},\, X_t^{1,n} = \xi_{1,n}\big) = p_{Y_t}(\xi_{n+1} - x_{n+1}),$$

where

$$Y_t = \begin{cases} \int_0^t \left|\frac{t-s}{t} x_{1,n} + \frac{s}{t}\xi_{1,n} + W_s^{0,t}\right|^k, & \text{for (3.70);} \\ \int_0^t \sum_{i=1}^n \left(\frac{t-s}{t} x_i + \frac{s}{t}\xi_i + W_s^{0,t,i}\right)^k, & \text{for (3.71),} \end{cases}$$

and $(W^{0,t})_s$, $s \in [0,t]$ stands for the standard d-dimensional Brownian bridge on $[0,t]$, i.e. starting and ending at 0.

The problem of getting upper and lower bounds for density by combining analytic or probabilistic results with optimization procedure was widely discussed in Delarue-Menozzi [25]. In their work the authors focused on *two-sided bounds* for the joint density of some degenerate Itô Stochastic processes with possibly sub-linear drifts. The processes that they considered are

$$\mathscr{L}u := \sum_{j,k=1}^{d} a_{jk}(x,t)\partial_{x_j x_k} u + \sum_{j=1}^{nd} F_j(x,t)\partial_{x_j} u - \partial_t u. \tag{3.72}$$

Here d, n are positive integers and $(x,t) \in \mathbb{R}^{nd} \times \mathbb{R}$. The main assumptions they made concerns are the following

- the spectrum of the diffusion matrix $\big(a_{jk}(x,t)\big)$ with $j,k = 1,\dots,d$ is included in a compact and positive interval;

- the functions $F_j(x,t),\ j = 1,...,nd$ are spatial Lipschitz continuous (uniformly in t) and the diffusion matrix $a(t,x)$ is spatial α-Hölder continuous (uniformly in t), with $\alpha \in (0,1)$.

The estimate that they gave are of Gaussian type with the same function from the above and below. The techniques which they used are related to the parametrix representation of the density combined with a Stochastic Control approach. We want to clarify that the result given by Delarue and Menozzi does not applies to our operator $\mathscr{L}$. The reason is that, even if $\mathscr{L}$ can be write in the form (3.72)

$$\mathscr{L} = a(x,y,t)x^2\partial_{xx} + \big(a_x(x,y,t)x + a(x,y,t) + b(x,y,t)\big)x\partial_x + x\partial_y - \partial_t, \quad (3.73)$$

it does not satisfy the assumption of boundedness on the diffusion matrix. Indeed, in our case we have

$$\tfrac{1}{2}[\sigma\sigma^*](x,y,t) = x^2 a(x,y,t),$$

which does not belong to a compact and positive interval for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times [0,T]$ (the coefficient x^2 is unbounded and has minimum equal to zero).

On the other hand, by means of a suitable change of variable, it is possible to rewrite the operator $\mathscr{L}$ in (3.73) in the following form

$$\mathscr{L}_2 = a(x,y,t)\partial_{xx} + (b(x,y,t) + x\partial a_x(x,y,t))\partial_x + e^x\partial_y - \partial_t. \quad (3.74)$$

For the operator $\mathscr{L}_2$ we have that the coefficient $a(x,y,t)$ is bounded from above and has positive infimum, but the function e^x is globally not Lipschitz.

Chapter 4

Arithmetic Asian Option type Operators

In this chapter we discuss the hypoelliptic operator

$$\mathscr{L}u := x\partial_x\big(a(x,y,t)x\partial_x u\big) + x\,b(x,y,t)\partial_x u + x\partial_y u - \partial_t u, \tag{4.1}$$

with $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$.

The main goal of this chapter is establish bounds analogous to (2.25), (2.66) given above. Specifically, we prove the following inequalities for the fundamental solution Γ of $\mathscr{L}$

$$\frac{c_\varepsilon^-}{t^2}\exp\big(-C^-\Psi(x,y+\varepsilon t,t-\varepsilon t)\big) \le \Gamma(x,y,t) \le \frac{C_\varepsilon^+}{t^2}\exp\big(-c^+\Psi(x,y-\varepsilon,t+\varepsilon)\big)\,, \tag{4.2}$$

for every $(x,y,t), \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ with $y + \varepsilon t < 0$, where $\varepsilon \in (0, \frac{1}{4T})$ is arbitrary. Here Ψ is the *value function* of the optimal control problem (2.40), which in this case formalizes as follows

$$\Psi(x,y,t) := \inf_{\omega \in L^1([0,t])} \int_0^t \omega^2(s)ds \quad \text{subject to constraint} \tag{4.3}$$

$$\begin{cases} \dot{q}_1(s) = \omega(s)q_1(s), & q_1(0) = x, \quad q_1(t) = 1, \\ \dot{q}_2(s) = q_1(s), & q_2(0) = y, \quad q_2(t) = 0. \end{cases}$$

We reach our goal by extending the method used in chapter 2. Note that, Theorem 4.2, provides just estimates for the fundamental solution with pole at $(1,0,0)$. As we have done before, we pass to the general case just by using invariance properties of the operator $\mathscr{L}$ with respect to a suitable translation group. In the next section we will go through the main results of this chapter, we briefly describe the involved techniques to achieve our goals. In particular we will list several geometric properties of the constant operator $\mathscr{L}_0$ related to $\mathscr{L}$, we will give the precise assumptions which we make on $\mathscr{L}$ and the statements of our results. Theorem 4.1.3 below exhibits the precise bounds for $\Gamma(x, y, t; \xi, \eta, \tau)$ at any point (x, y, t) belonging to a specific subset of $\mathbb{R}^+ \times \mathbb{R} \times [0, T]$.

The PDE approach adopted in this chapter allows us to consider more general problems. Among them, we can consider an option on a basket containing n assets $S_t = \left(S_t^1, \dots, S_t^n\right)$ whose dynamic is

$$dS_t^j = S_t^j \mu_j(S_t, A_t, t) + S_t^j \sum_{k=1}^{n} \sigma_{jk}(S_t, A_t, t) dW_t^k, \qquad j = 1, \dots, n, \tag{4.4}$$

where $\left(W_t^1, \dots, W_t^n\right)_{t \geq 0}$ is a n-dimensional Wiener process and $\left(A_t\right)_{t \geq 0}$ is an average of the assets. If we chose

$$A_t^j = \int_0^t S_\tau^j d\tau, \quad j = 1, \dots, n, \qquad \text{or} \qquad A_t = \sum_{j=1}^{n} \int_0^t S_\tau^j d\tau,$$

in analogy with (3.63), we are led to consider the following operators

$$\widetilde{\mathscr{L}_1} u := \sum_{j,k=1}^{n} x_j \partial_{x_j} \left(a_{jk}(x, y, t) x_k \partial_{x_k} u\right) + \sum_{j=1}^{n} x_j b_j(x, y, t) \partial_{x_j} u + \sum_{j=1}^{n} x_j \partial_{y_j} u - \partial_t u, \tag{4.5}$$

with $(x, y, t) \in (\mathbb{R}^+)^n \times \mathbb{R}^n \times]0, T]$, and

$$\widetilde{\mathscr{L}_2} u := \sum_{j,k=1}^{n} x_j \partial_{x_j} \left(a_{jk}(x, y, t) x_k \partial_{x_k} u\right) + \sum_{j=1}^{n} x_j b_j(x, y, t) \partial_{x_j} u + \sum_{j=1}^{n} x_j \partial_{y} u - \partial_t u, \tag{4.6}$$

with $(x, y, t) \in (\mathbb{R}^+)^n \times \mathbb{R} \times]0, T]$, respectively. In these examples, denoting by $\sigma(x, y, t)$ the matrix $\left(\sigma(x, y, t)\right)_{j,k=1,\dots,n}$, we have

$$\left(a_{jk}(x, y, t)\right)_{j,k=1,\dots,n} = \tfrac{1}{2} \left[\sigma(x, y, t) \sigma(x, y, t)^*\right].$$

The coefficients $b_j(x, y, t)$, $j = 1, \dots, n$ depend on the coefficients $\mu_1, \dots, \mu_n$ and on the derivatives of the a_{jk}.

The extension to the ideal case of no-correlated assets is immediate. Indeed, in this case, we are led to consider operators of the form

$$\bar{\mathscr{L}_1} u := \sum_{j=1}^{n} x_j \partial_{x_j} \left(a_{jj}(x, y, t) x_j \partial_{x_j} u\right) + \sum_{j,k=1}^{n} x_j b_j(x, y, t) \partial_{x_j} u + \sum_{j=1}^{n} x_j \partial_{y_j} u - \partial_t u,$$

with $(x, y, t) \in (\mathbb{R}^+)^n \times \mathbb{R}^n \times]0, T]$, and

$$\bar{\mathscr{L}_2} u := \sum_{j=1}^{n} x_j \partial_{x_j} \big(a_{jj}(x, y, t) x_j \partial_{x_j} u \big) + \sum_{j=1}^{n} x_j b_j(x, y, t) \partial_{x_j} u + \sum_{j=1}^{n} x_j \partial_y u - \partial_t u,$$

with $(x, y, t) \in (\mathbb{R}^+)^n \times \mathbb{R} \times]0, T]$.

In these cases, a fundamental solution $\bar{\Gamma}_1(x_1, ..., x_n, y_1, ..., y_n, t)$ of $\bar{\mathscr{L}_1}$ can be written as

$$\bar{\Gamma}_1(x_1, ..., x_n, y_1, ..., y_n, t) = \prod_{i=1}^{n} \Gamma_i(x_i, y_i, t),$$

where $\Gamma_i(x_i, y_i, t)$ is a fundamental solution of the equation

$$x_i \partial_{x_i} \big(a_{ii}(x, y, t) x_i \partial_{x_i} u \big) + x_i b_i(x, y, t) \partial_{x_i} u + x_i \partial_{y_i} u - \partial_t u = 0,$$

and respectively a fundamental solution $\bar{\Gamma}_2(x_1, ..., x_n, y, t)$ of $\bar{\mathscr{L}_2}$ can be written as

$$\bar{\Gamma}_2(x_1, ..., x_n, y, t) = \prod_{i=1}^{n} \Gamma_i(x_i, y, t),$$

where $\Gamma_i(x_i, y, t)$ is a fundamental solution of the equation

$$x_i \partial_{x_i} \big(a_{ii}(x, y, t) x_i \partial_{x_i} u \big) + x_i b_i(x, y, t) \partial_{x_i} u + x_i \partial_y u - \partial_t u = 0.$$

In this chapter we focus on the simplest case (4.1) for the sake of simplicity.

4.1 Invariance properties and main results

This section contains the precise statement of our assumptions and our main results. Since the operator $\mathscr{L}$ in (4.1) appears with variable coefficients, a crucial point is *how many regularity* we assume on them. We will study the operator $\mathscr{L}$ by assuming that its coefficients $a(x, y, t)$, $b(x, y, t)$ are bounded Hölder continuous function. In order to introduce the correct meaning of Hölder continuity of the coefficients a and b of $\mathscr{L}$ in this setting, we recall some properties of the related constant coefficient operator

$$\mathscr{L}_0 = x^2 \partial_{xx} + x \partial_x + x \partial_y - \partial_t.$$

An invariance property was given by Monti and Pascucci, which observe in [59] that $\mathscr{L}_0$ is invariant with respect to the following group operation on $\mathbb{R}^+ \times \mathbb{R}^2$:

$$(x_0, y_0, t_0) \circ (x, y, t) = (x_0 x, y_0 + x_0 y, t_0 + t). \tag{4.7}$$

Indeed, if we set

$$v(x,y,t) = u(x_0 x, y_0 + x_0 y, t_0 + t), \tag{4.8}$$

then $\mathscr{L}_0 v = 0$ if, and only if $\mathscr{L}_0 u = 0$. We also note that

$$\mathbb{G} := \left(\mathbb{R}^+ \times \mathbb{R}^2, \circ\right) \tag{4.9}$$

is a Lie group, its identity $\mathbf{1}_{\mathbb{G}}$ and the inverse of (x,y,t) are defined as

$$\mathbf{1}_{\mathbb{G}} = (1,0,0), \qquad (x,y,t)^{-1} = \left(\frac{1}{x}, -\frac{y}{x}, -t\right). \tag{4.10}$$

Then, in particular, we have

$$(x_0, y_0, t_0)^{-1} \circ (x,y,t) = \left(\frac{x}{x_0}, \frac{y - y_0}{x_0}, t - t_0\right), \tag{4.11}$$

so that (4.8) is equivalent to $u(x,y,t) = v\left(\frac{x}{x_0}, \frac{y-y_0}{x_0}, t - t_0\right)$.
For the operator $\mathscr{L}_0$ a dilation group of the form

$$(\delta_r)_{r>0} : (x,y,t) \mapsto (r^\alpha x, r^\beta y, r^\gamma t),$$

where α, β, γ are positive parameters, under which the following invariance property holds true

$$\mathscr{L}_0\big(u(r^\alpha x, r^\beta y, r^\gamma t)\big) = r^\gamma \left(\mathscr{L}_0 u\right)(r^\alpha x, r^\beta y, r^\gamma t),$$

for any $u(x,y,t)$ belonging to $C^{2,1}(\mathbb{R}^+ \times \mathbb{R} \times [0,T])$, cannot exist. Then the operator $\mathscr{L}_0$ does not satisfy the Hypothesis [H1]-*ii)* made in chapter 2.

We now introduce a further notation based on the invariance property of $\mathscr{L}_0$ with respect to $\mathbb{G}$. As the zero of the group $(\mathbb{R}^+ \times \mathbb{R}^2, \circ)$ is $(1,0,0)$, in the sequel we use the simplified notation

$$\Gamma(x,y,t) := \Gamma(x,y,t;1,0,0). \tag{4.12}$$

Then, thanks to the invariance with respect to the left translation of $\mathbb{G}$, we have

$$\Gamma(x,y,t;x_0,y_0,t_0) = \Gamma((x_0,y_0,t_0)^{-1} \circ (x,y,t);1,0,0) = \Gamma\left(\frac{x}{x_0}, \frac{y-y_0}{x_0}, t - t_0\right).$$

Analogously, we denote by $\Psi(x,y,t;x_0,y_0,t_0)$ the function defined in (4.3), with the *end point* $(1,0)$ replaced by (x_0,y_0), and t replaced by $t - t_0$. Note that, in analogy with (4.12), we have

$$\Psi(x,y,t) = \Psi(x,y,t;1,0,0).$$

The definition of Ψ is explicitly written in (4.59) below and is well posed only when $t > t_0$ and $y_0 > y$, otherwise problem (4.3) has no solution. In this case we agree to set $\Psi(x, y, t; x_0, y_0, t_0) = +\infty$. The following Proposition (which will be proven by means of formulas (4.60) and (4.70) below) states its invariance properties with respect to the operation on $\mathbb{G}$.

Proposition 4.1.1. *For every $(x, y, t), (x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t_0 < t$ and $y_0 > y$, and for every $r > 0$ we have*

$$\Psi(x, y, t; x_0, y_0, t_0) = \Psi\left(\tfrac{x}{x_0}, \tfrac{y-y_0}{x_0}, t - t_0\right); \tag{4.13}$$

$$\Psi(x, y, t; x_0, y_0, t_0) = \tfrac{1}{r}\Psi\left(x, \tfrac{y}{r}, \tfrac{t}{r}; x_0, \tfrac{y_0}{r}, \tfrac{t_0}{r}\right). \tag{4.14}$$

In particular, for $r = t - t_0$, we find

$$\Psi(x, y, t; x_0, y_0, t_0) = \tfrac{1}{t-t_0}\Psi\left(\tfrac{x}{x_0}, \tfrac{y-y_0}{(t-t_0)x_0}, 1\right).$$

In this frame, the scaling property (4.14) for the value function Ψ, replaces the Hypothesis [H1]-*ii)* introduced in chapter 2.

We assume the following conditions on the coefficients a and b: there exists $\lambda > 0$ such that

$$a(x, y, t) \geq \lambda \quad \text{for every} \quad (x, y, t), (\xi, \eta, \tau) \in \mathbb{R}^+ \times \mathbb{R}^+ \times]0, T]. \tag{4.15}$$

Moreover, $a, b, \partial_x(xa)$ and $\partial_x(xb)$ are bounded and Hölder continuous functions in accordance with the following definition: there exist $\overline{M} \geq 0$ and $\alpha \in]0, 1]$ such that

$$|a(x, y, t) - a(\xi, \eta, \tau)| \leq \overline{M}\left(\left|\tfrac{x-\xi}{\xi}\right| + \left|\tfrac{y-\eta}{\xi} + t - \tau\right|^{1/3} + |t-\tau|^{1/2}\right)^{\alpha}, \tag{4.16}$$

for every $(x, y, t), (\xi, \eta, \tau) \in \mathbb{R}^+ \times \mathbb{R}^+ \times]0, T]$. As said above, the same condition is assumed on $b, \partial_x(xa)$ and $\partial_x(xb)$.

We briefly discuss our definition (4.16) of Hölder continuity. With this aim, we first note that $\mathscr{L}$ can be written in the form

$$\mathscr{L}_0 = X^2 + Y,$$

where

$$Xu(x, y, t) := x\partial_x u(x, y, t), \quad Yu(x, y, t) := x\partial_y u(x, y, t) - \partial_t u(x, y, t). \tag{4.17}$$

As usual in the study of parabolic operators, $\mathscr{L}_0$ is a second order operator provided that we consider Y as a second order derivative. The commutator of X and Y plays a crucial role in the study of the regularity of $\mathscr{L}_0$. In particular, we note that $x\partial_y$ is obtained as a *commutator* of X and Y, that is $x\partial_y = [X, Y] = XY - YX$; if we consider X and Y as first and second order derivative, respectively, then $x\partial_y$ is a *third order derivative*. This explains the exponent $1/3$ appearing in (4.16). Then our definition of Hölder continuity is completely natural in view of (4.11).

Remark 4.1.2. *Unlike $\mathscr{L}_0$, the operator $\mathscr{L}$ is not invariant with respect to the left translation* (4.7)*. Indeed, as we apply the change of variable* (4.8) *to a solution u of $\mathscr{L}u = 0$, then v is a solution of $\mathscr{L}_{z_0}v = 0$, where $z_0 = (x_0, y_0, t_0)$ and*

$$\begin{aligned}\mathscr{L}_{z_0}v =& x\partial_x\big(a(x_0x, y_0 + x_0y, t_0 + t)x\partial_x v\big)\\ &+ x\, b(x_0x, y_0 + x_0y, t_0 + t)\partial_x v + x\partial_y v - \partial_t v. \end{aligned} \tag{4.18}$$

However, even if $\mathscr{L}_{z_0}$ does not agree with $\mathscr{L}$, it satisfies assumptions (4.15) *and* (4.16)*, with the same constants M, λ and α used for $\mathscr{L}$. This property will be often used in the sequel and is the basis of the invariant nature of our bounds* (4.20) *for the fundamental solution of $\mathscr{L}$.*

We point out that (4.16) is required for the validity of Harnack inequality, which is the main tool in the proof of the lower bound of Γ. Even if we rely on some regularity properties of the coefficients a, b in our proof of the upper bound of Γ, a method based on the Moser iteration would lead to the same results assuming a, b measurable only. The existence of a fundamental solution is guaranteed by Malliavin Calculus if the coefficients are smooth and satisfy further conditions. In this work we prove upper and lower bounds for Γ in terms of quantities only depending on the constants appearing in (4.15), (4.16) and on the L^∞-norm of the coefficients. In a future study we plan to prove the existence of a fundamental solution of $\mathscr{L}$ only requiring (4.15), (4.16), by using the bounds (4.2).
An alternative approach for proving the existence of the fundamental solution Γ of $\mathscr{L}$ might be to construct Γ by using the parametrix method following the techniques appearing in [26] and [57].
For this reason, in our main result we assume the existence of a fundamental solution Γ of $\mathscr{L}$. We prove uniform bounds for Γ, that only depend on the constants λ, M and α appearing in (4.15), (4.16) and on the L^∞ norms of $a, b, \partial_x(xa)$ and $\partial_x(xb)$.

The main result of this chapter is the following

Theorem 4.1.3. *Let Γ be the fundamental solution of $\mathscr{L}$. Then for every (x_0, y_0, t_0), (x, y, t) belonging to $\mathbb{R}^+ \times \mathbb{R} \times [0, T]$, with $t > t_0$, we have*

$$\Gamma(x, y, t, x_0, y_0, t_0) = 0 \qquad \forall\ (x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2 \setminus \big\{]-\infty, y_0[\times]t_0, T[\big\}. \tag{4.19}$$

Moreover, for arbitrary $\varepsilon \in]0, \frac{1}{4T}[$, there exist two positive constants $c_\varepsilon^-, C_\varepsilon^+$ depending on ε, on T and on the operator $\mathscr{L}$, and two positive constants C^-, c^+, only depending on the operator $\mathscr{L}$ such that

$$\begin{aligned} &\frac{c_\varepsilon^-}{x_0^2 (t-t_0)^2} \exp\left(-C^- \Psi(x, y + x_0 \varepsilon (t-t_0), t - \varepsilon(t-t_0); x_0, y_0, t_0) \right) \leq \\ &\quad \Gamma(x, y, t; x_0, y_0, t_0) \leq \\ &\qquad \frac{C_\varepsilon^+}{x_0^2 (t-t_0)^2} \exp\left(-c^+ \Psi(x, y - x_0 \varepsilon, t + \varepsilon; x_0, y_0, t_0) \right), \end{aligned} \tag{4.20}$$

for every $(x, y, t) \in \mathbb{R}^+ \times]-\infty, y_0 - x_0 \varepsilon (t-t_0)[\times]t_0, T]$. Here Ψ is the value function of the optimal control problem

$$\Psi(x, y, t; x_0, y_0, t_0) := \inf_{\omega \in L^1([0, t-t_0])} \int_0^{t-t_0} \omega^2(s) ds \quad \textit{subject to constraint} \tag{4.21}$$

$$\begin{cases} \dot{q_1}(s) = \omega(s) q_1(s), & q_1(0) = x, \quad q_2(t-t_0) = x_0, \\ \dot{q_2}(s) = q_2(s), & q_2(0) = y, \quad q_2(t-t_0) = y_0. \end{cases}$$

If we agree to set

$$e^{\left(-c^\pm \Psi(x, y, t; x_0, y_0, t_0) \right)} = 0, \quad \text{if} \quad \Psi(x, y, t; x_0, y_0, t_0) = +\infty,$$

then (4.20) holds for every $(x_0, y_0, t_0), (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$.

Clearly, the knowledge of the function Ψ in (4.21) is crucial for the application of our Theorem 4.1.3. We summarize here some of the quantitative information about Ψ, that are written in terms of the function g defined as follows

$$g(r) = \begin{cases} \frac{\sinh(\sqrt{r})}{\sqrt{r}}, & r > 0, \\ 1, & r = 0, \\ \frac{\sin(\sqrt{-r})}{\sqrt{-r}}, & -\pi^2 < r < 0. \end{cases} \tag{4.22}$$

Proposition 4.1.4. *For every $(x, y, t), (x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t_0 < t$ and $y_0 > y$, we have*

$$\begin{cases} \Psi(x, y, t; x_0, y_0, t_0) = E(t-t_0) + \frac{4(x+x_0)}{y_0 - y} - 4\sqrt{E + \frac{4 x x_0}{(y_0-y)^2}}, \\ \textit{if } E \geq -\frac{\pi^2}{t-t_0}; \\ \Psi(x, y, t; x_0, y_0, t_0) = E(t-t_0) + \frac{4(x+x_0)}{y_0 - y} + 4\sqrt{E + \frac{4 x x_0}{(y_0-y)^2}}, \\ \textit{if } -\frac{4\pi^2}{t-t_0} < E < -\frac{\pi^2}{t-t_0}. \end{cases} \tag{4.23}$$

where

$$E = E(x,y,t;x_0,y_0,t_0) = \frac{4}{(t-t_0)^2} g^{-1}\left(\frac{y_0-y}{(t-t_0)\sqrt{xx_0}}\right). \tag{4.24}$$

Moreover,

$$\frac{\Psi(x,y,t;x_0,y_0,t_0)}{\frac{4}{(t-t_0)}\log^2\left(\frac{y_0-y}{(t-t_0)\sqrt{xx_0}}\right)+\frac{4(x_0+x)}{y_0-y}} \to 1, \qquad as \qquad \frac{y_0-y}{(t-t_0)\sqrt{x_0x}} \to +\infty; \tag{4.25}$$

$$\frac{\Psi(x,y,t;x_0,y_0,t_0)}{\frac{4(\sqrt{x}+\sqrt{x_0})^2}{y_0-y}-\frac{4\pi^2}{(t-t_0)}} \to 1, \qquad as \qquad \frac{y_0-y}{(t-t_0)\sqrt{x_0x}} \to 0. \tag{4.26}$$

The proof of the lower bound is based on a Harnack inequality for positive solutions of $\mathscr{L}u = 0$. The repeated application of the Harnack inequality, combined with a suitable optimization procedure, provides us with the lower bound of the fundamental solution. The upper bound for Γ follows from the fact that the value function Ψ is a solution of the relevant Hamilton-Jacobi-Bellman equation.

As a corollary of Theorem 4.1.3, by applying (4.2) to Γ and to the fundamental solutions $\Gamma^{\pm}$ of the operators

$$\mathscr{L}^{\pm}u = \lambda^{\pm}x^2\partial_{xx}u + x\partial_x u + x\partial_y u - \partial_t u, \quad (x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T], \tag{4.27}$$

for some strictly positive constants $\lambda^{\pm}$, we obtain the following result which says that the fundamental solutions of $\mathscr{L}$ and $\mathscr{L}_0$ have the same behavior.

Proposition 4.1.5. *For every* $\varepsilon \in]0, \frac{1}{4T}[$, *there exist* $\Gamma^{\pm}$ *in the form* (4.27), *and positive constants* $k^{\pm}$ *such that*

$$\begin{aligned} &k^-\Gamma^-(x, y + x_0\varepsilon(t-t_0+1), t-\varepsilon(t-t_0+1); x_0,y_0,t_0) \leq \\ &\quad \Gamma(x,y,t,x_0,y_0,t_0) \leq \\ &\qquad k^+\Gamma^+\left(x, y - x_0\frac{\varepsilon}{1-\varepsilon}(t-t_0+1), t + \frac{\varepsilon}{1-\varepsilon}(t-t_0+1), x_0,y_0,t_0\right), \end{aligned}$$

for every $(x,y,t),(x_0,y_0,t_0) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ *with* $y + x_0\varepsilon(t-t_0+1) < y_0$ *and* $t > t_0 + \varepsilon/(1-\varepsilon)$.

4.2 Harnack inequality and Green function

In this section we will introduce the Harnack inequality for $\mathscr{L}$ in (4.1). For any $z_0 = (x_0,y_0,t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$ and $r \in]0,1/2]$, we set

$$
\begin{aligned}
H_r(z_0) = \Big\{(x,y,t) \in \mathbb{R}^3 : |x - x_0| < r x_0, -r^2 < t - t_0 < 0, \\
|y - y_0 + x_0(t - t_0)| < r^3 x_0\Big\} \\
S_r(z_0) = \Big\{(x,y,t) \in \mathbb{R}^3 : |x - x_0| \le r x_0, -r^2 \le t - t_0 \le -\frac{r^2}{2}, \\
|y - y_0 + x_0(t - t_0)| \le r^3 x_0\Big\}.
\end{aligned} \tag{4.28}
$$

Notice that the circular segments defined in (4.28) are the most natural geometric sets which can be defined taking into account the group operation (4.7).

Proposition 4.2.1. *Let $z_0 \in \mathbb{R}^+ \times \mathbb{R}^2$ and $r \in]0, 1/2]$. If u is a non negative solution of $\mathscr{L} u = 0$ in $H_r(z_0)$, then*

$$u(z) \le M\, u(z_0)$$

for every $z \in S_{\theta r}(z_0)$. The two constants $\theta \in]0,1[$ and $M > 0$ only depend on the operator $\mathscr{L}$.

The proof of Proposition 4.2.1 relies on the Theorem 2.3.3. For the sake of completeness, we recall the statement suitable for our operator $\mathscr{L}$.

Let Ω be an open subset of $\mathbb{R}^3$. Consider the following operator

$$\mathcal{K} v = \widetilde{a}(x,y,t)\partial_{xx} v + \widetilde{b}(x,y,t)\partial_x v + x\partial_y v - \partial_t v, \quad (x,y,t) \in \Omega. \tag{4.29}$$

Assume that $\widetilde{a}$ and $\widetilde{b}$ are bounded continuous functions such that $\inf_\Omega \widetilde{a}(x,y,t) > 0$. Suppose also that $\widetilde{a}$ and $\widetilde{b}$ satisfy the following Hölder continuity condition

$$|\widetilde{a}(x,y,t) - \widetilde{a}(\xi,\eta,\tau)| \le \widetilde{M}\left(|x - \xi| + |y - \eta + (t-\tau)\xi|^{1/3} + |t - \tau|^{1/2}\right)^\alpha,$$

for every $(x,y,t), (\xi,\eta,\tau) \in \Omega$. Let $(1,0,0) \in \Omega$, $r \in]0,1/2]$ be such that $H_r(1,0,0) \subseteq \Omega$. Then there exist two positive constants θ and M, only depending on the operator $\mathcal{K}$, such that

$$v(z) \le M\, v(1,0,0), \qquad \text{for every } z \in S_{\theta r}(1,0,0), \tag{4.30}$$

and for every non-negative solution v of $\mathcal{K} v = 0$ in Ω.

Proof of Proposition 4.2.1. Let u be a positive solution of $\mathscr{L} u = 0$ in $H_r(z_0)$, with $r \in]0, 1/2]$. We first consider the case $z_0 = (1,0,0)$. We plan to apply (4.30) to u. With this aim, we write $\mathscr{L}$ in its non-divergence form (4.29) by setting

$$
\begin{aligned}
\widetilde{a}(x,y,t) &= x^2 a(x,y,t), \\
\widetilde{b}(x,y,t) &= x\left(a(x,y,t) + x\partial_x a(x,y,t) + b(x,y,t)\right).
\end{aligned} \tag{4.31}
$$

As the coefficients $\widetilde{a}$ and $\widetilde{b}$ are not bounded, $\mathscr{L}$ does not satisfy the conditions required for the validity of (4.30). We overcome this problem by modifying them out of the cylinder $H_r(z_0)$ as follows. We set

$$\begin{aligned} \widetilde{a}(x,y,t) &:= \varphi^2(x) a(x,y,t), \\ \widetilde{b}(x,y,t) &:= \varphi(x)\left(a(x,y,t) + \varphi(x)\partial_x a(x,y,t) + b(x,y,t)\right), \end{aligned} \tag{4.32}$$

where

$$\varphi(x) = \begin{cases} 1/2 & \text{for } x \in]0,1/2], \\ x & \text{for } x \in]1/2, 3/2[, \\ 3/2 & \text{for } x \in [3/2, \infty[. \end{cases} \tag{4.33}$$

Then, it is easy to check that our assumption (4.15) and (4.16) on $\mathscr{L}$ imply the conditions on $\mathcal{K}$ for the validity of (4.30). In particular, our claim is proven for $z_0 = (1,0,0)$ and for every $r \in]0,1/2]$, since in this case $\mathscr{L}$ agrees with $\mathcal{K}$ in the cylinder $H_r(z_0)$.

An argument similar to that used above would give the proof of Proposition 4.2.1 with a constant M that may depend on z_0. In order to prove our claim as stated, with M independent on z_0, we rely on the left translation (4.7). As we apply the change of variable (4.8) to a solution u of $\mathscr{L}u = 0$ in $H_r(z_0)$, then v is a solution of $\mathscr{L}_{z_0} v = 0$ in $H_r(1,0,0)$ where $\mathscr{L}_{z_0}$ is defined in (4.18). Note that, as we have noticed in Remark 4.1.2, $\mathscr{L}_{z_0}$ satisfies assumptions (4.15) and (4.16), with the same constants used for $\mathscr{L}$. In particular, the Harnack inequality (4.30) holds for v, and implies

$$u(x,y,t) = v\left(\tfrac{x}{x_0}, \tfrac{y-y_0}{x_0}, t - t_0\right) \le M\, v(1,0,0) = M\, u(x_0,y_0,t_0),$$

for every $(x,y,t) \in S_{\theta r}(x_0,y_0,t_0)$. This concludes the proof. $\square$

As a direct consequence, we obtain the following

Corollary 4.2.2. *If u is a non negative solution of $\mathscr{L}u = 0$ in $H_r(z_0)$, where $0 < r \le 1/2$, then*

$$u(z) \le M\, u(z_0)$$

for every z in the set

$$\begin{aligned} \mathscr{P}_r(z_0) = \Big\{ (x,y,t) \in \mathbb{R}^3 : 0 < t_0 - t \le \theta^2 r^2, |x - x_0| \le (t_0 - t)^{\frac{1}{2}} x_0, \\ |y - y_0 - (t_0 - t)x_0| \le (t_0 - t)^{\frac{3}{2}} x_0 \Big\}. \end{aligned} \tag{4.34}$$

In the proof of our lower bound we will also use an estimate of a Green function for the operator defined in (4.29), that has been proved in [27]. We introduce here a simplified notation useful for our purpose. We first define a cylinder analogous to $H_r(z_0)$. For any $r, \delta \in]0, 1/2]$, we set

$$\begin{aligned} H_r^0(1,0,0) &= \Big\{(x,y,t) \in \mathbb{R}^3 : \tfrac{(x-1)^2}{r^2} + \tfrac{|x-1|}{r} + \tfrac{(y+t)^2}{r^6} < 1, 0 < t < r^2\Big\}, \\ H_{r,\delta}^+(1,0,0) &= \Big\{(x,y,t) \in \mathbb{R}^3 : \tfrac{(x-1)^2}{r^2} + \tfrac{|x-1|}{r} + \tfrac{(y+t)^2}{r^6} \le \delta, \tfrac{r^2}{2} \le t < r^2\Big\}, \\ S_{r,\delta}^0(1,0,0) &= \Big\{(x,y,t) \in \mathbb{R}^3 : \tfrac{(x-1)^2}{r^2} + \tfrac{|x-1|}{r} + \tfrac{y^2}{r^6} \le \delta, t = 0\Big\}. \end{aligned} \tag{4.35}$$

Note that $H_r^0(1,0,0) \subset \big\{1-r < x < r\big\}$. In particular, if we define $\widetilde{a}$ and $\widetilde{b}$ according to (4.32) and (4.33), then $\mathcal{K}$ agrees with $\mathscr{L}$ in the cylinder $H_r^0(1,0,0)$. Also note that the geometry of $H_r^0(1,0,0)$ is more complicated than the one of $H_r(1,0,0)$, due to the fact the the Dirichlet problem for $\mathcal{K}$ in (4.29) is well posed in $H_r^0(1,0,0)$.

In [27, Section 4] it is proven the existence of a *Green function* $G_r : \overline{H_r^0(1,0,0)} \times H_r^0(1,0,0) \to [0,+\infty[$ with the following property. For every $f \in C_0^\infty(H_r^0(1,0,0))$, the function

$$v_r(x,y,t) := \int_{H_r^0(1,0,0)} G_r(x,y,t;\xi,\eta,\tau) f(\xi,\eta,\tau) d\xi\, d\eta\, d\tau, \tag{4.36}$$

is a classical solution of the Dirichlet problem

$$\begin{cases} \mathscr{L}u = -f & \text{in } H_r^0(1,0,0), \\ u = 0 & \text{in } \partial(H_r^0(1,0,0)) \cap \big\{t < r^2\big\}. \end{cases} \tag{4.37}$$

Theorem 4.3 in [27] states that there exist positive δ and κ such that

$$G_r(x,y,t;\xi,\eta,\tau) \ge \frac{\kappa}{r^4},$$

for every $r \in]0,1/2], (x,y,t) \in H_{r,\delta}^+(1,0,0)$ and $(\xi,\eta,\tau) \in S_{r,\delta}^0(1,0,0)$. In particular, if we choose $0 < s < \frac{1}{4}$ and we set $r = \sqrt{s}$, we have

$$G_{\sqrt{s}}(1,-s,s;1,0,0) \ge \frac{\kappa}{s^2}, \quad \text{for every} \quad s \in]0,1/4[. \tag{4.38}$$

4.2.1 Harnack chains

Any $\mathscr{L}$-admissible path $\gamma(s) = (x(s), y(s), t(s))$ for $\mathscr{L}$ is the solution of the Cauchy problem

$$\begin{cases} \dot{x}(s) = \omega(s)x(s) & x(0) = x_0, \\ \dot{y}(s) = x(s) & y(0) = y_0, \\ \dot{t}(s) = -1, & t(0) = t_0, \end{cases} \tag{4.39}$$

where $\omega \in L^1([0, t_0 - t])$. In this setting, we refer to the function ω as the *control* of the problem (4.39).

We next compute the attainable set $\mathscr{A}_{(x_0,y_0,t_0)}(\mathbb{R}^+ \times \mathbb{R}^+ \times]0,T])$ introduced in (2.15). With this aim, we introduce the function

$$f : \mathbb{R} \to \mathbb{R}^+, \qquad f(r) = \frac{e^r - 1}{r}, \tag{4.40}$$

as $r \neq 0, f(0) = 1$ and we note that it is bijective. Moreover $f^{-1}(r) < 0$ only if $r < 1$ and $f^{-1}(r) > 0$ only if $r > 1$ and

$$\lim_{r \to 0} r f^{-1}(r) = -1, \qquad \lim_{r \to +\infty} \frac{f^{-1}(r)}{\log(r)} = 1. \tag{4.41}$$

Proposition 4.2.3. *For every $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T[$ it holds:*

$$\mathscr{A}_{(x_0,y_0,t_0)} =]0, +\infty[\times]y_0, +\infty[\times]0, t_0[. \tag{4.42}$$

PROOF. From (4.39) it follows that

$$\mathscr{A}_{(x_0,y_0,t_0)}(\mathbb{R}^+ \times \mathbb{R}^+ \times]0,T]) \subseteq]0, +\infty[\times]y_0, +\infty[\times]0, t_0[$$

In order to prove the opposite inclusion, we fix an ending point (x_1, y_1, t_1) belonging to $]0, +\infty[\times]y_0, +\infty[\times]0, t_0[$ and we show that there exists a piecewise constant control $\omega(s)$, with $s \in [0, t_0 - t_1]$, such that $\gamma(t_0 - t_1) = (x_1, y_1, t_1)$.

We first construct an admissible path γ such that $x(t_0 - t_1) = x_0$. We choose the control as follows

$$\begin{cases} \omega(s) = \omega_0, & 0 < s \leq \frac{t_0 - t_1}{2}; \\ \omega(s) = -\omega_0, & \frac{t_0 - t_1}{2} < s \leq t_0 - t_1. \end{cases} \tag{4.43}$$

for some constant ω_0 that will be specified in the sequel. Note that we have $\int_0^{t_0 - t_1} \omega(s) ds = 0$ so that $x(t_0 - t_1) = x_0$, and

$$\gamma(t_0 - t_1) = \left(x_0, y_0 + \frac{2}{\omega_0} x_0 \left(e^{\omega_0 \frac{t_0 - t_1}{2}} - 1 \right), t_1 \right)$$

We choose

$$\omega_0 := \frac{2}{t_0 - t_1} f^{-1} \left(\frac{y_1 - y_0}{(t_0 - t_1) x_0} \right), \tag{4.44}$$

and we obtain

$$y_1 = y_0 + \frac{2}{\omega_0} x_0 \left(e^{\omega_0 \frac{t_0 - t_1}{2}} - 1 \right),$$

then $\gamma(t_0 - t_1) = (x_0, y_1, t_1)$. In particular, our claim is proven in the case $x_1 = x_0$.

We adopt a similar criterium when $x_0 \neq x_1$. We consider two different cases.

i) If $y_1 > y_0 + \frac{t_0-t_1}{2}\left(\frac{x_1-x_0}{\log(x_1)-\log(x_0)}\right)$, we define the control ω as follows.

$$\omega(s) = \begin{cases} \omega_1, & 0 \le s \le \frac{t_0-t_1}{2}; \\ \omega_2, & \frac{t_0-t_1}{2} < s < \frac{3(t_0-t_1)}{4}; \\ -\omega_2, & \frac{3(t_0-t_1)}{4} < s \le t_0 - t_1. \end{cases} \tag{4.45}$$

We choose ω_1 in order to have $x\left(\frac{t_0-t_1}{2}\right) = x_1$. We recall that $x\left(\frac{t_0-t_1}{2}\right) = x_0 e^{\omega_1 \frac{t_0-t_1}{2}}$, and we set

$$\omega_1 = \frac{2}{t_0-t_1} \log\left(\frac{x_1}{x_0}\right). \tag{4.46}$$

With this choice of ω_1 we find

$$\gamma\left(\frac{t_0-t_1}{2}\right) = \left(x_1, y_0 + \frac{t_0-t_1}{2}\left(\frac{x_1-x_0}{\log(x_1)-\log(x_0)}\right), \frac{t_0+t_1}{2}\right). \tag{4.47}$$

Note that, by our assumption, we have $y_1 > y\left(\frac{t_0-t_1}{2}\right)$, then we can choose ω_2 arguing as in the previous case. Specifically we set

$$\omega_2 = \frac{4}{t_0-t_1} f^{-1}\left(\frac{2(y_1 - y\left(\frac{t_0-t_1}{2}\right))}{(t_0-t_1)x_1}\right), \tag{4.48}$$

and we obtain $\gamma(t_0-t_1) = (x_1, y_1, t_1)$.

ii) Suppose that $y_0 < y_1 \le y_0 + \frac{t_0-t_1}{2}\left(\frac{x_1-x_0}{\log(x_1)-\log(x_0)}\right)$. In order to accomplish the proof also in this case, we first assume that $x_1 > x_0$, and we introduce two auxiliary admissible paths $\bar{\gamma} = (\bar{x}, \bar{y}, \bar{t})$ and $\widetilde{\gamma} = (\widetilde{x}, \widetilde{y}, \widetilde{t})$ such that $\bar{\gamma}(0) = \widetilde{\gamma}(0) = (x_1, y_0, t_0)$. The control ω of $\bar{\gamma}$ is defined as follows

$$\begin{cases} \omega(s) = \bar{\omega}_1, & 0 < s \le \frac{t_0-t_1}{2}; \\ \omega(s) = -\bar{\omega}_1, & \frac{t_0-t_1}{2} < s \le t_0 - t_1. \end{cases} \tag{4.49}$$

with

$$\bar{\omega}_1 = \frac{2}{t_0-t_1} f^{-1}\left(\frac{y_1-y_0}{(t_0-t_1)x_1}\right). \tag{4.50}$$

Note that, with this choice of $\bar{\omega}_1$, we have $\bar{y}(t_0-t_1) = y_1$, then $\bar{\gamma}(t_0-t_1) = (x_1, y_1, t_1)$. The control ω of $\widetilde{\gamma}$ is defined as

$$\begin{cases} \omega(s) = -\omega_1, & 0 < s \le \frac{t_0-t_1}{2}; \\ \omega(s) = \omega_1, & \frac{t_0-t_1}{2} < s \le t_0 - t_1. \end{cases} \tag{4.51}$$

with ω_1 as in (4.46). Also note that $\widetilde{x}(\frac{t_0-t_1}{2}) = x_0$.

We claim that $\bar{x}(\frac{t_0-t_1}{2}) < x_0$. To prove our assertion, we note that

$$\bar{y}\left(\frac{t_0-t_1}{2}\right) = y_1 - \frac{t_0-t_1}{2}x_1 f\left(-\bar{\omega}_1\frac{t_0-t_1}{2}\right),$$

and, in view of (4.47)

$$\widetilde{y}\left(\tfrac{t_0-t_1}{2}\right) = y_0 + \tfrac{t_0-t_1}{2}x_1 f\left(-\omega_1\tfrac{t_0-t_1}{2}\right) = y_0 + \tfrac{t_0-t_1}{2}\left(\tfrac{x_1-x_0}{\log(x_1)-\log(x_0)}\right).$$

By our assumption, we have $y_1 < \widetilde{y}\left(\frac{t_0-t_1}{2}\right)$. Moreover $y_0 < \bar{y}\left(\frac{t_0-t_1}{2}\right)$, then we have $\widetilde{y}\left(\frac{t_0-t_1}{2}\right) - y_0 > y_1 - \bar{y}\left(\frac{t_0-t_1}{2}\right)$, that is equivalent to

$$\frac{t_0-t_1}{2}x_1 f\left(-\omega_1\frac{t_0-t_1}{2}\right) > \frac{t_0-t_1}{2}x_1 f\left(-\bar{\omega}_1\frac{t_0-t_1}{2}\right).$$

Since f is increasing, the above inequality yields $-\omega_1 > -\bar{\omega}_1$. We then conclude that.

$$\bar{x}\left(\frac{t_0-t_1}{2}\right) = x_1 e^{-\bar{\omega}_1\frac{t_0-t_1}{2}} < x_1 e^{-\omega_1\frac{t_0-t_1}{2}} = x_0.$$

Now we complete the construction of the admissible path steering (x_0, y_0, t_0) to (x_1, y_1, t_1). From the continuity of $\bar{x}(t)$, it follows that there exists $t_m \in \left]\frac{t_0-t_1}{2}, t_0-t_1\right[$ such that $\bar{x}(t_m) = x_0$. We set $\bar{y}(t_m) = y_m$ and we note that $\frac{y_1-y_0}{2} < y_m < y_1$. We define the control function ω as follows

$$\begin{cases} \omega(s) = -\bar{\omega}_2, & 0 < s < \frac{t_m}{2}; \\ \omega(s) = \bar{\omega}_2, & \frac{t_m}{2} < s < t_m; \\ \omega(s) = -\bar{\omega}_1, & t_m < s < t_0 - t_1. \end{cases} \tag{4.52}$$

where

$$\bar{\omega}_2 = \frac{2}{t_m} f^{-1}\left(\frac{y_m - y_0}{t_m x_0}\right). \tag{4.53}$$

This concludes the proof when $y_0 < y_1 < y_0 + \frac{t_0-t_1}{2}\left(\frac{x_1-x_0}{\log(x_1)-\log(x_0)}\right)$ and $x_1 > x_0$. The proof in the case $x_1 < x_0$ is analogous and will be omitted. □

Remark 4.2.4. *We note that, since we are able to solve the optimal control problem* (4.3) *in section 4.3, the inclusion* $]0,+\infty[\times]y_0,+\infty[\times]0,t_0[\supseteq \mathscr{A}_{(x_0,y_0,z_0)}$ *will simply follow once we have computed the optimal trajectories of* (4.3). *In this frame, a* handmade *construction of* smart *paths which connect a general point* $(x,y,t) \in]0,+\infty[\times]y_0,+\infty[\times]0,t_0[$ *with* (x_0,y_0,t_0) *is not required.*

As we will properly explain in section 4.3, we are able to exhibit the optimal $\mathscr{L}$*-admissible paths from* (x_0,y_0,t_0) *and we will see that they steer* (x_0,y_0,t_0) *to any given point* $(x,y,t) \in]0,+\infty[\times]y_0,+\infty[\times]0,t_0[$.

The following result provides a bound of any positive solution u of $\mathscr{L}u = 0$ at the end point $\gamma(t_0 - t)$ of an $\mathscr{L}$-admissible path γ.

Proposition 4.2.5. *There exist four positive constants θ, h, β and M, with $\theta < 1$ and $M > 1$, only depending on the operator $\mathscr{L}$ such that the following property holds.*

Let $T_0 < t < t_0 < T_1$ be fixed. Fix (x_0, y_0) and let $\omega \in L^1([t, t_0], \mathbb{R})$ be a control, with $\gamma : [t, t_0] \to \mathbb{R}^3$ the corresponding $\mathscr{L}$-admissible path of (4.39) *starting from (x_0, y_0, t_0). Denote with $(x, y, t) = \gamma(t_0)$ its end-point. Then, for every positive solution $u : \mathbb{R}^+ \times \mathbb{R} \times]T_0, T_1[$ of $\mathscr{L}u = 0$ it holds*

$$u(x, y, t) \leq \left(\tfrac{t - T_0}{t_0 - T_0}\right)^{\beta} M^{1 + \frac{\Phi(\omega)}{h} + \frac{4(t_0 - t)}{\theta^2}} u(x_0, y_0, t_0),$$

where

$$\Phi(\omega) = \int_t^{t_0} \omega^2(s)\, ds. \tag{4.54}$$

Proof. If $\omega \in L^1([t, t_0]) \setminus L^2([t, t_0])$, then our claim reads as $u(x, y, t) \leq +\infty$, that is clearly true. We now assume $\omega \in L^2([t, t_0])$. The proof of the proposition is based on the construction of a Harnack chain, by applying several times Corollary 4.2.2. We then first fix $\theta \in]0, 1[$ as in Corollary 4.2.2, and we also fix the constant $h = 4 \log^2(3/2)$.

Step 1. We fix three restrictive assumptions:

- it holds $t_0 - T_0 \leq \frac{1}{4}$;
- the path γ is defined on the time interval $[0, t_0 - t]$ with $t_0 - t \leq \theta^2(t_0 - T_0)$;
- the function $\Phi(\omega)$ satisfies $\Phi(\omega) \leq h$.

We first claim that, under such hypotheses, it holds

$$\gamma(t + s) \in \mathscr{P}_r(x_0, y_0, t_0) \quad \text{for every } s \in [0, t_0 - t], \tag{4.55}$$

with $r := \sqrt{t_0 - T_0} \leq \frac{1}{2}$. Indeed, Hölder inequality implies

$$\left| \int_t^{t+s} \omega(\tau) d\tau \right| \leq \sqrt{s} \left(\int_t^{t+s} \omega^2(\tau) d\tau \right)^{\frac{1}{2}} \leq \sqrt{h}\sqrt{s} \leq \log(1 + \sqrt{s}),$$

for every $s \in [0, t_0 - t] \subset [0, \frac{1}{4}]$. The last inequality follows from the concavity of $\log(1 + a)$, that implies $\log(1 + a) \geq 2\log(3/2)a$ for $a \in [0, 1/2]$ and from the definition of h. We then find

$$\left| e^{\int_t^{t+s} \omega(\tau)d\tau} - 1 \right| \leq e^{\left|\int_t^{t+s} \omega(\tau)d\tau\right|} - 1 \leq \sqrt{s}$$

for every $s \in [0, t_0 - t]$. Thus, integrating the system (4.39), we obtain

$$|x(s) - x_0| \leq \sqrt{s}x_0, \quad \text{and} \quad |y(s) - y_0 - sx_0| \leq \tfrac{2}{3}s^{\frac{3}{2}}x_0 < s^{\frac{3}{2}}x_0$$

for every $s \in [0, t_0 - t]$, and (4.55) is proven. Since $H_r(x_0, y_0, t_0) \subset \mathbb{R}^+ \times \mathbb{R}\times]T_0, T_1[$ for the definition of r, then Corollary 4.2.2 can be applied, and it holds $u(x, y, t) \leq Mu(x_0, y_0, t_0)$ with M given in Proposition 4.2.1.

Step 2. We now remove the three hypotheses of Step 1 and prove the main statement. Consider any control $\omega \in L^2([t, t_0])$ and the corresponding curve $\gamma(.)$. Define the sequence of times $t < t_k < t_{k-1} < \ldots < t_2 < t_1 < t_0$ recursively starting from t_0 as follows

$$t_{j+1} = \max\left\{t,\ t_j - \theta^2/4, t_j - \theta^2(t_j - T_0),\ \inf\left\{s \ \text{ s.t. } \int_s^{t_j} |\omega(\tau)|^2\, d\tau \leq h\right\}\right\}. \tag{4.56}$$

The recursive formula terminates when the lower boundary t is reached. For simplicity of notation, we denote $t_{k+1} = t$.

We now define $r_j = \sqrt{t_j - t_{j+1}}/\theta$, then we note that $r_j \leq 1/2$ and

$$H_{r_j}(x(t_0 - t_j), y(t_0 - t_j), t_j) \subset \mathbb{R}^+ \times \mathbb{R} \times [T_0, T_1],$$

by (4.56). Moreover, we clearly have $t_j - t_{j+1} \leq \theta^2 r_j^2$. By applying Step 1 on the $k + 1$ intervals $[t_{j+1}, t_j]$, it holds

$$u(x, y, t) \leq M^{1+k}u(x_0, y_0, t_0).$$

We point out that the points $(x(t_0 - t_j), y(t_0 - t_j), t_j)$, $j = 1, \ldots k + 1$, selected on the path $\gamma(.)$, form a Harnack chain. Since (4.56) implies

$$k \leq \frac{\int_t^{t_0} |\omega(\tau)|^2 d\tau}{h} + 4\frac{t_0 - t}{\theta^2} + \frac{1}{|\log(1 - \theta^2)|}\log\left(\tfrac{t-T_0}{t_0-T_0}\right),$$

this concluses the proof of Proposition 4.2.5, by setting $\beta := \frac{\log(M)}{|\log(1-\theta^2)|}$. □

Remark 4.2.6. *Even if $\mathscr{L}$ does not write in the form* (2.7)*, the lower bound in Proposition 4.2.5 basically depends on γ, that in turns depends on the vector fields X and Y that define $\mathscr{L}_0$. This feature depends on the fact that γ is contained in the set $\mathscr{P}_r(z_0)$, where the Harnack inequality holds for both operators $\mathscr{L}_0$ and $\mathscr{L}$.*

According to the paths constructed in Proposition 4.2.3, which rely on pathwise constant controls, we are able to compute explicitly their costs $\Phi(\omega)$.

Proposition 4.2.7. *According to the controls in* (4.45) *and* (4.52)*, it holds the following equalities for the function $\Phi(\omega)$ in* (4.54)*.*

i) if $x_0 \neq x_1$, and

$$\frac{2(y_1 - y_0)}{t_0 - t_1} > \frac{x_1 - x_0}{\log(x_1) - \log(x_0)} + x_1, \tag{4.57}$$

then

$$\Phi(x_0, y_0, t_0, x_1, y_1, t_1) = \tfrac{2}{t_0 - t_1} \left(\log^2 \left(\tfrac{x_1}{x_0} \right) + 8 \log^2 \left(\tfrac{2(y_1 - y_0)}{(t_0 - t_1) x_1} - \tfrac{x_1 - x_0}{x_1 (\log(x_1) - \log(x_0))} \right) \right).$$

If $x_0 = x_1$, and $\frac{y_1 - y_0}{t_0 - t_1} > x_0$, then

$$\Phi(x_0, y_0, t_0, x_0, y_1, t_1) = \tfrac{8}{t_0 - t_1} \log^2 \left(\tfrac{y_1 - y_0}{(t_0 - t_1) x_0} \right).$$

Here C_1 is a positive constant depending only on f.

ii) if $x_0 \neq x_1$, and

$$0 < \frac{2(y_1 - y_0)}{t_0 - t_1} < x_0, \tag{4.58}$$

then

$$\Phi(x_0, y_0, t_0, x_1, y_1, t_1) = 16 \frac{(x_0^2 + x_1^2)(t_0 - t_1)}{(y_1 - y_0)^2}.$$

The same estimate holds if $x_0 = x_1$, and $\frac{y_1 - y_0}{t_0 - t_1} < x_0$.

PROOF. We give here an explicit bound of $\Psi(x_0, y_0, t_0, x_1, y_1, t_1)$ based on the paths considered in Proposition 4.2.3, and on the asymptotic behavior of f^{-1} given in (4.41).

i) If $x_0 \neq x_1$, and (4.57) holds, then

$$\Phi(x_0, y_0, t_0, x_1, y_1, t_1) = \frac{t_0 - t_1}{2} \left(\omega_1^2 + \omega_2^2 \right).$$

where ω_1 and ω_2 are defined in (4.46) and (4.48), respectively. Note that, by (4.57) and (4.47), we have $\omega_2 > 0$. The conclusion then follows form the second assertion (4.41). The proof in the case $x_0 = x_1$ is analogous.

ii) If $x_0 \neq x_1$, and (4.58) holds, we consider the path defined by the control function introduced in (4.52). We then find

$$\Phi(x_0, y_0, t_0, x_1, y_1, t_1) = t_m \bar{\omega}_2^2 + (t_0 - t_1 - t_m)\,\bar{\omega}_1^2 \leq (t_0 - t_1)\,\bar{\omega}_2^2 + \left(\tfrac{t_0 - t_1}{2}\right)\bar{\omega}_1^2.$$

We note that the control $\bar{\omega}_1$ and $\bar{\omega}_2$ defined in (4.50) and (4.53) respectively, are both negative. The conclusion of the proof then follows from the first assertion (4.41). □

4.3 Application of the Pontryagin Maximum Principle

In this section we apply the Pontryagin Maximum Principle to our problem (4.3). Note that the ending point of the $\mathscr{L}$-admissible path considered in (4.3) is $(1, 0, 0)$, we give here the formulation for any end-point $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$. In accordance with the notation used for the fundamental solution of $\mathscr{L}$, we denote the starting point of the path by $(x_1, y_1, t_1) \in \mathbb{R}^+ \times \mathbb{R}^2$, with $t_1 > t_0$.

$$\begin{cases} \dot{x}(s) = \omega(s)x(s) & \\ \dot{y}(s) = x(s) & 0 \leq s \leq T, \\ \dot{t}(s) = -1, & \end{cases} \tag{4.59}$$
$$(x, y, t)(0) = (x_1, y_1, t_1), \qquad (x, y, t)(T) = (x_0, y_0, t_0).$$

where $T = t_1 - t_0$. We first observe that such optimal control problem is invariant on the Lie group $\mathbb{R}^+ \times \mathbb{R}^2$ endowed with the operation (4.7). We recall that optimal control problems on Lie group with invariant vector fields satisfy useful invariance properties, that permit to have simpler solutions of the Pontryagin Maximum Principle, eventually leading to complete synthesis for specific problems, see e.g. [17]. In our specific problem, it is sufficient to observe the following invariance property for the solution of (4.59). Consider a control $\omega(\cdot)$ steering (x_1, y_1, t_1) to (x_0, y_0, t_0) with the trajectory $(x(s), y(s), t(s))$. Then the same control $\omega(.)$ steers $(x_0, y_0, t_0)^{-1} \circ (x_1, y_1, t_1)$ to $(1, 0, 0)$. This can be proved by observing that the trajectory $(x_0, y_0, t_0)^{-1} \circ (x(s), y(s), t(s))$ is a solution of (4.59) with the same control $\omega(.)$. Since the cost depends on the control only, then the two trajectories have the same cost, hence

$$\Psi(x_1, y_1, t_1; x_0, y_0, t_0) = \Psi((x_0, y_0, t_0)^{-1} \circ (x_1, y_1, t_1); 1, 0, 0). \tag{4.60}$$

As a consequence, we will now fix the final condition $(x_0, y_0, t_0) = (1, 0, 0)$ in the optimal control problem (4.59), then using the invariance property to solve it with a general initial condition.

The constraint $\dot{t} = -1$ implies that $\mathscr{L}$-admissible paths satisfy $t(s) = t_1 - s$, hence $T = t_1 - t_0$. Then, in the sequel we drop the time variable, we set $T := t_1 - t_0$, and we denote

$$\Psi(x_1, y_1, t_1; x_0, y_0, t_0) = \inf_{\omega \in L^1([0,T])} \int_0^T \omega^2(\tau) d\tau, \tag{4.61}$$

where $\omega \in L^1([0, T])$ is such that (4.59) holds true.

For the above reasons, the optimal control problem (4.59), (2.38) now reads as follows:

$$\Psi(x_1, y_1, t_1; 1, 0, 0) = \min_{\omega \in L^1([0,t_1])} \int_0^{t_1} \omega^2(\tau) d\tau \quad \text{subject to constraint} \tag{4.62}$$

$$\begin{cases} \dot{x}(s) = \omega(s)x(s), & x(0) = x_1, \quad x(t_1) = 1, \\ \dot{y}(s) = x(s), & y(0) = y_1, \quad y(t_1) = 0. \end{cases}$$

To simplify the notation, in the sequel we agree to set

$$\Psi(x_1, y_1, t_1) := \Psi(x_1, y_1, t_1; 1, 0, 0).$$

We now solve this problem. As a by-product, we show that we can always steer (x_1, y_1) to (x_0, y_0) in time T, when $y_1 < y_0$. This implies that there exists a control ω steering (x_1, y_1, t_1) to $\gamma(t_1 - t_0) = (x_0, y_0, t_0)$, as we pointed out in Remark 4.2.4.

We now apply the Pontryagin Maximum Principle to problem (4.62). The Hamiltonian of the problem (4.62) is

$$H(x, y, \lambda_1, \lambda_2, p_0, \omega) = \lambda_1 x\omega + \lambda_2 x + p_0\omega^2, \tag{4.63}$$

where (λ_1, λ_2) are the coordinates of the covector λ.

We first remark that Problem (4.62) admits no abnormal extremals. Indeed, assume by contradiction $p_0 = 0$ in (4.63). Then

$$H(x, y, \lambda_1, \lambda_2, p_0, \omega) = \lambda_1 x\omega + \lambda_2 x$$

Recall that $x > 0$. Hence, the maximization of the Hamiltonian is equivalent to

$$\frac{\partial H}{\partial \omega}(x, y, \lambda_1, \lambda_2, p_0, \omega) = 0 \quad \Rightarrow \quad \lambda_1(s) = 0, \quad \forall s \in [0, t_1].$$

Moreover, using the fact that $\lambda_1(s) = 0$ for all $s \in [0, t_1]$, it holds

$$\dot{\lambda}_1(s) = -\frac{\partial H}{\partial x}(x, y, \lambda_1, \lambda_2, p_0, \omega) = -\lambda_1(s)\omega(s) - \lambda_2(s) = 0,$$

hence $\lambda_2(s) = 0$ for every $s \in [0, t_1]$. We conclude that

$$(\lambda_1(s), \lambda_2(s), p_0) = (0, 0, 0) \quad \text{for every} \quad s \in [0, t_1].$$

This is in contradiction with the fact that $(\lambda_1(s), \lambda_2(s), p_0)$ is always non-vanishing.

Since no abnormal extremals occur, we choose $p_0 = -\frac{1}{2}$. We then compute the optimal control as the unique minimizer of $H\left(x, y, \lambda_1, \lambda_2, -\frac{1}{2}, \omega\right)$, that is

$$\omega(s) = \lambda_1(s)x(s), \tag{4.64}$$

and the maximized Hamiltonian is

$$H^*(x, y, \lambda_1, \lambda_2, p_0) = \frac{1}{2}\lambda_1^2 x^2 + \lambda_2 x. \tag{4.65}$$

The corresponding Hamiltonian system reads as

$$\begin{cases} \dot{x}(s) = \lambda_1(s)x^2(s) \\ \dot{y}(s) = x(s) \\ \dot{\lambda}_1(s) = -\lambda_1^2(s)x(s) - \lambda_2(s) \\ \dot{\lambda}_2(s) = 0 \end{cases} \tag{4.66}$$

In the sequel, we choose the parameters

$$k := \lambda_1(t_1) \qquad \text{and} \qquad c := \lambda_2(t_1)$$

as the final condition for each extremal, that is uniquely determined by being the solution of (4.66) with final condition $(x, y, \lambda_1, \lambda_2)(t_1) = (1, 0, k, c)$. Note that, by the last equation in (4.66), we have $\lambda_2(s) = c$ for every $s \in [0, t_1]$. Furthermore, the value of the Hamiltonian is a constant of motion, fixed by the final data. From now on, we then fix

$$E := \lambda_1^2(s)x^2(s) + 2\lambda_2(s)x(s) = k^2 + 2c. \tag{4.67}$$

Moreover, by recalling the explicit expression for the optimal control (4.64) and $\dot{y} = x$, we have the following expression of the cost for extremals:

$$C(\omega(\,\cdot\,)) = \int_0^{t_1} \omega(s)^2\, ds = \int_0^{t_1} \lambda_1^2(s)x^2(s)\, ds = \int_0^{t_1} (E - 2c\dot{y}(s))\, ds = Et_1 + 2cy_1. \tag{4.68}$$

We now describe the explicit solutions to (4.66), as a function of the final value of the Hamiltonian $E = k^2 + 2c$. For simplicity, we consider the space variable (x, y) only. We have three cases:

1. **$E = 0$:** it holds $(x(s), y(s)) = \left(\frac{4}{(k(t_1-s)+2)^2}, -\frac{2(t_1-s)}{k(t_1-s)+2}\right)$;

2. **$E > 0$:** it holds

$$x(s) = \frac{E}{\left(\sqrt{E}\cosh\left(\frac{t_1-s}{2}\sqrt{E}\right) + k\sinh\left(\frac{t_1-s}{2}\sqrt{E}\right)\right)^2},$$
$$y(s) = \frac{-2\sinh\left(\frac{t_1-s}{2}\sqrt{E}\right)}{\sqrt{E}\cosh\left(\frac{t_1-s}{2}\sqrt{E}\right) + k\sinh\left(\frac{t_1-s}{2}\sqrt{E}\right)};$$

3. **$E < 0$:** it holds

$$x(s) = \frac{-E}{\left(\sqrt{-E}\cos\left(\frac{t_1-s}{2}\sqrt{-E}\right) + k\sin\left(\frac{t_1-s}{2}\sqrt{-E}\right)\right)^2},$$
$$y(s) = \frac{-2\sin\left(\frac{t_1-s}{2}\sqrt{-E}\right)}{\sqrt{-E}\cos\left(\frac{t_1-s}{2}\sqrt{-E}\right) + k\sin\left(\frac{t_1-s}{2}\sqrt{-E}\right)},$$

   where the trajectory is defined on the whole time interval $s \in [0, t_1]$ when $E > -\frac{\pi^2}{T^2}$ only.

The three cases can be unified by using the function g defined in (4.22) and observing that it always holds

$$y(s) = -g\left(\frac{E(t_1-s)^2}{4}\right)(t_1 - s)\sqrt{x(s)}. \tag{4.69}$$

We are now ready to prove the invariance properties of Ψ.

Proof of Proposition 4.1.1. The proof of (4.13) is a direct consequence of (4.60). In order to prove (4.14) we introduce another symmetry of the problem. Consider an extremal of (4.62) steering (x, y) to $(1, 0)$ in time t, with a final covector parametrized by (k, c), hence with Hamiltonian $E = k^2 + 2c$ and cost $C = ET + 2cy_1$. Fix now $r > 0$: the extremal ending to $(1, 0)$ with final covector (rk, r^2c) steers $\left(x, \frac{y}{r}\right)$ to $(1, 0)$ in time $\frac{t}{r}$. Moreover, the Hamiltonian is r^2E and the cost is $r\,C$. The proof is a direct consequence of the explicit expression of solutions of (4.66). As a consequence, a trajectory parametrized by (k, c) steering (x, y) to $(1, 0)$ in time t is optimal if and only if the trajectory parametrized by (rk, r^2c) steering $\left(x, \frac{y}{r}\right)$ to $(1,0)$ in time $\frac{t}{r}$ is optimal too. Combining this with (4.60) we get the property

$$\begin{aligned}\Psi(x_1,y_1,t_1;x_0,y_0,t_0) &= \Psi\left(\tfrac{x_1}{x_0}, \tfrac{y_1-y_0}{x_0}, t_1-t_0; 1,0,0\right) \\ &= \tfrac{1}{r}\Psi\left(\tfrac{x_1}{x_0}, \tfrac{y_1-y_0}{rx_0}, \tfrac{t_1-t_0}{r}; 1,0,0\right) = \tfrac{1}{r}\Psi\left(x_1, \tfrac{y_1}{r}, \tfrac{t_1}{r}; x_0, \tfrac{y_0}{r}, \tfrac{t_0}{r}\right).\end{aligned} \tag{4.70}$$

This proves (4.14) in Proposition 4.1.1. □

In view of (4.13) and (4.14), with no loss of generality, from now on we consider the problem of steering (x_1, y_1) to $(1,0)$ with fixed final time $t_1 = 2$. First observe that, since g is a C^∞, strictly increasing function, from (4.69) we find the unique value for the prime integral E for which it holds $(x(0), y(0)) = (x_1, y_1)$, that is

$$E = \frac{4}{t_1^2} g^{-1}\left(-\frac{y_1}{t_1\sqrt{x_1}}\right) = g^{-1}\left(-\frac{y_1}{2\sqrt{x_1}}\right). \tag{4.71}$$

It also clearly gives the basic relation $c = \frac{E-k^2}{2}$, hence c is uniquely determined by k. Then, the cost of the corresponding extremal is

$$C = 2E + y_1(E-k^2) = (2+y_1)E - y_1k^2. \tag{4.72}$$

We now compute the value of k by imposing the initial condition on the second component only, i.e. $y(0) = y_1$. It holds:

- for $y_1 = -2\sqrt{x_1}$, the unique extremal satisfying $y(0) = y_1$ has final covector $k = -\frac{y_1+2}{y_1}$ and the optimal cost is $C = \frac{(y_1+2)^2}{y_1}$.
- for $y_1 > 2\sqrt{x_1}$, the unique extremal satisfying $y(0) = y_1$ has final covector

  $$k = -\sqrt{E}\left(\coth(\sqrt{E})\right) - \tfrac{2}{y_1} = \frac{\sqrt{Ey_1^2 + 4x_1} - 2}{y_1}$$

  and the optimal cost is $C = 2\frac{Ey_1 - 2x_1 - 2 + 2\sqrt{4x_1 + Ey_1^2}}{y_1}$.
- for $y_1 < 2\sqrt{x_1}$, the unique extremal satisfying $y(0) = y_1$ has final covector

  $$k = -\sqrt{-E}\left(\cot(\sqrt{-E})\right) - \tfrac{2}{y_1}$$

  Since $-\pi^2 < E < 0$, we find

  $$\begin{cases} k = \frac{\sqrt{Ey_1^2+4x_1}-2}{y_1}, & \text{if } -\pi^2/4 \leq E < 0; \\ k = -\frac{\sqrt{Ey_1^2+4x_1}+2}{y_1}, & \text{if } -\pi^2 < E < -\pi^2/4, \end{cases}$$

and the expression of the optimal cost is

$$\begin{cases} C = 2\frac{Ey_1-2x_1-2+2\sqrt{4x_1+Ey_1^2}}{y_1}, & \text{if } -\pi^2/4 \le E < 0; \\ C = 2\frac{Ey_1-2x_1-2-2\sqrt{4x_1+Ey_1^2}}{y_1}, & \text{if } -\pi^2 < E < -\pi^2/4. \end{cases}$$

In conclusion, we have that the unique extremal satisfying $y(0) = y_1$ has final covector

$$\begin{cases} k = \frac{\sqrt{Ey_1^2+4x_1}-2}{y_1}, & \text{if } E \ge -\pi^2/4; \\ k = -\frac{\sqrt{Ey_1^2+4x_1}+2}{y_1}, & \text{if } -\pi^2 < E < -\pi^2/4, \end{cases} \tag{4.73}$$

and the optimal cost is

$$\begin{cases} C = 4\frac{\frac{E}{2}y_1-x_1-1+\sqrt{4x_1+g^{-1}\left(-\frac{y_1}{2\sqrt{x1}}\right)y_1^2}}{y_1}, & \text{if } E \ge -\pi^2/4; \\ C = 4\frac{\frac{E}{2}y_1-x_1-\sqrt{4x_1+g^{-1}\left(-\frac{y_1}{2\sqrt{x1}}\right)y_1^2}}{y_1}, & \text{if } -\pi^2 < E < -\pi^2/4. \end{cases} \tag{4.74}$$

We are now left to prove that, with the previous choice of k, one also has $x(0) = x_1$ and $x(t_1) = 1$. With this goal, it is sufficient to observe the following interesting geometric feature of solutions of (4.66): the quantity $\lambda_1(s)x(s)+\lambda_2(s)y(s)$ is another constant of motion for (4.66), whose value set at $s = t_1$ is k. Merging this information with (4.67), we have

$$2cx(s) = E - (k - cy(s))^2$$

for all points $(x(s), y(s))$ of the solution of (4.66). In other terms, the trajectory $(x(s), y(s))$ always belongs to the parabola

$$x(s) = -\frac{c}{2}y^2(s) + ky(s) + 1.$$

Then, when the trajectory reaches $y(0) = y_1$ and $t_1 = 2$, it holds

$$x(0) = \frac{k^2 - E}{4}y_1^2 + ky_1 + 1 = x_1, \tag{4.75}$$

by plugging the explicit expression (4.73) of k.

Summing up, the optimal trajectory steering (x_1, y_1) to $(1, 0)$ in time $t_1 = 2$ is the unique solution of (4.66) with final covector $(k, \frac{k^2-E}{2})$, where k and E are given by (4.73) and (4.71). We next prove Proposition 4.1.4 by applying the symmetry inverse transformations (4.13) and (4.14).

Proof of Proposition 4.1.4. By (4.14) with $r = \frac{t_1-t_0}{2}$ we find

$$\Psi(x_1, y_1, t_1; x_0, y_0, t_0) = \frac{2}{t_1-t_0}\Psi\left(\frac{x_1}{x_0}, \frac{2(y_1-y_0)}{x_0(t_0-t_1)}, 2; 1, 0, 0\right).$$

Moreover, the Hamiltonian of the optimal trajectory of (4.66) corresponding to the right hand side of the above equation is $\frac{(t_1-t_0)^2}{4}E$, where E is the Hamiltonian of the optimal trajectory steering (x_1, y_1, t_1) to (x_0, y_0, t_0). From (4.71) we obtain $\frac{(t_1-t_0)^2}{4}E = g^{-1}\left(\frac{y_0-y_1}{(t_1-t_0)\sqrt{x_0 x_1}}\right)$, that gives (4.24). By using the first expression in (4.74) of the $\Psi\left(\frac{x_1}{x_0}, \frac{2(y_1-y_0)}{x_0(t_1-t_0)}, 2; 1, 0, 0\right)$, we obtain

$$\begin{aligned}\Psi(x_1, y_1, t_1; x_0, y_0, t_0) = & \frac{2}{t_1-t_0} 4\Bigg(\tfrac{(t_1-t_0)^2}{4}\tfrac{E}{2} \cdot \tfrac{2(y_1-y_0)}{x_0(t_0-t_1)} - \tfrac{x_1}{x_0} - 1 + \\ & + \sqrt{4\tfrac{x_1}{x_0} + \tfrac{(t_1-t_0)^2}{4}E\left(\tfrac{2(y_1-y_0)}{x_0(t_1-t_0)}\right)^2}\Bigg) \frac{x_0(t_1-t_0)}{2(y_1-y_0)},\end{aligned}$$

which, recalling that $y_0 > y_1$, agrees with (4.23). The proof of the second one is analogous.

In order to prove (4.25), we claim that, for every $\varepsilon \in]0, 1[$ there exists a positive E_ε such that

$$\frac{4}{(t_1-t_0)^2}\log^2\left(\tfrac{y_0-y_1}{(t_1-t_0)\sqrt{x_0x_1}}\right) < E < \frac{4}{(1-\varepsilon)^2(t_1-t_0)^2}\log^2\left(\tfrac{y_0-y_1}{(t_1-t_0)\sqrt{x_0x_1}}\right), \quad (4.76)$$

where E is the function defined in (4.71), for every $E > E_\varepsilon$. To prove the claim, we fix $\varepsilon \in]0, 1[$ and we note that

$$\exp((1-\varepsilon)x) < \frac{\sinh(x)}{x} < \exp(x), \quad (4.77)$$

for every sufficiently large positive x. Recalling (4.24), since $\frac{y_0-y_1}{(t_1-t_0)\sqrt{x_0x_1}} \to +\infty$, we consider $g(r)$ in (4.22) with $r > 0$. Then, from (4.77) it follows that

$$\exp\left(\frac{(1-\varepsilon)(t_1-t_0)\sqrt{E}}{2}\right) < \frac{y_0-y_1}{(t_1-t_0)\sqrt{x_0x_1}} < \exp\left(\frac{(t_1-t_0)\sqrt{E}}{2}\right),$$

for any positive E big enough. This proves (4.76). Moreover, for E big enough, we have, for every arbitrary $\varepsilon > 0$

$$0 \le \frac{4x_1x_0}{(y_0-y_1)^2} = \frac{E}{\sinh^2\left(\frac{(t_1-t_0)}{2}\sqrt{E}\right)} < \varepsilon. \quad (4.78)$$

We next consider the value function Ψ as a function of $\frac{y_0-y_1}{(t_1-t_0)\sqrt{x_1x_0}}$. From the first expression in (4.23) and (4.78), we obtain the following inequality

$$\Psi(x_1, y_1, t_1; x_0, y_0, t_0) \le \frac{4}{(1-\varepsilon)^2(t_1-t_0)}\log^2\left(\frac{y_0-y_1}{(t_0-t)\sqrt{x_0x_1}}\right) + \frac{4(x_1+x_0)}{y_0-y_1}$$

for every $E > E_\varepsilon$. On the other hand, modifying if necessary the choice of E_ε, we also have

$$\Psi(x_1, y_1, t_1; x_0, y_0, t_0) \geq \frac{4(1-\varepsilon)^2}{(t_1 - t_0)} \log^2\left(\frac{y_0 - y_1}{(t_1 - t_0)\sqrt{x_0 x_1}}\right) + \frac{4(x_1 + x_0)}{y_0 - y_1} - 2\varepsilon$$

for every $E > E_\varepsilon$. This concludes the proof of (4.25).

The proof of (4.26) is easier. It suffices to note that since, $\frac{y_0 - y_1}{(t_1 - t_0)\sqrt{x_1 x_0}} \to 0$, we consider $g(r)$ in (4.22) with $r < 0$, then $E \to -\frac{4\pi^2}{(t_1 - t_0)^2}$. From the second expression in (4.23) we have

$$\lim_{E \to -\frac{4\pi^2}{(t_1 - t_0)^2}} \frac{\Psi(x_1, y_1, t_1; x_0, y_0, t_0)}{\frac{4(x_1 + x_0) + 4\sqrt{4x_1 x_0}}{y_0 - y_1} - \frac{4\pi^2}{(t_1 - t_0)}} = 1.$$

□

4.3.1 Lower Estimates for Fundamental Solution

In this section we give the proof of the lower bound in Theorem 4.1.3 for a preliminary choice of the pole $z_0 = (x_0, y_0, t_0) = (1, 0, 0)$. We postpone the general case at the end of section 4.4. We first prove the following

Lemma 4.3.1. *There exists a positive constant κ such that*

$$\Gamma(1, -t, t; 1, 0, 0) \geq \frac{\kappa}{t^2},$$

for every $t \in]0, 1/4]$.

PROOF. We claim that, for every $r \in]0, 1/2]$ we have

$$\Gamma(x, y, t; \xi, \eta, \tau) \geq G_r(x, y, t; \xi, \eta, \tau),$$

for every $(x, y, t; \xi, \eta, \tau) \in \overline{H_r^0(1, 0, 0)} \times H_r^0(1, 0, 0)$, where $G_r(x, y, t; \xi, \eta, \tau)$ is the Green function appearing in (4.36). The proof of Lemma 4.3.1 then follows from (4.38).

In order to prove our claim, we fix $r \in]0, 1/2]$. For every non-negative $f \in C_0^\infty(H_r^0(1, 0, 0))$ and for every $(x, y, t) \in \overline{H_r^0(1, 0, 0)}$ we set

$$v_f(x, y, t) := \int_{H_r^0(1,0,0)} G_r(x, y, t; \xi, \eta, \tau) f(\xi, \eta, \tau) d\xi\, d\eta\, d\tau,$$
$$u_f(x, y, t) := \int_{H_r^0(1,0,0)} \Gamma(x, y, t; \xi, \eta, \tau) f(\xi, \eta, \tau) d\xi\, d\eta\, d\tau.$$

Both v_f and u_f are solution of $\mathscr{L}u = -f$ in $H^0_r(1,0,0)$. Moreover $u_f(x,y,t) \geq 0$ for every $(x,y,t) \in \partial(H^0_r(1,0,0)) \cap \{t < r^2\}$. From (4.37) and from the comparison principle we then find $u_f \geq v_f$ in $H^0_r(1,0,0)$. In other words, we have

$$\int_{H^0_r(1,0,0)} \big(\Gamma(x,y,t;\xi,\eta,\tau) - G_r(x,y,t;\xi,\eta,\tau)\big) f(\xi,\eta,\tau) d\xi\, d\eta\, d\tau \geq 0,$$

for every non-negative $f \in C^\infty_0(H^0_r(1,0,0))$ and for every $(x,y,t) \in \overline{H^0_r(1,0,0)}$. This proves our claim. □

We next state and prove the main result of this section.

Proposition 4.3.2. *Let $0 < \varepsilon < \frac{1}{4T}$ be fixed arbitrarily. There exists a positive constant $c^-_{\varepsilon,T}$ only depending on the operator $\mathscr{L}$, on ε and on T such that for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ with $y < -\varepsilon t$ it holds*

$$\Gamma\left(x,y,t;1,0,0\right) \geq \frac{c^-_{\varepsilon,T}}{t^2} \exp\left(-C\Psi(x,y+\varepsilon t, t-\varepsilon t; 1,0,0)\right). \tag{4.79}$$

Proof. Let $\varepsilon \in]0, \frac{1}{4T}[$ be fixed, by Proposition 4.2.5 and Lemma 4.3.1 we have

$$\begin{aligned}\Gamma(x,y,t;1,0,0) &\geq \varepsilon^{-\beta} M^{-1-\frac{4(1-\varepsilon)t}{\theta^2} - \frac{\Psi(x,y,t;1,-\varepsilon t,\varepsilon t)}{h}} \Gamma(1,-\varepsilon t, \varepsilon t; 1,0,0) \\ &\geq \varepsilon^{-\beta} M^{-1-\frac{4T}{\theta^2} - \frac{\Psi(x,y,t;1,-\varepsilon t,\varepsilon t)}{h}} \frac{\kappa}{(\varepsilon t)^2},\end{aligned} \tag{4.80}$$

for every $(x,y,t) \in \mathbb{R}^+ \times \mathbb{R} \times]0,T]$ with $y < -\varepsilon t$. This proves (4.79) for $(x_0,y_0,t_0) = (1,0,0)$, with $c^-_{\varepsilon,T} = \frac{\kappa}{\varepsilon^{2+\beta}} M^{-1-\frac{4T}{\theta^2}}$.

Remark 4.3.3. *The presence of ε is necessary to keep the constant $c^-_{\varepsilon,T}$ in (4.79) bounded. Therefore, we are led to consider a smaller time range $(1-\varepsilon)(t-t_0)$ than $t-t_0$. Moreover, according to the fact that*

$$x(s) > 0, \quad y(s) = y_0 + \int_{t_0}^{s} x(\tau) d\tau, \quad s \in [0,T],$$

the parameter ε appears also in the spatial component y of Ψ.

4.4 Upper Bound and Proof of the Main Theorem

In this section we prove the Theorem 4.1.3. For the scopes of this section it is more convenient to write $\mathscr{L}$ in its divergence form

$$\mathscr{L}u = -X^*(aXu) + (b-a)Xu + Yu, \tag{4.81}$$

where

$$Xu(x,y,t) := x\partial_x u(x,y,t), \quad X^*u(x,y,t) := -Xu(x,y,t) - u(x,y,t),$$
$$Yu(x,y,t) = x\partial_y u(x,y,t) - \partial_t u(x,y,t). \tag{4.82}$$

We first prove the upper bound in (4.2) for the fundamental solution of $\mathscr{L}$, which in turns is the main result of this section:

Proposition 4.4.1 (Upper Bound). *Let T_0, T_1 be fixed and consider the set $\mathbb{R}^+ \times \mathbb{R} \times]T_0, T_1[$. Let $\mathscr{L}$ be the operator in (4.84), and let $\Gamma(x,y,t;1,0,0)$ its fundamental solution. Denoting by M_1 the L^∞-norm of $a(x,y,t)$ and $T = T_1 - T_0$, then for every positive ε, there exists a positive constant C_ε^+, depending on the vector fields X, Y, on ε, T and on the L^∞-norm of $a(x,y,t)$ such that*

$$\Gamma(x,y,t;1,0,0) \leq \frac{C_\varepsilon^+}{t^2} \exp\Big(-\tfrac{1}{16M_1}\Psi(x, y-\varepsilon, t+\varepsilon; 1,0,0)\Big) \tag{4.83}$$

for every $(x,y,t) \in \mathbb{R}^+ \times]-\infty, 0[\times]0, T]$.

We prove Proposition 4.4.1 by requiring less restrictive regularity assumptions on the coefficients than the ones needed for the analogous lower bounds. In particular, in this setting, we only need that Γ is a *distributional solution* of $\mathscr{L}u = 0$.

To achieve the proof of Proposition 4.4.1, we need to introduce some preliminary results on non-negative weak solution u to $\mathscr{L}u = 0$ in $\mathbb{R}^+ \times \mathbb{R} \times]T_0, T_1[$ and on non-negative weak solution u to its formal adjoint $\mathscr{L}^*u = 0$ in $\mathbb{R}^+ \times \mathbb{R} \times]T_0, T_1[$.
For this reason, we consider operators with a *zero order term*, namely

$$\mathscr{L}_1 u(x,y,t) = -X^*\big(aXu\big) + (b-a)Xu + cu + Yu. \tag{4.84}$$

with the notation used in (4.82). Clearly, $\mathscr{L}$ is the particular case of $\mathscr{L}_1$ that we obtain with $c = 0$. With the same notation, its formal adjoint $\mathscr{L}_1^*$ is

$$\mathscr{L}_1^* u(x,y,t) = -X^*\big(aXu\big) + X^*\big((b-a)u\big) + cu - Yu. \tag{4.85}$$

We assume that

$$a, b, c, \partial_x(xa), \partial_x(xb) \quad \text{are bounded Hölder continuous functions} \tag{4.86}$$

in the sense of (4.16) and that a satisfies the condition (4.15). Note that the same assumptions holds for $\mathscr{L}_1^*$.

The proof of Theorem 4.4.1 is based on a local L^∞ a priori estimate for non-negative solution u of $\mathscr{L}_1 u = 0$. In order to state precisely this estimate, we introduce some notation.

For every $(x_0, y_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^2$ and $r \in]0, 1/2]$ we consider the set $H_r(x_0, y_0, t_0)$ introduced in (4.28)

$$H_r(x_0, y_0, t_0) := \Big\{(x, y, t) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid |x - x_0| < r x_0,$$
$$|y - y_0 + x_0(t - t_0)| < r^3 x_0, \quad -r^2 < t - t_0 < 0\Big\}.$$

Proposition 4.4.2. *Let (x_0, y_0, t_0) be any point of $\mathbb{R}^+ \times \mathbb{R}^2$, and let r, ρ with $0 < r/2 \le \rho < r \le 1/2$. Let u be a non-negative weak solution of $\mathscr{L}_1 u(x, y, t) = 0$ in $H_r(x_0, y_0, t_0)$ and let $u \in L^p(H_r(x_0, y_0, t_0))$, with $p \ge 1$. Then*

$$\sup_{H_\rho(x_0,y_0,t_0)} u^p \le \frac{\bar{c}}{(r-\rho)^6} \int_{H_r(x_0,y_0,t_0)} u^p, \tag{4.87}$$

where the constant $\bar{c} > 0$ depends only on $\mathscr{L}_1, p$ and on the L^∞ norm of a, b, c.

The proof of Proposition 4.4.2 relies on the analogous result proven in [22, Theorem 1.4] for the Kolmogorov equation with bounded coefficients. For the sake of simplicity we recall here its statement for a particular operator strongly related to $\mathscr{L}_1$. For every (x_0, y_0, t_0) and $r > 0$ we denote

$$\widetilde{H}_r^-(x_0, y_0, t_0) := \Big\{(x, y, t) \in \mathbb{R}^3 \mid |x - x_0| < r,$$
$$|y - y_0 + x_0(t - t_0)| < r^3, \ -r^2 < t - t_0 < 0\Big\}.$$

Let Ω be an open subset of $\mathbb{R}^3, (x, y, t) \in \Omega$ and consider $v(x, y, t)$ a positive weak solution in Ω of the following equation

$$\partial_x(\widetilde{a}(x, y, t)\partial_x v) + \widetilde{b}(x, y, t)\partial_x v + x\partial_y v + \widetilde{c}(x, y, t)v - \partial_t v = 0. \tag{4.88}$$

Assume that $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ are measurable bounded continuous functions such that

$$\inf_{\Omega} a(x, y, t) > 0.$$

Let $(x_0, y_0, t_0) \in \Omega$, let ρ, r be real parameters such that $0 < r/2 \le \rho < r \le 1$ and $\widetilde{H}_r(x_0, y_0, t_0) \subseteq \Omega$. Then, there exists a positive constant c depending on the L^∞ norm of $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ and on p such that

$$\sup_{\widetilde{H}_\rho(x_0,y_0,t_0)} v^p \le \frac{c}{(r-\rho)^6} \int_{\widetilde{H}_r(x_0,y_0,t_0)} v^p. \tag{4.89}$$

for every $v \in L^p(\widetilde{H}_r(x_0, y_0, t_0))$.

PROOF OF PROPOSITION 4.4.2. We first note that $\mathscr{L}_1 u = 0$ reads as follows

$$\partial_x(x^2 a(x,y,t)\partial_x u) + (b(x,y,t) - a(x,y,t))x\partial_x u + c(x,y,t)u + x\partial_y u - \partial_t u = 0 \quad (4.90)$$

so that it has the form (4.88). Even if coefficents of $\mathscr{L}_1$ are unbounded and $\inf_{\mathbb{R}^+ \times \mathbb{R}^2} x^2 a(x,y,t) = 0$, estimate (4.89) holds on compact cylinders contained in $\mathbb{R}^+ \times \mathbb{R}^2$. However, we need to show that the constant $\bar{c}$ in (4.87) does not depend on (x_0, y_0, t_0) and r.

We first fix $(x_0, y_0, t_0) = (1, 0, 0)$, so that the cylinders $H_r(1, 0, 0)$ and $\widetilde{H}_r(1, 0, 0)$ coincide. We modify the functions $a(x,y,t)$, $b(x,y,t)$ and $c(x,y,t)$ as we have done in chapter 2

$$\begin{aligned}
&\widetilde{a}(x,y,t) = \varphi^2(x)a(x,y,t), \ \widetilde{b}(x,y,t) = \varphi(x)(b(x,y,t) - a(x,y,t)), \\
&\widetilde{c}(x,y,t) = \varphi(x)c(x,y,t)
\end{aligned}$$

where $\varphi(x)$ is the function defined in (4.33). Then the functions $\widetilde{a}, \widetilde{b}$ and $\widetilde{c}$ are uniformly bounded, $\inf \widetilde{a}$ is strictly positive and (4.89) implies (4.87) if $(x_0, y_0, t_0) = (1, 0, 0)$.

For a general (x_0, y_0, t_0), we consider the function

$$w(x,y,t) := u\big((x_0, y_0, t_0) \circ (x,y,t)\big)$$

and we conclude the proof by the argument used in the proof of Proposition 4.2.1. □

We next introduce a result that, combined with Proposition 4.4.2, provides us with the asymptotic upper bound of the fundamental solution of $\mathscr{L}_1$. We first introduce a suitable *cut-off function.* Let choose $R > 1$ and consider the following function

$$\chi_R(x,y) = g_R(x)h_R(y), \quad (x,y) \in \mathbb{R}^+ \times \mathbb{R}, \quad (4.91)$$

where

- $g_R(x) = \varphi\left(\frac{\log^2(x)+1}{\log^2(R)+1}\right)$;
- $\varphi(s)$ is a continuous function such that $\varphi(s) = 1$ if $s \in [0, 1/2]$ and $\varphi(s) = 0$ if $s \in [1, +\infty[$;

- $h(y)$ is a continuous function such that

  - $h(y) = 1$ if $y \in [-R, R]$;
  - $h(y) = 0$ if $y \in]-\infty, -R^2] \cup [R^2, +\infty[$;
  - $h(y)$ is a C^2 spline function with derivative bounded by $\frac{2}{R^2-R}$, if $y \in [-R^2; -R] \cup [R, R^2]$.

We first observe that $g_R(x) \neq 0$ only if $x \in [1/R, R]$ and

$$\begin{aligned}|x\partial_y \chi_R(x,y)| &\leq x|g_R(x)||\partial_y h_R(y)| \leq \frac{2}{R-1},\\ |x\partial_x \chi_R(x,y)| &\leq x|h_R(y)|\|\varphi'\|_{L^\infty(\mathbb{R})}\frac{2\log(x)}{x(\log^2(R)+1)}\\ &\leq \|\varphi'\|_{L^\infty(\mathbb{R})}\frac{2\log(x)}{(\log^2(R)+1)}.\end{aligned}$$

Therefore

$$\begin{aligned}|X\chi_R| &\leq C\frac{\log R}{\log^2 R+1} \to 0 \quad \text{as } R \to +\infty\\ |Y\chi_R| &\leq |x\partial_y \chi_R| \leq \frac{2}{R-1} \to 0 \quad \text{as } R \to +\infty.\end{aligned}$$

Now we are ready to state the following

Proposition 4.4.3. *Let $u \in L^2(\mathbb{R}^+ \times \mathbb{R}^2)$ be a weak solution of $\mathscr{L}_1 u = 0$, and let Ψ be the value function of the control problem (2.38). Then there exist two positive constants m, M_1 depending on the L^∞ norm of a, b, c, $x\partial_x a$, $x\partial_x b$, such that*

$$\int_{\mathbb{R}^+\times\mathbb{R}} e^{-\frac{\Psi(x_1,y_1,s;x,y,t_1)}{8M_1}-mt_1}u^2(x,y,t_1)dx\,dy \leq \int_{\mathbb{R}^+\times\mathbb{R}} e^{-\frac{\Psi(x_1,y_1,s;x,y,t_0)}{8M_1}-mt_0}u^2(x,y,t_0)dx\,dy, \tag{4.92}$$

for every t_0, t_1 with $t_0 < t_1$, and $(x_1, y_1, s) \in \mathbb{R}^+ \times \mathbb{R} \times]t_1, +\infty[$.

Proof. Fix $(x_1, y_1, t_1) \in \mathbb{R}^+ \times \mathbb{R}^2$, and $t_0 < t_1$, and recall that, for any $(x_0, y_0, t_0) \in \mathbb{R}^3$, in view of (4.61) the function $(x, y, t) \mapsto \Psi(x_0, y_0, t_0; x, y, t)$ is a classical solution of the Hamilton-Jacobi-Bellman equation (see [9])

$$Y\Psi + \frac{1}{4}(X\Psi)^2 = 0.$$

We set $v(x,y,t) := \frac{1}{16M_1}\Psi(x_0,y_0,t_0;x,y,t)$ where M_1 is the L^∞-norm of a. Then v satisfies

$$Yv + 4M_1(Xv)^2 = 0. \tag{4.93}$$

We prove (4.92) by showing that

$$\lim_{R\to+\infty}\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \frac{d}{dt}\chi_R^2 e^{-2v-mt}u^2 \le 0, \tag{4.94}$$

where χ_R is the cut-off function introduced above and the constant m will be specified in the sequel. Let u be a positive solution of $\mathscr{L}_1$ in the domain $\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]$. We note that

$$\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} x\partial_y\big(\chi_R^2 e^{-2v-mt}u^2\big) = 0$$

since the function $\chi_R(x,y)$ has compact support in $\mathbb{R}^+\times\mathbb{R}$. Therefore we obtain

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \frac{d}{dt}\chi_R^2 e^{-2v-mt}u^2 =& -\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} Y\big(\chi_R^2 e^{-2v-mt}u^2\big)\\ =& \int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} e^{-2v-mt}u^2\Big(-Y\big(\chi_R^2\big) + 2\chi_R^2 Yv - m\chi_R^2\Big)\\ &- 2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \chi_R^2 e^{-2v-mt}uYu. \end{aligned} \tag{4.95}$$

We first focus on the last term of (4.95). By using the fact that u is weak solution of $\mathscr{L}_1 u = 0$ one gets

$$\begin{aligned} A :=& -2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \chi_R^2 e^{-2v-mt}uYu\\ =& -2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} aX\big(\chi_R^2 e^{-2v-mt}u\big)Xu\\ &+2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \big(\chi_R^2 e^{-2v-mt}u\big)(b-a)Xu\\ &+2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} c\chi_R^2 e^{-2v-mt}u^2 =: A_1 + A_2 + A_3. \end{aligned} \tag{4.96}$$

Consider the first term in (4.96) and compute the derivatives

$$\begin{aligned} A_1 =& -2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} aX\big(\chi_R^2 e^{-2v-mt}u\big)Xu\\ =& -4\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R e^{-2v-mt}uXuX\chi_R\\ &+4\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R^2 e^{-2v-mt}uXuXv\\ &-2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R^2 e^{-2v-mt}(Xu)^2 =: B_1 + B_2 + B_3. \end{aligned} \tag{4.97}$$

By using Young inequality, it follows

$$\begin{aligned}
B_1 =& -4\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R e^{-2v-mt} uXuX\chi_R \\
\leq& 4\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R e^{-2v-mt} |Xu|\,|uX\chi_R| \\
\leq& \int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R^2 e^{-2v-mt}(Xu)^2 \\
&+ 4\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a e^{-2v-mt} u^2 (X\chi_R)^2 =: C_1 + C_2,
\end{aligned} \tag{4.98}$$

Merging the inequalities (4.97) and (4.98), since $B_3 = -2C_1$, we conclude

$$\begin{aligned}
A_1 =& -\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R^2 e^{-2v-mt}(Xu)^2 + B_2 + C_2 \\
\leq& 4\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} a\chi_R^2 e^{-2v-mt} u^2 (Xv)^2 + C_2 \\
\leq& 4M_1\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \chi_R^2 e^{-2v-mt} u^2 (Xv)^2 + C_2.
\end{aligned} \tag{4.99}$$

Now consider the second term in (4.96). Start from integration by parts formula

$$A_2 = 2\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} uX^*\left((b-a)\big(\chi_R^2 e^{-2v-mt}u\big)\right).$$

Reminding that $X^* = -X-1$, similarly to (4.97), (4.98) and (4.99) we have

$$\begin{aligned}
A_2 \leq& -\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} (b-a)\chi_R^2 e^{-2v-mt}u^2 \\
&+ 4M_1\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \chi_R^2 e^{-2v-mt}u^2(Xv)^2 \\
&+ \frac{1}{4M_1}\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} (b-a)^2\chi_R^2 e^{-2v-mt}u^2 \\
&+ \int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} X\big(\chi_R^2\big) e^{-2v-mt}u^2 + \int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} X(b-a)\chi_R^2 e^{-2v-mt}u^2 \\
\leq& 4M_1\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \chi_R^2 e^{-2v-mt}u^2(Xv)^2 \\
&+ \int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} X\big(\chi_R^2\big) e^{-2v-mt}u^2 + \frac{3}{4}m\int_{\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1]} \chi_R^2 e^{-2v-mt}u^2.
\end{aligned} \tag{4.100}$$

by setting

$$\begin{aligned}
m := 4\max\Big\{&\tfrac{1}{4M_1}\|(b-a)^2\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1])},\ \|X(b-a)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1])}, \\
&2\|c\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1])},\ \|b-a\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}\times[t_0,t_1])}\Big\}.
\end{aligned} \tag{4.101}$$

Going back to (4.96), combining (4.99), (4.100) and

$$A_3 \le \frac{1}{4} m \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} \chi_R^2 e^{-2v-mt} u^2,$$

we have

$$\begin{aligned} A \le & 8M_1 \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} \chi_R^2 e^{-2v-mt} u^2 (Xv)^2 \\ & + 4 \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} a e^{-2v-mt} u^2 (X\chi_R)^2 \\ & + \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} X(\chi_R^2) e^{-2v-mt} u^2 + m \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} \chi_R^2 e^{-2v-mt} u^2. \end{aligned} \tag{4.102}$$

Merging (4.102) with (4.96) ad (4.95) we conclude that

$$\begin{aligned} & \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} \frac{d}{dt} \chi_R^2 e^{-2v-mt} u^2 \\ & \qquad \le 2 \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} \left(\chi_R^2 e^{-2v-mt} u^2 \right) \left[Yv + 4M_1 (Xv)^2 \right] \\ & \qquad\quad + \int_{\mathbb{R}^+ \times \mathbb{R} \times [t_0,t_1]} \left(e^{-2v-mt} u^2 \right) \left(- Y\left(\chi_R^2\right) + 4a (X\chi_R)^2 + X\left(\chi_R^2\right) \right). \end{aligned}$$

The first integral is zero since v satisfies the Hamilton-Jacobi-Bellman equation (4.93), and (4.92) simply follows by letting $R \to +\infty$. □

The next Lemma is crucial to prove Theorem 4.4.1.

Lemma 4.4.4. *Let ε be a fixed positive constants. Then there exists a constant $c_\varepsilon > 0$, only depending on $\mathscr{L}_1$ and ε such that*

$$u^2\left(1, 0, t/2\right) \le c_\varepsilon \int_{\mathbb{R}^+ \times \mathbb{R}} e^{-\frac{1}{8M} \Psi\left(1, -\varepsilon, \frac{t}{2} + \varepsilon, \xi, \eta, 0\right)} u^2(\xi, \eta, 0) d\xi d\eta \tag{4.103}$$

for every non-negative weak solution u of $\mathscr{L}_1 u = 0$ in $\mathbb{R}^+ \times \mathbb{R} \times]T_0, T_1[$.

Proof. Let $\varepsilon > 0$ be fixed and let $r \in]0, 1/2]$ be such that $r^3 < \varepsilon$. By Proposition 4.4.2, with $p = 2$, we have

$$u^2\left(1, 0, t/2\right) \le \sup_{H^-_{r/2}\left(1,0,t/2\right)} u^2(\xi, \eta, \tau) \le \frac{c}{(r/2)^6} \int_{H^-_r\left(1,0,t/2\right)} u^2(\xi, \eta, \tau) d\xi d\eta d\tau \tag{4.104}$$

for every $t \in]T_0, T_1[$. Multiply and divide the integrand of the above inequality by the quantity

$$e^{\frac{1}{8M_1} \Psi\left(1, -\varepsilon, \frac{t}{2} + \varepsilon; \xi, \eta, \tau\right) + m\tau}. \tag{4.105}$$

Note that, as $r^3 < \varepsilon$, the function $(\xi,\eta,\tau) \mapsto \Psi\big(1,-\varepsilon,\frac{t}{2}+\varepsilon;\xi,\eta,\tau\big)$ is well defined, continuous and bounded in the set $\overline{H_{r/2}\big(1,0,t/2\big)}$.

Therefore, we denote by C_ε the maximum of the function in (4.105) in the set $\overline{H_{r/2}\big(1,0,t/2\big)}$, which is uniform with respect to $t \in]T_0,T_1[$. We then find

$$
\begin{aligned}
u^2\big(1,0,t/2\big) &\le \frac{c\,C_\varepsilon}{(r/2)^6}\int_{H_r^-\left(1,0,t/2\right)} e^{-\frac{1}{8M_1}\Psi\left(1,-\varepsilon,\frac{t}{2}+\varepsilon;\xi,\eta,\tau\right)-m\tau}u^2(\xi,\eta,\tau)d\xi d\eta d\tau \\
&\le \tfrac{c\,C_\varepsilon}{(r/2)^6}\int_{t/2-r^2}^{t/2}\int_{\mathbb{R}^+\times\mathbb{R}} e^{-\frac{1}{8M_1}\Psi\left(1,-\varepsilon,\frac{t}{2}+\varepsilon;\xi,\eta,\tau\right)-m\tau}u^2(\xi,\eta,\tau)d\xi d\eta d\tau \\
&\quad \text{(by Proposition 4.4.3, with } t_0 = 0 \text{ and } t_1 = \tau) \\
&\le \frac{c\,C_\varepsilon}{(r/2)^6}\int_{t/2-r^2}^{t/2}\int_{\mathbb{R}^+\times\mathbb{R}} e^{-\frac{1}{8M_1}\Psi\left(1,-\varepsilon,\frac{t}{2}+\varepsilon;\xi,\eta,0\right)}u^2(\xi,\eta,0)d\xi d\eta \\
&\le \frac{c\,C_\varepsilon}{(r/2)^6}\int_{\mathbb{R}^+\times\mathbb{R}} e^{-\frac{1}{8M_1}\Psi\left(1,-\varepsilon,\frac{t}{2}+\varepsilon;\eta,0\right)}u^2(\xi,\eta,0)d\xi d\eta.
\end{aligned}
$$

which gives (4.103) by setting $c_\varepsilon := \frac{c\,C_\varepsilon}{(r/2)^6}$. □

We finally introduce a last results we need in order to prove the Proposition (4.4.1). The following Proposition is a direct consequence of Proposition 4.4.2 and involves the fundamental solution Γ of $\mathscr{L}$ in (4.81):

Proposition 4.4.5. *Let fix (x,y,t), (x_0,y_0,t_0) in $\mathbb{R}^+\times\mathbb{R}^2$ with $y<y_0$ and $T_0 \le t_0 < t \le T_1$ and let Γ be a fundamental solution of $\mathscr{L}$ in (4.81). Let denote by $T = T_1 - T_0$, then there exist a positive constant C_T depending on the operator $\mathscr{L}$ and on T such that the following upper bounds hold for Γ*

i) $\Gamma(x,y,t;x_0,y_0,t_0) \le \frac{C_T}{(t-t_0)^2}$;

ii) $\int_{\mathbb{R}^+\times\mathbb{R}}\Gamma^2(x,y,t;x_0,y_0,t_0)dx_0dy_0 \le \frac{C_T}{(t-t_0)^2}$;

Proof. We only prove **i)**, since **ii)** is its direct consequence reminding that

$$\int_{\mathbb{R}^+\times\mathbb{R}}\Gamma(x,y,t;x_0,y_0,t_0)dx_0dy_0 = 1.$$

We first fix $0 < t - t_0 < 1$ and, by using Proposition 4.4.2, we have

$$\begin{aligned}
\Gamma(x, y, t; x_0, y_0, t_0) &\leq \sup_{H_{\sqrt{t-t_0}/2}(x,y,t)} \Gamma(\cdot, \cdot, \cdot; x_0, y_0, t_0) \\
&\leq \frac{\bar{C}}{(t-t_0)^3} \int_{H_{\sqrt{t-t_0}}(x,y,t)} \Gamma(\xi, \eta, \tau; x_0, y_0, t_0) d\xi d\eta d\tau \\
&\leq \frac{\bar{C}}{(t-t_0)^3} \int_{t-(t-t_0)}^{t} d\tau \int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma(\xi, \eta, \tau; x_0, y_0, t_0) d\xi d\eta \\
&= \frac{\bar{C}}{(t-t_0)^2} \qquad (4.106)
\end{aligned}$$

since $\int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma(\xi, \eta, \tau; x_0, y_0, t_0) d\xi d\eta < +\infty$. If $t - t_0 \geq 1$ we set $\nu = \frac{t-t_0}{T} < 1$, and starting from the reproduction property (3.68), we have

$$\begin{aligned}
\Gamma(x, y, t; x_0, y_0, t_0) &= \int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma(x, y, t; \xi, \eta, t_0 + \nu) \Gamma(\xi, \eta, t_0 + \nu; x_0, y_0, t_0) d\xi d\eta \\
&\leq \frac{C_T}{(t-t_0)^2} \int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma(x, y, t; \xi, \eta, t_0 + \nu) d\xi d\eta \leq \frac{C_T}{(t-t_0)^2}
\end{aligned}$$

by (4.106) where $C_T = \bar{C} T^2$ and $\int_{\mathbb{R}^+ \times \mathbb{R}} \Gamma(x, y, t; \xi, \eta, t_0 + \nu) d\xi d\eta = 1$. □

We are now ready to prove the main proposition of this Section.

Proof of Proposition 4.4.1. Let $\varepsilon > 0$ be fixed and let $\Gamma(x, y, t; 1, 0, 0)$ be the fundamental solution of $\mathscr{L}$ (4.81) and $(x, y, t) \in \mathbb{R}^+ \times] - \infty, 0[\times]T_0, T_1[$. We define

$$\begin{aligned}
D_1 = &\big\{(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^- \mid \Psi\big(x, y - \varepsilon, t + \varepsilon/2, \xi, \eta, t/2\big) \\
&\qquad \leq \Psi(\xi, \eta, t/2; 1, 0, -\varepsilon/2)\big\}, \\
D_2 = &\big\{(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^- \mid \Psi(x, y - \varepsilon, t + \varepsilon/2, \xi, \eta, t/2) \\
&\qquad > \Psi(\xi, \eta, t/2; 1, 0, -\varepsilon/2)\big\},
\end{aligned}$$

Starting from the reproduction property (3.68) of Γ

$$\begin{aligned}
\Gamma(x, y, t; 1, 0, 0) &= \int_{\mathbb{R}^+ \times \mathbb{R}^-} \Gamma(x, t, y; \xi, \eta, t/2) \Gamma(\xi, \eta, t/2, 1, 0, 0) d\xi d\eta \\
&= \int_{D_1} \Gamma(x, t, y; \xi, \eta, t/2) \Gamma(\xi, \eta, t/2, 1, 0, 0) d\xi d\eta + \\
&\quad + \int_{D_2} \Gamma(x, t, y; \xi, \eta, t/2) \Gamma(\xi, \eta, t/2, 1, 0, 0) d\xi d\eta \\
&\leq \frac{C_T}{t^2} \Bigg(\left(\int_{D_1} \Gamma^2(\xi, \eta, t/2, 1, 0, 0) d\xi d\eta \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{D_2} \Gamma^2(x, y, t; \xi, \eta, t/2) d\xi d\eta \right)^{\frac{1}{2}} \Bigg)
\end{aligned}$$

where $T = T_1 - T_0$ and the last inequality follows from Hölder inequality and (4.106). We now introduce the sets

$$\begin{aligned}\widetilde{D}_1 = \ &\big\{(\xi,\eta) \in \mathbb{R}^+ \times \mathbb{R}^- \mid \Psi\big(x,y-\varepsilon, t+\varepsilon/2, 1, 0, -\varepsilon/2\big) \\ &\qquad\qquad \le 2\Psi(\xi,\eta,t/2,1,0,-\varepsilon/2)\big\},\\ \widetilde{D}_2 = \ &\big\{(\xi,\eta) \in \mathbb{R}^+ \times \mathbb{R}^- \mid \Psi(x,y-\varepsilon,t+\varepsilon/2,1,0,-\varepsilon/2) \\ &\qquad\qquad \le 2\Psi(x,y-\varepsilon,t+\varepsilon/2;\xi,\eta,t/2)\big\},\end{aligned}$$

and we note that $D_1 \subseteq \widetilde{D}_1$ and $D_2 \subseteq \widetilde{D}_2$ as a consequence of the *triangular inequality* of the value function:

$$\Psi(x_0,y_0,t_0;x,y,t) \le \Psi(x_0,y_0,t_0;\xi,\eta,\tau) + \Psi(\xi,\eta,\tau;x,y,t)$$

for arbitrary points $(x_0,y_0,t_0), (\xi,\eta,\tau), (x,y,t)$ belonging to $\mathbb{R}^+ \times \mathbb{R} \times]T_0,T_1[$ with $y > \eta > y_0$ and $T_0 \le t < \tau < t_0 \le T_1$. Hence

$$\begin{aligned}\Gamma(x,y,t;1,0,0) \le &\frac{C_T}{t^2}\Big(\Big(\int_{\widetilde{D}_1}\Gamma^2(\xi,\eta,t/2,1,0,0)d\xi d\eta\Big)^{\frac12} \\ &+\Big(\int_{\widetilde{D}_2}\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta\Big)^{\frac12}\Big)\end{aligned}$$

We now claim that:

$$\int_{\widetilde{D}_1}\Gamma^2(\xi,\eta,t/2,1,0,0)d\xi d\eta \le c_\varepsilon e^{-\frac{1}{16M_1}\Psi(x,y-\varepsilon,t+\varepsilon/2;1,0,-\varepsilon/2)} \tag{4.107}$$

$$\int_{\widetilde{D}_2}\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta \le c_\varepsilon e^{-\frac{1}{16M_1}\Psi(x,y-\varepsilon,t+\varepsilon/2;1,0,-\varepsilon/2)} \tag{4.108}$$

where c_ε is a positive constant depending on $\mathscr{L}$ and ε. We first prove (4.108) and we define the functions

$$\begin{aligned}v(z,w,s) &= \int_{\widetilde{D}_2}\Gamma(z,w,s;\xi,\eta,t/2)\Gamma(x,y,t;\xi,\eta,t/2)d\xi d\eta,\\ u(z,w,s) &= v((x,y,t/2)\circ(z,w,s)).\end{aligned}$$

We further note that the functions u and v satisfy the following properties:

i) $v(z,w,s)$ is a solution of $\mathscr{L}v(z,w,s) = 0$ in $\mathbb{R}^+ \times \mathbb{R} \times [t/2,T_1[$. Then $u(z,w,s)$ is a solution of $\mathscr{L}_{\bar z}u(z,w,s) = 0$ in $\mathbb{R}^+ \times \mathbb{R} \times]0,T_1[$ where $\bar z = (x,y,t/2)$ and

$$\begin{aligned}\mathscr{L}_{\bar z}u(z,w,s) = &z\partial_z\big(a(xz,y+xw,t/2+s)z\partial_z u\big) \\ &+ z\,b(xz,y+xw,t/2+s)\partial_z u + \\ &+ z\partial_w u + c(xz,y+xw,t/2+s)u - \partial_t u.\end{aligned} \tag{4.109}$$

ii) the function v satisfies the initial condition

$$v(z,w,t/2)=\Gamma(x,y,t;z,w,t/2)\mathbf{1}_{\widetilde{D}_2}(z,w);$$

iii) it holds $u(1,0,t/2)=\int_{\widetilde{D}_2}\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta$.

where $\mathbf{1}_{\widetilde{D}_2}(z,w)$ denotes the characteristic function of the set $\widetilde{D}_2$. In virtue of Lemma 4.4.4 we have

$$u^2\big(1,0,t/2\big)\le c_\varepsilon\int_{\mathbb{R}^+\times\mathbb{R}}e^{-\frac{1}{8M_1}\Psi\left(1,-\varepsilon,\frac{t}{2}+\varepsilon/2,z,w,0\right)}u^2(z,w,0)dzdw.$$

By observing that

$$\Psi\big(1,-\varepsilon,\tfrac{t}{2}+\varepsilon/2,z,w,0\big)=\Psi\big(x,y-\varepsilon,t+\varepsilon/2;x,y,t/2)\circ(z,w,0)\big),$$

by the change of variable $(\xi,\eta,t/2)=(x,y,t/2)\circ(z,w,0)$ and by properties **ii)** and **iii)**, we get

$$\begin{aligned}\left(\int_{\widetilde{D}_2}\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta\right)^2&=u^2\big(1,0,t/2\big)\\ \le c_\varepsilon\int_{\widetilde{D}_2}e^{-\frac{1}{8M_1}\Psi\left(x,y-\varepsilon,t+\varepsilon/2;\xi,\eta,t/2\right)}&\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta\end{aligned}$$

We finally obtain (4.108) by recalling the definition of $\widetilde{D}_2$

$$\left(\int_{\widetilde{D}_2}\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta\right)^2\le$$
$$c_\varepsilon e^{-\frac{1}{16M_1}\Psi\left(x,y-\varepsilon,t+\varepsilon/2,1,0,-\varepsilon/2\right)}\int_{\widetilde{D}_2}\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta$$

and the result immediately follows by dividing by $\int_{\widetilde{D}_2}\Gamma^2(x,y,t;\xi,\eta,t/2)d\xi d\eta$ and by recalling that $\Psi\big(x,y-\varepsilon,t+\varepsilon/2,1,0,-\varepsilon/2\big)=\Psi\big(x,y-\varepsilon,t+\varepsilon,1,0,0\big)$

The proof of inequality (4.107) is analogous to (4.108). Indeed, consider the function

$$v_2(z,w,s)=\int_{\widetilde{D}_1}\Gamma(\xi,\eta,t/2;z,w,s)\Gamma(\xi,\eta,t/2;1,0,0)d\xi d\eta,$$

which is a non-negative solution to $\mathscr{L}^*v_2=0$ with final data $v_2(z,w,t/2)=\Gamma(z,w,t/2;1,0,0)$ if $(z,w)\in\widetilde{D}_1$ and $v_2(z,w,t/2)=0$ if $(z,w)\notin\widetilde{D}_1$. Notice that the coefficients of $\mathscr{L}^*$ satisfy the same assumptions (4.15) and (4.86) made on $\mathscr{L}_1$ in (4.84) and on $\mathscr{L}_1^*$ in (4.85), then all the properties shown for the function $(x,y,t)\mapsto\Gamma(x,y,t;\xi,\eta,\tau)$ and used to prove (4.108), also hold for the function

$(x,y,t) \mapsto \Gamma(\xi,\eta,\tau;x,y,t)$ (which is the fundamental solution of $\mathscr{L}^* u = 0$) and they can be used to prove (4.107). This proves the claim. $\square$

We are now ready to prove the main result announced in Theorem 4.1.3 .

PROOF OF THEOREM 4.1.3. Let $\Gamma(x,y,t;x_0,y_0,t_0)$ denote the fundamental solution of $\mathscr{L}$ in (4.1) and $(x,y,t),(x_0,y_0,t_0)$ in $\mathbb{R}^+ \times \mathbb{R} \times [0,T]$ with $y < y_0$ and $t > t_0$. If $(x_0,y_0,t_0) = (1,0,0)$, the lower bound of Γ follows from Proposition 4.3.2, whereas the upper bound follows from Proposition 4.4.1. For a general choice of $z_0 = (x_0,y_0,t_0)$ it suffices to note that the function

$$\Gamma_{z_0}(x,y,t;1,0,0) = x_0^2\,\Gamma((x_0,y_0,t_0)\circ(x,y,t);x_0,y_0,t_0) \tag{4.110}$$

is the fundamental solution of the operator $\mathscr{L}_{z_0}$ defined in (4.18), with singularity at $(1,0,0)$. As noticed in Remark 4.1.2, it satisfies assumptions (4.15) and (4.16), with the same constants M,λ and α used for $\mathscr{L}$, then (4.79) and (4.83) applies to Γ_{z_0}. If one consider the lower estimates, we find

$$\Gamma((x_0,y_0,t_0)\circ(x,y,t);x_0,y_0,t_0) \geq \frac{c^-_{\varepsilon,T}}{x_0^2 t^2}\exp\left(-C^-\Psi(x,y,t;1,-\varepsilon t,\varepsilon t)\right),$$

that can be written equivalently as follows

$$\Gamma((x,y,t;x_0,y_0,t_0) \geq$$
$$\frac{c^-_{\varepsilon,T}}{x_0^2(t-t_0)^2}\exp\left(-C^-\Psi((x_0,y_0,t_0)^{-1}\circ(x,y,t);1,-\varepsilon(t-t_0),\varepsilon(t-t_0))\right).$$

The conclusion follows by applying the invariance property (4.60) of Ψ:

$$\begin{aligned}
&\Psi((x_0,y_0,t_0)^{-1}\circ(x,y,t);1,-\varepsilon(t-t_0),\varepsilon(t-t_0))\\
&= \Psi(x,y,t;(x_0,y_0,t_0)\circ(1,-\varepsilon(t-t_0),\varepsilon(t-t_0)))\\
&= \Psi(x,y,t;x_0,y_0-\varepsilon(t-t_0)x_0,t_0+\varepsilon(t-t_0))\\
&= \Psi(x,y+\varepsilon(t-t_0)x_0,t-\varepsilon(t-t_0);x_0,y_0,t_0).
\end{aligned}$$

The proof of the upper bound is analogous. $\square$

Conclusions and Future Perpespectives

In this thesis we have derived the two sided bounds for the fundamental solution Γ of the operator

$$\mathscr{L}u = x\partial_x(a(x,y,t)x\partial_x u) + b(x,y,t)x\partial_x u + x\partial_y u - \partial_t u.$$

We have seen in the previous chapter that the operator $\mathscr{L}$ cannot be globally seen as a Kolmogorov operator with bounded and variable coefficients since the coefficient x^2 is unbonded and its infimum equals zero in $\mathbb{R}^+$.

The key ingredients needed to achieve our result are the invariant Harnack inequality for non negative solution of $\mathscr{L}u = 0$ to obtain the lower bound and the Moser iteration for the upper one.
This properties has been derived by the fact that $\mathscr{L}$ is *locally* well approximated by a Kolmogorov operator with variable and bounded coefficients. Indeed, the proof of the main theorem is based on local estimates of non negative the solution of $\mathscr{L}u = 0$ and the operator $\mathscr{L}$ can be locally seen as a Kolmogorov operator with a variable coefficients in a compact set K included in $\mathbb{R}^+ \times \mathbb{R} \times [0,T]$. This fact allowed us to deduce a Harnack inequality and a result analogous to the Moser iteration for $\mathscr{L}$. Furthermore, the invariance of such operator with respect to a suitable left translation yields that the constants involved in the Harnack inequality and in the Moser iteration are idependent of the choice of K.

The first advantage of this strategy is to acquire the diagonal bounds

$$\Gamma(x_0, y_0 + x_0(t-t_0), t; x_0, y_0, t_0) \approx \frac{C_T}{x_0(t-t_0)^2}, \quad t > t_0,$$

which agrees with the diagonal term of the fundamental solution

$$\Gamma_0(x,y,t;x_0,y_0,t_0) = \frac{\sqrt{3}}{2\pi(t-t_0)^2} \exp\Big(- \tfrac{(x-x_0)^2}{4(t-t_0)} - \tfrac{3}{(t-t_0)^3}\big(y - y_0 - (t-t_0)\tfrac{(x+x_0)}{2}\big)^2\Big)$$

of the operator

$$\mathcal{L}_0 u = \partial_{xx} u + x\partial_y u - \partial_t u.$$

This bound has an interesting probabilistic parallel since the diagonal estimate corresponds to the product of the standard deviations in short time t of the random variables

$$X_t = x_0 \exp\left(\sigma W_t + (\mu - \frac{\sigma^2}{2})t\right),$$
$$A_t = y_0 + x_0 \int_0^t \exp\left(\sigma W_s + (\mu - \frac{\sigma^2}{2})s\right) ds$$

defined in (3.2) introduced by Yor and Geman. Indeed, when $\mu = 1$, $\sigma = \sqrt{2}$ we have

$$\mathrm{Var}(X_t) = x_0^2 e^{2t}\left(e^{2t} - 1\right) = 2x_0^2 t + o(t), \quad \text{as } t \to 0,$$
$$\mathrm{Var}(A_t) = x_0^2\left(\tfrac{1}{6}\left(e^{4t} - 1\right) - \tfrac{2}{3}\left(e^t - 1\right) - \left(e^t - 1\right)^2\right) = \tfrac{2}{3} x_0^2 t^3 + o(t^3), \quad \text{as } t \to 0.$$

The second advantage is that the technique allows us to obtain optimal upper and lower bound for the off-diagonal term. Indeed, in connection with control theory, the basic idea is that, to obtain the bounds between two points, there must exist an admissible path between those points, along which the Harnack inequality holds. For the non-degenerate case in $\mathbb{R}^N$ this path is simply the straight line as it is shown in Moser's articles [60, 62], whereas in degenerate problem it is the geodesic curve joining the two points (when the optimal control problem is well-posed).

For the off-diagonal upper bound, we have developed the techniques appearing in Aronson [2]. However, in order to obtain a similar upper bounds for the fundamental solution Γ of $\mathscr{L}$ in terms of the cost associated to the geodesic curve joining the two points, we has been led to consider the Hamilton-Jacobi-Bellmann equation related to our value function.

Concerning the existence of the fundamental solution of $\mathscr{L}$, the only known result is the representation formula proven by Geman and Yor (3.27) and in its general expression (3.39), which has been obtained by probabilistic techniques for the simplest case of operators related to Asian options with arithmetic average.
The existence of the fundamental solution in the more general case of smooth variable coefficients is proved in this thesis by means of the Theory of stochastic processes and, in particular, by Malliavin Calculus.
Because the estimates of the fundamental solution apply both to operators with

variable coefficients $\mathscr{L}$ and to operators $\mathscr{L}_0$ considered by Geman and Yor, we get, in particular, a comparison between the two fundamental solutions. Consequently, the expression in integral form shown by Geman and Yor provides a further estimate of the fundamental solution Γ of the operator $\mathscr{L}$ with variable coefficients.

We plan to pursue the research work considering the problem of the existence of the fundamental solution of the operator $\mathscr{L}$ relaxing the smoothness assumption on the coefficients, using a method different from the classic parametrix method, but that is based on the approximation of the coefficients with regular functions. The existence of the limit should be ensured by the estimates proven in the thesis, which depend only on the module of the Hölder continuity of the coefficients and not on their further regularity.

In the case of uniformly parabolic equations, this set of information provides criteria of existence and identifies the classes of uniqueness solution of the problem of initial values. The extension of this study to the operator of the arithmetic average Asian options is also a future research work. We also plan to extend this program to more general operators, following the lines of research classically used in the study of uniformly parabolic operators.

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