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DIOPHANTINE APPROXIMATION WITH PRIME VARIABLES

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1 Introduction

1.1 General description of the problem

The name "Diophantine equation" derives from Diophantus of Alexandria (about A.D. 250), author of a series of books called *Arithmetica*, who was the first mathematician to study solution of equations involving only integers numbers. One of the first problems was to know how well a real number can be approximated by rational numbers. We can talk in this case of Diophantine approximation.

We define a Diophantine inequality $|\mathscr{P}(\mathbf{n})| < \eta$, $\eta \in \mathbb{R}^+$, as an inequality for a real generalized polynomial $\mathscr{P}$ in r integer variables $\mathbf{n} = (n_1, \dots, n_r)$; however, in our research, we focus only on sums of monomials of r different variables.

Let $\omega \in \mathbb{R}$; the aim is to approximate ω with $F(\mathbf{x}, \mathbf{k}, \lambda)$ where

$$F(\mathbf{x}, \mathbf{k}, \boldsymbol{\lambda}) = F(\mathbf{x}) = F(x_1, \dots, x_r, k_1, \dots, k_r, \lambda_1, \dots, \lambda_r) = \lambda_1 x_1^{k_1} + \dots + \lambda_r x_r^{k_r} \tag{1.1}$$

with

- $\mathbf{x} = (x_1, \dots, x_r)$ where $x_1, \dots, x_r$ are integer variables,
- $\mathbf{k} = (k_1, \dots, k_r)$ with $k_i \in \mathbb{R}^+$ fixed real exponents,
- $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$ with $\lambda_i \in \mathbb{R}^*$ fixed real constants.

In our dissertation $\mathbf{k} = (k_1, \dots, k_r)$ will have at most one non integer component. More precisely, $k_1 \dots, k_{r-1}$, will be fixed positive integers, while $k_r > 1$ will be a real parameter. One of our goals will be to prove non-trivial results in as wide a range for k_r as possible. Moreover we will only look for solutions in prime variables which we denote by $\mathbf{x} = \mathbf{p} = (p_1, \dots, p_r)$.

The exact value of ω may be not reachable in general for obvious reasons but our goal is to prove that $\forall \omega \in \mathbb{R}$ and $\forall \eta > 0$ fixed, there exist infinitely many solutions of the Diophantine inequality

$$|F(\mathbf{p}) - \omega| < \eta \tag{1.2}$$

with some conditions on the coefficients λ_j. In fact, some further hypothesis is necessary:

- on signs: otherwise it would be impossible to approximate all real numbers.
- at least on one ratio λ_i/λ_j: in fact, if all coefficients λ_j and all exponents k_j were integers, then F would only take on integral values and it would be impossible to approximate non integers; more generally, if all exponents were integers and all the coefficients λ_j were rational, the values of F would all be multiple of some fixed rational number and, again, it would be impossible to approximate other real numbers.

The difficulties of the problem depend both on the number of variables r and on the exponents k_j. We call "density" of the function F the quantity

$$\rho = \rho(F) = \frac{1}{k_1} + \cdots + \frac{1}{k_r}, \tag{1.3}$$

in agreement with the definition of density in typical additive problems with integers.

The "smaller" is ρ the harder the problem. For small value of the density the goal is to prove that $|F(\mathbf{p}) - \omega| < \eta$ has infinitely many solutions for some fixed $\eta > 0$, but if ρ is larger we can prove stronger results taking η as a small negative power of $\mathrm{M}(\mathbf{p},\mathbf{k}) := (\max_j(p_1^{k_1}, \ldots, p_r^{k_r}))$, say

$$\eta = (\mathrm{M}(\mathbf{p},\mathbf{k}))^{-\psi(\mathbf{k})}, \tag{1.4}$$

where $\mathbf{k}$ is the parameter referred to above and $\psi(\mathbf{k}) > 0$. The number of variables r also plays a role in determining whether a problem is easy or difficult: having more variables complicates the computations but it is possible to apply the Cauchy-Schwartz and the Hölder inequalities in several different ways and thus make the problem "easier".

In these kinds of problems it is enough to prove that there is at least one solution of inequality (1.2) with $p_j^{k_j} \in [\delta X_n, X_n]$ for $j = 1, \ldots, r$, where δ is a small positive constant and X_n is a suitable sequence with $\lim\limits_{n\to+\infty} X_n = +\infty$. Actually X (we are dropping the useless suffix n), will be related to the convergents of the continued fraction of the ratio λ_i/λ_j, that we supposed to be irrational, and we can define all parameters in terms of X. In other words, as we supposed that λ_i/λ_j is irrational, there exist infinitely many solutions of the inequality

$$\left| \frac{\lambda_i}{\lambda_j} - \frac{a}{q} \right| < \frac{1}{q^2}$$

with $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a,q) = 1$. We will take some fixed positive power of the denominator $q > 0$ as a sequence of X that therefore tends to infinity.

1.2 Main known results

Many recent results are known with various types of assumptions and conclusions. The first approach to deal with these kinds of problems is due to Davenport & Heilbronn in 1946 (see [10]): their technique is a variant of the circle method as we will explain later in the next section. In their problem $\mathbf{k} = (2,2,2,2,2)$ and all the five variables are integers numbers.

Following the Davenport & Heilbronn variant, many authors faced similar problems where variables were prime numbers: Vaughan in 1974 (see [44]) dealt with a ternary linear form, $\mathbf{k} = (1,1,1)$ while Brüdern, Cook & Perelli in [5] dealt with a binary linear form in prime arguments with $\mathbf{k} = (1,1)$ in 1997. Cook & Fox in [7] dealt with a ternary form with primes squared, $\mathbf{k} = (2,2,2)$, that was improved in term of approximation by Harman in 2004 (see [22]). Furthermore, Cook in [6] gave a deeper description of general the problem with r primes $p_1,\dots,p_r$ and this work was improved by Cook & Harman in 2006 (see [8]).

Some authors dealt with the number of *exceptional* real numbers ω that cannot be well approximated with (1.1). In detail, we call a well-spaced set $\mathscr{V}$ a set of positive reals where there is a $c > 0$ such that $u,v \in \mathscr{V}, \quad u \neq v \ \Rightarrow \ |u-v| > c$. The idea is to study the set $E(\mathscr{V},X,\delta)$ that denotes the number of $\omega \in \mathscr{V}$ with $\omega \leq X$ such that the inequality

$$|F(\mathbf{p}) - \omega| < \omega^{-\delta}$$

has no solutions in primes $p_1,\dots,p_r$.

There are some differences between the results quoted above and our purpose: in our case the value of η does depend on the primes p_j and it will be actually a negative power of the maximum of the p_j, see (1.4), while in the papers quoted above η is a small negative power of ω and the assumption that the coefficients λ_j are all positive is not a restriction. Moreover k_j is the same positive integer for all j. Nevertheless the assumptions that λ_1/λ_2, say, must be irrational is still the heart of the matter and it is necessary, if one wants to approximate to all real numbers and not only some proper subset; in fact we remark that if all λ_i/λ_j were rational $F(\mathbf{p})$ would be multiple of some fixed, positive real number, if all k_j are integers, as we pointed out above.

Vaughan in [44] followed another approach, that is the same we are using in our dissertation: dealing with a ternary linear form in prime arguments with $\mathbf{k} = (1,1,1)$ and assuming some more suitable conditions on the λ_j, he proved that there are infinitely many solutions of the problem (1.2) when η is defined as in (1.4); in his case $\psi(\mathbf{k}) = \psi = \frac{1}{10}$. Such result was improved by Baker & Harman in [2] with $\psi = \frac{1}{6}$, later by Harman in [21] with $\psi = \frac{1}{5}$ and finally by Matomäki in [34] with $\psi = \frac{2}{9}$. Baker & Harman [2] also proved that under the generalized Riemann hypothesis it is possible to reach $\psi = \frac{1}{4}$.

Piatetski-Shapiro in [39] was the first to approach a Diophantine problem, analogue of the Waring-Goldbach problem, in which a non-integral exponent appears. Later Tolev in [42] improved the results of Piatetski-Shapiro on the ternary problem with all $\lambda_j = 1$ and all the exponents k_j equal to a rational constant $k \notin \mathbb{N}$.

Languasco & Zaccagnini in [29] and [30] dealt with ternary problems with different powers k_j, one of them depending on a real parameter k, that is, $\mathbf{k} = (1,2,k)$ and $\mathbf{k} = (1,1,k)$, respectively. The idea is to get the best approximation reaching the widest range of values for k whereby the inequality holds. In both cases, the choice of the values of the different parameters, and hence the range for k and the value of $\psi(\mathbf{k})$, depends on the solution of an optimization problem; see in particular Sections 2.3 and 3.3. A result of this dissertation is the improvement of the ternary problem in [30] as one can see in Chapter 4 and [15]. The case $\mathbf{k} = (1,1,1)$ has not been treated here since, as described above, has already been dealt with by several authors; in fact for integral values of the exponents, stronger estimates are available for the exponential sums we will define in the next sections.

Languasco & Zaccagnini also dealt with a quaternary form, $\mathbf{k} = (1,2,2,2)$ in [28] with a prime and 3 squares of primes obtaining $\psi = \frac{1}{18}$; this was improved by Liu & Sun in [33] with $\psi = \frac{1}{16}$ using the Harman technique and recently refined by Wang & Yao in [48] with $\psi = \frac{1}{14}$ using a better estimation on the minor arc due to Harman [22] and Harman & Kumchev [23]. Another result of this dissertation is the generalization such a quaternary problem as one can see in Chapter 2 and [13].

Recently Mu in [36] dealt with a problem in five variables with four squares of primes and a k-th integer power of a prime, $k \geq 3$; in this case $\mathbf{k} = (2,2,2,2,k)$. Ge & Li in [16] used a quaternary form with different integer powers k_j. Mu & Lü dealt also with a quaternary problem in [37] with two square of primes, a cube of a prime and a k-th power of a prime with $k \geq 3$ integer: $\mathbf{k} = (2,2,3,k)$. Finally, as we will show in this dissertation in Chapter 4, the results of the ternary problem [30] is improved by Languasco, Zaccagnini & the author of this dissertation in [15] widening the k-range to $k \in (1,3]$ and giving a stronger bound for the approximation in the common range, combining Harman's technique on the minor arc with the L^4-norm of the relevant Weyl sum over primes S_k.

Another approach in Diophantine approximation considers a real analogue of the Goldbach-Linnik problem, i.e. adding to the polynomial a combination of primes and s powers of 2, where s is a fixed integer:

$$\lambda_1 p_1^{k_1} + \cdots + \lambda_r p_r^{k_r} + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s}.$$

The first result on this matter is due to Linnik himself in [31] and [32] but for more recent results proved on the topic we cite Parsell [38], Languasco & Zaccagnini [27], Languasco & Settimi [26], Wang [47]. The approach is similar to the one without powers of two and the heart of the matter is again the assumption that some ratio between two of the λ_j must be irrational.

We remark that the presence of the powers of 2 does not change the density of the problem.

Some authors have faced similar problems with primes and almost-prime[1] numbers (see Harman [20] and Yang [50]) or the special prime numbers: see Dimitrov [11] and Tolev [43], where a special prime number is a prime p that satisfy that $p+2$ has at most n prime factors where n varies according to the different papers.

1.3 The Circle Method

The Circle Method is an idea developed by Hardy & Ramanujan [18] in investigations on the partition function and later applied in additive problems like the Waring Problem and the Goldbach Conjecture. Actually various problems in additive number theory motivated the development of the Circle Method and many problems were solved with this method by Hardy & Littlewood in the beginning of the XX century; the standard reference for this method is Vaughan, [46].

In general a typical additive problem has the following nature:

Given $s \geq 2$ subsets of $\mathbb{N}$, $A_1, \ldots, A_s$, not necessarily distinct and a positive integer n, determine how many solutions has the equation

$$n = a_1 + \cdots + a_s$$

where $a_j \in A_j$ for $j = 1, \ldots, s$.

Sometimes for large values of n is sufficient to show that such an equation has at least one solution. Nevertheless, even for large n, there can be arithmetic conditions that force some restriction in the choice of n: for instance, in the binary Goldbach problem n must be even otherwise the conjecture is usually false: it is not possible to write an odd number as a sum of two primes because every prime greater than 2 is odd; on the other hand in ternary Goldbach problem n must be odd. In the Waring problem - solved by D. Hilbert in 1909 in ”Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches Problem)” - all the subsets $A_1, \ldots, A_s$ are equal to the k-th powers of the natural numbers and the goal is to represent every natural number n as a sum of at most s k-th powers.

[1] P_r is an almost-prime number is it is an integer number with at most r prime factor

In order to study these problems we start by the original Hardy, Littlewood & Ramanujan ideas: we assume that $A_1 = A_2 = \cdots A_n = A$ is an infinite set and we define a **generating function**

$$f_A(z) := \sum_{n=0}^{+\infty} a(n)z^n \qquad a_n = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise.} \end{cases}$$

f is a power series with radius of convergence 1. Now, setting $r_s(n)$ the number of ways of writing n as the sum of s elements of A,

$$r_s(n) := |\{(a_1, \dots, a_s) \in A^s : n = a_1 + \cdots + a_s\}|$$

we have

$$f^s(z) = \sum_{n=0}^{+\infty} r_s(n)^n z^n.$$

Consequently,

$$r_s(n) = \frac{1}{2\pi i} \oint_\gamma \frac{f_A(z)^s}{z^{n+1}} \mathrm{d}z$$

where γ is the circle whose center is at the origin and radius is $\rho < 1$.

Vinogradov in the 1930's introduced some simplifications using finite sums instead of infinite sums: the problem we considered since the beginning is interesting only if A is an infinite set, otherwise we could enumerate the different values of $a_1 + \ldots + a_s$ in a finite numbers of steps. If we define $A_N = \{a \in A : a \leq N\}$, A_N is a finite set for every N and the A_N is an increasing sequence of subsets of A. For each N it is possible to consider the **truncated generating function** for every A_N without any singularities and without any problem of convergence.

Also with Vinogradov's approach the problem can be transformed in a complex analysis problem where the circle is replaced by the interval $[0, 1]$.

The idea of the Circle Method is to split the interval $[0, 1]$ into disjoint pieces called *major arcs* and *minor arcs*: on the major arcs we will find the main term of the contribution bounded away from zero; they are the union of suitable neighborhoods of rational numbers with "small" denominators. The minor arcs are the complement of the major arcs and we should prove that their contribution is smaller than the major arcs.

The Circle Method was used to attack problems involving prime numbers so it needs several results concerning analytic number theory. For those reason we will use the exponential sums defined in Appendix B and their weighted version as basic tools also for the problems we will discuss in the next Chapters.

More details on the Circle Method, it is possible to consult the classical literature; this section is adapted from Davenport [9] and Miller & Takloo-Bighash [35].

1.4 The Davenport-Heilbronn variant

A variant of the classical circle method was introduced by Davenport & Heilbronn in 1946 (see [10]) in order to attack Diophantine problems; in particular it permits the investigation of some given type of inequalities as (1.2). The integration on a circle, or equivalently on the interval $[0,1]$, is now replaced by integration on the whole real line. The problem we are dealing with is not to count the exact hits like in Goldbach or Waring's problems but to find how close it is possible to be at any real number ω as described in section 1.1.

For this purpose we need a measure of "proximity" that can be chosen in many ways: in these Diophantine problems there is only one major arc, which is a suitable neighborhood of 0. The most natural way is to choose the characteristic function $\chi_{[-\eta,\eta]}$ of the interval $[-\eta,\eta]$. Nevertheless, as we will see later, we need a function whose essential feature is that the rate of the decay at infinity of its inverse Fourier transform is $\mathscr{O}\left(|\alpha|^{-1}\right)$. We want to point out that since we are considering only even functions, it makes no difference to speak of Fourier transform or inverse Fourier transform. To keep the same notation of the references in bibliography we will call K_η the function and $\widehat{K}_\eta$ its Fourier transform. In this case $\widehat{K}_\eta(\alpha) = \chi_{[-\eta,\eta]}(\alpha)$ is not suitable as its inverse Fourier transform behaves at infinity like $|\alpha|^{-1}$.

We need a continuous function as $\widehat{K}_\eta(\alpha)$ (see Fig.1.4), so we introduce

$$\widehat{K}_\eta(\alpha) := \max(0, \eta - |\alpha|) \quad \text{where} \quad \eta > 0$$

whose inverse Fourier transform is

$$K_\eta(\alpha) = \left(\frac{\sin(\pi\alpha\eta)}{\pi\alpha}\right)^2$$

for $\alpha \neq 0$ and, by continuity, $K_\eta(0) = \eta^2$. It vanishes at infinity like α^{-2} and it is trivial to prove that

$$K_\eta(\alpha) \ll \min(\eta^2, |\alpha|^{-2}). \tag{1.5}$$

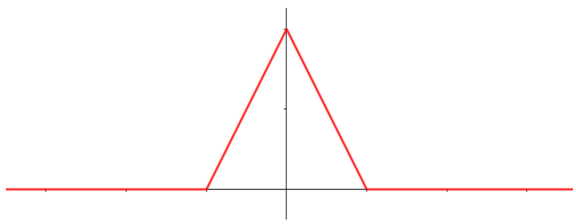

Figure 1.1: Graph of function $\widehat{K}_\eta(\alpha)$

1.5 L^2-norm estimation approach

First of all, we need to define an exponential sum related to the primes, similarly to the Goldbach problem: for $k \geq 1$, let

$$S_k(\alpha) = \sum_{\delta X \leq p^k \leq X} \log p \, e(p^k \alpha). \tag{1.6}$$

where δ is a small, fixed positive constant; the choice of starting from δX instead of 1 or 2 is needed for technical reasons but it does not alter the final result. δ may also depend on the coefficients λ_j.

We will approximate the exponential sum S_k with both the corresponding exponential sum with the coefficient $\log p$ replaced by its average 1 and the exponential integral:

$$U_k(\alpha) = \sum_{\delta X \leq n^k \leq X} e(n^k \alpha) \tag{1.7}$$

$$T_k(\alpha) = \int_{(\delta X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\alpha t^k) \, \mathrm{d}t. \tag{1.8}$$

The reader can find in Appendix B some details on the bounds for the exponential sums and integrals $S_k(\alpha)$, $U_k(\alpha)$ and $T_k(\alpha)$.

The original works of Davenport & Heillbronn in [10] and later Vaughan in [44] and [45] approximate directly the difference $|S_k(\alpha) - T_k(\alpha)|$ and estimate it in a trivial way with $O(1)$. The idea of Brüdern, Cook & Perelli in [5] and Languasco & Zaccagnini in [30], is to improve these estimations taking the L^2-norm of $|S_k(\alpha) - T_k(\alpha)|$ leading to significantly better conditions and to have a wider major arc (we will define the major arc for our problems in Section 1.6 similarly to the definition of major arcs for the Goldbach problem) compared to the original DH approach.

In fact, setting the generalized version of the Selberg integral

$$\mathscr{J}_k(X,h) = \int_X^{2X} \left(\theta((x+h)^{\frac{1}{k}}) - \theta(x^{\frac{1}{k}}) - ((x+h)^{\frac{1}{k}} - x^{\frac{1}{k}}) \right)^2 \mathrm{d}x,$$

where $\theta(x) = \sum_{p \leq x} \log p$ is the first Chebyshev function, Gallagher's Lemma (Lemma 1 of [12]) allows us to connect the mean-square average of $S_k - U_k$ to the Selberg integral $\mathscr{J}_k$, instead of using a point-wise bound (see [5]). We will have then Lemmas B.11 and B.12 that allow us to get a proper bound for the quantity

$$\int_{-Y}^{Y} |S_k(\alpha) - U_k(\alpha)|^2 \mathrm{d}\alpha$$

for $0 < Y < \frac{1}{2}$.

The Davenport & Heilbronn method is related to the maximum of the quantity $\theta(x) - x$, while Brüdern, Cook & Perelli and Languasco & Zaccagnini are related to the variance of $\theta(x) - x$: as we know that the behavior of such a function is better when we take the average, we gain something in terms of width of the major arc.

1.6 Setting the problem

We decompose $\mathbb{R}$ into subsets such that $\mathbb{R} = \mathfrak{M} \cup m \cup t$ where $\mathfrak{M}$ is the major arc, m is the minor arc and t is the trivial arc. In contrast to Goldbach problem the major arc is a small interval centered in 0 that becomes smaller as our parameter X increases. Moreover in Goldbach the number of the arcs increases as the parameter considered grows while in our problems we have only 3 or 4 arcs.

We expect to have on $\mathfrak{M}$ the main term with the right order of magnitude without any special hypothesis on the coefficients λ_j. It is necessary to prove that the contributions of the trivial arc and the minor arc are small compared to the major arc: the contribution of the trivial arc will not be a problem because we will prove its contribution is "tiny" with respect to the main term because of the Fourier transform property. The real problem is on the minor arc where we will need the full force of the hypothesis on the λ_j and the theory of continued fractions. There is another difference to the Goldbach problem: we will be able to prove that the terms on the minor arc and on the trivial arc are a little-o of the main term only on a suitable sequence where the exponential sums are small taking advantage of the fact that the ratio λ_i/λ_j is irrational.

Sometimes it may happen that for matters of choice of the parameters we have a gap between the end of the major arc and the beginning of the minor arc. This kind of intermediate arc can be filled, depending on the case, with standard estimates such as those that have been used for instance by Liu & Sun in [33] and by Languasco, Zaccagnini & the author of this dissertation in [15] (see Chapter 4).

$\mathcal{N}(X)$ will always denote the number of solutions of our inequality and we expect that a lower bound for $\mathcal{N}(X)$ is $\eta X^{\rho-1}$, where ρ is the density of the problem defined in (1.3). This is a reason why the smaller is the density the more difficult is the problem. For the same reason every problem presented in this dissertation has got a density greater than 1.

In general, we expect that a Diophantine problem, once that the necessary conditions have been discussed, is soluble $\forall\eta > 0$ fixed, as soon as $\rho > 1$; it is just because today's techniques are not strong enough that we need more restrictive conditions, that is, to take larger values of ρ. For the same reason, any numerical simulation about the theorems that will be enunciated suggests that less restrictive conditions may be taken or that the density may be reduced.

Another clue suggesting that we may take less restrictive conditions is that the term p_4^k is always irrational if k is not integer and one might argue that the hypothesis of irrationality might become redundant. Though p^k is uniformly distributed (mod 1) it is not of any help as we are approximating **any** real number and not within an equivalence class. Today's techniques do not allow us to have better estimates on all $\mathbb{R}$ than those in the Appendix B.

One thing that allows us to have better estimates is to increase the number r of variables as it happens for the Goldbach ternary case, which, unlike the binary case, has been solved. In fact, having more variables complicates the computations, but on the other hand with more variables it is possible to apply the Cauchy-Schwartz and the Hölder inequalities in several different ways getting better results. For instance, this fact can be seen clearly in Section 4.5 where the Hölder inequality is used in three different ways depending on the specific needs.

Remark: from now on, anytime we use the Vinogradov symbol $\ll$ or $\gg$ we drop the dependence of the approximation from the constants λ_j, δ and k. Furthermore ε stands for a sufficiently small positive number whose value could vary depending on the occurrences. We use the notation $f = \infty(g)$ for $g = o(f)$.

1.7 Results

Theorem 1.1 ([13], Theorem 1)**.** *Assume that* $1 < k < 14/5$*,* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *be non-zero real numbers, not all of the same sign, that* λ_1/λ_2 *is irrational and let* ω *be a real number. The inequality*

$$\left|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega\right| \leq (\mathrm{M}(\mathbf{p}, \mathbf{k}))^{-\psi(k)+\varepsilon}$$

has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 *for any* $\varepsilon > 0$*, where*

$$\psi(k) = \min\left(\frac{1}{14}, \frac{14-5k}{28k}\right).$$

The first problem we are dealing with in Chapter 2 is a quaternary problem that can be seen as a generalization of Languasco & Zaccagnini [28], Liu & Sun [33] and Wang & Yao [48]. In particular we recover [48] for $k = 2$ obtaining the same result (neglecting the log powers) but we also enlarged the k-range reaching $k = \frac{14}{5} - \varepsilon$.

Theorem 1.2 ([14], Theorem 1)**.** *Assume that* $1 < k < 7/6$*,* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *be non-zero real numbers, not all of the same sign, that* λ_1/λ_2 *is irrational and let* ω *be a real number. The inequality*

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega\right| \leq (\mathrm{M}(\mathbf{p}, \mathbf{k}))^{-\frac{7-6k}{14k}+\varepsilon}$$

has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 for any $\varepsilon > 0$.

The second quaternary problem outlined in Chapter 3 has a lower density and it leads to a narrower range for k. It is very similar to Harman's ternary work in [22], but the addition of a variable makes it possible to raise the density and then to get infinitely many solutions for $k < 7/6$.

Theorem 1.3 ([15], Theorem 1)**.** *Assume that $1 < k \le 3$, λ_1, λ_2 and λ_3 are non-zero real numbers, not all of the same sign, that λ_1/λ_2 is irrational and let ω be a real number. The inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega| \le \left(\mathrm{M}(\mathbf{p},\mathbf{k})\right)^{-\psi(k)+\varepsilon}$$

has infinitely many solutions in prime variables p_1, p_2, p_3 for any $\varepsilon > 0$, where

$$\psi(k) = \begin{cases} (3-2k)/(6k) & \text{if } 1 < k \le \frac{6}{5}, \\ 1/12 & \text{if } \frac{6}{5} < k \le 2, \\ (3-k)/(6k) & \text{if } 2 < k < 3, \\ 1/24 & \text{if } k = 3. \end{cases} \tag{1.9}$$

As we remarked above, for integer values of k stronger estimates are available for exponential sums (see Appendix B) and they allow us to prove Theorem 1.3 with $\psi(3) = \frac{1}{24}$.

Problems in Theorem 1.1 and Theorem 1.3 have the same density but as we will see we can reach a wider range for k in Theorem 1.3. This fact can be explained noting that it is easier to handle the distribution of one prime than the distribution of the sum of two squares of primes although the density is the same. Theorem 1.3 is also a refining of a result of [30] which was only proved to hold for $1 < k < 4/3$. We point out that in the common range $1 < k < 4/3$ we have a stronger bound (see Chapter 4 here and [15]).

All three theorems involve a Diophantine approximation with a real k-th power of one of the primes.

2 One prime, two squares of primes and one k-th power of prime

The first problem is a generalization of Languasco & Zaccagnini [28], Liu & Sun [33] and Wang & Yao [48] who found the best approximation for $k = 2$ with $\psi = \frac{1}{14}$.

We want to investigate the following problem: let $k > 1$ be a real number and assume that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are non-zero real numbers, not all of the same sign and with the ratio λ_1/λ_2 irrational. Let ω be a real number. We would like to find a range for k where

$$|F(p_1, p_2, p_3, p_4, 1, 2, 2, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4) - \omega| = |\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega| \leq \eta \tag{2.1}$$

has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 for some $\eta \to 0$ a small negative power of the largest prime as in (1.4).

We will prove the following:

Theorem 2.1. *Assume that $1 < k < 14/5$, $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be non-zero real numbers, not all of the same sign, that λ_1/λ_2 is irrational and let ω be a real number. The inequality*

$$\left|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega\right| \leq (\mathrm{M}(\mathbf{p}, \mathbf{k}))^{-\psi(k)+\varepsilon}$$

has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 for any $\varepsilon > 0$, where

$$\psi(k) = \min\left(\frac{1}{14}, \frac{14-5k}{28k}\right).$$

Let

$$\mathscr{P}(X) = \{(p_1, p_2, p_3, p_4) : \delta X < p_1 < X,\ \delta X < p_2^2, p_3^2 < X,\ \delta X < p_4^k < X\}$$

and let us define

$$\mathscr{I}(\eta, \omega, \mathfrak{X}) = \int_{\mathfrak{X}} S_1(\lambda_1 \alpha) S_2(\lambda_2 \alpha) S_2(\lambda_3 \alpha) S_k(\lambda_4 \alpha) K_\eta(\alpha) e(-\omega\alpha) \mathrm{d}\alpha$$

where $\mathfrak{X}$ is a measurable subset of $\mathbb{R}$.

From the definitions of $S_j(\alpha)$ and performing the Fourier transform for $K_\eta(\alpha)$, we get

$$\begin{aligned}\mathscr{I}(\eta,\omega,\mathbb{R}) &= \sum_{(p_1,\dots,p_4)\in\mathscr{P}(X)} \log p_1 \log p_2 \log p_3 \log p_4 \ \max(0,\eta-|F(p_1,p_2,p_3,p_4,1,2,2,k)-\omega|)\\ &\le \eta(\log X)^4\mathscr{N}(X),\end{aligned}$$

where $\mathscr{N}(X)$ denotes the number of solutions of the inequality (2.1) with $(p_1,p_2,p_3,p_4)\in\mathscr{P}(X)$. In other words $\mathscr{I}(\eta,\omega,\mathbb{R})$ provides a lower bound for the quantity we are interested in and therefore for the proof of the theorem it is sufficient to prove that $\mathscr{I}(\eta,\omega,\mathbb{R})>0$ on a suitable sequence of values of X with limit infinity.

We now decompose $\mathbb{R}$ into subsets such that $\mathbb{R}=\mathfrak{M}\cup\mathfrak{m}\cup\mathfrak{t}$ where $\mathfrak{M}$ is the major arc, $\mathfrak{m}$ is the minor arc and $\mathfrak{t}$ is the trivial arc. The decomposition is the following:

$$\mathfrak{M}=\left[-\frac{P}{X},\frac{P}{X}\right] \qquad \mathfrak{m}=\left[\frac{P}{X},R\right]\cup\left[-R,-\frac{P}{X}\right] \qquad \mathfrak{t}=\mathbb{R}\backslash(\mathfrak{M}\cup\mathfrak{m})$$

so that

$$\mathscr{I}(\eta,\omega,\mathbb{R})=\mathscr{I}(\eta,\omega,\mathfrak{M})+\mathscr{I}(\eta,\omega,\mathfrak{m})+\mathscr{I}(\eta,\omega,\mathfrak{t}).$$

The parameters $P=P(X)>1$ and $R=R(X)>1/\eta$ are chosen later (see (2.7) and (2.10)) as well as $\eta=\eta(X)$, that, as we explained before, will be a small negative power of $\mathrm{M}(\mathbf{p},\mathbf{k})$ (and so of X, see (2.26)).

It is necessary to prove that $\mathscr{I}(\eta,\omega,\mathfrak{m})$ and $\mathscr{I}(\eta,\omega,\mathfrak{t})$ are both $o(\mathscr{I}(\eta,\omega,\mathfrak{M}))$. As we will see, in this first case we are dealing with, we do not have any gap between the major arc and the minor arc.

2.1 The major arc

Let us start from the major arc and the computation of the main term. We replace all S_k defined in (1.6) with the corresponding T_k defined in (1.8). This replacing brings up some errors that we must estimate by means of Lemma B.11, the Cauchy-Schwarz and the Hölder inequalities.

We write

$$\begin{aligned}\mathscr{I}(\eta,\omega,\mathfrak{M}) &= \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha\\ &= \int_{\mathfrak{M}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha\\ &\quad+\int_{\mathfrak{M}}(S_1(\lambda_1\alpha)-T_1(\lambda_1\alpha))T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathfrak{M}} S_1(\lambda_1\alpha)(S_2(\lambda_2\alpha)-T_2(\lambda_2\alpha))T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha\\
&+\int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_2(\lambda_3\alpha)-T_2(\lambda_3\alpha))T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha\\
&+\int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)(S_k(\lambda_4\alpha)-T_k(\lambda_4\alpha))K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha\\
=&J_1+J_2+J_3+J_4+J_5, \qquad (2.2)
\end{aligned}
$$

say.

J_1: Main Term

As the reader might expect the main term is given by the summand J_1.

Let $H(\alpha)=T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)$ so that

$$
J_1=\int_{\mathbb{R}} H(\alpha)\mathrm{d}\alpha+\mathscr{O}\left(\int_{P/X}^{+\infty}|H(\alpha)|\mathrm{d}\alpha\right).
$$

Using inequalities (1.5) and Theorem B.1,

$$
\begin{aligned}
\int_{P/X}^{+\infty}|H(\alpha)|\mathrm{d}\alpha \ll& X^{-\frac{1}{2}}X^{-\frac{1}{2}}X^{\frac{1}{k}-1}\eta^2\int_{P/X}^{+\infty}\frac{\mathrm{d}\alpha}{\alpha^4}\\
\ll& X^{\frac{1}{k}+1}\eta^2P^{-3}=o\left(X^{\frac{1}{k}+1}\eta^2\right)
\end{aligned}
$$

provided that $P\to+\infty$.

Let $D=[\delta X,X]\times[(\delta X)^{\frac{1}{2}},X^{\frac{1}{2}}]^2\times[(\delta X)^{\frac{1}{k}},X^{\frac{1}{k}}]$; we have

$$
\begin{aligned}
\int_{\mathbb{R}} H(\alpha)\mathrm{d}\alpha&=\int\cdots\int_D\int_{\mathbb{R}} e((\lambda_1t_1+\lambda_2t_2^2+\lambda_3t_3^2+\lambda_4t_4^k-\omega)\alpha)K_\eta(\alpha)\mathrm{d}\alpha\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4\\
&=\int\cdots\int_D \max(0,\eta-|\lambda_1t_1+\lambda_2t_2^2+\lambda_3t_3^2+\lambda_4t_4^k-\omega)|)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4.
\end{aligned}
$$

Apart from trivial changes of sign, there are essentially three cases as in [28]:

1. $\lambda_1>0$, $\lambda_2>0$, $\lambda_3>0$, $\lambda_4<0$
2. $\lambda_1>0$, $\lambda_2>0$, $\lambda_3<0$, $\lambda_4<0$
3. $\lambda_1>0$, $\lambda_2<0$, $\lambda_3<0$, $\lambda_4<0$.

We deal with the second case, the other ones being similar. We warn the reader that here it may be necessary to adjust the value of δ in order to guarantee the necessary set inclusions. Let us perform the following change of variables: $u_1=t_1-\frac{\omega}{\lambda_1}$ (in order to make the dependence on ω disappear), $u_2=t_2^2$, $u_3=t_3^2$, $u_4=t_4^k$ so that the set D becomes essentially $[\delta X,X]^4$ and let us define

$D' = [\delta X, (1-\delta)X]^4$ for large X. The Jacobian determinant of the change of variables above is $\frac{1}{4k} u_2^{-\frac{1}{2}} u_3^{-\frac{1}{2}} u_4^{\frac{1}{k}-1}$. Then

$$\begin{aligned}\int_{\mathbb{R}} H(\alpha)\mathrm{d}\alpha \gg& \int \cdots \int_{D'} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4)|) \frac{\mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \mathrm{d}u_4}{u_2^{\frac{1}{2}} u_3^{\frac{1}{2}} u_4^{1-\frac{1}{k}}} \\ \gg& X^{\frac{1}{k}-2} \int \cdots \int_{D'} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4)|) \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \mathrm{d}u_4.\end{aligned}$$

Apart from sign, the computation is essentially symmetrical with respect to the coefficients λ_j: we assume, as we may, that $|\lambda_4| \geq \max(\lambda_1, \lambda_2, \lambda_3)$, the other cases being similar.

Now, for $j = 1,2,3$ let $a_j = \frac{4|\lambda_4|}{|\lambda_j|}$, $b_j = \frac{3}{2} a_j$ and $\mathscr{D}_j = [a_j \delta X, b_j \delta X]$; if $u_j \in \mathscr{D}_j$ for $j = 1,2,3$ then

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in [2|\lambda_4|\delta X, 8|\lambda_4|\delta X]$$

so that, for every choice of (u_1, u_2, u_3) the interval

$$[a,b] = \left[\frac{1}{|\lambda_4|}\left(-\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)\right), \frac{1}{|\lambda_4|}\left(\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)\right)\right]$$

is contained in $[\delta X, (1-\delta)X]$. In other words, for $u_4 \in [a,b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ cover the whole interval $[-\eta, \eta]$. Hence for any $(u_1, u_2, u_3) \in \mathscr{D}_1 \times \mathscr{D}_2 \times \mathscr{D}_3$ we have

$$\begin{aligned}&\int_{\delta X}^{(1-\delta)X} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|)\mathrm{d}u_4 \\ &\geq |\lambda_4|^{-1} \int_{-\eta}^{\eta} \max(0, \eta - |u|)\mathrm{d}u \gg \eta^2.\end{aligned}$$

Finally,

$$\int_{\mathbb{R}} H(\alpha)\mathrm{d}\alpha \gg \eta^2 X^{\frac{1}{k}-2} \iiint_{\mathscr{D}_1 \times \mathscr{D}_2 \times \mathscr{D}_3} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \gg \eta^2 X^{\frac{1}{k}-2} X^3 = \eta^2 X^{\frac{1}{k}+1}.$$

It means that the lower bound for J_1 is $\eta^2 X^{\frac{1}{k}+1}$, as expected.

Bound for J_2

We expect the main term to have the dominant asymptotic behavior, then we shall prove that all the remaining terms of the sum (2.2) are $o\left(\eta^2 X^{\frac{1}{k}+1}\right)$.

Retrieving (1.5) and using the triangle inequality,

$$\begin{aligned}J_2 =& \int_{\mathfrak{M}} (S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha)) T_2(\lambda_2\alpha) T_2(\lambda_3\alpha) T_k(\lambda_4\alpha) K_\eta(\alpha) e(-\omega\alpha)\mathrm{d}\alpha \\ \ll& \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ \ll& \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha\end{aligned}$$

$$
\begin{aligned}
&+\eta^2\int_{\mathfrak{M}}|U_1(\lambda_1\alpha)-T_1(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha\\
=&\eta^2(A_2+B_2),
\end{aligned}
$$

say, where $U_1(\lambda_1\alpha)$ is given by (1.7).

Using the Cauchy-Schwarz inequality and remembering the definition of $\mathfrak{M}$,

$$
A_2 \ll \left(\int_{-P/X}^{P/X}|S_1(\lambda_1\alpha)-U_1(\lambda_1\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{-P/X}^{P/X}|T_2(\lambda_2\alpha)|^2|T_2(\lambda_3\alpha)|^2|T_k(\lambda_4\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}};
$$

then using Lemma B.11 and Lemma B.12,

$$
\begin{aligned}
\int_{-P/X}^{P/X}|S_1(\lambda_1\alpha)-U_1(\lambda_1\alpha)|^2\mathrm{d}\alpha \ll& \frac{P^2}{X^2}\mathscr{J}_1\left(X,\frac{X}{2P}\right)\\
&\ll X\exp\left(-c_1\left(\frac{\log X}{\log\log X}\right)^{\frac{1}{3}}\right)\ll\frac{X}{(\log X)^A}
\end{aligned}
$$

for any fixed $A>0$ and $X^{1-\frac{5}{6}+\varepsilon}\le\frac{X}{P}\le X$ (conditions of Lemma B.12).

Let us point out that in the estimation above we have neglected the two terms

$$
\frac{X^{2/k-2}\log^2 X}{P/X}+P^2/X
$$

of Lemma B.11. This can be explained by the fact that the hypotheses of Lemma B.12 must be met and therefore these terms become negligible.

The first condition on P (with $k=1$) is then the following:

$$
\frac{X}{P}\ge X^{\frac{1}{6}+\varepsilon}\quad\Rightarrow\quad P\le X^{\frac{5}{6}-\varepsilon}. \tag{2.3}
$$

Let us complete the estimate for A_2:

$$
\begin{aligned}
A_2 \ll \left(\frac{X}{(\log X)^A}\right)^{\frac{1}{2}}\Bigg(&\int_0^{1/X}|T_2(\lambda_2\alpha)|^2|T_2(\lambda_3\alpha)|^2|T_k(\lambda_4\alpha)|^2\mathrm{d}\alpha\\
&+\int_{1/X}^{P/X}|T_2(\lambda_2\alpha)|^2|T_2(\lambda_3\alpha)|^2|T_k(\lambda_4\alpha)|^2\mathrm{d}\alpha\Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Remembering Theorem B.1,

$$
\begin{aligned}
A_2 &\ll \left(\frac{X}{(\log X)^A}\right)^{\frac{1}{2}}\left(\int_0^{1/X}(X^{\frac{1}{2}})^2(X^{\frac{1}{2}})^2(X^{\frac{1}{k}})^2\mathrm{d}\alpha+\int_{1/X}^{P/X}\frac{(X^{-\frac{1}{2}})^2(X^{-\frac{1}{2}})^2(X^{\frac{1}{k}-1})^2}{\alpha^6}\mathrm{d}\alpha\right)^{\frac{1}{2}}\\
&\ll \left(\frac{X}{(\log X)^A}\right)^{\frac{1}{2}}\left(\int_0^{1/X}X^{2+\frac{2}{k}}\mathrm{d}\alpha+\int_{1/X}^{P/X}\frac{X^{\frac{2}{k}-4}}{\alpha^6}\mathrm{d}\alpha\right)^{\frac{1}{2}}\\
&\ll \left(\frac{X}{(\log X)^A}\right)^{\frac{1}{2}}\left(X^{1+\frac{2}{k}}+X^{\frac{2}{k}-4}\cdot\left(\frac{1}{X}\right)^{-5}\right)^{\frac{1}{2}}\\
&\ll \left(\frac{X}{(\log X)^A}\right)^{\frac{1}{2}}\left(X^{\frac{2}{k}+1}\right)^{\frac{1}{2}}=\frac{X^{\frac{1}{2}}}{(\log X)^{\frac{A}{2}}}X^{\frac{1}{2}+\frac{1}{k}}=\frac{X^{1+\frac{1}{k}}}{(\log X)^{\frac{A}{2}}}=o(X^{1+\frac{1}{k}})
\end{aligned}
$$

for all $A > 0$.

Now we need an estimation for B_2: noting that

$$1+|\alpha|X \ll \begin{cases} 1 & \text{if } |\alpha| \le \frac{1}{X} \\ |\alpha|X & \text{if } |\alpha| \ge \frac{1}{X}, \end{cases}$$

in this case we use Theorem B.2,

$$\begin{aligned} B_2 &= \int_{\mathfrak{M}} |U_1(\lambda_1\alpha) - T_1(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\ll \int_0^{1/X} |T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha + X\int_{1/X}^{P/X} \alpha|T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\ll \int_0^{1/X} X^{\frac{1}{2}}X^{\frac{1}{2}}X^{\frac{1}{k}}\mathrm{d}\alpha + XX^{-\frac{1}{2}}X^{-\frac{1}{2}}X^{\frac{1}{k}-1}\int_{1/X}^{P/X}\frac{\alpha}{\alpha^3}\mathrm{d}\alpha \\ &\ll X^{\frac{1}{k}} = o\left(X^{\frac{1}{k}+1}\right). \end{aligned}$$

Bound for J_4

The computations on J_3 are similar to and simpler than the corresponding one on J_4, so we will skip them. Using the triangle inequality and (1.5),

$$\begin{aligned} J_4 &= \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha))T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\ &\ll \eta^2\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\le \eta^2\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\quad + \eta^2\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||U_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &= \eta^2(A_4 + B_4), \end{aligned}$$

say. Using the trivial inequality $|S_2(\alpha)| \ll X^{\frac{1}{2}}$, Theorem B.1 and then the Cauchy-Schwarz inequality,

$$\begin{aligned} A_4 &\ll X^{\frac{1}{2}}X^{\frac{1}{k}}\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|\mathrm{d}\alpha \\ &\ll X^{\frac{1}{2}}X^{\frac{1}{k}}\left(\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{M}} |S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}. \end{aligned}$$

Using Theorem B.3 and Lemmas B.11-B.12, for any fixed A,

$$A_4 \ll X^{\frac{1}{2}+\frac{1}{k}}(X\log X)^{\frac{1}{2}}(\log X)^{-\frac{A}{2}} = X^{1+\frac{1}{k}}(\log X)^{\frac{1}{2}-\frac{A}{2}} = o\left(X^{\frac{1}{k}+1}\right)$$

as long as $A > 1$.

As for A_2 we used in the estimation above Lemma B.11 that has two more terms, but also in this case these terms are negligible if we want to meet the hypothesis of Lemma B.12: in fact it requires that

$$X^{1-\frac{5}{12}+\varepsilon} \le \frac{X}{P} \le X$$

and this is consistent with the choice we will make in (2.4).

Again using Theorem B.2,

$$\begin{aligned}B_4 &= \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||U_2(\lambda_3\alpha)-T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha\\ &\ll \int_0^{1/X} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha + X\int_{1/X}^{P/X} \alpha|S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha.\end{aligned}$$

Remembering that $|\alpha| \le \frac{P}{X}$ on $\mathfrak{M}$ and using the Cauchy inequality, trivial bounds, Theorem B.3 and Lemma B.4, we have

$$\begin{aligned}B_4 \ll & X X^{\frac{1}{2}} X^{\frac{1}{k}} \frac{1}{X} + X X^{\frac{1}{k}} \left(\int_{1/X}^{P/X} |S_1(\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_{1/X}^{P/X} \alpha^2|S_2(\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}\\ \ll & X^{\frac{1}{2}+\frac{1}{k}} + X^{1+\frac{1}{k}}(X\log X)^{\frac{1}{2}} \left(\int_{1/X}^{P/X} \alpha^4\mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\int_{1/X}^{P/X} |S_2(\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}}\\ \ll & X^{\frac{1}{2}+\frac{1}{k}} + X^{\frac{3}{2}+\frac{1}{k}}(\log X)^{\frac{1}{2}} \left(\frac{P}{X}\right)^{\frac{5}{4}} X^{\frac{1}{4}}(\log X)^{\frac{1}{2}} = X^{\frac{1}{2}+\frac{1}{k}} P^{\frac{5}{4}} \log X.\end{aligned}$$

We assume

$$P \le X^{\frac{2}{5}-\varepsilon}, \tag{2.4}$$

so that $P^{\frac{5}{4}} = o(X^{\frac{1}{2}}/\log X)$ which, with the upper bound for B_4 here above, ensures that

$$B_4 = o(X^{1+1/k}).$$

Bound for J_5

In order to provide an estimation for J_5, we use (1.5),

$$J_5 \ll \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)-T_k(\lambda_4\alpha)|\mathrm{d}\alpha$$

and then the arithmetic-geometric inequality ($ab \le a^2+b^2$):

$$J_5 \ll \eta^2 \sum_{j=2}^{3} \left(\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_j\alpha)|^2|S_k(\lambda_4\alpha)-T_k(\lambda_4\alpha)|\mathrm{d}\alpha\right).$$

The two terms may be estimated in the same way and produce the same upper bound. We show the details of the bound only for the case $j=2$:

$$\eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)|^2|S_k(\lambda_4\alpha)-T_k(\lambda_4\alpha)|\mathrm{d}\alpha$$

$$\begin{aligned}
\ll & \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)|^2|S_k(\lambda_4\alpha)-U_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
& + \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)|^2|U_k(\lambda_4\alpha)-T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
= & \eta^2(A_5+B_5),
\end{aligned}$$

say. Using trivial estimates,

$$A_5 \ll X \int_{\mathfrak{M}} |S_2(\lambda_2\alpha)|^2|S_k(\lambda_4\alpha)-U_k(\lambda_4\alpha)|\mathrm{d}\alpha$$

then using the Cauchy–Schwarz inequality, for any fixed $A>2$, by Lemmas B.4, B.11 and B.12 we have

$$\begin{aligned}
A_5 \ll & X\left(\int_{\mathfrak{M}} |S_2(\lambda_2\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{M}} |S_k(\lambda_4\alpha)-U_k(\lambda_4\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}} \\
\ll & X X^{\frac{1}{2}} \log X \frac{P}{X}\mathscr{J}_k\left(X,\frac{X}{P}\right)^{\frac{1}{2}} \ll_A X^{1+\frac{1}{k}}(\log X)^{1-\frac{A}{2}} = o\left(X^{\frac{1}{k}+1}\right)
\end{aligned}$$

provided that $\frac{X}{P} \geq X^{1-\frac{5}{6k}+\varepsilon}$ (condition of Lemma B.12), that is,

$$(\log X)^A \ll_A P \leq X^{\frac{5}{6k}-\varepsilon}. \tag{2.5}$$

Now we turn to B_5, using Theorem B.2:

$$B_5 \ll \int_0^{1/X} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)|^2\mathrm{d}\alpha + X\int_{1/X}^{P/X} \alpha|S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)|^2\mathrm{d}\alpha.$$

Using trivial estimates and Lemma B.4

$$\begin{aligned}
B_5 \ll & \frac{1}{X}(X\cdot X) + X\frac{P}{X}\left(\int_{1/X}^{P/X} |S_1(\lambda_1\alpha)|^2\mathrm{d}\alpha \cdot \int_{1/X}^{P/X} |S_2(\lambda_2\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{2}} \\
\ll & X + P(X\log X \cdot X\log^2 X)^{\frac{1}{2}} \\
= & X + PX(\log X)^{\frac{3}{2}}.
\end{aligned}$$

The case $j=3$ can be estimated in the same way, then we need

$$P = o\left(X^{\frac{1}{k}-\varepsilon}\right),$$

and, summing up with (2.5),

$$P \leq X^{\frac{5}{6k}-\varepsilon}. \tag{2.6}$$

Collecting all the bounds for P, that is, (2.3), (2.4), (2.6) we can take

$$P \leq \min\left(X^{\frac{2}{5}-\varepsilon}, X^{\frac{5}{6k}-\varepsilon}\right). \tag{2.7}$$

In fact, if we consider (2.4) and (2.6), we should choose the most restrictive condition between the two: if $k \leq \frac{25}{12}$, $P = X^{\frac{2}{5}-\varepsilon}$, otherwise, if $\frac{25}{12} < k < \frac{8}{3}$, $P = X^{\frac{5}{6k}-\varepsilon}$.

2.2 Trivial arc

By the arithmetic-geometric mean inequality and the trivial bound for $S_k(\lambda_4\alpha)$, we see that

$$\begin{aligned}|\mathscr{I}(\eta,\omega,\mathfrak{t})| \ll &\int_R^{+\infty} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\eta(\alpha)|\mathrm{d}\alpha\\ \ll &X^{\frac{1}{k}}\sum_{j=2}^{3}\int_R^{+\infty}|S_1(\lambda_1\alpha)|\,|S_2(\lambda_j\alpha)|^2K_\eta(\alpha)\mathrm{d}\alpha.\end{aligned}$$

The two terms may be estimated in the same way and produce the same upper bound. We show the details of the bound only for the case $j=2$:

$$\begin{aligned}&X^{\frac{1}{k}}\int_R^{+\infty}|S_1(\lambda_1\alpha)|\,|S_2(\lambda_j\alpha)|^2K_\eta(\alpha)\mathrm{d}\alpha\\ &\quad\ll X^{\frac{1}{k}}\left(\int_R^{+\infty}|S_1(\lambda_1\alpha)|^2K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_R^{+\infty}|S_2(\lambda_2\alpha)|^4K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}}\\ &\quad\ll X^{\frac{1}{k}}\left(\int_R^{+\infty}\frac{|S_1(\lambda_1\alpha)|^2}{\alpha^2}\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_R^{+\infty}\frac{|S_2(\lambda_2\alpha)|^4}{\alpha^2}\mathrm{d}\alpha\right)^{\frac{1}{2}}=X^{\frac{1}{k}}C_1^{\frac{1}{2}}C_2^{\frac{1}{2}},\end{aligned}$$

say. Using the PNT and the periodicity of $S_1(\alpha)$, we have

$$\begin{aligned}C_1=&\int_R^{+\infty}\frac{|S_1(\lambda_1\alpha)|^2}{\alpha^2}\mathrm{d}\alpha\ll\int_{\lambda_1 R}^{+\infty}\frac{|S_1(\alpha)|^2}{\alpha^2}\mathrm{d}\alpha\\ \ll&\sum_{n\geq|\lambda_1|R}\frac{1}{(n-1)^2}\int_{n-1}^{n}|S_1(\alpha)|^2\mathrm{d}\alpha\ll\frac{X\log X}{R}.\end{aligned}\tag{2.8}$$

Now using Lemma B.4,

$$\begin{aligned}C_2=&\int_R^{+\infty}\frac{|S_2(\lambda_2\alpha)|^4}{\alpha^2}\mathrm{d}\alpha\ll\int_{\lambda_1 R}^{+\infty}\frac{|S_2(\alpha)|^4}{\alpha^2}\mathrm{d}\alpha\\ \ll&\sum_{n\geq|\lambda_1|R}\frac{1}{(n-1)^2}\int_{n-1}^{n}|S_2(\alpha)|^4\mathrm{d}\alpha\ll\frac{X\log^2 X}{R}.\end{aligned}\tag{2.9}$$

Collecting (2.8) and (2.9),

$$|\mathscr{I}(\eta,\omega,\mathfrak{t})|\ll X^{\frac{1}{k}}\left(\frac{X\log X}{R}\right)^{\frac{1}{2}}\left(\frac{X\log^2 X}{R}\right)^{\frac{1}{2}}\ll\frac{X^{1+\frac{1}{k}}(\log X)^{\frac{3}{2}}}{R}.$$

Hence, remembering that $|\mathscr{I}(\eta,\omega,\mathfrak{t})|$ must be $o\left(\eta^2X^{\frac{1}{k}+1}\right)$, i.e. little-o of the main term, the choice

$$R=\frac{\log^2 X}{\eta^2}\tag{2.10}$$

is admissible[1].

[1] We could also take $R=\frac{\log^{\frac{3}{2}+\varepsilon}X}{\eta^2}$ but (2.10) is sufficient for our purpose.

2.3 The minor arc

In [29] Lemma 2.3 it is proven that the measure of the set where $|S_1(\lambda_1\alpha)|^{\frac{1}{2}}$ and $|S_2(\lambda_2\alpha)|$ are both large for $\alpha \in \mathfrak{m}$ is small, exploiting the fact that the ratio λ_1/λ_2 is irrational. The idea is to reach the widest k-range, assuming k as a parameter from which all other parameters depend and adjusting all them retrospectively.

In this section we have chosen to leave the parameters free. We will call $(1,d)$ the k-range and $a(k)$, $b(k)$, $c(k)$ the parameters that will determine the choices of η and the link between the choice of X and the convergent of the fraction λ_1/λ_2. This choice reflects the work procedure followed to find the values of the parameters a posteriori and may help the reader to understand the choice. The following lemma is not true for each parameter choice, but only for a subset of them where the optimum values are chosen. The parameters are:

$$d = \frac{8}{3}, \quad \frac{1}{a(k)} = \max\left(\frac{5}{9}, \frac{4+3k}{9k}\right), \quad b(k) = \max\left(\frac{2}{9}, \frac{3k-2}{9k}\right), \quad c(k) = \min\left(\frac{1}{18}, \frac{8-3k}{18k}\right). \tag{2.11}$$

Now, following the lines of Lemma 2.3 of [29], we have

Lemma 2.2. *Let $1 < k < d$. Assume that λ_1/λ_2 is irrational and let $X = q^{a(k)}$, where q is the denominator of a convergent of the continued fraction for λ_1/λ_2. Let*

$$V(\alpha) = \min(|S_1(\lambda_1\alpha)|, |S_2(\lambda_2\alpha)|^2).$$

Then

$$\sup_{\alpha\in\mathfrak{m}} V(\alpha) \ll X^{1-\frac{b(k)}{2}+\varepsilon}.$$

Proof. Let $\alpha \in m$ and $Q = X^{b(k)} \leq P$. By rational approximation Dirichlet's theorem (Theorem A.3), there exist integers a_i, q_i with $1 \leq q_i \leq \frac{X}{Q}$ and $(a_i, q_i) = 1$, such that

$$|\lambda_i\alpha q_i - a_i| \leq \frac{Q}{X}$$

for $i = 1,2$. We remark that $a_1a_2 \neq 0$, for otherwise we would have $\alpha \in \mathfrak{M}$. Now suppose that $q_i \leq Q$ for $i = 1,2$. In this case we get

$$a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2 = (\lambda_1\alpha q_1 - a_1)\frac{a_2}{\lambda_2\alpha} - (\lambda_2\alpha q_2 - a_2)\frac{a_1}{\lambda_2\alpha}$$

and hence

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \leq 2\left(1+\left|\frac{\lambda_1}{\lambda_2}\right|\right)\frac{Q^2}{X} < \frac{1}{2q} \tag{2.12}$$

for sufficiently large X. In fact, the last inequality is true due to the choice of the parameters in (2.11): being $q < Q$, we need

$$Q^3 < X \quad \Rightarrow \quad X^{3b(k)} < X \quad \Rightarrow \quad 3b(k) < 1,$$

and this is always true: $\max\left(\frac{2}{3}, \frac{3k-2}{3k}\right) < 1$.

Then, from the law of best approximation (Theorem A.4)

$$X^{\frac{1}{a(k)}} = q \leq |a_2 q_1|$$

so, recalling the definition of $\mathfrak{m}$, we obtain

$$|a_2 q_1| \ll q_1 q_2 R \leq Q^2 R \leq X^{\frac{1}{a(k)} - \varepsilon} \tag{2.13}$$

which is absurd[2] if $\eta = X^{-c(k)}$ and $R = \frac{\log^2 X}{\eta^2}$. Hence either $q_1 > Q$ or $q_2 > Q$.

Assuming that $q_1 > Q$ and using Lemma B.13:

$$V(\alpha) \leq |S_1(\lambda_1 \alpha)| \ll \sup_{Q < q_1 < \frac{X}{Q}} \left(\frac{X}{\sqrt{q_1}} + \sqrt{X q_1} + X^{\frac{4}{5}} \right) \log^4 X \ll X^{1 - \frac{b(k)}{2} + \varepsilon}. \tag{2.14}$$

Assuming that $q_2 > Q$ and using Lemma B.14:

$$V(\alpha) \leq |S_2(\lambda_2 \alpha)|^2 \ll X^{1+2\varepsilon} \sup_{Q < q_2 < \frac{X}{Q}} \left(\frac{1}{q_2} + \frac{1}{X^{\frac{1}{4}}} + \frac{q_2}{X} \right)^{\frac{1}{2}} \ll X^{1 - \frac{b(k)}{2} + \varepsilon}. \tag{2.15}$$

Again, both inequalities in (2.14) and (2.15) are true due to the appropriate choice of the parameters in (2.11), and this completes the proof of the Lemma. $\square$

If we want that the last inequalities in both (2.12) and (2.13) are true, we need the following conditions on $a(k)$, $b(k)$ and $c(k)$:

$$2b(k) - 1 \leq -\frac{1}{a(k)} \tag{2.16}$$

$$2b(k) + 2c(k) \leq \frac{1}{a(k)}. \tag{2.17}$$

Now let

$$\mathscr{X}_1 = \left\{ \alpha \in \left[\frac{P}{X}, R\right] : |S_1(\lambda_1 \alpha)| \leq |S_2(\lambda_2 \alpha)|^2 \right\}$$

$$\mathscr{X}_2 = \left\{ \alpha \in \left[\frac{P}{X}, R\right] : |S_1(\lambda_1 \alpha)| \geq |S_2(\lambda_2 \alpha)|^2 \right\}$$

so that $\left[\frac{P}{X}, R\right] = \mathscr{X}_1 \cup \mathscr{X}_2$ and

$$|\mathscr{I}(\eta, \omega, \mathfrak{m})| \ll \left(\int_{\mathscr{X}_1} + \int_{\mathscr{X}_2} \right) |S_1(\lambda_1 \alpha)| |S_2(\lambda_2 \alpha)| |S_2(\lambda_3 \alpha)| |S_k(\lambda_4 \alpha)| K_\eta(\alpha) \mathrm{d}\alpha.$$

[2] The choice of the parameters in (2.11) is made in order to get a contradiction.

If $\alpha \in \mathscr{X}_1$, we have $|S_1(\lambda_1\alpha)| \le |S_2(\lambda_2\alpha)|^2$, then

$$
\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathscr{X}_1)| \ll & \int_{\mathscr{X}_1} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\
\ll & (\max|S_1(\lambda_1\alpha)|)^{\frac{1}{2}} \left(\int_{\mathscr{X}_1} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \\
& \prod_{i=2}^{3}\left(\int_{\mathscr{X}_1} |S_2(\lambda_i\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4}\left(\int_{\mathscr{X}_1} |S_k(\lambda_4\alpha)|^4 K_\eta(\alpha)|\mathrm{d}\alpha\right)^{1/4} \\
\ll & X^{\frac{1}{2}-\frac{b(k)}{4}+\varepsilon}(\eta X\log X)^{\frac{1}{4}}(\eta X\log^2 X)^{\frac{1}{4}}(\eta X\log^2 X)^{\frac{1}{4}}\left(\eta X^{\varepsilon}\max(X^{\frac{2}{k}},X^{\frac{4}{k}-1})\right)^{\frac{1}{4}} \\
= & \eta X^{\frac{5}{4}-\frac{b(k)}{4}+\varepsilon}\max(X^{\frac{1}{2k}},X^{\frac{1}{k}-\frac{1}{4}})
\end{aligned}
$$

using Lemma B.10.

In this case $|\mathscr{I}(\eta,\omega,\mathfrak{m})| = o\left(X^{1+\frac{1}{k}-\varepsilon}\right)$ only if

$$\eta = \infty\left(\max(X^{\frac{1}{4}-\frac{1}{2k}-\frac{b(k)}{4}+\varepsilon},X^{-\frac{b(k)}{4}})\right)$$

then we can add the following condition on $c(k)$:

$$-c(k) \ge \max\left(\frac{1}{4}-\frac{1}{2k}-\frac{b(k)}{4},-\frac{b(k)}{4}\right).$$

It is clear that for $1<k<2$, η is a negative power of X independently from the value of k as $b(k)>0$. The we have the following most restrictive condition for $k \ge 2$:

$$-c(k) \ge \frac{1}{4}-\frac{1}{2k}-\frac{b(k)}{4}.$$

If $\alpha \in \mathscr{X}_2$, we have $|S_1(\lambda_1\alpha)| \ge |S_2(\lambda_2\alpha)|^2$, then

$$
\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathscr{X}_2)| \ll & \int_{\mathscr{X}_2} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\
\ll & \max|S_2(\lambda_2\alpha)|\left(\int_{\mathscr{X}_2} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/2} \\
& \left(\int_{\mathscr{X}_2} |S_2(\lambda_3\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4}\left(\int_{\mathscr{X}_2} |S_k(\lambda_4\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \\
\ll & X^{\frac{1}{2}-\frac{b(k)}{4}+\varepsilon}(\eta X\log X)^{\frac{1}{2}}(\eta X\log^2 X)^{\frac{1}{4}}\left(\eta X^{\varepsilon}\max(X^{\frac{2}{k}},X^{\frac{4}{k}-1})\right)^{\frac{1}{4}} \\
= & \eta X^{\frac{5}{4}-\frac{b(k)}{4}+2\varepsilon}\max(X^{\frac{1}{2k}},X^{\frac{1}{k}-\frac{1}{4}}).
\end{aligned}
$$

As in the previous case $|\mathscr{I}(\eta,\omega,\mathfrak{m})| = o\left(X^{1+\frac{1}{k}-\varepsilon}\right)$ only if

$$\eta = \infty\left(\max(X^{\frac{1}{4}-\frac{1}{2k}-\frac{b(k)}{4}+\varepsilon},X^{-\frac{b(k)}{4}})\right) \tag{2.18}$$

then we can add the following condition on $c(k)$:

$$-c(k) \ge \max\left(\frac{1}{4}-\frac{1}{2k}-\frac{b(k)}{4},-\frac{b(k)}{4}\right).$$

essentially the same condition we got for $\alpha \in \mathscr{X}_1$.

Collecting (2.16), (2.17) and (2.18) we get $d = \frac{8}{3}$, the maximum value we can have for k. This is justified as follows: neglecting the log-powers and recalling the choice of $P(X)$ in (2.7) and $R(X)$ in (2.10), we must maximize k subject to the constraints:

$$\begin{cases} a(k) \geq 1 \\ 0 \leq b(k) \leq \min\left(\frac{2}{5}, \frac{5}{6k}\right) \\ c(k) > 0 \\ 2b(k) - 1 \leq -\frac{1}{a(k)} \\ 2b(k) + 2c(k) \leq \frac{1}{a(k)} \\ -c(k) \geq \max\left(\frac{1}{4} - \frac{1}{2k} - \frac{b(k)}{4}, -\frac{b(k)}{4}\right) \end{cases}$$

which is a linear optimization problem in the variables $\frac{1}{k}$, $\frac{1}{a(k)}$, $b(k)$, $c(k)$. The solution of this problem is

$$\frac{1}{a(k)} = \max\left(\frac{5}{9}, \frac{4+3k}{9k}\right), \qquad b(k) = \max\left(\frac{2}{9}, \frac{3k-2}{9k}\right), \qquad c(k) = \min\left(\frac{1}{18}, \frac{8-3k}{18k}\right),$$

for $k < \frac{8}{3}$.

2.4 The minor arc using the Harman technique

We start again from the idea that the measure of the set where $|S_1(\lambda_1\alpha)|^{\frac{1}{2}}$ and $|S_2(\lambda_2\alpha)|$ are both large for $\alpha \in m$ is suitably bounded, if the ratio λ_1/λ_2 is irrational. We now state some considerations about Lemmas B.13 and B.14 that allow us to introduce two more corollaries to those Lemmas.

By B.13, $S_1(\alpha) \ll \left(\frac{X}{\sqrt{q}} + \sqrt{Xq} + X^{\frac{4}{5}}\right)\log^4 X$, then it is easy to check out that:

- if $q \leq X^{\frac{2}{5}}$ the main term is $X^{1+\varepsilon}q^{-\frac{1}{2}}$
- if $X^{\frac{2}{5}} \leq q \leq X^{\frac{3}{5}}$, the main term is $X^{\frac{4}{5}+\varepsilon}$
- if $q \geq X^{\frac{3}{5}}$, the main term is $X^{\frac{1}{2}+\varepsilon}q^{\frac{1}{2}}$.

By B.14, $S_2(\alpha) \ll_\varepsilon X^{\frac{1}{2}+\varepsilon}\left(\frac{1}{q} + \frac{1}{X^{\frac{1}{4}}} + \frac{q}{X}\right)^{\frac{1}{4}}$, then it is easy to check out that:

- if $q \leq X^{\frac{1}{4}}$ the main term is $X^{\frac{1}{2}+\varepsilon}q^{-\frac{1}{4}}$
- if $X^{\frac{1}{4}} \leq q \leq X^{\frac{3}{4}}$, the main term is $X^{\frac{7}{16}+\varepsilon}$
- if $q \geq X^{\frac{3}{4}}$, the main term is $X^{\frac{1}{4}+\varepsilon}q^{\frac{1}{4}}$.

From the considerations above and by the fact that, as we will see later, we will need an estimate for low values of q we need corollaries B.15 and B.16. Let us now split $\mathfrak{m}$ into subsets $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}^* = \mathfrak{m} \backslash (\mathfrak{m}_1 \cup \mathfrak{m}_2)$, where

$$\mathfrak{m}_1 = \{\alpha \in \mathfrak{m} : |S_1(\lambda_1 \alpha)| \leq X^{1-t+\varepsilon}\}$$
$$\mathfrak{m}_2 = \{\alpha \in \mathfrak{m} : |S_2(\lambda_2 \alpha)| \leq X^{\frac{1}{2}-u+\varepsilon}\}$$

remembering that Corollaries B.15 and B.16 hold respectively for $0 \leq t \leq \frac{1}{5}$ and $0 \leq u \leq \frac{1}{16}$. We make again the choice to follow the work procedure leaving the parameters t and u free.

Using the Hölder inequalities, Lemma B.10 and the definition of $\mathfrak{m}_1$ we obtain

$$\begin{aligned}
|\mathscr{I}(\eta, \omega, \mathfrak{m}_1)| &\ll \int_{\mathfrak{m}_1} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\
&\ll \left(\max_{\alpha\in\mathfrak{m}_1} |S_1(\lambda_1\alpha)|\right)^{\frac{1}{2}} \left(\int_{\mathfrak{m}_1} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \\
&\quad \prod_{i=2}^{3} \left(\int_{\mathfrak{m}_1} |S_2(\lambda_i\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \left(\int_{\mathfrak{m}_1} |S_k(\lambda_4\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \\
&\ll X^{\frac{1}{2}-\frac{t}{2}+\varepsilon} (\eta X \log X)^{\frac{1}{4}} (\eta X \log^2 X)^{\frac{1}{2}} \left(\eta X^\varepsilon \max(X^{\frac{2}{k}}, X^{\frac{4}{k}-1})\right)^{\frac{1}{4}} \\
&= \eta X^{\frac{1}{2}-\frac{t}{2}+\frac{3}{4}+\varepsilon} \max(X^{\frac{1}{2k}}, X^{\frac{1}{k}-\frac{1}{4}}).
\end{aligned} \tag{2.19}$$

by Corollaries B.15 and B.16.

Using the Hölder inequalities, Lemma B.10 and the definition of m_2 we obtain

$$\begin{aligned}
|\mathscr{I}(\eta, \omega, \mathfrak{m}_2)| &\ll \int_{\mathfrak{m}_2} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\
&\ll \max_{\alpha\in\mathfrak{m}_2} |S_2(\lambda_2\alpha)| \left(\int_{\mathfrak{m}_2} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/2} \\
&\quad \left(\int_{\mathfrak{m}_2} |S_2(\lambda_3\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \left(\int_{\mathfrak{m}_2} |S_k(\lambda_4\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \\
&\ll X^{\frac{1}{2}-u+\varepsilon} (\eta X \log X)^{\frac{1}{2}} (\eta X \log^2 X)^{\frac{1}{4}} \left(\eta X^\varepsilon \max(X^{\frac{2}{k}}, X^{\frac{4}{k}-1})\right)^{\frac{1}{4}} \\
&= \eta X^{\frac{1}{2}-u+\frac{3}{4}+\varepsilon} \max(X^{\frac{1}{2k}}, X^{\frac{1}{k}-\frac{1}{4}}).
\end{aligned} \tag{2.20}$$

Both (2.19) and (2.20) must be $o\left(\eta^2 X^{1+\frac{1}{k}}\right)$, consequently, as $t > 0$ and $u > 0$, it is clear that for $1 < k < 2$, η is a negative power of X independently from the value of k. Then we have the following conditions for $k \geq 2$:

$$\eta = \infty\left(X^{\frac{1}{4}-\frac{t}{2}-\frac{1}{2k}+\varepsilon}\right)$$
$$\eta = \infty\left(X^{\frac{1}{4}-u-\frac{1}{2k}+\varepsilon}\right).$$

It remains to discuss the set $\mathfrak{m}^*$ in which the following bounds hold simultaneously

$$|S_1(\lambda_1\alpha)| > X^{1-t+\varepsilon}, \qquad |S_2(\lambda_2\alpha)| > X^{\frac{1}{2}-u+\varepsilon}, \qquad \frac{P}{X} = \min\left(X^{-\frac{3}{5}}, X^{\frac{5}{6k}-1}\right) < |\alpha| \le \frac{\log^2 X}{\eta^2}.$$

Following the dyadic dissection argument shown by Harman in [22] we divide $\mathfrak{m}^*$ into disjoint sets $E(Z_1,Z_2,y)$ in which, for $\alpha \in E(Z_1,Z_2,y)$, we have

$$Z_1 < |S_1(\lambda_1\alpha)| \le 2Z_1, \qquad Z_2 < |S_2(\lambda_2\alpha)| \le 2Z_2, \qquad y < |\alpha| \le 2y$$

where $Z_1 = 2^{k_1}X^{1-t+\varepsilon}$, $Z_2 = 2^{k_2}X^{\frac{1}{2}-u+\varepsilon}$ and $y = 2^{k_3}X^{-\frac{3}{5}-\varepsilon}$ for some non-negative integers k_1,k_2 and k_3.

It follows that the disjoint sets are, at most, $\ll \log^3 X$. Let us define $\mathscr{A}$ a shorthand for the sets $E(Z_1,Z_2,y)$; we have the following result about the Lebesgue measure of $\mathscr{A}$ following the same lines of Lemma 6 in [36].

In the following Lemma, it is crucial that both the integers a_1 and a_2 appearing in (2.21) below do not vanish: in fact, if $a_1 = 0$, say, then $q_1 = 1$ and $|\alpha|$ is so small that it can not belong to $\mathfrak{m}^*$. In order to use the Harman technique then, we are forced to be far from the major arc in which we would have $a_1a_2 = 0$. As we will see later, after setting the parameters t and u, we will not have a gap between the major and the minor arc, so it will not be necessary to introduce an intermediate arc.

Lemma 2.3. *We have*

$$\mu(\mathscr{A}) \ll yX^{2+4u+2t+3\varepsilon}Z_1^{-2}Z_2^{-4}$$

where $\mu(\cdot)$ denotes the Lebesgue measure.

Proof. If $\alpha \in \mathscr{A}$, by Corollaries B.15 and B.16 there are coprime integers (a_1,q_1) and (a_2,q_2) such that

$$\begin{aligned} 1 \le q_1 &\ll \left(\frac{X^{1+\varepsilon/2}}{Z_1}\right)^2, \qquad & |q_1\lambda_1\alpha - a_1| &\ll \left(\frac{X^{\frac{1}{2}+\varepsilon/2}}{Z_1}\right)^2 \\ 1 \le q_2 &\ll \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4, \qquad & |q_2\lambda_2\alpha - a_2| &\ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4. \end{aligned} \tag{2.21}$$

We remark that $a_1a_2 \ne 0$ otherwise we would have $\alpha \in \mathfrak{M}$. In fact, if $a_1 = 0$ recalling the definitions of Z_1 and (2.21), we get

$$\alpha \ll q_1^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/2}}{Z_1}\right)^2 \ll \frac{X}{X^{2-2t+\varepsilon}} = X^{-1+2t-\varepsilon}; \tag{2.22}$$

otherwise, if $a_2 = 0$ recalling the definitions of Z_2 and (2.21), we get

$$|\alpha| \ll q_2^{-1} X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2} \right)^4 \ll \frac{X}{X^{2-4u+3\varepsilon}} = X^{-1+4u-3\varepsilon}.$$

It means that, on the minor arc

$$|\alpha| \gg \max\left(X^{-1+2t-\varepsilon}, X^{-1+4u-3\varepsilon} \right).$$

We wonder now if there is a gap between the end of the major arc and the beginning of the minor arc, i.e. if

$$X^{-1+2t-\varepsilon} < \frac{P}{X} = \min\left(X^{-\frac{3}{5}}, X^{\frac{5}{6k}-1} \right) \quad \text{or} \quad X^{-1+4u-3\varepsilon} < \frac{P}{X} = \min\left(X^{-\frac{3}{5}}, X^{\frac{5}{6k}-1} \right). \tag{2.23}$$

This possibility can be evaluated only afterwards, after finding the parameters t and u and the maximum value for k. Both with the results that we get in this section (2.27), and with those we will get in (2.29) in which we improve the estimate, we can state that there will be no gap between the two arcs.

Now, we can further split $\mathfrak{m}^*$ into sets $I(Z_1, Z_2, y, Q_1, Q_2)$ where $Q_j \le q_j \le 2Q_j$ on each set. Note that a_i and q_i are uniquely determined by α. In the opposite direction, for a given quadruple a_1, q_1, a_2, q_2 the inequalities (2.21) define an interval of α of length

$$\mu(I) \ll \min\left(Q_1^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon/2}}{Z_1} \right)^2, Q_2^{-1} X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2} \right)^4 \right).$$

Taking the geometric mean ($\min(a,b) \le \sqrt{a}\sqrt{b}$) we can write

$$\mu(I) \ll Q_1^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} X^{-\frac{1}{2}} \left(\frac{X^{\frac{1}{2}+\varepsilon/2}}{Z_1} \right) \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2} \right)^2 \ll \frac{X^{1+\varepsilon}}{Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} Z_1 Z_2^2}. \tag{2.24}$$

Now we need a lower bound for $Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}}$: by (2.21)

$$\begin{aligned} \left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| &= \left| \frac{a_2}{\lambda_2 \alpha} (q_1 \lambda_1 \alpha - a_1) - \frac{a_1}{\lambda_2 \alpha} (q_2 \lambda_2 \alpha - a_2) \right| \\ &\ll q_2 |q_1 \lambda_1 \alpha - a_1| + q_1 |q_2 \lambda_2 \alpha - a_2| \\ &\ll Q_2 \left(\frac{X^{\frac{1}{2}+\varepsilon/2}}{Z_1} \right)^2 + Q_1 X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2} \right)^4. \end{aligned}$$

Remembering that $Q_1 \ll \left(\frac{X^{1+\varepsilon/2}}{Z_1} \right)^2$, $Q_2 \ll \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2} \right)^4$, $Z_1 \gg X^{1-t+\varepsilon}$, $Z_2 \gg X^{\frac{1}{2}-u+\varepsilon}$,

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| \ll \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{X^{\frac{1}{2}-u+\varepsilon}} \right)^4 \left(\frac{X^{\frac{1}{2}+\varepsilon/2}}{X^{1-t+\varepsilon}} \right)^2 + \left(\frac{X^{1+\varepsilon/2}}{X^{1-t+\varepsilon}} \right)^2 X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{X^{\frac{1}{2}-u+\varepsilon}} \right)^4$$

$$\ll \frac{X^{2+\varepsilon}X^{1+\varepsilon}}{X^{2-4u+4\varepsilon}X^{2-2t+2\varepsilon}} \ll X^{-1+4u+2t-4\varepsilon}. \tag{2.25}$$

We recall that $q = X^{1-4u-2t}$ is a denominator of a convergent of λ_1/λ_2. Hence by (2.25) Legendre's law of best approximation (see Appendix A.4) implies that $|a_2q_1| \geq q$ and by the same token, for any pair α, α' having distinct associated products a_2q_1 (see [49], Lemma 2),

$$|a_2(\alpha)q_1(\alpha) - a_2(\alpha')q_1(\alpha')| \geq q;$$

thus, by the pigeon-hole principle, there is at most one value of a_2q_1 in the interval $[rq,(r+1)q)$ for any positive integer r. Hence a_2q_1 determines a_2 and q_1 to within X^{ε} possibilities (see the upper bound for the divisor function in Appendix, Theorem A.8) and consequently also a_2q_1 determines a_1 and q_2 to within X^{ε} possibilities from (2.25).

Hence we got a lower bound for q_1q_2, remembering that in our shorthand $Q_j \leq q_j \leq 2Q_j$:

$$q_1q_2 = a_2q_1\frac{q_2}{a_2} \gg \frac{rq}{|\alpha|} \gg rqy^{-1}$$

for the quadruple under consideration. As a consequence we obtain from (2.24), that the total length of the interval $I(Z_1,Z_2,y,Q_1,Q_2)$ with $a_2q_1 \in [rq,(r+1)q)$ does not exceed

$$\mu(I) \ll X^{1+2\varepsilon}Z_1^{-1}Z_2^{-2}r^{-\frac{1}{2}}q^{-\frac{1}{2}}y^{\frac{1}{2}}.$$

Now we need a bound for r: inside the interval $[rq,(r+1)q)$, $rq \leq |a_2q_1|$ and, in turn from (2.21), $a_2 \ll q_2|\alpha|$, then

$$rq \ll q_1q_2|\alpha| \ll \left(\frac{X^{1+\varepsilon/2}}{Z_1}\right)^2\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4 y \ll yX^{4+2\varepsilon}Z_1^{-2}Z_2^{-4}$$
$$\Rightarrow r \ll q^{-1}yX^{4+2\varepsilon}Z_1^{-2}Z_2^{-4}.$$

Now, we sum on every interval to get an upper bound for the measure of $\mathscr{A}$:

$$\mu(\mathscr{A}) \ll X^{1+2\varepsilon}Z_1^{-1}Z_2^{-2}q^{-\frac{1}{2}}y^{\frac{1}{2}} \sum_{1\leq r\ll q^{-1}yX^{4+2\varepsilon}Z_1^{-2}Z_2^{-4}} r^{-\frac{1}{2}}$$

By standard estimation we obtain

$$\sum_{1\leq r\ll q^{-1}yX^{4+2\varepsilon}Z_1^{-2}Z_2^{-4}} r^{-\frac{1}{2}} \ll (q^{-1}yX^{4+2\varepsilon}Z_1^{-2}Z_2^{-4})^{\frac{1}{2}}$$

then

$$\mu(\mathscr{A}) \ll yX^{3+3\varepsilon}Z_1^{-2}Z_2^{-4}q^{-1} \ll yX^{3+3\varepsilon}Z_1^{-2}Z_2^{-4}X^{-1+4u+2t} \ll yX^{2+4u+2t+3\varepsilon}Z_1^{-2}Z_2^{-4}.$$

This concludes the proof of Lemma 2.3. $\square$

Using Lemma 2.3 we finally are able to get a bound for $\mathscr{I}(\eta,\omega,\mathscr{A})$:

$$\begin{aligned}
\mathscr{I}(\eta,\omega,\mathscr{A}) &= \int_{\mathscr{A}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\
&\ll \left(\int_{\mathscr{A}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)|^2 K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_{\mathscr{A}} |S_2(\lambda_3\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}} \\
&\quad \left(\int_{\mathscr{A}} |S_k(\lambda_4\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}} \\
&\ll \left(\min\left(\eta^2,\frac{1}{y^2}\right)\right)^{\frac{1}{2}} \left((Z_1Z_2)^2\mu(\mathscr{A})\right)^{\frac{1}{2}} \left(\eta X\log^2 X\right)^{\frac{1}{4}} \left(\eta X^{\varepsilon}\max(X^{\frac{2}{k}},X^{\frac{4}{k}-1})\right)^{\frac{1}{4}} \\
&\ll \left(\min\left(\eta^2,\frac{1}{y^2}\right)\right)^{\frac{1}{2}} Z_1Z_2(yX^{2+4u+2t+4\varepsilon}Z_1^{-2}Z_2^{-4})^{\frac{1}{2}}\eta^{\frac{1}{2}}X^{\frac{1}{4}+2\varepsilon}\max(X^{\frac{1}{2k}},X^{\frac{1}{k}-\frac{1}{4}}) \\
&\ll \left(\min\left(\eta^2,\frac{1}{y^2}\right)\right)^{\frac{1}{2}} y^{\frac{1}{2}}Z_2^{-1}X^{1+2u+t+2\varepsilon}\eta^{\frac{1}{2}}X^{\frac{1}{4}+2\varepsilon}\max(X^{\frac{1}{2k}},X^{\frac{1}{k}-\frac{1}{4}}) \\
&\ll \left(\min\left(\eta^2,\frac{1}{y^2}\right)\right)^{\frac{1}{2}} y^{\frac{1}{2}}X^{\frac{1}{2}+3u+t+2\varepsilon}\eta^{\frac{1}{2}}X^{\frac{1}{4}+2\varepsilon}\max(X^{\frac{1}{2k}},X^{\frac{1}{k}-\frac{1}{4}}) \\
&\ll \eta X^{\frac{3}{4}+3u+t+4\varepsilon}\max(X^{\frac{1}{2k}},X^{\frac{1}{k}-\frac{1}{4}}).
\end{aligned}$$

Remark: if $\eta < \frac{1}{y}$, $y^{\frac{1}{2}}\eta^{\frac{3}{2}} < \eta$; if $\eta > \frac{1}{y}$, $y^{-\frac{1}{2}}\eta^{\frac{1}{2}} < \eta$.

Again, (2.26) must be $o\left(X^{1+\frac{1}{k}-\varepsilon}\right)$ and even in this case, if $1<k<2$, η is a negative power of X independently from the value of k. The last condition on η is:

$$\eta = \infty\left(X^{-\frac{1}{4}-\frac{1}{2k}+3u+t+\varepsilon}\right). \tag{2.26}$$

Collecting all the conditions (2.19), (2.20), (2.26) and the condition given by corollaries B.15 and B.16, we get the following linear optimization system: setting $x=\frac{1}{k}$ and let w be the exponent of η we would like to optimize,

$$\begin{cases}
x\le 1;\ w\ge 0;\ t\le \frac{1}{5};\ u\le\frac{1}{16} \\
-w\ge \frac{1}{4}-\frac{x}{2}-\frac{t}{2} \\
-w\ge \frac{1}{4}-\frac{x}{2}-u \\
-w\ge -\frac{1}{4}-\frac{x}{2}+3u+t.
\end{cases}$$

Solving the system, it turns out that

$$u=\frac{1}{16} \qquad t=\frac{1}{8} \tag{2.27}$$

(and consequently $X=q^2$) are the optimal values; the maximum k-range is $\left(1,\frac{8}{3}\right)$. In this case we do not get any improvement using Harman technique for the range of k, as the conditions (2.19), (2.20) are more restrictive than (2.26).

However, if we consider the value of $c(k)$ in (2.11) and the condition on η by specifying the parameters t and u in (2.19) and (2.20) we can notice an improvement in estimation:

$$\eta = \infty\left(\max\left(X^{-\frac{1}{16}+\varepsilon}, X^{\frac{3k-8}{16k}+\varepsilon}\right)\right)$$

then

$$\psi(k) = \min\left(\frac{1}{16}, \frac{8-3k}{16k}\right) > \min\left(\frac{1}{18}, \frac{8-3k}{18k}\right).$$

2.5 Further improvements

We tried to improve the estimations on (2.19) and (2.20) using the bounds of Bourgain [3], Theorem 10, and Hölder inequalities in a different way, but we could not get any better estimation. In fact, by Bourgain, Theorem 10,

$$\int_0^1 |S_2(\alpha)|^6 \mathrm{d}\alpha \ll X^{2+\varepsilon}$$

but we could reach the same result using trivial estimation for $S_2(\alpha)$ (see (B.1)) and (B.4):

$$\int_0^1 |S_2(\alpha)|^6 \mathrm{d}\alpha \ll X \cdot X \log^2 X.$$

This fact does not surprise us because Bourgain himself states that Theorem 10 of [3] improves Hua's Lemma only for greater powers of $S_2(\alpha)$.

In a recent work of Wang & Yao [48] concerning a prime a three square of primes, the authors shows that it is possible for the special case $k = 2$ reach the exponent $-\frac{1}{14}$ for η.

In their paper they make use of a new estimation formula firstly introduced by Harman ([22], sections 8 and 9), where the exponent $7/16$ in the estimation of $S_2(\alpha)$ is improved to $3/7$. The idea is to use $\rho(m)$ as a characteristic function for the set of primes, in place of $\log n$ or $\Lambda(n)$, that satisfy:

$$\rho(m) \ll \begin{cases} 1 & \text{if } m \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{m \le T} \rho(m) \gg \frac{T}{\log X}$$

for $X^{\frac{1}{4}} \le T \le X^{\frac{1}{2}}$.

This approach leads to the construction of two functions, using Buchstab's identity, that can be used as a function and do not change the treatment of the major arc. Wang and Yao in their work proved [Lemma 1,[48]], following the proof of Lemma 1 in [23] where $g(\alpha)$ is the exponential sum defined with the characteristic function above:

$$g(\alpha) = \sum_{m \in T} \rho(m) e(\alpha m^2)$$

where $T = \left[\frac{1}{3}X^{\frac{1}{2}}, \frac{2}{3}X^{\frac{1}{2}}\right)$.

The estimation in our notation is:

$$S_2(\alpha) \ll X^{\frac{3}{7}+\varepsilon} + X^{\frac{1}{2}+\varepsilon}\left(\frac{1}{q} + \frac{q}{X}\right)^{\frac{1}{4}} \tag{2.28}$$

and in this situation we can replace Corollary B.16 with the following:

Corollary 2.4. *Suppose that $X^{\frac{1}{2}} \geq Z \geq X^{\frac{1}{2}-\frac{1}{14}+\varepsilon}$ and $|S_2(\lambda_2\alpha)| > Z$. Then there are coprime integers $(a,q) = 1$ satisfying*

$$1 \leq q \leq \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^4, \qquad |q\lambda_2\alpha - a| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^4.$$

The proof follows the same steps of Corollary B.16 but it will use the estimation (2.28). Using such corollary (2.19) and (2.20) can be improved choosing

$$u = \frac{1}{14} \qquad t = \frac{1}{7} \tag{2.29}$$

in place of $u = \frac{1}{16}$ and $t = \frac{1}{8}$ and after a linear optimization we get a wider k-range and we can improve the approximation: the maximum k-range became $\left(1, \frac{14}{5}\right)$ and the exponent of η can be replaced with

$$\psi(k) = \min\left(\frac{1}{14}, \frac{14-5k}{28k}\right) > \min\left(\frac{1}{16}, \frac{8-3k}{16k}\right).$$

Corollary 2.4 does not affect Lemma 2.3 as the Harman technique on the arc $\mathfrak{m}^*$ can be used with $X = q^{\frac{7}{3}}$ in place of $X = q^2$. However, after the new parameter choices in (2.29) and the new k-range, we may wonder whether a gap has been created between the major and the minor arc. Also in this case, there is no need to introduce an intermediate arc because from (2.23),

$$X^{-1+\frac{2}{7}+2\varepsilon} < \frac{P}{X} = \min\left(X^{-\frac{3}{5}}, X^{\frac{5}{6k}-1}\right)$$

for every choice of k in $\left(1, \frac{14}{5}\right)$.

Finally, not only the range for k can be enlarged, but also that for the value of k between 1 and $\frac{8}{3}$ we have a better estimation.

Using the work of Kumchev [25] it is also possible to replace the exponents 4 in the right side of both inequalities of Corollary 2.4 with 2, but it does not lead to an improvement on the range of k because one still requires $Z \geq X^{3/7}$.

Wang & Yao in [48] have also shown that assuming the hypothesis in $k = 2$ that both λ_1/λ_2 and λ_1/λ_3 are irrational and algebraic, managed to get $\psi(2) = 3/40$. This is due to the fact those assumptions enable them to use the functions constructed in Harman and Kumchev [24].

These techniques could also be applied to our case and may be the beginning of further investigations.

3 Three squares of primes and one k-th power of prime

The second problem studied is very similar to the first with the only difference of having three squares of primes and a k-th power. This little change lowers the density of the problem making it slightly more complicated: let $k > 1$ be a real number and assume that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are non-zero real numbers, not all of the same sign and with the ratio λ_1/λ_2 irrational. Let ω be a real number. We would like to find a range for k where

$$|F(p_1, p_2, p_3, p_4, 2, 2, 2, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4) - \omega| = |\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega| \le \eta \tag{3.1}$$

has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 for some $\eta \to 0$ as a small negative power of the largest prime, as in (1.4).

We will prove the following:

Theorem 3.1. *Assume that $1 < k < 7/6$, $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be non-zero real numbers, not all of the same sign, that λ_1/λ_2 is irrational and let ω be a real number. The inequality*

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega\right| \le (\mathrm{M}(\mathbf{p}, \mathbf{k}))^{-\frac{7-6k}{14k}+\varepsilon}$$

has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 for any $\varepsilon > 0$.

Let

$$\mathscr{P}(X) = \{(p_1, p_2, p_3, p_4) : \delta X < p_1^2, p_2^2, p_3^2 < X,\ \delta X < p_4^k < X\}$$

and let us define

$$\mathscr{I}(\eta, \omega, \mathfrak{X}) = \int_{\mathfrak{X}} S_2(\lambda_1\alpha) S_2(\lambda_2\alpha) S_2(\lambda_3\alpha) S_k(\lambda_4\alpha) K_\eta(\alpha) e(-\omega\alpha) \mathrm{d}\alpha$$

where $\mathfrak{X}$ is a measurable subset of $\mathbb{R}$.

From the definitions of the $S_j(\lambda_i\alpha)$ and performing the Fourier transform for $K_\eta(\alpha)$, we get

$$\mathscr{I}(\eta, \omega, \mathbb{R}) = \sum_{(p_1, \dots, p_4) \in \mathscr{P}(X)} \log p_1 \log p_2 \log p_3 \log p_4\ \max(0, \eta - |F(p_1, p_2, p_3, p_4, 2, 2, 2, k) - \omega|)$$

$$\leq \eta(\log X)^4 \mathscr{N}(X),$$

where $\mathscr{N}(X)$ actually denotes the number of solutions of the inequality (3.1) with $(p_1p_2p_3p_4) \in \mathscr{P}(X)$. In other words $\mathscr{I}(\eta,\omega,\mathbb{R})$ provides a lower bound for the quantity we are interested in, therefore for the proof of the theorem it is sufficient to prove that $\mathscr{I}(\eta,\omega,\mathbb{R}) > 0$ on a suitable sequence of values of X with limit infinity.

As in the first problem, we decompose $\mathbb{R}$ into subsets such that $\mathbb{R} = \mathfrak{M} \cup m \cup t$ where $\mathfrak{M}$ is the major arc, m is the minor arc and t is the trivial arc. The decomposition is the following:

$$\mathfrak{M} = \left[-\frac{P}{X},\frac{P}{X}\right] \qquad \mathfrak{m} = \left[\frac{P}{X},R\right] \cup \left[-R,-\frac{P}{X}\right] \qquad \mathfrak{t} = \mathbb{R} \backslash (\mathfrak{M} \cup \mathfrak{m})$$

so that

$$\mathscr{I}(\eta,\omega,\mathbb{R}) = \mathscr{I}(\eta,\omega,\mathfrak{M}) + \mathscr{I}(\eta,\omega,\mathfrak{m}) + \mathscr{I}(\eta,\omega,\mathfrak{t}).$$

The parameters $P = P(X) > 1$ and $R = R(X) > 1/\eta$ are chosen later (see (3.7) and (3.10)) as well as $\eta = \eta(X)$, that, as we explained before, we will take a small negative power of $\mathrm{M}(\mathbf{p},\mathbf{k})$ (and so of X, see (3.23)).

It is necessary to prove that $\mathscr{I}(\eta,\omega,\mathfrak{m})$ and $\mathscr{I}(\eta,\omega,\mathfrak{t})$ are both $o(\mathscr{I}(\eta,\omega,\mathfrak{M}))$. As we will see, also in this second case we are dealing with, we do not have any gap between the major arc and the minor arc.

3.1 The major arc

Let us start from the major arc and the computation of the main term. We replace all S_n defined in (1.6) with the corresponding T_n defined in (1.8). This replacing brings up some errors that we must estimate by means of Lemma B.11, the Cauchy-Schwarz and the Hölder inequalities.

We have

$$\begin{aligned}
\mathscr{I}(\eta,\omega,\mathfrak{M}) &= \int_{\mathfrak{M}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\
&= \int_{\mathfrak{M}} T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\
&\quad + \int_{\mathfrak{M}} (S_2(\lambda_1\alpha)-T_2(\lambda_1\alpha))T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\
&\quad + \int_{\mathfrak{M}} S_2(\lambda_1\alpha)(S_2(\lambda_2\alpha)-T_2(\lambda_2\alpha))T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\
&\quad + \int_{\mathfrak{M}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_2(\lambda_3\alpha)-T_2(\lambda_3\alpha))T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\
&\quad + \int_{\mathfrak{M}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)(S_k(\lambda_4\alpha)-T_k(\lambda_4\alpha))K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha
\end{aligned}$$

$$=J_1+J_2+J_3+J_4+J_5, \tag{3.2}$$

say.

J_1: Main Term

As the reader might expect the main term is given by the summand J_1.

Let $H(\alpha)=T_2(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)$ so that

$$J_1=\int_{\mathbb{R}}H(\alpha)\mathrm{d}\alpha+\mathscr{O}\left(\int_{P/X}^{+\infty}|H(\alpha)|\mathrm{d}\alpha\right).$$

Using the inequality (1.5) and the Theorem (B.1),

$$\begin{aligned}\int_{P/X}^{+\infty}|H(\alpha)|\mathrm{d}\alpha \ll& X^{-\frac12}X^{-\frac12}X^{-\frac12}X^{\frac1k-1}\eta^2\int_{P/X}^{+\infty}\frac{\mathrm{d}\alpha}{\alpha^4}\\ \ll& X^{\frac1k-\frac52}\eta^2\frac{X^3}{P^3}=X^{\frac1k+\frac12}\eta^2P^{-3}=o\left(X^{\frac1k+\frac12}\eta^2\right)\end{aligned}$$

provided that $P\to+\infty$.

Let $D=[(\delta X)^{\frac12},X^{\frac12}]^3\times[(\delta X)^{\frac1k},X^{\frac1k}]$; we have

$$\begin{aligned}\int_{\mathbb{R}}H(\alpha)\mathrm{d}\alpha=&\int\cdots\int_D\int_{\mathbb{R}}e((\lambda_1t_1^2+\lambda_2t_2^2+\lambda_3t_3^2+\lambda_4t_4^k-\omega)\alpha)K_\eta(\alpha)\mathrm{d}\alpha\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4\\ =&\int\cdots\int_D\max(0,\eta-|\lambda_1t_1^2+\lambda_2t_2^2+\lambda_3t_3^2+\lambda_4t_4^k-\omega)|)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4.\end{aligned}$$

Apart from trivial changes of sign, there are essentially three cases as in [28]:

1. $\lambda_1>0$, $\lambda_2>0$, $\lambda_3>0$, $\lambda_4<0$
2. $\lambda_1>0$, $\lambda_2>0$, $\lambda_3<0$, $\lambda_4<0$
3. $\lambda_1>0$, $\lambda_2<0$, $\lambda_3<0$, $\lambda_4<0$.

We deal with the second case, the other ones being similar (we warn the reader that here it may be necessary to adjust the value of δ in order to guarantee the necessary set inclusions): let us perform the following change of variables: $u_1=t_1^2-\frac{\omega}{\lambda_1}$ (in order to make ω disappear), $u_2=t_2^2$, $u_3=t_3^2$, $u_4=t_4^k$. The set D becomes essentially $[\delta X,X]^4$ and let us define $D'=[\delta X,(1-\delta)X]^4$ for large X. The Jacobian determinant of the change of variables above is $\dfrac{1}{4k}u_1^{-\frac12}u_2^{-\frac12}u_3^{-\frac12}u_4^{\frac1k-1}$. Then

$$\begin{aligned}\int_{\mathbb{R}}H(\alpha)\mathrm{d}\alpha\gg&\int\cdots\int_{D'}\max(0,\eta-|\lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\lambda_4u_4)|)\frac{\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4}{u_1^{\frac12}u_2^{\frac12}u_3^{\frac12}u_4^{1-\frac1k}}\\ \gg& X^{\frac1k-\frac52}\int\cdots\int_{D'}\max(0,\eta-|\lambda_1u_1+\lambda_2u_2+\lambda_3u_3+\lambda_4u_4)|)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4.\end{aligned}$$

Apart from sign, the computation is essentially symmetrical with respect to the coefficients λ_j: we assume, as we may, that $|\lambda_4| \geq \max(\lambda_1,\lambda_2,\lambda_3)$, the other cases being similar. Now, for $j=1,2,3$ let $a_j = \dfrac{4|\lambda_4|}{|\lambda_j|}$, $b_j = \dfrac{3}{2}a_j$ and $\mathscr{D}_j = [a_j\delta X, b_j\delta X]$; if $u_j \in \mathscr{D}_j$ for $j=1,2,3$ then

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in [2|\lambda_4|\delta X, 8|\lambda_4|\delta X]$$

so that, for every choice of (u_1,u_2,u_3) the interval

$$[a,b] = \left[\frac{1}{|\lambda_4|}(-\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)), \frac{1}{|\lambda_4|}(\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3))\right]$$

is contained in $[\delta X, (1-\delta)X]$. In other words, for $u_4 \in [a,b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ cover the whole interval $[-\eta,\eta]$. Hence for any $(u_1,u_2,u_3) \in \mathscr{D}_1 \times \mathscr{D}_2 \times \mathscr{D}_3$ we have

$$\begin{aligned}
&\int_{\delta X}^{(1-\delta)X} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|)\mathrm{d}u_4 \\
&\geq |\lambda_4|^{-1} \int_{-\eta}^{\eta} \max(0, \eta - |u|)\mathrm{d}u \gg \eta^2.
\end{aligned}$$

Finally,

$$\int_{\mathbb{R}} H(\alpha)\mathrm{d}\alpha \gg \eta^2 X^{\frac{1}{k}-\frac{5}{2}} \iiint_{\mathscr{D}_1\times\mathscr{D}_2\times\mathscr{D}_3} \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 \gg \eta^2 X^{\frac{1}{k}-2} X^3 = \eta^2 X^{\frac{1}{k}+\frac{1}{2}}.$$

It means that the lower bound for J_1 is $\eta^2 X^{\frac{1}{k}+\frac{1}{2}}$, as expected.

Bound for J_2

We expect the main term to have the dominant asymptotic behavior, then we shall prove that all the remaining terms of the sum (3.2) are $o\left(\eta^2 X^{\frac{1}{k}+\frac{1}{2}}\right)$.

Retrieving (1.5) and using the triangle inequality,

$$\begin{aligned}
J_2 =& \int_{\mathfrak{M}} (S_2(\lambda_1\alpha) - T_2(\lambda_1\alpha))T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\
\ll& \eta^2 \int_{\mathfrak{M}} |S_2(\lambda_1\alpha) - T_2(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
\ll& \eta^2 \int_{\mathfrak{M}} |S_2(\lambda_1\alpha) - U_2(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&+ \eta^2 \int_{\mathfrak{M}} |U_2(\lambda_1\alpha) - T_2(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
=& \eta^2(A_2 + B_2),
\end{aligned}$$

say. Using the Cauchy-Schwartz inequality and remembering the definition of $\mathfrak{M}$,

$$A_2 \ll \left(\int_{-P/X}^{P/X} |S_2(\lambda_1\alpha) - U_2(\lambda_1\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_{-P/X}^{P/X} |T_2(\lambda_2\alpha)|^2|T_2(\lambda_3\alpha)|^2|T_k(\lambda_4\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}};$$

then using Lemma B.11 and Lemma B.12,

$$\int_{-P/X}^{P/X} |S_2(\lambda_1\alpha) - U_2(\lambda_1\alpha)|^2 \mathrm{d}\alpha \ll \frac{P^2}{X^2}\mathscr{J}_2\left(X, \frac{X}{2P}\right)$$
$$\ll \exp\left(-c_1\left(\frac{\log X}{\log\log X}\right)^{\frac{1}{3}}\right) \ll \frac{1}{(\log X)^A}$$

for any fixed $A > 0$ and $X^{1-\frac{5}{12}+\varepsilon} \leq \frac{X}{P} \leq X$ (conditions of Lemma B.12).

Also in this case, in the estimation above we have neglected the two terms

$$\frac{X^{2/k-2}\log^2 X}{P/X} + P^2/X$$

of Lemma B.11 as these terms become negligible in order to meet the hypotheses of Lemma B.12.

The first condition on P (with $k = 2$) is then the following:

$$\frac{X}{P} \geq X^{\frac{7}{12}+\varepsilon} \quad \Rightarrow \quad P \leq X^{\frac{5}{12}-\varepsilon}. \tag{3.3}$$

Let us complete the estimate for A_2:

$$A_2 \ll \left(\frac{1}{(\log X)^A}\right)^{\frac{1}{2}} \left(\int_0^{1/X} |T_2(\lambda_2\alpha)|^2 |T_2(\lambda_3\alpha)|^2 |T_k(\lambda_4\alpha)|^2 \mathrm{d}\alpha\right.$$
$$\left. + \int_{1/X}^{P/X} |T_2(\lambda_2\alpha)|^2 |T_2(\lambda_3\alpha)|^2 |T_k(\lambda_4\alpha)|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}}.$$

Remembering Theorem B.1,

$$A_2 \ll \left(\frac{1}{(\log X)^A}\right)^{\frac{1}{2}} \left(\int_0^{1/X} (X^{\frac{1}{2}})^2 (X^{\frac{1}{2}})^2 (X^{\frac{1}{k}})^2 \mathrm{d}\alpha\right.$$
$$\left. + \int_{1/X}^{P/X} \frac{(X^{-\frac{1}{2}})^2 (X^{-\frac{1}{2}})^2 (X^{\frac{1}{k}-1})^2}{\alpha^6} \mathrm{d}\alpha\right)^{\frac{1}{2}}$$
$$\ll \left(\frac{1}{(\log X)^A}\right)^{\frac{1}{2}} \left(\int_0^{1/X} X^{2+\frac{2}{k}} \mathrm{d}\alpha + \int_{1/X}^{P/X} \frac{X^{\frac{2}{k}-4}}{\alpha^6} \mathrm{d}\alpha\right)^{\frac{1}{2}}$$
$$\ll \left(\frac{1}{(\log X)^A}\right)^{\frac{1}{2}} \left(X^{1+\frac{2}{k}} + X^{\frac{2}{k}-4} \cdot \left(\frac{1}{X}\right)^{-5}\right)^{\frac{1}{2}}$$
$$\ll \left(\frac{1}{(\log X)^A}\right)^{\frac{1}{2}} \left(X^{\frac{2}{k}+1}\right)^{\frac{1}{2}} = \frac{1}{(\log X)^{\frac{A}{2}}} X^{\frac{1}{2}+\frac{1}{k}} = \frac{X^{\frac{1}{2}+\frac{1}{k}}}{(\log X)^{\frac{A}{2}}} = o(X^{\frac{1}{2}+\frac{1}{k}})$$

for all $A > 0$.

Now we need an estimation for B_2: noting that

$$1 + |\alpha|X \ll \begin{cases} 1 & \text{if } |\alpha| \leq \frac{1}{X} \\ |\alpha|X & \text{if } |\alpha| \geq \frac{1}{X}, \end{cases}$$

in this case we use the Theorem B.2,

$$\begin{aligned}
B_2 &= \int_{\mathfrak{M}} |U_2(\lambda_1\alpha) - T_2(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&\ll \int_0^{1/X} |T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&\quad + X\int_{1/X}^{P/X} \alpha|T_2(\lambda_2\alpha)||T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&\ll \int_0^{1/X} X^{\frac{1}{2}}X^{\frac{1}{2}}X^{\frac{1}{k}}\mathrm{d}\alpha + XX^{-\frac{1}{2}}X^{-\frac{1}{2}}X^{\frac{1}{k}-1}\int_{1/X}^{P/X}\frac{\alpha}{\alpha^3}\mathrm{d}\alpha \\
&\ll X^{\frac{1}{k}} = o\left(X^{\frac{1}{k}+\frac{1}{2}}\right).
\end{aligned}$$

Bound for J_4

The computations on J_3 are similar and simpler than the corresponding one on J_4, so we will skip them. Using the triangle inequality and (1.5),

$$\begin{aligned}
J_4 &= \int_{\mathfrak{M}} S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha))T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\
&\ll \eta^2 \int_{\mathfrak{M}} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&\le \eta^2 \int_{\mathfrak{M}} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&\quad + \eta^2 \int_{\mathfrak{M}} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||U_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&= \eta^2(A_4 + B_4),
\end{aligned}$$

say. Using Theorem B.1 and the Hölder inequality,

$$\begin{aligned}
A_4 &\ll X^{\frac{1}{k}} \int_{\mathfrak{M}} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|\mathrm{d}\alpha \\
&\ll X^{\frac{1}{k}} \left(\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\int_{\mathfrak{M}} |S_2(\lambda_2\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\int_{\mathfrak{M}} |S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}.
\end{aligned}$$

Using Lemmas B.4-B.11-B.12, for any fixed A,

$$A_4 \ll X^{\frac{1}{k}}(X\log^2 X)^{\frac{1}{2}}(\log X)^{-\frac{A}{2}} = X^{\frac{1}{2}+\frac{1}{k}}(\log X)^{1-\frac{A}{2}} = o\left(X^{\frac{1}{k}+\frac{1}{2}}\right)$$

as long as $A > 2$.

As for A_2 we used in the estimation above Lemma B.11 that has two more terms, but also in this case these terms are negligible if we want to meet the hypothesis of Lemma B.12: in fact it requires that

$$X^{1-\frac{5}{12}+\varepsilon} \le \frac{X}{P} \le X$$

and this is consistent with the choice we will make in (3.4).

Again using Theorem B.2,

$$\begin{aligned}B_4 =& \int_{\mathfrak{M}} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||U_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ \ll& \int_0^{1/X} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &+ X\int_{1/X}^{P/X} \alpha|S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha.\end{aligned}$$

Remembering that $|\alpha| \le \frac{P}{X}$ on $\mathfrak{M}$ and using the Hölder inequality, trivial bounds and Lemma B.4 we have

$$\begin{aligned}B_4 \ll& X^{\frac{1}{2}}X^{\frac{1}{2}}X^{\frac{1}{k}}\frac{1}{X} + XX^{\frac{1}{k}}\left(\int_{1/X}^{P/X}\alpha^2\right)^{\frac{1}{2}}\left(\int_{1/X}^{P/X}|S_2(\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{1/X}^{P/X}|S_2(\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}} \\ \ll& X^{\frac{1}{k}} + X^{1+\frac{1}{k}}\left(X\log^2 X\right)^{\frac{1}{2}}\left(\int_{1/X}^{P/X}\alpha^2\mathrm{d}\alpha\right)^{\frac{1}{2}} \\ \ll& X^{\frac{1}{k}} + X^{\frac{3}{2}+\frac{1}{k}}\log X\left(\frac{P}{X}\right)^{\frac{3}{2}} = X^{\frac{1}{k}}P^{\frac{3}{2}}\log X.\end{aligned}$$

We assume

$$P \le X^{\frac{1}{3}-\varepsilon}, \tag{3.4}$$

so that $P^{\frac{3}{2}} = o(X^{\frac{1}{2}}/\log X)$ which, with the upper bound for B_4 here above, ensures that

$$B_4 = o(X^{1/2+1/k}).$$

Bound for J_5

In order to provide an estimation for J_5, we use (1.5),

$$J_5 \ll \eta^2\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha) - T_k(\lambda_4\alpha)|\mathrm{d}\alpha$$

and then the arithmetic-geometric inequality:

$$J_5 \ll \eta^2\sum_{j=1}^{3}\left(\int_{\mathfrak{M}} |S_2(\lambda_j\alpha)|^3|S_k(\lambda_4\alpha) - T_k(\lambda_4\alpha)|\mathrm{d}\alpha\right).$$

The three terms may be estimated in the same way and produce the same upper bound. We show the details of the bound only for the case $j = 1$:

$$\begin{aligned}&\eta^2\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^3|S_k(\lambda_4\alpha) - T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\quad\ll \eta^2\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^3|S_k(\lambda_4\alpha) - U_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\quad\quad+ \eta^2\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^3|U_k(\lambda_4\alpha) - T_k(\lambda_4\alpha)|\mathrm{d}\alpha\end{aligned}$$

$$=\eta^2(A_5+B_5),$$

say. Using trivial estimates,

$$A_5 \ll X^{\frac{1}{2}} \int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^2 |S_k(\lambda_4\alpha)-U_k(\lambda_4\alpha)| \mathrm{d}\alpha$$

then using the Cauchy-Schwartz inequality, for any fixed $A>2$, by Lemmas B.4, B.11 and B.12 we have

$$\begin{aligned} A_5 \ll & X^{\frac{1}{2}} \left(\int_{\mathfrak{M}} |S_2(\lambda_1\alpha)|^4 \mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_{\mathfrak{M}} |S_k(\lambda_4\alpha)-U_k(\lambda_4\alpha)|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \\ \ll & X^{\frac{1}{2}} X^{\frac{1}{2}} \log X \frac{P}{X} \mathscr{J}_k\left(X, \frac{X}{P}\right)^{\frac{1}{2}} \ll_A X^{\frac{1}{2}+\frac{1}{k}} (\log X)^{1-\frac{A}{2}} = o\left(X^{\frac{1}{2}+\frac{1}{k}}\right) \end{aligned}$$

provided that $\frac{X}{P} \geq X^{1-\frac{5}{6k}+\varepsilon}$ (condition of Lemma B.12), that is,

$$(\log X)^A \ll_A P \leq X^{\frac{5}{6k}-\varepsilon}. \tag{3.5}$$

Now we turn to B_5, using Theorem B.2:

$$B_5 \ll \int_0^{1/X} |S_2(\lambda_1\alpha)|^3 \mathrm{d}\alpha + X \int_{1/X}^{P/X} \alpha |S_2(\lambda_1\alpha)|^3 \mathrm{d}\alpha.$$

Using trivial estimates and Lemma B.4,

$$\begin{aligned} B_5 \ll & X^{\frac{3}{2}} \frac{1}{X} + X \cdot X^{\frac{1}{2}} \left(\int_{1/X}^{P/X} \alpha^2 \mathrm{d}\alpha \cdot \int_{1/X}^{P/X} |S_2(\lambda_1\alpha)|^4 \mathrm{d}\alpha\right)^{\frac{1}{2}} \\ \ll & X^{\frac{1}{2}} + X^{\frac{3}{2}} (P/X)^{\frac{3}{2}} X^{\frac{1}{2}} \log X = X^{\frac{1}{2}} + P^{\frac{3}{2}} X^{\frac{1}{2}} \log X. \end{aligned}$$

The case $j=2$ and $j=3$ can be estimated in the same way. We need

$$P = o\left(X^{\frac{2}{3k}-\varepsilon}\right).$$

Summing up with (3.5),

$$P \leq X^{\frac{2}{3k}-\varepsilon}. \tag{3.6}$$

Collecting all the bounds for P, that is, (3.3), (3.4), (3.6) we can take

$$P \leq X^{\frac{1}{3}-\varepsilon}. \tag{3.7}$$

In fact, if we consider (3.4), (3.5) and (3.6) we should choose the most restrictive condition among the three but as we expect that the value of k is smaller than 2, (3.4) is the most restrictive: $\frac{2}{3k} \leq \frac{5}{6k}$ and $\frac{1}{3} \geq \frac{2}{3k}$ only if $k \geq 2$.

3.2 Trivial arc

By the arithmetic-geometric mean inequality and the trivial bound for $S_2(\lambda_1\alpha)$, we see that

$$\begin{aligned}|\mathscr{I}(\eta,\omega,\mathrm{t})| &\ll \int_R^{+\infty} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\eta(\alpha)|\mathrm{d}\alpha \\ &\ll X^{\frac{1}{2}}\sum_{j=1}^{3}\int_R^{+\infty}|S_2(\lambda_j\alpha)|^2|S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha.\end{aligned}$$

The three terms may be estimated in the same way and produce the same upper bound. We show the details of the bound only for the case $j=1$:

$$\begin{aligned}&X^{\frac{1}{2}}\int_R^{+\infty}|S_2(\lambda_j\alpha)|^2|S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\ &\quad\ll X^{\frac{1}{2}}\left(\int_R^{+\infty}|S_2(\lambda_1\alpha)|^4K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_R^{+\infty}|S_k(\lambda_4\alpha)|^2K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}} \\ &\quad\ll X^{\frac{1}{2}}\left(\int_R^{+\infty}\frac{|S_2(\lambda_1\alpha)|^4}{\alpha^2}\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_R^{+\infty}\frac{|S_k(\lambda_4\alpha)|^2}{\alpha^2}\mathrm{d}\alpha\right)^{\frac{1}{2}} = X^{\frac{1}{2}}C_1^{\frac{1}{2}}C_2^{\frac{1}{2}},\end{aligned}$$

say. Using Lemma B.4, we have

$$\begin{aligned}C_1 &= \int_R^{+\infty}\frac{|S_2(\lambda_2\alpha)|^4}{\alpha^2}\mathrm{d}\alpha \ll \int_{\lambda_1 R}^{+\infty}\frac{|S_2(\alpha)|^4}{\alpha^2}\mathrm{d}\alpha \\ &\ll \sum_{n\geq|\lambda_1|R}\frac{1}{(n-1)^2}\int_{n-1}^{n}|S_2(\alpha)|^4\mathrm{d}\alpha \ll \frac{X\log^2 X}{R}.\end{aligned} \tag{3.8}$$

Now using Lemma B.6,

$$\begin{aligned}C_2 &= \int_R^{+\infty}\frac{|S_k(\lambda_4\alpha)|^4}{\alpha^2}\mathrm{d}\alpha \ll \int_{\lambda_4 R}^{+\infty}\frac{|S_k(\alpha)|^2}{\alpha^2}\mathrm{d}\alpha \\ &\ll \sum_{n\geq|\lambda_4|R}\frac{1}{(n-1)^2}\int_{n-1}^{n}|S_k(\alpha)|^2\mathrm{d}\alpha \ll \frac{X^{\frac{1}{k}}\log^3 X}{R}.\end{aligned} \tag{3.9}$$

Collecting (3.8) and (3.9),

$$|\mathscr{I}(\eta,\omega,\mathrm{t})| \ll X^{\frac{1}{2}}\left(\frac{X\log^2 X}{R}\right)^{\frac{1}{2}}\left(\frac{X^{\frac{1}{k}}\log^3 X}{R}\right)^{\frac{1}{2}} \ll \frac{X^{1+\frac{1}{2k}}(\log X)^{\frac{5}{2}}}{R}.$$

Hence, remembering that $|\mathscr{I}(\eta,\omega,t)|$ must be $o\left(\eta^2X^{\frac{1}{k}+1}\right)$, i.e. little-o of the main term, the choice

$$R = \frac{X^{\frac{1}{2}-\frac{1}{2k}}\log^3 X}{\eta^2} \tag{3.10}$$

is admissible.

3.3 The minor arc

The measure of the set where $|S_2(\lambda_1\alpha)|$ and $|S_2(\lambda_2\alpha)|$ are both large for $\alpha \in \mathfrak{m}$ is suitably bounded, exploiting the fact that the ratio λ_1/λ_2 is irrational. The idea is to reach the widest k-range assuming k as a parameter from which all other parameters depend and adjusting all them retrospectively.

Also in this section we have chosen to leave the parameters free then we proceed as we did in Chapter 2. The following lemma is not true for each parameter choice, but only for a subset of them where the optimum values are chosen. The parameters are:

$$d = \frac{8}{7}, \qquad \frac{1}{a(k)} = \frac{k+4}{9k}, \qquad b(k) = \frac{4k-2}{9k}, \qquad c(k) = \frac{8-7k}{18k}. \tag{3.11}$$

Now, following the lines of Lemma 2.3 of [29], we have

Lemma 3.2. *Let $1 < k < d$. Assume that λ_1/λ_2 is irrational and let $X = q^{a(k)}$, where q is the denominator of a convergent of the continued fraction for λ_1/λ_2. Let*

$$V(\alpha) = \min(|S_1(\lambda_1\alpha)|, |S_2(\lambda_2\alpha)|).$$

Then

$$\sup_{\alpha\in\mathfrak{m}} V(\alpha) \ll X^{1-\frac{b(k)}{2}+\varepsilon}.$$

Proof. Let $\alpha \in m$ and $Q = X^{b(k)} \le P$. By rational approximation Dirichlet's (Theorem A.3), there exist integers a_i, q_i with $1 \le q_i \le \frac{X}{Q}$ and $(a_i, q_i) = 1$, such that

$$|\lambda_i\alpha q_i - a_i| \le \frac{Q}{X}$$

for $i = 1, 2$. We remark that $a_1 a_2 \neq 0$, for otherwise we would have $\alpha \in \mathfrak{M}$. Now suppose that $q_i \le Q$ for $i = 1, 2$. In this case we get

$$a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 = (\lambda_1\alpha q_1 - a_1)\frac{a_2}{\lambda_2\alpha} - (\lambda_2\alpha q_2 - a_2)\frac{a_1}{\lambda_2\alpha}$$

and hence

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| \le 2\left(1 + \left|\frac{\lambda_1}{\lambda_2}\right|\right)\frac{Q^2}{X} < \frac{1}{2q} \tag{3.12}$$

for sufficiently large X. In fact, the last inequality is true due to the choice of the parameters in (3.11): being $q < Q$, we need

$$Q^3 < X \quad \Rightarrow \quad X^{3b(k)} < X \quad \Rightarrow \quad 3b(k) < 1,$$

this is always true: $\frac{4k-2}{9k} < 1 \quad \Leftrightarrow \quad 4k - 2 < 9k \quad \Leftrightarrow \quad k > -\frac{2}{5}.$

Then, from the law of best approximation (Theorem A.4)

$$X^{\frac{1}{a(k)}} = q \le |a_2 q_1|$$

so, recalling the definition of $\mathfrak{m}$, we obtain

$$|a_2 q_1| \ll q_1 q_2 R \le Q^2 R \le X^{\frac{1}{a(k)}-\varepsilon} \tag{3.13}$$

which is absurd[1] if $\eta = X^{-c(k)}$ and $R = \frac{\log^2 X}{\eta^2}$. Hence either $q_1 > Q$ or $q_2 > Q$.

If $\eta = X^{-c(k)}$ and $R = \frac{\log^2 X}{\eta^2}$, hence either $q_1 > Q$ or $q_2 > Q$.

Assuming that $q_i > Q$ and using Lemma B.14:

$$V(\alpha) \le |S_2(\lambda_i \alpha)|^2 \ll X^{1+2\varepsilon} \sup_{Q<q_i<\frac{X}{Q}} \left(\frac{1}{q_i} + \frac{1}{X^{\frac{1}{4}}} + \frac{q_i}{X} \right)^{\frac{1}{2}} \ll X^{1-\frac{b(k)}{2}+\varepsilon}. \tag{3.14}$$

The inequality for $i = 1,2$ in (3.14) is true due to the appropriate choice of the parameters in (3.11), and this completes the proof of the Lemma.

□

If we want that the last inequalities in both (3.12) and (3.13) are true, we need the following conditions on $a(k)$, $b(k)$ and $c(k)$:

$$2b(k) - 1 \le -\frac{1}{a(k)} \tag{3.15}$$

$$2b(k) + 2c(k) \le \frac{1}{a(k)}. \tag{3.16}$$

Now let

$$\mathscr{X}_1 = \left\{ \alpha \in \left[\frac{P}{X}, R \right] : |S_2(\lambda_1 \alpha)| \le |S_2(\lambda_2 \alpha)| \right\}$$

$$\mathscr{X}_2 = \left\{ \alpha \in \left[\frac{P}{X}, R \right] : |S_2(\lambda_1 \alpha)| \ge |S_2(\lambda_2 \alpha)| \right\}$$

so that $\left[\frac{P}{X}, R\right] = \mathscr{X}_1 \cup \mathscr{X}_2$ and

$$|\mathscr{I}(\eta,\omega,m)| \ll \left(\int_{\mathscr{X}_1} + \int_{\mathscr{X}_2} \right) |S_2(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||S_2(\lambda_3 \alpha)||S_k(\lambda_4 \alpha)| K_\eta(\alpha) \mathrm{d}\alpha$$

If $\alpha \in \mathscr{X}_1$, we have $|S_2(\lambda_1 \alpha)| \le |S_2(\lambda_2 \alpha)|$, then

$$\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathscr{X}_1)| &\ll \int_{\mathscr{X}_1} |S_2(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||S_2(\lambda_3 \alpha)||S_k(\lambda_4 \alpha)| K_\eta(\alpha) \mathrm{d}\alpha \\
&\ll \left(\max_{\alpha \in \mathscr{X}_1} |S_2(\lambda_1 \alpha)| \right) \left(\int_{\mathscr{X}_1} |S_2(\lambda_2 \alpha)|^4 K_\eta(\alpha) \mathrm{d}\alpha \right)^{1/4} \\
&\quad \left(\int_{\mathscr{X}_1} |S_2(\lambda_3 \alpha)|^4 K_\eta(\alpha) \mathrm{d}\alpha \right)^{1/4} \left(\int_{\mathscr{X}_1} |S_k(\lambda_4 \alpha)|^2 K_\eta(\alpha) \mathrm{d}\alpha \right)^{1/2}
\end{aligned}$$

[1] In this Chapter too, the choice of the parameters in (3.11) is made in order to get a contradiction.

$$\begin{aligned}&\ll X^{\frac{1}{2}-\frac{b}{4}+\varepsilon}(\eta X\log^2 X)^{\frac{1}{4}}(\eta X\log^2 X)^{\frac{1}{4}}\left(\eta X^{\frac{1}{k}}\log^3 X\right)^{\frac{1}{2}}\\ &=X^{1-\frac{b}{4}+\frac{1}{2k}+\varepsilon}\eta\log^2 X\end{aligned}$$

In this case $|\mathscr{I}(\eta,\omega,\mathfrak{m})| = o\left(X^{\frac{1}{2}+\frac{1}{k}-\varepsilon}\right)$ only if

$$\eta = \infty\left(X^{\frac{1}{2}-\frac{1}{2k}-\frac{b(k)}{4}+\varepsilon}\right)$$

then we can add the following condition on $c(k)$:

$$-c(k) \geq \frac{1}{2}-\frac{1}{2k}-\frac{b(k)}{4} \tag{3.17}$$

The other case with $\alpha \in \mathscr{X}_2$ is totally similar.

Collecting (3.15), (3.16) and (3.17) we get $d = 8/7$ the maximum value we can have for k. This is justified as follows: neglecting the log-powers and recalling the choice of $P(X)$ in (3.7) and $R(X)$ in (3.10),we have to maximize k subject to the constraints:

$$\begin{cases} a(k) \geq 1\\ 0 \leq b(k) \leq \frac{1}{3}\\ c(k) > 0\\ 2b(k)-1 \leq -\frac{1}{a(k)}\\ 2b(k)+2c(k) \leq \frac{1}{a(k)}\\ -c(k) \geq \frac{1}{2}-\frac{1}{2k}-\frac{b(k)}{4}\end{cases}$$

which is a linear optimization problem in the variables $\frac{1}{k}$, $\frac{1}{a(k)}$, $b(k)$, $c(k)$. The solution of this problem is

$$\frac{1}{a(k)}=\frac{k+4}{9k}, \qquad b(k)=\frac{4k-2}{9k}, \qquad c(k)=\frac{8-7k}{18k},$$

for $k < \frac{8}{7}$.

3.4 The minor arc using the Harman technique

We start again from the idea that the measure of the set where $|S_2(\lambda_1\alpha)|^{\frac{1}{2}}$ and $|S_2(\lambda_2\alpha)|$ are both large for $\alpha \in \mathfrak{m}$ is suitably bounded, if the ratio λ_1/λ_2 is irrational. In this case we need only Corollary B.16. Let us split $\mathfrak{m}$ into subsets $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}^* = \mathfrak{m}\backslash(\mathfrak{m}_1\cup\mathfrak{m}_2)$ where

$$\mathfrak{m}_i = \{\alpha\in\mathfrak{m} : |S_2(\lambda_i\alpha)| \leq X^{\frac{1}{2}-u+\varepsilon}\}$$

remembering that Corollary B.16 holds for $0 \leq u \leq \frac{1}{16}$. In this case we leave only the parameter u free.

Using the Hölder inequalities and the definition of m_i we obtain

$$
\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathfrak{m}_i)| \ll & \int_{\mathfrak{m}_i} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\
\ll & \max|S_2(\lambda_1\alpha)| \left(\int_{\mathfrak{m}_i} |S_2(\lambda_2\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \\
& \left(\int_{\mathfrak{m}_i} |S_2(\lambda_3\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/4} \left(\int_{\mathfrak{m}_i} |S_k(\lambda_4\alpha)|^2 K_\eta(\alpha)\mathrm{d}\alpha\right)^{1/2} \\
\ll & X^{\frac{1}{2}-u+\varepsilon}(\eta X\log^2 X)^{\frac{1}{4}}(\eta X\log^2 X)^{\frac{1}{4}}\left(\eta X^{\frac{1}{k}}\log^3 X\right)^{\frac{1}{2}} \\
= & \eta X^{1-u+\frac{1}{2k}+\varepsilon}\log^{\frac{5}{2}} X. \qquad (3.18)
\end{aligned}
$$

by Corollary B.16.

The bound (3.18) must be $o\left(\eta^2 X^{\frac{1}{2}+\frac{1}{k}}\right)$, consequently we have the following condition:

$$\eta = \infty\left(X^{\frac{1}{2}-\frac{1}{2k}-u+\varepsilon}\right).$$

It remains to discuss the set $\mathfrak{m}^*$ in which the following bounds hold simultaneously

$$|S_2(\lambda_i\alpha)| > X^{\frac{1}{2}-u+\varepsilon}, \qquad \frac{P}{X} = X^{-\frac{2}{3}} < |\alpha| \leq \frac{\log^2 X}{\eta^2}.$$

Following the dyadic dissection argument shown by Harman in [22] we divide $\mathfrak{m}^*$ into disjoint sets $E(Z_1,Z_2,y)$ in which, for $\alpha \in E(Z_1,Z_2,y)$, we have

$$Z_1 < |S_2(\lambda_1\alpha)| \leq 2Z_1, \qquad Z_2 < |S_2(\lambda_2\alpha)| \leq 2Z_2, \qquad y < |\alpha| \leq 2y$$

where $Z_i = 2^{k_i}X^{\frac{1}{2}-u+\varepsilon}$ for $i=1,2$, and $y = 2^{k_3}X^{-\frac{2}{3}-\varepsilon}$ for some non-negative integers k_1,k_2,k_3.

It follows that the disjoint sets are, at most, $\ll \log^3 X$. Let us define $\mathscr{A}$ a shorthand for the sets $E(Z_1,Z_2,y)$; we have the following result about the Lebesgue measure of $\mathscr{A}$ following the same lines of Lemma 6 in [36].

In the following Lemma, it is crucial that both the integers a_1 and a_2 appearing in (2.21) below do not vanish: in fact, if $a_1 = 0$, say, then $q_1 = 1$ and $|\alpha|$ is so small that it can not belong to $\mathfrak{m}^*$. In order to use the Harman technique then, we are forced to be far from the major arc in which we would have $a_1a_2 = 0$. As we will see later, after setting the parameter u, we will not have a gap between the major and the minor arc, so it will not be necessary to introduce an intermediate arc.

Lemma 3.3. *We have*

$$\mu(\mathscr{A}) \ll yX^{2+8u+3\varepsilon}Z_1^{-4}Z_2^{-4}$$

where $\mu(\cdot)$ denotes the Lebesgue measure.

Proof. If $\alpha \in \mathscr{A}$, by Corollary B.16 there are coprime integers (a_1, q_1) and (a_2, q_2) such that

$$1 \le q_2 \ll \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4, \qquad |q_2\lambda_2\alpha - a_2| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4 \tag{3.19}$$

We remark that $a_1a_2 \neq 0$ otherwise we would have $\alpha \in \mathfrak{M}$. In fact, if $a_i = 0$ recalling the definitions of Z_1 and (3.19), we get

$$|\alpha| \ll q_2^{-1}X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4 \ll \frac{X}{X^{2-4u+3\varepsilon}} = X^{-1+4u-3\varepsilon}.$$

It means that, on the minor arc

$$|\alpha| \gg X^{-1+4u-3\varepsilon}.$$

We wonder now if there is a gap between the end of the major arc and the beginning of the minor arc: from Corollary B.16 we are sure that $u \le \frac{1}{16}$; furthermore, from the previous lower bound for α, we need to check whether $\frac{P}{X}$ is greater than it:

$$X^{-1+4u-3\varepsilon} < \frac{P}{X} = X^{-\frac{2}{3}} \quad \Rightarrow \quad u < \frac{1}{12}.$$

It is clear that we can choose any parameter u with the condition given by Corollary B.16 without leaving any gap from the two arcs.

Now, we can further split $\mathfrak{m}^*$ into sets $I(Z_1, Z_2, y, Q_1, Q_2)$ where $Q_j \le q_j \le 2Q_j$ on each set. Note that a_i and q_i are uniquely determined by α. In the opposite direction, for a given quadruple a_1, q_1, a_2, q_2 the inequalities (3.19) define an interval of α of length

$$\mu(I) \ll \min\left(Q_1^{-1}X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_1}\right)^4, Q_2^{-1}X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4\right).$$

Taking the geometric mean ($\min(a,b) \le \sqrt{a}\sqrt{b}$) we can write

$$\mu(I) \ll Q_1^{-\frac{1}{2}}Q_2^{-\frac{1}{2}}X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_1}\right)^2\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^2 \ll \frac{X^{1+\varepsilon}}{Q_1^{\frac{1}{2}}Q_2^{\frac{1}{2}}Z_1^2Z_2^2}. \tag{3.20}$$

Now we need a lower bound for $Q_1^{\frac{1}{2}}Q_2^{\frac{1}{2}}$: by (3.19)

$$\begin{aligned} \left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| &= \left|\frac{a_2}{\lambda_2\alpha}(q_1\lambda_1\alpha - a_1) - \frac{a_1}{\lambda_2\alpha}(q_2\lambda_2\alpha - a_2)\right| \\ &\ll q_2|q_1\lambda_1\alpha - a_1| + q_1|q_2\lambda_2\alpha - a_2| \\ &\ll Q_2X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_1}\right)^4 + Q_1X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4. \end{aligned} \tag{3.21}$$

Remembering that $Q_i \ll \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_i}\right)^4$, $Z_i \gg X^{\frac{1}{2}-u+\varepsilon}$,

$$\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| \ll \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{X^{\frac{1}{2}-u+\varepsilon}}\right)^4 X^{-1} \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{X^{\frac{1}{2}-u+\varepsilon}}\right)^4 \ll \frac{X^{3+2\varepsilon}}{X^{4-8u+8\varepsilon}} \ll X^{-1+8u-6\varepsilon}. \tag{3.22}$$

We recall that $q = X^{1-8u}$ is a denominator of a convergent of λ_1/λ_2. Hence by (3.22) Legendre's law of best approximation (see Appendix A.4) implies that $|a_2 q_1| \geq q$ and by the same token, for any pair α, α' having distinct associated products $a_2 q_1$ (see [49], Lemma 2),

$$|a_2(\alpha)q_1(\alpha) - a_2(\alpha')q_1(\alpha')| \geq q;$$

thus, by the pigeon-hole principle, there is at most one value of $a_2 q_1$ in the interval $[rq, (r+1)q)$ for any positive integer r. Hence $a_2 q_1$ determines a_2 and q_1 to within X^ε possibilities (see the upper bound for the divisor function in Appendix, Theorem A.8) and consequently also $a_2 q_1$ determines a_1 and q_2 to within X^ε possibilities from (3.22).

Hence we got a lower bound for $q_1 q_2$, remembering that in our shorthand $Q_j \leq q_j \leq 2Q_j$:

$$q_1 q_2 = a_2 q_1 \frac{q_2}{a_2} \gg \frac{rq}{|\alpha|} \gg rqy^{-1}$$

for the quadruple under consideration. As a consequence we obtain from (3.20), that the total length of the interval $I(Z_1, Z_2, y, Q_1, Q_2)$ with $a_2 q_1 \in [rq, (r+1)q)$ does not exceed

$$\mu(I) \ll X^{1+2\varepsilon} Z_1^{-2} Z_2^{-2} r^{-\frac{1}{2}} q^{-\frac{1}{2}} y^{\frac{1}{2}}.$$

Now we need a bound for r: inside the interval $[rq, (r+1)q)$, $rq \leq |a_2 q_1|$ and, in turn from (3.19), $a_2 \ll q_2|\alpha|$, then

$$rq \ll q_1 q_2 |\alpha| \ll \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_1}\right)^4 \left(\frac{X^{\frac{1}{2}+\varepsilon/4}}{Z_2}\right)^4 y \ll yX^{4+2\varepsilon} Z_1^{-4} Z_2^{-4}$$
$$\Rightarrow r \ll q^{-1} y X^{4+2\varepsilon} Z_1^{-4} Z_2^{-4}.$$

Now, we sum on every interval to get an upper bound for the measure of $\mathscr{A}$:

$$\mu(\mathscr{A}) \ll X^{1+2\varepsilon} Z_1^{-2} Z_2^{-2} q^{-\frac{1}{2}} y^{\frac{1}{2}} \sum_{1 \leq r \ll q^{-1} y X^{4+2\varepsilon} Z_1^{-4} Z_2^{-4}} r^{-\frac{1}{2}}.$$

By standard estimation we obtain

$$\sum_{1 \leq r \ll q^{-1} y X^{4+2\varepsilon} Z_1^{-4} Z_2^{-4}} r^{-\frac{1}{2}} \ll (q^{-1} y X^{4+2\varepsilon} Z_1^{-4} Z_2^{-4})^{\frac{1}{2}}$$

then

$$\mu(\mathscr{A}) \ll yX^{3+3\varepsilon} Z_1^{-4} Z_2^{-4} q^{-1} \ll yX^{3+3\varepsilon} Z_1^{-4} Z_2^{-4} X^{-1+8u} \ll yX^{2+8u+3\varepsilon} Z_1^{-4} Z_2^{-4}.$$

This concludes the proof of Lemma 3.3. □

Using Lemma 3.3 we finally are able to get a bound for $\mathscr{I}(\eta,\omega,\mathscr{A})$:

$$\begin{aligned}
\mathscr{I}(\eta,\omega,\mathscr{A}) =& \int_{m^*} |S_2(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\mathrm{d}\alpha \\
&\ll \left(\int_{\mathscr{A}} |S_2(\lambda_1\alpha)S_2(\lambda_2\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\int_{\mathscr{A}} |S_2(\lambda_3\alpha)|^4 K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}} \\
&\quad \left(\int_{\mathscr{A}} |S_k(\lambda_4\alpha)|^2 K_\eta(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}} \\
&\ll \left(\min\left(\eta^2,\frac{1}{y^2}\right)\right)^{\frac{1}{4}} \left((Z_1Z_2)^4\mu(\mathscr{A})\right)^{\frac{1}{4}} \left(\eta X\log^2 X\right)^{\frac{1}{4}} \left(\eta X^{\frac{1}{k}}\log^3 X\right)^{\frac{1}{2}} \\
&\ll \left(\min\left(\eta^2,\frac{1}{y^2}\right)\right)^{\frac{1}{4}} Z_1Z_2(yX^{2+8u+4\varepsilon}Z_1^{-4}Z_2^{-4})^{\frac{1}{4}}\eta^{\frac{3}{4}}X^{\frac{1}{4}+\frac{1}{2k}+\varepsilon} \\
&\ll \left(\min\left(\eta^2,\frac{1}{y^2}\right)\right)^{\frac{1}{4}} y^{\frac{1}{4}}\eta^{\frac{3}{4}}X^{\frac{3}{4}+2u+\frac{1}{2k}+\varepsilon} \\
&\ll \eta X^{\frac{3}{4}+u+\frac{1}{2k}+\varepsilon}
\end{aligned}$$

and this must be $o\left(X^{1+\frac{1}{k}-\varepsilon}\right)$.

The condition on η is

$$\eta = \infty\left(X^{\frac{1}{4}-\frac{1}{2k}+2u+\varepsilon}\right). \tag{3.23}$$

Collecting all the conditions (3.18), (3.23) and the condition given by Corollary B.16, we get the following linear optimization system: setting $x=\frac{1}{k}$ and let w be the exponent of η we would like to optimize,

$$\begin{cases}
x\le 1;\ w\ge 0;\ u\le \frac{1}{16} \\
-w \ge \frac{1}{4}-\frac{x}{2}-u \\
-w \ge -\frac{1}{4}-\frac{x}{2}+2u.
\end{cases}$$

Solving the system, it turns out that $u=\frac{1}{16}$ (and consequently $X=q^2$) are the optimal values; the maximum k-range is $\left(1,\frac{8}{7}\right)$ and even in this second case we do not get any improvement using Harman technique if we stop on the range of k, as the condition (3.18), is more restrictive than (3.23). However, if we consider the value of $c(k)$ in (3.11) and the condition on η by specifying the parameter u in (3.18) we can notice again an improvement in estimation:

$$\eta = \infty\left(X^{\frac{7k-8}{16k}+\varepsilon}\right)$$

then

$$\psi(k) = \frac{8-7k}{16k} > \frac{8-7k}{18k}.$$

3.5 Further improvements

In this case the mean value of $|S_k(\alpha)|^4$ does not give us any improvement. Nevertheless using the same argument of Chapter 2, the bound on $S_2(\alpha)$ showed in (2.28) and Corollary 2.4, we are able to replace $8/7$ with $7/6$.

The proof follows the same steps of Corollary B.16 but it will use the estimation (2.28). Using such corollary, (3.18) can be improved choosing $u = \frac{1}{14}$ in place of $u = \frac{1}{16}$ and after a linear optimization we get a wider k-range and we can improve the approximation: the maximum k-range became $\left(1, \frac{7}{6}\right)$ and the exponent of η can be replaced with

$$\psi(k) = \frac{7-6k}{14k}.$$

Also in this case, Corollary 2.4 does not affect Lemma 3.3 as the Harman technique on the arc $\mathfrak{m}^*$ can be used with $X = q^{\frac{7}{3}}$ in place of $X = q^2$. Moreover the condition (3.21) assures us that we will not have a gap between the major and the minor arc.

4 Two primes and one k-th power of a prime

4.1 Introduction

The last problem of this dissertation deals with an improvement of the result contained in Languasco & Zaccagnini [30]. Such improvements are contained in [15], due to Languasco, Zaccagnini and the author of this dissertation, and they are shown in this Chapter.

The problem tackled in [30] had $r = 3$, $k_1 = k_2 = 1$, $k_3 = k \in (1, 4/3)$:

$$|F(p_1, p_2, p_3, 1, 1, k) - \omega| = |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega| \leq \eta.$$

Assuming that λ_1/λ_2 is irrational and that the coefficients λ_j are not all of the same sign, Languasco and Zaccagnini proved that one can take $\eta = \big(\mathrm{M}(\mathbf{p}, \mathbf{k})\big)^{-\phi(k)+\varepsilon}$ for any fixed $\varepsilon > 0$, where $\phi(k) = (4 - 3k)/(10k)$. Our purpose in this dissertation is to improve on this result both in the admissible range for k and in the exponent, replacing $\phi(k)$ by a larger value in the common range. More precisely, we prove the following Theorem.

Theorem 4.1. *Assume that $1 < k \leq 3$, λ_1, λ_2 and λ_3 are non-zero real numbers, not all of the same sign, that λ_1/λ_2 is irrational and let ω be a real number. The inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega| \leq \big(\mathrm{M}(\mathbf{p}, \mathbf{k})\big)^{-\psi(k)+\varepsilon} \tag{4.1}$$

has infinitely many solutions in prime variables p_1, p_2, p_3 for any $\varepsilon > 0$, where

$$\psi(k) = \begin{cases} (3-2k)/(6k) & \text{if } 1 < k \leq \frac{6}{5}, \\ 1/12 & \text{if } \frac{6}{5} < k \leq 2, \\ (3-k)/(6k) & \text{if } 2 < k < 3, \\ 1/24 & \text{if } k = 3. \end{cases}$$

We point out that in the common range $1 < k < 4/3$ we have $\psi(k) > \phi(k)$.

We also remark that the strong bounds for the exponential sum S_k, defined in (1.7), that recently became available for integral k (see Bourgain [3] and Bourgain, Demeter & Guth [4]) are not useful in our problem. Our improvement is due to the use of the Harman technique on the minor arc and to the fourth-power average for the exponential sum S_k.

4.2 Outline of the proof

In order to prove that (4.1) has infinitely many solutions, it is sufficient to construct an increasing sequence X_n that tends to infinity such that (4.1) has at least one solution with $\mathrm{M}(\mathbf{p},\mathbf{k}) \in [\delta X_n, X_n]$, with a fixed $\delta > 0$ which depends only on the choice of λ_1, λ_2 and λ_3. Let q be a denominator of a convergent to λ_1/λ_2 and let $X_n = X$ (dropping the suffix n) run through the sequence $X = q^3$. The main quantities we will use are again S_k with T_k and U_k. We will use the bounds of Theorem B.1 and Theorem B.2 as well. We need K_η as a continuous function we will use to detect the solutions of (4.1).

Let now

$$\mathscr{P}(X) = \{(p_1, p_2, p_3) : \delta X < p_1, p_2 \le X,\ \delta X < p_3^k \le X\}$$

and

$$\mathscr{I}(\eta, \omega, \mathfrak{X}) = \int_{\mathfrak{X}} S_1(\lambda_1\alpha) S_1(\lambda_2\alpha) S_k(\lambda_3\alpha) K_\eta(\alpha) e(-\omega\alpha)\, \mathrm{d}\alpha,$$

where $\mathfrak{X}$ is a measurable subset of $\mathbb{R}$. From (1.7) and using the Fourier transform of $K_\eta(\alpha)$, we get

$$\begin{aligned}\mathscr{I}(\eta, \omega, \mathbb{R}) &= \sum_{(p_1,p_2,p_3)\in\mathscr{P}(X)} \log p_1 \log p_2 \log p_3\ \max\big(0; \eta - |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega|\big)\\ &\le \eta (\log X)^3 \mathscr{N}(X),\end{aligned}$$

where $\mathscr{N}(X)$ actually denotes the number of solutions of the inequality (4.1) with $(p_1, p_2, p_3) \in \mathscr{P}(X)$. In other words $\mathscr{I}(\eta, \omega, \mathbb{R})$ provides a lower bound for the quantity we are interested in; therefore it is sufficient to prove that $\mathscr{I}(\eta, \omega, \mathbb{R}) > 0$.

In this problem we need to introduce the intermediate arc: we now decompose $\mathbb{R}$ into subsets such that $\mathbb{R} = \mathfrak{M} \cup \mathfrak{M}^* \cup \mathfrak{m} \cup \mathfrak{t}$ where $\mathfrak{M}$ is the major arc, $\mathfrak{M}^*$ is the intermediate arc (which is non-empty only for some values of k, see section 4.6), $\mathfrak{m}$ is the minor arc and $\mathfrak{t}$ is the trivial arc. The decomposition is the following: if $1 < k < 5/2$ we consider

$$\begin{aligned}&\mathfrak{M} = [-P/X, P/X], && \mathfrak{M}^* = \emptyset,\\ &\mathfrak{m} = [P/X, R] \cup [-R, -P/X], && \mathfrak{t} = \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{M}^* \cup \mathfrak{m}),\end{aligned} \tag{4.2}$$

while, for $5/2 \le k \le 3$, we set

$$\mathfrak{M} = [-P/X, P/X], \qquad \mathfrak{M}^* = [P/X, X^{-3/5}] \cup [-X^{-3/5}, -P/X],$$
$$\mathfrak{m} = [X^{-3/5}, R] \cup [-R, -X^{-3/5}], \qquad \mathfrak{t} = \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{M}^* \cup \mathfrak{m}), \tag{4.3}$$

where the parameters $P = P(X) > 1$ and $R = R(X) > 1/\eta$ are chosen later (see (4.4) and (4.5)) as well as $\eta = \eta(X)$, that, as we explained before, we will take as a small negative power of $\mathrm{M}(\mathbf{p}, \mathbf{k})$ (and so of X). We have to distinguish two cases in the previous decomposition of the real line because eq. (4.12) implies that we are able to apply Harman technique only if $|\alpha| \gg X^{-2/3}$, then we are forced to choose as minor arc, an interval that may leave a gap the major arc for some values of k, see also section 4.6. We remark that, after the choice in (4.4), the inequality $P/X < X^{-3/5}$ holds for $k > 25/12$. However, as we will see later in section 4.6, we need to introduce the intermediate arc only for $k > 5/2$.

The constraints on η are in (4.7), (4.9) and (4.10), according to the value of k. In any case, we have $\mathscr{I}(\eta, \omega, \mathbb{R}) = \mathscr{I}(\eta, \omega, \mathfrak{M}) + \mathscr{I}(\eta, \omega, \mathfrak{M}^*) + \mathscr{I}(\eta, \omega, \mathfrak{m}) + \mathscr{I}(\eta, \omega, \mathfrak{t})$. We expect that $\mathfrak{M}$ provides the main term with the right order of magnitude without any special hypothesis on the coefficients λ_j. It is necessary to prove that $\mathscr{I}(\eta, \omega, \mathfrak{M}^*)$, $\mathscr{I}(\eta, \omega, \mathfrak{m})$ and $\mathscr{I}(\eta, \omega, t)$ are $o\big(\mathscr{I}(\eta, \omega, \mathfrak{M})\big)$.

4.3 The Major arc

We recall the definitions in (4.2) and (4.3). The major arc computation is the same as in [30]:

$$\begin{aligned}
\mathscr{I}(\eta, \omega, \mathfrak{M}) &= \int_{\mathfrak{M}} S_1(\lambda_1 \alpha) S_1(\lambda_2 \alpha) S_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega\alpha)\, \mathrm{d}\alpha \\
&= \int_{\mathfrak{M}} T_1(\lambda_1 \alpha) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega\alpha)\, \mathrm{d}\alpha \\
&\qquad + \int_{\mathfrak{M}} (S_1(\lambda_1 \alpha) - T_1(\lambda_1 \alpha)) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega\alpha)\, \mathrm{d}\alpha \\
&\qquad + \int_{\mathfrak{M}} S_1(\lambda_1 \alpha)(S_1(\lambda_2 \alpha) - T_1(\lambda_2 \alpha)) T_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega\alpha)\, \mathrm{d}\alpha \\
&\qquad + \int_{\mathfrak{M}} S_1(\lambda_1 \alpha) S_1(\lambda_2 \alpha)(S_k(\lambda_3 \alpha) - T_k(\lambda_3 \alpha)) K_\eta(\alpha) e(-\omega\alpha)\, \mathrm{d}\alpha \\
&= J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

say.

Main Term: lower bound for J_1

As the reader might expect the main term is given by the summand J_1. Let

$$H(\alpha) = T_1(\lambda_1 \alpha) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega\alpha)$$

so that

$$J_1 = \int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha + \mathcal{O}\Big(\int_{P/X}^{+\infty} |H(\alpha)|\,\mathrm{d}\alpha\Big).$$

Using (1.5) and Theorem B.1, we get

$$\int_{P/X}^{+\infty} |H(\alpha)|\,\mathrm{d}\alpha \ll \eta^2 X^{1/k-1} \int_{P/X}^{+\infty} \frac{\mathrm{d}\alpha}{\alpha^3} \ll \eta^2 \frac{X^{1+1/k}}{P^2} = o\big(\eta^2 X^{1+1/k}\big),$$

provided that $P \to +\infty$. Let now $D = [\delta X, X]^2 \times [(\delta X)^{1/k}, X^{1/k}]$. We obtain

$$\begin{aligned}\int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha &= \iiint_D \int_{\mathbb{R}} e((\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \omega)\alpha) K_\eta(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \\ &= \iiint_D \max(0; \eta - |\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \omega)|)\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3.\end{aligned}$$

Apart from trivial changes of sign, there are essentially two cases:

1. $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$

2. $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$.

After a suitable change of variables, letting $D' = [\delta X, (1-\delta)X]^3$,

$$\begin{aligned}\int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha &\gg \iiint_{D'} \max(0; \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)|)\, u_3^{1/k-1}\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 \\ &\gg X^{1/k-1} \iiint_{D'} \max(0; \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)|)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3.\end{aligned}$$

We deal with the first one. We warn the reader that here it may be necessary to adjust the value of δ in order to guarantee the necessary set inclusions. Apart from sign, the computation is essentially symmetrical with respect to the coefficients λ_j: we assume, as we may, that $|\lambda_3| \geq \max(\lambda_1, \lambda_2)$, the other cases being similar. Now, for $j = 1,2$ let $a_j = \dfrac{2\delta|\lambda_3|}{|\lambda_j|}$, $b_j = \dfrac{3}{2}a_j$ and $\mathscr{D}_j = [a_j X, b_j X]$; if $u_j \in \mathscr{D}_j$ for $j = 1,2$ then

$$\lambda_1 u_1 + \lambda_2 u_2 \in [4|\lambda_3|\delta X, 6|\lambda_3|\delta X]$$

so that, for every choice of (u_1, u_2) the interval $[a,b]$ with endpoints $\pm\eta/|\lambda_3| + (\lambda_1 u_1 + \lambda_2 u_2)/|\lambda_3|$ is contained in $[\delta X, (1-\delta)X]$. In other words, for $u_3 \in [a,b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ cover the whole interval $[-\eta, \eta]$. Hence for any $(u_1, u_2) \in \mathscr{D}_1 \times \mathscr{D}_2$ we have

$$\int_{\delta X}^{(1-\delta)X} \max(0; \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|)\,\mathrm{d}u_3 = |\lambda_3|^{-1} \int_{-\eta}^{\eta} \max(0; \eta - |u|)\,\mathrm{d}u \gg \eta^2.$$

Summing up we get

$$J_1 \gg \eta^2 X^{1/k-1} \iint_{\mathscr{D}_1 \times \mathscr{D}_2} \mathrm{d}u_1\mathrm{d}u_2 \gg \eta^2 X^{1/k-1} X^2 = \eta^2 X^{1+1/k},$$

which is the expected lower bound.

Bound for J_2, J_3 and J_4

The computations for J_2 and J_3 are similar to and simpler than the corresponding one for J_4; moreover the most restrictive condition on P arises from J_4; hence we will skip the computation for both J_2 and J_3. Using the triangle inequality and (1.5),

$$\begin{aligned} J_4 &\ll \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha) - T_k(\lambda_3\alpha)|\,\mathrm{d}\alpha \\ &\le \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha) - U_k(\lambda_3\alpha)|\,\mathrm{d}\alpha \\ &\qquad + \eta^2 \int_{\mathfrak{M}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||U_k(\lambda_3\alpha) - T_k(\lambda_3\alpha)|\,\mathrm{d}\alpha \\ &= \eta^2(A_4 + B_4), \end{aligned}$$

say, where $U_k(\lambda_3\alpha)$ and $T_k(\lambda_3\alpha)$ are defined in (1.7). Using the Cauchy-Schwarz inequality, Lemmas B.11-B.12 and trivial bounds yields, for any fixed $A > 0$,

$$\begin{aligned} A_4 &\ll X\Big(\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{\mathfrak{M}} |S_k(\lambda_3\alpha) - U_k(\lambda_3\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \\ &\ll X^{1+1/k}(\log X)^{(1-A)/2} = o\big(X^{1+1/k}\big) \end{aligned}$$

as long as $A > 1$, provided that $P \le X^{5/(6k)-\varepsilon}$. Using again the Cauchy-Schwarz inequality, Theorem B.2 and trivial bounds, we see that

$$\begin{aligned} B_4 &\ll \int_0^{1/X} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)|\,\mathrm{d}\alpha + X\int_{1/X}^{P/X} \alpha|S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)|\,\mathrm{d}\alpha \\ &\ll X + P\Big(\int_{1/X}^{P/X} |S_1(\lambda_1\alpha)|^2\,\mathrm{d}\alpha \int_{1/X}^{P/X} |S_1(\lambda_2\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \ll PX\log X. \end{aligned}$$

Taking $P = o\big(X^{1/k}(\log X)^{-1}\big)$ we get $\eta^2 B_4 = o\big(\eta^2 X^{1+1/k}\big)$. We may therefore choose

$$P = X^{5/(6k)-\varepsilon}. \tag{4.4}$$

4.4 The trivial arc

We recall that the trivial arc is defined in (4.2) and (4.3). Using the Cauchy-Schwarz inequality and the Theorem B.1, we see that

$$\begin{aligned} |\mathscr{I}(\eta,\omega,\mathrm{t})| &\ll \int_R^{+\infty} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)S_k(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\ &\ll X^{1/k}\Big(\int_R^{+\infty} |S_1(\lambda_1\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_R^{+\infty} |S_1(\lambda_2\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\ &\ll X^{1/k} C_1^{1/2} C_2^{1/2}, \end{aligned}$$

say. Using the PNT and the periodicity of $S_1(\alpha)$, for every $j = 1,2$ we have that

$$C_j = \int_R^{+\infty} |S_1(\lambda_j\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^2} \ll \int_{|\lambda_j|R}^{+\infty} |S_1(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^2} \ll \sum_{n \geq |\lambda_j|R} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_1(\alpha)|^2 \, \mathrm{d}\alpha \ll \frac{X \log X}{|\lambda_j|R}.$$

Hence, recalling that $|\mathscr{I}(\eta,\omega,\mathfrak{t})|$ has to be $o(\eta^2 X^{1+1/k})$, the choice

$$R = \eta^{-2}(\log X)^{3/2} \tag{4.5}$$

is admissible.

4.5 The minor arc

Here we use Harman's technique as described in [21]. The minor arc is defined in (4.2) and (4.3), according to the value of k. Using Lemma B.15 we now split $\mathfrak{m}$ into subsets $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}^* = \mathfrak{m} \setminus (\mathfrak{m}_1 \cup \mathfrak{m}_2)$ where

$$\mathfrak{m}_i = \{\alpha \in \mathfrak{m} \colon |S_1(\lambda_i \alpha)| \leq X^{5/6+\varepsilon}\} \quad \text{for } i = 1,2.$$

In order to obtain the optimization, we chose to split the range for k into two intervals in which to take advantage of the L^2-norm of $S_k(\alpha)$ in one case (Lemma B.7) and the L^4-norm of $S_k(\alpha)$ in the other one (Lemma B.10). The same choice will be made later in the discussion of the arc $\mathfrak{m}^*$. We will see that it is not possible to split the minor arc in another way in order to get a better result.

Bounds on $\mathfrak{m}_1 \cup \mathfrak{m}_2$

Using Hölder's inequality and Lemma B.6, for $1 < k \leq 6/5$ we obtain

$$\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathfrak{m}_i)| &\ll \int_{\mathfrak{m}_i} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\max_{\alpha\in\mathfrak{m}_i} |S_1(\lambda_1\alpha)|\Big)\Big(\int_{\mathfrak{m}_i} |S_1(\lambda_2\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\
&\quad \times \Big(\int_{\mathfrak{m}_i} |S_k(\lambda_3\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll X^{5/6+\varepsilon}(\eta X \log X)^{1/2}(\eta X^{1/k}(\log X)^3)^{1/2} \\
&\ll \eta X^{4/3+1/(2k)+\varepsilon}.
\end{aligned} \tag{4.6}$$

The estimate in (4.6) should be $o(\eta^2 X^{1+1/k})$; hence this leads to the constraint

$$\eta = \infty(X^{1/3-1/(2k)+\varepsilon}). \tag{4.7}$$

Using Hölder's inequality and Lemmas B.3 and B.10, for $6/5 < k < 3$ we obtain

$$|\mathscr{I}(\eta,\omega,\mathfrak{m}_i)| \ll \int_{\mathfrak{m}_i} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha$$

$$
\begin{aligned}
&\ll \Big(\max_{\alpha\in\mathfrak{m}_i}|S_1(\lambda_1\alpha)|^{1/2}\Big)\Big(\int_{\mathfrak{m}_i}|S_1(\lambda_1\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/4}\\
&\qquad\times\Big(\int_{\mathfrak{m}_i}|S_k(\lambda_3\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/4}\Big(\int_{\mathfrak{m}_i}|S_1(\lambda_2\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2}\\
&\ll X^{5/12+\varepsilon}(\eta X\log X)^{1/4}(\eta\max(X^{2/k};X^{4/k-1}))^{1/4}(\eta X\log X)^{1/2}\\
&\ll \eta\max\big(X^{7/6+1/(2k)};X^{11/12+1/k}\big)X^{\varepsilon}. \qquad (4.8)
\end{aligned}
$$

The estimate in (4.8) should be $o(\eta^2X^{1+1/k})$; hence this leads to

$$\eta=\infty\big(\max(X^{1/6-1/(2k)+\varepsilon};X^{-1/12+\varepsilon})\big). \qquad (4.9)$$

If $k=3$ we use Lemmas B.3 and Lemma B.5 with $\ell=3$, thus getting

$$
\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathfrak{m}_i)| &\ll \int_{\mathfrak{m}_i}|S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_3(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha\\
&\ll \Big(\max_{\alpha\in\mathfrak{m}_i}|S_1(\lambda_1\alpha)|^{1/4}\Big)\Big(\int_{\mathfrak{m}_i}|S_1(\lambda_1\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{3/8}\\
&\qquad\times\Big(\int_{\mathfrak{m}_i}|S_3(\lambda_3\alpha)|^8K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/8}\Big(\int_{\mathfrak{m}_i}|S_1(\lambda_2\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2}\\
&\ll \eta X^{31/24+\varepsilon}.
\end{aligned}
$$

This bound leads to the constraint

$$\eta=\infty\big(X^{-1/24+\varepsilon}\big), \qquad (4.10)$$

which justifies the last line of (1.9)

Bound on $\mathfrak{m}^*$

It remains to discuss the set $\mathfrak{m}^*$ where the following bounds hold simultaneously

$$|S_1(\lambda_1\alpha)|>X^{5/6+\varepsilon},\quad |S_1(\lambda_2\alpha)|>X^{5/6+\varepsilon},\quad T<|\alpha|\le\eta^{-2}(\log X)^{3/2}=R.$$

where $T=P/X=X^{5/(6k)-1-\varepsilon}$ by our choice in (4.4) if $k<5/2$, and $T=X^{-3/5}$ otherwise. Using a dyadic dissection, we split $\mathfrak{m}^*$ into disjoint sets $E(Z_1,Z_2,y)$ like in the previous Chapters in which, for $\alpha\in E(Z_1,Z_2,y)$, we have

$$Z_i<|S_1(\lambda_i\alpha)|\le 2Z_i,\qquad y<|\alpha|\le 2y,$$

where $Z_i=2^{k_i}X^{5/6+\varepsilon}$ and $y=2^{k_3}X^{5/(6k)-1-\varepsilon}$ for some non-negative integers k_1,k_2,k_3. The range of α is given by (4.4) and (4.5).

It follows that the number of disjoint sets is, at most, $\ll(\log X)^3$. Let us write $\mathscr{A}$ as a shorthand for the set $E(Z_1,Z_2,y)$. We have the following result about the Lebesgue measure of $\mathscr{A}$.

In the following Lemma, it is crucial that both the integers a_1 and a_2 appearing in (4.11) below do not vanish: in fact, if $a_1 = 0$, say, then $q_1 = 1$ and $|\alpha|$ is so small that it can not belong to $\mathfrak{m}^*$. In order to use the Harman technique then, we are forced to be far from the major arc in which we would have $a_1a_2 = 0$. Notice that $a_1 = 0$ or $a_2 = 0$ implies that $|\alpha| \ll X^{-2/3}$ from (4.11), which means that, on the minor arc, we are working on $|\alpha| \gg X^{-2/3}$ as showed in (4.12). In some cases, for high values of k, there would be a gap between the major arc and the minor arc, because P/X becomes smaller as k grows and it can become smaller than $X^{-2/3}$; this is the reason why we introduce the intermediate arc, when k exceeds a certain threshold, in Section 4.6. In any case, we will show that we are able to control the contribution of such intermediate arc.

Lemma 4.2. *We have that $\mu(\mathscr{A}) \ll yX^{8/3+3\varepsilon}Z_1^{-2}Z_2^{-2}$, where $\mu(\cdot)$ denotes the Lebesgue measure.*

Proof. If $\alpha \in \mathscr{A}$, by Lemma B.15 there are coprime integers (a_1, q_1) and (a_2, q_2) such that

$$1 \le q_i \ll \left(\frac{X^{1+\varepsilon/2}}{Z_i}\right)^2, \qquad |q_i\lambda_i\alpha - a_i| \ll \left(\frac{X^{1/2+\varepsilon/2}}{Z_i}\right)^2. \tag{4.11}$$

We remark that $a_1a_2 \neq 0$ otherwise we would have $\alpha \in \mathfrak{M}$. In fact, if $a_1 = 0$ or $a_2 = 0$, recalling the definitions of Z_i and (4.11) we get

$$|\alpha| \ll q_i^{-1}\left(\frac{X^{1/2+\varepsilon/2}}{Z_i}\right)^2 \ll X^{-2/3}. \tag{4.12}$$

Now, we can further split $\mathfrak{m}^*$ into sets $I = I(Z_1, Z_2, y, Q_1, Q_2)$ where, on each set, $Q_j \le q_j \le 2Q_j$. Note that a_i and q_i are uniquely determined by α. In the opposite direction, for a given quadruple a_1, q_1, a_2, q_2 the inequalities (4.11) define an interval of α of length

$$\begin{aligned}\mu(I) \ll& \min\left(Q_1\left(\frac{X^{1/2+\varepsilon/2}}{Z_1}\right)^2, Q_2\left(\frac{X^{1/2+\varepsilon/2}}{Z_2}\right)^2\right)\\ \ll& Q_1^{-1/2}Q_2^{-1/2}\left(\frac{X^{1/2+\varepsilon/2}}{Z_1}\right)\left(\frac{X^{1/2+\varepsilon/2}}{Z_2}\right)\\ \ll& \frac{X^{1+\varepsilon}}{Q_1^{1/2}Q_2^{1/2}Z_1Z_2},\end{aligned}$$

by taking the geometric mean.

Now we need a lower bound for $Q_1^{1/2}Q_2^{1/2}$: by (4.11) we obtain

$$\begin{aligned}\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| &= \left|\frac{a_2}{\lambda_2\alpha}(q_1\lambda_1\alpha - a_1) - \frac{a_1}{\lambda_2\alpha}(q_2\lambda_2\alpha - a_2)\right|\\ &\ll q_2|q_1\lambda_1\alpha - a_1| + q_1|q_2\lambda_2\alpha - a_2|\\ &\ll Q_2\left(\frac{X^{1/2+\varepsilon/2}}{Z_1}\right)^2 + Q_1\left(\frac{X^{1/2+\varepsilon/2}}{Z_2}\right)^2.\end{aligned}$$

Recalling that $Q_i \ll (X^{1+\varepsilon/2}/Z_i)^2$, $Z_i \gg X^{5/6+\varepsilon}$,

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \ll \left(\frac{X^{1+\varepsilon/2}}{X^{5/6+\varepsilon}}\right)^2\left(\frac{X^{1/2+\varepsilon/2}}{X^{5/6+\varepsilon}}\right)^2 \ll X^{-1/3-2\varepsilon} < \frac{1}{4q}. \tag{4.13}$$

We recall that $q = X^{1/3}$ is a denominator of a convergent of λ_1/λ_2. Hence by (4.13) Legendre's law of best approximation (see Appendix A.4) implies that $|a_2 q_1| \geq q$ and by the same token, for any pair α, α' having distinct associated products $a_2 q_1$ (see [49], Lemma 2),

$$|a_2(\alpha)q_1(\alpha) - a_2(\alpha')q_1(\alpha')| \geq q;$$

thus, by the pigeon-hole principle, there is at most one value of $a_2 q_1$ in the interval $[rq, (r+1)q)$ for any positive integer r. Hence $a_2 q_1$ determines a_2 and q_1 to within X^{ε} possibilities (see the upper bound for the divisor function in Appendix, Theorem A.8) and consequently also $a_2 q_1$ determines a_1 and q_2 to within X^{ε} possibilities from (4.13).

Hence we got a lower bound for $q_1 q_2$, since, using $Q_j \leq q_j \leq 2Q_j$, we get

$$q_1 q_2 = a_2 q_1 \frac{q_2}{a_2} \gg \frac{rq}{|\alpha|} \gg rqy^{-1}$$

for the quadruple under consideration.

As a consequence we obtain that the total length of the interval $I(Z_1, Z_2, y, Q_1, Q_2)$, with $a_2 q_1 \in [rq, (r+1)q)$ does not exceed

$$\mu(I) \ll X^{\varepsilon} X^{1+\varepsilon} Z_1^{-1} Z_2^{-1} r^{-1/2} q^{-1/2} y^{1/2}.$$

Now we need a bound for r: since $a_2 q_1 \in [rq, (r+1), q)$, we have

$$rq \leq |a_2 q_1| \ll q_1 q_2 |\alpha| \ll y \left(\frac{X^{1+\varepsilon/2}}{Z_1} \right)^2 \left(\frac{X^{1+\varepsilon/2}}{Z_2} \right)^2 \ll yX^{4+2\varepsilon} Z_1^{-2} Z_2^{-2}$$

and hence we get

$$r \ll q^{-1} yX^{4+2\varepsilon} Z_1^{-2} Z_2^{-2}.$$

Next, we sum on every interval to get an upper bound for the measure of $\mathscr{A}$: we get

$$\mu(\mathscr{A}) \ll X^{1+2\varepsilon} Z_1^{-1} Z_2^{-1} q^{-1/2} y^{1/2} \sum_{1 \leq r \ll q^{-1} yX^{4+2\varepsilon} Z_1^{-2} Z_2^{-2}} r^{-1/2}.$$

By standard estimation we obtain

$$\sum_{1 \leq r \ll q^{-1} yX^{4+2\varepsilon} Z_1^{-2} Z_2^{-2}} r^{-1/2} \ll (q^{-1} yX^{4+2\varepsilon} Z_1^{-2} Z_2^{-2})^{1/2}$$

and hence we can finally write

$$\mu(\mathscr{A}) \ll yX^{3+3\varepsilon} Z_1^{-2} Z_2^{-2} q^{-1} \ll yX^{3+3\varepsilon} Z_1^{-2} Z_2^{-2} X^{-1/3} \ll yX^{8/3+3\varepsilon} Z_1^{-2} Z_2^{-2}.$$

This proves the Lemma. $\square$

4.6 The intermediate arc: $5/2 \le k \le 3$

In section 4.5 we apply Harman's technique to the minor arc, using Lemma B.15 as the starting point. We remark that in the course of the proof of Lemma 4.2 it is crucial that both the integers a_1 and a_2 appearing in (4.11) below do not vanish; in fact, if $a_1 = 0$, say, then α is very small ($\alpha \ll X^{-2/3}$) and, according to our definitions above, it belongs to $\mathfrak{M} \cup \mathfrak{M}^*$.

For small k we do not need an intermediate arc, because the major arc is wide enough to rule out the possibility that $a_1 a_2 = 0$ for $\alpha \in \mathfrak{m}$. For larger values of k, the constraint (4.4) implies that there is a gap between the major arc and the minor arc which we need to fill: see the definition in (4.3). Using the intermediate arc $\mathfrak{M}^*$, we are able to cover more than needed.

Lemma 4.3. *Let* $X^{-1} \ll \alpha \ll X^{-3/5}$. *Then* $|S_1(\alpha)| \ll X^{1/2}|\alpha|^{-1/2}(\log X)^4$.

Proof. It follows immediately from Lemma B.13 by choosing $q = \lfloor 1/\alpha \rfloor$ and $a = 1$. □

Using (1.5), Lemma 4.3, the Cauchy-Schwarz inequality and (4.4) we get

$$\begin{aligned}
\mathscr{I}(\eta,\omega,\mathfrak{M}^*) &\ll \eta^2 \int_{P/X}^{X^{-3/5}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)|\,\mathrm{d}\alpha \\
&\ll \eta^2 X(\log X)^8 \int_{P/X}^{X^{-3/5}} |S_k(\lambda_3\alpha)| \frac{\mathrm{d}\alpha}{\alpha} \\
&\ll \eta^2 X(\log X)^8 \Big(\int_{-X^{-3/5}}^{X^{-3/5}} |S_k(\lambda_3\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \Big(\int_{P/X}^{X^{-3/5}} \frac{\mathrm{d}\alpha}{\alpha^2}\Big)^{1/2} \\
&\ll \eta^2 X (X^{1/k-3/5})^{1/2} (X^{1-5/(6k)})^{1/2} X^{\varepsilon} \ll \eta^2 X^{6/5+1/(12k)+\varepsilon},
\end{aligned}$$

where we also used Lemma B.6 with $\tau = X^{-3/5}$. The last estimate is $o\big(\eta^2 X^{1+1/k}\big)$ for every $5/2 \le k < 55/12$.

4.7 Conclusion

Here we finally justify the choice of the function ψ in the statement of the main Theorem. Using Lemmas B.6-B.10-4.2 we are now able to estimate $\mathscr{I}(\eta,\omega,\mathscr{A})$ for $1 < k \le 3$.

If $1 < k \le 6/5$ we proceed as follows:

$$\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathscr{A})| &\ll \int_{\mathscr{A}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)| K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\int_{\mathscr{A}} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \Big(\int_{\mathscr{A}} |S_k(\lambda_3\alpha)|^2 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll \big(\min(\eta^2;y^{-2})\big)^{1/2} \big((Z_1Z_2)^2\mu(\mathscr{A})\big)^{1/2} \big(\eta X^{1/k+\varepsilon}\big)^{1/2} \\
&\ll \big(\min(\eta^2;y^{-2})\big)^{1/2} Z_1Z_2 \big(yX^{8/3}Z_1^{-2}Z_2^{-2}\big)^{1/2} \eta^{1/2} X^{1/(2k)+\varepsilon}
\end{aligned}$$

$$\ll \eta X^{4/3+1/(2k)+\varepsilon}.$$

Hence we need $\eta = \infty\big(X^{1/3-1/(2k)+\varepsilon}\big)$, which is the same condition we got in (4.7).

If $6/5 < k < 3$,

$$\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathscr{A})| &\ll \int_{\mathscr{A}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_k(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\int_{\mathscr{A}} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)|^{4/3}K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{3/4}\Big(\int_{\mathscr{A}} |S_k(\lambda_3\alpha)|^4 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/4} \\
&\ll \big(\min\big(\eta^2;y^{-2}\big)\big)^{3/4}\big((Z_1Z_2)^{4/3}\mu(\mathscr{A})\big)^{3/4}\big(\eta\max(X^{2/k};X^{4/k-1})X^{\varepsilon}\big)^{1/4} \\
&\ll \big(\min\big(\eta^2;y^{-2}\big)\big)^{3/4}Z_1Z_2\big(yX^{8/3}Z_1^{-2}Z_2^{-2}\big)^{3/4}\eta^{1/4}\max(X^{1/(2k)};X^{1/k-1/4})X^{\varepsilon} \\
&\ll \eta Z_1^{-1/2}Z_2^{-1/2}X^{2+\varepsilon}\max(X^{1/(2k)};X^{1/k-1/4}) \\
&\ll \eta\max(X^{7/6+1/(2k)};X^{11/12+1/k})X^{\varepsilon}.
\end{aligned}$$

Hence we need $\eta = \infty\big(\max(X^{1/6-1/(2k)+\varepsilon};X^{-1/12+\varepsilon})\big)$, which is the same condition we got in (4.9). If $k=3$, using Lemmas B.5 with $\ell = 3$ and 4.2 we obtain

$$\begin{aligned}
|\mathscr{I}(\eta,\omega,\mathscr{A})| &\ll \int_{\mathscr{A}} |S_1(\lambda_1\alpha)||S_1(\lambda_2\alpha)||S_3(\lambda_3\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \Big(\int_{\mathscr{A}} |S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)|^{8/7}K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{7/8}\Big(\int_{\mathscr{A}} |S_3(\lambda_3\alpha)|^8 K_\eta(\alpha)\,\mathrm{d}\alpha\Big)^{1/8} \\
&\ll \eta Z_1^{-3/4}Z_2^{-3/4}X^{7/3+5/24+\varepsilon} \ll \eta X^{31/24+\varepsilon}.
\end{aligned}$$

This leads to the same constraint for η that we had in (4.10).

A Elementary results

A.1 Continued fractions

This thesis needs some important tools coming from continued fractions theory, in particular it is closely linked to the Legendre law of best approximation: therefore we will give some basic details of the theory of continued fractions.

A finite continued fraction is a representation of the $n+1$ variables $a_0, a_1, \ldots, a_n$

$$f(a_0, a_1, \ldots, a_n) := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\cdots}{\cdots + \frac{1}{a_n}}}}.$$

There is a simpler form to represent a continued fraction using square brackets:

$$f(a_0, a_1, \ldots, a_n) = [a_0, a_1, \ldots, a_n].$$

A *finite regular continued fraction* $[a_0, a_1, \ldots, a_n]$ is a repeated quotient with integers a_i satisfying $a_i > 0$ and $a_n > 1$. Every rational number has got a unique representation with a regular finite continued fraction (if we do not assume $a_n > 1$, the representation of a rational number by continued fractions is actually not unique (see [19], section 10.5) but if two simple continued fractions have the same value $x = [a_0, \ldots, a_n] = [b_0, \ldots, b_m]$ with $m = n$ and $a_n > 1$, $b_m > 1$ then the fractions are identical.

If we take only a truncated representation we will get a *convergent*, $\frac{p_m}{q_m}$ to the continued fraction:

$$\begin{aligned}
&[a_0] = a_0 \qquad [a_0, a_1] = \frac{a_1 a_0 + 1}{a_1} = a_0 + \frac{1}{[a_1]} \\
&[a_0, a_1, a_2] = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1} = a_0 + \frac{1}{[a_1, a_2]} \\
&[a_0, a_1, \ldots, a_m] = a_0 + \frac{1}{[a_1, a_2, \ldots, a_m]} = \frac{p_m}{q_m} \quad \forall m \leq n
\end{aligned}$$

Two continued fractions are identical if they are formed by the same sequence of partial quotients.

Theorem A.1 (see [19])**.** *If $x = [a_0, \dots, a_n]$ is a rational number, $n > 1$ and $m > 0$ then the difference*

$$\left| x - \frac{p_m}{q_m} \right| \leq \frac{1}{q_m q_{m+1}} < \frac{1}{q_m^2} \qquad \forall m \leq n-1$$

where $x_m = \frac{p_m}{q_m} = [a_0, a_1, \dots, a_m]$ is called a convergent of x for $m < n-1$.

The main interest of continued fractions lies in the representation of irrational numbers, so in this case we need to talk about *infinite* continued fractions. The most important feature we will use is the difference between the irrational number and its convergents. It is proven that every infinite simple continued fraction is convergent and every irrational x can be represented by an infinite continued fraction $x = [a_0, a_1, a_2, \dots]$. Also in this case two simple infinite continued fractions which have the same value are identical and every irrational can be expressed just in one way using simple infinite continued fractions.

The results of the previous theorem hold also for infinite continued fractions and in this case we can talk about *best approximation* with a rational number: $\frac{p_m}{q_m}$ is the fraction, among all fractions whose denominator does not exceed q_m, that provides the best approximation for the irrational number $x = [a_0, a_1, \dots]$. Legendre's law of best approximation states that the convergents to an irrational number give a sequence of best approximations. In other words,

Theorem A.2. *If $x \notin \mathbb{Q}$, $m > 1$, $0 < q \leq q_m$ and $\frac{p}{q} \neq \frac{p_m}{q_m}$,*

$$\left| \frac{p_m}{q_m} - x \right| < \left| \frac{p}{q} - x \right|$$

or, equivalently,

$$|p_m - q_m x| < |p - qx|.$$

If x is an irrational number there is an infinity of fractions satisfying

$$\left| \frac{p}{q} - x \right| < \frac{1}{q^2}. \tag{A.1}$$

Dirichlet proved a more general statement:

Theorem A.3 (Dirichlet)**.** *For any real number x there exists integers $Q \geq 1$, a and q with $(a, q) = 1$ and $1 \leq q \leq Q$, such that*

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{q(Q+1)}.$$

It is also a fact that the *simplest* numbers in term of continued fractions like the golden ratio $\phi = [1,1,1,\ldots]$ are the worst in terms of approximation: in other words, it is not possible to improve the approximation with a power of q greater than 2 in (A.1):

Theorem A.4. *Any irrational x has an infinity of approximations which satisfy*

$$\left|\frac{p}{q} - x\right| < \frac{1}{Aq^2}$$

where $A = \sqrt{5}$. If we take $A > \sqrt{5}$ the previous inequalities may have only a finite number of solutions.

In particular for all algebraic numbers the best approximations is actually $\frac{1}{q^2}$:

Theorem A.5 (Roth)**.** *Let x be a real algebraic number. For any $\varepsilon > 0$ and $(p,q) = 1$, the inequality*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

has only finitely many solutions in p and q.

This last theorem will give us a natural limit in the choice of the exponent of the denominator of the continued fraction.

Dirichlet Theorem A.3 and the law of best approximation A.4 are both used in the study of the minor arc in all three cases discussed in this dissertation. For instance, in Chapter 2, the Dirichet theorem is used in Lemma 2.2 while the law best approximation is used in both Lemma 2.2 and Lemma 2.3 in which Harman technique is applied.

A.2 Elementary tools

Abel's partial summation formula

Theorem A.6 (Abel summation formula)**.** *Let a_n be a sequence of real or complex numbers and $\phi : \mathbb{R}^+ \to \mathbb{C}$ a $\mathscr{C}^1$ function. If we define*

$$A(x) := \sum_{n \le x} a_n,$$

then

$$\sum_{n \le x} a_n \phi(n) = A(x)\phi(x) - \int_1^x A(t)\phi'(t)\mathrm{d}t.$$

Euler summation formula

Theorem A.7 (Euler summation formula)**.** *Let $f:(x,y]\to\mathbb{C}$ be a derivable function.*

$$\sum_{x<n\leq y} f(n) = \int_x^y f(t)\mathrm{d}t + \int_x^y \{t\}f'(t)\mathrm{d}t - \{y\}f(y) + \{x\}f(x)$$

where $\{x\}$ denotes the fractional part of a real number x.

See Apostol [1] Theorem 3.1 for the proof.

Bound on the divisor function

Theorem A.8. *Let $d(n)$ denote the number of divisor of a given positive integer n:*

$$d(n) := \sum_{d|n} 1 = |\{d\in\mathbb{N}^* : d\,|\,n\}|\,.$$

Then for large n, $d(n) \ll n^{\varepsilon}$, or, more precisely,

$$d(n) \ll \exp\left(\frac{\log n}{\log\log n}\right).$$

See Apostol [1] Theorem 13.12 for the proof.

In this section we have presented some elementary results of analytic number theory that can be found on classical literature such as Hardy-Wright [19] and Apostol [1]

B Exponential sums

Exponential sums are objects defined in the following way

$$S(\chi, f) := \sum_{x \in \chi} \rho(x) e(f(x)),$$

where χ is an arbitrary set, f is a function on χ, $e(\alpha) = \exp(2\pi i \alpha)$ and $\rho(x)$ is a suitable weight.

In analytic number theory we are dealing with infinite sets like $\mathbb{N}$ or the set of the prime numbers that we can denote with the letter $\mathbb{P}$, and it may be useful to deal with truncated exponential sums:

$$S(N, f) := \sum_{x \leq N} \rho(x) e(f(x)).$$

The definition of $U_k(\alpha)$ in (1.7) is a truncated exponential sum.

Sometimes it can be useful to treat weighted exponential sums when we are summing over prime numbers:

$$S(X, \alpha) := \sum_{p \leq X} \log p \, e(p\alpha)$$

$$S'(X, \alpha) := \sum_{n \leq X} \Lambda(n) \, e(n\alpha).$$

where

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

is the von Mangoldt function. The first definition is used in the circle method and in a more generalized version in (1.6), more suitable for our purpose (dropping the argument X):

$$S_k(\alpha) = \sum_{\delta X \leq p^k \leq X} \log p \; e(p^k \alpha),$$

where $k \geq 1$ is a real parameter and δ is a small positive constant. We always use this last definition in which the choice of starting from δX instead of 1 or 2 is needed for technical reasons but it does not alter the final result.

It is possible to use the weighted sum with both the logarithm or the von Mangoldt function as weights; in the latter case the weights are exactly the coefficients of the logarithmic derivative of the Riemann ζ function. It is well known that weighted exponential sums are extremely useful in analytic number theory.

The weighted exponential sum over the prime numbers can be approximated with a sum without weights or with the exponential integral:

$$U_k(\alpha) = \sum_{\delta X \le n^k \le X} e(n^k \alpha)$$
$$T_k(\alpha) = \int_{(\delta X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\alpha t^k)\, \mathrm{d}t.$$

These definitions are those used in this dissertation and are already defined respectively in (1.6)-(1.7)-(1.8).

Sometimes, we do not need to know the exact behavior of the exponential sum but it is important to have an upper bound; in all cases a trivial upper bound is the number of element of the truncated set:

$$|S_k(\alpha)| \ll X^{\frac{1}{k}}, \qquad |U_k(\alpha)| \ll X^{\frac{1}{k}}, \qquad |T_k(\alpha)| \ll X^{\frac{1}{k}} \tag{B.1}$$

in which, in the estimation of $|S_k(\alpha)|$, we used the Prime Number Theorem.

The estimations above are trivial estimates that often do not lead to satisfactory results. In this dissertation, in order to be able to exploit the Cauchy-Schwartz and the Hölder inequalities, we need estimates we need stronger bounds on the exponential sums such as L^n-norm estimations (in almost all cases L^2-norm or L^4-norm). Moreover, we need sometimes to estimate the difference between two of the objects defined above, sometimes also in L^2-norm.

Theorem B.1. *For $k \ge 1$ we have*

$$T_k(\alpha) \ll_{k,\delta} X^{\frac{1}{k}-1} \min(X, |\alpha|^{-1}).$$

Proof. For all $\alpha \in \mathbb{R}$,

$$\int_{(\delta X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(t^k \alpha)\mathrm{d}t \ll \int_{(\delta X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} \mathrm{d}t \ll X^{\frac{1}{k}}.$$

Using an appropriate change of variables followed by an integration by parts we have

$$\int_{(\delta X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(t^k \alpha)\mathrm{d}t = \int_{\delta X}^{X} \frac{1}{k} s^{\frac{1}{k}-1} e(s\alpha)\mathrm{d}s = \left[\frac{1}{k} s^{\frac{1}{k}-1} \frac{e(s\alpha)}{\alpha}\right]_{\delta X}^{X} + \frac{k-1}{k^2} \int_{\delta X}^{X} \frac{e(s\alpha)}{\alpha} s^{\frac{1}{k}-2}\mathrm{d}s$$

If $|\alpha| > X^{-1}$ it is easy to see that this is $\ll_{k,\delta} X^{\frac{1}{k}-1}|\alpha|^{-1}$.

□

Theorem B.2.

$$|T_k(\alpha) - U_k(\alpha)| \ll 1 + |\alpha| X.$$

It can be proved using Euler summation formula (Theorem A.7).

B.1 Mean-value of $|S_1(\alpha)|^2$

Theorem B.3.

$$\int_0^1 |S_1^2(\alpha)| \mathrm{d}\alpha \ll X \log X.$$

Proof.

$$\begin{aligned}
\int_0^1 |S_1^2(\alpha)| \mathrm{d}\alpha &= \int_0^1 S_1(\alpha) S_1(-\alpha) \mathrm{d}\alpha \\
&= \int_0^1 \sum_{p \le X} \log p \cdot e(p\alpha) \sum_{q \le X} \log q \cdot e(-q\alpha) \mathrm{d}\alpha \\
&= \sum_{p \le X} \sum_{q \le X} \log p \log q \int_0^1 e((p-q)\alpha) \mathrm{d}\alpha.
\end{aligned}$$

If $p = q$ the integral is 1, otherwise the integral is 0, therefore

$$\int_0^1 |S_1^2(\alpha)| \mathrm{d}\alpha = \sum_{p \le X} \log^2 p = X \log X + o(X \log X)$$

using partial summation (Theorem A.6) and the Prime Number Theorem. □

B.2 Mean-value of $|S_2(\alpha)|^4$

Lemma B.4.

$$\int_0^1 |S_2(\alpha)|^4 \mathrm{d}\alpha \ll_\varepsilon X \log^2 X.$$

The proof is due to Rieger [40] p. 94 satz 3.

B.3 Mean-value of $|S_3(\alpha)|^4$ and $|S_j(\alpha)|^{2^j}$

Lemma B.5 (Hua's Lemma)**.** *Let* $k \ge 1$ *integer and* $1 \le \ell \le k$*; we have*

$$\int_0^1 |S_k(\alpha)|^{2^\ell} \mathrm{d}\alpha \ll X^{\frac{2^\ell - \ell}{k} + \varepsilon}.$$

The proof of Hua's Lemma can be found in [46] Lemma 2.5.

B.4 Mean-value of $|S_k(\alpha)|^2$

Lemma B.6 ([42], Lemma 7)**.** *Let* $k>1$, $\tau>0$,

$$\int_{-\tau}^{\tau}|S_k(\alpha)|^2\mathrm{d}\alpha \ll \max(\tau X^{1/k+\varepsilon}, X^{2/k-1+\varepsilon}).$$

Proof. It comes directly from [42] Lemma 7, in fact, although in Tolev's paper $c\in\left(1,\frac{15}{14}\right)$, the Lemma is true for every $c>1$. In our case we use the definition of $S_k(\alpha)$ in (1.6):

$$\begin{aligned}\int_{-\tau}^{\tau}|S_k(\alpha)|^2\mathrm{d}\alpha &\ll \sum_{\delta X<p_1^k,p_2^k\le X}(\log p_1)(\log p_2)\int_{-\tau}^{\tau}e((p_1^k-p_2^k)\alpha)\mathrm{d}\alpha\\ &\ll \log^2 X\sum_{\delta X<p_1^k,p_2^k\le X}\min\left(\tau,\frac{1}{|p_1^k-p_2^k|}\right)\\ &\ll \log^2 X\sum_{\delta X<n_1^k,n_2^k\le X}\min\left(\tau,\frac{1}{|n_1^k-n_2^k|}\right)\\ &\ll U\tau\log^2 X+V\log^2 X,\end{aligned}$$

where

$$U=\sum_{\substack{\delta X<n_1^k,n_2^k\le X\\ |n_1^k-n_2^k|\le 1/\tau}}1, \qquad V=\sum_{\substack{\delta X<n_1^k,n_2^k\le X\\ |n_1^k-n_2^k|>1/\tau}}\frac{1}{|n_1^k-n_2^k|}.$$

We have

$$U\ll \mathscr{A}(X^{1/k};k;1/\tau)\ll\left(\frac{1}{\tau}X^{2/k-1}+X^{1/k}\right)X^{\varepsilon}$$

then

$$U\tau\log^2 X\ll\left(X^{2/k-1}+\tau X^{1/k}\right)X^{\varepsilon}.$$

On the other hand, using a dyadic argument on $u=|n_1^k-n_2^k|^{-1}$, $V\le\sum_l V_l$ where

$$V_l=\sum_{\substack{\delta X<n_1^k,n_2^k\le X\\ l<|n_1^k-n_2^k|\le 2l}}\frac{1}{u}$$

and l takes the values $\dfrac{2^i}{\tau}$, $i=0,1,2,\ldots$, with $l\le X$. We have

$$V_l\ll\frac{1}{l}\sum_{\delta X<n_1^k\le X}\sum_{\substack{\delta X<n_2^k\le X\\ (n_1^k+l)^{1/k}\le n_2\le(n_1^k+2l)^{1/k}}}1.$$

For $l\ge\dfrac{1}{\tau}$ and $\delta X<n_1\le X$ it is easy to see that

$$(n_1^k+2l)^{\frac{1}{k}}-(n_1^k+l)^{\frac{1}{k}}>1,$$

hence

$$V_l \ll \frac{1}{l} \sum_{\delta X < n_1^k \le X} \left((n_1^k + 2l)^{1/k} - (n_1^k + l)^{1/k} \right) \ll X^{2/k-1}$$

by the mean-value Theorem. Collecting all bounds we get

$$\int_{-\tau}^{\tau} |S_k(\alpha)|^2 \mathrm{d}\alpha \ll \left(\tau X^{1/k} + X^{2/k-1} \right) X^{\varepsilon}.$$

□

Lemma B.7 ([30], Lemma 5)**.** *Let* $\lambda \in \mathbb{R} \setminus \{0\}$*,* $k > 1$*,* $0 < \eta < 1$*,* $R > 1/\eta$ *and* $1 < P < X$*. We have*

$$\int_{P/X}^{R} |S_k(\lambda\alpha)|^2 K_\eta(\alpha) \, \mathrm{d}\alpha \ll \eta X^{1/k} (\log X)^3.$$

B.5 Mean-value of $|S_k(\alpha)|^4$

Lemma B.8 ([15], Lemma 3)**.** *Let* $\varepsilon > 0$*,* $k > 1$ *and* $\gamma > 0$*. Let further* $\mathscr{A}(X^{1/k}; k; \gamma)$ *denote the number of solutions of the inequalities*

$$|n_1^k + n_2^k - n_3^k - n_4^k| < \gamma, \qquad X^{1/k} < n_1, n_2, n_3, n_4 \le 2X^{1/k}.$$

Then

$$\mathscr{A}(X^{1/k}; k; \gamma) \ll \left(X^{2/k} + \gamma X^{4/k-1} \right) X^{\varepsilon}.$$

Proof. This is an immediate consequence of Theorem 2 of Robert & Sargos [41]; we just need to choose $M = X^{1/k}$, $\alpha = k$ and $\gamma = \delta M^k$ there. □

Lemma B.9 ([15], Lemma 4)**.** *Let* $\varepsilon > 0$*,* $\delta > 0$*,* $k > 1$*,* $n \in \mathbb{N}$ *and* $\tau > 0$*. Then we have*

$$\int_{-\tau}^{\tau} |S_k(\alpha)|^4 \, \mathrm{d}\alpha \ll \left(\tau X^{2/k} + X^{4/k-1} \right) X^{\varepsilon} \quad \textit{and} \quad \int_{n}^{n+1} |S_k(\alpha)|^4 \, \mathrm{d}\alpha \ll \left(X^{2/k} + X^{4/k-1} \right) X^{\varepsilon}.$$

Proof. A direct computation gives

$$\begin{aligned}
\int_{-\tau}^{\tau} |S_k(\alpha)|^4 \, \mathrm{d}\alpha &= \sum_{\delta X < p_1^k, p_2^k, p_3^k, p_4^k \le X} (\log p_1) \cdots (\log p_4) \int_{-\tau}^{\tau} e((p_1^k + p_2^k - p_3^k - p_4^k)\alpha) \, \mathrm{d}\alpha \\
&\ll (\log X)^4 \sum_{\delta X < p_1^k, p_2^k, p_3^k, p_4^k \le X} \min\left(\tau; \frac{1}{|p_1^k + p_2^k - p_3^k - p_4^k|} \right) \\
&\ll (\log X)^4 \sum_{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \le X} \min\left(\tau; \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|} \right) \\
&\ll U\tau (\log X)^4 + V (\log X)^4, \qquad \text{(B.2)}
\end{aligned}$$

where

$$U := \sum_{\substack{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \le X \\ |n_1^k + n_2^k - n_3^k - n_4^k| \le 1/\tau}} 1, \quad \text{and} \quad V := \sum_{\substack{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \le X \\ |n_1^k + n_2^k - n_3^k - n_4^k| > 1/\tau}} \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|},$$

say. Using Lemma B.8 on U we get

$$U \ll \mathscr{A}(X^{1/k}; k; 1/\tau) \ll \Big(X^{2/k} + \frac{1}{\tau} X^{4/k-1}\Big) X^{\varepsilon}.$$

Concerning V, by a dyadic argument we get

$$\begin{aligned} V &\ll \log X \Big(\max_{1/\tau < W \ll X} \sum_{\substack{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \le X \\ W < |n_1^k + n_2^k - n_3^k - n_4^k| \le 2W}} \frac{1}{u} \Big) \ll \log X \max_{1/\tau < W \ll X} \Big(\frac{1}{W} \mathscr{A}(X^{1/k}; k; 2W) \Big) \\ &\ll \max_{1/\tau < W \ll X} \Big(\frac{1}{W} (2W X^{4/k-1} + X^{2/k}) \Big) X^{\varepsilon} \ll \max_{1/\tau < W \ll X} \Big(X^{4/k-1} + \frac{X^{2/k}}{W} \Big) X^{\varepsilon} \\ &\ll (\tau X^{2/k} + X^{4/k-1}) X^{\varepsilon}. \end{aligned} \tag{B.3}$$

Combining (B.2)-(B.3), the first part of the Lemma follows. The second part can be proved in a similar way. □

Lemma B.10 ([15], Lemma 10)**.** *Let* $\lambda \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$, $k > 1$, $0 < \eta < 1$, $R > 1/\eta$ *and* $1 < P < X$. *Then*

$$\int_{P/X}^{R} |S_k(\lambda\alpha)|^4 K_\eta(\alpha) \, \mathrm{d}\alpha \ll \eta \max(X^{2/k}; X^{4/k-1}) X^{\varepsilon}.$$

Proof. Using (1.5) we obtain

$$\int_{P/X}^{R} |S_k(\lambda\alpha)|^4 K_\eta(\alpha) \, \mathrm{d}\alpha \ll \eta^2 \int_{P/X}^{1/\eta} |S_k(\lambda\alpha)|^4 \, \mathrm{d}\alpha + \int_{1/\eta}^{R} |S_k(\lambda\alpha)|^4 \frac{\mathrm{d}\alpha}{\alpha^2} = A + B, \tag{B.4}$$

say. By Lemma B.9, we immediately get

$$A \ll \eta^2 \int_{-|\lambda|/\eta}^{|\lambda|/\eta} |S_k(\alpha)|^4 \, \mathrm{d}\alpha \ll \eta \max(X^{2/k}; \eta X^{4/k-1}) X^{\varepsilon}.$$

Moreover, again by Lemma B.9, we have that

$$\begin{aligned} B &\ll_\lambda \int_{|\lambda|/\eta}^{+\infty} |S_k(\alpha)|^4 \frac{\mathrm{d}\alpha}{\alpha^2} \ll \sum_{n \ge |\lambda|/\eta} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_k(\alpha)|^4 \, \mathrm{d}\alpha \\ &\ll \eta \max(X^{2/k}; X^{4/k-1}) X^{\varepsilon}. \end{aligned} \tag{B.5}$$

Combining (B.4)-(B.5) and using $0 < \eta < 1$, the Lemma follows. □

B.6 Generalized Selberg integrals

Let us define the generalized version of the Selberg integral:

$$\mathscr{J}_k(X,h) = \int_X^{2X} \left(\theta((x+h)^{\frac{1}{k}}) - \theta(x^{\frac{1}{k}}) - ((x+h)^{\frac{1}{k}} - x^{\frac{1}{k}})\right)^2 \mathrm{d}x,$$

where $\theta(x) = \sum_{p\le x} \log p$ is the usual Chebyshev function.

Lemma B.11 ([29], Theorem 3.1)**.** *Let $k \ge 1$ be a real number. For $0 < Y < \frac{1}{2}$ we have*

$$\int_{-Y}^{Y} |S_k(\alpha) - U_k(\alpha)|^2 \mathrm{d}\alpha \ll_k \frac{X^{\frac{2}{k}-2}\log^2 X}{Y} + Y^2 X + Y^2 \mathscr{J}_k\left(X, \frac{1}{2Y}\right).$$

Lemma B.12 ([29], Theorem 3.2)**.** *Let $k \ge 1$ be a real number and ε be an arbitrarily small positive constant. There exists a positive constant $c_1(\varepsilon)$, which does not depend on k, such that*

$$\mathscr{J}_k(X,h) \ll_k h^2 X^{\frac{2}{k}-1} \exp\left(-c_1\left(\frac{\log X}{\log\log X}\right)^{\frac{1}{3}}\right)$$

uniformly for $X^{1-\frac{5}{6k}+\varepsilon} \le h \le X$.

B.7 The theorems of Vaughan and Ghosh

Lemma B.13 (Vaughan [46], Theorem 3.1)**.** *Let α be a real number and a,q be positive integers satisfying $(a,q) = 1$ and $\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}$. Then*

$$S_1(\alpha) \ll \left(\frac{X}{\sqrt{q}} + \sqrt{Xq} + X^{\frac{4}{5}}\right)\log^4 X.$$

Lemma B.14 (Ghosh [17], Theorem 2)**.** *Let α be a real number and a,q be positive integers satisfying $(a,q) = 1$ and $\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}$. Let moreover $\varepsilon > 0$, then*

$$S_2(\alpha) \ll_\varepsilon X^{\frac{1}{2}+\varepsilon}\left(\frac{1}{q} + \frac{1}{X^{\frac{1}{4}}} + \frac{q}{X}\right)^{\frac{1}{4}}.$$

Corollary B.15. *Let $\lambda \in \mathbb{R}\setminus\{0\}$, $X \ge Z \ge X^{4/5}(\log X)^5$ and $|S_1(\lambda\alpha)| > Z$. Then there are coprime integers $(a,q) = 1$ satisfying*

$$1 \le q \ll \left(\frac{X(\log X)^4}{Z}\right)^2, \qquad |q\lambda\alpha - a| \ll \frac{X(\log X)^{10}}{Z^2}.$$

Proof. Let Q be a parameter that we will choose later. By Dirichlet's theorem A.3 there exist coprime integers $(a,q) = 1$ such that $1 \le q \le Q$ and $|q\lambda\alpha - a| \ll Q^{-1} \le q^{-1}$. The choice

$$Q = \frac{Z^2}{X(\log X)^{10}}$$

allows us to prove the second part of the statement and to neglect some terms in the estimations of $|S_1(\lambda\alpha)|$. Using Lemma B.13, knowing that $Z \geq X^{4/5}(\log X)^5$ and $|S_1(\lambda\alpha)| > Z$, we can rewrite the bound for $|S_1(\lambda\alpha)|$ neglecting the term $X^{4/5}$:

$$Z < |S_1(\lambda\alpha)| \ll (Xq^{-1/2} + X^{1/2}q^{1/2})(\log X)^4.$$

The condition $q \leq Q$ allows us to neglect the term $X^{1/2}q^{1/2}$ and deal with small values of q; in fact, if $q > X^{1/2}$ then we would have a contradiction

$$Z < |S_1(\lambda\alpha)| \ll X^{1/2}q^{1/2}(\log X)^4 \leq X^{1/2}\frac{Z}{X^{1/2}(\log X)^5}(\log X)^4 = o\left(Z\right).$$

Then $q \leq \min(X^{1/2}, Q) = X^{1/2}$, since $Z \geq X^{4/5}(\log X)^5 > X^{3/4}(\log X)^5$. Moreover, we can rewrite the inequality on $|S_1(\lambda\alpha)|$ as

$$Z < |S_1(\lambda\alpha)| \ll Xq^{-1/2}(\log X)^4$$

and finally we get $q^{1/2}Z \ll X(\log X)^4$, which completes the proof.

□

Corollary B.16. *Let* $\lambda \in \mathbb{R} \setminus \{0\}$, $X^{\frac{1}{2}} \geq Z \geq X^{\frac{1}{2}-\frac{1}{16}+\varepsilon}$ *and* $|S_2(\lambda\alpha)| > Z$. *Then there are coprime integers* $(a,q) = 1$ *satisfying*

$$1 \leq q \leq \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^4, \qquad |q\lambda\alpha - a| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^4.$$

Proof. Let Q be a parameter that we will choose later. By Dirichlet's Theorem A.3 there exist coprime integers $(a,q) = 1$ such $1 \leq q \leq Q$ such that $|q\lambda\alpha - a| \ll Q^{-1} \leq q^{-1}$. The choice

$$Q = X\left(\frac{Z}{X^{\frac{1}{2}+\varepsilon}}\right)^4$$

allows us to prove the second part of the statement and to neglect some terms in the estimations of $|S_2(\lambda\alpha)|$. Using Lemma B.14, knowing that $Z \geq X^{\frac{7}{16}+\varepsilon}$ and $|S_2(\lambda_1\alpha)| > Z$, we can rewrite the bound for $|S_2(\lambda\alpha)|$ neglecting the term $X^{\frac{7}{16}+\varepsilon}$:

$$Z < |S_2(\lambda\alpha)| \ll X^{\frac{1}{2}+\varepsilon}q^{-\frac{1}{4}} + X^{\frac{1}{4}+\varepsilon/2}q^{\frac{1}{4}}.$$

The condition $q \leq Q$ allows us to neglect the term $X^{\frac{1}{4}+\varepsilon/2}q^{\frac{1}{4}}$ and deal with small value of q; in fact, if $q > X^{\frac{1}{2}}$ then we would have a contradiction:

$$Z < |S_2(\lambda\alpha)| \ll X^{\frac{1}{4}+\varepsilon/2}q^{\frac{1}{4}} \leq X^{\frac{1}{4}+\varepsilon/2}X^{\frac{1}{4}}\frac{Z}{X^{\frac{1}{2}+\varepsilon}} = o(Z).$$

Then $q \leq \min(X^{\frac{1}{2}}, Q) = X^{\frac{1}{2}}$, since $Z \geq X^{\frac{7}{16}+\varepsilon} > X^{\frac{3}{8}+\varepsilon}$. Moreover we can rewrite the inequality for $|S_2(\lambda\alpha)|$ as

$$Z < |S_2(\lambda\alpha)| \ll X^{\frac{1}{2}+\varepsilon} q^{-\frac{1}{4}}$$

and get

$$q^{\frac{1}{4}} Z \leq X^{\frac{1}{2}+\varepsilon} \quad \Rightarrow \quad q \leq \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z} \right)^4$$

which proves the Lemma. □

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