Tensor products and perturbations of BiHom-Novikov-Poisson algebras

Ling Liu
College of Mathematics and Computer Science,
Zhejiang Normal University,
Jinhua 321004, China
e-mail: ntliulin@zjnu.cn

Abdenacer Makhlouf
Université de Haute Alsace,
IRIMAS-Département de Mathématiques,
6 bis, rue des frères Lumière, F-68093 Mulhouse, France
e-mail: Abdenacer.Makhlouf@uha.fr

Claudia Menini
University of Ferrara,
Department of Mathematics and Computer Science
Via Machiavelli 30, Ferrara, I-44121, Italy
e-mail: men@unife.it

Florin Panaite
Institute of Mathematics of the Romanian Academy,
PO-Box 1-764, RO-014700 Bucharest, Romania
e-mail: florin.panaite@imar.ro

Abstract

We study BiHom-Novikov-Poisson algebras, which are twisted generalizations of Novikov-Poisson algebras and Hom-Novikov-Poisson algebras, and find that BiHom-Novikov-Poisson algebras are closed under tensor products and several kinds of perturbations. Necessary and sufficient conditions are given under which BiHom-Novikov-Poisson algebras give rise to BiHom-Poisson algebras.

Keywords: BiHom-Novikov-Poisson algebra, BiHom-Novikov algebra, BiHom-Poisson algebra, BiHom-commutative algebra.
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Introduction

Hom-type algebras appeared in the Physics literature of the 1990's, when looking for quantum deformations of some algebras of vector fields, like Witt and Virasoro algebras ([2], [11]). It was observed that algebras obtained by deforming certain Lie algebras no longer satisfied the Jacobi identity, but a modified version of it involving a homomorphism. An axiomatization of this type of algebras (called Hom-Lie algebras) was given in [10], [12]. The associative counterpart of

Hom-Lie algebras (called Hom-associative algebras) has been introduced in [17], where it was proved also that the commutator bracket defined by the multiplication in a Hom-associative algebra gives rise to a Hom-Lie algebra. Since then, Hom analogues of many other classical algebraic structures have been introduced and studied (such as Hom-bialgebras, Hom-pre-Lie algebras, Hom-dendriform algebras etc.).

A categorical approach to Hom-type algebras was considered in [4]. An attempt to generalize the construction in [4] by including a group action led in [8] to the observation that Hom-structures have a very natural generalization, in which a classical algebraic identity is twisted by two (commuting) homomorphisms, instead of just one. This new type of algebras has been called BiHom-algebras (examples include BiHom-associative algebras, BiHom-Lie algebras, BiHom-bialgebras etc.). The main tool to obtain examples of Hom-algebras from classical algebras, the so-called "Yau twisting", works perfectly fine also in the BiHom-type case. There is a growing literature on Hom and BiHom-type algebras, let us just mention the very recent papers [13], [14], [15] (see also references therein).

Novikov algebras appeared in connection with the Poisson brackets of hydrodynamic type ([3], [5], [6]) and Hamiltonian operators in the formal variational calculus ([7], [19]). They are a special class of pre-Lie algebras and are related to several branches of geometry and mathematical physics, such as Lie groups and algebras, affine manifolds, vertex and conformal algebras etc.

Novikov-Poisson algebras were introduced by Xu in [19], motivated by the study of simple Novikov algebras and irreducible modules. A Novikov-Poisson algebra is a Novikov algebra having an extra structure (a commutative associative product) together with some compatibility conditions between the two structures. As proved by Xu in [19], Novikov-Poisson algebras have the remarkable property (not shared by Novikov algebras) that they are closed under tensor products; also, Xu proved in [20] that they are also closed under certain types of perturbations.

Hom-type analogues of Novikov and Novikov-Poisson algebras have been introduced and studied by Yau in [21], [22]. Examples can be obtained by Yau twisting from their classical counterparts. Also, it turns out that many properties of Novikov and Novikov-Poisson algebras are shared by their Hom-type counterparts; in particular, Yau proved that Hom-Novikov-Poisson algebras are closed under tensor products and certain types of perturbations.

In our previous paper [15], we introduced and studied the concept of BiHom-Novikov algebra (which is different from the one introduced in [9]), in relation also with the so-called infinitesimal BiHom-bialgebras. In a short section of [15], we introduced as well the concept of BiHom-Novikov-Poisson algebra and provided some classes of examples. The aim of the present paper is to continue the study of BiHom-Novikov-Poisson algebras. More precisely, we show that BiHom-Novikov-Poisson algebras are closed under tensor products and under certain types of perturbations, extending thus the results mentioned above obtained by Xu in the classical case and by Yau in the Hom-type case.

The most tricky part in what we are doing is contained in Lemma 3.1, and we try now to describe its content. The perturbations used by Xu in the classical case are essentially based on the following obvious fact: if (A, μ) is a commutative associative algebra (with multiplication μ denoted by juxtaposition) and $a \in A$ is a fixed element, and we define a new multiplication on A by $x \diamond y = axy$, then $(A, \diamond)$ is also a commutative associative algebra. The Hom-type analogue of this result (Lemma 4.1 in Yau's paper [22]) reads as follows: if (A, μ, α) is a commutative Hom-associative algebra and $a \in A$ is an element such that $\alpha^2(a) = a$ and we define a new multiplication on A by $x \diamond y = a(xy)$, then $(A, \diamond, \alpha^2)$ is also a commutative Hom-associative algebra. We have tried to extend this result (by generalizing the formula for $\diamond$) to the BiHom-type case and we failed. Then we realized that actually the natural formula for $\diamond$ that has to be extended (both in the Hom-type and BiHom-type case) is not $x \diamond y = axy$, but $x \diamond y = xay$ (which is associative even if the original multiplication is noncommutative). Thus, our Lemma 3.1 reads as follows: if (A, μ, α, β) is a BiHom-associative algebra and $a \in A$ is an element satisfying $\alpha^2(a) = \beta^2(a) = a$ and we define a new operation on A by $x \diamond y = \alpha(x)(\alpha(a)y)$, then $(A, \diamond, \alpha^2, \beta^2)$ is also a BiHom-associative algebra; if moreover (A, μ, α, β) is BiHom-commutative, then $(A, \diamond, \alpha^2, \beta^2)$ is also BiHom-commutative. It turns out that Yau's result can be obtained as a particular case of this (see the comments after our Lemma 3.1).

In the last section of the paper we investigate a special class of BiHom-Novikov-Poisson algebras, called "left BiHom-associative"; in the case of bijective structure maps, these are exactly those BiHom-Novikov-Poisson algebras that give rise to so-called BiHom-Poisson algebras (a concept we introduce in this paper, where we consider a left-handed Leibniz identity, we refer to [1] for the concept dealing with a right-handed version) see Theorem 4.5. It turns out that left BiHom-associativity is preserved by Yau twisting, by tensor products and by the perturbations mentioned above.

1 Preliminaries

We work over a base field $\Bbbk$. All algebras, linear spaces etc. will be over $\Bbbk$; unadorned $\otimes$ means $\otimes_{\Bbbk}$. We denote by ${}_{\Bbbk}\mathcal{M}$ the category of linear spaces over $\Bbbk$. Unless otherwise specified, the algebras (associative or not) that will appear in what follows are *not* supposed to be unital, and a multiplication $\mu : A \otimes A \to A$ on a linear space A is denoted by juxtaposition: $\mu(v \otimes v') = vv'$. For the composition of two maps f and g, we write either $g \circ f$ or simply gf. For the identity map on a linear space A we use the notation id_A.

Definition 1.1 *([8]) A BiHom-associative algebra is a 4-tuple (A, μ, α, β), where A is a linear space and $\alpha, \beta : A \to A$ and $\mu : A \otimes A \to A$ are linear maps such that $\alpha \circ \beta = \beta \circ \alpha$, $\alpha(xy) = \alpha(x)\alpha(y)$, $\beta(xy) = \beta(x)\beta(y)$ and*

$$\alpha(x)(yz) = (xy)\beta(z), \quad \forall\, x, y, z \in A. \tag{1.1}$$

The maps α and β (in this order) are called the structure maps of A and condition (1.1) is called the BiHom-associativity condition. A morphism $f : (A, \mu_A, \alpha_A, \beta_A) \to (B, \mu_B, \alpha_B, \beta_B)$ of BiHom-associative algebras is a linear map $f : A \to B$ such that $\alpha_B \circ f = f \circ \alpha_A$, $\beta_B \circ f = f \circ \beta_A$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

If (A, μ) is an associative algebra and $\alpha, \beta : A \to A$ are two commuting algebra maps, then $A_{(\alpha,\beta)} := (A, \mu \circ (\alpha \otimes \beta), \alpha, \beta)$ is a BiHom-associative algebra, called the Yau twist of A given by the maps α and β.

Definition 1.2 *([15]) A BiHom-associative algebra (A, μ, α, β) is called BiHom-commutative if*

$$\beta(a)\alpha(b) = \beta(b)\alpha(a), \quad \forall\ a, b \in A. \tag{1.2}$$

Definition 1.3 *A left pre-Lie algebra is an algebra $(A, *)$ for which*

$$x * (y * z) - (x * y) * z = y * (x * z) - (y * x) * z, \quad \forall\ x, y, z \in A.$$

An algebra $(A, *)$ is called Novikov algebra if it is left pre-Lie and

$$(x * y) * z = (x * z) * y, \quad \forall\, x, y, z \in A.$$

A morphism of Novikov algebras from $(A, *)$ *to* $(A', *')$ *is a linear map* $f : A \to A'$ *satisfying* $f(x * y) = f(x) *' f(y)$*, for all* $x, y \in A$*.*

Definition 1.4 *([19], [20]) A Novikov-Poisson algebra is a triple* $(A, \mu, *)$ *such that* (A, μ) *is a commutative associative algebra,* $(A, *)$ *is a Novikov algebra and the following compatibility conditions hold, for all* $x, y, z \in A$*:*

$$(x * y)z - x * (yz) = (y * x)z - y * (xz), \tag{1.3}$$

$$(xy) * z = (x * z)y. \tag{1.4}$$

A morphism of Novikov-Poisson algebras from $(A, \mu, *)$ *to* $(A', \mu', *')$ *is a linear map* $f : A \to A'$ *satisfying* $f \circ \mu = \mu' \circ (f \otimes f)$ *and* $f(x * y) = f(x) *' f(y)$*, for all* $x, y \in A$*.*

Note that, by the commutativity of (A, μ), (1.4) is equivalent to

$$(xy) * z = x(y * z), \quad \forall\, x, y, z \in A. \tag{1.5}$$

Definition 1.5 *([15]) A BiHom-Novikov algebra is a 4-tuple* $(A, *, \alpha, \beta)$*, where* A *is a linear space,* $* : A \otimes A \to A$ *is a linear map and* $\alpha, \beta : A \to A$ *are commuting linear maps (called the structure maps of* A*), satisfying the following conditions, for all* $x, y, z \in A$*:*

$$\alpha(x * y) = \alpha(x) * \alpha(y), \quad \beta(x * y) = \beta(x) * \beta(y), \tag{1.6}$$

$$(\beta(x) * \alpha(y)) * \beta(z) - \alpha\beta(x) * (\alpha(y) * z) = (\beta(y) * \alpha(x)) * \beta(z) - \alpha\beta(y) * (\alpha(x) * z), \tag{1.7}$$

$$(x * \beta(y)) * \alpha\beta(z) = (x * \beta(z)) * \alpha\beta(y). \tag{1.8}$$

A morphism $f : (A, *_A, \alpha_A, \beta_A) \to (B, *_B, \alpha_B, \beta_B)$ *of BiHom-Novikov algebras is a linear map* $f : A \to B$ *such that* $\alpha_B \circ f = f \circ \alpha_A$*,* $\beta_B \circ f = f \circ \beta_A$ *and* $f(x *_A y) = f(x) *_B f(y)$*, for all* $x, y \in A$*.*

Definition 1.6 *([15]) A BiHom-Novikov-Poisson algebra is a 5-tuple* $(A, \mu, *, \alpha, \beta)$ *such that:*
(1) (A, μ, α, β) *is a BiHom-commutative algebra;*
(2) $(A, *, \alpha, \beta)$ *is a BiHom-Novikov algebra;*
(3) the following compatibility conditions hold for all $x, y, z \in A$*:*

$$(\beta(x) * \alpha(y))\beta(z) - \alpha\beta(x) * (\alpha(y)z) = (\beta(y) * \alpha(x))\beta(z) - \alpha\beta(y) * (\alpha(x)z), \tag{1.9}$$

$$(x\beta(y)) * \alpha\beta(z) = (x * \beta(z))\alpha\beta(y), \tag{1.10}$$

$$\alpha(x)(y * z) = (xy) * \beta(z). \tag{1.11}$$

The maps α *and* β *(in this order) are called the structure maps of* A*.*

A morphism $f : (A, \mu, *, \alpha, \beta) \to (A', \mu', *', \alpha', \beta')$ *of BiHom-Novikov-Poisson algebras is a map that is a morphism of BiHom-associative algebras from* (A, μ, α, β) *to* $(A', \mu', \alpha', \beta')$ *and a morphism of BiHom-Novikov algebras from* $(A, *, \alpha, \beta)$ *to* $(A', *', \alpha', \beta')$*.*

From Lemma 3.2 in [15], we know that if α and β are bijective, the conditions (1.10) and (1.11) are equivalent (assuming (1) and multiplicativity of α and β with respect to $*$).

2 Tensor products

Novikov algebras are not closed under tensor products in a non-trivial way. One reason Novikov-Poisson algebras were introduced in [19] was that the tensor product of two Novikov-Poisson algebras is a Novikov-Poisson algebra non-trivially ([19], Theorem 4.1). This fact was extended by Yau (see [22]) to the class of Hom-Novikov-Poisson algebras, and the aim of this section is to further extend it to BiHom-Novikov-Poisson algebras.

Theorem 2.1 *Let $(A_i, \cdot_i, *_i, \alpha_i, \beta_i)$ be BiHom-Novikov-Poisson algebras, for $i = 1, 2$. Then $A := (A_1 \otimes A_2, \cdot, *, \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a BiHom-Novikov-Poisson algebra, where the operations $\cdot$ and $*$ are defined as follows (for $x_i, y_i \in A_i$):*

$$\begin{aligned}
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) &= (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2), \\
(x_1 \otimes x_2) * (y_1 \otimes y_2) &= (x_1 *_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 *_2 y_2).
\end{aligned}$$

Proof. To improve readability, we omit the subscripts in $\cdot_i$ and $*_i$ in the proof.

(1) We show that $(A_1 \otimes A_2, \cdot, \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a BiHom-commutative algebra. For $x_1, y_1 \in A_1$, $x_2, y_2 \in A_2$, it is easy to check that $(\alpha_1 \otimes \alpha_2)[(x_1 \otimes x_2) \cdot (y_1 \otimes y_2)] = [(\alpha_1 \otimes \alpha_2)(x_1 \otimes x_2)] \cdot [(\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)]$ and $(\beta_1 \otimes \beta_2)[(x_1 \otimes x_2) \cdot (y_1 \otimes y_2)] = [(\beta_1 \otimes \beta_2)(x_1 \otimes x_2)] \cdot [(\beta_1 \otimes \beta_2)(y_1 \otimes y_2)]$.

We verify the BiHom-associativity condition (1.1):

$$\begin{aligned}
&[(\alpha_1 \otimes \alpha_2)(x_1 \otimes x_2)] \cdot [(y_1 \otimes y_2) \cdot (z_1 \otimes z_2)] \\
&\quad = (\alpha_1(x_1) \otimes \alpha_2(x_2)) \cdot [(y_1 \cdot z_1) \otimes (y_2 \cdot z_2)] = (\alpha_1(x_1) \cdot (y_1 \cdot z_1)) \otimes (\alpha_2(x_2) \cdot (y_2 \cdot z_2)) \\
&\quad \overset{(1.1)}{=} ((x_1 \cdot y_1) \cdot \beta_1(z_1)) \otimes ((x_2 \cdot y_2) \cdot \beta_2(z_2)) = [(x_1 \cdot y_1) \otimes (x_2 \cdot y_2)] \cdot (\beta_1(z_1) \otimes \beta_2(z_2)) \\
&\quad = [(x_1 \otimes x_2) \cdot (y_1 \otimes y_2)] \cdot [(\beta_1 \otimes \beta_2)(z_1 \otimes z_2)].
\end{aligned}$$

Now we check the BiHom-commutativity condition (1.2):

$$\begin{aligned}
[(\beta_1 \otimes \beta_2)(x_1 \otimes x_2)] \cdot [(\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)] &= (\beta_1(x_1) \cdot \alpha_1(y_1)) \otimes (\beta_2(x_2) \cdot \alpha_2(y_2)) \\
&\overset{(1.2)}{=} (\beta_1(y_1) \cdot \alpha_1(x_1)) \otimes (\beta_2(y_2) \cdot \alpha_2(x_2)) \\
&= [(\beta_1 \otimes \beta_2)(y_1 \otimes y_2)] \cdot [(\alpha_1 \otimes \alpha_2)(x_1 \otimes x_2)].
\end{aligned}$$

So indeed A is a BiHom-commutative algebra.

(2) We prove that $(A_1 \otimes A_2, *, \alpha = \alpha_1 \otimes \alpha_2, \beta = \beta_1 \otimes \beta_2)$ is a BiHom-Novikov algebra. It is obvious that (1.6) holds. We prove now (1.8). For $x = x_1 \otimes x_2, y = y_1 \otimes y_2$ and $z = z_1 \otimes z_2$ in A, we have to show that $(x * \beta(y)) * \alpha\beta(z)$ is symmetric in y and z. We have:

$$\begin{aligned}
&(x * \beta(y)) * \alpha\beta(z) \\
&\quad = [(x_1 * \beta_1(y_1)) \otimes (x_2 \cdot \beta_2(y_2)) + (x_1 \cdot \beta_1(y_1)) \otimes (x_2 * \beta_2(y_2))] * (\alpha_1\beta_1(z_1) \otimes \alpha_2\beta_2(z_2)) \\
&\quad = \underbrace{[(x_1 * \beta_1(y_1)) * \alpha_1\beta_1(z_1)] \otimes [(x_2 \cdot \beta_2(y_2)) \cdot \alpha_2\beta_2(z_2)]}_{a} \\
&\qquad + \underbrace{[(x_1 * \beta_1(y_1)) \cdot \alpha_1\beta_1(z_1)] \otimes [(x_2 \cdot \beta_2(y_2)) * \alpha_2\beta_2(z_2)]}_{b(x,y,z)} \\
&\qquad + \underbrace{[(x_1 \cdot \beta_1(y_1)) * \alpha_1\beta_1(z_1)] \otimes [(x_2 * \beta_2(y_2)) \cdot \alpha_2\beta_2(z_2)]}_{c(x,y,z)}
\end{aligned}$$

$$+\underbrace{[(x_1\cdot\beta_1(y_1))\cdot\alpha_1\beta_1(z_1)]\otimes[(x_2*\beta_2(y_2))*\alpha_2\beta_2(z_2)]}_{d}.$$

We prove that a is symmetric in y and z and leave to prove the same thing about d to the reader:

$$\begin{aligned}
a &= [(x_1*\beta_1(y_1))*\alpha_1\beta_1(z_1)]\otimes[(x_2\cdot\beta_2(y_2))\cdot\alpha_2\beta_2(z_2)]\\
&\overset{(1.1),(1.8)}{=} [(x_1*\beta_1(z_1))*\alpha_1\beta_1(y_1)]\otimes[\alpha_2(x_2)\cdot(\beta_2(y_2)\cdot\alpha_2(z_2))]\\
&\overset{(1.2)}{=} [(x_1*\beta_1(z_1))*\alpha_1\beta_1(y_1)]\otimes[\alpha_2(x_2)\cdot(\beta_2(z_2)\cdot\alpha_2(y_2))]\\
&\overset{(1.1)}{=} [(x_1*\beta_1(z_1))*\alpha_1\beta_1(y_1)]\otimes[(x_2\cdot\beta_2(z_2))\cdot\alpha_2\beta_2(y_2)].
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
b(x,y,z) &= [(x_1*\beta_1(y_1))\cdot\alpha_1\beta_1(z_1)]\otimes[(x_2\cdot\beta_2(y_2))*\alpha_2\beta_2(z_2)]\\
&\overset{(1.10)}{=} [(x_1\cdot\beta_1(z_1))*\alpha_1\beta_1(y_1)]\otimes[(x_2*\beta_2(z_2))\cdot\alpha_2\beta_2(y_2)] = c(x,z,y).
\end{aligned}$$

Thus

$$\begin{aligned}
(x*\beta(y))*\alpha\beta(z) &= a+b(x,y,z)+c(x,y,z)+d,\\
(x*\beta(z))*\alpha\beta(y) &= a+b(x,z,y)+c(x,z,y)+d\\
&= a+c(x,y,z)+b(x,y,z)+d=(x*\beta(y))*\alpha\beta(z).
\end{aligned}$$

This shows that $(x*\beta(y))*\alpha\beta(z)$ is symmetric in y and z, so formula (1.8) holds in A.

For (1.7) in A we must show that $(\beta(x)*\alpha(y))*\beta(z)-\alpha\beta(x)*(\alpha(y)*z)$ with respect to $*$ in A is symmetric in x and y. We compute:

$$(\beta(x)*\alpha(y))*\beta(z)-\alpha\beta(x)*(\alpha(y)*z)$$

$$\begin{aligned}
&= [(\beta_1(x_1)*\alpha_1(y_1))*\beta_1(z_1)]\otimes[(\beta_2(x_2)\cdot\alpha_2(y_2))\cdot\beta_2(z_2)]\\
&\quad +[(\beta_1(x_1)*\alpha_1(y_1))\cdot\beta_1(z_1)]\otimes[(\beta_2(x_2)\cdot\alpha_2(y_2))*\beta_2(z_2)]\\
&\quad +[(\beta_1(x_1)\cdot\alpha_1(y_1))*\beta_1(z_1)]\otimes[(\beta_2(x_2)*\alpha_2(y_2))\cdot\beta_2(z_2)]\\
&\quad +[(\beta_1(x_1)\cdot\alpha_1(y_1))\cdot\beta_1(z_1)]\otimes[(\beta_2(x_2)*\alpha_2(y_2))*\beta_2(z_2)]\\
&\quad -[\alpha_1\beta_1(x_1)*(\alpha_1(y_1)*z_1)]\otimes[\alpha_2\beta_2(x_2)\cdot(\alpha_2(y_2)\cdot z_2)]\\
&\quad -[\alpha_1\beta_1(x_1)\cdot(\alpha_1(y_1)*z_1)]\otimes[\alpha_2\beta_2(x_2)*(\alpha_2(y_2)\cdot z_2)]\\
&\quad -[\alpha_1\beta_1(x_1)*(\alpha_1(y_1)\cdot z_1)]\otimes[\alpha_2\beta_2(x_2)\cdot(\alpha_2(y_2)*z_2)]\\
&\quad -[\alpha_1\beta_1(x_1)\cdot(\alpha_1(y_1)\cdot z_1)]\otimes[\alpha_2\beta_2(x_2)*(\alpha_2(y_2)*z_2)]\\
&\underset{(1.11)}{\overset{(1.1)}{=}} [(\beta_1(x_1)*\alpha_1(y_1))*\beta_1(z_1)]\otimes[(\beta_2(x_2)\cdot\alpha_2(y_2))\cdot\beta_2(z_2)]\\
&\quad +[(\beta_1(x_1)*\alpha_1(y_1))\cdot\beta_1(z_1)]\otimes[(\beta_2(x_2)\cdot\alpha_2(y_2))*\beta_2(z_2)]\\
&\quad +[(\beta_1(x_1)\cdot\alpha_1(y_1))*\beta_1(z_1)]\otimes[(\beta_2(x_2)*\alpha_2(y_2))\cdot\beta_2(z_2)]\\
&\quad +[(\beta_1(x_1)\cdot\alpha_1(y_1))\cdot\beta_1(z_1)]\otimes[(\beta_2(x_2)*\alpha_2(y_2))*\beta_2(z_2)]\\
&\quad -[\alpha_1\beta_1(x_1)*(\alpha_1(y_1)*z_1)]\otimes[(\beta_2(x_2)\cdot\alpha_2(y_2))\cdot\beta_2(z_2)]\\
&\quad -[(\beta_1(x_1)\cdot\alpha_1(y_1))*\beta_1(z_1)]\otimes[\alpha_2\beta_2(x_2)*(\alpha_2(y_2)\cdot z_2)]\\
&\quad -[\alpha_1\beta_1(x_1)*(\alpha_1(y_1)\cdot z_1)]\otimes[(\beta_2(x_2)\cdot\alpha_2(y_2))*\beta_2(z_2)]\\
&\quad -[(\beta_1(x_1)\cdot\alpha_1(y_1))\cdot\beta_1(z_1)]\otimes[\alpha_2\beta_2(x_2)*(\alpha_2(y_2)*z_2)]
\end{aligned}$$

$$
\begin{aligned}
= \quad & [(\beta_1(x_1) * \alpha_1(y_1)) * \beta_1(z_1) - \alpha_1\beta_1(x_1) * (\alpha_1(y_1) * z_1)] \otimes [(\beta_2(x_2) \cdot \alpha_2(y_2)) \cdot \beta_2(z_2)] \\
& + [(\beta_1(x_1) \cdot \alpha_1(y_1)) \cdot \beta_1(z_1)] \otimes [(\beta_2(x_2) * \alpha_2(y_2)) * \beta_2(z_2) - \alpha_2\beta_2(x_2) * (\alpha_2(y_2) * z_2)] \\
& + [(\beta_1(x_1) * \alpha_1(y_1)) \cdot \beta_1(z_1) - \alpha_1\beta_1(x_1) * (\alpha_1(y_1) \cdot z_1)] \otimes [(\beta_2(x_2) \cdot \alpha_2(y_2)) * \beta_2(z_2)] \\
& + [(\beta_1(x_1) \cdot \alpha_1(y_1)) * \beta_1(z_1)] \otimes [(\beta_2(x_2) * \alpha_2(y_2)) \cdot \beta_2(z_2) - \alpha_2\beta_2(x_2) * (\alpha_2(y_2) \cdot z_2)] \\
\overset{(1.2),(1.7)}{\underset{(1.9)}{=}} \quad & [(\beta_1(y_1) * \alpha_1(x_1)) * \beta_1(z_1) - \alpha_1\beta_1(y_1) * (\alpha_1(x_1) * z_1)] \otimes [(\beta_2(y_2) \cdot \alpha_2(x_2)) \cdot \beta_2(z_2)] \\
& + [(\beta_1(y_1) \cdot \alpha_1(x_1)) \cdot \beta_1(z_1)] \otimes [(\beta_2(y_2) * \alpha_2(x_2)) * \beta_2(z_2) - \alpha_2\beta_2(y_2) * (\alpha_2(x_2) * z_2)] \\
& + [(\beta_1(y_1) * \alpha_1(x_1)) \cdot \beta_1(z_1) - \alpha_1\beta_1(y_1) * (\alpha_1(x_1) \cdot z_1)] \otimes [(\beta_2(y_2) \cdot \alpha_2(x_2)) * \beta_2(z_2)] \\
& + [(\beta_1(y_1) \cdot \alpha_1(x_1)) * \beta_1(z_1)] \otimes [(\beta_2(y_2) * \alpha_2(x_2)) \cdot \beta_2(z_2) - \alpha_2\beta_2(y_2) * (\alpha_2(x_2) \cdot z_2)] \\
= \quad & (\beta(y) * \alpha(x)) * \beta(z) - \alpha\beta(y) * (\alpha(x) * z).
\end{aligned}
$$

(3) We prove the relations (1.9)-(1.11). To prove (1.10) in A, we compute:

$$
\begin{aligned}
& (x \cdot \beta(y)) * \alpha\beta(z) \\
= \quad & [(x_1 \cdot \beta_1(y_1)) \otimes (x_2 \cdot \beta_2(y_2))] * (\alpha_1\beta_1(z_1) \otimes \alpha_2\beta_2(z_2)) \\
= \quad & [(x_1 \cdot \beta_1(y_1)) * \alpha_1\beta_1(z_1)] \otimes [(x_2 \cdot \beta_2(y_2)) \cdot \alpha_2\beta_2(z_2)] \\
& + [(x_1 \cdot \beta_1(y_1)) \cdot \alpha_1\beta_1(z_1)] \otimes [(x_2 \cdot \beta_2(y_2)) * \alpha_2\beta_2(z_2)] \\
\overset{(1.1)}{\underset{(1.10)}{=}} \quad & [(x_1 * \beta_1(z_1)) \cdot \alpha_1\beta_1(y_1)] \otimes [\alpha_2(x_2) \cdot (\beta_2(y_2) \cdot \alpha_2(z_2))] \\
& + [\alpha_1(x_1) \cdot (\beta_1(y_1) \cdot \alpha_1(z_1))] \otimes [(x_2 * \beta_2(z_2)) \cdot \alpha_2\beta_2(y_2)] \\
\overset{(1.2)}{=} \quad & [(x_1 * \beta_1(z_1)) \cdot \alpha_1\beta_1(y_1)] \otimes [\alpha_2(x_2) \cdot (\beta_2(z_2) \cdot \alpha_2(y_2))] \\
& + [\alpha_1(x_1) \cdot (\beta_1(z_1) \cdot \alpha_1(y_1))] \otimes [(x_2 * \beta_2(z_2)) \cdot \alpha_2\beta_2(y_2)] \\
\overset{(1.1)}{=} \quad & [(x_1 * \beta_1(z_1)) \cdot \alpha_1\beta_1(y_1)] \otimes [(x_2 \cdot \beta_2(z_2)) \cdot \alpha_2\beta_2(y_2)] \\
& + [(x_1 \cdot \beta_1(z_1)) \cdot \alpha_1\beta_1(y_1)] \otimes [(x_2 * \beta_2(z_2)) \cdot \alpha_2\beta_2(y_2)] \\
= \quad & [(x_1 * \beta_1(z_1)) \otimes (x_2 \cdot \beta_2(z_2)) + (x_1 \cdot \beta_1(z_1)) \otimes (x_2 * \beta_2(z_2))] \cdot (\alpha_1\beta_1(y_1) \otimes \alpha_2\beta_2(y_2)) \\
= \quad & [(x_1 \otimes x_2) * (\beta_1(z_1) \otimes \beta_2(z_2))] \cdot (\alpha_1\beta_1(y_1) \otimes \alpha_2\beta_2(y_2)) = (x * \beta(z)) \cdot \alpha\beta(y).
\end{aligned}
$$

To prove (1.9) in A, we compute:

$$
\begin{aligned}
& (\beta(x) * \alpha(y)) \cdot \beta(z) - \alpha\beta(x) * (\alpha(y) \cdot z) \\
= \quad & [(\beta_1(x_1) \cdot \alpha_1(y_1)) \cdot \beta_1(z_1)] \otimes [(\beta_2(x_2) * \alpha_2(y_2)) \cdot \beta_2(z_2)] \\
& + [(\beta_1(x_1) * \alpha_1(y_1)) \cdot \beta_1(z_1)] \otimes [(\beta_2(x_2) \cdot \alpha_2(y_2)) \cdot \beta_2(z_2)] \\
& - [\alpha_1\beta_1(x_1) \cdot (\alpha_1(y_1) \cdot z_1)] \otimes [\alpha_2\beta_2(x_2) * (\alpha_2(y_2) \cdot z_2)] \\
& - [\alpha_1\beta_1(x_1) * (\alpha_1(y_1) \cdot z_1)] \otimes [\alpha_2\beta_2(x_2) \cdot (\alpha_2(y_2) \cdot z_2)] \\
\overset{(1.1)}{=} \quad & [(\beta_1(x_1) \cdot \alpha_1(y_1)) \cdot \beta_1(z_1)] \otimes [(\beta_2(x_2) * \alpha_2(y_2)) \cdot \beta_2(z_2) - \alpha_2\beta_2(x_2) * (\alpha_2(y_2) \cdot z_2)] \\
& + [(\beta_1(x_1) * \alpha_1(y_1)) \cdot \beta_1(z_1) - \alpha_1\beta_1(x_1) * (\alpha_1(y_1) \cdot z_1)] \otimes [(\beta_2(x_2) \cdot \alpha_2(y_2)) \cdot \beta_2(z_2)] \\
\overset{(1.2)}{\underset{(1.9)}{=}} \quad & [(\beta_1(y_1) \cdot \alpha_1(x_1)) \cdot \beta_1(z_1)] \otimes [(\beta_2(y_2) * \alpha_2(x_2)) \cdot \beta_2(z_2) - \alpha_2\beta_2(y_2) * (\alpha_2(x_2) \cdot z_2)] \\
& + [(\beta_1(y_1) * \alpha_1(x_1)) \cdot \beta_1(z_1) - \alpha_1\beta_1(y_1) * (\alpha_1(x_1) \cdot z_1)] \otimes [(\beta_2(y_2) \cdot \alpha_2(x_2)) \cdot \beta_2(z_2)] \\
= \quad & (\beta(y) * \alpha(x)) \cdot \beta(z) - \alpha\beta(y) * (\alpha(x) \cdot z).
\end{aligned}
$$

Now we prove (1.11) in A. We compute:

$$
(x \cdot y) * \beta(z) \quad = \quad ((x_1 \otimes x_2) \cdot (y_1 \otimes y_2)) * (\beta_1(z_1) \otimes \beta_2(z_2))
$$

$$
\begin{aligned}
&= (x_1 \cdot y_1 \otimes x_2 \cdot y_2) * (\beta_1(z_1) \otimes \beta_2(z_2)) \\
&= (x_1 \cdot y_1) * \beta_1(z_1) \otimes (x_2 \cdot y_2) \cdot \beta_2(z_2) + (x_1 \cdot y_1) \cdot \beta_1(z_1) \otimes (x_2 \cdot y_2) * \beta_2(z_2) \\
&\overset{(1.1)}{\underset{(1.11)}{=}} \alpha_1(x_1) \cdot (y_1 * z_1) \otimes \alpha_2(x_2) \cdot (y_2 \cdot z_2) + \alpha_1(x_1) \cdot (y_1 \cdot z_1) \otimes \alpha_2(x_2) \cdot (y_2 * z_2) \\
&= (\alpha_1(x_1) \otimes \alpha_2(x_2)) \cdot (y_1 * z_1 \otimes y_2 \cdot z_2 + y_1 \cdot z_1 \otimes y_2 * z_2) \\
&= (\alpha_1(x_1) \otimes \alpha_2(x_2)) \cdot ((y_1 \otimes y_2) * (z_1 \otimes z_2)) = \alpha(x) \cdot (y * z).
\end{aligned}
$$

From the above, it follows that $A := (A_1 \otimes A_2, \cdot, *, \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ is a BiHom-Novikov-Poisson algebra. □

By taking in Theorem 2.1 $\alpha_i = \beta_i = id_A$ for $i = 1, 2$, we recover the following result, which is Theorem 4.1 in [19].

Corollary 2.2 *Let $(A_i, \cdot_i, *_i)$ be Novikov-Poisson algebras for $i = 1, 2$. Then $A := (A_1 \otimes A_2, \cdot, *)$ is a Novikov-Poisson algebra defined as follows (for $x_i, y_i \in A_i$):*

$$
\begin{aligned}
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) &= (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2), \\
(x_1 \otimes x_2) * (y_1 \otimes y_2) &= (x_1 *_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 *_2 y_2).
\end{aligned}
$$

By taking in Theorem 2.1 $\alpha_i = \beta_i$ for $i = 1, 2$, we recover the Hom-version of our result, which is Theorem 3.1 in [22].

Corollary 2.3 *Let $(A_i, \cdot_i, *_i, \alpha_i)$ be Hom-Novikov-Poisson algebras for $i = 1, 2$. Then $A := (A_1 \otimes A_2, \cdot, *, \alpha_1 \otimes \alpha_2)$ is a Hom-Novikov-Poisson algebra defined as follows (for $x_i, y_i \in A_i$):*

$$
\begin{aligned}
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) &= (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2), \\
(x_1 \otimes x_2) * (y_1 \otimes y_2) &= (x_1 *_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 *_2 y_2).
\end{aligned}
$$

By using Theorem 2.1, we can construct new BiHom-Novikov-Poisson algebras. For example, the next result is obtained by first using Proposition 3.6 in [15] and then Theorem 2.1.

Corollary 2.4 *Let $(A_i, \cdot_i, *_i, \alpha_i, \beta_i)$ be BiHom-Novikov-Poisson algebras for $i = 1, 2$ and $A = A_1 \otimes A_2$. For integers $n, m \geq 0$, define the linear maps $\alpha, \beta : A \to A$ and $\cdot, * : A \otimes A \to A$ by:*

$$
\begin{aligned}
\alpha = \alpha_1^{n+1} \otimes \alpha_2^{m+1} \quad &, \quad \beta = \beta_1^{n+1} \otimes \beta_2^{m+1}, \\
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) &= (\alpha_1^n(x_1) \cdot_1 \beta_1^n(y_1)) \otimes (\alpha_2^m(x_2) \cdot_2 \beta_2^m(y_2)), \\
(x_1 \otimes x_2) * (y_1 \otimes y_2) &= (\alpha_1^n(x_1) *_1 \beta_1^n(y_1)) \otimes (\alpha_2^m(x_2) \cdot_2 \beta_2^m(y_2)) \\
&\quad + (\alpha_1^n(x_1) \cdot_1 \beta_1^n(y_1)) \otimes (\alpha_2^m(x_2) *_2 \beta_2^m(y_2)),
\end{aligned}
$$

for $x_i, y_i \in A_i$. Then $(A, \cdot, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra.

Proof. Indeed, by applying Proposition 3.6 in [15] for $\tilde{\alpha} := \alpha_1^n$ and $\tilde{\beta} := \beta_1^n$ and respectively for $\tilde{\alpha} := \alpha_2^m$ and $\tilde{\beta} := \beta_2^m$, we obtain that

$$
\begin{aligned}
A_1^n &:= (A_1, \cdot_1 \circ (\alpha_1^n \otimes \beta_1^n), *_1 \circ (\alpha_1^n \otimes \beta_1^n), \alpha_1^{n+1}, \beta_1^{n+1}), \\
A_2^m &:= (A_2, \cdot_2 \circ (\alpha_2^m \otimes \beta_2^m), *_2 \circ (\alpha_2^m \otimes \beta_2^m), \alpha_2^{m+1}, \beta_2^{m+1})
\end{aligned}
$$

are BiHom-Novikov-Poisson algebras. From Theorem 2.1, it follows that the tensor product $(A, \cdot, *, \alpha, \beta)$ of A_1^n and A_2^m is also a BiHom-Novikov-Poisson algebra, where $A = A_1 \otimes A_2$. □

3 Perturbations of BiHom-Novikov-Poisson algebras

The main aim of this section is to show that BiHom-Novikov-Poisson algebra structures are preserved under certain "perturbations". We begin with a result of independent interest.

Lemma 3.1 *Let (A, μ, α, β) be a BiHom-associative algebra and let $a \in A$ satisfying*

$$\alpha^2(a) = \beta^2(a) = a. \tag{3.1}$$

Define a new operation on A by

$$\diamond : A \otimes A \to A, \quad x \diamond y = \alpha(x)(\alpha(a)y), \tag{3.2}$$

for all $x, y \in A$. Then $A' = (A, \diamond, \alpha^2, \beta^2)$ is also a BiHom-associative algebra. If moreover (A, μ, α, β) is BiHom-commutative, then $A' = (A, \diamond, \alpha^2, \beta^2)$ is also BiHom-commutative.

Proof. The fact that α^2 and β^2 are multiplicative with respect to $\diamond$ follows by an easy computation (using (3.1)) which is left to the reader. We prove now that $(A, \diamond, \alpha^2, \beta^2)$ is BiHom-associative. For $x, y, z \in A$ we compute:

$$\begin{aligned}
(x \diamond y) \diamond \beta^2(z) &= (\alpha(x)(\alpha(a)y)) \diamond \beta^2(z) = (\alpha^2(x)(\alpha^2(a)\alpha(y)))(\alpha(a)\beta^2(z)) \\
&\overset{(1.1)}{=} ((\alpha(x)\alpha^2(a))\alpha\beta(y))(\alpha(a)\beta^2(z)) \\
&\overset{(3.1)}{=} ((\alpha(x)\alpha^2(a))\alpha\beta(y))(\alpha\beta^2(a)\beta^2(z)) \\
&= ((\alpha(x)\alpha^2(a))\alpha\beta(y))\beta(\alpha\beta(a)\beta(z)) \\
&\overset{(1.1),(3.1)}{=} (\alpha^2(x)\alpha(a))(\alpha\beta(y)(\alpha\beta(a)\beta(z))) = (\alpha^2(x)\alpha(a))\beta(\alpha(y)(\alpha(a)z)) \\
&\overset{(1.1)}{=} \alpha^3(x)(\alpha(a)(\alpha(y)(\alpha(a)z))) \\
&= \alpha^2(x) \diamond (\alpha(y)(\alpha(a)z)) = \alpha^2(x) \diamond (y \diamond z).
\end{aligned}$$

Now we assume that (A, μ, α, β) is BiHom-commutative and we prove that $(A, \diamond, \alpha^2, \beta^2)$ is also BiHom-commutative:

$$\begin{aligned}
\beta^2(x) \diamond \alpha^2(y) &= \alpha\beta^2(x)(\alpha(a)\alpha^2(y)) \overset{(1.1)}{=} (\beta^2(x)\alpha(a))\alpha^2\beta(y) \\
&\overset{(1.2)}{=} (\beta(a)\alpha\beta(x))\alpha^2\beta(y) = \beta(a\alpha(x))\alpha(\alpha\beta(y)) \\
&\overset{(1.2)}{=} \beta(\alpha\beta(y))\alpha(a\alpha(x)) = \alpha(\beta^2(y))(\alpha(a)\alpha^2(x)) = \beta^2(y) \diamond \alpha^2(x),
\end{aligned}$$

finishing the proof. □

Let us see now what Lemma 3.1 gives in the Hom-associative case. Let (A, μ, α) be a commutative Hom-associative algebra, i.e. $\alpha(x)(yz) = (xy)\alpha(z)$ and $xy = yx$, for all $x, y, z \in A$. Assume that α is multiplicative with respect to μ and let $a \in A$ such that $\alpha^2(a) = a$. Then obviously (A, μ, α, α) is also a BiHom-commutative algebra and we are in the hypotheses of Lemma 3.1; in this particular case of Lemma 3.1, the multiplication (3.2) may be rewritten as:

$$\begin{aligned}
x \diamond y &= \alpha(x)(\alpha(a)y) = (x\alpha(a))\alpha(y) \\
&= (\alpha(a)x)\alpha(y) = \alpha^2(a)(xy) = a(xy).
\end{aligned}$$

What we obtained, $x \diamond y = a(xy)$, is exactly formula (4.1.1) in [22]. Thus, Lemma 3.1 is indeed a generalization of the corresponding result (Lemma 4.1 in [22]) for the multiplicative commutative Hom-associative case. However, note that formula (4.1.1) in [22] gives a Hom-associative multiplication only in the commutative case; for the noncommutative case, it is clear that the proper formula is (3.2).

We can now use Lemma 3.1 to prove the main result of this section.

Theorem 3.2 *Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ with $\alpha^2(a) = \beta^2(a) = a$. Then $A' = (A, \diamond, *_{\alpha,\beta}, \alpha^2, \beta^2)$ is also a BiHom-Novikov-Poisson algebra, where*

$$x \diamond y = \alpha(x)(\alpha(a)y), \quad x *_{\alpha,\beta} y = \alpha(x) * \beta(y), \quad \forall\, x, y \in A.$$

Proof. By Lemma 3.1 we know that $(A, \diamond, \alpha^2, \beta^2)$ is a BiHom-commutative algebra. By Corollary 3.9 in [15] we get that $(A, *_{\alpha,\beta}, \alpha^2, \beta^2)$ is a BiHom-Novikov algebra (the $n = 1$ case). It remains to prove the compatibility conditions for A'. Let $x, y, z \in A$, we prove (1.10):

$$\begin{aligned}
&(x \diamond \beta^2(y)) *_{\alpha,\beta} \alpha^2\beta^2(z)\\
&\quad= (\alpha(x)(\alpha(a)\beta^2(y))) *_{\alpha,\beta} \alpha^2\beta^2(z) = \alpha(\alpha(x)(\alpha(a)\beta^2(y))) * \beta\alpha^2\beta^2(z)\\
&\quad\overset{(3.1)}{=} (\alpha^2(x)(a\alpha\beta^2(y))) * \alpha\beta(\alpha\beta^2(z)) \overset{(3.1)}{=} (\alpha^2(x)\,\beta(\beta(a)\,\alpha\beta(y))) * \alpha\beta(\alpha\beta^2(z))\\
&\quad\overset{(1.10)}{=} (\alpha^2(x) * \beta(\alpha\beta^2(z)))\alpha\beta(\beta(a)\,\alpha\beta(y)) = (\alpha^2(x) * \alpha\beta^3(z))\left(\alpha\beta^2(a)\,\alpha^2\beta^2(y)\right)\\
&\quad\overset{(3.1)}{=} \alpha\left(\alpha(x) * \beta^3(z)\right)\left(\alpha(a)\,\alpha^2\beta^2(y)\right) = \left(\alpha(x) * \beta^3(z)\right) \diamond \alpha^2\beta^2(y)\\
&\quad= (x *_{\alpha,\beta} \beta^2(z)) \diamond \alpha^2\beta^2(y).
\end{aligned}$$

To prove the compatibility condition (1.9) for A', we compute:

$$\begin{aligned}
&(\beta^2(x) *_{\alpha,\beta} \alpha^2(y)) \diamond \beta^2(z) - \alpha^2\beta^2(x) *_{\alpha,\beta} (\alpha^2(y) \diamond z)\\
&\quad= \left(\alpha\beta^2(x) * \beta\alpha^2(y)\right) \diamond \beta^2(z) - \alpha^3\beta^2(x) * \beta\left(\alpha^2(y) \diamond z\right)\\
&\quad= \alpha\left(\alpha\beta^2(x) * \beta\alpha^2(y)\right)\left(\alpha(a)\,\beta^2(z)\right) - \alpha^3\beta^2(x) * \beta\left(\alpha^3(y)\,(\alpha(a)\,z)\right)\\
&\quad= \left(\alpha^2\beta^2(x) * \alpha\beta\alpha^2(y)\right)\left(\alpha(a)\,\beta^2(z)\right) - \alpha^3\beta^2(x) * \left(\beta\alpha^3(y)\,(\beta\alpha(a)\,\beta(z))\right)\\
&\quad\overset{(3.1)}{=} (\beta(\alpha^2\beta(x)) * \alpha(\beta\alpha^2(y)))\beta(\alpha\beta(a)\,\beta(z))\\
&\qquad -\alpha\beta(\alpha^2\beta(x)) * (\alpha(\beta\alpha^2(y))(\alpha\beta(a)\,\beta(z)))\\
&\quad\overset{(1.9)}{=} (\beta(\beta\alpha^2(y)) * \alpha(\alpha^2\beta(x)))\beta(\alpha\beta(a)\,\beta(z))\\
&\qquad -\alpha\beta(\beta\alpha^2(y)) * (\alpha(\alpha^2\beta(x))(\alpha\beta(a)\,\beta(z)))\\
&\quad\overset{(3.1)}{=} \left(\alpha^2\beta^2(y) * \beta\alpha^3(x)\right)\left(\alpha(a)\,\beta^2(z)\right) - \alpha^3\beta^2(y) * \left(\alpha^3\beta(x)\,(\beta\alpha(a)\,\beta(z))\right)\\
&\quad= \alpha\left(\alpha\beta^2(y) * \beta\alpha^2(x)\right)\left(\alpha(a)\,\beta^2(z)\right) - \alpha^3\beta^2(y) * \beta\left(\alpha^3(x)\,(\alpha(a)\,z)\right)\\
&\quad= \left(\alpha\beta^2(y) * \beta\alpha^2(x)\right) \diamond \beta^2(z) - \alpha^3\beta^2(y) * \beta(\alpha^2(x) \diamond z)\\
&\quad= (\beta^2(y) *_{\alpha,\beta} \alpha^2(x)) \diamond \beta^2(z) - \alpha^2\beta^2(y) *_{\alpha,\beta} (\alpha^2(x) \diamond z).
\end{aligned}$$

Finally, we prove (1.11):

$$\begin{aligned}
\alpha^2(x) \diamond (y *_{\alpha,\beta} z) &= \alpha^2(x) \diamond (\alpha(y) * \beta(z)) = \alpha(\alpha^2(x))(\alpha(a)(\alpha(y) * \beta(z)))\\
&\overset{(1.1)}{=} (\alpha^2(x)\alpha(a))\beta(\alpha(y) * \beta(z)) = \alpha(\alpha(x)\,a)(\beta\alpha(y) * \beta^2(z))\\
&\overset{(1.11)}{=} ((\alpha(x)\,a)\beta\alpha(y)) * \beta^3(z) \overset{(1.11)}{=} (\alpha^2(x)\,(a\alpha(y))) * \beta^3(z)\\
&\overset{(3.1)}{=} (\alpha^2(x)(\alpha^2(a)\alpha(y))) * \beta^3(z) = \alpha(\alpha(x)(\alpha(a)y)) * \beta^3(z)
\end{aligned}$$

$$= \quad \alpha\,(x \diamond y) * \beta^3\,(z) = (x \diamond y) *_{\alpha,\beta} \beta^2(z),$$

finishing the proof. □

The next result is the Hom-version of Theorem 3.2 with $\alpha = \beta$, which is Theorem 4.2 in [22].

Corollary 3.3 *Let $(A, \mu, *, \alpha)$ be a multiplicative Hom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = a$. Then $A' = (A, \diamond, *_\alpha, \alpha^2)$ is also a multiplicative Hom-Novikov-Poisson algebra, where $x \diamond y = a(xy)$ and $x *_\alpha y = \alpha(x) * \alpha(y)$, for all $x, y \in A$.*

The next result is similar to Theorem 3.2; it uses, as in Theorem 3.2, a suitable element in a BiHom-Novikov-Poisson algebra, but this time to perturb the other multiplication.

Theorem 3.4 *Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ with $\alpha^2(a) = \beta^2(a) = a$. Then $\overline{A} = (A, \cdot_{\alpha,\beta}, \times, \alpha^2, \beta^2)$ is also a BiHom-Novikov-Poisson algebra, where*

$$x \cdot_{\alpha,\beta} y = \alpha(x)\beta(y), \quad x \times y = \alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y), \quad \forall\, x, y \in A.$$

Proof. By Corollary 3.8 in [15], we get that $(A, \cdot_{\alpha,\beta}, \alpha^2, \beta^2)$ is a BiHom-commutative algebra (the $n = 1$ case). We show that $(A, \times, \alpha^2, \beta^2)$ is a BiHom-Novikov algebra in Lemma 3.5 below. The relations (1.9), (1.10) and (1.11) for $\overline{A}$ are proved in Lemma 3.6 below. □

Lemma 3.5 *In the hypotheses of Theorem 3.4, $(A, \times, \alpha^2, \beta^2)$ is a BiHom-Novikov algebra.*

Proof. We prove first condition (1.6):

$$\begin{aligned}
\alpha^2(x \times y) &= \alpha^2(\alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y)) \overset{(3.1)}{=} \alpha^3(x) * \alpha^2\beta(y) + \alpha^3(x)(\alpha(a)\alpha^2(y)) \\
&= \alpha(\alpha^2(x)) * \beta(\alpha^2(y)) + \alpha(\alpha^2(x))(\alpha(a)\alpha^2(y)) = \alpha^2(x) \times \alpha^2(y).
\end{aligned}$$

Similarly one can prove that $\beta^2(x \times y) = \beta^2(x) \times \beta^2(y)$.

To check condition (1.8), we compute:

$$\begin{aligned}
(x \times \beta^2(y)) \times \alpha^2\beta^2(z) &= (\alpha(x) * \beta^3(y) + \alpha(x)(\alpha(a)\beta^2(y))) \times \alpha^2\beta^2(z) \\
&\overset{(3.1)}{=} \underbrace{(\alpha^2(x) * \alpha\beta^3(y)) * \alpha^2\beta^3(z)}_{u} + \underbrace{(\alpha^2(x)(a\alpha\beta^2(y))) * \alpha^2\beta^3(z)}_{v(x,y,z)} \\
&\quad + \underbrace{(\alpha^2(x) * \alpha\beta^3(y))(\alpha(a)\alpha^2\beta^2(z))}_{s(x,y,z)} \\
&\quad + \underbrace{(\alpha^2(x)(a\alpha\beta^2(y)))(\alpha(a)\alpha^2\beta^2(z))}_{t}.
\end{aligned}$$

The term $u = (\alpha^2(x) * \beta(\alpha\beta^2(y))) * \alpha\beta(\alpha\beta^2(z))$ is symmetric in y and z by (1.8) in $(A, *, \alpha, \beta)$.

Now we compute:

$$\begin{aligned}
v(x, y, z) &\overset{(3.1)}{=} (\alpha^2(x)(\beta^2(a)\alpha\beta^2(y))) * \alpha^2\beta^3(z) = (\alpha^2(x)\beta(\beta(a)\alpha\beta(y))) * \alpha\beta(\alpha\beta^2(z)) \\
&\overset{(1.10)}{=} (\alpha^2(x) * \alpha\beta^3(z))\alpha\beta(\beta(a)\alpha\beta(y)) \\
&\overset{(3.1)}{=} (\alpha^2(x) * \alpha\beta^3(z))(\alpha(a)\alpha^2\beta^2(y)) = s(x, z, y).
\end{aligned}$$

Now we prove that t is symmetric in y and z:

$$\begin{aligned} t &\overset{(3.1)}{=} (\alpha^2(x)(a\alpha\beta^2(y)))(\alpha\beta^2(a)\alpha^2\beta^2(z)) = (\alpha^2(x)(a\alpha\beta^2(y)))\beta(\alpha\beta(a)\alpha^2\beta(z)) \\ &\overset{(1.1)}{=} \alpha^3(x)((a\alpha\beta^2(y))(\alpha\beta(a)\alpha^2\beta(z))) \overset{(3.1)}{=} \alpha^3(x)((\beta^2(a)\alpha\beta^2(y))(\alpha\beta(a)\alpha^2\beta(z))) \\ &= \alpha^3(x)(\beta(\beta(a)\alpha\beta(y))\alpha(\beta(a)\alpha\beta(z))) \overset{(1.2)}{=} \alpha^3(x)(\beta(\beta(a)\alpha\beta(z))\alpha(\beta(a)\alpha\beta(y))) \\ &= \alpha^3(x)((\beta^2(a)\alpha\beta^2(z))(\alpha\beta(a)\alpha^2\beta(y))), \end{aligned}$$

and the last expression coincides with the one obtained three steps before it with y and z interchanged, proving that indeed t is symmetric in y and z.
From the above we get that $(x \times \beta^2(y)) \times \alpha^2\beta^2(z)$ is symmetric in y and z, proving (1.8).
Now we prove (1.7). We need to check that the following expression is symmetric in x and y:

$$\begin{aligned} &(\beta^2(x) \times \alpha^2(y)) \times \beta^2(z) - \alpha^2\beta^2(x) \times (\alpha^2(y) \times z) \\ &\quad= (\alpha\beta^2(x) * \alpha^2\beta(y)) \times \beta^2(z) + (\alpha\beta^2(x)(\alpha(a)\alpha^2(y))) \times \beta^2(z) \\ &\qquad -\alpha^2\beta^2(x) \times (\alpha^3(y) * \beta(z)) - \alpha^2\beta^2(x) \times (\alpha^3(y)(\alpha(a)z)) \\ &\quad\overset{(3.1)}{=} (\alpha^2\beta^2(x) * \alpha^3\beta(y)) * \beta^3(z) + (\alpha^2\beta^2(x) * \alpha^3\beta(y))(\alpha(a)\beta^2(z)) \\ &\qquad +(\alpha^2\beta^2(x)(a\alpha^3(y))) * \beta^3(z) + (\alpha^2\beta^2(x)(a\alpha^3(y)))(\alpha(a)\beta^2(z)) \\ &\qquad -\alpha^3\beta^2(x) * (\alpha^3\beta(y) * \beta^2(z)) - \alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y) * \beta(z))) \\ &\qquad -\alpha^3\beta^2(x) * (\alpha^3\beta(y)(\alpha\beta(a)\beta(z))) - \alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y)(\alpha(a)z))). \end{aligned}$$

We denote the eight terms in this expression respectively by t, v, u, w, t', u', v', w', so the expression reads

$$t + v + u + w - t' - u' - v' - w'.$$

We prove that $t - t'$ is symmetric in x and y and $w - w' = 0 = u - u'$. We compute:

$$\begin{aligned} t - t' &= (\alpha^2\beta^2(x) * \alpha^3\beta(y)) * \beta^3(z) - \alpha^3\beta^2(x) * (\alpha^3\beta(y) * \beta^2(z)) \\ &= (\beta(\alpha^2\beta(x)) * \alpha(\alpha^2\beta(y))) * \beta(\beta^2(z)) - \alpha\beta(\alpha^2\beta(x)) * (\alpha(\alpha^2\beta(y)) * \beta^2(z)) \\ &\overset{(1.7)}{=} (\beta(\alpha^2\beta(y)) * \alpha(\alpha^2\beta(x))) * \beta(\beta^2(z)) - \alpha\beta(\alpha^2\beta(y)) * (\alpha(\alpha^2\beta(x)) * \beta^2(z)) \\ &= (\alpha^2\beta^2(y) * \alpha^3\beta(x)) * \beta^3(z) - \alpha^3\beta^2(y) * (\alpha^3\beta(x) * \beta^2(z)). \end{aligned}$$

So clearly $t - t'$ is symmetric in x and y. Next we compute:

$$\begin{aligned} w - w' &= (\alpha^2\beta^2(x)(a\alpha^3(y)))(\alpha(a)\beta^2(z)) - \alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y)(\alpha(a)z))) \\ &\overset{(1.1)}{=} (\alpha^2\beta^2(x)(a\alpha^3(y)))(\alpha(a)\beta^2(z)) - \alpha^3\beta^2(x)((a\alpha^3(y))(\alpha\beta(a)\beta(z))) \\ &\overset{(1.1)}{=} (\alpha^2\beta^2(x)(a\alpha^3(y)))(\alpha(a)\beta^2(z)) - (\alpha^2\beta^2(x)(a\alpha^3(y)))(\alpha\beta^2(a)\beta^2(z))) \overset{(3.1)}{=} 0, \end{aligned}$$

$$\begin{aligned} u - u' &= (\alpha^2\beta^2(x)(a\alpha^3(y))) * \beta^3(z) - \alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y) * \beta(z))) \\ &\overset{(1.11)}{=} \alpha^3\beta^2(x)((a\alpha^3(y)) * \beta^2(z)) - \alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y) * \beta(z))) \\ &\overset{(1.11)}{=} \alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y) * \beta(z))) - \alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y) * \beta(z))) = 0. \end{aligned}$$

The last thing to prove is that $v - v'$ is symmetric in x and y, that is, we need to prove that

$$(\alpha^2\beta^2(x)*\alpha^3\beta(y))(\alpha(a)\beta^2(z))-\alpha^3\beta^2(x)*(\alpha^3\beta(y)(\alpha\beta(a)\beta(z)))$$
$$= (\alpha^2\beta^2(y)*\alpha^3\beta(x))(\alpha(a)\beta^2(z))-\alpha^3\beta^2(y)*(\alpha^3\beta(x)(\alpha\beta(a)\beta(z))).$$

We prove this identity as follows:

$$\begin{aligned}
&(\alpha^2\beta^2(x)*\alpha^3\beta(y))(\alpha(a)\beta^2(z))-(\alpha^2\beta^2(y)*\alpha^3\beta(x))(\alpha(a)\beta^2(z))\\
&\overset{(3.1)}{=} (\alpha^2\beta^2(x)*\alpha^3\beta(y))(\alpha\beta^2(a)\beta^2(z))-(\alpha^2\beta^2(y)*\alpha^3\beta(x))(\alpha\beta^2(a)\beta^2(z))\\
&= (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\alpha\beta(a)\beta(z))-(\beta(\alpha^2\beta(y))*\alpha(\alpha^2\beta(x)))\beta(\alpha\beta(a)\beta(z))\\
&\overset{(1.9)}{=} \alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))(\alpha\beta(a)\beta(z)))-\alpha\beta(\alpha^2\beta(y))*(\alpha(\alpha^2\beta(x))(\alpha\beta(a)\beta(z)))\\
&= \alpha^3\beta^2(x)*(\alpha^3\beta(y)(\alpha\beta(a)\beta(z)))-\alpha^3\beta^2(y)*(\alpha^3\beta(x)(\alpha\beta(a)\beta(z))).
\end{aligned}$$

From all the above, it follows that indeed the expression $(\beta^2(x)\times\alpha^2(y))\times\beta^2(z)-\alpha^2\beta^2(x)\times(\alpha^2(y)\times z)$ is symmetric in x and y, finishing the proof. □

Lemma 3.6 *In the hypotheses of Theorem 3.4, $\overline{A}$ satisfies the compatibility conditions (1.9), (1.10) and (1.11).*

Proof. To prove (1.10) for $\overline{A}$, we compute as follows:

$$\begin{aligned}
&(x\cdot_{\alpha,\beta}\beta^2(y))\times\alpha^2\beta^2(z)\\
&= (\alpha(x)\beta^3(y))\times\alpha^2\beta^2(z)\\
&= (\alpha^2(x)\alpha\beta^3(y))*\alpha^2\beta^3(z)+(\alpha^2(x)\alpha\beta^3(y))(\alpha(a)\alpha^2\beta^2(z))\\
&= (\alpha^2(x)\beta(\alpha\beta^2(y)))*\alpha\beta(\alpha\beta^2(z))+\alpha(\alpha(x)\beta^3(y))(\alpha(a)\alpha^2\beta^2(z))\\
&\overset{(1.1),(1.10)}{=} (\alpha^2(x)*\beta(\alpha\beta^2(z)))\alpha\beta(\alpha\beta^2(y))+((\alpha(x)\beta^3(y))\alpha(a))\alpha^2\beta^3(z)\\
&\overset{(3.1)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+((\alpha(x)\beta^3(y))\alpha\beta^2(a))\alpha^2\beta^3(z)\\
&\overset{(1.1)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+(\alpha^2(x)(\beta^3(y)\alpha\beta(a)))\alpha^2\beta^3(z)\\
&\overset{(1.2)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+(\alpha^2(x)(\beta^2(a)\alpha\beta^2(y)))\alpha^2\beta^3(z)\\
&\overset{(1.1),(3.1)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+((\alpha(x)a)\alpha\beta^3(y))\alpha^2\beta^3(z)\\
&\overset{(1.1)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+(\alpha^2(x)\alpha(a))(\alpha\beta^3(y)\alpha^2\beta^2(z))\\
&\overset{(1.2)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+(\alpha^2(x)\alpha(a))(\alpha\beta^3(z)\alpha^2\beta^2(y))\\
&\overset{(1.1)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+((\alpha(x)a)\alpha\beta^3(z))\alpha^2\beta^3(y)\\
&\overset{(1.1)}{=} (\alpha^2(x)*\alpha\beta^3(z))\alpha^2\beta^3(y)+(\alpha^2(x)(a\alpha\beta^2(z)))\alpha^2\beta^3(y)\\
&\overset{(3.1)}{=} (\alpha(x)*\beta^3(z)+\alpha(x)(\alpha(a)\beta^2(z)))\cdot_{\alpha,\beta}\alpha^2\beta^2(y)=(x\times\beta^2(z))\cdot_{\alpha,\beta}\alpha^2\beta^2(y).
\end{aligned}$$

To prove (1.9) for $\overline{A}$, we compute:

$$\begin{aligned}
&(\beta^2(x)\times\alpha^2(y))\cdot_{\alpha,\beta}\beta^2(z)-\alpha^2\beta^2(x)\times(\alpha^2(y)\cdot_{\alpha,\beta}z)\\
&= (\alpha\beta^2(x)*\alpha^2\beta(y)+\alpha\beta^2(x)(\alpha(a)\alpha^2(y)))\cdot_{\alpha,\beta}\beta^2(z)\\
&\quad -\alpha^2\beta^2(x)\times(\alpha^3(y)\beta(z))
\end{aligned}$$

$$\begin{aligned}
&\overset{(3.1)}{=} && (\alpha^2\beta^2(x)*\alpha^3\beta(y))\beta^3(z)+(\alpha^2\beta^2(x)(a\alpha^3(y)))\beta^3(z)\\
& && -\alpha^3\beta^2(x)*(\alpha^3\beta(y)\beta^2(z))-\alpha^3\beta^2(x)(\alpha(a)(\alpha^3(y)\beta(z)))\\
&\overset{(1.1),(3.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +(\alpha^2\beta^2(x)(\beta^2(a)\alpha^3(y)))\beta^3(z)-\alpha^3\beta^2(x)((a\alpha^3(y))\beta^2(z))\\
&\overset{(1.2),(3.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +(\alpha^2\beta^2(x)(\alpha^2\beta(y)\alpha\beta(a)))\beta^3(z)-\alpha^3\beta^2(x)((\beta^2(a)\alpha^3(y))\beta^2(z))\\
&\overset{(1.2),(3.1),(1.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +((\alpha\beta^2(x)\alpha^2\beta(y))\alpha(a))\beta^3(z)-\alpha^3\beta^2(x)((\alpha^2\beta(y)\alpha\beta(a))\beta^2(z))\\
&\overset{(1.2),(1.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +((\alpha\beta^2(y)\alpha^2\beta(x))\alpha(a))\beta^3(z)-\alpha^3\beta^2(x)(\alpha^3\beta(y)(\alpha\beta(a)\beta(z)))\\
&\overset{(3.1),(1.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +((\alpha\beta^2(y)\alpha^2\beta(x))\alpha(a))\beta^3(z)-(\alpha^2\beta^2(x)\alpha^3\beta(y))(\alpha(a)\beta^2(z))\\
&\overset{(3.1),(1.2)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +((\alpha\beta^2(y)\alpha^2\beta(x))\alpha\beta^2(a))\beta^3(z)-(\alpha^2\beta^2(y)\alpha^3\beta(x))(\alpha(a)\beta^2(z))\\
&\overset{(3.1),(1.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +(\alpha^2\beta^2(y)(\alpha^2\beta(x)\alpha\beta(a)))\beta^3(z)-(\alpha^2\beta^2(y)\alpha^3\beta(x))(\alpha\beta^2(a)\beta^2(z))\\
&\overset{(1.2),(1.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +(\alpha^2\beta^2(y)(\beta^2(a)\alpha^3(x)))\beta^3(z)-\alpha^3\beta^2(y)(\alpha^3\beta(x)(\alpha\beta(a)\beta(z)))\\
&\overset{(3.1),(1.1)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +(\alpha^2\beta^2(y)(a\alpha^3(x)))\beta^3(z)-\alpha^3\beta^2(y)((\alpha^2\beta(x)\alpha\beta(a))\beta^2(z))\\
&\overset{(3.1),(1.2)}{=} && (\beta(\alpha^2\beta(x))*\alpha(\alpha^2\beta(y)))\beta(\beta^2(z))-\alpha\beta(\alpha^2\beta(x))*(\alpha(\alpha^2\beta(y))\beta^2(z))\\
& && +(\alpha^2\beta^2(y)(a\alpha^3(x)))\beta^3(z)-\alpha^3\beta^2(y)((a\alpha^3(x))\beta^2(z))\\
&\overset{(1.9),(1.1)}{=} && (\alpha^2\beta^2(y)*\alpha^3\beta(x))\beta^3(z)-\alpha^3\beta^2(y)*(\alpha^3\beta(x)\beta^2(z))\\
& && +(\alpha^2\beta^2(y)(a\alpha^3(x)))\beta^3(z)-\alpha^3\beta^2(y)(\alpha(a)(\alpha^3(x)\beta(z)))\\
&= && (\alpha^2\beta^2(y)*\alpha^3\beta(x)+\alpha^2\beta^2(y)(a\alpha^3(x)))\beta^3(z)\\
& && -\alpha^3\beta^2(y)*(\alpha^3\beta(x)\beta^2(z))-\alpha^3\beta^2(y)(\alpha(a)(\alpha^3(x)\beta(z)))\\
&\overset{(3.1)}{=} && (\alpha\beta^2(y)*\alpha^2\beta(x)+\alpha\beta^2(y)(\alpha(a)\alpha^2(x)))\cdot_{\alpha,\beta}\beta^2(z)\\
& && -\alpha(\alpha^2\beta^2(y))*\beta(\alpha^3(x)\beta(z))-\alpha(\alpha^2\beta^2(y))(\alpha(a)(\alpha^3(x)\beta(z)))\\
&= && (\beta^2(y)\times\alpha^2(x))\cdot_{\alpha,\beta}\beta^2(z)-\alpha^2\beta^2(y)\times(\alpha^3(x)\beta(z))\\
&= && (\beta^2(y)\times\alpha^2(x))\cdot_{\alpha,\beta}\beta^2(z)-\alpha^2\beta^2(y)\times(\alpha^2(x)\cdot_{\alpha,\beta}z).
\end{aligned}$$

Finally, to prove (1.11) we compute:

$$\begin{aligned}
\alpha^2(x)\cdot_{\alpha,\beta}(y\times z) &= \alpha^2(x)\cdot_{\alpha,\beta}(\alpha(y)*\beta(z)+\alpha(y)(\alpha(a)z))\\
&= \alpha^3(x)(\alpha\beta(y)*\beta^2(z))+\alpha^3(x)(\alpha\beta(y)(\alpha\beta(a)\beta(z)))\\
&\overset{(3.1),(1.1),(1.11)}{=} (\alpha^2(x)\alpha\beta(y))*\beta^3(z)+(\alpha^2(x)\alpha\beta(y))(\alpha(a)\beta^2(z))\\
&= \alpha(\alpha(x)\beta(y))*\beta(\beta^2(z))+\alpha(\alpha(x)\beta(y))(\alpha(a)\beta^2(z))
\end{aligned}$$

$$= \qquad (\alpha(x)\beta(y)) \times \beta^2(z) = (x \cdot_{\alpha,\beta} y) \times \beta^2(z),$$

finishing the proof. □

The next result is the Hom-version of Theorem 3.4 with $\alpha = \beta$, which is Theorem 4.4 in [22].

Corollary 3.7 *Let $(A, \mu, *, \alpha)$ be a multiplicative Hom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = a$. Then $\overline{A} = (A, \cdot_\alpha, \times, \alpha^2)$ is also a multiplicative Hom-Novikov-Poisson algebra, where $x \cdot_\alpha y = \alpha(x)\alpha(y)$ and $x \times y = \alpha(x) * \alpha(y) + a(xy)$, for all $x, y \in A$.*

Forgetting about the BiHom-associative product $\cdot_{\alpha,\beta}$ in Theorem 3.4, we obtain a non-trivial way to construct a BiHom-Novikov algebra from a BiHom-Novikov-Poisson algebra:

Corollary 3.8 *Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = \beta^2(a) = a$. Then $(A, \times, \alpha^2, \beta^2)$ is a BiHom-Novikov algebra, where $x \times y = \alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y)$, for all $x, y \in A$.*

The following perturbation result is obtained by combining Theorem 3.2 and Theorem 3.4.

Corollary 3.9 *Let $(A, \mu, *, \alpha, \beta)$ be a BiHom-Novikov-Poisson algebra and $a, b \in A$ elements such that $\alpha^2(a) = \beta^2(a) = a$ and $\alpha^4(b) = \beta^4(b) = b$. Then $\widetilde{A} = (A, \diamond, \boxtimes, \alpha^4, \beta^4)$ is also a BiHom-Novikov-Poisson algebra, where*

$$\begin{aligned} x \diamond y &= \alpha^3(x)(\alpha^3\beta(b)\beta^2(y)), \quad \forall\, x, y \in A, \\ x \boxtimes y &= \alpha^3(x) * \beta^3(y) + \alpha^3(x)(\alpha(a)\beta^2(y)), \quad \forall\, x, y \in A. \end{aligned}$$

Proof. By Theorem 3.4, we get that $\overline{A} = (A, \cdot_{\alpha,\beta}, \times, \alpha^2, \beta^2)$ is a BiHom-Novikov-Poisson algebra. Now apply Theorem 3.2 to $\overline{A}$ and the element $b \in A$, which satisfies $(\alpha^2)^2(b) = (\beta^2)^2(b) = b$. We obtain a BiHom-Novikov-Poisson algebra $(\overline{A})'$, which is $\widetilde{A}$ above. □

The following result is a special case of Corollary 3.9.

Corollary 3.10 *Let (A, μ) be a commutative and associative algebra, $\alpha, \beta : A \to A$ two commuting algebra morphisms, and $D : A \to A$ a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. Let $a, b \in A$ be elements such that $\alpha^2(a) = \beta^2(a) = a$ and $\alpha^4(b) = \beta^4(b) = b$. Then $(A, \lozenge, \boxdot, \alpha^4, \beta^4)$ is a BiHom-Novikov-Poisson algebra, where*

$$\begin{aligned} x \lozenge y &= \alpha^4(x)\beta^2(b)\beta^4(y), \quad \forall\, x, y \in A, \\ x \boxdot y &= \alpha^4(x)D(\beta^4(y)) + \alpha^4(x)\beta(a)\beta^4(y), \quad \forall\, x, y \in A. \end{aligned}$$

Proof. By Corollary 3.10 in [15], we get that $A_{\alpha,\beta} = (A, \bullet, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra, where $x \bullet y = \alpha(x)\beta(y)$ and $x * y = \alpha(x)D(\beta(y))$. Now apply Corollary 3.9 to $A_{\alpha,\beta}$ and the elements a and b. The outcome is the BiHom-Novikov-Poisson algebra $\widetilde{A_{\alpha,\beta}}$, which is exactly $(A, \lozenge, \boxdot, \alpha^4, \beta^4)$. □

4 From BiHom-Novikov-Poisson algebras to BiHom-Poisson algebras

The purpose of this section is to investigate a special class of BiHom-Novikov-Poisson algebras, called left BiHom-associative, having the property that, in case of bijective structure maps, they provide BiHom-Poisson algebras.

Definition 4.1 *([16]) A BiHom-Lie algebra* $(L, [\cdot,\cdot], \alpha, \beta)$ *is a 4-tuple in which* L *is a linear space,* $\alpha, \beta : L \to L$ *are linear maps and* $[\cdot,\cdot] : L \times L \to L$ *is a bilinear map, such that*

$$\alpha \circ \beta = \beta \circ \alpha, \tag{4.1}$$

$$\alpha([x,y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta([x,y]) = [\beta(x), \beta(y)], \tag{4.2}$$

$$[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \qquad \text{(BiHom-skew-symmetry)} \tag{4.3}$$

$$\left[\beta^2(x), [\beta(y), \alpha(z)]\right] + \left[\beta^2(y), [\beta(z), \alpha(x)]\right] + \left[\beta^2(z), [\beta(x), \alpha(y)]\right] = 0, \tag{4.4}$$

(BiHom-Jacobi condition)

for all $x, y, z \in L$*. The maps* α *and* β *(in this order) are called the structure maps of* L*.*

Definition 4.2 *A BiHom-Poisson algebra is a 5-tuple* $(A, \mu, [\cdot,\cdot], \alpha, \beta)$*, with the property that*

(1) (A, μ, α, β) *is a BiHom-commutative algebra;*

(2) $(A, [\cdot,\cdot], \alpha, \beta)$ *is a BiHom-Lie algebra;*

(3) the following BiHom-Leibniz identity holds for all $x, y, z \in A$*:*

$$[\alpha\beta(x), yz] = [\beta(x), y]\,\beta(z) + \beta(y)\,[\alpha(x), z]. \tag{4.5}$$

BiHom-Poisson algebras are defined as a BiHom-type generalization of Poisson algebras (the case $\alpha = \beta = id_A$) and Hom-Poisson algebras (the case $\alpha = \beta$, see [18] and [22]).

Let us mention that (4.5) is actually a left-handed BiHom-Leibniz identity, so the concept introduced in Definition 4.2 is actually a sort of left-handed BiHom-Poisson algebra, as compared to the right-handed versions of both that are introduced in [1].

We are interested now in the following question: under what conditions one can construct a BiHom-Poisson algebra from a BiHom-Novikov-Poisson algebra by taking the commutator bracket of the BiHom-Novikov product (which automatically requires bijectivity of the structure maps)? We begin by introducing the following definitions.

Definition 4.3 *Let* $(A, \mu, *, \alpha, \beta)$ *be a BiHom-Novikov-Poisson algebra. Then* A *is called left BiHom-associative if the following condition holds for all* $x, y, z \in A$*:*

$$\alpha(x) * (yz) = (xy) * \beta(z). \tag{4.6}$$

By using (1.11), it is easy to see that (4.6) holds if and only if

$$\alpha(x)(y * z) = \alpha(x) * (yz), \quad \forall\, x, y, z \in A. \tag{4.7}$$

Definition 4.4 *Let* $(A, \mu, *, \alpha, \beta)$ *be a BiHom-Novikov-Poisson algebra with* α, β *bijective. Then* A *is called admissible if* $A^- := (A, \mu, [\cdot,\cdot], \alpha, \beta)$ *is a BiHom-Poisson algebra, where*

$$[x, y] = x * y - \alpha^{-1}\beta(y) * \alpha\beta^{-1}(x), \quad \forall\, x, y \in A.$$

We now give a necessary and sufficient condition under which a BiHom-Novikov-Poisson algebra is admissible. The Hom-type version of this result may be found in [22].

Theorem 4.5 *Let* $(A, \mu, *, \alpha, \beta)$ *be a BiHom-Novikov-Poisson algebra with* α, β *bijective. Then* A *is admissible if and only if it is left BiHom-associative.*

Proof. We know from the definition that (A, μ, α, β) is a BiHom-commutative algebra, while the fact that $(A, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie algebra follows from [16], Proposition 2.4.
The left-hand side of the BiHom-Leibniz identity (4.5) for A^- is:

$$\begin{aligned}
[\alpha\beta(x), yz] &= \alpha\beta(x) * (yz) - \alpha^{-1}\beta(yz) * \alpha^2(x) \\
&= \alpha\beta(x) * (\alpha(\alpha^{-1}(y))z) - (\alpha^{-1}\beta(y)\beta(\alpha^{-1}(z))) * \alpha\beta(\alpha\beta^{-1}(x)) \\
&\overset{(1.9),(1.10)}{=} (\beta(x) * y)\beta(z) - (\alpha^{-1}\beta(y) * \alpha(x))\beta(z) \\
&\quad +\beta(y) * (\alpha(x)z) - \beta(y)(\alpha^{-1}\beta(z) * \alpha^2\beta^{-1}(x)).
\end{aligned}$$

The right-hand side of (4.5) for A^- is:

$$\begin{aligned}
[\beta(x), y]\,\beta(z) + \beta(y)\,[\alpha(x), z] &= (\beta(x) * y)\beta(z) - (\alpha^{-1}\beta(y) * \alpha(x))\beta(z) \\
&\quad +\beta(y)(\alpha(x) * z) - \beta(y)(\alpha^{-1}\beta(z) * \alpha^2\beta^{-1}(x)).
\end{aligned}$$

It follows that A^- satisfies (4.5) if and only if $\beta(y) * (\alpha(x)z) = \beta(y)(\alpha(x) * z)$. Since

$$\beta(y)(\alpha(x) * z) = \alpha(\alpha^{-1}\beta(y))(\beta(\alpha\beta^{-1}(x)) * z) \overset{(1.11)}{=} (\alpha^{-1}\beta(y)\alpha(x)) * \beta(z)$$

and $x, y, z \in A$ are arbitrary, we obtain that A^- is a BiHom-Poisson algebra if and only if $\alpha(y) * (xz) = (yx) * \beta(z)$ holds, that is, if and only if A is left BiHom-associative. □

Example 4.6 *Let (A, μ) be a commutative and associative algebra, $\alpha, \beta : A \to A$ two commuting algebra morphisms, and $D : A \to A$ a derivation such that $D \circ \alpha = \alpha \circ D$ and $D \circ \beta = \beta \circ D$. By Corollary 3.10 in [15], $A_{\alpha,\beta} = (A, \bullet, *, \alpha, \beta)$ is a BiHom-Novikov-Poisson algebra, where $x \bullet y = \alpha(x)\beta(y)$ and $x * y = \alpha(x)D(\beta(y))$, for all $x, y \in A$. Then $A_{\alpha,\beta}$ is left BiHom-associative if and only if $\alpha^2(x)D(\alpha\beta(y))\beta^2(z) = 0$, for all $x, y, z \in A$.*

Indeed, $A_{\alpha,\beta}$ is left BiHom-associative if and only if (4.7) holds for $A_{\alpha,\beta}$, i.e. if and only if

$$\begin{aligned}
0 &= \alpha(x) * (y \bullet z) - \alpha(x) \bullet (y * z) \\
&= \alpha(x) * (\alpha(y)\beta(z)) - \alpha(x) \bullet (\alpha(y)D(\beta(z))) \\
&= \alpha^2(x)D(\alpha\beta(y)\beta^2(z)) - \alpha^2(x)\alpha\beta(y)D(\beta^2(z)) \\
&= \alpha^2(x)D(\alpha\beta(y))\beta^2(z) + \alpha^2(x)\alpha\beta(y)D(\beta^2(z)) - \alpha^2(x)\alpha\beta(y)D(\beta^2(z)) \\
&= \alpha^2(x)D(\alpha\beta(y))\beta^2(z).
\end{aligned}$$

Now we prove that left BiHom-associativity is preserved by Yau twisting.

Proposition 4.7 *Let $(A, \mu, *, \alpha, \beta)$ be a left BiHom-associative BiHom-Novikov-Poisson algebra and $\tilde{\alpha}, \tilde{\beta} : A \to A$ two morphisms of BiHom-Novikov-Poisson algebras such that any two of the maps $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ commute. Then $A_{(\tilde{\alpha},\tilde{\beta})} := (A, \tilde{\cdot} := \mu \circ (\tilde{\alpha} \otimes \tilde{\beta}), \tilde{*} := * \circ (\tilde{\alpha} \otimes \tilde{\beta}), \alpha \circ \tilde{\alpha}, \beta \circ \tilde{\beta})$ is also a left BiHom-associative BiHom-Novikov-Poisson algebra.*

Proof. From [15], we know that $A_{(\tilde{\alpha},\tilde{\beta})}$ is a BiHom-Novikov-Poisson algebra. For all $x, y, z \in A$,

$$\begin{aligned}
&(x \tilde{\cdot} y) \tilde{*} \beta\tilde{\beta}(z) - \alpha\tilde{\alpha}(x) \tilde{*} (y \tilde{\cdot} z) \\
&= (\tilde{\alpha}(x)\tilde{\beta}(y)) \tilde{*} \beta\tilde{\beta}(z) - \alpha\tilde{\alpha}^2(x) * \tilde{\beta}(y \tilde{\cdot} z) \\
&= (\tilde{\alpha}^2(x)\tilde{\alpha}\tilde{\beta}(y)) * \beta\tilde{\beta}^2(z) - \alpha\tilde{\alpha}^2(x) * (\tilde{\alpha}\tilde{\beta}(y)\tilde{\beta}^2(z)) \\
&= (\tilde{\alpha}^2(x)\tilde{\alpha}\tilde{\beta}(y)) * \beta(\tilde{\beta}^2(z)) - \alpha(\tilde{\alpha}^2(x)) * (\tilde{\alpha}\tilde{\beta}(y)\tilde{\beta}^2(z)) \overset{(4.6)}{=} 0,
\end{aligned}$$

thus proving (4.6) for $A_{(\tilde{\alpha},\tilde{\beta})}$ and finishing the proof. □

In the context of Proposition 4.7 and assuming moreover that $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ are bijective, the BiHom-Lie bracket in the BiHom-Poisson algebra $(A_{(\tilde{\alpha},\tilde{\beta})})^-$ is given by $\widetilde{[\cdot,\cdot]} = [\cdot,\cdot] \circ (\widetilde{\alpha} \otimes \widetilde{\beta})$, where $[\cdot,\cdot]$ is the BiHom-Lie bracket in the BiHom-Poisson algebra A^-.

The next result is a special case of Proposition 4.7.

Corollary 4.8 *Let $(A, \mu, *, \alpha, \beta)$ be a left BiHom-associative BiHom-Novikov-Poisson algebra. Then so is $A^n := (A, \mu \circ (\alpha^n \otimes \beta^n), * \circ (\alpha^n \otimes \beta^n), \alpha^{n+1}, \beta^{n+1})$, for each $n \geq 0$.*

Proof. Apply Proposition 4.7 for $\tilde{\alpha} := \alpha^n$ and $\tilde{\beta} := \beta^n$. □

In the rest of this section, we show that left BiHom-associativity is compatible with the constructions in the previous sections. First we deal with tensor products.

Proposition 4.9 *Let $(A_i, \cdot_i, *_i, \alpha_i, \beta_i)$ be a left BiHom-associative BiHom-Novikov-Poisson algebra, for $i = 1, 2$, and let $A = A_1 \otimes A_2$ be the BiHom-Novikov-Poisson algebra constructed in Theorem 2.1. Then A is left BiHom-associative.*

Proof. Pick $x = x_1 \otimes x_2, y = y_1 \otimes y_2$ and $z = z_1 \otimes z_2$ in A. Then we have:

$$
\begin{aligned}
&[(x_1 \otimes x_2) \cdot (y_1 \otimes y_2)] * (\beta_1(z_1) \otimes \beta_2(z_2)) \\
&\quad = [(x_1 \cdot y_1) \otimes (x_2 \cdot y_2)] * (\beta_1(z_1) \otimes \beta_2(z_2)) \\
&\quad = ((x_1 \cdot y_1) * \beta_1(z_1)) \otimes ((x_2 \cdot y_2) \cdot \beta_2(z_2)) + ((x_1 \cdot y_1) \cdot \beta_1(z_1)) \otimes ((x_2 \cdot y_2) * \beta_2(z_2)) \\
&\quad \overset{(1.1),(4.6)}{=} (\alpha_1(x_1) * (y_1 \cdot z_1)) \otimes (\alpha_2(x_2) \cdot (y_2 \cdot z_2)) + (\alpha_1(x_1) \cdot (y_1 \cdot z_1)) \otimes (\alpha_2(x_2) * (y_2 \cdot z_2)) \\
&\quad = (\alpha_1(x_1) \otimes \alpha_2(x_2)) * [(y_1 \otimes y_2) \cdot (z_1 \otimes z_2)],
\end{aligned}
$$

thus proving (4.6) for A and finishing the proof. □

In the context of Proposition 4.9 and assuming moreover that all structure maps are bijective, the BiHom-Lie bracket in the BiHom-Poisson algebra A^- is given by

$$
\begin{aligned}
&[x_1 \otimes x_2, y_1 \otimes y_2] \\
&\quad = (x_1 \otimes x_2) * (y_1 \otimes y_2) - (\alpha_1^{-1}\beta_1(y_1) \otimes \alpha_2^{-1}\beta_2(y_2)) * (\alpha_1\beta_1^{-1}(x_1) \otimes \alpha_2\beta_2^{-1}(x_2)) \\
&\quad = (x_1 * y_1) \otimes (x_2 \cdot y_2) + (x_1 \cdot y_1) \otimes (x_2 * y_2) \\
&\qquad -(\beta_1(\alpha_1^{-1}(y_1)) \cdot \alpha_1(\beta_1^{-1}(x_1))) \otimes (\alpha_2^{-1}\beta_2(y_2) * \alpha_2\beta_2^{-1}(x_2)) \\
&\qquad -(\alpha_1^{-1}\beta_1(y_1) * \alpha_1\beta_1^{-1}(x_1)) \otimes (\beta_2(\alpha_2^{-1}(y_2)) \cdot \alpha_2(\beta_2^{-1}(x_2))) \\
&\quad \overset{(1.2)}{=} (x_1 * y_1) \otimes (x_2 \cdot y_2) + (x_1 \cdot y_1) \otimes (x_2 * y_2) \\
&\qquad -(x_1 \cdot y_1) \otimes (\alpha_2^{-1}\beta_2(y_2) * \alpha_2\beta_2^{-1}(x_2)) - (\alpha_1^{-1}\beta_1(y_1) * \alpha_1\beta_1^{-1}(x_1)) \otimes (x_2 \cdot y_2) \\
&\quad = (x_1 * y_1 - \alpha_1^{-1}\beta_1(y_1) * \alpha_1\beta_1^{-1}(x_1)) \otimes (x_2 \cdot y_2) \\
&\qquad +(x_1 \cdot y_1) \otimes (x_2 * y_2 - \alpha_2^{-1}\beta_2(y_2) * \alpha_2\beta_2^{-1}(x_2)) \\
&\quad = [x_1, y_1] \otimes (x_2 \cdot y_2) + (x_1 \cdot y_1) \otimes [x_2, y_2].
\end{aligned}
$$

Here $[x_i, y_i]$ is the BiHom-Lie bracket in the BiHom-Poisson algebra A_i^-.

Next we prove that left BiHom-associativity is preserved by perturbations as in Theorem 3.2.

Proposition 4.10 *Let $(A, \mu, *, \alpha, \beta)$ be a left BiHom-associative BiHom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = \beta^2(a) = a$. Then the BiHom-Novikov-Poisson algebra $A' = (A, \diamond, *_{\alpha,\beta}, \alpha^2, \beta^2)$ constructed in Theorem 3.2 is also left BiHom-associative.*

Proof. We need to prove (4.6) for A'. We compute:

$$\begin{aligned}
&\alpha^2(x) *_{\alpha,\beta} (y \diamond z) \\
&\quad= \alpha^2(x) *_{\alpha,\beta} (\alpha(y)(\alpha(a)z)) = \alpha^3(x) * (\alpha\beta(y)(\alpha\beta(a)\beta(z))) \\
&\quad\overset{(4.6)}{=} (\alpha^2(x)\alpha\beta(y)) * \beta(\alpha\beta(a)\beta(z)) \overset{(3.1)}{=} (\alpha^2(x)\alpha\beta(y)) * (\alpha(a)\beta^2(z)) \\
&\quad= \alpha(\alpha(x)\beta(y)) * (\alpha(a)\beta^2(z)) \overset{(4.6)}{=} ((\alpha(x)\beta(y))\alpha(a)) * \beta^3(z) \\
&\quad\overset{(3.1)}{=} ((\alpha(x)\beta(y))\alpha\beta^2(a)) * \beta^3(z) \overset{(1.1)}{=} (\alpha^2(x)(\beta(y)\alpha\beta(a))) * \beta^3(z) \\
&\quad\overset{(1.2)}{=} (\alpha^2(x)(\beta^2(a)\alpha(y))) * \beta^3(z) \overset{(3.1)}{=} (\alpha^2(x)(\alpha^2(a)\alpha(y))) * \beta^3(z) \\
&\quad= \alpha(\alpha(x)(\alpha(a)y)) * \beta^3(z) = \alpha(x \diamond y) * \beta^3(z) = (x \diamond y) *_{\alpha,\beta} \beta^2(z),
\end{aligned}$$

finishing the proof. □

In the context of Proposition 4.10 and assuming moreover that α and β are bijective, the BiHom-Lie bracket in the BiHom-Poisson algebra $(A')^-$ is given by

$$\begin{aligned}
x *_{\alpha,\beta} y - \alpha^{-2}\beta^2(y) *_{\alpha,\beta} \alpha^2\beta^{-2}(x) &= \alpha(x) * \beta(y) - \alpha^{-1}\beta^2(y) * \alpha^2\beta^{-1}(x) \\
&= \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) \\
&= [\alpha(x), \beta(y)] = [\cdot, \cdot] \circ (\alpha \otimes \beta)(x \otimes y),
\end{aligned}$$

where $[\cdot, \cdot]$ is the BiHom-Lie bracket in the BiHom-Poisson algebra A^-.

Finally, we prove a similar result for perturbations as in Theorem 3.4.

Proposition 4.11 *Let $(A, \mu, *, \alpha, \beta)$ be a left BiHom-associative BiHom-Novikov-Poisson algebra and $a \in A$ an element satisfying $\alpha^2(a) = \beta^2(a) = a$. Then the BiHom-Novikov-Poisson algebra $\overline{A} = (A, \cdot_{\alpha,\beta}, \times, \alpha^2, \beta^2)$ constructed in Theorem 3.4 is also left BiHom-associative.*

Proof. We need to prove (4.6) for $\overline{A}$. We compute:

$$\begin{aligned}
\alpha^2(x) \times (y \cdot_{\alpha,\beta} z) &= \alpha^2(x) \times (\alpha(y)\beta(z)) \\
&= \alpha^3(x) * (\alpha\beta(y)\beta^2(z)) + \alpha^3(x)(\alpha(a)(\alpha(y)\beta(z))) \\
&\overset{(4.6),(1.1)}{=} (\alpha^2(x)\alpha\beta(y)) * \beta^3(z) + \alpha^3(x)((a\alpha(y))\beta^2(z)) \\
&\overset{(3.1)}{=} (\alpha^2(x)\alpha\beta(y)) * \beta^3(z) + \alpha^3(x)((\beta^2(a)\alpha(y))\beta^2(z)) \\
&\overset{(1.2)}{=} (\alpha^2(x)\alpha\beta(y)) * \beta^3(z) + \alpha^3(x)((\beta(y)\alpha\beta(a))\beta^2(z)) \\
&\overset{(1.1)}{=} (\alpha^2(x)\alpha\beta(y)) * \beta^3(z) + \alpha^3(x)(\alpha\beta(y)(\alpha\beta(a)\beta(z))) \\
&\overset{(3.1),(1.1)}{=} (\alpha^2(x)\alpha\beta(y)) * \beta^3(z) + (\alpha^2(x)\alpha\beta(y))(\alpha(a)\beta^2(z)) \\
&= \alpha(\alpha(x)\beta(y)) * \beta^3(z) + \alpha(\alpha(x)\beta(y))(\alpha(a)\beta^2(z)) \\
&= (\alpha(x)\beta(y)) \times \beta^2(z) = (x \cdot_{\alpha,\beta} y) \times \beta^2(z),
\end{aligned}$$

finishing the proof. □

In the context of Proposition 4.11 and assuming moreover that α and β are bijective, the BiHom-Lie bracket in the BiHom-Poisson algebra $(\overline{A})^-$ is given by

$$
\begin{aligned}
& x \times y - \alpha^{-2}\beta^2(y) \times \alpha^2\beta^{-2}(x) \\
& = \alpha(x) * \beta(y) + \alpha(x)(\alpha(a)y) - \alpha^{-1}\beta^2(y) * \alpha^2\beta^{-1}(x) - \alpha^{-1}\beta^2(y)(\alpha(a)\alpha^2\beta^{-2}(x)) \\
& = \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) + \alpha(x)(\alpha(a)y) - \alpha(\alpha^{-2}\beta^2(y))(\alpha(a)\alpha^2\beta^{-2}(x)) \\
& \overset{(1.1)}{=} \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) + \alpha(x)(\alpha(a)y) - (\alpha^{-2}\beta^2(y)\alpha(a))\alpha^2\beta^{-1}(x) \\
& \overset{(1.2)}{=} \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) + \alpha(x)(\alpha(a)y) - (\beta(a)\alpha^{-1}\beta(y))\alpha^2\beta^{-1}(x) \\
& = \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) + \alpha(x)(\alpha(a)y) - \beta(a\alpha^{-1}(y))\alpha(\alpha\beta^{-1}(x)) \\
& \overset{(1.2)}{=} \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) + \alpha(x)(\alpha(a)y) - \alpha(x)(\alpha(a)y) \\
& = \alpha(x) * \beta(y) - \alpha^{-1}\beta(\beta(y)) * \alpha\beta^{-1}(\alpha(x)) = [\alpha(x), \beta(y)] = [\cdot,\cdot] \circ (\alpha \otimes \beta)(x \otimes y),
\end{aligned}
$$

where $[\cdot,\cdot]$ is the BiHom-Lie bracket in the BiHom-Poisson algebra A^-.

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