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Abstract

In this paper we study the Aw-Rascle-Zhang (ARZ) model with non-conservative local point constraint on the density flux introduce in [Garavello, M., and Goatin, P. The Aw-Rascle traffic model with locally constrained flow. Journal of Mathematical Analysis and Applications 378, 2 (2011), 634-648], its motivation being, for instance, the modeling of traffic across a toll gate. We prove the existence of weak solutions under assumptions that result to be more general than those required in [Garavello, M., and Villa, S. The Cauchy problem for the Aw-Rascle-Zhang traffic model with locally constrained flow. Journal of Hyperbolic Differential Equations 14, 03 (2017), 393-414]. More precisely, we do not require that the waves of the first characteristic family have strictly negative speeds of propagation. The result is achieved by showing the convergence of a sequence of approximate solutions constructed via the wave-front tracking algorithm. The case of solutions attaining values at the vacuum is considered. We also present an explicit numerical example to describe some qualitative features of the solutions.

Keywords: Hyperbolic systems of conservation laws; local flux constraint; wave-front tracking; traffic flow modeling.
AMS Subject Classification: 35L65, 90B20

1 Introduction

Macroscopic models of traffic flows are nowadays a consolidated and nonetheless continuously in ferment field of mathematical research from both theoretical and applied points of view, as the surveys [5, 17, 18, 20] and the books [7, 13, 19] demonstrate.

In this paper we consider the model introduced in [12] for vehicular traffic. It is obtained by coupling the Aw-Rascle-Zhang (ARZ) model, see [4, 21], with a fixed local point constraint on the (density) flux. Conservation laws with unilateral constraints on the flux were first introduced in [11] and then studied in [1, 3, 8, 9] in the case of first order traffic flow models (consisting in scalar conservation laws) to describe the situations in which "obstacles" like toll gates, traffic lights or construction sites are present, see [10, 19]. We recall that in this case the presence of the constraint may enforce the appearance of *undercompressive* stationary jump discontinuities, which do not satisfy the Kruzhkov entropy condition [16] even though the first Rankine-Hugoniot condition remains valid, ensuring mass conservation.

The generalization to systems proposed in [12] for the 2×2 ARZ model is based on the assumption, motivated by applications to traffic flow modeling, that in general only the mass is conserved across the point constraint. As a consequence, physically reasonable Riemann solvers must guarantee the conservation of the density, but not necessarily that of the generalized momentum. In particular, in [12] the authors propose both a fully conservative Riemann solver $\mathcal{RS}_1^q$ and a partially conservative Riemann solver $\mathcal{RS}_2^q$, which satisfies the Rankine-Hugoniot condition for the conservation law of the density, but in general not for that of the generalized momentum.

The existence of $\mathcal{RS}_1^q$-solutions to general constrained Cauchy problems for the ARZ system has been proved in [2] for the case of initial data with bounded variation and piece-wise constant constraint. The existence of $\mathcal{RS}_2^q$-solutions has been addressed by [14] in the case of constant constraint functions and under assumptions on the initial data ensuring that the waves of the first family have negative (propagation) speeds.

Here we focus on $\mathcal{RS}_2^q$-solutions as in [14], but we provide a general existence result for solutions corresponding to initial data with bounded variation and constant constraint, without any restriction on the speeds of the waves of the first family. In particular, we are able to handle the presence of waves of the first family having positive speed, which are excluded in [14]. Our proof is based on the wave-front tracking approximation technique (see [6, 15] and the references therein) and the use of a functional to control the solutions' variation. Such functional is proved to be decreasing through a careful analysis of all possible wave interactions, thus providing a unifom bound of the total variation of a sequence of approximate solutions.

The paper is organized as follows. In the next section we introduce the main notations, recall the definition of the constrained Riemann solver $\mathcal{RS}_2^q$ and state the main result of the paper in Theorem 2.1. In Section 3 we apply the model to simulate the traffic in presence of a point constraint, such as a toll gate. In Section 4 we describe the wave-front tracking algorithm used to construct a globally defined approximate solution. Finally, in Section 5 we give the proof of Theorem 2.1.

2 Statement of the problem and main result

We first introduce some useful notations. The road is parametrized by the coordinate $x \in \mathbb{R}$ and the vehicles move in the direction of increasing x. We denote by $\rho = \rho(t,x) \geq 0$, $v = v(t,x) \geq 0$, $w = w(t,x) \geq 0$, $y = y(t,x) \geq 0$, $f = f(t,x) \geq 0$ the density, the mean velocity, the Langrangian marker, the generalized momentum and the flow of the vehicles at time $t \geq 0$ and position $x \in \mathbb{R}$, respectively. Away from the vacuum $\rho = 0$, these quantities are linked by the identities

$$Y \doteq \begin{pmatrix}\rho\\ y\end{pmatrix} = \begin{pmatrix}p^{-1}(w-v)\\ p^{-1}(w-v)w\end{pmatrix} \doteq \Psi\left(\begin{pmatrix}v\\ w\end{pmatrix}\right), \quad U \doteq \begin{pmatrix}v\\ w\end{pmatrix} = \begin{pmatrix}\frac{y}{\rho} - p(\rho)\\ \frac{y}{\rho}\end{pmatrix} \doteq \Psi^{-1}\left(\begin{pmatrix}\rho\\ y\end{pmatrix}\right), \quad f(U) = v\rho,$$

where $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+$ denotes the change of variables function and $p \in \mathbf{C^0}(\mathbb{R}_+;\mathbb{R}_+) \cap \mathbf{C^2}((0,\infty);\mathbb{R}_+)$ represents the anticipation factor accounting for drivers' reaction to the state of traffic downstream. We assume that

$$p(0) = 0 \qquad \text{and} \qquad p(\rho) > 0, \qquad p'(\rho) > 0, \qquad 2p'(\rho) + \rho p''(\rho) > 0 \qquad \text{for any } \rho > 0. \tag{1}$$

A typical choice is $p(\rho) = \rho^\gamma$, $\gamma > 0$, see [4]. The physical domain in Riemann invariant coordinates U is denoted

$$\mathcal{D} \doteq \left\{U = (v,w)^T \in [0, v_{\max}] \times [0, w_{\max}] \colon v \leq w\right\},$$

where $v_{\max}$ and $w_{\max}$ are the maximal allowed speed and Lagrangian marker, respectively, with $w_{\max} \geq v_{\max}$. By definition, the vacuum state $\rho = 0$ corresponds to the set $\mathcal{D}_0 \doteq \{U \in \mathcal{D} \colon v = w\}$

and the non-vacuum states $\rho > 0$ to $\mathcal{D} \setminus \mathcal{D}_0 = \{U \in \mathcal{D} \colon v < w\}$. In particular, Ψ fails to be injective at the vacuum and is invertible only away from the vacuum.

The ARZ model can be written away from the vacuum (i.e. for $v < w$) in conservative variables $Y = \Psi(U)$ as the 2×2 system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x (v\rho) = 0, \\ \partial_t y + \partial_x (vy) = 0, \end{cases}$$

which may also be written as

$$\partial_t Y + \partial_x (vY) = 0. \tag{2}$$

Beside the initial condition

$$Y(0, x) = \Psi\left(U_0(x)\right), \tag{3}$$

we impose a constraint on the first component of the flux, namely on the density flow, at $x = 0$

$$f\left(U(t, 0)\right) \leq q, \tag{4}$$

where $q > 0$ is a given constant and corresponds to the maximal density flow allowed through the obstacle located at $x = 0$.

Solutions to (2), (3), (4) are defined in the weak sense as follows:

Definition 2.1. (Weak solution) *Let $U_0 \in \mathbf{BV}(\mathbb{R}; \mathcal{D})$. A function $U \in \mathbf{BV}(\mathbb{R}_+ \times \mathbb{R}; \mathcal{D}) \cap \mathbf{C^0}(\mathbb{R}_+; \mathbf{L^1}(\mathbb{R}; \mathcal{D}))$ is a weak solution of* (2), (3), (4) *if $Y = \Psi(U)$ satisfies the initial condition* (3) *for a.e. $x \in \mathbb{R}$ and for any test function $\phi \in \mathbf{C^\infty_c}((0, \infty) \times \mathbb{R}; \mathbb{R})$*

$$\iint_{\mathbb{R}^+ \times \mathbb{R}} \left(Y \partial_t \phi + vY \partial_x \phi\right) dx dt + q \int_{\mathbb{R}^+} \operatorname{sgn}\left(v(t, 0^+)\right) \begin{pmatrix} 0 \\ w(t, 0^+) - w(t, 0^-) \end{pmatrix} \phi(t, 0) dt = 0. \tag{5}$$

Moreover, the traces of Y at $x = 0$ satisfy

$$f\left(U(t, 0^+)\right) = f\left(U(t, 0^-)\right) \leq q \qquad \text{for a.e. } t > 0. \tag{6}$$

Note that the existence of the traces is ensured by the **BV** assumption and their equality in (6) is guaranteed by the first Rankine-Hugoniot condition derived by (5).

Remark 1. *Note that the fully conservative $\mathcal{RS}^q_1$-solutions studied in [2] satisfy Definition 2.1, indeed for such solutions the second term on the left-hand side is zero for a.e. $t > 0$. However, the class of solutions satisfying* (5), (6) *is wider, and includes the $\mathcal{RS}^q_2$-solutions, as we will show later on.*

To introduce the functional that will be used in the interaction estimates later on, we need to define the following key values. Recall that, in the (ρ, f)-plane, the first and second Lax curves of (2) through a point $U_\ell = (v_\ell, w_\ell)^T$ are respectively

$$\mathcal{L}_1(\rho; U_\ell) \doteq \left(w_\ell - p(\rho)\right)\rho, \qquad \mathcal{L}_2(\rho; U_\ell) \doteq v_\ell \rho.$$

We now set:

- If $q = 0$, then we let $\dot{U} \doteq (0, 0)^T$, otherwise $\dot{U} = (\dot{v}, \dot{w})^T$ is implicitly defined by

$$\dot{v}^2 / q = p'\left(q/\dot{v}\right), \qquad \dot{w} = \dot{v} + p\left(q/\dot{v}\right),$$

i.e. $\dot{U}$ is the unique state in $\mathcal{D}$ at wich $\rho \mapsto \mathcal{L}_1(\rho; \dot{U})$ has q as maximum value.

- Let $\overline{U}(w_\ell) \doteq (\bar{v}(w_\ell), w_\ell)^T \in \mathcal{D}$ correspond to the maximum point of $\rho \mapsto \mathcal{L}_1\left(\rho; (0, w_\ell)^T\right)$; clearly

$$w_\ell = p\left(\bar{\rho}(w_\ell)\right) + p'\left(\bar{\rho}(w_\ell)\right) \bar{\rho}(w_\ell).$$

• If $w_\ell > \dot{w}$, let $U^\flat(w_\ell) \doteq (v^\flat(w_\ell), w_\ell)^T$ and $\hat{U}(w_\ell) \doteq (\hat{v}(w_\ell), w_\ell)^T$ correspond to the intersection with lower and higher density, respectively, between $f = q$ and $\rho \mapsto \mathcal{L}_1(\rho; (0, w_\ell)^T)$, whereas if $w_\ell \leq \dot{w}$ let $U^\flat(w_\ell) \doteq \dot{U} \doteq \hat{U}(w_\ell)$; clearly

$$v^\flat(w_\ell) = \begin{cases} \max\{v \in (0, w_\ell) \colon v + p(q/v) = w_\ell\} & \text{if } w_\ell > \dot{w}, \\ \dot{v} & \text{if } w_\ell \leq \dot{w}, \end{cases}$$

$$\hat{v}(w_\ell) = \begin{cases} \min\{v \in (0, w_\ell) \colon v + p(q/v) = w_\ell\} & \text{if } w_\ell > \dot{w}, \\ \dot{v} & \text{if } w_\ell \leq \dot{w}. \end{cases}$$

Observe that $w \mapsto v^\flat(w)$ is non-decreasing and $w \mapsto \hat{v}(w)$ is non-increasing.

• Let $\tilde{U}(U_\ell, U_r) \doteq (\min\{v_r, w_\ell\}, w_\ell)^T$ correspond to the intersection with higher density between $\rho \mapsto \mathcal{L}_1(\rho; U_\ell)$ and $\rho \mapsto \mathcal{L}_2(\rho; U_r)$; clearly

$$\tilde{\rho}(U_\ell, U_r) = p^{-1}\left(\max\{0, w_\ell - v_r\}\right).$$

• If $v_r > 0$, let $\check{U}(v_r)$ correspond to the intersection between $f = q$ and $\rho \mapsto \mathcal{L}_2(\rho; U_r)$; clearly

$$\check{U}(v_r) = \left(v_r, v_r + p(q/v_r)\right)^T.$$

• If $\rho_\ell \neq \rho_r$, the Rankine-Hugoniot speed of the discontinuity (U_ℓ, U_r) is

$$\sigma(U_\ell, U_r) \doteq \frac{f_r - f_\ell}{\rho_r - \rho_\ell}. \tag{7}$$

Notice that if $w_\ell > \dot{w}$ and $v_r > 0$, then

$$\hat{f}(w_\ell) = \check{f}(v_r) = f^\flat(w_\ell) = q.$$

For any $U_\ell, U_r \in \mathcal{D}$, we consider the Riemann-like initial datum

$$U_0(x) = \begin{cases} U_\ell & \text{if } x < 0, \\ U_r & \text{if } x \geq 0. \end{cases} \tag{8}$$

Let $\mathcal{RS}$ be the classical Riemann solver for (2), (8), which is described in [4], see also [2, Section 2.1]. In general, the solution given by $\mathcal{RS}$ does not satisfy condition (4). For this reason, we introduce the following sets

$$\Omega_1 \doteq \left\{(U_\ell, U_r) \in \mathcal{D} \times \mathcal{D} \colon f\left(\mathcal{RS}[U_\ell, U_r]\right)(0^\pm) \leq q\right\},$$
$$\Omega_2 \doteq \left\{(U_\ell, U_r) \in \mathcal{D} \times \mathcal{D} \colon f\left(\mathcal{RS}[U_\ell, U_r]\right)(0^\pm) > q\right\}.$$

We recall the definition of the constrained Riemann solver $\mathcal{RS}_2^q$ given in [12].

Definition 2.2. *The constrained Riemann solver for* (2), (4), (8)

$$\mathcal{RS}_2^q : \mathcal{D} \times \mathcal{D} \to \mathbf{BV}(\mathbb{R}; \mathcal{D})$$

is defined as follows:

$$(U_\ell, U_r) \in \Omega_1 \quad \Rightarrow \quad \mathcal{RS}_2^q[U_\ell, U_r] \equiv \mathcal{RS}[U_\ell, U_r],$$

$$(U_\ell, U_r) \in \Omega_2 \quad \Rightarrow \quad \mathcal{RS}_2^q[U_\ell, U_r](\xi) = \begin{cases} \mathcal{RS}[U_\ell, \hat{U}(w_\ell)](\xi) & \textit{if } \xi < 0, \\ \mathcal{RS}[\check{U}(v_r), U_r](\xi) & \textit{if } \xi \geq 0. \end{cases}$$

Notice that if $(U_\ell, U_r) \in \Omega_2$, then $\mathcal{RS}_2^q$ displays a stationary *undercompressive* shock $(\hat{U}(w_\ell), \check{U}(v_r))$ and $\hat{v}(w_\ell) < \check{v}(v_r) = v_r$.

We are now ready to introduce our functional: for $t \geq 0$ we set

$$\begin{aligned}\Upsilon\left(U(t,\cdot)\right) \doteq{} & \mathrm{TV}\left(v(t,\cdot);\mathbb{R}\right) + \mathrm{TV}\left(w(t,\cdot);\mathbb{R}\right) + 3\mathrm{TV}\left(w(t,\cdot);\mathbb{R}_-\right) \\ & + 2\left[\mathrm{TV}\left(\hat{v}(t);\mathbb{R}_-\right) + \mathrm{TV}\left(v^\flat(t);\mathbb{R}_-\right) + w_{\max} - \gamma(t) + \mathrm{TV}_+\left(\tilde{\eta}\left(U(t,\cdot)\right)\right)\right],\end{aligned} \tag{9}$$

where

$$\hat{v}(t) \doteq \hat{v}\left(w(t,\cdot)\right), \qquad\qquad v^\flat(t) \doteq v^\flat\left(w(t,\cdot)\right),$$

and

$$\tilde{\eta}\left(U\right) \doteq \begin{cases}\check{w}\left(v\right) & \text{if } v \in [\hat{v}(w_{\max}), \dot{v}]\,,\\ 0 & \text{otherwise,}\end{cases}$$

$$\gamma(t) \doteq \begin{cases}\min\left\{v^\flat\left(w(t,0^-)\right), v(t,0^+)\right\} - v(t,0^-) & \text{if } (U(t,0^-), U(t,0^+)) \in \Omega_2,\\ 0 & \text{otherwise.}\end{cases}$$

Also, we denote by $\mathrm{TV}_+(\eta)$ the positive variation of the function η. Conventionally, we assume that U is left continuous in time, thus $U(t^-) = U(t)$, hence also $t \mapsto \mathrm{TV}(v(t,\cdot)) + \mathrm{TV}(w(t,\cdot))$ and $t \mapsto \Upsilon(U(t,\cdot))$ are left continuous.

Remark 2. *In* (9), $3\mathrm{TV}(w(t,\cdot);\mathbb{R}_-) + 2\mathrm{TV}(\hat{v}(t);\mathbb{R}_-) + 2\mathrm{TV}(v^\flat(t);\mathbb{R}_-)$ *measures the strength of the contact discontinuities upstream the constraint,* $\mathrm{TV}_+(\tilde{\eta}(U(t,\cdot)))$ *measures the strength of the rarefaction waves (and possible undercompressive shocks) between states having* v*-coordinate in* $[\hat{v}(w_{\max}), \dot{v}]$ *and* γ *takes into account the presence of undercompressive shocks. Note also that* $\Upsilon(t) \geq 0$.

Our main result states the existence of weak solutions of the constrained Cauchy problem (2), (3), (4) corresponding to initial data $U_0 \in \mathcal{D}$ with bounded total variation for which $\Upsilon(U_0)$ is finite.

Theorem 2.1. *Let* $U_0 \in \mathbf{BV}(\mathbb{R};\mathcal{D})$ *be such that* $\Upsilon(U_0)$ *is finite. Then constrained Cauchy problem* (2), (3), (4) *admits a weak solution* $U \in \mathbf{BV}(\mathbb{R}_+ \times \mathbb{R};\mathcal{D}) \cap \mathbf{C^0}(\mathbb{R}_+;\mathbf{L^1}(\mathbb{R};\mathcal{D}))$ *in the sense of Definition 2.1. Moreover for any* $t,\, s \geq 0$ *we have*

$$\mathrm{TV}\left(v(t,\cdot)\right) + \mathrm{TV}\left(w(t,\cdot)\right) \leq \Upsilon(U_0), \qquad\qquad \|U(t,\cdot) - U(s,\cdot)\|_{\mathbf{L^1}(\mathbb{R})} \leq L|t-s|,$$

with $L \doteq \Upsilon(U_0)\max\{v_{\max}, p^{-1}(w_{\max})p'(p^{-1}(w_{\max}))\}$.

Remark 3. *Regarding the functional* Υ *defined in* (9), *we observe that* $\tilde{\eta}$ *is discontinuous both at* $v = \dot{v}$ *and* $v = \hat{v}(\max)$. *The functions* $\hat{v}$ *and* $v^\flat$ *fail to be Lipschitz continuous close to* $w = \dot{w}$. *This implies that the functional is not controlled by the* $\mathrm{TV}(U_0)$. *For this reason in Theorem 2.1 we have to assume that the initial datum is such that* $\Upsilon(U_0)$ *is bounded, as it was the case in [2].*

3 A case study

In this section we apply the the Riemann solver $\mathcal{RS}_2^q$ to describe the evolution of traffic through a point constraint representing, for instance, a toll gate. More precisely, we consider the constrained Cauchy problem (2), (3), (4) with a piecewise constant initial datum

$$U_0(x) = \begin{cases}(w_1, w_1) & \text{if } x < x_A,\\ (0, w_1) & \text{if } x_A \leq x < x_B,\\ (0, w_2) & \text{if } x_B \leq x < 0,\\ (w_2, w_2) & \text{if } x \geq x_B,\end{cases}$$

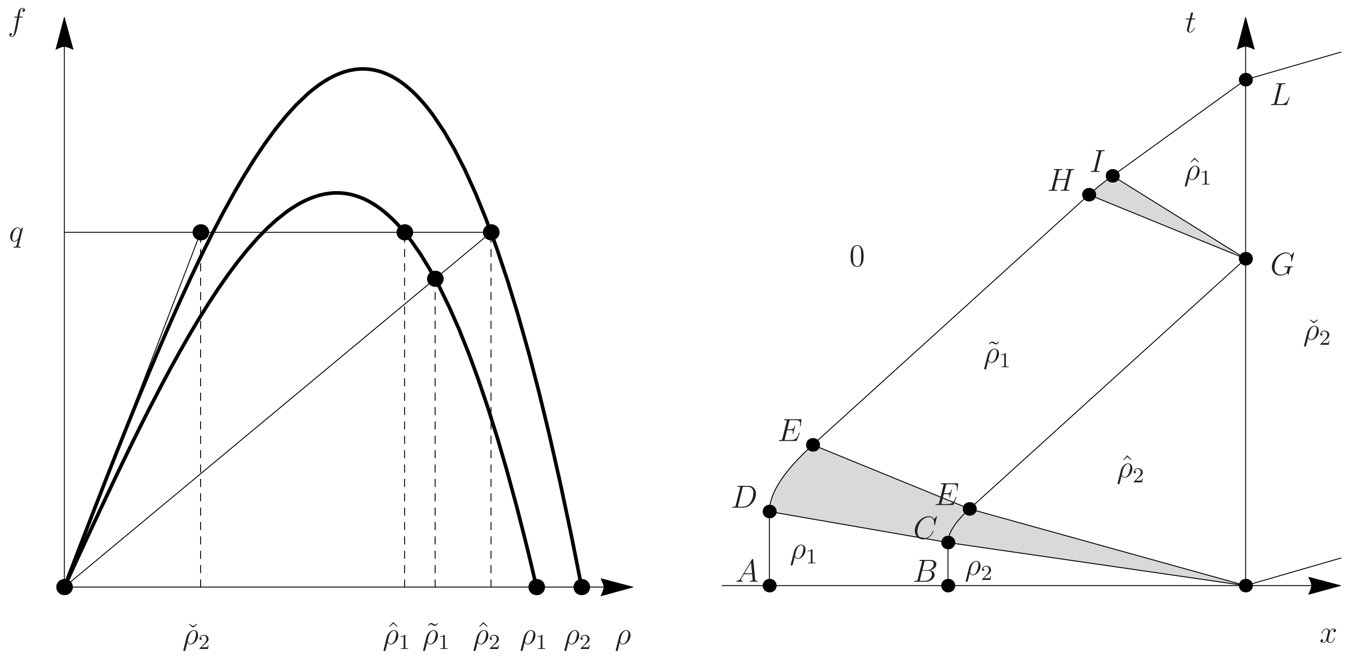

Figure 1: The solution constructed in Section 3. The shaded areas correspond to rarefactions.

where $x_A < x_B < 0$ and $w_1 < w_2$, see Figure 1. In other words, we consider in $[x_A, 0[$ vehicles that are bumper to bumper, hence with zero velocity, but with different maximal densities $\rho_1 = p^{-1}(w_1)$ and $\rho_2 = p^{-1}(w_2)$ in $[x_A, x_B[$ and $[x_B, 0[$, respectively. Let $\hat{U}_i \doteq \hat{U}(w_i)$ and $\tilde{U}_1 \doteq \tilde{U}_1(\hat{U}_1, \hat{U}_2)$. Assume that q belongs to $]0, \bar{f}(w_1)[$. The corresponding solution can be constructed as follows. We first solve the Riemann problems at $A(0, x_A)$ and $B(0, x_B)$ by applying the classical $\mathcal{RS}$ and the one at $(0,0)$ by applying $\mathcal{RS}_2^q$. As a result, we obtain two stationary contact discontinuities CD_A and CD_B starting from A and B, while from $(0,0)$ we have a backward rarefaction R_0, a stationary undercompressive shock US_0 and a forward contact discontinuity CD_0. Let C and E be the first and last interaction points between CD_B and R_0. Observe that CD_B accelerates during its interaction with R_0. Moreover R_0 crosses CD_B. Let D and F be the first and last interaction points between CD_A and R_0. Also CD_A accelerates during its interaction with R_0, however R_0 does not cross CD_A and expires. Both CD_A and CD_B move with velocity $\tilde{v}_1 = \hat{v}_2$ after their interaction with R_0. Once CD_B reaches $x = 0$ at $G = (t_G, 0)$, a backward rarefaction R_G appears and the left state of US_0 changes from $\hat{U}_2$ to $\hat{U}_1$. Let H and I be the first and last interaction points between CD_A and R_G. Again CD_A accelerates during its interaction with R_G, moreover R_G does not cross CD_A and expires. After time $t = t_I$ we have that CD_A moves with velocity $\hat{v}_1$, reaches $x = 0$ at $L = (t_L, 0)$ and finally continues in $]0, \infty[$. Since CD_0 and CD_A move with velocities w_2 and $\check{v}_2$, respectively, we have that they will not interact.

Such solution has the following physical interpretation. At time $t = 0$, the rightmost vehicle starts to move with constant velocity w_2 and the other vehicles start to accelerate as soon as the distance from the vehicle ahead increases. This acceleration corresponds to the rarefaction R_0. Due to the presence of the toll gate, which hinders the flow at $x = 0$, the vehicles initially in $[x_B, 0[$ stop to accelerate once they reach the velocity $\hat{v}_2$ and flow q, which is the maximal capacity of the toll gate. On the other hand, the vehicles initially in $[x_A, x_B[$ stop to accelerate once they reach the velocity $\tilde{v}_1 = \hat{v}_2$ of the other vehicles in the upstream of the toll gates. Once all the vehicles initially in $[x_B, 0[$ have crossed the toll gate, namely at time $t = t_G$, the upstream vehicles accelerate and reach the velocity $\hat{v}_1$ and flow q. This acceleration corresponds to the rarefaction R_G. Finally, after time t_L all the vehicles have passed the toll gates.

In Figure 1 we represent the initial datum and an overall overview of the solution corresponding to

$$w_1 = 1, \qquad w_2 = 6/5, \qquad p(\rho) = \rho^2, \qquad q = \sqrt{3}/5, \qquad x_A = -8, \qquad x_B = -5, \qquad t_L \approx 24.4716.$$

The quantitative evolution of the corresponding solution is represented in different coordinates in

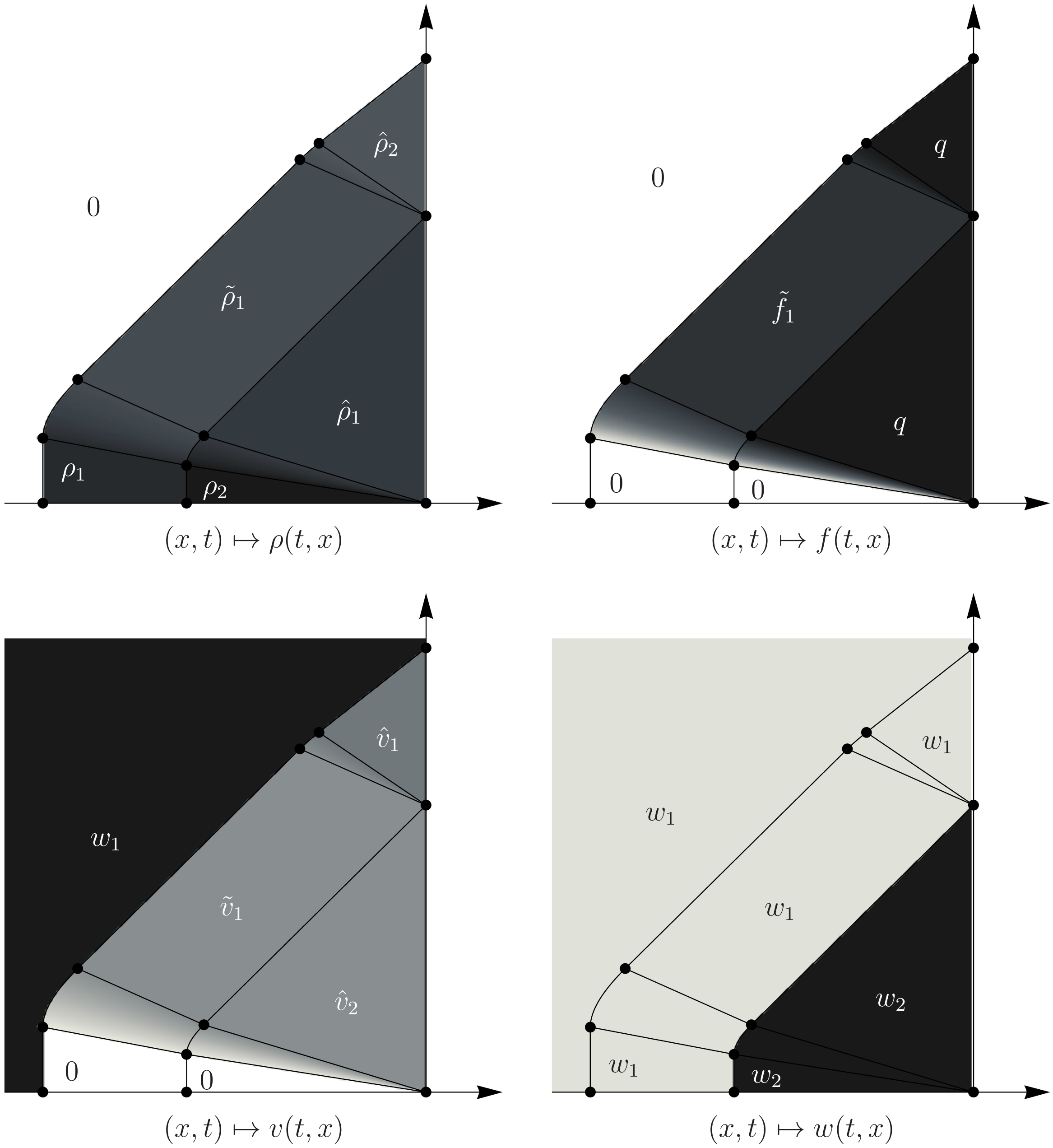

Figure 2: The solution constructed in Section 3. Darker colors correspond to higher values.

Figure 2. We finally observe that, once the overall picture of the solution is known, it is possible to express in a closed form the time at which the last vehicle passes through $x = 0$, indeed $t_L = [(x_B - x_A)p^{-1}(w_1) - x_B p^{-1}(w_2)]/q$.

4 Wave-front tracking

In this section we apply the wave-front tracking method (see [6, 15] and the references therein) to construct a piecewise constant approximate solution U^n to the constrained Cauchy problem (2), (3), (4).

The approximate solution U^n is obtained as follows. We approximate U_0 by U_0^n belonging to the set of piecewise constant functions with a finite number of jumps, denoted below by **PC**. At time

$t = 0$, we solve the Riemann problems at every discontinuity point of U_0^n by applying $\mathcal{RS}^n$ away from the $x = 0$ and $\mathcal{RS}_2^{q,n}$ at $x = 0$. Here $\mathcal{RS}_2^{q,n}$ and $\mathcal{RS}^n$ are approximate Riemann solvers of the Riemann solvers $\mathcal{RS}_2^q$ and $\mathcal{RS}$, respectively, defined below and obtained by approximating every rarefaction wave by a fan of rarefaction shocks. With this choice the strength of each rarefaction shock has upper and lower bounds.

The solution can then be prolonged until time $t_1 > 0$, when two waves interact or a wave reaches $x = 0$. In both cases we solve the Riemann problem at the interaction point by applying the appropriate approximate Riemann solver. Since we can bound the total variation of U^n and the number of waves, we can iterate this procedure and extend the approximate solution U^n globally in time.

4.1 Grid construction and approximate Riemann solvers

In this section we introduce a grid $\mathcal{D}_n$ in $\mathcal{D}$. Assume that $q \in]0, \bar{f}(w_{\max})[$, the cases $q = 0$ and $q = \bar{f}(w_{\max})$ being straightforward generalizations. We let $\mathcal{D}_n = (\mathcal{W}_n \times \mathcal{W}_n) \cap \mathcal{D}$, where $\mathcal{W}_n = \{\omega_i^j\}_{i=0,\dots,n}^{j=-3-J,\dots,J}$ is a finite subset of $[0, w_{\max}]$ constructed as follows, see Figure 3:

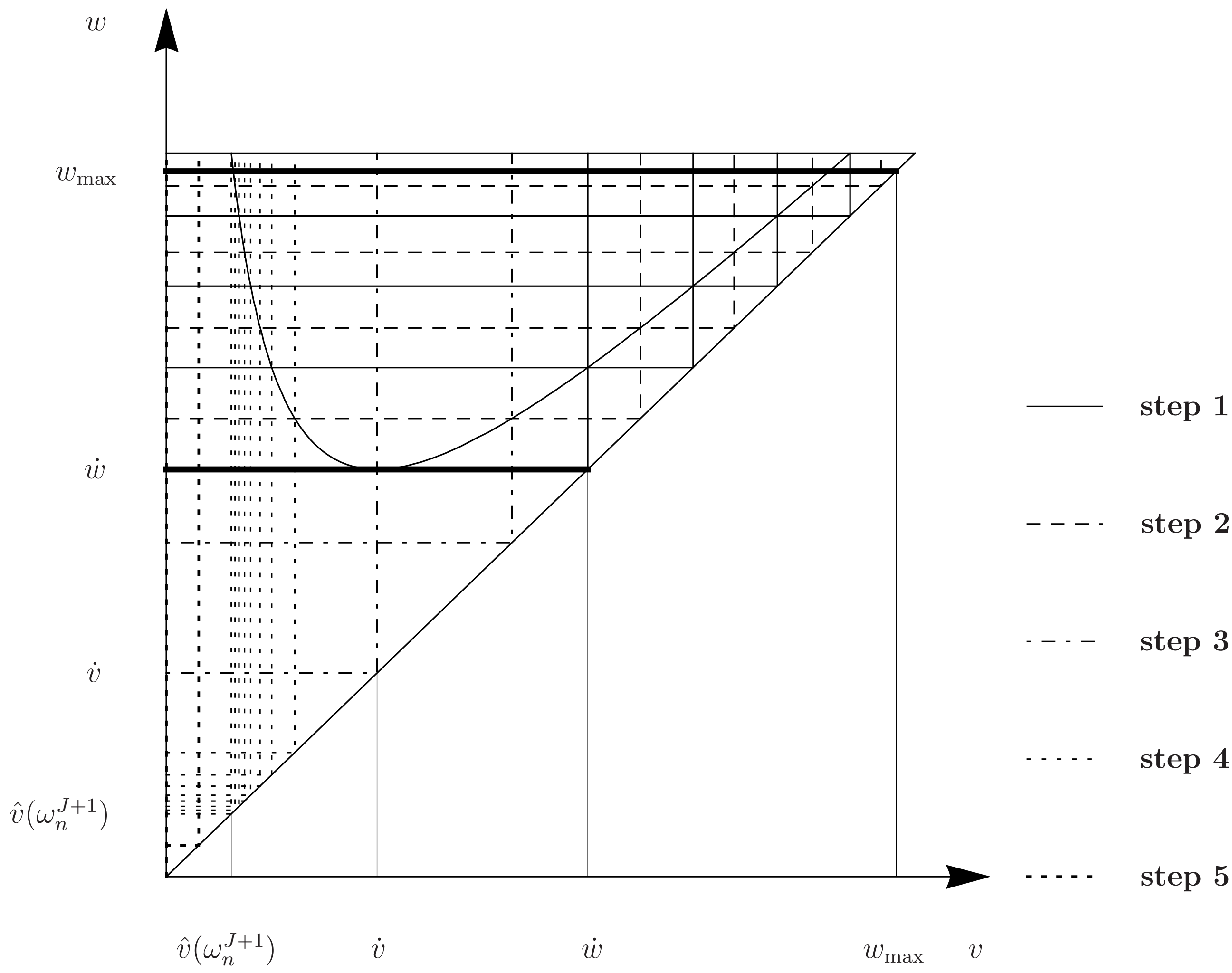

Figure 3: The grid $\mathcal{D}_n$ for $n = 2$.

step 1 Let $\omega^0 = \dot{w}$ and consider the recursive sequence $\omega^j = \omega^{j-1} + p(q/\omega^{j-1})$ for $j \in \mathbb{N}$ (see continuous lines in Figure 3). In the (v, w)-plane, ω^j is the w-coordinate of the intersection between $v = \omega^{j-1}$ and the curve $w = v + p(q/v)$. We observe that there exists $J \leq (w_{\max} - \dot{w})/p(q/w_{\max})$

such that $\omega^J \leq w_{\max} <$. Indeed, by the monotonicity of p we have

$$\omega^j - \omega^{j-1} = p(q/\omega^{j-1}) \geq p(q/w_{\max}).$$

step 2 Divide $[\omega^0, \omega^1]$ into n intervals of length $n^{-1}(\omega^1 - \omega^0) = n^{-1} p(q/\dot{w})$ with endpoints

$$\omega_i^0 = \omega^0 + i\frac{p(q/\dot{w})}{n}, \qquad i \in \{0, \ldots, n\}.$$

Consider then the recursive sequence $\omega_i^j = \omega_i^{j-1} + p(q/\omega_i^{j-1}) \in [\dot{w}, \omega^{J+1}]$ for $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, J\}$ (see dashed lines in Figure 3). Notice that $\omega_n^j = \omega_0^{j+1} = \omega^{j+1}$.

step 3 In $[\dot{v}, \dot{w}]$ we introduce

$$\omega_i^{-1} = v^\flat(\omega_i^0), \qquad i \in \{0, \ldots, n\},$$

(see dot-dashed lines in Figure 3).

step 4 In $[\hat{v}(\omega^{J+1}), \dot{v}]$ we introduce

$$\omega_i^{-2-j} = \hat{v}(\omega_{n-i}^{J-j}), \qquad i \in \{0, \ldots, n\},\ j \in \{0, \ldots, J\},$$

(see thin dotted lines in Figure 3).

step 5 Divide $[0, \hat{v}(\omega^{J+1})]$ into n intervals of length $n^{-1}\hat{v}(\omega^{J+1})$ and endpoints

$$\omega_i^{-3-J} = i\frac{\hat{v}(\omega^{J+1})}{n}, \qquad i \in \{0, \ldots, n\},$$

(see thick dotted lines in Figure 3).

For notational simplicity we let $\mathcal{W}_n = \{w_1, \ldots, w_N\}$, where $w_i < w_{i+1}$.

In the next lemma we prove that the grid $\mathcal{D}_n$ is well defined and that the distance between two points of $\mathcal{D}_n$ has a lower bound.

Lemma 4.1. *For any $n \in \mathbb{N}$, $q \in]0, \bar{f}(w_{\max})[$, the grid $\mathcal{D}_n$ is well defined and*

$$\varepsilon_n = \min_{i \in \{1,\ldots,N\}} (w_{i+1} - w_i) > 0.$$

Proof. It is sufficient to prove that the function $h_q(v) \doteq v + p(q/v)$ is Lipschitz and convex in $I = [\hat{v}(w_N), v^\flat(w_N)] \subset]0, \infty[$. For any $v \in I$ we have that

$$h_q'(v) = 1 - \frac{q}{v^2} p'\left(\frac{q}{v}\right), \qquad h_q''(v) = \frac{q}{v^3}\left(2p'\left(\frac{q}{v}\right) + \frac{q}{v} p''\left(\frac{q}{v}\right)\right).$$

By (1) we have that $h_q''(v) > 0$, therefore

$$h_q'(v) \in \left[1 - \frac{q}{\hat{v}(w_N)^2} p'\left(\frac{q}{\hat{v}(w_N)}\right), 1 - \frac{q}{v^\flat(w_N)^2} p'\left(\frac{q}{v^\flat(w_N)}\right)\right],$$

hence h_q is Lipschitz. Moreover, the inverse of the restrictions of h_q to both $[\hat{v}(w_N), \hat{v}(\omega_1^0)]$ and $[v^\flat(\omega_1^0), v^\flat(w_N)]$ are Lipschitz. □

In the cases $q = 0$ and $q = \bar{f}(w_{\max})$, the grid $\mathcal{D}_n$ can be defined by adapting the above construction.

At last, we define the approximate Riemann solvers $\mathcal{RS}^n$, $\mathcal{RS}_2^{q,n} : \mathcal{D}_n \times \mathcal{D}_n \to \mathbf{PC}(\mathbb{R}; \mathcal{D}_n)$ by splitting the rarefactions. More precisely, for any $(U_\ell, U_r) \in \mathcal{D}_n \times \mathcal{D}_n$ such that $w_\ell = w_r$ and $v_\ell = w_h < v_r = w_{h+k}$, we let

$$\mathcal{RS}^n[U_\ell, U_r](\xi) = \begin{cases} U_\ell & \text{if } \xi \leq \sigma(U_\ell, U_1), \\ U_j & \text{if } \sigma(U_{j-1}, U_j) < \xi \leq \sigma(U_j, U_{j+1}),\ 1 \leq j \leq k-1, \\ U_r & \text{if } \xi \geq \sigma(u_{k-1}, U_r), \end{cases}$$

where $U_0 = U_\ell, U_k = U_r, U_j = (w_{h+j}, w_\ell)^T$ and σ is defined in (7). The Riemann solver $\mathcal{RS}_2^{q,n}$ is defined as follows:

$$f\left(\mathcal{RS}^n[U_\ell, U_r](0^\pm)\right) \leq q \qquad \Rightarrow \qquad \mathcal{RS}^n[U_\ell, U_r] \equiv \mathcal{RS}_2^{q,n}[U_\ell, U_r],$$

$$f\left(\mathcal{RS}^n[U_\ell, U_r](0^\pm)\right) > q \qquad \Rightarrow \qquad \mathcal{RS}_2^{q,n}[U_\ell, U_r](\xi) = \begin{cases} \mathcal{RS}^n[U_\ell, \hat{U}(w_\ell)](\xi) & \text{if } \xi < 0, \\ \mathcal{RS}^n[\check{U}(v_r), U_r](\xi) & \text{if } \xi \geq 0. \end{cases}$$

4.2 Interaction estimates

In this subsection we study all possible interactions of the wave-fronts of the approximate solution U^n and study the map $t \mapsto \Gamma^n(t)$ defined below. In particular we show that Γ^n does not increase after any interaction and it decreases by at least $\varepsilon_n > 0$ each time the number of the waves increases. As a consequence, U^n can be extended globally in time and $\mathrm{TV}(U^n(t)) \le \Gamma^n(t) \le \Gamma^n(0) \le \Upsilon(U_0)$.

For notational simplicity we omit the dependence on n and write, for instance, U in place of U^n and ε for ε_n.

Assume that at time $t_i > 0$ an interaction occurs, namely either two waves interact (the general case is similar) or a wave reaches $x = 0$. We introduce the non-negative function

$$\begin{aligned}\Gamma(U(t)) \doteq{}& \mathrm{TV}\,(U(t)) + 3\mathrm{TV}\,(w(t);\mathbb{R}_-) + 2\mathrm{TV}\,(\hat{v}(t);\mathbb{R}_-) + 2\mathrm{TV}\left(v^\flat(t);\mathbb{R}_-\right)\\ &+ 2\left[w_{\max} - \gamma(t) + \sum_{x\in\mathsf{J}(t)} \eta\left(U(t,x^-),U(t,x^+)\right)\right],\end{aligned}$$

where $\mathsf{J}(t) \subset \mathbb{R}$ is the (finite) set of discontinuity points of $U(t,\cdot)$, $\hat{v}(t) = \hat{v}(w(t))$, $v^\flat(t) = v^\flat(w(t))$,

$$\eta\left(U_-,U_+\right) = \begin{cases} \check{w}\left(v_-\right) - \check{w}\left(v_+\right) & \text{if } w_- = w_+ \text{ and } \hat{v}(w_{\max}) \le v_- < v_+ \le \dot{v},\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 4.2. *For any $n \in \mathbb{N}$ and $U_0^n \in \mathbf{PC}(\mathbb{R};\mathcal{D}_n)$, let U^n be the corresponding approximate solution constructed by wave-front tracking. Then the map $t \mapsto \Gamma^n(U^n(t))$ either decreases by at least ε_n or remains constant and the number of waves does not increase.*

Proof. We denote by CD a contact discontinuity, R a rarefaction shock, S a shock and US an undercompressive shock. For completeness, we compute below

$$\begin{aligned}
\Delta_v &\doteq \mathrm{TV}\left(v(t_i^+)\right) - \mathrm{TV}\left(v(t_i^-)\right), & \Delta_{\hat{v}}^- &\doteq \mathrm{TV}\left(\hat{v}(t_i^+);\mathbb{R}_-\right) - \mathrm{TV}\left(\hat{v}(t_i^-);\mathbb{R}_-\right),\\
\Delta_w &\doteq \mathrm{TV}\left(w(t_i^+)\right) - \mathrm{TV}\left(w(t_i^-)\right), & \Delta_w^- &\doteq \mathrm{TV}\left(w(t_i^+);\mathbb{R}_-\right) - \mathrm{TV}\left(w(t_i^-);\mathbb{R}_-\right),\\
\Delta\gamma &\doteq \gamma(t_i^+) - \gamma(t_i^-), & \Delta_{v^\flat}^- &\doteq \mathrm{TV}\left(v^\flat(t_i^+);\mathbb{R}_-\right) - \mathrm{TV}\left(v^\flat(t_i^-);\mathbb{R}_-\right),\\
\Delta\Gamma &\doteq \Gamma(t_i^+) + \Gamma(t_i^-), & \Delta\sharp &\doteq \sharp\mathsf{J}(t_i^+) - \sharp\mathsf{J}(t_i^-),
\end{aligned}$$

and

$$\Delta_\eta \doteq \sum_{x\in\mathsf{J}(t_i^+)} \eta\left(U(t_i^+,x^-),U(t_i^+,x^+)\right) - \sum_{x\in\mathsf{J}(t_i^-)} \eta\left(U(t_i^-,x^-),U(t_i^-,x^+)\right).$$

A Assume now that at time $t_i > 0$ exactly one interaction between two waves (U_ℓ, U_m) and (U_m, U_r) occurs at $x = 0$. For notational simplicity let $\tilde{U} = \tilde{U}(U_\ell, U_r)$, $f_\ell = f(U_\ell)$, $\hat{v}_\ell = \hat{v}(w_\ell)$, $v_\ell^\flat = v^\flat(w_\ell)$ and so on. We distinguish the following cases:

A1 Assume that (U_ℓ, U_m) is a S and (U_m, U_r) an US. Clearly $w_\ell = w_m \ge \dot{w}$, $v_m < \min\{v_\ell, v_r\}$ and $f_\ell < f_m = f_r = q$. As a consequence $(U_\ell, U_r) \in \Omega_1$, see Figure 4.

A1.a If $v_r > v_\ell$, then $\Delta\sharp \ge 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a fan of Rs $(U_\ell, \tilde{U})$ and a CD $(\tilde{U}, U_r)$, see Figure 4. Obviously

$$\Delta_w^- = 0, \qquad \Delta_{\hat{v}}^- = 0, \qquad \Delta_{v^\flat}^- = 0.$$

Since $v_m \le \dot{v} < v_\ell < \tilde{v} = v_r$, $\dot{w} \le \tilde{w} = w_\ell = w_m \le w_r$ and $v_r > v_\ell^\flat = v_m^\flat$, we have

$$\Delta_v = -2(v_\ell - v_m) < 0, \qquad \Delta\gamma = -(v_m^\flat - v_m) < 0, \qquad \Delta_w = 0, \qquad \Delta_\eta = 0.$$

In conclusion we have $\Delta\Gamma = -2(v_\ell - v_m^\flat) \le -2\varepsilon$.

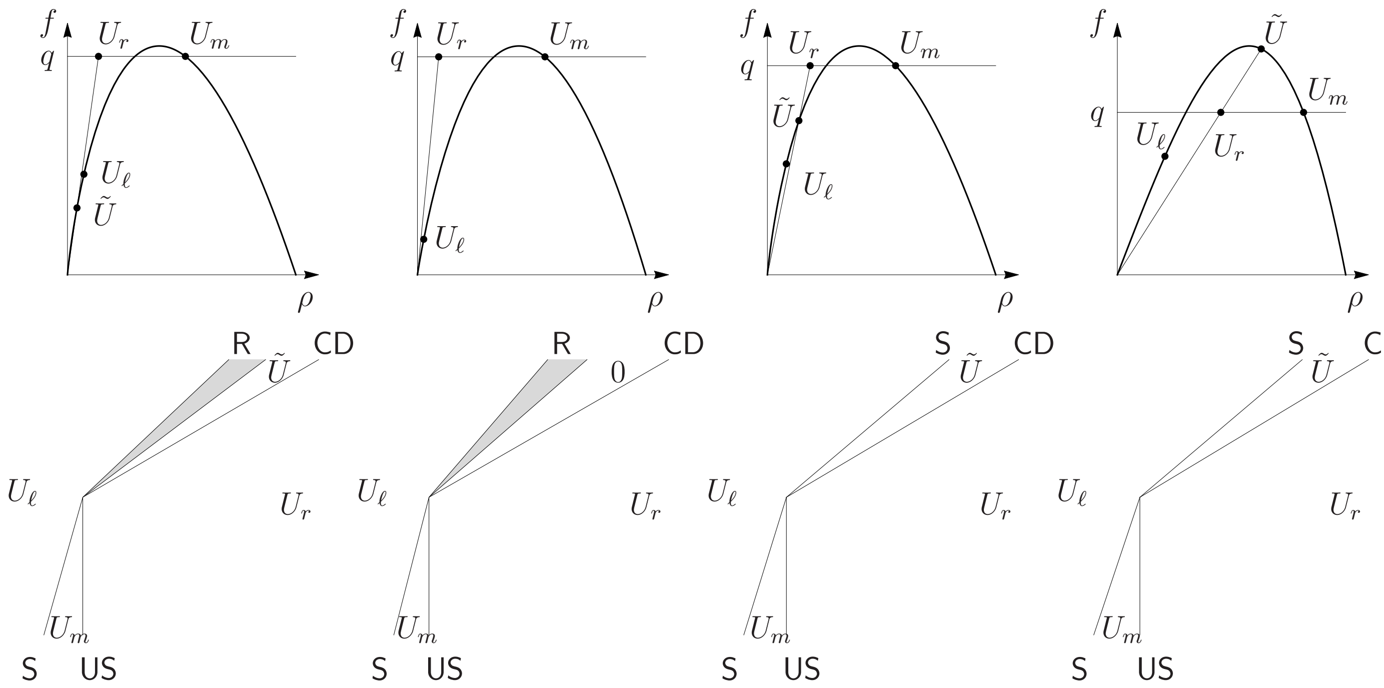

Figure 4: Case **A1**.

A1.b If $v_r \leq v_\ell$, then $\Delta\sharp \leq 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a possibly null S $(U_\ell, \tilde{U})$ and a possibly null CD $(\tilde{U}, U_r)$ (but not both null), see Figure 4. Obviously

$$0 = \Delta_w^- = \Delta_{\hat{v}}^- = \Delta_{v^\flat}^- = \Delta_\eta.$$

Since $v_m < \tilde{v} = v_r < v_\ell$, $\dot{w} \leq \tilde{w} = w_\ell = w_m$, $\dot{w} \leq w_r$ and $v_\ell^\flat = v_m^\flat$, we have

$$\Delta_v = -2(v_r - v_m) < 0, \qquad \Delta_w = 0, \qquad \Delta\gamma = -\left(\min\left\{v_m^\flat, v_r\right\} - v_m\right) < 0.$$

In conclusion we have

$$\Delta\Gamma = -2\left(v_r - \min\left\{v_m^\flat, v_r\right\}\right) \leq 0.$$

A2 Assume that (U_ℓ, U_m) is a CD and (U_m, U_r) an US. Clearly $v_\ell = v_m = \hat{v}_m < v_r$, $\dot{w} \leq \min\{w_m, w_r\}$ and $f_m = f_r = q$.

A2.a If $(U_\ell, U_r) \in \Omega_1$, then $\Delta\sharp \geq -1$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a fan of Rs $(U_\ell, \tilde{U})$ and a possibly null CD $(\tilde{U}, U_r)$, see Figure 5. Since $\tilde{v} = v_r > v_\ell = v_m$, $\breve{w}_\ell = w_m$ and $\breve{w}(v_r) = w_r$ we have

$$\Delta_v = 0, \qquad \Delta_\eta = \begin{cases} \breve{w}_\ell - \breve{w}(\tilde{v}) & \text{if } \tilde{v} \leq \dot{v} \\ \breve{w}_\ell - \dot{w} & \text{if } \tilde{v} > \dot{v} \end{cases} = \begin{cases} w_m - w_r & \text{if } v_r \leq \dot{v}, \\ w_m - \dot{w} & \text{if } v_r > \dot{v}. \end{cases}$$

Since $\tilde{w} = w_\ell < w_m$, $w_\ell \leq w_r$ and $\dot{w} \leq \min\{w_m, w_r\}$ we have that

$$\Delta_w = (w_r - w_m) - |w_r - w_m| \leq 0, \qquad \Delta_w^- = -(w_m - w_\ell) < 0.$$

Since $w_\ell < w_m$ and $f_m = q$, we have that $v_m^\flat \geq v_\ell^\flat \geq \dot{v} \geq \hat{v}_\ell \geq \hat{v}_m = v_m$ we have

$$\Delta_{\hat{v}}^- = -(\hat{v}_\ell - v_m) \leq 0, \qquad \Delta_{v^\flat}^- = -(v_m^\flat - v_\ell^\flat) \leq 0,$$
$$\Delta\gamma = -\left(\min\left\{v_m^\flat, v_r\right\} - v_m\right) \leq 0.$$

In conclusion we have that if $v_r \in (v_m, \dot{v}]$ then $\hat{v}_\ell \geq v_r$, $w_\ell \leq w_r < w_m$ and

$$\Delta\Gamma \leq -w_m - 2w_r + 3w_\ell - 2\left(\hat{v}_\ell - \min\left\{v_m^\flat, v_r\right\}\right) - 2\left(v_m^\flat - v_\ell^\flat\right) \leq -3\varepsilon,$$

while if $v_r > \dot{v}$ then $w_\ell \leq \dot{w}$, $w_\ell < w_m$, $\hat{v}_\ell = \dot{v} = v_\ell^\flat$ and

$$\Delta\Gamma \leq -w_m - 2\dot{w} + 3w_\ell - 2\left(v_m^\flat - \min\left\{v_m^\flat, v_r\right\}\right) \leq -\varepsilon.$$

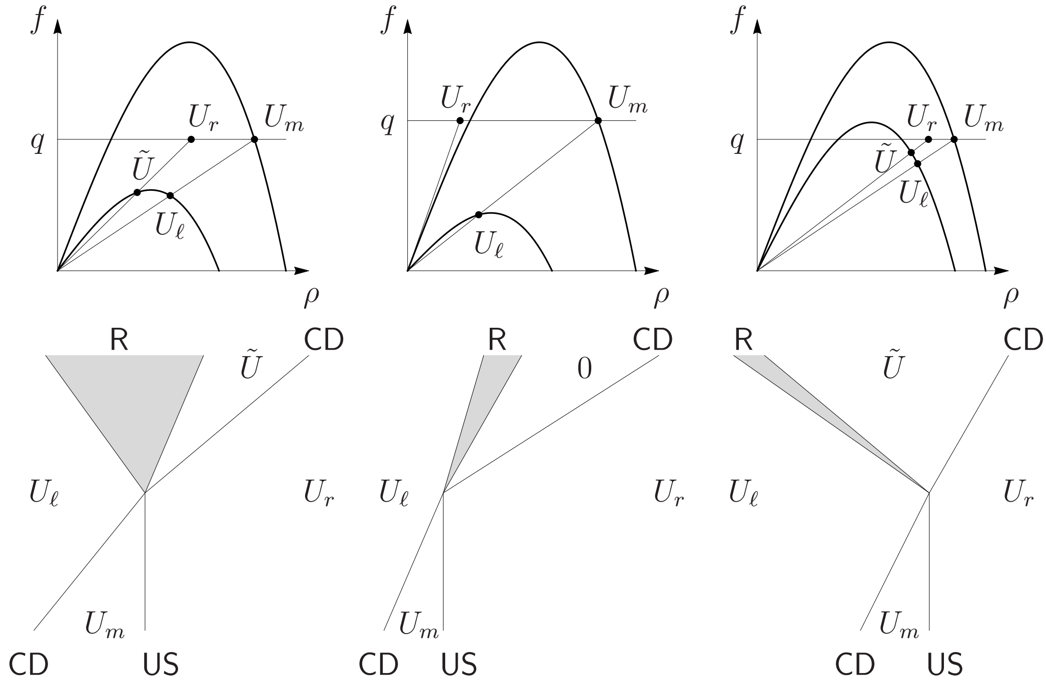

Figure 5: Case **A2.a**.

A2.b If $(U_\ell, U_r) \in \Omega_2$ and $\dot{w} < w_\ell < w_m$, then $\Delta\sharp \geq 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a fan of Rs $(U_\ell, \hat{U}_\ell)$ and an US $(\hat{U}_\ell, U_r)$, see Figure 6, left. Since $v_r > \hat{v}_\ell > v_\ell = v_m$ and $\breve{w}(\hat{v}_\ell) = \hat{w}_\ell = w_\ell < \breve{w}_\ell = w_m$, we have

$$\Delta_v = 0, \qquad\qquad \Delta_\eta = (w_m - w_\ell) > 0.$$

Since $w_\ell = \hat{w}_\ell < w_m$ we have that

$$\Delta_w = |w_r - w_\ell| - ((w_m - w_\ell) + |w_r - w_m|) \leq 0, \qquad\qquad \Delta_w^- = -(w_m - w_\ell) < 0.$$

Since $\dot{w} < w_\ell < w_m$ and $f_m = q$, we have that $v_m^\flat > v_\ell^\flat > \hat{v}_\ell > v_\ell = v_m = \hat{v}_m$ we have

$$\Delta_{\hat{v}}^- = -(\hat{v}_\ell - v_\ell) < 0, \qquad \Delta_{v^\flat}^- = -(v_m^\flat - v_\ell^\flat) < 0,$$
$$\Delta\gamma = -\left(\hat{v}_\ell - v_\ell\right) - \left(\min\left\{v_m^\flat, v_r\right\} - \min\left\{v_\ell^\flat, v_r\right\}\right) < 0.$$

In conclusion we have that $\Delta\sharp \geq 0$ and

$$\begin{aligned}\Delta\Gamma &\leq 3\Delta_w^- + 2(\Delta_\eta + \Delta_{\hat{v}}^- + \Delta_{v^\flat}^- - \Delta\gamma)\\ &\leq -(w_m - w_\ell) - 2(v_m^\flat - v_\ell^\flat) + 2\left(\min\left\{v_m^\flat, v_r\right\} - \min\left\{v_\ell^\flat, v_r\right\}\right)\\ &\leq -(w_m - w_\ell) \leq -\varepsilon.\end{aligned}$$

A2.c If $(U_\ell, U_r) \in \Omega_2$ and $\dot{w} < w_m < w_\ell$, then $\Delta\sharp = 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a S $(U_\ell, \hat{U}_\ell)$ and an US $(\hat{U}_\ell, U_r)$, see Figure 6, right. Since $v_r > v_m = v_\ell > \hat{v}_\ell$ and no R is involved, we have

$$\Delta_v = 2(v_\ell - \hat{v}_\ell) > 0, \qquad\qquad \Delta_\eta = 0.$$

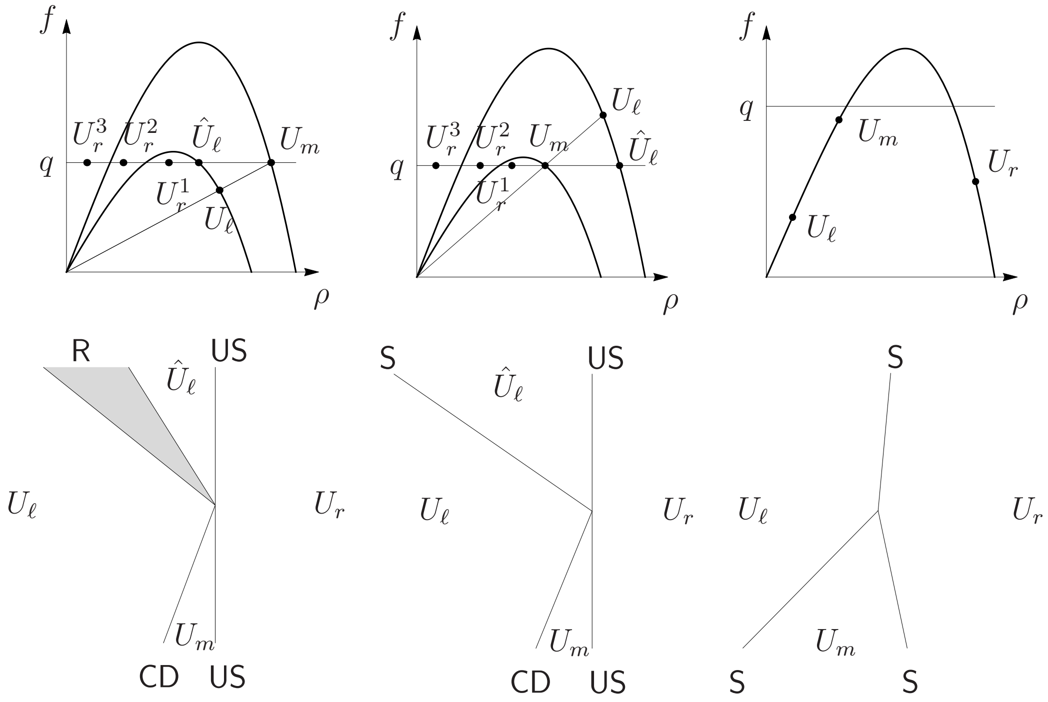

Figure 6: Cases **A2.b**, **A2.c** and **A3**.

Since $w_m < w_\ell = \hat{w}_\ell$ we have that

$$\Delta_w = |w_r - w_\ell| - ((w_\ell - w_m) + |w_r - w_m|) \leq 0, \qquad \Delta_{\check{w}}^- = -(w_\ell - w_m) < 0.$$

Since $\dot{w} < w_m < w_\ell$ and $f_m = q$, we have that $v_\ell^\flat > v_m^\flat > v_m = \hat{v}_m = v_\ell > \hat{v}_\ell$ we have

$$\Delta_{\hat{v}}^- = -(v_\ell - \hat{v}_\ell) < 0, \qquad \Delta_{v^\flat}^- = -(v_\ell^\flat - v_m^\flat) < 0,$$
$$\Delta\gamma = (v_\ell - \hat{v}_\ell) + \Big(\min\Big\{v_\ell^\flat, v_r\Big\} - \min\Big\{v_m^\flat, v_r\Big\}\Big) \geq 0.$$

In conclusion we have that $\Delta\sharp = 0$ and $\Delta\Gamma < \Delta_v + 2\Delta_{\hat{v}}^- = 0$.

A3 Assume that (U_ℓ, U_m) and (U_m, U_r) are two shocks. Clearly $v_r < v_m < v_\ell$, $w_\ell = w_m = w_r$ and $\max\{f_\ell, f_m, f_r\} \leq q$. Hence, the solution consists of a shock (U_ℓ, U_r), see Figure 6. In this case we have

$$\Delta_v = 0, \quad \Delta_w = 0, \quad \Delta_{\hat{v}}^- = 0, \quad \Delta_{v^\flat}^- = 0, \quad \Delta\gamma = 0, \quad \Delta_{\check{w}}^- = 0, \quad \Delta_\eta = 0, \quad \Delta\Gamma = 0.$$

A4 Assume that (U_ℓ, U_m) is a CD and (U_m, U_r) is a S. Clearly $w_m = w_r$, $v_\ell = v_m > v_r = \tilde{v}$ and $\max\{f_m, f_r\} \leq q$.

A4.a If $(U_\ell, U_r) \in \Omega_1$, namely $\min\{f_\ell, \tilde{f}\} \leq q$, then $\Delta\sharp = 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a S $(U_\ell, \tilde{U})$ and a CD $(\tilde{U}, U_r)$, see Figure 7. Since $v_\ell = v_m > v_r = \tilde{v}$, $w_\ell = \tilde{w}$, $w_m = w_r$ and neither R or US are involved, we have

$$0 = \Delta_v = \Delta_\eta = \Delta_w = \Delta\gamma, \qquad \Delta_{\check{w}}^- = -|w_\ell - w_r| < 0,$$
$$\Delta_{\hat{v}}^- = -|\hat{v}_\ell - \hat{v}_r| < 0, \qquad \Delta_{v^\flat}^- = -|v_\ell^\flat - v_r^\flat| < 0.$$

In conclusion we have that $\Delta\sharp = 0$ and $\Delta\Gamma < 0$.

A4.b If $(U_\ell, U_r) \in \Omega_2$, namely $\min\{f_\ell, \tilde{f}\} > q$, then $w_\ell = \tilde{w} > w_m = w_r$, $v_r > \hat{v}_\ell$ and $\Delta\sharp \geq 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a S $(U_\ell, \hat{U}_\ell)$, an US $(\hat{U}_\ell, \check{U}_r)$ and a possibly null CD $(\check{U}_r, U_r)$, see Figure 7. Since $w_m = w_r < \check{w}_r < w_\ell = \tilde{w}$, $\hat{v}_\ell < \tilde{v} = v_r = \check{v}_r < v_\ell = v_m < v_\ell^\flat$ and no R is involved, we have $\hat{v}_\ell < \hat{v}_r \leq v_r^\flat < v_\ell^\flat$ and

$$\Delta_v = 2(v_r - \hat{v}_\ell) > 0, \qquad \Delta_{\hat{v}}^- = -(\hat{v}_r - \hat{v}_\ell) < 0, \qquad \Delta_{v^\flat}^- = -(v_\ell^\flat - v_r^\flat) < 0,$$
$$\Delta_w = 0, \qquad \Delta_{\check{w}}^- = -(w_\ell - w_m) < 0, \qquad \Delta\gamma = (v_r - \hat{v}_\ell) > 0, \qquad \Delta_\eta = 0.$$

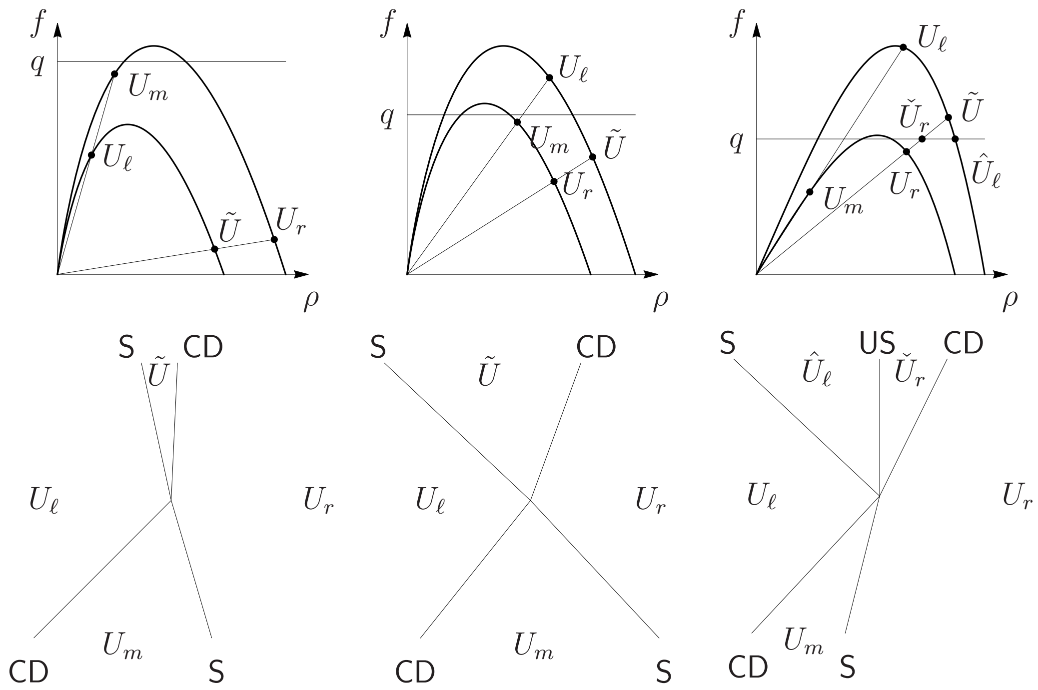

Figure 7: Cases **A4.a** and **A4.b**.

In conclusion we have that $\Delta\sharp \geq 0$ and $\Delta\Gamma < \Delta_v + 2(\Delta^-_{\hat v} - \Delta\gamma) = -2(\hat v_r - \hat v_\ell) \leq -2\varepsilon$.

A5 Assume that (U_ℓ, U_m) is an US and (U_m, U_r) is a S. Clearly $w_m = w_r$, $v_m > \max\{v_\ell, v_r\}$ and $f_r < f_m = q = f_\ell$.

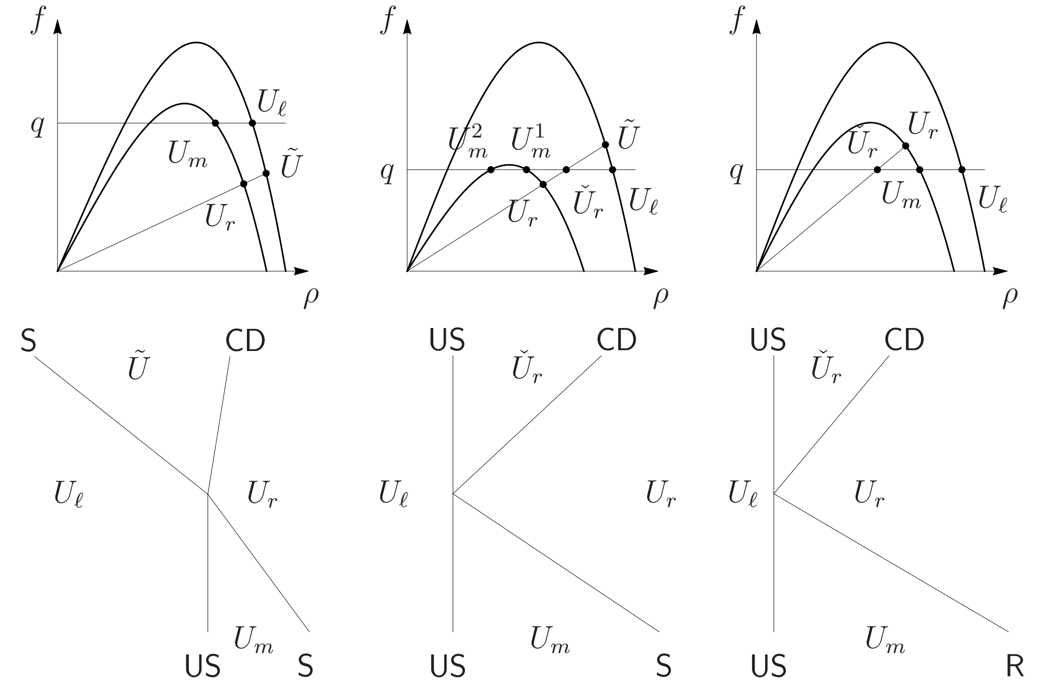

Figure 8: Cases **A5.a**, **A5.b** and **A6**.

A5.a If $(U_\ell, U_r) \in \Omega_1$, then $v_r < v_\ell$, $\max\{f_r, \tilde f\} < f_\ell = q = f_m$. Moreover $\Delta\sharp \leq 0$ because $\mathcal{RS}^q_2[U_\ell, U_r]$ has a S $(U_\ell, \tilde U)$ and a possibly null CD $(\tilde U, U_r)$, see Figure 8. Clearly

$$0 = \Delta^-_{\hat v} = \Delta^-_{v^\flat} = \Delta^-_w = \Delta_\eta.$$

Since $v_r = \tilde{v} < v_\ell < v_m$, $v_\ell \le v_\ell^\flat$, $w_\ell = \tilde{w}$ and $w_m = w_r$, we have

$$\Delta_v = -2(v_m - v_\ell) < 0, \qquad \Delta\gamma = -\left[\min\{v_\ell^\flat, v_m\} - v_\ell\right] \le 0, \qquad \Delta_w = 0.$$

In conclusion we have that $\Delta\sharp \le 0$ and $\Delta\Gamma = -2\left[v_m - \min\{v_\ell^\flat, v_m\}\right] \le 0$.

A5.b If $(U_\ell, U_r) \in \Omega_2$, then $v_\ell < v_r$, $f_r < f_\ell = q = f_m < \tilde{f}$. Moreover $\Delta\sharp = 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has an US $(U_\ell, \check{U}_r)$ and a CD $(\check{U}_r, U_r)$, see Figure 8. Clearly

$$0 = \Delta_{\hat{v}}^- = \Delta_{v^\flat}^- = \Delta_w^- = \Delta_\eta.$$

Since $v_\ell < v_r = \check{v}_r < v_m < v_\ell^\flat$ and $w_m = w_r < \check{w}_r < w_\ell$, we have

$$\Delta_v = -2(v_m - v_r) < 0, \qquad \Delta_w = 0, \qquad \Delta\gamma = -(v_m - v_r) < 0.$$

In conclusion we have that $\Delta\sharp = 0$ and $\Delta\Gamma = 0$.

A6 Assume that (U_ℓ, U_m) is an US and (U_m, U_r) a R. Clearly $w_m = w_r < w_\ell$, $v_m \le v_r - \varepsilon$ and $q = f_\ell = f_m < f_r$. Moreover $\Delta\sharp = 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has an US $(U_\ell, \check{U}_r)$ and a CD $(\check{U}_r, U_r)$, see Figure 8. Clearly

$$0 = \Delta_{\hat{v}}^- = \Delta_{v^\flat}^- = \Delta_w^-.$$

Since $v_\ell^\flat > \check{v}_r = v_r > v_m > v_\ell$ and $w_\ell > w_m = \check{w}_m = w_r > \check{w}_r$, we have

$$\Delta_v = 0, \qquad \Delta_w = 2(w_r - \check{w}_r) > 0, \qquad \Delta\gamma = (v_r - v_m) > 0, \qquad \Delta_\eta = -(w_r - \check{w}_r) < 0.$$

In conclusion we have that $\Delta\sharp = 0$ and $\Delta\Gamma = -2(v_r - v_m) < -2\varepsilon$.

A7 Assume that (U_ℓ, U_m) is a R and (U_m, U_r) a S. Clearly $w_\ell = w_m = w_r$ and $v_\ell \ge v_m - \varepsilon$. It is easy to check that $(U_\ell, U_r) \in \Omega_1$. Thus $\mathcal{RS}_2^q[U_\ell, U_r]$ is a S (U_ℓ, U_r) and $\Delta\sharp = -1$, see Figure 9. Clearly

$$0 = \Delta_w = \Delta_{\hat{v}}^- = \Delta_{v^\flat}^- = \Delta_w^- = \Delta\gamma, \qquad \Delta_\eta \le 0.$$

Since $v_m > v_\ell > v_r$ we have

$$\Delta_v = -2(v_m - v_\ell) < 0.$$

In conclusion we have that $\Delta\sharp = -1$ and $\Delta\Gamma < 0$.

A8 Assume that (U_ℓ, U_m) is a CD and (U_m, U_r) is a R. Clearly $v_\ell = v_m < v_r$, $w_m = w_r$ and $\min\{f_m, f_r\} \le q$.

A8.a If $(U_\ell, U_r) \in \Omega_1$, namely $\max\{f_\ell, \tilde{f}\} \le q$, then $\mathcal{RS}_2^q[U_\ell, U_r]$ has a single R $(U_\ell, \tilde{U})$ and a CD $(\tilde{U}, U_r)$, see Figure 9. Clearly

$$\Delta_{\hat{v}}^- = \begin{Bmatrix} -|\hat{v}_\ell - \hat{v}_m| & \text{if } v_\ell > 0 \\ 0 & \text{if } v_\ell = 0 \end{Bmatrix} \le 0, \qquad \Delta_{v^\flat}^- = \begin{Bmatrix} -|v_\ell^\flat - v_m^\flat| & \text{if } v_\ell > 0 \\ 0 & \text{if } v_\ell = 0 \end{Bmatrix} \le 0,$$

$$\Delta_w^- = \begin{Bmatrix} -|w_\ell - w_m| & \text{if } v_\ell > 0 \\ 0 & \text{if } v_\ell = 0 \end{Bmatrix} \le 0, \qquad \Delta\gamma = 0.$$

Since $v_\ell = v_m < v_r = \tilde{v}$, $\tilde{w} = w_\ell$ and $w_m = w_r$, we have

$$\Delta_v = 0, \qquad \Delta_w = 0, \qquad \Delta_\eta = 0.$$

In conclusion we have that $\Delta\sharp = 0$ and $\Delta\Gamma \le 0$.

A8.b If $(U_\ell, U_r) \in \Omega_2$, namely $\max\{f_\ell, \tilde{f}\} > q$, then $\Delta\sharp \ge -1$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a possibly null S $(U_\ell, \hat{U}_\ell)$, an US $(\hat{U}_\ell, \check{U}_r)$ and a possibly null CD $(\check{U}_r, U_r)$, see Figure 9. Since $\check{w}_r \ge w_r - \varepsilon$, $w_m = w_r < w_\ell = \hat{w}_\ell$, $\hat{v}_\ell \le v_\ell = v_m < \check{v}_r = v_r = \tilde{v} \le v_\ell^\flat$, $\hat{v}_\ell < \hat{v}_m = \hat{v}_r$ and $v_\ell^\flat > v_m^\flat = v_r^\flat$, we have

$$\Delta_v = 2(v_\ell - \hat{v}_\ell) \ge 0, \qquad \Delta_w = \begin{Bmatrix} 2(w_r - \check{w}_r) & \text{if } f_m = q < f_r \\ 0 & \text{otherwise} \end{Bmatrix} \ge 0,$$

$$\Delta_{\hat{v}}^- = -(\hat{v}_r - \hat{v}_\ell) < 0, \qquad \Delta_w^- = -(w_\ell - w_r) < 0,$$

$$\Delta_{v^\flat}^- = -(v_\ell^\flat - v_r^\flat) < 0, \qquad \Delta_\eta = \begin{Bmatrix} -(\check{w}_m - \check{w}_r) & \text{if } \hat{v}(w_{\max}) \le v_m < v_r \le \dot{v} \\ 0 & \text{otherwise} \end{Bmatrix} \le 0,$$

$$\Delta\gamma = (v_r - \hat{v}_\ell) > 0.$$

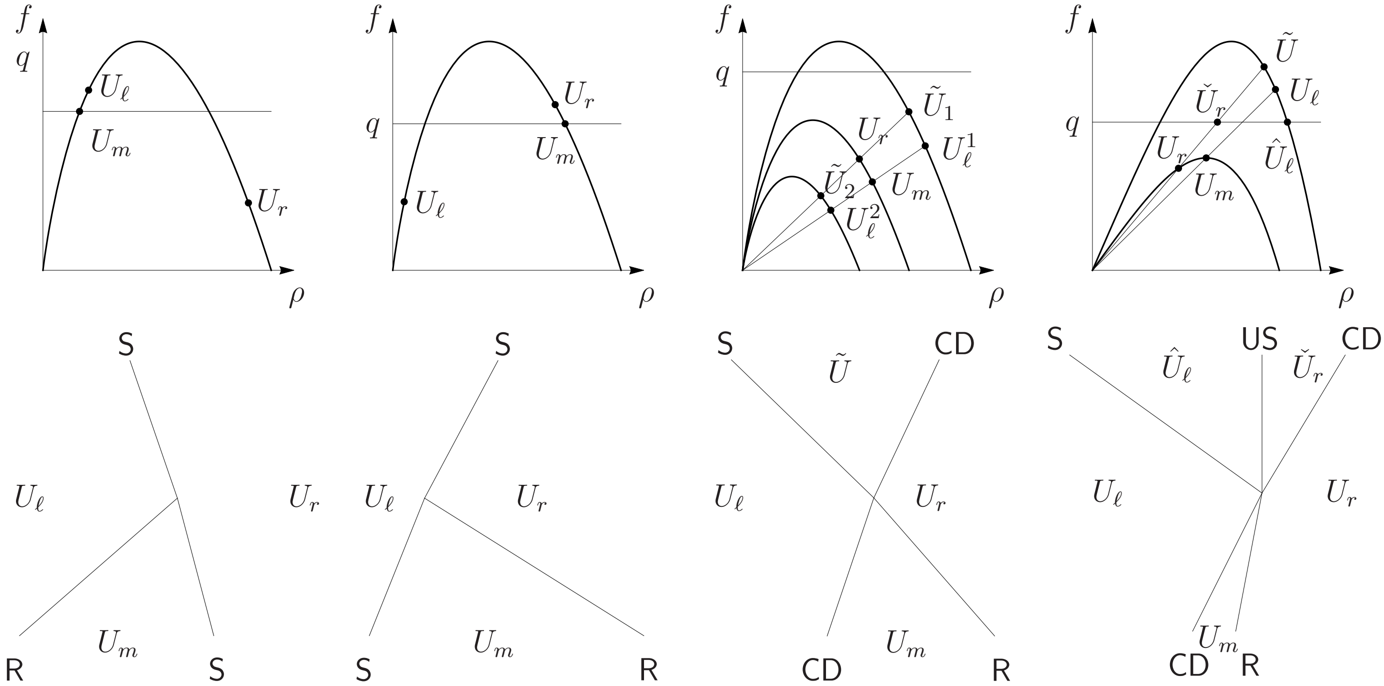

Figure 9: Cases **A7**, **A8.a** and **A8.b**.

In conclusion we have that $\Delta\sharp \geq -1$ and

$$\begin{aligned}\Delta\Gamma &< \Delta_v - 2\Delta\gamma + \Delta_w + 2\Delta_\eta \\ &= -2(v_r - v_\ell) + \begin{Bmatrix} 2(w_r - \check{w}_m) = 0 & \text{if } f_m = q < f_r \\ 0 & \text{otherwise} \end{Bmatrix} \leq -2\varepsilon.\end{aligned}$$

A9 Assume that (U_ℓ, U_m) is a S and (U_m, U_r) is a R. Clearly $v_m < v_r < v_\ell$ and $w_\ell = w_m = w_r$. By observing that $f(\mathcal{RS}[U_\ell, U_r])(0^\pm) = \min\{f_\ell, f_r\} \leq q$, we have that $(U_\ell, U_r) \in \Omega_1$. Hence $\mathcal{RS}_2^q[U_\ell, U_r] \equiv \mathcal{RS}[U_\ell, U_r]$ has a S (U_ℓ, U_r) and $\Delta\sharp = -1$, see Figure 10. As a consequence $\Delta_\eta \leq 0$, $\Delta\gamma = 0$ because no US is involved, $\Delta_w = \Delta_w^- = \Delta_{\hat{v}}^- = \Delta_{v^\flat}^- = 0$ because no CD is involved. Since $v_m < v_m + \varepsilon \leq v_r < v_\ell$ we have $\Delta_v = -2(v_r - v_m) < 0$. In conclusion $\Delta\sharp = -1$ and $\Delta\Gamma < 0$.

B Assume now that at time $t_i > 0$ exactly one wave (U_ℓ, U_r) reaches $x = 0$. We distinguish the following cases:

B1 Assume (U_ℓ, U_r) is a CD. Clearly $v_\ell = v_r$ and $f_r \leq q$.

B1.a If $(U_\ell, U_r) \in \Omega_1$, namely $f_\ell \leq q$, then $\Delta\sharp = 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a CD (U_ℓ, U_r), see Figure 10, left. Clearly $0 = \Delta_v = \Delta_w = \Delta\gamma = \Delta_\eta$ and

$$\Delta_w^- = -|w_\ell - w_r| \leq -\varepsilon, \qquad \Delta_{\hat{v}}^- = -|\hat{v}_\ell - \hat{v}_r| \leq -\varepsilon, \qquad \Delta_{v^\flat}^- = -|v_\ell^\flat - v_r^\flat| \leq -\varepsilon,$$

therefore $\Delta\Gamma = 3\Delta_w^- + 2(\Delta_{\hat{v}}^- + \Delta_{v^\flat}^-) \leq -7\varepsilon \leq 0$.

B1.b If $(U_\ell, U_r) \in \Omega_2$, namely $f_\ell > q$, then $\Delta\sharp > 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a shock $(U_\ell, \hat{U}_\ell)$, an US $(\hat{U}_\ell, \check{U}_r)$ and a possibly null CD $(\check{U}_r, U_r)$, see Figure 10, center. Since $v_\ell = v_r = \check{v}_r > \hat{v}_\ell$ and no R is involved, we have

$$\Delta_v = 2(v_\ell - \hat{v}_\ell) > 0, \qquad \Delta_\eta = 0.$$

Since $w_r \leq \check{w}_r < w_\ell = \hat{w}_\ell$ we have that

$$\Delta_w = 0, \qquad \Delta_w^- = -(w_\ell - w_r) < 0.$$

Since $w_r < w_\ell$, we have that $v_\ell^\flat > v_r^\flat > \hat{v}_r > \hat{v}_\ell$, $\check{v}_r = v_r = v_\ell < v_\ell^\flat$ and

$$\Delta_{\hat{v}}^- = -(\hat{v}_r - \hat{v}_\ell) < 0, \qquad \Delta_{v^\flat}^- = -(v_\ell^\flat - v_r^\flat) < 0, \qquad \Delta\gamma = v_\ell - \hat{v}_\ell > 0.$$

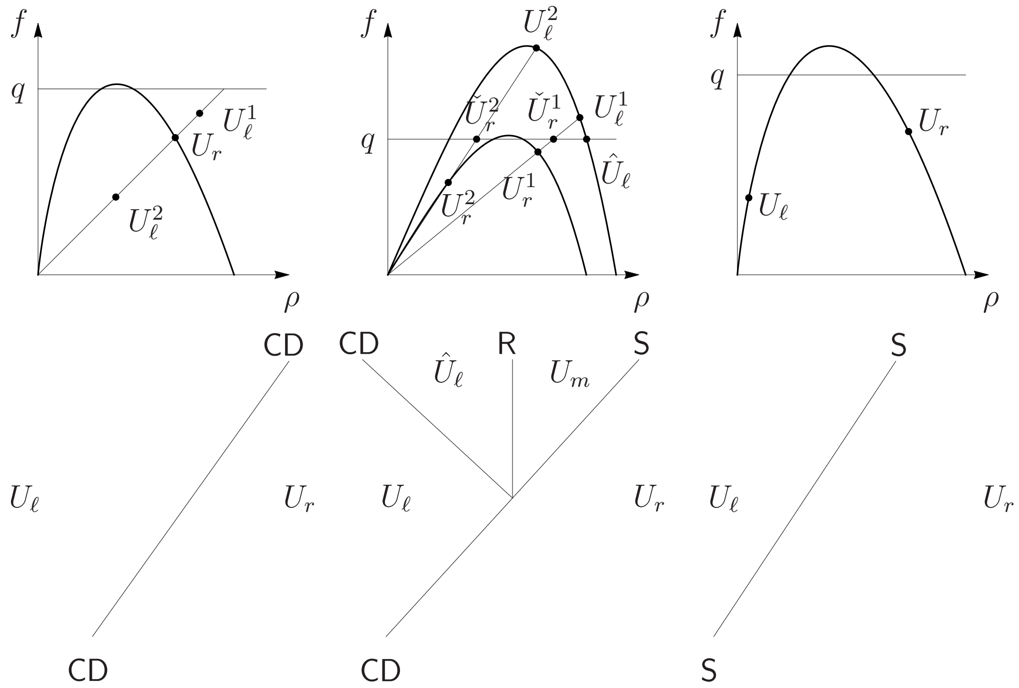

Figure 10: Cases **A9**, **B1.a**, **B1.b** and **B2**.

In conclusion we have that $\Delta\sharp > 0$ and

$$\Delta\Gamma = \Delta_v + 2(\Delta^-_{\hat{v}} + \Delta^-_{v^\flat} - \Delta\gamma) + 3\Delta^-_w = -2(v^\flat_\ell - v^\flat_r) - 2(\hat{v}_r - \hat{v}_\ell) - 3(w_\ell - w_r) \leq -7\varepsilon.$$

B2 Assume that (U_ℓ, U_r) is a S. Clearly $w_\ell = w_r$ and $\max\{f_\ell, f_r\} \leq q$. Hence, $\Delta\sharp = 0$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a S (U_ℓ, U_r), see Figure 10, right. In this case we have

$$\Delta\sharp = 0 = \Delta_v = \Delta_w = \Delta^-_{\hat{v}} = \Delta^-_{v^\flat} = \Delta\gamma = \Delta^-_w = \Delta_\eta = \Delta\Gamma.$$

B3 Assume that (U_ℓ, U_r) is a R. Clearly $w_\ell = w_r$ and $v_\ell \leq v_r - \varepsilon$.

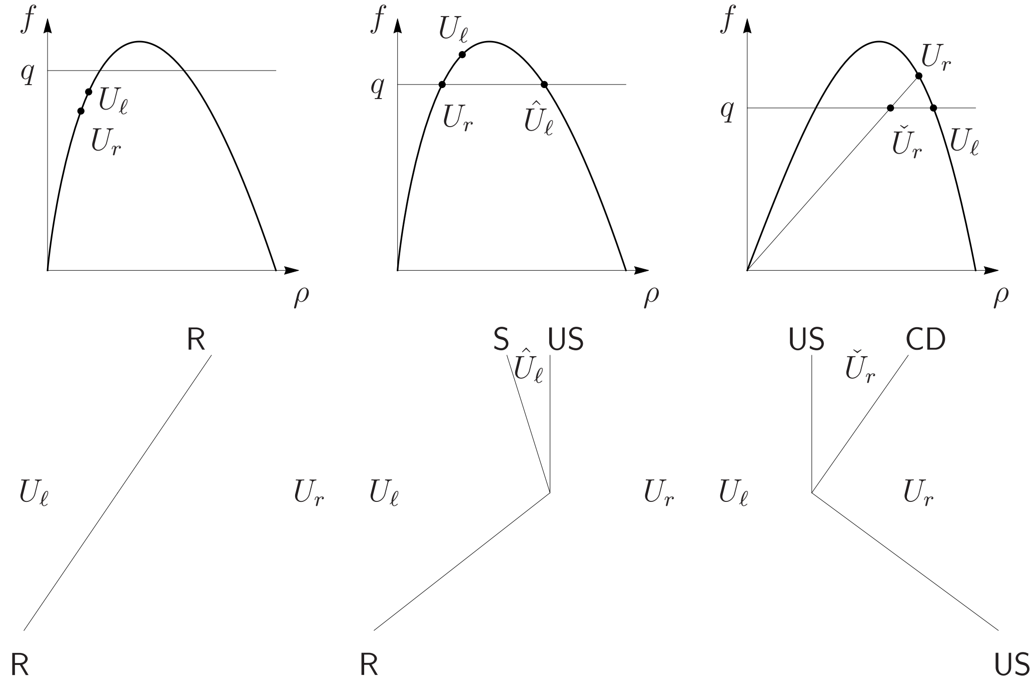

Figure 11: Cases **B3.a**, **B3.b** and **B3.c**.

B3.a If $(U_\ell, U_r) \in \Omega_1$, namely $\max\{f_\ell, f_r\} \le q$, then $\mathcal{RS}_2^q[U_\ell, U_r]$ has a single R (U_ℓ, U_r), see Figure 11, center. In this case we have

$$\Delta\sharp = 0 = \Delta_v = \Delta_w = \Delta^-_{\hat{v}} = \Delta^-_{v^\flat} = \Delta\gamma = \Delta^-_{\check{w}} = \Delta_\eta = \Delta\Gamma.$$

B3.b If $(U_\ell, U_r) \in \Omega_2$ and $f_r = q < f_\ell$, then $\Delta\sharp = 1$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has a S $(U_\ell, \hat{U}_\ell)$ and an US $(\hat{U}_\ell, U_r)$, see Figure 11, right. Since $w_\ell = \hat{w}_\ell = w_r$ and $\hat{v}_\ell < \dot{v} < v_\ell < v_r = v_\ell^\flat$, we have

$$\Delta_v = 2(v_\ell - \hat{v}_\ell) > 0, \qquad 0 = \Delta_w = \Delta^-_{\hat{v}} = \Delta^-_{v^\flat} = \Delta^-_{\check{w}} = \Delta_\eta, \qquad \Delta\gamma = (v_r - \hat{v}_\ell) > 0.$$

In conclusion we have $\Delta\sharp = 1$ and $\Delta\Gamma = -2(v_r - v_\ell) \le -2\varepsilon$.

B3.c If $(U_\ell, U_r) \in \Omega_2$ and $f_\ell = q < f_r$, then $\Delta\sharp = 1$ because $\mathcal{RS}_2^q[U_\ell, U_r]$ has an US $(U_\ell, \check{U}_r)$ and a CD $(\check{U}_r, U_r)$, see Figure 9. Since $\check{w}_r < w_r = w_\ell = \check{w}_\ell$ and $v_\ell < \check{v}_r = v_r < v_\ell^\flat$, we have

$$0 = \Delta_v = \Delta^-_{\hat{v}} = \Delta^-_{v^\flat} = \Delta^-_{\check{w}}, \qquad \Delta_\eta = -(w_\ell - \check{w}_r) < 0,$$
$$\Delta\gamma = (v_r - v_\ell) > 0, \qquad \Delta_w = 2(w_r - \check{w}_r) = -2\Delta_\eta > 0.$$

In conclusion we have $\Delta\sharp = 1$ and $\Delta\Gamma = -2(v_r - v_\ell) \le -2\varepsilon$.

C At last, assume that at time $t_i > 0$ exactly one interaction between two waves occurs at $x \neq 0$. In this case $\Delta_v \le 0$, the number of the waves does not increase, $\Delta\sharp \le 0$, the size of the jumps in the w-coordinate does not change, hence $0 = \Delta_w = \Delta^-_{\check{w}} = \Delta^-_{\hat{v}} = \Delta^-_{v^\flat}$, no R is created, hence $\Delta_\eta \le 0$, and clearly no USs are involved, hence $\Delta\gamma$. As a consequence $\Delta\sharp \le 0$ and $\Delta\Gamma \le 0$. □

5 Proof of Theorem 2.1

We approximate U_0 with $U_0^n \in \mathbf{PC}(\mathbb{R}; \mathcal{D}_n)$ such that:

$$\|U_0^n\|_{\mathbf{L^\infty}(\mathbb{R})} \le \|U_0\|_{\mathbf{L^\infty}(\mathbb{R})}, \qquad \lim_{n\to\infty} \|U_0 - U_0^n\|_{\mathbf{L^1}(\mathbb{R})} = 0, \qquad \Gamma(U_0^n) \le \Upsilon(U_0). \tag{10}$$

By Lemma 4.2, the approximate solution U^n can be constructed for any time $t > 0$ and the map $t \to \Gamma(U^n(t))$ is non-increasing. Therefore

$$\mathrm{TV}\left(U^n(t,\cdot)\right) \le \Gamma^n\left(U^n(t,\cdot)\right) \le \Gamma^n(U_0^n) \le \Upsilon(U_0)$$

and a classical application of Helly's Theorem allows to infer that and only finitely many interaction may occur in finite time. Hence, the construction of U^n may be extended globally in time.

It is straightforward to see that $\|U^n(t)\|_{\mathbf{L^\infty}(\mathbb{R})} \le \|U_0\|_{\mathbf{L^\infty}(\mathbb{R})}$ and

$$\|U^n(t,\cdot) - U^n(s,\cdot)\|_{\mathbf{L^1}(\mathbb{R})} \le L|t-s| \tag{11}$$

with $L \doteq \Upsilon(U_0)\max\{v_{\max}, p^{-1}(w_{\max})p'(p^{-1}(w_{\max}))\}$. Indeed, if no interaction occurs for times between t and s, then

$$\begin{aligned}&\|U^n(t,\cdot) - U^n(s,\cdot)\|_{\mathbf{L^1}(\mathbb{R})}\\ &= \sum_{x\in\mathsf{J}(t^+)} \left\|(t-s)\sigma\left(U(t^+,x^-), U(t^+,x^+)\right)\left(U(t^+,x^-) - U(t^+,x^+)\right)\right\| \le L|t-s|,\end{aligned}$$

where $\mathsf{J}(t^+) \subset \mathbb{R}$ is the (finite) set of discontinuity points of $U(t^+,\cdot)$. Moreover, the map $t \mapsto U^n(t,\cdot)$ is $\mathbf{L^1}$-continuous across interaction times.

Therefore, by standard application of Helly's theorem, see Theorem 2.4 in [6], there exists a subsequence, still denoted by $(U^n)_n$, which converges to some function U in $\mathbf{L^1_{loc}}(\mathbb{R}_+ \times \mathbb{R}; \mathcal{D})$ as we let n go to infinity. Moreover this limit function satisfies:

$$\mathrm{TV}(U(t,\cdot)) \le \Upsilon(U_0), \quad \|U(t,\cdot) - U(s,\cdot)\|_{\mathbf{L^1}} \le L|t-s|, \quad \|U(t,\cdot)\|_{\mathbf{L^\infty}} \le \|U_0\|_{\mathbf{L^\infty}}.$$

We are now left to show that the limit function U is indeed a weak solution of (2), (3), (4) in the sense of Definition 2.1.

The initial condition (3) holds by (10), (11) and by the $\mathbf{L^1_{loc}}$-convergence of U_n to U. To prove that U is a weak solution of the Cauchy problem (2), (3), we have to show that it satisfies (5), namely that for any test function $\phi \in \mathbf{C_c^\infty}((0,\infty)\times\mathbb{R};\mathbb{R})$

$$\iint_{\mathbb{R}^+\times\mathbb{R}} (Y\partial_t\phi + vY\partial_x\phi)\, dxdt + q\int_{\mathbb{R}^+} \operatorname{sgn}\left(v(t,0^+)\right)\begin{pmatrix} 0 \\ w(t,0^+)-w(t,0^-)\end{pmatrix}\phi(t,0)dt = 0.$$

Since $Y^n = \Psi(U^n)$ and v^nY^n are uniformly bounded, it is sufficient to prove that

$$\lim_{n\to\infty}\left[\iint_{\mathbb{R}^+\times\mathbb{R}} (Y^n\partial_t\phi + v^nY^n\partial_x\phi)\, dxdt + q\int_{\mathbb{R}^+} \operatorname{sgn}\left(v^n(t,0^+)\right)\begin{pmatrix} 0 \\ w(t,0^+)-w(t,0^-)\end{pmatrix}\phi(t,0)dt\right] = 0. \tag{12}$$

Choose $T>0$ such that $\phi(t,x)=0$ whenever $t\geq T$. By the Green-Gauss formula the double integral in (12) can be written as

$$\int_0^T \sum_{x\in \mathtt{J}(t)} \left[\sigma\left(U^n(t,x^-),U^n(t,x^+)\right)\Delta Y^n(t,x) - \Delta F^n(t,x)\right]\phi(t,x)dt,$$

where

$$\Delta Y^n(t,x) \doteq Y^n(t,x^+) - Y^n(t,x^-), \qquad \Delta F^n(t,x) \doteq v^n(t,x^+)Y^n(t,x^+) - v^n(t,x^-)Y^n(t,x^-).$$

We may observe that if the discontinuity at the point (t,x) is not an undercompressive shock, then it satisfies Rankine-Hugoniot condition and

$$\sigma\left(U^n(t,x^-),U^n(t,x^+)\right)\Delta Y^n(t,x) - \Delta F^n(t,x) = 0,$$

otherwise

$$\sigma\left(U^n(t,0^-),U^n(t,0^+)\right)\Delta Y^n(t,0) - \Delta F^n(t,0) = -\Delta F^n(t,0).$$

We point out that the second integrand in (12) is not equal to 0 if and only if the stationary wave at $x=0$ is an undercompressive shock and $q>0$. If $q=0$ then undercompressive shocks coincide with contact discontinuities and satisfy the Rankine-Hugoniot condition. As a consequence we have

$$\int_0^T \sum_{x\in \mathtt{J}(t)} \left[\sigma\left(U^n(t,x^-),U^n(t,x^+)\right)\Delta Y^n(t,x) - \Delta F^n(t,x)\right]\phi(t,x)dt =$$

$$-\int_0^T \sum_{\substack{w(t,0^+)\neq w(t,0^-)\\ v(t,0^+)>v(t,0^-)}} \Delta F^n(t,0)\phi(t,0)dt = -\int_0^T \operatorname{sgn}\left(v^n(t,0^+)\right)\Delta F^n(t,0)\phi(t,0)dt$$

and then the Eq. (12) holds and concludes the proof that U is a weak solution of (2), (3).

It remains to show that U satisfies the constraint (4). To show this, we exploit the fact that both U and U^n are weak solutions of (2), (3) in the half planes $\mathbb{R}_+\times\mathbb{R}_-$ and $\mathbb{R}_+\times\mathbb{R}_+$ and by construction $f(U^n(t,0^-)) = f(U^n(t,0^+)) \leq q$ for all $t>0$. Therefore, by applying Gauss-Green formula to the weak formulation of the first equation in (2), $\partial_t\rho^n + \partial_x f(U^n) = 0$, we find

$$q\int_{\mathbb{R}_+}\psi(t)dt \geq \int_{\mathbb{R}_+} f\left(U^n(t,0^-)\right)\psi(t)dt = \iint_{\mathbb{R}^+\times\mathbb{R}^-}\left(\rho^n(t,x)\dot\psi(t)\xi(x) + f\left(U^n(t,x)\right)\psi(t)\dot\xi(x)\right)dxdt,$$

where ψ is an arbitrary $\mathbf{C_c^\infty}$ test function of time with compact support in $]0,\infty[$ and ξ is some fixed $\mathbf{C_c^\infty}$ test function of space such that $\xi(0)=1$. Passing to the limit $n\to\infty$ in the right-hand side and applying again the Green-Gauss formula we obtain

$$q\int_{\mathbb{R}_+}\psi(t)dt \geq \lim_{n\to\infty}\int_{\mathbb{R}_+} f\left(U^n(t,0^-)\right)\psi(t)dt = \int_{\mathbb{R}_+} f\left(U(t,0^-)\right)\psi(t)dt.$$

Therefore the trace $f(U^n(t,0^-))$ weakly converges to the trace $f(U(t,0^-))$ and we have $f(U(t,0^-))\leq q$ for a.e. $t>0$. At last, since U is a weak solution in the whole $\mathbb{R}_+\times\mathbb{R}$, we have that $f(U(t,0^+)) = f(U(t,0^+)) \leq q$ for a.e. $t>0$, namely also (4) is satisfied.

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