# ADJUNCTIONS AND BRAIDED OBJECTS 

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#### Abstract

In this paper we investigate the categories of braided objects, algebras and bialgebras in a given monoidal category, some pairs of adjoint functors between them and their relations. In particular we construct a braided primitive functor and its left adjoint, the braided tensor bialgebra functor, from the category of braided objects to the one of braided bialgebras. The latter is obtained by a specific elaborated construction introducing a braided tensor algebra functor as a left adjoint of the forgetful functor from the category of braided algebras to the one of braided objects. The behaviour of these functors in the case when the base category is braided is also considered.


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## Introduction

Let $B$ be a braided bialgebra over a field $\mathbb{k}$. This means that $B$ is both an algebra and a coalgebra and these structures are suitably compatible with a braiding $c: B \otimes B \rightarrow B \otimes B$ of $B$. It is well-known that $c$ induces a braiding $c_{P}$ on the space $P=P(B)$ of primitive elements of $B$, see e.g. Kh, page 4] in the connected case. It is natural to wonder whether this result remains true for braided bialgebras in a monoidal category $\mathcal{M}$. Note that $\mathcal{M}$ needs not to be braided, a priori exactly as the above braiding $c$ needs not to be the evaluation of a braiding defined on the whole category of vector spaces. On the other hand it is also well-known that, under mild assumptions, the forgetful functor $\Omega: \operatorname{Alg}_{\mathcal{M}} \rightarrow \mathcal{M}$ from the category of algebras into $\mathcal{M}$ has a left adjoint $T: \mathcal{M} \rightarrow \operatorname{Alg}_{\mathcal{M}}$ given by the tensor algebra functor, see Remark 1.3.

In this paper we prove that, under mild assumptions, the forgetful functor $\Omega_{\mathrm{Br}}: \operatorname{BrAlg} \mathcal{M}_{\mathcal{M}} \rightarrow \mathrm{Br}_{\mathcal{M}}$ from the category $\operatorname{BrAlg}_{\mathcal{M}}$ of braided algebras in $\mathcal{M}$ to the category $\mathrm{Br}_{\mathcal{M}}$ of braided objects in $\mathcal{M}$ has a left adjoint $T_{\mathrm{Br}}$, see Proposition 3.1, which is induced by $T$. This is achieved by a rather technical tool which makes use of suitable morphisms $c_{T}^{m, n}$ constructed in Proposition 2.7 by means of Lemma 2.6, where the Braid Category plays a central role. We also introduce a braided primitive functor $P_{\mathrm{Br}}: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \mathrm{Br}_{\mathcal{M}}$ where $\operatorname{BrBialg}_{\mathcal{M}}$ denotes the category of braided bialgebras in $\mathcal{M}$, see Lemma 3.3. We prove that this functor $P_{\mathrm{Br}}$ has also a left adjoint, namely the functor $\bar{T}_{\mathrm{Br}}$ which is induced by the functor $T_{\mathrm{Br}}$.

Another problem is to investigate the case when the monoidal category $\mathcal{M}$ is braided. In this case one can also consider the category $\operatorname{Bialg}_{\mathcal{M}}$ of bialgebras in $\mathcal{M}$. Moreover, the categories $\mathcal{M}$,

[^0]$A \lg _{\mathcal{M}}$ and $\operatorname{Bialg} \mathcal{M}_{\mathcal{M}}$ are related to their braided analogues by means of the functors
$$
J: \mathcal{M} \rightarrow \operatorname{Br}_{\mathcal{M}}, \quad J_{\mathrm{Alg}}: \operatorname{Alg}_{\mathcal{M}} \rightarrow \operatorname{BrAlg}_{\mathcal{M}} \quad \text { and } \quad J_{\text {Bialg }}: \operatorname{Bialg}_{\mathcal{M}} \rightarrow \operatorname{BrBialg}_{\mathcal{M}}
$$
see Proposition 4.4. Using these functors we investigate the relation between $T$ and $T_{\mathrm{Br}}$ (Proposition 4.5). We also show that $P_{\mathrm{Br}}$ gives rise to a primitive functor $P: \operatorname{Bialg}_{\mathcal{M}} \rightarrow \mathcal{M}$, see Proposition 4.7. We prove that even this functor $P$ has a left adjoint $\bar{T}$ (Theorem 4.9) which is related to the functor $\bar{T}_{\mathrm{Br}}$ as in (71).

In this paper we also investigate the behaviour of the primitive functors mentioned above when the base category changes through an arbitrary monoidal functors $F: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$. This is done in Proposition 2.5. Proposition 3.6 and Proposition 4.10.
The adjunctions considered above will be studied in connection with monadic decomposition of functors in a forthcoming paper where the particular cases when $\mathcal{M}$ or $\mathcal{M}^{\prime}$ are the category of vector spaces or the category of (co)modules over a not necessarily finite-dimensional (dual) quasibialgebra will be investigated.

## 1. Preliminaries

In this section, we shall fix some basic notation and terminology.
Notation 1.1. Throughout this paper $\mathfrak{k}$ will denote a field. All vector spaces will be defined over $\mathbb{k}$. The unadorned tensor product $\otimes$ will denote the tensor product over $\mathbb{k}$ if not stated otherwise.
1.2. Monoidal Categories. Recall that (see Ka, Chap. XI]) a monoidal category is a category $\mathcal{M}$ endowed with an object $1 \in \mathcal{M}$ (called unit), a functor $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ (called tensor product), and functorial isomorphisms $a_{X, Y, Z}:(X \otimes Y) \otimes Z \rightarrow X \otimes(Y \otimes Z), l_{X}: \mathbf{1} \otimes X \rightarrow X$, $r_{X}: X \otimes \mathbf{1} \rightarrow X$, for every $X, Y, Z$ in $\mathcal{M}$. The functorial morphism $a$ is called the associativity constraint and satisfies the Pentagon Axiom, that is the equality

$$
\left(U \otimes a_{V, W, X}\right) \circ a_{U, V \otimes W, X} \circ\left(a_{U, V, W} \otimes X\right)=a_{U, V, W \otimes X} \circ a_{U \otimes V, W, X}
$$

holds true, for every $U, V, W, X$ in $\mathcal{M}$. The morphisms $l$ and $r$ are called the unit constraints and they obey the Triangle Axiom, that is $\left(V \otimes l_{W}\right) \circ a_{V, \mathbf{1}, W}=r_{V} \otimes W$, for every $V, W$ in $\mathcal{M}$.

A monoidal functor (also called strong monoidal in the literature)

$$
\left(F, \phi_{0}, \phi_{2}\right):(\mathcal{M}, \otimes, \mathbf{1}, a, l, r) \rightarrow\left(\mathcal{M}^{\prime}, \otimes^{\prime}, \mathbf{1}^{\prime}, a^{\prime}, l^{\prime}, r^{\prime}\right)
$$

between two monoidal categories consists of a functor $F: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$, an isomorphism $\phi_{2}(U, V)$ : $F(U) \otimes^{\prime} F(V) \rightarrow F(U \otimes V)$, natural in $U, V \in \mathcal{M}$, and an isomorphism $\phi_{0}: \mathbf{1}^{\prime} \rightarrow F(\mathbf{1})$ such that the diagram

is commutative, and the following conditions are satisfied:

$$
F\left(l_{U}\right) \circ \phi_{2}(\mathbf{1}, U) \circ\left(\phi_{0} \otimes^{\prime} F(U)\right)=l_{F(U)}^{\prime}, \quad F\left(r_{U}\right) \circ \phi_{2}(U, \mathbf{1}) \circ\left(F(U) \otimes^{\prime} \phi_{0}\right)={r^{\prime}}_{F(U)}
$$

The monoidal functor is called strict if the isomorphisms $\phi_{0}, \phi_{2}$ are identities of $\mathcal{M}^{\prime}$.
The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories.

From now on we will omit the associativity and unity constraints unless needed to clarify the context.

Let $V$ be an object in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. Define iteratively $V^{\otimes n}$ for all $n \in \mathbb{N}$ by setting $V^{\otimes 0}:=1$ for $n=0$ and $V^{\otimes n}:=V^{\otimes(n-1)} \otimes V$ for $n>0$.

Remark 1.3. Let $\mathcal{M}$ be a monoidal category. Denote by $\operatorname{Alg}_{\mathcal{M}}$ the category of algebras in $\mathcal{M}$ and their morphisms. Assume that $\mathcal{M}$ has denumerable coproducts and that the tensor products (i.e. $M \otimes(-): \mathcal{M} \rightarrow \mathcal{M}$ and $(-) \otimes M: \mathcal{M} \rightarrow \mathcal{M}$, for every object $M$ in $\mathcal{M}$ ) preserve such coproducts. By M, Theorem 2, page 172], the forgetful functor

$$
\Omega: \operatorname{Alg}_{\mathcal{M}} \rightarrow \mathcal{M}
$$

has a left adjoint $T: \mathcal{M} \rightarrow \operatorname{Alg}_{\mathcal{M}}$. By construction $\Omega T V=\oplus_{n \in \mathbb{N}} V^{\otimes n}$ for every $V \in \mathcal{M}$. For every $n \in \mathbb{N}$, we will denote by

$$
\alpha_{n} V: V^{\otimes n} \rightarrow \Omega T V
$$

the canonical injection. Given a morphism $f: V \rightarrow W$ in $\mathcal{M}$, we have that $T f$ is uniquely determined by the following equality

$$
\begin{equation*}
\Omega T f \circ \alpha_{n} V=\alpha_{n} W \circ f^{\otimes n}, \text { for every } n \in \mathbb{N} \tag{1}
\end{equation*}
$$

The multiplication $m_{\Omega T V}$ and the unit $u_{\Omega T V}$ are uniquely determined by

$$
\begin{align*}
m_{\Omega T V} \circ\left(\alpha_{m} V \otimes \alpha_{n} V\right) & =\alpha_{m+n} V, \text { for every } m, n \in \mathbb{N},  \tag{2}\\
u_{\Omega T V} & =\alpha_{0} V . \tag{3}
\end{align*}
$$

REMARK 1.4. The unit $\eta$ and the counit $\epsilon$ of the adjunction $(T, \Omega)$ are uniquely determined, for all $V \in \mathcal{M}$ and $\left(A, m_{A}, u_{A}\right) \in \operatorname{Alg}_{\mathcal{M}}$ by the following equalities

$$
\begin{equation*}
\eta V:=\alpha_{1} V \quad \text { and } \quad \Omega \epsilon\left(A, m_{A}, u_{A}\right) \circ \alpha_{n} A:=m_{A}^{n-1} \text { for every } n \in \mathbb{N} \tag{4}
\end{equation*}
$$

where $m_{A}^{n-1}: A^{\otimes n} \rightarrow A$ is the iterated multiplication of $A$ defined by $m_{A}^{-1}:=u_{A}, m_{A}^{0}:=\operatorname{Id}_{A}$ and, for $n \geq 2, m_{A}^{n-1}=m_{A}\left(m_{A}^{n-2} \otimes A\right)$.

## 2. BRAIDED OBJECTS

Definition 2.1. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category (as usual we omit the brackets although we are not assuming the constraints are trivial).

1) Let $V$ be an object in $\mathcal{M}$. A morphism $c=c_{V}: V \otimes V \rightarrow V \otimes V$ is called a braiding (see Ka, Definition XIII.3.1] where it is called a Yang-Baxter operator) if it satisfies the quantum Yang-Baxter equation

$$
\begin{equation*}
(c \otimes V)(V \otimes c)(c \otimes V)=(V \otimes c)(c \otimes V)(V \otimes c) \tag{5}
\end{equation*}
$$

on $V \otimes V \otimes V$. We further assume that $c$ is invertible. The pair $(V, c)$ will be called a braided object in $\mathcal{M}$. A morphism of braided objects $\left(V, c_{V}\right)$ and $\left(W, c_{W}\right)$ in $\mathcal{M}$ is a morphism $f: V \rightarrow W$ such that $c_{W}(f \otimes f)=(f \otimes f) c_{V}$. This defines the category $\operatorname{Br}_{\mathcal{M}}$ of braided objects and their morphisms.
2) B] A quadruple $(A, m, u, c)$ is called a braided algebra if

- $(A, m, u)$ is an algebra in $\mathcal{M}$;
- $(A, c)$ is a braided object in $\mathcal{M}$;
- $m$ and $u$ commute with $c$, that is the following conditions hold:

$$
\begin{gather*}
c(m \otimes A)=(A \otimes m)(c \otimes A)(A \otimes c),  \tag{6}\\
c(A \otimes m)=(m \otimes A)(A \otimes c)(c \otimes A),  \tag{7}\\
c(u \otimes A) l_{A}^{-1}=(A \otimes u) r_{A}^{-1}, \quad c(A \otimes u) r_{A}^{-1}=(u \otimes A) l_{A}^{-1} . \tag{8}
\end{gather*}
$$

A morphism of braided algebras is, by definition, a morphism of algebras which, in addition, is a morphism of braided objects. This defines the category $\mathrm{BrAlg}_{\mathcal{M}}$ of braided algebras and their morphisms.
3) A quadruple $(C, \Delta, \varepsilon, c)$ is called a braided coalgebra if

- $(C, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{M}$;
- $(C, c)$ is a braided object in $\mathcal{M}$;
- $\Delta$ and $\varepsilon$ commute with $c$, that is the following relations hold:

$$
\begin{gather*}
(\Delta \otimes C) c=(C \otimes c)(c \otimes C)(C \otimes \Delta),  \tag{9}\\
(C \otimes \Delta) c=(c \otimes C)(C \otimes c)(\Delta \otimes C),  \tag{10}\\
l_{C}(\varepsilon \otimes C) c=r_{C}(C \otimes \varepsilon), \quad r_{C}(C \otimes \varepsilon) c=l_{C}(\varepsilon \otimes C) . \tag{11}
\end{gather*}
$$

A morphism of braided coalgebras is, by definition, a morphism of coalgebras which, in addition, is a morphism of braided objects. This defines the category $\operatorname{BrCoalg}_{\mathcal{M}}$ of braided coalgebras and their morphisms.
4) [1] Definition 5.1] A sextuple ( $B, m, u, \Delta, \varepsilon, c$ ) is a called a braided bialgebra if

- $(B, m, u, c)$ is a braided algebra;
- $(B, \Delta, \varepsilon, c)$ is a braided coalgebra;
- the following relations hold:

$$
\begin{align*}
\Delta m & =(m \otimes m)(B \otimes c \otimes B)(\Delta \otimes \Delta)  \tag{12}\\
\Delta u & =(u \otimes u) \Delta_{\mathbf{1}}  \tag{13}\\
\varepsilon m & =m_{\mathbf{1}}(\varepsilon \otimes \varepsilon)  \tag{14}\\
\varepsilon u & =\operatorname{Id}_{\mathbf{1}} \tag{15}
\end{align*}
$$

A morphism of braided bialgebras is both a morphism of braided algebras and coalgebras. This defines the category $\mathrm{BrBialg}_{\mathcal{M}}$ of braided bialgebras.
Proposition 2.2. Let $\mathcal{M}$ be a monoidal category.

1) Consider a datum $\left(A_{1}, A_{2}, c_{2,1}\right)$ where $A_{1}=\left(A_{1}, m_{1}, u_{1}\right)$ and $A_{2}=\left(A_{2}, m_{2}, u_{2}\right)$ are algebras in $\mathcal{M}$ and $c_{2,1}: A_{2} \otimes A_{1} \rightarrow A_{1} \otimes A_{2}$ is a morphism in $\mathcal{M}$ such that

$$
\begin{gather*}
c_{2,1}\left(m_{2} \otimes A_{1}\right)=\left(A_{1} \otimes m_{2}\right)\left(c_{2,1} \otimes A_{2}\right)\left(A_{2} \otimes c_{2,1}\right)  \tag{16}\\
c_{2,1}\left(A_{2} \otimes m_{1}\right)=\left(m_{1} \otimes A_{2}\right)\left(A_{1} \otimes c_{2,1}\right)\left(c_{2,1} \otimes A_{1}\right)  \tag{17}\\
c_{2,1}\left(u_{2} \otimes 1\right) l_{A_{1}}^{-1}=\left(A_{1} \otimes u_{2}\right) r_{A_{1}}^{-1}, \quad c_{2,1}\left(A_{2} \otimes u_{1}\right) r_{2}^{-1}=\left(u_{1} \otimes A_{2}\right) l_{A_{2}}^{-1} . \tag{18}
\end{gather*}
$$

Then $\left(A_{1} \otimes A_{2}, m_{A_{1} \otimes A_{2}}, u_{A_{1} \otimes A_{2}}\right)$ is an algebra in $\mathcal{M}$ where

$$
\begin{align*}
m_{A_{1} \otimes A_{2}} & :=\left(m_{1} \otimes m_{2}\right)\left(A_{1} \otimes c_{2,1} \otimes A_{2}\right)  \tag{19}\\
u_{A_{1} \otimes A_{2}} & :=\left(u_{1} \otimes u_{2}\right) \Delta_{1} \tag{20}
\end{align*}
$$

2) Let $\left(A_{1}, m_{1}, u_{1}\right), A_{2}=\left(A_{2}, m_{2}, u_{2}\right) \in \operatorname{Alg}_{\mathcal{M}}$. Assume that, for all $i, j \in\{1,2\}$, there are isomorphisms $c_{i, j}: A_{i} \otimes A_{j} \rightarrow A_{j} \otimes A_{i}$ such that the following equalities are fulfilled

$$
\begin{gather*}
c_{i, j}\left(m_{i} \otimes A_{j}\right)=\left(A_{j} \otimes m_{i}\right)\left(c_{i, j} \otimes A_{i}\right)\left(A_{i} \otimes c_{i, j}\right),  \tag{21}\\
c_{i, j}\left(A_{i} \otimes m_{j}\right)=\left(m_{j} \otimes A_{i}\right)\left(A_{j} \otimes c_{i, j}\right)\left(c_{i, j} \otimes A_{j}\right),  \tag{22}\\
c_{i, j}\left(u_{i} \otimes A_{j}\right) l_{A_{j}}^{-1}=\left(A_{j} \otimes u_{i}\right) r_{A_{j}}^{-1}, \quad c_{i, j}\left(A_{i} \otimes u_{j}\right) r_{A_{i}}^{-1}=\left(u_{j} \otimes A_{i}\right) l_{A_{i}}^{-1}  \tag{23}\\
\left(A_{k} \otimes c_{i, j}\right)\left(c_{i, k} \otimes A_{j}\right)\left(A_{i} \otimes c_{j, k}\right)=\left(c_{j, k} \otimes A_{i}\right)\left(A_{j} \otimes c_{i, k}\right)\left(c_{i, j} \otimes A_{k}\right) \tag{24}
\end{gather*}
$$

for all $i, j, k \in\{1,2\}$.
Then $\left(A_{1}, m_{1}, u_{1}, c_{1,1}\right),\left(A_{2}, m_{2}, u_{2}, c_{2,2}\right) \in \operatorname{BrAlg}_{\mathcal{M}}$.
Moreover, for all $i, j \in\{1,2\},\left(A_{i} \otimes A_{j}, m_{A_{i} \otimes A_{j}}, u_{A_{i} \otimes A_{j}}, c_{A_{i} \otimes A_{j}}\right) \in \operatorname{BrAlg} \mathcal{M}$ where $m_{A_{i} \otimes A_{j}}$ and $u_{A_{i} \otimes A_{j}}$ are as in 1) and

$$
c_{A_{i} \otimes A_{j}}=\left(A_{i} \otimes c_{i, j} \otimes A_{j}\right)\left(c_{i, i} \otimes c_{j, j}\right)\left(A_{i} \otimes c_{j, i} \otimes A_{j}\right)
$$

3) Let $A_{1}, A_{2}$ (respectively $A_{1}^{\prime}, A_{2}^{\prime}$ ) be objects in $\operatorname{Alg}_{\mathcal{M}}$ that fulfil the requirements in 2). Let $f_{1}: A_{1} \rightarrow A_{1}^{\prime}$ and $f_{2}: A_{2} \rightarrow A_{2}^{\prime}$ be morphisms in $\operatorname{Alg}_{\mathcal{M}}$ such that

$$
\begin{equation*}
\left(f_{i} \otimes f_{j}\right) c_{j, i}=c_{j, i}^{\prime}\left(f_{j} \otimes f_{i}\right) \tag{25}
\end{equation*}
$$

for all $i, j \in\{1,2\}$. Then,for all $i, j \in\{1,2\}, f_{i}$ and $f_{i} \otimes f_{j}$ are morphisms in $\operatorname{BrAlg}_{\mathcal{M}}$.
Proof. It is straightforward.

Lemma 2.3. Let $\left(A, m_{A}, u_{A}, c_{A}\right) \in \operatorname{BrAlg}_{\mathcal{M}}$. Then $\left(A_{1}, m_{1}, u_{1}\right),\left(A_{2}, m_{2}, u_{2}\right), c_{i, j}$ fulfil the requirements of Proposition 2.8, where $\left(A_{1}, m_{1}, u_{1}\right):=\left(A, m_{A}, u_{A}\right), c_{1,1}:=c_{A}$,

$$
\begin{aligned}
A_{2} & =A \otimes A, \quad m_{2}:=\left(m_{A} \otimes m_{A}\right)\left(A \otimes c_{A} \otimes A\right), \quad u_{2}:=\left(u_{A} \otimes u_{A}\right) \Delta_{\mathbf{1}}, \\
c_{2,2} & :=\left(A \otimes c_{A} \otimes A\right)\left(c_{A} \otimes c_{A}\right)\left(A \otimes c_{A} \otimes A\right), \\
c_{2,1} & :=\left(c_{A} \otimes A\right)\left(A \otimes c_{A}\right): A_{2} \otimes A_{1} \rightarrow A_{1} \otimes A_{2}, \\
c_{1,2} & :=\left(A \otimes c_{A}\right)\left(c_{A} \otimes A\right): A_{1} \otimes A_{2} \rightarrow A_{2} \otimes A_{1} .
\end{aligned}
$$

In particular $\left(E, m_{E}, u_{E}, c_{E}\right) \in \operatorname{BrAlg}_{\mathcal{M}}$, where

$$
\begin{aligned}
E & :=A_{1} \otimes A_{2}, \quad m_{E}:=\left(m_{1} \otimes m_{2}\right)\left(A_{1} \otimes c_{2,1} \otimes A_{2}\right), \quad u_{E}:=\left(u_{1} \otimes u_{2}\right) \Delta_{\mathbf{1}}, \\
c_{E} & :=\left(A_{1} \otimes c_{1,2} \otimes A_{2}\right)\left(c_{1,1} \otimes c_{2,2}\right)\left(A_{1} \otimes c_{2,1} \otimes A_{2}\right)
\end{aligned}
$$

Proof. It is straightforward.
Definition 2.4. A functor is called conservative if it reflects isomorphisms.
Proposition 2.5. Let $\mathcal{M}$ and $\mathcal{M}^{\prime}$ be monoidal categories. Let $\left(F, \phi_{0}, \phi_{2}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ be a monoidal functor. Then $F$ induces functors

$$
\begin{gathered}
\operatorname{Br} F: \operatorname{Br}_{\mathcal{M}} \rightarrow \operatorname{Br}_{\mathcal{M}^{\prime}}, \quad \operatorname{Alg} F: \operatorname{Alg}_{\mathcal{M}} \rightarrow \operatorname{Alg}_{\mathcal{M}^{\prime}} \\
\operatorname{BrAlg} F: \operatorname{BrAlg}_{\mathcal{M}} \rightarrow \operatorname{BrAlg}_{\mathcal{M}^{\prime}}, \quad \operatorname{BrBialg} F: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \operatorname{BrBialg}_{\mathcal{M}^{\prime}}
\end{gathered}
$$

which act as $F$ on morphisms and defined on objects by

$$
\begin{aligned}
(\mathrm{Br} F)\left(V, c_{V}\right) & :=\left(F V, c_{F V}\right), \\
(\mathrm{Alg} F)\left(A, m_{A}, u_{A}\right) & :=\left(F A, m_{F A}, u_{F A}\right), \\
(\mathrm{BrAlg} F)\left(A, m_{A}, u_{A}, c_{A}\right) & :=\left(F A, m_{F A}, u_{F A}, c_{F A}\right), \\
(\operatorname{BrBialg} F)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right) & :=\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, c_{F B}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
c_{F V} & :=\phi_{2}^{-1}(V, V) \circ F c_{V} \circ \phi_{2}(V, V): F V \otimes F V \rightarrow F V \otimes F V \\
m_{F A} & :=F m_{A} \circ \phi_{2}(A, A): F A \otimes F A \rightarrow F A \\
u_{F A} & :=F u_{A} \circ \phi_{0}: \mathbf{1} \rightarrow F A \\
\Delta_{F B} & :=\phi_{2}^{-1}(B, B) \circ F \Delta_{B}: F B \rightarrow F B \otimes F B \\
\varepsilon_{F B} & :=\phi_{0}^{-1} \circ F \varepsilon_{B}: F B \rightarrow \mathbf{1}
\end{aligned}
$$

and the following diagrams commute, where the vertical arrows denote the obvious forgetful functors.



## Moreover

1) The functors $H, \Omega, H_{\mathrm{Alg}}, \Omega_{\mathrm{Br}}, \mho_{\mathrm{Br}}$ are conservative.
2) $\operatorname{Br} F, \operatorname{Alg} F, \operatorname{BrAlg} F$ and $\operatorname{BrBialg} F$ are equivalences (resp. isomorphisms or conservative) whenever $F$ is.

Proof. Let $\left(V, c_{V}\right)$ be a braided object in $\mathcal{M}$. Let us check that $\left(F V, c_{F V}\right)$ is a braided object in $\mathcal{M}^{\prime}$. We have

$$
\phi_{2}(V \otimes V, V) \circ\left(\phi_{2}(V, V) \otimes F V\right)=\phi_{2}(V, V \otimes V) \circ\left(F V \otimes \phi_{2}(V, V)\right)
$$

Call $\omega: F V \otimes F V \otimes F V \rightarrow F(V \otimes V \otimes V)$ this composition. Using the definition of $c_{F V}$ and the naturality of $\phi_{2}$ one easily gets

$$
\begin{equation*}
\omega \circ\left(c_{F V} \otimes F V\right)=F\left(c_{V} \otimes V\right) \circ \omega, \quad \omega \circ\left(F V \otimes c_{F V}\right)=F\left(V \otimes c_{V}\right) \circ \omega \tag{26}
\end{equation*}
$$

Thus we obtain

$$
\begin{aligned}
& \omega \circ\left(c_{F V} \otimes F V\right) \circ\left(F V \otimes c_{F V}\right) \circ\left(c_{F V} \otimes F V\right) \\
= & F\left[\left(c_{V} \otimes V\right) \circ\left(V \otimes c_{V}\right) \circ\left(c_{V} \otimes V\right)\right] \circ \omega \\
= & F\left[\left(V \otimes c_{V}\right) \circ\left(c_{V} \otimes V\right) \circ\left(V \otimes c_{V}\right)\right] \circ \omega \\
= & \omega \circ\left(F V \otimes c_{F V}\right) \circ\left(c_{F V} \otimes F V\right) \circ\left(F V \otimes c_{F V}\right) .
\end{aligned}
$$

Since $\omega$ is an isomorphism, we conclude that $c_{F V}$ is a braiding. Thus ( $F V, c_{F V}$ ) is a braided object. Let $f:\left(V, c_{V}\right) \rightarrow\left(V^{\prime}, c_{V^{\prime}}\right)$ be a morphism of braided objects in $\mathcal{M}$. Using the definition of $c_{F V^{\prime}}$, the naturality of $\phi_{2}$, that $f$ is compatible with the braiding, one easily gets $c_{F V^{\prime}} \circ(F f \otimes F f)=$ $(F f \otimes F f) \circ c_{F V}$. Thus the functor $\operatorname{Br} F: \operatorname{Br}_{\mathcal{M}} \rightarrow \operatorname{Br}_{\mathcal{M}^{\prime}}$ of the statement is well-defined. By construction one easily checks that $H^{\prime} \circ \operatorname{Br} F=F \circ H$.

Let $\left(A, m_{A}, u_{A}\right) \in \operatorname{Alg}_{\mathcal{M}}$. By AMS, Proposition 1.5], we have that $\left(F A, m_{F A}, u_{F A}\right)$ is in $\operatorname{Alg}_{\mathcal{M}^{\prime}}$. Let $f:\left(A, m_{A}, u_{A}\right) \rightarrow\left(A^{\prime}, m_{A^{\prime}}, u_{A^{\prime}}\right)$ be a morphism of algebras in $\mathcal{M}$. Using the definition of $m_{F A^{\prime}}$, the naturality of $\phi_{2}$ and the multiplicativity of $f$ one gets $m_{F A^{\prime}} \circ(F f \otimes F f)=F f \circ m_{F A}$. Moreover, using the definition of $u_{F A}$ and the unitarity of $f$ one has $F f \circ u_{F A}=u_{F A^{\prime}}$. Thus the functor $\operatorname{Alg} F: \operatorname{Alg}_{\mathcal{M}} \rightarrow \operatorname{Alg}_{\mathcal{M}^{\prime}}$ is well-defined. It is clear that $F \circ \Omega=\Omega^{\prime} \circ \operatorname{Alg} F$.

Let $\left(A, m_{A}, u_{A}, c_{A}\right)$ be an object in $\operatorname{BrAlg}_{\mathcal{M}}$. Then $\left(A, c_{A}\right) \in \operatorname{Br}_{\mathcal{M}}$ and $\left(A, m_{A}, u_{A}\right) \in \operatorname{Alg}_{\mathcal{M}}$ so that, by the foregoing, we get that $\left(F A, c_{F A}\right) \in \operatorname{Br}_{\mathcal{M}^{\prime}}$ and $\left(F A, m_{F A}, u_{F A}\right) \in \operatorname{Alg}_{\mathcal{M}^{\prime}}$. We have

$$
\begin{gathered}
\phi_{2}(A, A) \circ\left(F A \otimes m_{F A}\right) \circ\left(c_{F A} \otimes F A\right) \circ\left(F A \otimes c_{F A}\right) \\
\stackrel{(*)}{=} F\left(A \otimes m_{A}\right) \circ \omega \circ\left(c_{F A} \otimes F A\right) \circ\left(F A \otimes c_{F A}\right) \\
\stackrel{\text { 26 }}{=} F\left(A \otimes m_{A}\right) \circ F\left(c_{A} \otimes A\right) \circ F\left(A \otimes c_{A}\right) \circ \omega
\end{gathered}
$$

$$
\text { (6) } F\left[c_{A} \circ\left(m_{A} \otimes A\right)\right] \circ \omega \stackrel{(*)}{=} F c_{A} \circ \phi_{2}(A, A) \circ\left(m_{F A} \otimes F A\right)=\phi_{2}(A, A) \circ c_{F A} \circ\left(m_{F A} \otimes F A\right)
$$

where in $\left(^{*}\right)$ we used the definition of $m_{F A}$, the naturality of $\phi_{2}(A, A)$ and the definition of $\omega$. Thus

$$
\left(F A \otimes m_{F A}\right) \circ\left(c_{F A} \otimes F A\right) \circ\left(F A \otimes c_{F A}\right)=c_{F A} \circ\left(m_{F A} \otimes F A\right)
$$

Similarly one proves that $\left(m_{F A} \otimes F A\right) \circ\left(F A \otimes c_{F A}\right) \circ\left(c_{F A} \otimes F A\right)=c_{F A} \circ\left(F A \otimes m_{F A}\right)$. Moreover

$$
\begin{aligned}
& \phi_{2}(A, A) \circ c_{F A} \circ\left(u_{F A} \otimes F A\right) \circ l_{F A}^{-1} \stackrel{(* *)}{=} F\left[c_{A} \circ\left(u_{A} \otimes A\right) \circ l_{A}^{-1}\right] \\
& \stackrel{(8)}{=} F\left[\left(A \otimes u_{A}\right) \circ r_{A}^{-1}\right] \stackrel{(* *)}{=} \phi_{2}(A, A) \circ\left(F A \otimes u_{F A}\right) \circ r_{F A}^{-1}
\end{aligned}
$$

where in $\left({ }^{* *}\right)$ we used the definitions of $c_{F A}$ and $u_{F A}$, the naturality of $\phi_{2}$ and the definition of monoidal functor. Thus $c_{F A} \circ\left(u_{F A} \otimes F A\right) \circ l_{F A}^{-1}=\left(F A \otimes u_{F A}\right) \circ r_{F A}^{-1}$. Similarly one proves that $c_{F A} \circ\left(F A \otimes u_{F A}\right) \circ r_{F A}^{-1}=\left(u_{F A} \otimes F A\right) \circ l_{F A}^{-1}$. We have so proved that $\left(F A, m_{F A}, u_{F A}, c_{F A}\right)$ is a braided algebra in $\mathcal{M}^{\prime}$. Since, by definition, a morphism of braided algebras is just a morphism of braided objects and of algebras, it is clear, by the foregoing, that $F$ preserves morphisms of braided algebras so that the functor $\operatorname{BrAlg} F: \operatorname{BrAlg}_{\mathcal{M}} \rightarrow \mathrm{BrAlg}_{\mathcal{M}^{\prime}}$ is well-defined. It is clear that $\mathrm{Alg} F \circ H_{\mathrm{Alg}}=H_{\mathrm{Alg}}^{\prime} \circ \mathrm{BrAlg} F$ and $\mathrm{Br} F \circ \Omega_{\mathrm{Br}}=\Omega_{\mathrm{Br}}^{\prime} \circ \mathrm{BrAlg} F$.

Let us define the functor $\operatorname{BrBialg} F$. Let $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right)$ be a braided bialgebra in $\mathcal{M}$. Then $\left(B, m_{B}, u_{B}, c_{B}\right)$ is a braided algebra in $\mathcal{M}$, so that, by the foregoing, $\left(F B, m_{F B}, u_{F B}, c_{F B}\right)$ is a braided algebra in $\mathcal{M}^{\prime}$. A dual argument proves that $\left(F B, \Delta_{F B}, \varepsilon_{F B}, c_{F B}\right)$ is a braided coalgebra in $\mathcal{M}^{\prime}$.

We compute

$$
\begin{aligned}
& \phi_{2}(B, B) \circ\left(m_{F B} \otimes m_{F B}\right) \circ\left(F B \otimes c_{F B} \otimes F B\right) \circ\left(\Delta_{F B} \otimes \Delta_{F B}\right) \\
& \stackrel{(* * *)}{=} F\left(m_{B} \otimes m_{B}\right) \circ \phi_{2}(B \otimes B \otimes B, B) \circ(\omega \otimes F B) \circ\left(F B \otimes c_{F B} \otimes F B\right) \circ\left(\Delta_{F B} \otimes \Delta_{F B}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \stackrel{(26)}{=} F\left(m_{B} \otimes m_{B}\right) \circ \phi_{2}(B \otimes B \otimes B, B) \circ\left(F\left(B \otimes c_{B}\right) \otimes F B\right) \circ(\omega \otimes F B) \circ\left(\Delta_{F B} \otimes \Delta_{F B}\right) \\
& \stackrel{(* * *)^{\prime}}{=} F\left[\left(m_{B} \otimes m_{B}\right) \circ\left(B \otimes c_{B} \otimes B\right) \circ\left(\Delta_{B} \otimes \Delta_{B}\right)\right] \circ \phi_{2}(B, B) \\
& \stackrel{12)}{=} F\left(\Delta_{B} \circ m_{B}\right) \circ \phi_{2}(B, B)=F \Delta_{B} \circ m_{F B}=\phi_{2}(B, B) \circ \Delta_{F B} \circ m_{F B}
\end{aligned}
$$

where in $\left(^{* * *}\right)$ we used the definition of $m_{F B}$, the naturality of $\phi_{2}$, the fact that $F$ is a monoidal functor and the definition of $\omega$ while in $\left(^{* * *}\right)^{\prime}$ we used the naturality of $\phi_{2}$, the definition of $\omega$, the definition of $\Delta_{F B}$ (the one on the left of the tensor), the fact that $F$ is monoidal, again the definition of $\Delta_{F B}$ (the one on the right of the tensor) and the naturality of $\phi_{2}$. Thus

$$
\left(m_{F B} \otimes m_{F B}\right) \circ\left(F B \otimes c_{F B} \otimes F B\right) \circ\left(\Delta_{F B} \otimes \Delta_{F B}\right)=\Delta_{F B} \circ m_{F B}
$$

We calculate
definitions of $\Delta_{F B}$ and $u_{F B}$, the unitarity of $\Delta_{B}$, the equality $\Delta_{\mathbf{1}}=l_{\mathbf{1}}^{-1}$, the monoidality of $F$, the naturality of the left unit constraint, the naturality of $\phi_{2}$ and again the definition of $u_{F B}$, one gets $\phi_{2}(B, B) \circ \Delta_{F B} \circ u_{F B}=\phi_{2}(B, B) \circ\left(u_{F B} \otimes u_{F B}\right) \circ \Delta_{\mathbf{1}}$ so that $\Delta_{F B} \circ u_{F B}=\left(u_{F B} \otimes u_{F B}\right) \circ \Delta_{\mathbf{1}}$. Dually one gets $\varepsilon_{F B} \circ m_{F B}=m_{\mathbf{1}} \circ\left(\varepsilon_{F B} \otimes \varepsilon_{F B}\right)$. Finally we have

$$
\varepsilon_{F B} \circ u_{F B}=\phi_{0}^{-1} \circ F \varepsilon_{B} \circ F u_{B} \circ \phi_{0}=\phi_{0}^{-1} \circ F\left(\varepsilon_{B} \circ u_{B}\right) \circ \phi_{0}=\phi_{0}^{-1} \circ \phi_{0}=\mathrm{Id}_{\mathbf{1}}
$$

We have so proved that $\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, c_{F B}\right)$ is a braided bialgebra. Let $f$ be a morphism of braided bialgebras from $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right)$ to ( $B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}, c_{B^{\prime}}$ ). Then $f:\left(B, m_{B}, u_{B}\right) \rightarrow\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}\right)$ is a morphism of algebras and $f:\left(B, \Delta_{B}, \varepsilon_{B}\right) \rightarrow\left(B^{\prime}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right)$ is a morphism of coalgebras. Thus $F f:\left(F B, m_{F B}, u_{F B}\right) \rightarrow\left(F B^{\prime}, m_{F B^{\prime}}, u_{F B^{\prime}}\right)$ is a morphism of algebras and $F f:\left(F B, \Delta_{F B}, \varepsilon_{F B}\right) \rightarrow\left(F B^{\prime}, \Delta_{F B^{\prime}}, \varepsilon_{F B^{\prime}}\right)$ is a morphism of coalgebras. Moreover we know that $F f:\left(F B, c_{F B}\right) \rightarrow\left(F B^{\prime}, c_{F B^{\prime}}\right)$ is a morphism of braided objects. We have so proved that $F f$ is a morphism of braided bialgebras. Thus the functor $\operatorname{BrBialg} F: \operatorname{BrBialg} \mathcal{M}_{\mathcal{M}} \rightarrow \operatorname{BrBialg}_{\mathcal{M}^{\prime}}$ of the statement is well-defined.

Let $\left(F, \phi_{0}^{F}, \phi_{2}^{F}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ and $\left(F^{\prime}, \phi_{0}^{F^{\prime}}, \phi_{2}^{F^{\prime}}\right): \mathcal{M}^{\prime} \rightarrow \mathcal{M}^{\prime \prime}$ be monoidal functors. Then $\left(F^{\prime} F, \phi_{2}^{F^{\prime} F}, \phi_{0}^{F^{\prime} F}\right)$ is a monoidal functor where $\phi_{2}^{F^{\prime} F}(U, V):=F^{\prime}\left(\phi_{2}^{F}\right)(U, V) \circ \phi_{2}^{F^{\prime}}(F U, F V)$ and $\phi_{0}^{F^{\prime} F}:=F^{\prime}\left(\phi_{0}^{F}\right) \circ \phi_{0}^{F^{\prime}}$. We compute
$\left(\mathrm{Br} F^{\prime} \circ \mathrm{Br} F\right)\left(V, c_{V}\right)=\operatorname{Br} F^{\prime}\left(F V, c_{F V}\right)=\left(F^{\prime} F V, c_{F^{\prime}(F V)}\right) \stackrel{(\bullet)}{=}\left(F^{\prime} F V, c_{\left(F^{\prime} F\right) V}\right)=\operatorname{Br}\left(F^{\prime} F\right)\left(V, c_{V}\right)$. where $(\bullet)$ follows from the following computation

$$
\begin{aligned}
c_{F^{\prime}(F V)} & =\left(\phi_{2}^{F^{\prime}}\right)^{-1}(F V, F V) \circ F^{\prime} c_{F V} \circ \phi_{2}^{F^{\prime}}(F V, F V) \\
& =\left(\phi_{2}^{F^{\prime}}\right)^{-1}(F V, F V) \circ F^{\prime}\left(\left(\phi_{2}^{F}\right)^{-1}(V, V) \circ F c_{V} \circ\left(\phi_{2}^{F}\right)(V, V)\right) \circ \phi_{2}^{F^{\prime}}(F V, F V) \\
& =\left(\phi_{2}^{F^{\prime}}\right)^{-1}(F V, F V) \circ F^{\prime}\left(\phi_{2}^{F}\right)^{-1}(V, V) \circ F^{\prime} F c_{V} \circ F^{\prime}\left(\phi_{2}^{F}\right)(V, V) \circ \phi_{2}^{F^{\prime}}(F V, F V) \\
& =\phi_{2}^{F^{\prime} F}(V, V)^{-1} \circ F^{\prime} F c_{V} \circ \phi_{2}^{F^{\prime} F}(V, V)=c_{\left(F^{\prime} F\right) V}
\end{aligned}
$$

Thus we have $\operatorname{Br} F^{\prime} \circ \operatorname{Br} F=\operatorname{Br}\left(F^{\prime} F\right)$. We compute

$$
\begin{aligned}
\left(\operatorname{Alg} F^{\prime} \circ \operatorname{Alg} F\right)\left(A, m_{A}, u_{A}\right) & =\operatorname{Alg} F^{\prime}\left(F A, m_{F A}, u_{F A}\right)=\left(F^{\prime} F A, m_{F^{\prime}(F A)}, u_{F^{\prime}(F A)}\right) \\
\stackrel{(\bullet)}{=}\left(F^{\prime} F A, m_{\left(F^{\prime} F\right) A}, u_{\left(F^{\prime} F\right) A}\right) & =\operatorname{Alg}\left(F^{\prime} F\right)\left(A, m_{A}, u_{A}\right)
\end{aligned}
$$

where ( $\bullet \bullet$ ) follows from the following computations

$$
\begin{aligned}
m_{F^{\prime}(F A)} & =F^{\prime} m_{F A} \circ \phi_{2}^{F^{\prime}}(F A, F A)=F^{\prime} F m_{A} \circ F^{\prime} \phi_{2}^{F}(A, A) \circ \phi_{2}^{F^{\prime}}(F A, F A) \\
& =F^{\prime} F m_{A} \circ \phi_{2}^{F^{\prime}} F(A, A)=m_{\left(F^{\prime} F\right) A}, \\
u_{F^{\prime}(F A)} & =F^{\prime} u_{F A} \circ \phi_{0}^{F^{\prime}}=F u_{A} \circ F^{\prime} \phi_{0}^{F} \circ \phi_{0}^{F^{\prime}}=F u_{A} \circ \phi_{0}^{F^{\prime} F}=u_{\left(F^{\prime} F\right) A} .
\end{aligned}
$$

Thus $\operatorname{Alg} F^{\prime} \circ \mathrm{Alg} F=\operatorname{Alg}\left(F^{\prime} F\right)$. By the foregoing it is clear that $\operatorname{BrAlg} F^{\prime} \circ \mathrm{Br} \mathrm{Alg} F=\operatorname{BrAlg}\left(F^{\prime} F\right)$.

By the foregoing and a dual argument on the comultiplication and counit, one also gets that $\operatorname{BrBialg} F^{\prime} \circ \operatorname{BrBialg} F=\operatorname{BrBialg}\left(F^{\prime} F\right)$.

Consider the strict monoidal functor $\operatorname{Id}_{\mathcal{M}}$. A direct computation shows

$$
\begin{gathered}
\operatorname{Br}\left(\operatorname{Id}_{\mathcal{M}}\right)=\operatorname{Id}_{\mathrm{Br}_{\mathcal{M}}}, \quad \operatorname{Alg}\left(\operatorname{Id}_{\mathcal{M}}\right)=\operatorname{Id}_{\mathrm{Alg}_{\mathcal{M}}} \\
\operatorname{BrAlg}\left(\operatorname{Id}_{\mathcal{M}}\right)=\operatorname{Id}_{\operatorname{BrAlg}_{\mathcal{M}}}, \quad \operatorname{BrBialg}\left(\operatorname{Id}_{\mathcal{M}}\right)=\operatorname{Id}_{\operatorname{BrBialg}_{\mathcal{M}}}
\end{gathered}
$$

Let $\left(F, \phi_{0}^{F}, \phi_{2}^{F}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ and $\left(F^{\prime}, \phi_{0}^{F^{\prime}}, \phi_{2}^{F^{\prime}}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ be monoidal functors. Let $\xi:$ $\left(F, \phi_{0}^{F}, \phi_{2}^{F}\right) \rightarrow\left(F^{\prime}, \phi_{0}^{F^{\prime}}, \phi_{2}^{F^{\prime}}\right)$ be a morphism of monoidal functors i.e. a natural transformation $\xi: F \rightarrow F^{\prime}$ such that $\xi \mathbf{1} \circ \phi_{0}^{F}=\phi_{0}^{F^{\prime}}$ and $\xi(U \otimes V) \circ \phi_{2}^{F}(U, V)=\phi_{2}^{F^{\prime}}(U, V) \circ(\xi U \otimes \xi V)$. Let us define a natural transformation $\operatorname{Br} \xi: \operatorname{Br} F \rightarrow \operatorname{Br} F^{\prime}$. First we have to define a morphism $\operatorname{Br} \xi(V, c)$ : $\operatorname{Br} F(V, c) \rightarrow \operatorname{Br} F^{\prime}(V, c)$. in $\operatorname{Br}_{\mathcal{M}^{\prime}}$. Now $\operatorname{Br} F(V, c)=\left(F V, c_{F V}\right)$ and $\operatorname{Br} F^{\prime}(V, c)=\left(F^{\prime} V, c_{F^{\prime} V}\right)$ so that a natural candidate is $\xi V$. We have to check it is a morphism of braided objects i.e. $(\xi V \otimes \xi V) \circ c_{F V}=c_{F^{\prime} V} \circ(\xi U \otimes \xi V)$ but this is achieved by means of the definition of $c_{F V}$, the fact that $\xi$ is a morphism of monoidal functors and the naturality of $\xi$. Thus $\xi V$ really induces a unique morphism $\operatorname{Br} \xi(V, c): \operatorname{Br} F(V, c) \rightarrow \operatorname{Br} F^{\prime}(V, c)$ such that $H^{\prime} \operatorname{Br} \xi(V, c)=\xi V$. Let us check that $\operatorname{Br} \xi(V, c)$ is natural in $(V, c)$. Let $f:(V, c) \rightarrow\left(V^{\prime}, c^{\prime}\right)$ be a morphism of braided object in $\mathcal{M}$. Then

$$
\begin{aligned}
& H^{\prime}\left(\operatorname{Br} \xi\left(V^{\prime}, c^{\prime}\right) \circ \operatorname{Br} F(f)\right)=H^{\prime} \operatorname{Br} \xi\left(V^{\prime}, c^{\prime}\right) \circ H^{\prime} \operatorname{Br} F(f) \\
= & \xi V^{\prime} \circ F H(f)=F^{\prime} H(f) \circ \xi V=H^{\prime}\left(\operatorname{Br} F^{\prime}(f) \circ \operatorname{Br} \xi(V, c)\right)
\end{aligned}
$$

so that $\operatorname{Br} \xi\left(V^{\prime}, c^{\prime}\right) \circ \operatorname{Br} F(f)=\operatorname{Br} F^{\prime}(f) \circ \operatorname{Br} \xi(V, c)$ and hence we get a natural transformation $\mathrm{Br} \xi: \mathrm{Br} F \rightarrow \mathrm{Br} F^{\prime}$.

We have to define a morphism $\operatorname{Alg} \xi\left(A, m_{A}, u_{A}\right): \operatorname{Alg} F\left(A, m_{A}, u_{A}\right) \rightarrow \operatorname{Alg} F^{\prime}\left(A, m_{A}, u_{A}\right)$ in $\operatorname{Alg}_{\mathcal{M}^{\prime}}$. Now $\operatorname{Alg} F\left(A, m_{A}, u_{A}\right)=\left(F A, m_{F A}, u_{F A}\right)$ and $\operatorname{Alg} F^{\prime}\left(A, m_{A}, u_{A}\right)=\left(F^{\prime} A, m_{F^{\prime} A}, u_{F^{\prime} A}\right)$ so that a natural candidate is again $\xi A: F A \rightarrow F^{\prime} A$. We have to check it is a morphism of algebras in $\mathcal{M}^{\prime}$ i.e. that $\xi A \circ m_{F A}=m_{F^{\prime} A} \circ(\xi A \otimes \xi A)$ and $\xi A \circ u_{F A}=u_{F^{\prime} A}$ but these equalities follow by definition of $m_{F A}\left(\right.$ resp. $\left.u_{F A}\right)$ the naturality of $\xi$ and the fact that $\xi$ is a morphism of monoidal functors. Hence there is a unique morphism $\operatorname{Alg} \xi\left(A, m_{A}, u_{A}\right)$ such that $\Omega^{\prime} \operatorname{Alg} \xi\left(A, m_{A}, u_{A}\right)=\xi A$. We check it is natural in $\left(A, m_{A}, u_{A}\right)$. For an algebra morphism $f:\left(A, m_{A}, u_{A}\right) \rightarrow\left(A^{\prime}, m_{A^{\prime}}, u_{A^{\prime}}\right)$, we get

$$
\begin{aligned}
& \Omega^{\prime}\left[\operatorname{Alg} \xi\left(A^{\prime}, m_{A^{\prime}}, u_{A^{\prime}}\right) \circ \operatorname{Alg} F(f)\right]=\xi A^{\prime} \circ F \Omega(f) \\
& =F^{\prime} \Omega(f) \circ \xi A=\Omega^{\prime}\left[\operatorname{Alg} F^{\prime}(f) \circ \operatorname{Alg} \xi\left(A, m_{A}, u_{A}\right)\right]
\end{aligned}
$$

so that we get a natural transformation $\operatorname{Alg} \xi: \operatorname{Alg} F \rightarrow \operatorname{Alg} F^{\prime}$. By the foregoing and the definition of $\operatorname{BrAlg} F$ we can define a natural transformation $\operatorname{BrAlg} \xi: \operatorname{BrAlg} F \rightarrow \operatorname{BrAlg} F^{\prime}$ using again $\xi A$. Similarly one gets a natural transformation BrBialg $\xi: \operatorname{BrBialg} F \rightarrow \operatorname{BrBialg} F^{\prime}$.

Let $\left(F, \phi_{0}^{F}, \phi_{2}^{F}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime},\left(F^{\prime}, \phi_{0}^{F^{\prime}}, \phi_{2}^{F^{\prime}}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ and $\left(F^{\prime \prime}, \phi_{0}^{F^{\prime \prime}}, \phi_{2}^{F^{\prime \prime}}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ be monoidal functors. Let $\xi:\left(F, \phi_{0}^{F}, \phi_{2}^{F}\right) \rightarrow\left(F^{\prime}, \phi_{0}^{F^{\prime}}, \phi_{2}^{F^{\prime}}\right)$ and $\xi^{\prime}:\left(F^{\prime}, \phi_{0}^{F^{\prime}}, \phi_{2}^{F^{\prime}}\right) \rightarrow\left(F^{\prime \prime}, \phi_{0}^{F^{\prime \prime}}, \phi_{2}^{F^{\prime \prime}}\right)$ be morphisms of monoidal functors. Thus

$$
\begin{aligned}
H^{\prime}\left(\operatorname{Br} \xi^{\prime} \circ \operatorname{Br} \xi\right) & =H^{\prime} \operatorname{Br} \xi^{\prime} \circ H^{\prime} \operatorname{Br} \xi=\xi^{\prime} H^{\prime} \circ \xi H^{\prime}=\left(\xi^{\prime} \xi\right) H^{\prime}=H^{\prime}\left(\operatorname{Br}\left(\xi^{\prime} \xi\right)\right) \\
\Omega^{\prime}\left(\operatorname{Alg} \xi^{\prime} \circ \operatorname{Alg} \xi\right) & =\Omega^{\prime} \operatorname{Alg} \xi^{\prime} \circ \Omega^{\prime} \operatorname{Alg} \xi=\xi^{\prime} \Omega^{\prime} \circ \xi \Omega^{\prime}=\left(\xi^{\prime} \xi\right) \Omega^{\prime}=\Omega^{\prime}\left(\operatorname{Alg}\left(\xi^{\prime} \xi\right)\right)
\end{aligned}
$$

so that $\operatorname{Br} \xi^{\prime} \circ \operatorname{Br} \xi=\operatorname{Br}\left(\xi^{\prime} \xi\right), \operatorname{Alg} \xi^{\prime} \circ \operatorname{Alg} \xi=\operatorname{Alg}\left(\xi^{\prime} \xi\right)$ and hence $\operatorname{BrAlg} \xi^{\prime} \circ \operatorname{BrAlg} \xi=\operatorname{BrAlg}\left(\xi^{\prime} \xi\right)$ and $\operatorname{BrBialg} \xi^{\prime} \circ \operatorname{BrBialg} \xi=\operatorname{BrBialg}\left(\xi^{\prime} \xi\right)$. Moreover

$$
\begin{aligned}
H^{\prime}\left(\operatorname{Br}\left(\operatorname{Id}_{F}\right)\right) & =\left(\operatorname{Id}_{F}\right) H=\operatorname{Id}_{F H}=H^{\prime} \operatorname{Id}_{\mathrm{Br} F} \\
\Omega\left(\operatorname{Alg}\left(\operatorname{Id}_{F}\right)\right) & =\left(\operatorname{Id}_{F}\right) \Omega=\operatorname{Id}_{F \Omega}=\Omega\left(\operatorname{Id}_{\mathrm{Alg} F}\right)
\end{aligned}
$$

so that $\operatorname{Br}\left(\operatorname{Id}_{F}\right)=\operatorname{Id}_{\operatorname{Br} F}, \operatorname{Alg}\left(\operatorname{Id}_{F}\right)=\operatorname{Id}_{A l g} F$ and hence $\operatorname{BrAlg}\left(\operatorname{Id}_{F}\right)=\operatorname{Id}_{\operatorname{BrAlg} F}$ and $\operatorname{BrBialg}\left(\operatorname{Id}_{F}\right)=$ $\operatorname{Id}_{\text {BrBialg } F}$. Now, if $\left(F, \phi_{0}^{F}, \phi_{2}^{F}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ is a monoidal equivalence, i.e. $F: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ is a
monoidal functor and there is a monoidal functor $\left(G, \phi_{0}^{G}, \phi_{2}^{G}\right): \mathcal{M}^{\prime} \rightarrow \mathcal{M}$ and monoidal isomorphisms of functors

$$
\alpha: \operatorname{Id}_{\mathcal{M}^{\prime}} \rightarrow F G \quad \beta: G F \rightarrow \operatorname{Id}_{\mathcal{M}}
$$

Then

$$
\begin{aligned}
& \operatorname{Br} \alpha \circ \operatorname{Br}\left(\alpha^{-1}\right)=\operatorname{Br}\left(\alpha \circ \alpha^{-1}\right)=\operatorname{Br}\left(\operatorname{Id}_{F G}\right)=\operatorname{Id}_{\operatorname{Br}(F G)}=\operatorname{Id}_{\operatorname{Br}(F) \circ \operatorname{Br}(G)} \\
& \operatorname{Br}\left(\alpha^{-1}\right) \circ \operatorname{Br} \alpha=\operatorname{Br}\left(\alpha^{-1} \circ \alpha\right)=\operatorname{Br}\left(\operatorname{Id}_{\operatorname{Id}_{\mathcal{M}^{\prime}}}\right)=\operatorname{Id}_{\operatorname{Br}\left(\operatorname{Id}_{\mathcal{M}^{\prime}}\right)}=\operatorname{Id}_{\mathrm{Id}_{\mathrm{Br}^{\prime}}}
\end{aligned}
$$

so that $\operatorname{Br} \alpha: \operatorname{Id}_{\mathrm{Br}^{\mathcal{M}}} \rightarrow \operatorname{Br}(F) \circ \operatorname{Br}(G)$ ( and similarly $\operatorname{Br} \beta$ ) is a functorial isomorphism. This means that $\operatorname{Br} F: \operatorname{Br}_{\mathcal{M}} \rightarrow \operatorname{Br}_{\mathcal{M}^{\prime}}$ is an equivalence. Analogously $\operatorname{Alg} F: \operatorname{Alg}_{\mathcal{M}} \rightarrow \operatorname{Alg}_{\mathcal{M}^{\prime}}$ is an equivalence and hence also $\operatorname{BrAlg} F: \operatorname{BrAlg}_{\mathcal{M}} \rightarrow \operatorname{BrAlg}_{\mathcal{M}^{\prime}}$ and $\operatorname{BrBialg} F: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \operatorname{BrBialg}_{\mathcal{M}^{\prime}}$ are equivalences.

If $F$ is a category isomorphism there is a monoidal functor $\left(G, \phi_{0}^{G}, \phi_{2}^{G}\right): \mathcal{M}^{\prime} \rightarrow \mathcal{M}$ such that $F G=\operatorname{Id}_{\mathcal{M}^{\prime}}$ and $G F=\operatorname{Id}_{\mathcal{M}}$. Hence $\operatorname{Br}(F) \circ \operatorname{Br}(G)=\operatorname{Br}(F G)=\operatorname{Br}\left(\operatorname{Id}_{\mathcal{M}^{\prime}}\right)=\operatorname{Id}_{\mathrm{Br}_{\mathcal{M}^{\prime}}}$ and similarly $\operatorname{Br}(G) \circ \operatorname{Br}(F)=\operatorname{Id}_{\mathrm{Br}_{\mathcal{M}}}$ so that $\operatorname{Br}(F)$ is a category isomorphism. Analogously $\operatorname{Alg} F: \operatorname{Alg}_{\mathcal{M}} \rightarrow \operatorname{Alg}_{\mathcal{M}^{\prime}}$ is an isomorphism and hence also $\operatorname{BrAlg} F: \operatorname{BrAlg}_{\mathcal{M}} \rightarrow \operatorname{BrAlg}_{\mathcal{M}^{\prime}}$ and BrBialg $F: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow$ BrBialg $_{\mathcal{M}^{\prime}}$ are isomorphisms.

The proof of 1) is straightforward. If $F$ is conservative, using 1), one easily check that so are $\operatorname{Br} F, \operatorname{Alg} F, \operatorname{BrAlg} F$ and $\mathrm{BrBialg} F$. For instance, $F$ and $H$ conservative implies $F H=H^{\prime}(\mathrm{Br} F)$ conservative and hence, since $H^{\prime}$, as any functor, preserves isomorphisms, we obtain that $\operatorname{Br} F$ is conservative.

The following result is essentially Ka, Lemma XII.3.5, page 327] in case the monoidal category is strict. We prove that it holds for any monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ using the monoidal equivalence $\mathcal{M}^{\text {str }} \rightarrow \mathcal{M}$ described in Ka, Theorem XI.5.3, page 291], where $\mathcal{M}^{\text {str }}$ is a strict monoidal category.

Lemma 2.6. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category and let $(V, c) \in \operatorname{Br}_{\mathcal{M}}$. There there exists a unique monoidal functor $\left(F, \varphi_{2}, \varphi_{0}\right): \mathcal{B} \rightarrow \mathcal{M}$ such that, for all $a, b \in \mathbb{N}$,

$$
F(0)=1, \quad F(1)=V, \quad F(a \otimes 1)=F(a) \otimes V, \quad F\left(c_{1,1}\right)=c
$$

and

$$
\begin{align*}
\varphi_{2}(0, b) & =l_{F(b)},  \tag{27}\\
\varphi_{2}(a, 0) & =r_{F(a)},  \tag{28}\\
\varphi_{2}(a, 1) & =\operatorname{Id}_{F(a) \otimes V}, \quad a \geq 1,  \tag{29}\\
\varphi_{2}(a, b \otimes 1) & =\left(\varphi_{2}(a, b) \otimes V\right) \circ a_{F(a), F(b), V}^{-1}, \quad a, b \geq 1,  \tag{30}\\
\varphi_{0} & =\operatorname{Id}_{\mathbf{1}} . \tag{31}
\end{align*}
$$

Here $\mathcal{B}$ denotes the Braid Category, see Ka, page 321], which is a strict monoidal category. Its braiding is denoted by $c_{m, n}: m \otimes n \rightarrow n \otimes m$. Moreover $F(n):=V^{\otimes n}$ for every $n \in \mathbb{N}$.

Proof. By Ka, Theorem XI.5.3, page 291], there is a monoidal equivalence $F^{\prime}: \mathcal{M}^{\text {str }} \rightarrow \mathcal{M}$, where $\mathcal{M}^{\text {str }}$ is a suitable strict monoidal category. Recall that objects in $\mathcal{M}^{\text {str }}$ are all finite sequences $S=\left(V_{1}, \ldots, V_{k}\right)$ of objects of $\mathcal{M}$ including the empty sequence $\emptyset$. The integer $k$ is by definition the length of the sequence and is denoted by $l(S)$. This category is strict monoidal with unit $\emptyset$ and tensor product given by

$$
\begin{aligned}
\emptyset \otimes S & : \\
\left(V_{1}, \ldots, V_{k}\right) \otimes\left(V_{1}^{\prime}, \ldots, V_{s}^{\prime}\right) & :=\left(V_{1}, \ldots, V_{k}, V_{1}^{\prime}, \ldots, V_{s}^{\prime}\right)
\end{aligned}
$$

To any sequence $S$ one assigns an object $F^{\prime}(S)$ defined recursively as follows: $F^{\prime}(\emptyset):=1$, $F^{\prime}\left(\left(V_{1}\right)\right):=V_{1}, F^{\prime}\left(\left(V_{1}, \ldots, V_{k}, V_{k+1}\right)\right)=F^{\prime}\left(\left(V_{1}, \ldots, V_{k}\right)\right) \otimes V_{k+1}$. The morphisms in $\mathcal{M}^{\text {str }}$ are given by

$$
\mathcal{M}^{\text {str }}\left(S, S^{\prime}\right):=\mathcal{M}\left(F^{\prime}(S), F^{\prime}\left(S^{\prime}\right)\right)
$$

Defining $F^{\prime}$ as the identity on morphisms, we get the functor $F^{\prime}: \mathcal{M}^{\text {str }} \rightarrow \mathcal{M}$. For arbitrary objects $S, S^{\prime} \in \mathcal{M}^{\text {str }}$ there is an isomorphism $\varphi_{2}^{\prime}\left(S, S^{\prime}\right): F^{\prime}(S) \otimes F^{\prime}\left(S^{\prime}\right) \rightarrow F^{\prime}\left(S \otimes S^{\prime}\right)$ defined iteratively as follows

$$
\begin{aligned}
& \varphi_{2}^{\prime}\left(\emptyset, S^{\prime}\right): \\
& \varphi_{2}^{\prime}(S,(Z)): \\
& l_{F^{\prime}\left(S^{\prime}\right)}, \quad \varphi_{2}^{\prime}(S, \emptyset):=r_{F^{\prime}(S)} \\
& \varphi_{2}^{\prime}\left(S, S^{\prime}(S) \otimes Z, l(S) \geq 1\right. \\
& \prime(Z))
\end{aligned}:=\left(\varphi_{2}^{\prime}\left(S, S^{\prime}\right) \otimes Z\right) \circ a_{F^{\prime}(S), F^{\prime}\left(S^{\prime}\right), Z}^{-1}, l(S) \geq 1, l\left(S^{\prime}\right) \geq 1 .
$$

Define $\varphi_{0}^{\prime}=\operatorname{Id}_{\mathbf{1}}: \mathbf{1} \rightarrow F^{\prime}(\emptyset)$. Then $\left(F^{\prime}, \varphi_{2}^{\prime}, \varphi_{0}^{\prime}\right)$ is the claimed monoidal functor. This comes out to be an equivalence. Its right adjoint $G^{\prime}: \mathcal{M} \rightarrow \mathcal{M}^{\text {str }}$ is given by $G^{\prime}(Z):=(Z)$ and is the identity on morphisms too. Note that $F^{\prime} G^{\prime}=\operatorname{Id}_{\mathcal{M}}$. The counit of this adjunction is the identity $\epsilon Z=\operatorname{Id}_{Z}: F^{\prime} G^{\prime} Z \rightarrow Z$. The unit is $\eta S=\operatorname{Id}_{F^{\prime} S}: S \rightarrow G^{\prime} F^{\prime} S$. By AMS, Proposition 1.4.], we have that $\left(G^{\prime}, \gamma_{2}^{\prime}, \gamma_{0}^{\prime}\right)$ is a monoidal functor where

$$
\begin{aligned}
& \gamma_{0}^{\prime}: \\
& \gamma_{2}^{\prime}(X, Y): \\
&=G^{\prime}\left(\left(\varphi_{0}^{\prime}\right)^{-1}\right) \circ \eta \emptyset: \emptyset \rightarrow G^{\prime} 1 \\
& \hline(\epsilon \in Y) \circ G^{\prime}\left(\left(\varphi_{2}^{\prime}\right)^{-1}\left(G^{\prime} X, G^{\prime} Y\right)\right) \circ \eta\left(G^{\prime} X \otimes G^{\prime} Y\right): G^{\prime} X \otimes G^{\prime} Y \rightarrow G^{\prime}(X \otimes Y)
\end{aligned}
$$

Hence $\gamma_{0}^{\prime}=\operatorname{Id}_{\mathbf{1}}: \emptyset \rightarrow(\mathbf{1})$ and $\gamma_{2}^{\prime}(X, Y)=\operatorname{Id}_{X \otimes Y}:(X, Y) \rightarrow(X \otimes Y)$ as

$$
\left(\varphi_{2}^{\prime}\right)^{-1}\left(G^{\prime} X, G^{\prime} Y\right)=\left(\varphi_{2}^{\prime}\right)^{-1}((X),(Y))=\operatorname{Id}_{F^{\prime}((X)) \otimes Y}=\operatorname{Id}_{X \otimes Y}
$$

Therefore we have a functor $\mathrm{Br}^{\prime}: \mathrm{Br}_{\mathcal{M}} \rightarrow \mathrm{Br}_{\mathcal{M}^{\text {str }}}$. By construction

$$
\operatorname{Br} G^{\prime}(V, c)=\left(G^{\prime}(V),\left(\gamma_{2}^{\prime}\right)^{-1}(V, V) \circ G^{\prime} c \circ \gamma_{2}^{\prime}(V, V)\right)=((V), c)
$$

Thus $((V), c)=\operatorname{Br} G^{\prime}(V, c) \in \operatorname{Br}_{\mathcal{M}^{\text {str }}}$. By Ka, Lemma XII.3.5, page 327], there is a unique strict monoidal functor $F^{\prime \prime}: \mathcal{B} \rightarrow \mathcal{M}^{\text {str }}$ such that $F^{\prime \prime}(1)=(V)$ and $F^{\prime \prime}\left(c_{1,1}\right)=c$. Define $F:=F^{\prime} \circ F^{\prime \prime}$ : $\mathcal{B} \rightarrow \mathcal{M}$. Hence $F(1)=F^{\prime} F^{\prime \prime}(1)=F^{\prime}((V))=V$ and $F\left(c_{1,1}\right)=F^{\prime} F^{\prime \prime}\left(c_{1,1}\right)=F^{\prime}(c)=c$. Let us compute the monoidal structure of $F$. By AMS, 1.3], we have that $\left(F, \varphi_{2}, \varphi_{0}\right)$ is monoidal where

$$
\begin{aligned}
\varphi_{2}(a, b) & : \quad=\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), F^{\prime \prime}(b)\right): F(a) \otimes F(b) \rightarrow F^{\prime}\left(F^{\prime \prime}(a) \otimes F^{\prime \prime}(b)\right)=F(a \otimes b) \\
\varphi_{0} & =\varphi_{0}^{\prime}: \mathbf{1} \rightarrow F^{\prime}(\emptyset)=F(0)
\end{aligned}
$$

We get

$$
\begin{aligned}
\varphi_{2}(0, b) & =\varphi_{2}^{\prime}\left(F^{\prime \prime}(0), F^{\prime \prime}(b)\right)=\varphi_{2}^{\prime}\left(\emptyset, F^{\prime \prime}(b)\right)=l_{F(b)}, \\
\varphi_{2}(a, 0) & =\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), F^{\prime \prime}(0)\right)=\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), \emptyset\right)=r_{F(a)}, \\
\varphi_{2}(a, 1) & =\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), F^{\prime \prime}(1)\right)=\varphi_{2}^{\prime}\left(F^{\prime \prime}(a),(V)\right)=\operatorname{Id}_{F(a) \otimes V} \\
\varphi_{2}(a, b \otimes 1) & =\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), F^{\prime \prime}(b \otimes 1)\right)=\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), F^{\prime \prime}(b) \otimes F^{\prime \prime}(1)\right) \\
& =\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), F^{\prime \prime}(b) \otimes(V)\right)=\left(\varphi_{2}^{\prime}\left(F^{\prime \prime}(a), F^{\prime \prime}(b)\right) \otimes V\right) \circ a_{F^{\prime} F^{\prime \prime}(a), F^{\prime} F^{\prime \prime}(b), V}^{-1} \\
& =\left(\varphi_{2}(a, b) \otimes V\right) \circ a_{F(a), F(b), V}^{-1}
\end{aligned}
$$

Thus (27), (28), (29), (30) and (31) hold true for $\left(F, \varphi_{2}, \varphi_{0}\right)$.
Let $\left(\widetilde{F}, \widetilde{\varphi}_{2}, \widetilde{\varphi}_{0}\right): \mathcal{B} \rightarrow \mathcal{M}$ be another monoidal functor such that $\widetilde{F}(0)=\mathbf{1}, \widetilde{F}(1)=V, \widetilde{F}(a \otimes 1)=$ $\widetilde{F}(a) \otimes V, \widetilde{F}\left(c_{1,1}\right)=c$ and the analogue equations (27), (28), (29), (30) and (31) hold true.

In order to prove that $\left(F, \varphi_{2}, \varphi_{0}\right)=\left(\widetilde{F}, \widetilde{\varphi}_{2}, \widetilde{\varphi}_{0}\right)$ it suffices to check that $F(a)=\widetilde{F}(a)$ for every $a \in \mathbb{N}$ (in fact the constraints are then uniquely determined by the equalities they fulfil).

Let us check that $\widetilde{F}(n)=V^{\otimes n}$ for $n \geq 0$. We proceed by induction on $n$. For $n=0$, by assumption we have $\widetilde{F}(0)=1$. Let $n>0$ and assume that $\widetilde{F}(n-1):=V^{\otimes(n-1)}$. Then

$$
\widetilde{F}(n)=\widetilde{F}((n-1) \otimes 1)=\widetilde{F}(n-1) \otimes V=V^{\otimes(n-1)} \otimes V=V^{\otimes n}
$$

Similarly $F(n)=V^{\otimes n}$ for $n \geq 0$.

Proposition 2.7. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category and let $(V, c) \in \operatorname{Br}_{\mathcal{M}}$. Then there is a unique morphism

$$
c_{T}^{a, b}: V^{\otimes a} \otimes V^{\otimes b} \rightarrow V^{\otimes b} \otimes V^{\otimes a}
$$

such that for all $l, m, n \in \mathbb{N}$

$$
\begin{gather*}
\left(V^{\otimes n} \otimes c_{T}^{l, m}\right) \circ\left(c_{T}^{l, n} \otimes V^{\otimes m}\right) \circ\left(V^{\otimes l} \otimes c_{T}^{m, n}\right)=\left(c_{T}^{m, n} \otimes V^{\otimes l}\right) \circ\left(V^{\otimes m} \otimes c_{T}^{l, n}\right) \circ\left(c_{T}^{l, m} \otimes V^{\otimes n}\right),  \tag{32}\\
\left(c_{T}^{l, n} \otimes V^{\otimes m}\right) \circ\left(V^{\otimes l} \otimes c_{T}^{m, n}\right)=c_{T}^{l+m, n}, \quad l \neq 0, m \neq 0,  \tag{33}\\
\left(V^{\otimes m} \otimes c_{T}^{l, n}\right) \circ\left(c_{T}^{l, m} \otimes V^{\otimes n}\right)=c_{T}^{l, m+n}, \quad m \neq 0, n \neq 0,  \tag{34}\\
c_{T}^{0, n} \circ l_{V}^{-1} \otimes n=r_{V \otimes n}^{-1},  \tag{35}\\
c_{T}^{n, 0} \circ r_{V \otimes n}^{-1}=l_{V \otimes n}^{-1},  \tag{36}\\
c_{T}^{1,1}=c . \tag{37}
\end{gather*}
$$

Proof. Consider the monoidal functor $\left(F, \varphi_{2}, \varphi_{0}\right): \mathcal{B} \rightarrow \mathcal{M}$ of Lemma 2.6. Consider for $a, b \in \mathbb{N}$, the isomorphism $\varphi_{2}(a, b): F(a) \otimes F(b) \rightarrow F(a+b)$ where $F(n)=V^{\otimes n}$ for every $n \in \mathbb{N}$. Set

$$
c_{T}^{a, b}:=\varphi_{2}(b, a)^{-1} \circ F\left(c_{a, b}\right) \circ \varphi_{2}(a, b): V^{\otimes a} \otimes V^{\otimes b} \rightarrow V^{\otimes b} \otimes V^{\otimes a}
$$

Thus

$$
\begin{equation*}
\varphi_{2}(b, a) \circ c_{T}^{a, b}=F\left(c_{a, b}\right) \circ \varphi_{2}(a, b) . \tag{38}
\end{equation*}
$$

Note that, since $\varphi_{2}(1,1)=\operatorname{Id}_{V \otimes V}$, we get $c_{T}^{1,1}=F\left(c_{1,1}\right)=c$ so that (37) holds. Since $c_{m, n}$ : $m \otimes n \rightarrow n \otimes m$ is the braiding in $\mathcal{B}$, we have that

$$
\begin{equation*}
\left(n \otimes c_{l, m}\right) \circ\left(c_{l, n} \otimes m\right) \circ\left(l \otimes c_{m, n}\right)=\left(c_{m, n} \otimes l\right) \circ\left(m \otimes c_{l, n}\right) \circ\left(c_{l, m} \otimes n\right) \tag{39}
\end{equation*}
$$

Thus,

$$
\begin{aligned}
& {\left[\begin{array}{c}
\varphi_{2}(n, m \otimes l) \circ\left(F(n) \otimes \varphi_{2}(m, l)\right) \circ\left(F(n) \otimes c_{T}^{l, m}\right) \circ a_{F(n), F(l), F(m)} \\
\circ\left(c_{T}^{l, n} \otimes F(m)\right) \circ a_{F(l), F(n), F(m)}^{-1} \circ\left(F(l) \otimes c_{T}^{m, n}\right)
\end{array}\right]} \\
& \stackrel{(*)}{=} F\left(n \otimes c_{l, m}\right) \circ F\left(c_{l, n} \otimes m\right) \circ F\left(l \otimes c_{m, n}\right) \circ \varphi_{2}(l, m \otimes n) \circ\left(F(l) \otimes \varphi_{2}(m, n)\right) \\
& \stackrel{(* *)}{=}\left[\begin{array}{c}
\varphi_{2}(n, m \otimes l) \circ\left(F(n) \otimes \varphi_{2}(m, l)\right) \circ a_{F(n), F(m), F(l)} \circ\left(c_{T}^{m, n} \otimes F(l)\right) \circ a_{F(m), F(n), F(l)}^{-1} \\
\circ\left(F(m) \otimes c_{T}^{l, n}\right) \circ a_{F(m), F(l), F(n)} \circ\left(c_{T}^{l, m} \otimes F(n)\right) \circ a_{F(l), F(m), F(n)}^{-1}
\end{array}\right]
\end{aligned}
$$

where in $\left(^{*}\right.$ ) we used in the order (38), the naturality of $\varphi_{2}$, the monoidality of $F$, (38), the naturality of $\varphi_{2}$, the monoidality of $F,(38)$ and the naturality of $\varphi_{2}$, while in $\left({ }^{* *}\right)$ we used in the order the monoidality of $F$, the naturality of $\varphi_{2}$, (38), the monoidality of $F$, repeated three times.

Since $\varphi_{2}(n, m \otimes l) \circ\left(F(n) \otimes \varphi_{2}(m, l)\right)$ is an isomorphism, from the computation above we deduce

$$
\begin{aligned}
& \left(F(n) \otimes c_{T}^{l, m}\right) \circ a_{F(n), F(l), F(m)} \circ\left(c_{T}^{l, n} \otimes F(m)\right) \circ a_{F(l), F(n), F(m)}^{-1} \circ\left(F(l) \otimes c_{T}^{m, n}\right) \\
= & {\left[\begin{array}{c}
a_{F(n), F(m), F(l)} \circ\left(c_{T}^{m, n} \otimes F(l)\right) \circ a_{F(m), F(n), F(l)}^{-1} \circ\left(F(m) \otimes c_{T}^{l, n}\right) \\
\circ a_{F(m), F(l), F(n)} \circ\left(c_{T}^{l, m} \otimes F(n)\right) \circ a_{F(l), F(m), F(n)}^{-1}
\end{array}\right] . }
\end{aligned}
$$

This is (32) with all the constraints. Since $c_{-,-}$is a braiding, we have

$$
\begin{equation*}
\left(c_{l, n} \otimes m\right) \circ\left(l \otimes c_{m, n}\right)=c_{l+m, n} \tag{40}
\end{equation*}
$$

We compute

$$
\begin{gathered}
\varphi_{2}(n, l \otimes m) \circ\left(F(n) \otimes \varphi_{2}(l, m)\right) \circ a_{F(n), F(l), F(m)} \circ\left(c_{T}^{l, n} \otimes F(m)\right) \circ a_{F(l), F(n), F(m)}^{-1} \circ\left(F(l) \otimes c_{T}^{m, n}\right) \\
\stackrel{(\bullet)}{=} F\left(c_{l, n} \otimes m\right) \circ F\left(l \otimes c_{m, n}\right) \circ \varphi_{2}(l, m \otimes n) \circ\left(F(l) \otimes \varphi_{2}(m, n)\right) \\
\stackrel{40)}{=} F\left(c_{l+m, n}\right) \circ \varphi_{2}(l, m \otimes n) \circ\left(F(l) \otimes \varphi_{2}(m, n)\right)
\end{gathered}
$$

$$
\stackrel{(\bullet \bullet)}{=} \varphi_{2}(n, l \otimes m) \circ c_{T}^{l+m, n} \circ\left(\varphi_{2}(l, m) \otimes F(n)\right) \circ a_{F(l), F(m), F(n)}^{-1} .
$$

where in $(\bullet)$ we used monoidality of $F,(38)$, naturality of $\varphi_{2}$, repeated twice, while in ( $\bullet \bullet$ ) we used monoidality of $F$ and (38). Since $\varphi_{2}(n, l \otimes m)$ is an isomorphism, we obtain

$$
\begin{aligned}
& \left(F(n) \otimes \varphi_{2}(l, m)\right) \circ a_{F(n), F(l), F(m)} \circ\left(c_{T}^{l, n} \otimes F(m)\right) \circ a_{F(l), F(n), F(m)}^{-1} \circ\left(F(l) \otimes c_{T}^{m, n}\right) \\
= & c_{T}^{l+m, n} \circ\left(\varphi_{2}(l, m) \otimes F(n)\right) \circ a_{F(l), F(m), F(n)}^{-1} .
\end{aligned}
$$

If $l \neq 0$ and $m \neq 0$ this formula is (33) with all the constraints.
Equation (34) follows analogously. Since $c_{-,-}$is a braiding, we have

$$
\begin{equation*}
c_{0, n} \circ l_{n}^{-1}=r_{n}^{-1} \tag{41}
\end{equation*}
$$

We get

$$
\begin{aligned}
& \varphi_{2}(n, 0) \circ c_{T}^{0, n} \circ\left(\varphi_{0} \otimes F(n)\right) \circ l_{F(n)}^{-1} F\left(c_{0, n}\right) \circ \varphi_{2}(0, n) \circ\left(\varphi_{0} \otimes F(n)\right) \circ l_{F(n)}^{-1} \\
= & F\left(c_{0, n}\right) \circ F\left(l_{n}^{-1}\right) \stackrel{41}{=} F\left(r_{n}^{-1}\right)=\varphi_{2}(n, 0) \circ\left(F(n) \otimes \varphi_{0}\right) \circ r_{F(n)}^{-1}
\end{aligned}
$$

so that

$$
c_{T}^{0, n} \circ\left(\varphi_{0} \otimes F(n)\right) \circ l_{F(n)}^{-1}=\left(F(n) \otimes \varphi_{0}\right) \circ r_{F(n)}^{-1}
$$

This is (35) with all the constraints. Equation (36) follows analogously.
We now deal with uniqueness. Suppose there exists another $c_{T^{\prime}}^{a, b}: V^{\otimes a} \otimes V^{\otimes b} \rightarrow V^{\otimes b} \otimes V^{\otimes a}$ that fulfills the analogue of the equalities that $c_{T}^{a, b}$ does for all $a, b \in \mathbb{N}$. Since $c_{T^{\prime}}^{a, b}$ fulfills the analogue of (33) we have

$$
\left(c_{T^{\prime}}^{l, n} \otimes V^{\otimes m}\right) \circ\left(V^{\otimes l} \otimes c_{T^{\prime}}^{m, n}\right)=c_{T^{\prime}}^{l+m, n}, \text { for all } l, m, n \in \mathbb{N}, l \neq 0, m \neq 0
$$

For $l=1$ we get

$$
\left(c_{T^{\prime}}^{1, n} \otimes V^{\otimes m}\right) \circ\left(V \otimes c_{T^{\prime}}^{m, n}\right)=c_{T^{\prime}}^{1+m, n}, \text { for all } l, m, n \in \mathbb{N}, m \neq 0
$$

Hence, an induction process tells that $c_{T^{\prime}}^{a, b}$ is uniquely determined by $c_{T}^{1, n}$ and $c_{T}^{0, n}$ for $n \in \mathbb{N}$. Since $c_{T}^{a, b}$ fulfills the same equalities, in order to prove $c_{T^{\prime}}^{a, b}=c_{T}^{a, b}$ it suffices to check that this is true for $a=0,1$. Analogously, using the analogue of (34) we can further reduce to the case $a, b=0,1$. The equality in these cases follows by (35), (36), (37) and their analogous.

## 3. Braided Adjunctions

Proposition 3.1. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category. Consider the category $\operatorname{BrAlg}_{\mathcal{M}}$ of braided algebras in $\mathcal{M}$ and their morphisms. Assume that $\mathcal{M}$ has denumerable coproducts and that the tensor functors preserve such coproducts for every object $M$ in $\mathcal{M}$. Denote by $\Omega_{\mathrm{Br}}: \operatorname{BrAlg}_{\mathcal{M}} \rightarrow$ $\operatorname{Br}_{\mathcal{M}}, H: \operatorname{Br}_{\mathcal{M}} \rightarrow \mathcal{M}$ and $H_{\mathrm{Alg}}: \operatorname{BrAlg}_{\mathcal{M}} \rightarrow \operatorname{Alg}_{\mathcal{M}}$ the obvious forgetful functors. Then the functor $\Omega_{\mathrm{Br}}: \mathrm{BrAlg}_{\mathcal{M}} \rightarrow \mathrm{Br}_{\mathcal{M}}$ has a left adjoint

$$
T_{\mathrm{Br}}: \mathrm{Br}_{\mathcal{M}} \rightarrow \operatorname{BrAlg}_{\mathcal{M}}
$$

Given $(V, c) \in \operatorname{Br}_{\mathcal{M}}$ one has that $T_{\mathrm{Br}}(V, c)=\left(T V, c_{T}\right)$ where $c_{T}:=c_{\Omega T V}$ is uniquely determined by

$$
\begin{equation*}
c_{T} \circ\left(\alpha_{m} V \otimes \alpha_{n} V\right)=\left(\alpha_{n} V \otimes \alpha_{m} V\right) \circ c_{T}^{m, n} \tag{42}
\end{equation*}
$$

and $c_{T}^{m, n}$ are the morphisms of Proposition 2.7. For $f$ a morphism in $\mathrm{Br}_{\mathcal{M}}$, one has $T_{\mathrm{Br}}(f):=$ $T(f)$.

The unit $\eta_{\mathrm{Br}}$ and the counit $\epsilon_{\mathrm{Br}}$ are uniquely determined by the following equations

$$
\begin{equation*}
H \eta_{\mathrm{Br}}=\eta H, \quad H_{\mathrm{Alg}} \epsilon_{\mathrm{Br}}=\epsilon H_{\mathrm{Alg}} \tag{43}
\end{equation*}
$$

where $\eta$ and $\epsilon$ denote the unit and counit of the adjunction $(T, \Omega)$ of Remark 1.3. Moreover the following diagrams commute.


Proof. Let $(V, c) \in \operatorname{Br}_{\mathcal{M}}$. By Proposition 2.7 we can consider, for $m, n \in \mathbb{N}$, the morphisms $c_{T}^{m, n}: V^{\otimes m} \otimes V^{\otimes n} \rightarrow V^{\otimes n} \otimes V^{\otimes m}$. By Remark 1.3, we can consider the tensor algebra $T V \in \operatorname{Alg}_{\mathcal{M}}$. Let us define a braiding $c_{T}$ on $T=\Omega T V$ using $c_{T}^{m, n}$. Let $\alpha_{n} V: V^{\otimes n} \rightarrow T$ be the canonical morphism. Since the tensor functors preserves denumerable coproducts, there is a unique morphism $c_{T}: T \otimes T \rightarrow T \otimes T$ such that (42).

Let us check that $\left(T, m_{T}, u_{T}, c_{T}\right)$ is a braided algebra. We know that $\left(T, m_{T}, u_{T}\right)=T V$ is an algebra. We compute

$$
\begin{aligned}
& \left(T \otimes c_{T}\right) \circ\left(c_{T} \otimes T\right) \circ\left(T \otimes c_{T}\right) \circ\left(\alpha_{l} V \otimes \alpha_{m} V \otimes \alpha_{n} V\right) \\
& \left(\alpha_{n} V \otimes \alpha_{m} V \otimes \alpha_{l} V\right) \circ\left(V^{\otimes n} \otimes c_{T}^{l, m}\right) \circ\left(c_{T}^{l, n} \otimes V^{\otimes m}\right) \circ\left(V^{\otimes l} \otimes c_{T}^{m, n}\right) \\
& \left(\alpha_{n} V \otimes \alpha_{m} V \otimes \alpha_{l} V\right) \circ\left(c_{T}^{m, n} \otimes V^{\otimes l}\right) \circ\left(V^{\otimes m} \otimes c_{T}^{l, n}\right) \circ\left(c_{T}^{l, m} \otimes V^{\otimes n}\right) \\
& \left(c_{T} \otimes T\right) \circ\left(T \otimes c_{T}\right) \circ\left(c_{T} \otimes T\right) \circ\left(\alpha_{l} V \otimes \alpha_{m} V \otimes \alpha_{n} V\right) .
\end{aligned}
$$

By arbitrariness of $l, m$ and $n$ we obtain that $c_{T}$ is a braiding i.e. that $\left(T, c_{T}\right)$ is a braided object. We have

$$
\begin{aligned}
& \left(T \otimes m_{T}\right) \circ\left(c_{T} \otimes T\right) \circ\left(T \otimes c_{T}\right) \circ\left(\alpha_{l} V \otimes \alpha_{m} V \otimes \alpha V_{n}\right) \\
& \left(T \otimes m_{T}\right) \circ\left(\alpha_{n} V \otimes \alpha_{l} V \otimes \alpha_{m} V\right) \circ\left(c_{T}^{l, n} \otimes V^{\otimes m}\right) \circ\left(V^{\otimes l} \otimes c_{T}^{m, n}\right) \\
& \text { (2) }\left(\alpha_{n} V \otimes \alpha_{l+m} V\right) \circ\left(c_{T}^{l, n} \otimes V^{\otimes m}\right) \circ\left(V^{\otimes l} \otimes c_{T}^{m, n}\right) \\
& \text { (33) }\left(\alpha_{n} V \otimes \alpha_{l+m} V\right) \circ c_{T}^{l+m, n} c_{T} \circ\left(\alpha_{l+m} V \otimes \alpha_{n} V\right) \\
& \text { (22) } c_{T} \circ\left(m_{T} \otimes T\right) \circ\left(\alpha_{l} V \otimes \alpha_{m} V \otimes \alpha_{n} V\right) .
\end{aligned}
$$

By arbitrariness of $l, m$ and $n$ we obtain that $\left(T \otimes m_{T}\right) \circ\left(c_{T} \otimes T\right) \circ\left(T \otimes c_{T}\right)=c_{T} \circ\left(m_{T} \otimes T\right)$. Similarly, using (34), one gets $\left(m_{T} \otimes T\right) \circ\left(T \otimes c_{T}\right) \circ\left(c_{T} \otimes T\right)=c_{T} \circ\left(T \otimes m_{T}\right)$.

We have

$$
\begin{aligned}
c_{T} \circ\left(u_{T} \otimes T\right) \circ l_{T}^{-1} \circ \alpha V_{n} & =c_{T} \circ\left(u_{T} \otimes \alpha V_{n}\right) \circ l_{V \otimes n}^{-1} \stackrel{\text { (3) }}{=} c_{T} \circ\left(\alpha V_{0} \otimes \alpha V_{n}\right) \circ l_{V \otimes n}^{-1} \\
& \stackrel{(42)}{ }\left(\alpha V_{n} \otimes \alpha V_{0}\right) \circ c_{T}^{0, n} \circ l_{V \otimes n}^{-1} \stackrel{(35)}{=}\left(\alpha V_{n} \otimes u_{T}\right) \circ r_{V \otimes n}^{-1} \\
& =\left(T \otimes u_{T}\right) \circ\left(\alpha V_{n} \otimes \mathbf{1}\right) \circ r_{V \otimes n}^{-1}=\left(T \otimes u_{T}\right) \circ r_{T}^{-1} \circ \alpha V_{n} .
\end{aligned}
$$

By arbitrariness of $n$ we obtain that $c_{T} \circ\left(u_{T} \otimes T\right) \circ l_{T}^{-1}=\left(T \otimes u_{T}\right) \circ r_{T}^{-1}$. Similarly, using (36), one gets $c_{T} \circ\left(T \otimes u_{T}\right) \circ r_{T}^{-1}=\left(u_{T} \otimes T\right) \circ l_{T}^{-1}$. We have so proved that $\left(T, m_{T}, u_{T}, c_{T}\right)$ is a braided algebra in $\mathcal{M}$. Define $T_{\mathrm{Br}}(V, c)$ to be this braided algebra in $\mathcal{M}$.

Let $f:(V, c) \rightarrow\left(V^{\prime}, c^{\prime}\right)$ be a morphism of braided objects. In particular $f: V \rightarrow V^{\prime}$ is a morphism in $\mathcal{M}$ so that we can consider the algebra homomorphism $T(f):\left(T, m_{T}, u_{T}\right) \rightarrow$ ( $T^{\prime}, m_{T^{\prime}}, u_{T^{\prime}}$ ). Let us check by induction on $m \in \mathbb{N}$ that

$$
\begin{equation*}
\left(f^{\otimes n} \otimes f^{\otimes m}\right) \circ c_{T}^{m, n}=c_{T^{\prime}}^{m, n} \circ\left(f^{\otimes m} \otimes f^{\otimes n}\right) \tag{45}
\end{equation*}
$$

For $m=0$ and $n \in \mathbb{N}$ we have

$$
\begin{aligned}
\left(f^{\otimes n} \otimes f^{\otimes m}\right) \circ c_{T}^{m, n} & =\left(f^{\otimes n} \otimes f^{\otimes 0}\right) \circ c_{T}^{0, n} \stackrel{(35)}{=}\left(f^{\otimes n} \otimes \mathbf{1}\right) \circ r_{V^{\otimes n}}^{-1} \circ l_{V^{\otimes n}}=r_{\left(V^{\prime}\right)^{\otimes n}}^{-1} \circ f^{\otimes n} \circ l_{V^{\otimes n}} \\
& =r_{\left(V^{\prime}\right)^{\otimes n}}^{-1} \circ l_{\left(V^{\prime}\right)^{\otimes n} \circ\left(\mathbf{1} \otimes f^{\otimes n}\right) \stackrel{(35)}{=} c_{T^{\prime}}^{0, n} \circ\left(f^{\otimes 0} \otimes f^{\otimes n}\right)=c_{T^{\prime}}^{m, n} \circ\left(f^{\otimes m} \otimes f^{\otimes n}\right) .} .
\end{aligned}
$$

For $m=1$. For $n=0$, it follows in a similar way. For $n=1$, we have

$$
\begin{aligned}
\left(f^{\otimes n} \otimes f^{\otimes m}\right) \circ c_{T}^{m, n} & =(f \otimes f) \circ c_{T}^{1,1}(f \otimes f) \circ c \\
& =c^{\prime} \circ(f \otimes f) \stackrel{(37)}{=} c_{T^{\prime}}^{1,1} \circ(f \otimes f)=c_{T^{\prime}}^{m, n} \circ\left(f^{\otimes m} \otimes f^{\otimes n}\right) .
\end{aligned}
$$

Assume that the formula holds for $n \geq 1$ and let us check it for $n+1$. Using (34), (37), the fact that $f$ is braided, (37) and (34) we get $\left(f^{\otimes n+1} \otimes f\right) \circ c_{T}^{1, n+1}=c_{T^{\prime}}^{1, n+1} \circ\left(f \otimes f^{\otimes n+1}\right)$. We have so proved that the statement holds for $m=1$ and $n \in \mathbb{N}$.

For $m \in \mathbb{N}$ and $n=0$ the formula holds in analogy to the first case we considered above.
Assume that the equation holds for $m \geq 1$ and $n \geq 1$. Then the formula holds for $(m+1, n)$ by means of (33), induction hypothesis and (33). Thus the formula is proved for all $m, n \in \mathbb{N}$.

Now, using in this order (42), (1), (45), (42) and (1) we get
$(\Omega T(f) \otimes \Omega T(f)) \circ c_{T} \circ\left(\alpha_{m} V \otimes \alpha_{n} V\right)=c_{T} \circ(\Omega T(f) \otimes \Omega T(f)) \circ\left(\alpha_{m} V \otimes \alpha_{n} V\right)$. By arbitrariness of $m, n$ we obtain that $(\Omega T(f) \otimes \Omega T(f)) \circ c_{T}=c_{T} \circ(\Omega T(f) \otimes \Omega T(f))$ so that $T(f)$ yields a morphism of braided algebras in $\mathcal{M}$ that we denote by $T_{\mathrm{Br}}(f)$. Thus we have defined a functor $T_{\mathrm{Br}}: \mathrm{Br}_{\mathcal{M}} \rightarrow \mathrm{BrAlg}_{\mathcal{M}}$. It is clear, from the construction that diagrams (44) commute. Let us check that $\left(T_{\mathrm{Br}}, \Omega_{\mathrm{Br}}\right)$ is an adjunction. For $(V, c) \in \mathrm{Br}_{\mathcal{M}}$,

$$
c_{T} \circ(\eta V \otimes \eta V) \stackrel{1}{=} c_{T} \circ\left(\alpha_{1} V \otimes \alpha_{1} V\right) \stackrel{42}{=}\left(\alpha_{1} V \otimes \alpha_{1} V\right) \circ c_{T}^{1,1} \stackrel{\text { 37) , (1) }}{=}(\eta V \otimes \eta V) \circ c
$$

Thus $\eta V: V \rightarrow \Omega T V$ defines a morphism $\eta_{\mathrm{Br}}(V, c):(V, c) \rightarrow\left(\Omega_{\mathrm{Br}} \circ T_{\mathrm{Br}}\right)(V, c)$ such that $H \eta_{\mathrm{Br}}(V, c)=\eta V$. Since $\eta V$ is natural in $V$ we get that $\eta_{\mathrm{Br}}(V, c)$ is natural in $(V, c)$ so that we get a natural transformation $\eta_{\mathrm{Br}}: \operatorname{Id}_{\mathrm{Br}_{\mathcal{M}}} \rightarrow \Omega_{\mathrm{Br}} \circ T_{\mathrm{Br}}$. Let $\left(A, m_{A}, u_{A}, c_{A}\right) \in \operatorname{BrAlg}_{\mathcal{M}}$. Then $\epsilon\left(A, m_{A}, u_{A}\right): T \Omega\left(A, m_{A}, u_{A}\right) \rightarrow\left(A, m_{A}, u_{A}\right)$ is an algebra map. Let us check it commutes with braidings:

$$
\begin{aligned}
& \left(\Omega \epsilon\left(A, m_{A}, u_{A}\right) \otimes \Omega \epsilon\left(A, m_{A}, u_{A}\right)\right) \circ c_{\Omega T \Omega\left(A, m_{A}, u_{A}\right)} \circ\left(\alpha_{m} A \otimes \alpha_{n} A\right) \\
= & \left(\Omega \epsilon\left(A, m_{A}, u_{A}\right) \otimes \Omega \epsilon\left(A, m_{A}, u_{A}\right)\right) \circ c_{\Omega T A} \circ\left(\alpha_{m} A \otimes \alpha_{n} A\right) \\
\text { (42) } & \left(\Omega \epsilon\left(A, m_{A}, u_{A}\right) \otimes \Omega \epsilon\left(A, m_{A}, u_{A}\right)\right) \circ\left(\alpha_{n} A \otimes \alpha_{m} A\right) \circ c_{\Omega T A}^{m, n} \\
\text { (1) } & \left(m_{A}^{n-1} \otimes m_{A}^{m-1}\right) \circ c_{\Omega T A}^{m, n} \stackrel{(*)}{=} c_{A} \circ\left(m_{A}^{m-1} \otimes m_{A}^{n-1}\right) \\
\text { (1) } & c_{A} \circ\left(\Omega \epsilon\left(A, m_{A}, u_{A}\right) \otimes \Omega \epsilon\left(A, m_{A}, u_{A}\right)\right) \circ\left(\alpha A_{m} \otimes \alpha A_{n}\right) .
\end{aligned}
$$

Let us prove $\left(^{*}\right)$ by induction. For $m=0$ and $n \in \mathbb{N}$ we have

$$
\begin{aligned}
\left(m_{A}^{n-1} \otimes m_{A}^{m-1}\right) \circ c_{\Omega T A}^{m, n} & =\left(m_{A}^{n-1} \otimes m_{A}^{-1}\right) \circ c_{\Omega T A}^{0, n} \stackrel{35}{=}\left(m_{A}^{n-1} \otimes u_{A}\right) \circ r_{A \otimes n}^{-1} \circ l_{A \otimes n} \\
& =\left(A \otimes u_{A}\right) \circ\left(m_{A}^{n-1} \otimes \mathbf{1}\right) \circ r_{A \otimes n}^{-1} \circ l_{A \otimes n}=\left(A \otimes u_{A}\right) \circ r_{A}^{-1} \circ m_{A}^{n-1} \circ l_{A \otimes n} \\
& =\left(A \otimes u_{A}\right) \circ r_{A}^{-1} \circ l_{A} \circ\left(\mathbf{1} \otimes m_{A}^{n-1}\right) \stackrel{(\otimes)}{ } c_{A} \circ\left(u_{A} \otimes A\right) \circ\left(\mathbf{1} \otimes m_{A}^{n-1}\right) \\
& =c_{A} \circ\left(u_{A} \otimes m_{A}^{n-1}\right)=c_{A} \circ\left(m_{A}^{-1} \otimes m_{A}^{n-1}\right)=c_{A} \circ\left(m_{A}^{m-1} \otimes m_{A}^{n-1}\right)
\end{aligned}
$$

For $n=0$ and $m \in \mathbb{N}$, the proof is similar. In particular we get the case $m=1$ and $n=0$. For $m=n=1$, we have

$$
\left(m_{A}^{n-1} \otimes m_{A}^{m-1}\right) \circ c_{\Omega T A}^{m, n}=\left(m_{A}^{0} \otimes m_{A}^{0}\right) \circ c_{\Omega T A}^{1,1}=c_{\Omega T A}^{1,1} \stackrel{(37)}{=} c_{A}=c_{A} \circ\left(m_{A}^{m-1} \otimes m_{A}^{n-1}\right)
$$

For $m=1$, assume that the formula holds for $n \geq 1$ and let us check it for $n+1$. We have

$$
\begin{gathered}
\left(m_{A}^{n} \otimes m_{A}^{m-1}\right) \circ c_{\Omega T A}^{m, n+1}=\left(m_{A}^{n} \otimes A\right) \circ c_{\Omega T A}^{1, n+1}\left(m_{A}^{n} \otimes A\right) \circ\left(A^{\otimes n} \otimes c_{\Omega T A}^{1,1}\right) \circ\left(c_{\Omega T A}^{1, n} \otimes A^{\otimes 1}\right) \\
\text { (37)}\left(m_{A} \otimes A\right) \circ\left(m_{A}^{n-1} \otimes A \otimes A\right) \circ\left(A^{\otimes n} \otimes c_{A}\right) \circ\left(c_{\Omega T A}^{1, n} \otimes A\right) \\
=\left(m_{A} \otimes A\right) \circ\left(A \otimes c_{A}\right) \circ\left(m_{A}^{n-1} \otimes m_{A}^{0} \otimes A\right) \circ\left(c_{\Omega T A}^{1, n} \otimes A\right) \\
\\
=\left(m_{A} \otimes A\right) \circ\left(A \otimes c_{A}\right) \circ\left(c_{A} \otimes A\right) \circ\left(m_{A}^{0} \otimes m_{A}^{n-1} \otimes A\right) \\
\text { (日) } c_{A} \circ\left(A \otimes m_{A}\right) \circ\left(m_{A}^{0} \otimes m_{A}^{n-1} \otimes A\right)=c_{A} \circ\left(m_{A}^{m-1} \otimes m_{A}^{n}\right) .
\end{gathered}
$$

We have so proved that the statement holds for $m=1$ and $n \in \mathbb{N}$.
Assume that the equation holds for $m \geq 1$ and $n \geq 1$ and let us prove it for $(m+1, n)$. We have

$$
\begin{gathered}
\left(m_{A}^{n-1} \otimes m_{A}^{m}\right) \circ c_{\Omega T A}^{m+1, n} \stackrel{(33)}{=}\left(m_{A}^{n-1} \otimes m_{A}^{m}\right) \circ\left(c_{\Omega T A}^{m, n} \otimes A^{\otimes 1}\right) \circ\left(A^{\otimes m} \otimes c_{\Omega T A}^{1, n}\right) \\
=\left(A \otimes m_{A}\right) \circ\left(m_{A}^{n-1} \otimes m_{A}^{m-1} \otimes A\right) \circ\left(c_{\Omega T A}^{m, n} \otimes A\right) \circ\left(A^{\otimes m} \otimes c_{\Omega T A}^{1, n}\right) \\
=\left(A \otimes m_{A}\right) \circ\left(c_{A} \otimes A\right) \circ\left(m_{A}^{m-1} \otimes m_{A}^{n-1} \otimes A\right) \circ\left(A^{\otimes m} \otimes c_{\Omega T A}^{1, n}\right) \\
=\left(A \otimes m_{A}\right) \circ\left(c_{A} \otimes A\right) \circ\left(A \otimes c_{A}\right) \circ\left(m_{A}^{m-1} \otimes A \otimes m_{A}^{n-1}\right) \\
\quad \text { (6) } c_{A} \circ\left(m_{A} \otimes A\right) \circ\left(m_{A}^{m-1} \otimes A \otimes m_{A}^{n-1}\right)=c_{A} \circ\left(m_{A}^{m} \otimes m_{A}^{n-1}\right) .
\end{gathered}
$$

We have so proved that $\left(^{*}\right)$ holds. Hence $\epsilon\left(A, m_{A}, u_{A}\right): T \Omega\left(A, m_{A}, u_{A}\right) \rightarrow\left(A, m_{A}, u_{A}\right)$ induces a morphism $\epsilon_{\mathrm{Br}}\left(A, m_{A}, u_{A}, c_{A}\right): T_{\mathrm{Br}} \Omega_{\mathrm{Br}}\left(A, m_{A}, u_{A}, c_{A}\right) \rightarrow\left(A, m_{A}, u_{A}, c_{A}\right)$ such that

$$
H_{\mathrm{Alg}}\left(\epsilon_{\mathrm{Br}}\left(A, m_{A}, u_{A}, c_{A}\right)\right)=\epsilon\left(A, m_{A}, u_{A}\right)
$$

The morphism $\epsilon_{\operatorname{Br}}\left(A, m_{A}, u_{A}, c_{A}\right)$ is natural as $\epsilon\left(A, m_{A}, u_{A}\right)$ is natural. We have

$$
\begin{aligned}
& H_{\mathrm{Alg}}\left(\epsilon_{\mathrm{Br}} T_{\mathrm{Br}} \circ T_{\mathrm{Br}} \eta_{\mathrm{Br}}\right)=\epsilon H_{\mathrm{Alg}} T_{\mathrm{Br}} \circ H_{\mathrm{Alg}} T_{\mathrm{Br}} \eta_{\mathrm{Br}} \\
= & \epsilon T H \circ T H \eta_{\mathrm{Br}}=\epsilon T H \circ T \eta H=\operatorname{Id}_{T H}=H_{\mathrm{Alg}}\left(\operatorname{Id}_{T}\right) \\
& H\left(\Omega_{\mathrm{Br}} \epsilon_{\mathrm{Br}} \circ \eta_{\mathrm{Br}} \Omega_{\mathrm{Br}}\right)=H \Omega_{\mathrm{Br}} \epsilon_{\mathrm{Br}} \circ H \eta_{\mathrm{Br}} \Omega_{\mathrm{Br}} \\
= & \Omega H_{\mathrm{Alg}} \epsilon_{\mathrm{Br}} \circ \eta H \Omega_{\mathrm{Br}}=\Omega \epsilon H_{\mathrm{Alg}} \circ \eta \Omega H_{\mathrm{Alg}}=\mathrm{Id}_{H_{\mathrm{Alg}}}=H_{\mathrm{Alg}}\left(\operatorname{Id}_{\mathcal{M}}\right) .
\end{aligned}
$$

Since both $H_{\mathrm{Alg}}$ and $H$ are faithful, we get that $\left(T_{\mathrm{Br}}, \Omega_{\mathrm{Br}}\right)$ is an adjunction with unit $\eta_{\mathrm{Br}}$ and counit $\epsilon_{\mathrm{Br}}$.
Definition 3.2. Let $\mathcal{M}$ be a preadditive monoidal category with equalizers. Assume that the tensor functors are additive. Let $\mathbb{C}:=\left(C, \Delta_{C}, \varepsilon_{C}, u_{C}\right)$ be a coalgebra $\left(C, \Delta_{C}, \varepsilon_{C}\right)$ endowed with a coalgebra morphism $u_{C}: \mathbf{1} \rightarrow C$. In this setting we always implicitly assume that we can choose a specific equalizer


We will use the same symbol when $\mathbb{C}$ comes out to be enriched with an extra structure such us when $\mathbb{C}$ will denote a bialgebra or a braided bialgebra.

Next result should be compared with GV, Lemma 6.2]. Note that, in our case, the braiding of the primitive elements has not order two, in general. Also our proof of the existence of such a braiding follows different lines.

Lemma 3.3. Let $\mathcal{M}$ a preadditive monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. For any $\mathbb{A}:=\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right) \in \operatorname{BrBialg}_{\mathcal{M}}$, there is a unique morphism $c_{P(\mathbb{A})}: P(\mathbb{A}) \otimes P(\mathbb{A}) \rightarrow P(\mathbb{A}) \otimes P(\mathbb{A})$ such that

$$
\begin{equation*}
(\xi \mathbb{A} \otimes \xi \mathbb{A}) \circ c_{P(\mathbb{A})}=c_{A} \circ(\xi \mathbb{A} \otimes \xi \mathbb{A}) \tag{46}
\end{equation*}
$$

We have that $\left(P(\mathbb{A}), c_{P(\mathbb{A})}\right) \in \operatorname{Br}_{\mathcal{M}}$ and that $\xi \mathbb{A}: P(\mathbb{A}) \rightarrow A$ is a morphism of braided objects that will be denoted by $\xi \mathbb{A}:\left(P(\mathbb{A}), c_{P(\mathbb{A})}\right) \rightarrow\left(A, c_{A}\right)$. For any morphism $f: \mathbb{A} \rightarrow \mathbb{A}^{\prime}$ in $\operatorname{BrBialg}_{\mathcal{M}}$, there is a unique morphism $P(f): P(\mathbb{A}) \rightarrow P\left(\mathbb{A}^{\prime}\right)$ such that

$$
\begin{equation*}
\xi \mathbb{A}^{\prime} \circ P(f)=f \circ \xi \mathbb{A} \tag{47}
\end{equation*}
$$

The morphism $P(f)$ is indeed a morphism of braided objects. This way we get a functor

$$
P_{\mathrm{Br}}: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \operatorname{Br}_{\mathcal{M}}: \mathbb{A} \mapsto\left(P(\mathbb{A}), c_{P(\mathbb{A})}\right), f \mapsto P(f)
$$

Moreover

$$
\begin{equation*}
\xi \mathbb{A}^{\prime} \circ P_{\mathrm{Br}}(f)=\Omega_{\mathrm{Br}} \mho_{\mathrm{Br}}(f) \circ \xi \mathbb{A} \tag{48}
\end{equation*}
$$

for every morphism $f: \mathbb{A} \rightarrow \mathbb{A}^{\prime}$ in $\operatorname{BrBialg}_{\mathcal{M}}$ i.e. $\xi \mathbb{A}: P_{\mathrm{Br}} \mathbb{A} \rightarrow \Omega_{\mathrm{Br}} \mho_{\mathrm{Br}} \mathbb{A}$ is natural in $\mathbb{A}$.

Proof. Note that, using Definition 3.2, we have

$$
(P(\mathbb{A}), \xi \mathbb{A}):=\left(P\left(A, \Delta_{A}, \varepsilon_{A}, u_{A}\right), \xi\left(A, \Delta_{A}, \varepsilon_{A}, u_{A}\right)\right)
$$

For sake of shortness we just write $P$ instead of $P(\mathbb{A})$. Let us check that the braiding of $A$ induces a braiding on $P$. To this aim, first consider the following diagram.


Note that $\left(A, \Delta_{A}, \varepsilon_{A}, c_{A}\right)$ is a braided coalgebra whence we have that (10) holds. Moreover, using the equalities $r_{A}^{-1} \otimes A=A \otimes l_{A}^{-1}$ and $l_{A}^{-1} \otimes A=l_{A \otimes A}^{-1}$, (8), the naturality of $l^{-1}$, the equalities $A \otimes r_{A}^{-1}=r_{A \otimes A}^{-1}$ and $l_{A \otimes A}^{-1}=l_{A}^{-1} \otimes A$, the naturality of $r^{-1}$, (8), the equalities $r_{A \otimes A}^{-1}=A \otimes r_{A}^{-1}$ and $r_{A}^{-1} \otimes A=A \otimes l_{A}^{-1}$, we get

$$
\begin{aligned}
& \left(c_{A} \otimes A\right)\left(A \otimes c_{A}\right)\left\{\left[\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}\right] \otimes A\right\} \\
= & \left\{A \otimes\left[\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}\right]\right\} c_{A} .
\end{aligned}
$$

Hence the diagram above serially commutes. Since the tensor product preserves equalizers, the bottom fork of the diagram is an equalizer so that there is a unique morphism $c_{P, A}: P \otimes A \rightarrow A \otimes P$ such that

$$
\begin{equation*}
(A \otimes \xi \mathbb{A}) \circ c_{P, A}=c_{A} \circ(\xi \mathbb{A} \otimes A) \tag{49}
\end{equation*}
$$

Similarly there is a unique morphism $c_{A, P}: A \otimes P \rightarrow P \otimes A$ such that

$$
\begin{equation*}
(\xi \mathbb{A} \otimes A) \circ c_{A, P}=c_{A} \circ(A \otimes \xi \mathbb{A}) \tag{50}
\end{equation*}
$$

Consider the following diagram


Using (49), (50), the equalizer defining $\xi \mathbb{A}$, (50), (49) we get

$$
\begin{aligned}
& (A \otimes A \otimes \xi \mathbb{A}) \circ\left(\Delta_{A} \otimes P\right) \circ c_{P, A} \circ(P \otimes \xi \mathbb{A}) \\
= & (A \otimes A \otimes \xi \mathbb{A}) \circ\left\{\left[\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}\right] \otimes P\right\} \circ c_{P, A} \circ(P \otimes \xi \mathbb{A})
\end{aligned}
$$

so that

$$
\left(\Delta_{A} \otimes P\right) \circ c_{P, A} \circ(P \otimes \xi \mathbb{A})=\left\{\left[\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}\right] \otimes P\right\} \circ c_{P, A} \circ(P \otimes \xi \mathbb{A})
$$

Hence there is a unique morphism $c_{P}: P \otimes P \rightarrow P \otimes P$ such that

$$
\begin{equation*}
(\xi \mathbb{A} \otimes P) \circ c_{P}=c_{P, A} \circ(P \otimes \xi \mathbb{A}) \tag{51}
\end{equation*}
$$

Using (51) and (49) one gets $(\xi \mathbb{A} \otimes \xi \mathbb{A}) c_{P}=c_{A}(\xi \mathbb{A} \otimes \xi \mathbb{A})$ so that (46) holds. Note that, since $\xi \mathbb{A} \otimes \xi \mathbb{A}$ is a monomorphism, the morphism $c_{P}$ is uniquely determined by (46).

Since $\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right) \in \operatorname{BrBialg}_{\mathcal{M}}$ we have that (10), (9) and (8) holds.If we write these equalities with respect to $c_{A}^{-1}$, we get

$$
\begin{gathered}
\left(A \otimes c_{A}^{-1}\right)\left(c_{A}^{-1} \otimes A\right)\left(A \otimes \Delta_{A}\right)=\left(\Delta_{A} \otimes A\right) c_{A}^{-1} \\
\left(c_{A}^{-1} \otimes A\right)\left(A \otimes c_{A}^{-1}\right)\left(\Delta_{A} \otimes A\right)=\left(A \otimes \Delta_{A}\right) c_{A}^{-1} \\
\left(u_{A} \otimes A\right) l_{A}^{-1}=c_{A}^{-1}\left(A \otimes u_{A}\right) r_{A}^{-1}, \quad\left(A \otimes u_{A}\right) r_{A}^{-1}=c_{A}^{-1}\left(u_{A} \otimes A\right) l_{A}^{-1}
\end{gathered}
$$

Thus $c_{A}^{-1}$ fulfills the same equalities that we used for $c_{A}$ in the computations above. Hence, the same argument entails that there is a morphism $c_{P}^{\prime}: P \otimes P \rightarrow P \otimes P$ such that

$$
(\xi \mathbb{A} \otimes \xi \mathbb{A}) c_{P}^{\prime}=c_{A}^{-1}(\xi \mathbb{A} \otimes \xi \mathbb{A})
$$

Thus $(\xi \mathbb{A} \otimes \xi \mathbb{A}) c_{P}^{\prime} c_{P}=c_{A}^{-1}(\xi \mathbb{A} \otimes \xi \mathbb{A}) c_{P}=c_{A}^{-1} c_{A}(\xi \mathbb{A} \otimes \xi \mathbb{A})=(\xi \mathbb{A} \otimes \xi \mathbb{A})$ and hence $c_{P}^{\prime} c_{P}=$ $\operatorname{Id}_{P \otimes P}$. Similarly $c_{P} c_{P}^{\prime}=\operatorname{Id}_{P \otimes P}$ so that $c_{P}$ is invertible. Moreover using (46) repeatedly and the fact that $c_{A}$ is a braiding, one checks that

$$
(\xi \mathbb{A} \otimes \xi \mathbb{A} \otimes \xi \mathbb{A})\left(c_{P} \otimes P\right)\left(P \otimes c_{P}\right)\left(c_{P} \otimes P\right)=(\xi \mathbb{A} \otimes \xi \mathbb{A} \otimes \xi \mathbb{A})\left(P \otimes c_{P}\right)\left(c_{P} \otimes P\right)\left(P \otimes c_{P}\right)
$$

so that $\left(c_{P} \otimes P\right)\left(P \otimes c_{P}\right)\left(c_{P} \otimes P\right)=\left(P \otimes c_{P}\right)\left(c_{P} \otimes P\right)\left(P \otimes c_{P}\right)$ which means that $c_{P}$ is a braiding. Note that (46) means that $\xi \mathbb{A}: P \rightarrow A$ is a morphism of braided objects that will be denoted by $\xi \mathbb{A}:\left(P, c_{P}\right) \rightarrow\left(A, c_{A}\right)$. Let $f: \mathbb{A} \rightarrow \mathbb{A}^{\prime}$ be a morphism in BrBialg $\mathcal{M}_{\mathcal{M}}$ and consider the following diagram


Using $(f \otimes f)\left(A \otimes u_{A}\right) r_{A}^{-1}=\left(A^{\prime} \otimes u_{A^{\prime}}\right) r_{A^{\prime}}^{-1} f$ and $(f \otimes f)\left(u_{A} \otimes A\right) l_{A}^{-1}=\left(u_{A^{\prime}} \otimes A^{\prime}\right) l_{A^{\prime}}^{-1} f$ we get that the diagram above serially commutes. Therefore there is a unique morphism $P(f): P(\mathbb{A}) \rightarrow$ $P\left(\mathbb{A}^{\prime}\right)$ such that (47) holds. Using (47), (46), the fact that $f$ is braided, (47) and (46) we arrive at $\left(\xi \mathbb{A}^{\prime} \otimes \xi \mathbb{A}^{\prime}\right)(P(f) \otimes P(f)) c_{P(\mathbb{A})}=\left(\xi \mathbb{A}^{\prime} \otimes \xi \mathbb{A}^{\prime}\right) c_{P\left(\mathbb{A}^{\prime}\right)}(P(f) \otimes P(f))$ so that $(P(f) \otimes P(f)) c_{P}=$ $c_{P^{\prime}}(P(f) \otimes P(f))$ which means that $P(f): P(\mathbb{A}) \rightarrow P\left(\mathbb{A}^{\prime}\right)$ is a morphism of braided objects. This way we get a functor

$$
P_{\mathrm{Br}}: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \operatorname{Br}_{\mathcal{M}}: \mathbb{A} \mapsto\left(P(\mathbb{A}), c_{P(\mathbb{A})}\right), f \mapsto P(f)
$$

By the foregoing we have (48) holds.
We now investigate some properties of $T_{\mathrm{Br}}$.
Lemma 3.4. Let $\mathcal{M}$ a preadditive monoidal category with equalizers and denumerable coproducts. Assume that the tensor functors are additive and preserve equalizers and denumerable coproducts. By Proposition 3.1, the forgetful functor $\Omega_{\mathrm{Br}}: \operatorname{BrAlg}_{\mathcal{M}} \rightarrow \mathrm{Br}_{\mathcal{M}}$ has a left adjoint $T_{\mathrm{Br}}: \mathrm{Br}_{\mathcal{M}} \rightarrow$ $\operatorname{BrAlg}_{\mathcal{M}}$. For all $\mathbb{B} \in \operatorname{BrBialg}_{\mathcal{M}}$, since $T_{\mathrm{Br}}(V, c) \in \operatorname{BrAlg}_{\mathcal{M}}$ we can write it in the form $T_{\mathrm{Br}}(V, c)=$ $\left(A, m_{A}, u_{A}, c_{A}\right)$. Regard $A \otimes A$ as an algebra in $\mathcal{M}$ via $m_{A \otimes A}:=\left(m_{A} \otimes m_{A}\right)\left(A \otimes c_{A} \otimes A\right)$ and $u_{A \otimes A}:=\left(u_{A} \otimes u_{A}\right) \Delta_{\mathbf{1}}$. For every $n \in \mathbb{N}$, denote by $\alpha_{n} V: V^{\otimes n} \rightarrow \Omega T V$ the canonical injection. Then there are unique algebra morphisms $\Delta_{A}: A \rightarrow A \otimes A$ and $\varepsilon_{A}: A \rightarrow \mathbf{1}$ such that

$$
\begin{align*}
\Delta_{A} \circ \alpha_{1} V & =\delta_{V}^{l}+\delta_{V}^{r}  \tag{52}\\
\varepsilon_{A} \circ \alpha_{1} V & =0 \tag{53}
\end{align*}
$$

where $\delta_{V}^{l}:=\left(u_{A} \otimes \alpha_{1} V\right) \circ l_{V}^{-1}$ and $\delta_{V}^{r}:=\left(\alpha_{1} V \otimes u_{A}\right) \circ r_{V}^{-1}$. Moreover

$$
\begin{equation*}
\varepsilon_{A} \circ \alpha_{n} V=\delta_{n, 0} \mathrm{Id}_{\mathbf{1}}, \text { for every } n \in \mathbb{N} \tag{54}
\end{equation*}
$$

The datum $\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right)$ is a braided bialgebra. Moreover $T_{\mathrm{Br}}: \operatorname{Br}_{\mathcal{M}} \rightarrow \operatorname{BrAlg}_{\mathcal{M}}$ induces the functor

$$
\begin{aligned}
\bar{T}_{\mathrm{Br}} & : \operatorname{Br}_{\mathcal{M}} \rightarrow \operatorname{BrBialg}_{\mathcal{M}} \\
(V, c) & \mapsto\left(T=T_{\mathrm{Br}}(V, c), \Delta_{T}, \varepsilon_{T}\right)=\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right) \\
f & \mapsto T_{\mathrm{Br}}(f)
\end{aligned}
$$

so that the following diagram commutes.


Proof. By Proposition 3.1, the unit $\eta_{\mathrm{Br}}$ and the counit $\epsilon_{\mathrm{Br}}$ of the adjunction $\left(T_{\mathrm{Br}}, \Omega_{\mathrm{Br}}\right)$ are uniquely determined by (43). Moreover the diagrams (44) commute. Given $(V, c) \in \operatorname{Br}_{\mathcal{M}}$, then $T_{\mathrm{Br}}(V, c)$ becomes an object in $\operatorname{BrBialg}_{\mathcal{M}}$ as follows. For all $\mathbb{B} \in \operatorname{BrBialg}_{\mathcal{M}}$, consider the canonical isomorphism

$$
\Phi((V, c), \mathbb{B}): \operatorname{BrAlg}_{\mathcal{M}}\left(T_{\mathrm{Br}}(V, c), \mathbb{B}\right) \rightarrow \operatorname{Br}_{\mathcal{M}}\left((V, c), \Omega_{\mathrm{Br}}(\mathbb{B})\right): f \mapsto \Omega_{\mathrm{Br}}(f) \circ \eta_{\mathrm{Br}}(V, c)
$$

Since $T_{\mathrm{Br}}(V, c) \in \operatorname{BrAlg}_{\mathcal{M}}$ we can write it in the form $T_{\mathrm{Br}}(V, c)=\left(A, m_{A}, u_{A}, c_{A}\right)$. By Lemma 2.3, $\left(B, m_{B}, u_{B}, c_{B}\right) \in \operatorname{BrAlg}_{\mathcal{M}}$ where $B:=A \otimes A, m_{B}:=\left(m_{A} \otimes m_{A}\right)\left(A \otimes c_{A} \otimes A\right), u_{B}:=$ $\left(u_{A} \otimes u_{A}\right) \Delta_{1}$ and $c_{B}:=\left(A \otimes c_{A} \otimes A\right)\left(c_{A} \otimes c_{A}\right)\left(A \otimes c_{A} \otimes A\right)$.

In particular we have the morphisms

$$
\begin{aligned}
\Phi\left((V, c),\left(B, m_{B}, u_{B}, c_{B}\right)\right) & : \quad \operatorname{BrAlg}_{\mathcal{M}}\left(T_{\mathrm{Br}}(V, c),\left(B, m_{B}, u_{B}, c_{B}\right)\right) \rightarrow \operatorname{Br}_{\mathcal{M}}\left((V, c),\left(B, c_{B}\right)\right), \\
\Phi\left((V, c),\left(\mathbf{1}, m_{\mathbf{1}}, u_{\mathbf{1}}, c_{\mathbf{1}}\right)\right) & : \quad \operatorname{BrAlg}_{\mathcal{M}}\left(T_{\mathrm{Br}}(V, c), \mathbb{B}\right) \rightarrow \operatorname{Br}_{\mathcal{M}}\left((V, c),\left(\mathbf{1}, c_{\mathbf{1}}\right)\right) .
\end{aligned}
$$

where $c_{\boldsymbol{1}}=\operatorname{Id}_{\mathbf{1} \otimes \boldsymbol{1}}$. Note that

$$
\begin{equation*}
H \eta_{\mathrm{Br}}(V, c) \stackrel{43)}{=} \eta H(V, c)=\eta V=\alpha_{1} V . \tag{56}
\end{equation*}
$$

Let us check that $\delta_{V}^{l}$ is a morphism of braided objects. Using in the given order the definitions of $c_{B}$ and $\delta_{V}^{l},(8)$ twice, the equality $l_{A}^{-1} \otimes A=l_{A \otimes A}^{-1}$, the naturality of $l^{-1}$, the equality $l_{A \otimes A}^{-1}=l_{A}^{-1} \otimes A$, the equality $r_{A} \otimes A=A \otimes l_{A}$, the naturality of $l^{-1}$, (8), the equality $\mathbf{1} \otimes l_{A}=r_{1} \otimes A$, the equality $r_{\mathbf{1}}=l_{\mathbf{1}}$, the naturality of $r$, the equality $r_{V} \otimes V=V \otimes l_{V}$, the naturality of $l^{-1}$, the equality (56) twice, the equality $l_{A \otimes \mathbf{1}}^{-1}=l_{A}^{-1} \otimes \mathbf{1}$, the fact that $\eta_{\mathrm{Br}}(V, c)$ is braided, the equality (56) twice, the equality $r_{A}^{-1} \otimes A=A \otimes l_{A}^{-1}$, the naturality of $l^{-1}$ and the definition of $\delta_{V}^{l}$, we arrive at $c_{B}\left(\delta_{V}^{l} \otimes \delta_{V}^{l}\right)=\left(\delta_{V}^{l} \otimes \delta_{V}^{l}\right) c$. Analogously one gets the equalities $c_{B}\left(\delta_{V}^{r} \otimes \delta_{V}^{r}\right)=\left(\delta_{V}^{r} \otimes \delta_{V}^{r}\right) c$, $c_{B}\left(\delta_{V}^{l} \otimes \delta_{V}^{r}\right)=\left(\delta_{V}^{r} \otimes \delta_{V}^{l}\right) c$ and $c_{B}\left(\delta_{V}^{r} \otimes \delta_{V}^{l}\right)=\left(\delta_{V}^{r} \otimes \delta^{l}\right) c$. By means of these equalities one easily gets $c_{B}\left[\left(\delta_{V}^{l}+\delta_{V}^{r}\right) \otimes\left(\delta_{V}^{l}+\delta_{V}^{r}\right)\right]=\left[\left(\delta_{V}^{l}+\delta_{V}^{r}\right) \otimes\left(\delta_{V}^{l}+\delta_{V}^{r}\right)\right] c$. Thus $\delta_{V}^{l}+\delta_{V}^{r}: V \rightarrow B$ defines a morphism $\widehat{\delta_{V}^{l}+\delta_{V}^{r}}:(V, c) \rightarrow\left(B, c_{B}\right)$ in $\operatorname{Br}_{\mathcal{M}}$ such that $H\left(\widehat{\delta_{V}^{l}+\delta_{V}^{r}}\right)=\delta_{V}^{l}+\delta_{V}^{r}$. Hence we can set

$$
\begin{aligned}
\Delta_{T} & :=\Phi\left((V, c),\left(B, m_{B}, u_{B}, c_{B}\right)\right)^{-1}\left(\widehat{\delta_{V}^{l}+\delta_{V}^{r}}\right) \in \operatorname{BrAlg}_{\mathcal{M}}\left(T_{\mathrm{Br}}(V, c),\left(B, m_{B}, u_{B}, c_{B}\right)\right) \\
\varepsilon_{T} & :=\Phi\left((V, c),\left(\mathbf{1}, m_{\mathbf{1}}, u_{\mathbf{1}}, c_{\mathbf{1}}\right)\right)^{-1}(0) \in \operatorname{BrAlg}_{\mathcal{M}}\left(T_{\mathrm{Br}}(V, c),\left(\mathbf{1}, m_{\mathbf{1}}, u_{\mathbf{1}}, c_{\mathbf{1}}\right)\right)
\end{aligned}
$$

Moreover we set

$$
\Delta_{A}:=H \Omega_{\mathrm{Br}} \Delta_{T} \quad \text { and } \quad \varepsilon_{A}:=H \Omega_{\mathrm{Br}} \varepsilon_{T}
$$

We have 43

$$
\begin{gathered}
\Delta_{A} \circ \alpha_{1} V \stackrel{(56)}{=} H \Omega_{\mathrm{Br}} \Delta_{T} \circ H \eta_{\mathrm{Br}}(V, c)=H\left(\Omega_{\mathrm{Br}} \Delta_{T} \circ \eta_{\mathrm{Br}}(V, c)\right) \\
=H\left(\Phi\left((V, c),\left(B, m_{B}, u_{B}, c_{B}\right)\right)\left[\Delta_{T}\right]\right)=H\left(\delta_{V}^{l}+\delta_{V}^{r}\right)=\delta_{V}^{l}+\delta_{V}^{r}, \\
\varepsilon_{A} \circ \alpha_{1} V \stackrel{(56)}{=} H \Omega_{\mathrm{Br}} \varepsilon_{T} \circ H \eta_{\mathrm{Br}}(V, c)=H\left(\Omega_{\mathrm{Br}} \varepsilon_{T} \circ \eta_{\mathrm{Br}}(V, c)\right) \\
=H\left(\Phi\left((V, c),\left(\mathbf{1}, m_{\mathbf{1}}, u_{\mathbf{1}}, c_{\mathbf{1}}\right)\right)\left[\varepsilon_{T}\right]\right)=H(0)=0
\end{gathered}
$$

so that we get (52) and (53). Note that, since the tensor algebra functor is a left adjoint of the forgetful functor and $\alpha_{1} V=\eta V$, the unit of the adjunction, we have that the algebra morphisms $\Delta_{A}$ and $\varepsilon_{A}$ are uniquely determined by (52) and (53).

For every $n>0$, we have

$$
\varepsilon_{A} \circ \alpha_{n} V \stackrel{(2)}{=} \varepsilon_{A} \circ m_{\Omega T V} \circ\left(\alpha_{n-1} V \otimes \alpha_{1} V\right)=\left(\varepsilon_{A} \otimes \varepsilon_{A}\right) \circ\left(\alpha_{n-1} V \otimes \alpha_{1} V\right)=0
$$

where we used that $\varepsilon_{A}$ is an algebra morphism. Moreover $\varepsilon_{A} \circ \alpha_{0} V=\varepsilon_{A} \circ u_{A}=\operatorname{Id}_{1}$. Thus we get that (54) holds. Using (42), (54), the naturality of $r$, the equality $r_{V \otimes n} c_{A}^{0, n}=l_{V \otimes n}$ (which holds
since construction $c_{A}^{0, n}$ fulfills (35), the naturality of $l$ and (54), we get

$$
r_{A}\left(A \otimes \varepsilon_{A}\right) c_{A}\left(\alpha_{m} V \otimes \alpha_{n} V\right)=l_{A}\left(\varepsilon_{A} \otimes A\right)\left(\alpha_{m} V \otimes \alpha_{n} V\right)
$$

Since this holds true for every $m, n$, we obtain that $r_{A}\left(A \otimes \varepsilon_{A}\right) c_{A}=l_{A}\left(\varepsilon_{A} \otimes A\right)$. Analogously one gets $l_{A}\left(\varepsilon_{A} \otimes A\right) c_{A}=r_{A}\left(A \otimes \varepsilon_{A}\right)$ so that (11) is proved. Note that $\operatorname{Id}_{A}: A \rightarrow A$ and $\varepsilon_{A}: A \rightarrow \mathbf{1}$ are morphisms in $\operatorname{BrAlg}_{\mathcal{M}}$. Moreover $\left(A_{1}, A_{2}, c_{2,1}\right)$ and $\left(A_{2}, A_{1}, c_{1,2}\right)$ (respectively $\left(A_{1}^{\prime}, A_{2}^{\prime}, c_{2,1}^{\prime}\right)$ and $\left.\left(A_{2}^{\prime}, A_{1}^{\prime}, c_{1,2}^{\prime}\right)\right)$ fulfil the requirements of Proposition 2.2-2) for $A_{2}=A_{1}=A$ and $c_{i, j}=c_{A}, i, j \in$ $\{1,2\}$ (resp. for $A_{1}^{\prime}=A, A_{2}^{\prime}=\mathbf{1}, c_{2,1}^{\prime}=r_{A}^{-1} l_{A}, c_{1,2}^{\prime}=l_{A}^{-1} r_{A}, c_{2,2}^{\prime}=\operatorname{Id}_{\mathbf{1} \otimes \mathbf{1}}, c_{1,1}^{\prime}=c_{A}$ ). Moreover, by the foregoing, we have

$$
\left(A \otimes \varepsilon_{A}\right) c_{2,1}=c_{2,1}^{\prime}\left(\varepsilon_{A} \otimes A\right), \quad\left(\varepsilon_{A} \otimes A\right) c_{1,2}=c_{1,2}^{\prime}\left(A \otimes \varepsilon_{A}\right)
$$

Thus, by Proposition 2.2-3) applied to $f_{1}:=\operatorname{Id}_{A}$ and $f_{2}:=\varepsilon_{A}$, we can conclude that $T \otimes \varepsilon_{T}$ is a morphism in $\operatorname{BrAlg}_{\mathcal{M}}$. Thus we have that $\left(T \otimes \varepsilon_{T}\right) \Delta_{T}$ is a morphism in $\operatorname{BrAlg}_{\mathcal{M}}$. One also checks that $r_{A}: A \otimes \mathbf{1} \rightarrow A$ is a morphism in $\operatorname{BrAlg}_{\mathcal{M}}$. Thus we can denote by $r_{T}$ the morphism $r_{A}$ regarded as a morphism in $\operatorname{BrAlg}_{\mathcal{M}}$. In other words $H \Omega_{\mathrm{Br}} r_{T}=r_{A}$. Thus we have that $r_{T}\left(T \otimes \varepsilon_{T}\right) \Delta_{T}: T \rightarrow T$ is a morphism in $\operatorname{BrAlg} \mathcal{M}_{\mathcal{M}}$. We have to check that $r_{T}\left(T \otimes \varepsilon_{T}\right) \Delta_{T}=\operatorname{Id}_{T}$. Since the two sides are in $\operatorname{BrAlg}_{\mathcal{M}}\left(T_{\mathrm{Br}}(V, c), T_{\mathrm{Br}}(V, c)\right)$ we have to prove that

$$
\Phi\left((V, c), T_{\mathrm{Br}}(V, c)\right)\left[r_{T}\left(T \otimes \varepsilon_{T}\right) \Delta_{T}\right]=\Phi\left((V, c), T_{\mathrm{Br}}(V, c)\right)\left[\operatorname{Id}_{T}\right]
$$

or equivalently

$$
\begin{equation*}
H \Phi\left((V, c), T_{\mathrm{Br}}(V, c)\right)\left[r_{T}\left(T \otimes \varepsilon_{T}\right) \Delta_{T}\right]=H \Phi\left((V, c), T_{\mathrm{Br}}(V, c)\right)\left[\mathrm{Id}_{T}\right] \tag{57}
\end{equation*}
$$

Note that for any braided algebra morphism $\xi: T \rightarrow U$, we have

$$
\begin{gathered}
H \Phi((V, c), U)\left(\xi \Delta_{T}\right)=H\left\{\Omega_{\mathrm{Br}}\left(\xi \Delta_{T}\right) \eta_{\mathrm{Br}}(V, c)\right\}=H \Omega_{\mathrm{Br}}(\xi) H\left\{\Omega_{\mathrm{Br}}\left(\Delta_{T}\right) \eta_{\mathrm{Br}}(V, c)\right\} \\
=H \Omega_{\mathrm{Br}}(\xi) H\left[\widehat{\delta_{V}^{l}+\delta_{V}^{r}}\right]=H \Omega_{\mathrm{Br}}(\xi)\left(\delta_{V}^{l}+\delta_{V}^{r}\right)
\end{gathered}
$$

so that

$$
\begin{equation*}
H \Phi((V, c), U)\left(\xi \Delta_{T}\right)=H \Omega_{\operatorname{Br}}(\xi)\left(\delta_{V}^{l}+\delta_{V}^{r}\right), \text { for any } \xi: T \rightarrow U \text { in } \operatorname{BrAlg}_{\mathcal{M}} \tag{58}
\end{equation*}
$$

The left-hand side of (57) is

$$
\begin{aligned}
& H \Phi\left((V, c), T_{\mathrm{Br}}(V, c)\right)\left[r_{T}\left(T \otimes \varepsilon_{T}\right) \Delta_{T}\right] \stackrel{(8)}{ } H \Omega_{\mathrm{Br}}\left[r_{T}\left(T \otimes \varepsilon_{T}\right)\right]\left(\delta_{V}^{l}+\delta_{V}^{r}\right) \\
= & r_{A}\left(A \otimes \varepsilon_{A}\right)\left(\delta_{V}^{l}+\delta_{V}^{r}\right)=r_{A}\left(A \otimes \varepsilon_{A}\right)\left(u_{A} \otimes \alpha_{1} V\right) l_{V}^{-1}+r_{A}\left(A \otimes \varepsilon_{A}\right)\left(\alpha_{1} V \otimes u_{A}\right) r_{V}^{-1} \\
(53) & r_{A}\left(\alpha_{1} V \otimes \mathbf{1}\right) r_{V}^{-1}=\left(\alpha_{1} V\right) r_{V} r_{V}^{-1}=\alpha_{1} V \stackrel{56)}{=} H \eta_{\mathrm{Br}}(V, c) \\
= & H \Phi\left((V, c), T_{\mathrm{Br}}(V, c)\right)\left[\mathrm{Id}_{T}\right]
\end{aligned}
$$

so that $r_{T}\left(T \otimes \varepsilon_{T}\right) \Delta_{T}=\mathrm{Id}_{T}$. Similarly one proves that $l_{T}\left(\varepsilon_{T} \otimes T\right) \Delta_{T}=\mathrm{Id}_{T}$. By construction $\Delta_{A}$ is a morphism of braided algebras so that $\left(\Delta_{A} \otimes \Delta_{A}\right) c_{A}=c_{A \otimes A}\left(\Delta_{A} \otimes_{A}\right)$ i.e.

$$
\left(\Delta_{A} \otimes \Delta_{A}\right) c_{A}=\left(A \otimes c_{A} \otimes A\right)\left(c_{A} \otimes c_{A}\right)\left(A \otimes c_{A} \otimes A\right)\left(\Delta_{A} \otimes \Delta_{A}\right)
$$

If we apply $\left(A \otimes A \otimes r_{A}\left(A \otimes \varepsilon_{A}\right)\right)$ we get
$\left(\Delta_{A} \otimes r_{A}\left(A \otimes \varepsilon_{A}\right) \Delta_{A}\right) c_{A}=\left(A \otimes A \otimes r_{A}\left(A \otimes \varepsilon_{A}\right)\right)\left(A \otimes c_{A} \otimes A\right)\left(c_{A} \otimes c_{A}\right)\left(A \otimes c_{A} \otimes A\right)\left(\Delta_{A} \otimes \Delta_{A}\right)$.
The left hand side is $\left(\Delta_{A} \otimes A\right) c_{A}$. Using equality $A \otimes r_{A}=r_{A \otimes A}$, the naturality of $r$, the equality $r_{A \otimes A}=A \otimes r_{A}$, equality (11), equality $A \otimes l_{A}=r_{A} \otimes A$, equality (11), equality $A \otimes l_{A}=r_{A} \otimes A$ and equality $r_{A}\left(A \otimes \varepsilon_{A}\right) \Delta_{A}=\operatorname{Id}_{A}$, we see that the right hand side is $\left(A \otimes c_{A}\right)\left(c_{A} \otimes A\right)\left(A \otimes \Delta_{A}\right)$. Hence we get (9). Analogously one gets (10).

Note that $\operatorname{Id}_{A}: A \rightarrow A$ and $\Delta_{A}: A \rightarrow A \otimes A$ are morphisms in $\operatorname{BrAlg}_{\mathcal{M}}$. Moreover $\left(A_{1}, A_{2}, c_{2,1}\right)$ and $\left(A_{2}, A_{1}, c_{1,2}\right)$ (respectively $\left(A_{1}^{\prime}, A_{2}^{\prime}, c_{2,1}^{\prime}\right)$ and $\left(A_{2}^{\prime}, A_{1}^{\prime}, c_{1,2}^{\prime}\right)$ ) fulfil the requirements of Proposition 2.2-2) for $A_{2}=A_{1}=A$ and $c_{i, j}=c_{A}, i, j \in\{1,2\}$ (resp. for $A_{1}^{\prime}=A, A_{2}^{\prime}=A \otimes A$
and $c_{2,1}^{\prime}=\left(c_{A} \otimes A\right)\left(A \otimes c_{A}\right), c_{1,2}^{\prime}=\left(A \otimes c_{A}\right)\left(c_{A} \otimes A\right), c_{1,1}^{\prime}=c_{A}, c_{2,2}^{\prime}=c_{A \otimes A}$ see Lemma 2.3). Moreover, by the foregoing, we have

$$
\left(A \otimes \Delta_{A}\right) c_{2,1}=c_{2,1}^{\prime}\left(\Delta_{A} \otimes A\right) \quad \text { and } \quad\left(\Delta_{A} \otimes A\right) c_{1,2}=c_{1,1}^{\prime}\left(A \otimes \Delta_{A}\right)
$$

Thus, by Proposition $2.2-3)$ applied to $f_{1}:=\operatorname{Id}_{A}$ and $f_{2}:=\Delta_{A}$, we can conclude that $T \otimes \Delta_{T}$ is a morphism in $\operatorname{BrAlg}_{\mathcal{M}}$. Similarly $\Delta_{T} \otimes T$ is a morphism in $\mathrm{BrAlg}_{\mathcal{M}}$. We have to check that $\left(T \otimes \Delta_{T}\right) \Delta_{T}=\left(\Delta_{T} \otimes T\right) \Delta_{T}$. Equivalently we will prove that

$$
\begin{equation*}
H \Phi((V, c), T \otimes T \otimes T)\left[\left(T \otimes \Delta_{T}\right) \Delta_{T}\right]=H \Phi((V, c), T \otimes T \otimes T)\left[\left(\Delta_{T} \otimes T\right) \Delta_{T}\right] \tag{59}
\end{equation*}
$$

If we apply (58) for $\xi=T \otimes \Delta_{T}$, the left-hand side of (59) becomes

$$
\begin{aligned}
& H \Phi((V, c), T \otimes T \otimes T)\left[\left(T \otimes \Delta_{T}\right) \Delta_{T}\right] \stackrel{(58)}{=} H \Omega_{\mathrm{Br}}\left[\left(T \otimes \Delta_{T}\right)\right]\left(\delta_{V}^{l}+\delta_{V}^{r}\right) \\
= & \left(A \otimes \Delta_{A}\right)\left(\delta_{V}^{l}+\delta_{V}^{r}\right)=\left(A \otimes \Delta_{A}\right)\left(u_{A} \otimes \alpha_{1} V\right) l_{V}^{-1}+\left(A \otimes \Delta_{A}\right)\left(\alpha_{1} V \otimes u_{A}\right) r_{V}^{-1} \\
\text { (52) } & \left(u_{A} \otimes\left(\delta_{V}^{l}+\delta_{V}^{r}\right)\right) l_{V}^{-1}+\left(\alpha_{1} V \otimes\left(u_{A} \otimes u_{A}\right) \Delta_{\mathbf{1}}\right) r_{V}^{-1} \\
= & {\left[\begin{array}{c}
\left(u_{A} \otimes u_{A} \otimes \alpha_{1} V\right)\left(\mathbf{1} \otimes l_{V}^{-1}\right) l_{V}^{-1}+\left(u_{A} \otimes \alpha_{1} V \otimes u_{A}\right)\left(\mathbf{1} \otimes r_{V}^{-1}\right) l_{V}^{-1}+ \\
+\left(\alpha_{1} V \otimes u_{A} \otimes u_{A}\right)\left(V \otimes \Delta_{1}\right) r_{V}^{-1}
\end{array}\right] . }
\end{aligned}
$$

If we apply (58) for $\xi=\Delta_{T} \otimes T$, the right-hand side of (59) becomes

$$
\begin{aligned}
& H \Phi((V, c), T \otimes T \otimes T)\left[\left(\Delta_{T} \otimes T\right) \Delta_{T}\right] \stackrel{58}{=} H \Omega_{\mathrm{Br}}\left[\left(\Delta_{T} \otimes T\right)\right]\left(\delta_{V}^{l}+\delta_{V}^{r}\right) \\
= & \left(\Delta_{A} \otimes A\right)\left(\delta_{V}^{l}+\delta_{V}^{r}\right)=\left(\Delta_{A} \otimes A\right)\left(u_{A} \otimes \alpha_{1} V\right) l_{V}^{-1}+\left(\Delta_{A} \otimes A\right)\left(\alpha_{1} V \otimes u_{A}\right) r_{V}^{-1} \\
= & \left(\left(u_{A} \otimes u_{A}\right) \Delta_{\mathbf{1}} \otimes \alpha_{1} V\right) l_{V}^{-1}+\left(\left(\delta_{V}^{l}+\delta_{V}^{r}\right) \otimes u_{A}\right) r_{V}^{-1} \\
= & {\left[\begin{array}{c}
\left(u_{A} \otimes u_{A} \otimes \alpha_{1} V\right)\left(\mathbf{1} \otimes l_{V}^{-1}\right) l_{V}^{-1}+\left(u_{A} \otimes \alpha_{1} V \otimes u_{A}\right)\left(\mathbf{1} \otimes r_{V}^{-1}\right) l_{V}^{-1}+ \\
+\left(\alpha_{1} V \otimes u_{A} \otimes u_{A}\right)\left(V \otimes \Delta_{1}\right) r_{V}^{-1}
\end{array}\right] . }
\end{aligned}
$$

where the last equality depends on the definitions $\delta_{V}^{l}$ and $\delta_{V}^{r}$, and on the relations $\Delta_{\mathbf{1}} \otimes V=$ $r_{\mathbf{1}}^{-1} \otimes V=\mathbf{1} \otimes l_{V}^{-1},\left(l_{V}^{-1} \otimes \mathbf{1}\right) r_{V}^{-1}=r_{\mathbf{1} \otimes V}^{-1} l_{V}^{-1}=\left(\mathbf{1} \otimes r_{V}^{-1}\right) l_{V}^{-1}$ and $r_{V}^{-1} \otimes \mathbf{1}=V \otimes \Delta_{\mathbf{1}}$.

We have so proved that $\left(T \otimes \Delta_{T}\right) \Delta_{T}=\left(\Delta_{T} \otimes T\right) \Delta_{T}$. Thus $\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right)$ is a braided bialgebra.

Let $f:(V, c) \rightarrow\left(V^{\prime}, c^{\prime}\right)$ be a morphism in $\operatorname{Br}_{\mathcal{M}}$. Let us prove that $T_{\mathrm{Br}}(f)$ is a morphism of braided bialgebras. We know that $T_{\mathrm{Br}}(f)$ is a morphism in $\mathrm{BrAlg}_{\mathcal{M}}$. We have to check that $T(f)=H \Omega_{\mathrm{Br}} T_{\mathrm{Br}}(f)$ is a morphism of coalgebras i.e. that

$$
\begin{aligned}
(T(f) \otimes T(f)) \circ \Delta_{T(V)} & =\Delta_{T\left(V^{\prime}\right)} \circ T(f) \\
\varepsilon_{T\left(V^{\prime}\right)} \circ T(f) & =\varepsilon_{T(V)}
\end{aligned}
$$

Take $A:=T(V)$ and $A^{\prime}:=T\left(V^{\prime}\right)$. Note that $T(H f): T(V) \rightarrow T\left(V^{\prime}\right)$ and $T(H f):$ $T(V) \rightarrow T\left(V^{\prime}\right)$ are morphisms in $\operatorname{BrAlg} \mathcal{M}_{\mathcal{M}}$. Moreover $\left(A_{1}, A_{2}, c_{2,1}\right)$ and $\left(A_{2}, A_{1}, c_{1,2}\right)$ (respectively $\left(A_{1}^{\prime}, A_{2}^{\prime}, c_{2,1}^{\prime}\right)$ and $\left(A_{2}^{\prime}, A_{1}^{\prime}, c_{1,2}^{\prime}\right)$ ) fulfil the requirements of Proposition 2.2-2) for $A_{2}=A_{1}=A$ and $c_{i, j}=c_{A}, i, j \in\{1,2\}$ (resp. for $A_{2}^{\prime}=A_{1}^{\prime}=A^{\prime}$ and $c_{i, j}^{\prime}=c_{A^{\prime}}, i, j \in\{1,2\}$, see Proposition 2.2). Moreover, since $T(H f)=H \Omega_{\mathrm{Br}} T_{\mathrm{Br}}(f)$ and $\Omega_{\mathrm{Br}} T_{\mathrm{Br}}(f)$ is a morphism of braided objects, we have

$$
\begin{aligned}
& (T(H f) \otimes T(H f)) c_{2,1}=c_{2,1}^{\prime}(T(H f) \otimes T(H f)), \\
& (T(H f) \otimes T(H f)) c_{1,2}=c_{1,2}(T(f) \otimes T(H f))
\end{aligned}
$$

Thus we can conclude that $T_{\mathrm{Br}}(f) \otimes T_{\mathrm{Br}}(f)$ is a morphism in $\operatorname{BrAlg}_{\mathcal{M}}$. We have to check that

$$
\left(T_{\mathrm{Br}}(f) \otimes T_{\mathrm{Br}}(f)\right) \circ \Delta_{T}=\Delta_{T^{\prime}} \circ T_{\mathrm{Br}}(f)
$$

as morphisms in $\mathrm{BrAlg}_{\mathcal{M}}$. Equivalently we will check that

$$
H \Phi((V, c), T \otimes T)\left[\left(T_{\mathrm{Br}}(f) \otimes T_{\mathrm{Br}}(f)\right) \circ \Delta_{T}\right]=H \Phi((V, c), T \otimes T)\left[\Delta_{T^{\prime}} \circ T_{\mathrm{Br}}(f)\right]
$$

The left hand-side is

$$
\begin{aligned}
& H \Phi((V, c), T \otimes T)\left[\left(T_{\mathrm{Br}}(f) \otimes T_{\mathrm{Br}}(f)\right) \circ \Delta_{T}\right] \stackrel{58}{=} H \Omega_{\mathrm{Br}}\left[\left(T_{\mathrm{Br}}(f) \otimes T_{\mathrm{Br}}(f)\right)\right]\left(\delta_{V}^{l}+\delta_{V}^{r}\right) \\
= & (\Omega T(H f) \otimes \Omega T(H f))\left(u_{A} \otimes \alpha_{1} V\right) l_{V}^{-1}+(\Omega T(H f) \otimes \Omega T(H f))\left(\alpha_{1} V \otimes u_{A}\right) r_{V}^{-1} \\
\text { (1) } & \left(u_{A^{\prime}} \otimes \alpha_{1} V^{\prime}\right)(\mathbf{1} \otimes H f) l_{V}^{-1}+\left(\alpha_{1} V^{\prime} \otimes u_{A^{\prime}}\right)(H f \otimes \mathbf{1}) r_{V}^{-1} \\
= & \left(u_{A^{\prime}} \otimes \alpha_{1} V^{\prime}\right) l_{V^{\prime}}^{-1} H f+\left(\alpha_{1} V^{\prime} \otimes u_{A^{\prime}}\right) r_{V^{\prime}}^{-1} H f=\left(\delta_{V^{\prime}}^{l}+\delta_{V^{\prime}}^{r}\right) H f .
\end{aligned}
$$

The right hand-side is

$$
\begin{aligned}
& H \Phi((V, c), T \otimes T)\left[\Delta_{T^{\prime}} \circ T_{\mathrm{Br}}(f)\right]=H\left\{\Omega_{\mathrm{Br}}\left[\Delta_{T^{\prime} \circ} \circ T_{\mathrm{Br}}(f)\right] \eta_{\mathrm{Br}}(V, c)\right\} \\
= & H\left\{\Omega_{\mathrm{Br}}\left[\Delta_{T^{\prime}}\right]\right\} H\left\{\Omega_{\mathrm{Br}} T_{\mathrm{Br}}(f) \eta_{\mathrm{Br}}(V, c)\right\}=H\left\{\Omega_{\mathrm{Br}}\left[\Delta_{T^{\prime}}\right]\right\} H\left\{\eta_{\mathrm{Br}}\left(V^{\prime}, c^{\prime}\right)\right\} H f \\
= & \left(\delta_{V^{\prime}}^{l}+\delta_{V^{\prime}}^{r}\right) H f .
\end{aligned}
$$

Hence the two sides coincide. We have proved that $T_{\mathrm{Br}}(f)$ is comultiplicative. Let us check it is counitary i.e. that $\varepsilon_{T^{\prime}} \circ T_{\mathrm{Br}}(f)=\varepsilon_{T}$ holds in $\operatorname{BrAlg}_{\mathcal{M}}$. Equivalently we have to prove that $\Phi((V, c), \mathbf{1})\left[\varepsilon_{T^{\prime}} \circ T_{\mathrm{Br}}(f)\right]=\Phi((V, c), \mathbf{1})\left[\varepsilon_{T}\right]$. We have

$$
\begin{aligned}
\Phi((V, c), \mathbf{1})\left[\varepsilon_{T^{\prime}} \circ T_{\mathrm{Br}}(f)\right] & =\Omega_{\mathrm{Br}}\left[\varepsilon_{T^{\prime}} \circ T_{\mathrm{Br}}(f)\right] \eta_{\mathrm{Br}}(V, c) \\
& =\Omega_{\mathrm{Br}}\left[\varepsilon_{T^{\prime}}\right]\left\{\Omega_{\mathrm{Br}} T_{\mathrm{Br}}(f) \eta_{\mathrm{Br}}(V, c)\right\} \\
& =\Omega_{\mathrm{Br}}\left[\varepsilon_{T^{\prime}}\right] \eta_{\mathrm{Br}}\left(V^{\prime}, c^{\prime}\right) f=0= \\
& =\Omega_{\mathrm{Br}}\left[\varepsilon_{T}\right] \eta_{\mathrm{Br}}(V, c)=\Phi((V, c), \mathbf{1})\left[\varepsilon_{T}\right] .
\end{aligned}
$$

By construction we have that diagram (55) commutes.
Next aim is to check that the functor $P_{\mathrm{Br}}: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \mathrm{Br}_{\mathcal{M}}$ of Lemma 3.3 is a left adjoint of $\bar{T}_{\mathrm{Br}}$.

ThEOREM 3.5. Take the hypotheses and notations of Lemma 3.3 i.e let $\mathcal{M}$ be a preadditive monoidal category with equalizers and assume that the tensor functors are additive and preserve equalizers. Assume also that the monoidal category $\mathcal{M}$ has denumerable coproducts and that the tensor functors preserve such coproducts. Then

$$
\left(\bar{T}_{\mathrm{Br}}: \mathrm{Br}_{\mathcal{M}} \rightarrow \operatorname{BrBialg}_{\mathcal{M}}, P_{\mathrm{Br}}: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \operatorname{Br}_{\mathcal{M}}\right)
$$

is an adjunction. The unit $\bar{\eta}_{\mathrm{Br}}$ and the counit $\bar{\epsilon}_{\mathrm{Br}}$ are uniquely determined by the following equalities

$$
\begin{align*}
\xi \bar{T}_{\mathrm{Br}} \circ \bar{\eta}_{\mathrm{Br}} & =\eta_{\mathrm{Br}},  \tag{60}\\
\epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \circ T_{\mathrm{Br}} \xi & =\mho_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}}, \tag{61}
\end{align*}
$$

where $(V, c) \in \operatorname{Br}_{\mathcal{M}}, \mathbb{B} \in \operatorname{BrBialg}_{\mathcal{M}}$ while $\eta_{\mathrm{Br}}$ and $\epsilon_{\mathrm{Br}}$ denote the unit and counit of the adjunction $\left(T_{\mathrm{Br}}, \Omega_{\mathrm{Br}}\right)$ respectively. Moreover $\mho_{\mathrm{Br}}: \operatorname{BrBialg}_{\mathcal{M}} \rightarrow \mathrm{BrAlg}_{\mathcal{M}}$ denotes the forgetful functor.

Proof. Let $(V, c) \in \operatorname{Br}_{\mathcal{M}}$. Let $\mathbb{A}:=\bar{T}_{\operatorname{Br}}(V, c)$. Write $\mathbb{A}:=\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right)$. Consider the equalizer

$$
P(\mathbb{A}) \xrightarrow{\xi \mathbb{A}} A \underset{\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}}{\frac{\Delta_{A}}{\longrightarrow}} A \otimes A
$$

Note that the codomain of $\eta_{\mathrm{Br}}(V, c)$ is $\Omega_{\mathrm{Br}} T_{\mathrm{Br}}(V, c)=\left(A, c_{A}\right)$ so that it makes sense to check if $H \eta_{\mathrm{Br}}(V, c): V \rightarrow A$ is equalized by the pair $\left(\Delta_{A},\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}\right)$. We have

$$
\begin{array}{ll} 
& {\left[\left(A \otimes u_{A}\right) \circ r_{A}^{-1}+\left(u_{A} \otimes A\right) \circ l_{A}^{-1}\right] \circ H \eta_{\mathrm{Br}}(V, c)} \\
\stackrel{56}{ } & {\left[\left(A \otimes u_{A}\right) \circ r_{A}^{-1}+\left(u_{A} \otimes A\right) \circ l_{A}^{-1}\right] \circ \alpha_{1} V} \\
= & \left(\alpha_{1} V \otimes u_{A}\right) \circ r_{V}^{-1}+\left(u_{A} \otimes \alpha_{1} V\right) \circ l_{V}^{-1} \\
= & \delta_{V}^{r}+\delta_{V}^{l} \stackrel{(52}{=} \Delta_{A} \circ \alpha_{1} V \stackrel{56}{=} \Delta_{A} \circ H \eta_{\mathrm{Br}}(V, c) .
\end{array}
$$

By the universal property of equalizers, there is a unique morphism $\bar{\eta}_{\mathrm{Br}}(V, c): V \rightarrow P(\mathbb{A})$ such that

$$
\begin{equation*}
\xi \mathbb{A} \circ \bar{\eta}_{\mathrm{Br}}(V, c)=H \eta_{\mathrm{Br}}(V, c) \tag{62}
\end{equation*}
$$

We have

$$
\begin{aligned}
& (\xi \mathbb{A} \otimes \xi \mathbb{A}) \circ c_{P(\mathbb{A})} \circ\left(\bar{\eta}_{\mathrm{Br}}(V, c) \otimes \bar{\eta}_{\mathrm{Br}}(V, c)\right) \stackrel{4 区)}{ } c_{A} \circ(\xi \mathbb{A} \otimes \xi \mathbb{A}) \circ\left(\bar{\eta}_{\mathrm{Br}}(V, c) \otimes \bar{\eta}_{\mathrm{Br}}(V, c)\right) \\
= & c_{A} \circ\left(H \eta_{\mathrm{Br}}(V, c) \otimes H \eta_{\mathrm{Br}}(V, c)\right)=\left(H \eta_{\mathrm{Br}}(V, c) \otimes H \eta_{\mathrm{Br}}(V, c)\right) \circ c \\
= & (\xi \mathbb{A} \otimes \xi \mathbb{A}) \circ\left(\bar{\eta}_{\mathrm{Br}}(V, c) \otimes \bar{\eta}_{\mathrm{Br}}(V, c)\right) \circ c
\end{aligned}
$$

and hence

$$
c_{P(\mathbb{A})} \circ\left(\bar{\eta}_{\mathrm{Br}} A \otimes \bar{\eta}_{\mathrm{Br}} A\right)=\left(\bar{\eta}_{\mathrm{Br}} A \otimes \bar{\eta}_{\mathrm{Br}} A\right) \circ c
$$

Hence $\bar{\eta}_{\mathrm{Br}}(V, c)$ induces a morphism of braided objects that we denote with the same symbol, namely $\bar{\eta}_{\mathrm{Br}}(V, c):(V, c) \rightarrow\left(P(\mathbb{A}), c_{P(\mathbb{A})}\right)$.

Let us check that $\bar{\eta}_{\mathrm{Br}}(V, c)$ is natural in $(V, c)$. Let $f:(V, c) \rightarrow\left(V^{\prime}, c^{\prime}\right)$ be a morphism in $\mathrm{Br}_{\mathcal{M}}$. Then

$$
\begin{array}{ll} 
& \xi \bar{T}_{\mathrm{Br}}\left(V^{\prime}, c^{\prime}\right) \circ P_{\mathrm{Br}} \bar{T}_{\mathrm{Br}}(f) \circ \bar{\eta}_{\mathrm{Br}}(V, c) \\
\Omega_{\mathrm{Br}} \mho_{\mathrm{Br}} \bar{T}_{\mathrm{Br}}(f) \circ \xi \bar{T}_{\mathrm{Br}}(V, c) \circ \bar{\eta}_{\mathrm{Br}}(V, c) \\
\text { (62) } & \Omega_{\mathrm{Br}} T_{\mathrm{Br}}(f) \circ \eta_{\mathrm{Br}}(V, c) \\
= & \eta_{\mathrm{Br}}\left(V^{\prime}, c^{\prime}\right) \circ f \stackrel{62}{2} \xi \bar{T}_{\mathrm{Br}}\left(V^{\prime}, c^{\prime}\right) \circ \bar{\eta}_{\mathrm{Br}}\left(V^{\prime}, c^{\prime}\right) \circ f
\end{array}
$$

and hence $P_{\mathrm{Br}} \bar{T}_{\mathrm{Br}}(f) \circ \bar{\eta}_{\mathrm{Br}}(V, c)=\bar{\eta}_{\mathrm{Br}}\left(V^{\prime}, c^{\prime}\right) \circ f$ which means that $\bar{\eta}_{\mathrm{Br}}(V, c)$ is natural in $(V, c)$. A similar argument holds for $\xi \mathbb{A}$ so that we have proved (60).

The morphism $\bar{\eta}_{\mathrm{Br}}(V, c)$ will play the role of the unit of the adjunction $\left(\bar{T}_{\mathrm{Br}}, P_{\mathrm{Br}}\right)$. Let $\mathbb{B}:=$ $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right) \in \operatorname{BrBialg}_{\mathcal{M}}$ and consider the canonical isomorphism

$$
\begin{aligned}
\Phi\left(P_{\mathrm{Br}}(\mathbb{B}), \mho_{\mathrm{Br}}(\mathbb{B})\right) & : \operatorname{BrAlg}_{\mathcal{M}}\left(T_{\mathrm{Br}} P_{\mathrm{Br}}(\mathbb{B}), \mho_{\mathrm{Br}}(\mathbb{B})\right) \rightarrow \operatorname{Br}_{\mathcal{M}}\left(P_{\mathrm{Br}}(\mathbb{B}), \Omega_{\mathrm{Br}} \mho_{\mathrm{Br}}(\mathbb{B})\right) \\
f & \mapsto \Omega_{\mathrm{Br}}(f) \circ \eta_{\mathrm{Br}} P_{\mathrm{Br}}(\mathbb{B}) .
\end{aligned}
$$

Define the morphism $\zeta \mathbb{B}:=\Phi\left(P_{\mathrm{Br}}(\mathbb{B}), \mho_{\mathrm{Br}}(\mathbb{B})\right)^{-1}(\xi \mathbb{B})$. This means that

$$
\begin{equation*}
\Omega_{\mathrm{Br}}(\zeta \mathbb{B}) \circ \eta_{\mathrm{Br}} P_{\mathrm{Br}}(\mathbb{B})=\xi \mathbb{B} \tag{63}
\end{equation*}
$$

Set $\zeta B:=H \Omega_{\mathrm{Br}}(\zeta \mathbb{B})=\Omega H_{\mathrm{Alg}}(\zeta \mathbb{B}): T P(\mathbb{B}) \rightarrow B$. Note that

$$
\begin{aligned}
& \zeta B \circ \eta P(\mathbb{B}) \stackrel{56}{=} H \Omega_{\mathrm{Br}}(\zeta \mathbb{B}) \circ H \eta_{\mathrm{Br}} P_{\mathrm{Br}}(\mathbb{B}) \\
= & H\left[\Omega_{\mathrm{Br}}(\zeta \mathbb{B}) \circ \eta_{\mathrm{Br}} P_{\mathrm{Br}}(\mathbb{B})\right] \stackrel{(3)}{=} H \xi \mathbb{B}
\end{aligned}
$$

so that

$$
\begin{equation*}
\zeta B \circ \eta P(\mathbb{B})=H \xi \mathbb{B} \tag{64}
\end{equation*}
$$

We will check that

$$
\begin{equation*}
\Delta_{B} \circ \zeta B=(\zeta B \otimes \zeta B) \circ \Delta_{T P(\mathbb{B})} \tag{65}
\end{equation*}
$$

The morphisms above are in particular algebra maps. Since $(T, \Omega)$ is an adjunction, the equality above holds if

$$
\Delta_{B} \circ \zeta B \circ \eta P(\mathbb{B})=(\zeta B \otimes \zeta B) \circ \Delta_{T P(\mathbb{B})} \circ \eta P(\mathbb{B})
$$

The first term is

$$
\begin{aligned}
& \Delta_{B} \circ \zeta B \circ \eta P(\mathbb{B}) \stackrel{(64)}{=} \Delta_{B} \circ H \xi \mathbb{B} \\
= & \left(\left(B \otimes u_{B}\right) r_{B}^{-1}+\left(u_{B} \otimes B\right) l_{B}^{-1}\right) \circ H \xi \mathbb{B}
\end{aligned}
$$

On the other hand, using, in the given order, (4), (52), the definitions of $\delta_{P(\mathbb{B})}^{l}$ and $\delta_{P(\mathbb{B})}^{r}$, (4), (64) and the naturality of the unit constraints, we obtain that the second term is $\left(\left(B \otimes u_{B}\right) r_{B}^{-1}+\left(u_{B} \otimes B\right) l_{B}^{-1}\right) \circ$
$H \xi \mathbb{B}$. Thus (65) holds true. Now we will prove that $\varepsilon_{B} \circ \zeta B=\varepsilon_{\Omega T P(\mathbb{B})}$. Since $(T, \Omega)$ is an adjunction, the equality above holds if

$$
\varepsilon_{B} \circ \zeta B \circ \eta P(\mathbb{B})=\varepsilon_{\Omega T P(\mathbb{B})} \circ \eta P(\mathbb{B})
$$

We have

$$
\varepsilon_{B} \circ \zeta B \circ \eta P(\mathbb{B}) \stackrel{64}{=} \varepsilon_{B} \circ H \xi \mathbb{B} \stackrel{(*)}{=} 0 \stackrel{53),(\mathbb{B})}{=} \varepsilon_{\Omega T P(\mathbb{B})} \circ \eta P(\mathbb{B})
$$

In order to prove $\left(^{*}\right)$ we proceed as follows. Consider the equalizer

$$
P(\mathbb{B}) \xrightarrow{\xi \mathbb{B}} B \underset{\left(B \otimes u_{B}\right) r_{B}^{-1}+\left(u_{B} \otimes B\right) l_{B}^{-1}}{\underset{\Delta_{B}}{\Longrightarrow}} B \otimes B
$$

By applying $m_{\mathbf{1}} \circ\left(\varepsilon_{B} \otimes \varepsilon_{B}\right)$ we get

$$
m_{1} \circ\left(\varepsilon_{B} \otimes \varepsilon_{B}\right) \circ \Delta_{B} \circ \xi \mathbb{B}=m_{\mathbf{1}} \circ\left(\varepsilon_{B} \otimes \varepsilon_{B}\right) \circ\left[\left(B \otimes u_{B}\right) r_{B}^{-1}+\left(u_{B} \otimes B\right) l_{B}^{-1}\right] \circ \xi \mathbb{B}
$$

The left hand-side is

$$
\begin{aligned}
& m_{1} \circ\left(\varepsilon_{B} \otimes \varepsilon_{B}\right) \circ \Delta_{B} \circ \xi \mathbb{B}=m_{\mathbf{1}} \circ\left(\varepsilon_{B} \otimes \mathbf{1}\right) \circ\left(B \otimes \varepsilon_{B}\right) \circ \Delta_{B} \circ \xi \mathbb{B} \\
= & m_{\mathbf{1}} \circ\left(\varepsilon_{B} \otimes \mathbf{1}\right) \circ r_{B}^{-1} \circ \xi \mathbb{B}=m_{\mathbf{1}} \circ r_{\mathbf{1}}^{-1} \circ \varepsilon_{B} \circ \xi \mathbb{B}=\varepsilon_{B} \circ \xi \mathbb{B}
\end{aligned}
$$

The right hand-side is

$$
\begin{aligned}
& m_{\mathbf{1}} \circ\left(\varepsilon_{B} \otimes \varepsilon_{B}\right) \circ\left[\left(B \otimes u_{B}\right) r_{B}^{-1}+\left(u_{B} \otimes B\right) l_{B}^{-1}\right] \circ \xi \mathbb{B} \\
= & m_{\mathbf{1}} \circ\left[\left(\varepsilon_{B} \otimes \mathbf{1}\right) r_{B}^{-1}+\left(\mathbf{1} \otimes \varepsilon_{B}\right) l_{B}^{-1}\right] \circ \xi \mathbb{B} \\
= & m_{\mathbf{1}} \circ\left[r_{\mathbf{1}}^{-1} \varepsilon_{B}+l_{\mathbf{1}}^{-1} \varepsilon_{B}\right] \circ \xi \mathbb{B}=2 \varepsilon_{B} \circ \xi \mathbb{B}
\end{aligned}
$$

Hence we get $\varepsilon_{B} \circ \xi \mathbb{B}=2 \varepsilon_{B} \circ \xi \mathbb{B}$ and hence $\varepsilon_{B} \circ \xi \mathbb{B}=0$ as required. Thus $\left(^{*}\right)$ is proved. Summing up, we have proved that $\zeta B: T P(\mathbb{B}) \rightarrow B$ is a coalgebra morphism. Since $\zeta B:=H \Omega_{\mathrm{Br}}(\zeta \mathbb{B})$, we also know it is a morphism of algebras and braided objects so that there is a unique morphism $\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}: \bar{T} P(\mathbb{B}) \rightarrow \mathbb{B}$ in $\mathrm{BrBialg}_{\mathcal{M}}$ such that

$$
\mho_{\mathrm{Br}}\left(\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}\right)=\zeta \mathbb{B} .
$$

By definition of $\zeta \mathbb{B}$, we have

$$
\Omega_{\mathrm{Br}} \mho_{\mathrm{Br}}\left(\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}\right) \circ \eta_{\mathrm{Br}} P_{\mathrm{Br}}(\mathbb{B})=\Omega_{\mathrm{Br}} \zeta \mathbb{B} \circ \eta_{\mathrm{Br}} P_{\mathrm{Br}}(\mathbb{B}) \stackrel{63}{=} \xi \mathbb{B}
$$

Observe that $\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}$ is uniquely determined by the last equality. Note also that

$$
\begin{equation*}
\epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \mathbb{B} \circ T_{\mathrm{Br}} \xi \mathbb{B}=\Phi\left(P_{\mathrm{Br}}(\mathbb{B}), \mho_{\mathrm{Br}} \mathbb{B}\right)^{-1}(\xi \mathbb{B})=\zeta \mathbb{B}=\mho_{\mathrm{Br}}\left(\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}\right) \tag{66}
\end{equation*}
$$

Let us check that $\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}$ is natural in $\mathbb{B}$. Let $f: \mathbb{B} \rightarrow \mathbb{B}^{\prime}$ be a morphism in $\operatorname{BrBialg}_{\mathcal{M}}$. Then

$$
\begin{array}{ll} 
& \mho_{\mathrm{Br}}\left[\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}^{\prime} \circ \bar{T}_{\mathrm{Br}} P_{\mathrm{Br}}(f)\right] \stackrel{\left.(6)^{(6)}\right)}{\epsilon_{\mathrm{Br}}} \mho_{\mathrm{Br}} \mathbb{B}^{\prime} \circ T_{\mathrm{Br}} \xi \mathbb{B}^{\prime} \circ T_{\mathrm{Br}} P_{\mathrm{Br}}(f) \\
(\underline{\text { (6) })} & \epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \mathbb{B}^{\prime} \circ T_{\mathrm{Br}} \Omega_{\mathrm{Br}} \mho_{\mathrm{Br}}(f) \circ T_{\mathrm{Br}} \xi \mathbb{B} \\
= & \mho_{\mathrm{Br}}(f) \circ \epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \mathbb{B} \circ T_{\mathrm{Br}} \xi \mathbb{B} \stackrel{(66)}{=} \mho_{\mathrm{Br}}\left[f \circ \bar{\epsilon}_{\mathrm{Br}} \mathbb{B}\right]
\end{array}
$$

Since $\mho_{\mathrm{Br}}$ is faithful, we obtain $\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}^{\prime} \circ \bar{T}_{\mathrm{Br}} P_{\mathrm{Br}}(f)=f \circ \bar{\epsilon}_{\mathrm{Br}} \mathbb{B}$ so that $\bar{\epsilon}_{\mathrm{Br}} \mathbb{B}$ is natural in $\mathbb{B}$. Thus (66) implies that (61) holds.

Let us check that $\left(\bar{T}_{\mathrm{Br}}, P_{\mathrm{Br}}\right)$ is an adjunction with unit $\bar{\eta}_{\mathrm{Br}}$ and counit $\bar{\epsilon}_{\mathrm{Br}}$. We compute

$$
\begin{array}{ll} 
& \xi \mathbb{B} \circ P_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}} \mathbb{B} \circ \bar{\eta}_{\mathrm{Br}} P_{\mathrm{Br}} \mathbb{B} \stackrel{48}{ } \Omega_{\mathrm{Br}} \mho_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}} \mathbb{B} \circ \xi \bar{T}_{\mathrm{Br}} P_{\mathrm{Br}} \mathbb{B} \circ \bar{\eta}_{\mathrm{Br}} P_{\mathrm{Br}} \mathbb{B} \\
=\Omega_{\mathrm{Br}} \mho_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}} \mathbb{B} \circ \eta_{\mathrm{Br}} P_{\mathrm{Br}} \mathbb{B}+\Omega_{\mathrm{Br}} \epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \mathbb{B} \circ \Omega_{\mathrm{Br}} T_{\mathrm{Br}} \xi \mathbb{B} \circ \eta_{\mathrm{Br}} P_{\mathrm{Br}} \mathbb{B} \\
= & \Omega_{\mathrm{Br}} \epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \mathbb{B} \circ \eta_{\mathrm{Br}} \Omega_{\mathrm{Br}} \mho_{\mathrm{Br}} \mathbb{B} \circ \xi \mathbb{B}=\xi \mathbb{B}
\end{array}
$$

Since $\xi \mathbb{B}$ is a monomorphism, we get $P_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}} \mathbb{B} \circ \bar{\eta}_{\mathrm{Br}} P_{\mathrm{Br}} \mathbb{B}=\operatorname{Id}_{P_{\mathrm{Br}} \mathbb{B}}$. We have

$$
\begin{aligned}
& \mho_{\mathrm{Br}}\left[\bar{\epsilon}_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} \circ \bar{T}_{\mathrm{Br}} \bar{\eta}_{\mathrm{Br}}\right] \stackrel{\left(\operatorname{dr1}^{( }\right)}{=} \epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} \circ T_{\mathrm{Br}} \xi \bar{T}_{\mathrm{Br}} \circ T_{\mathrm{Br}} \bar{\eta}_{\mathrm{Br}} \\
& \stackrel{\text { (id) }}{=} \epsilon_{\mathrm{Br}} T_{\mathrm{Br}} \circ T_{\mathrm{Br}} \eta_{\mathrm{Br}}=\operatorname{Id}_{T_{\mathrm{Br}}}=\mho_{\mathrm{Br}}\left[\mathrm{Id}_{\bar{T}_{\mathrm{Br}}}\right] .
\end{aligned}
$$

Since $\mho_{\mathrm{Br}}$ is faithful, we get $\bar{\epsilon}_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} \circ \bar{T}_{\mathrm{Br}} \bar{\eta}_{\mathrm{Br}}=\operatorname{Id}_{\bar{T}_{\mathrm{Br}}}$.
Proposition 3.6. Let $\mathcal{M}$ and $\mathcal{M}^{\prime}$ be preadditive monoidal categories with equalizers. Assume that the tensor functors are additive and preserve equalizers in both categories. Let $\left(F, \phi_{0}, \phi_{2}\right): \mathcal{M} \rightarrow$ $\mathcal{M}^{\prime}$ be a monoidal functor which preserves equalizers. Then the following diagram commutes, where $\operatorname{BrBialg} F$ and $\mathrm{Br} F$ are the functors of Proposition 2.5.


Moreover we have

$$
\begin{equation*}
\xi^{\prime}(\operatorname{BrBialg} F)=(\operatorname{Br} F) \xi \tag{68}
\end{equation*}
$$

Proof. By Lemma 3.3, for any $\mathbb{A}:=\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right) \in \operatorname{BrBialg}_{\mathcal{M}}$ we have that $P_{\mathrm{Br}} \mathbb{A}=$ $\left(P(\mathbb{A}), c_{P(\mathbb{A})}\right)$ where $P(\mathbb{A})$ is the equalizer

$$
P(\mathbb{A}) \xrightarrow{\xi \mathbb{A}} A \underset{\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}}{\longrightarrow} A \otimes A
$$

and $c_{P(\mathbb{A})}$ is defined by (46). We have

$$
\begin{aligned}
& \left(P_{\mathrm{Br}}^{\prime} \circ \operatorname{BrBialg} F\right)(\mathbb{A})=P_{\mathrm{Br}}^{\prime}((\operatorname{BrBialg} F)(\mathbb{A})) \\
= & \left(P^{\prime}((\operatorname{BrBialg} F)(\mathbb{A})), c_{P^{\prime}((\operatorname{BrBialg} F)(\mathbb{A}))}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& \left(P^{\prime}((\operatorname{BrBialg} F)(\mathbb{A})), \xi^{\prime}(\operatorname{BrBialg} F)(\mathbb{A})\right) \\
= & \left(P^{\prime}\left(\left(F A, m_{F A}, u_{F A}, \Delta_{F A}, \varepsilon_{F A}, c_{F A}\right)\right), \xi^{\prime}\left(F A, m_{F A}, u_{F A}, \Delta_{F A}, \varepsilon_{F A}, c_{F A}\right)\right) \\
= & \operatorname{Equ}_{\mathcal{M}^{\prime}}\left(\Delta_{F A},\left(F A \otimes u_{F A}\right) r_{F A}^{-1}+\left(u_{F A} \otimes F A\right) l_{F A}^{-1}\right) \\
= & \operatorname{Equ}_{\mathcal{M}^{\prime}}\left(\phi_{2}(A, A) \Delta_{F A}, \phi_{2}(A, A)\left(F A \otimes u_{F A}\right) r_{F A}^{-1}+\phi_{2}(A, A)\left(u_{F A} \otimes F A\right) l_{F A}^{-1}\right) \\
= & \operatorname{Equ}_{\mathcal{M}^{\prime}}\left(F \Delta_{A}, \phi_{2}(A, A)\left(F A \otimes F u_{A}\right)\left(F A \otimes \phi_{0}\right) r_{F A}^{-1}+\phi_{2}(A, A)\left(F u_{A} \otimes F A\right)\left(\phi_{0} \otimes F A\right) l_{F A}^{-1}\right) \\
= & \operatorname{Equ}_{\mathcal{M}^{\prime}}\left(F \Delta_{A}, F\left(A \otimes u_{A}\right) \phi_{2}(A, \mathbf{1})\left(F A \otimes \phi_{0}\right) r_{F A}^{-1}+F\left(u_{A} \otimes A\right) \phi_{2}(\mathbf{1}, A)\left(\phi_{0} \otimes F A\right) l_{F A}^{-1}\right) \\
= & \operatorname{Equ}_{\mathcal{M}^{\prime}}\left(F \Delta_{A}, F\left(A \otimes u_{A}\right) F\left(r_{A}^{-1}\right)+F\left(u_{A} \otimes A\right) F\left(l_{A}^{-1}\right)\right) \\
= & F\left(\operatorname{Equ}_{\mathcal{M}}\left(\Delta_{A},\left(A \otimes u_{A}\right) r_{A}^{-1}+\left(u_{A} \otimes A\right) l_{A}^{-1}\right)\right) \\
= & \left((F \circ P)\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right), F \xi\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right)\right) \\
= & (F P(\mathbb{A}), F \xi \mathbb{A})
\end{aligned}
$$

and $c_{P((\operatorname{BrBialg} F)(\mathbb{A}))}$ fulfills

$$
\begin{array}{ll} 
& \left(\xi^{\prime}(\operatorname{BrBialg} F)(\mathbb{A}) \otimes \xi^{\prime}(\operatorname{BrBialg} F)(\mathbb{A})\right) \circ c_{P^{\prime}((\operatorname{BrBialg} F)(\mathbb{A}))} \\
(46) & c_{F A} \circ\left(\xi^{\prime}(\operatorname{BrBialg} F)(\mathbb{A}) \otimes \xi^{\prime}(\operatorname{BrBialg} F)(\mathbb{A})\right) \\
= & c_{F A} \circ(F \xi \mathbb{A} \otimes F \xi \mathbb{A})=\phi_{2}^{-1}(A, A) \circ F c_{A} \circ \phi_{2}(A, A) \circ(F \xi \mathbb{A} \otimes F \xi \mathbb{A}) \\
= & \phi_{2}^{-1}(A, A) \circ F c_{A} \circ F(\xi \mathbb{A} \otimes \xi \mathbb{A}) \circ \phi_{2}(P(\mathbb{A}), P(\mathbb{A})) \\
(\sqrt[46)]{ } & \phi_{2}^{-1}(A, A) \circ F(\xi \mathbb{A} \otimes \xi \mathbb{A}) \circ F c_{P(\mathbb{A})} \circ \phi_{2}(P(\mathbb{A}), P(\mathbb{A})) \\
= & (F \xi \mathbb{A} \otimes F \xi \mathbb{A}) \circ \phi_{2}^{-1}(P(\mathbb{A}), P(\mathbb{A})) \circ F c_{P(\mathbb{A})} \circ \phi_{2}(P(\mathbb{A}), P(\mathbb{A})) \\
= & \left(\xi^{\prime}(\operatorname{BrBialg} F)(\mathbb{A}) \otimes \xi^{\prime}(\operatorname{BrBialg} F)(\mathbb{A})\right) \circ \phi_{2}^{-1}(P(\mathbb{A}), P(\mathbb{A})) \circ F c_{P(\mathbb{A})} \circ \phi_{2}(P(\mathbb{A}), P(\mathbb{A}))
\end{array}
$$

so that

$$
c_{P^{\prime}((\operatorname{BrBialg} F)(\mathbb{A}))}=\phi_{2}^{-1}(P(\mathbb{A}), P(\mathbb{A})) \circ F c_{P(\mathbb{A})} \circ \phi_{2}(P(\mathbb{A}), P(\mathbb{A}))
$$

Summing up we get

$$
\left(P_{\mathrm{Br}}^{\prime} \circ \operatorname{BrBialg} F\right)(\mathbb{A})=\left(P^{\prime}((\operatorname{BrBialg} F)(\mathbb{A})), c_{P^{\prime}((\operatorname{BrBialg} F)(\mathbb{A}))}\right)
$$

$$
\begin{aligned}
& =\left(F P(\mathbb{A}), \phi_{2}^{-1}(P(\mathbb{A}), P(\mathbb{A})) \circ F c_{P(\mathbb{A})} \circ \phi_{2}(P(\mathbb{A}), P(\mathbb{A}))\right) \\
& =(\operatorname{Br} F)\left(P(\mathbb{A}), c_{P(\mathbb{A})}\right)=\left(\operatorname{Br} F \circ P_{\mathrm{Br}}\right)(\mathbb{A})
\end{aligned}
$$

We have so proved that $\left(P_{\mathrm{Br}}^{\prime} \circ \operatorname{BrBialg} F\right)(\mathbb{A})=\left(\operatorname{Br} F \circ P_{\mathrm{Br}}\right)(\mathbb{A})$. Let us check that the two functors coincide also on morphisms. Let $f: \mathbb{A} \rightarrow \mathbb{A}^{\prime}$ be a morphism in $\operatorname{BrBialg}_{\mathcal{M}}$. In view of Lemma 3.3, $P(f)$ makes the diagram

commutative. If we apply $F$ we get the commutative diagram


Composing as above with $\phi_{2}\left(A^{\prime}, A^{\prime}\right)$ we get the commutative diagram whose rows are equalizers


Hence

$$
\left(P_{\mathrm{Br}}^{\prime} \circ \operatorname{BrBialg} F\right)(f)=P^{\prime}(F(f))=F(P(f))=(F \circ P)(f)
$$

In conclusion, the diagram in the statement commutes. Moreover (68) holds.

## 4. Braided Categories

4.1. A braided monoidal category $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r, c)$ is a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ equipped with a braiding $c$, that is an isomorphism $c_{U, V}: U \otimes V \rightarrow V \otimes U$, natural in $U, V \in \mathcal{M}$, satisfying, for all $U, V, W \in \mathcal{M}$,

$$
\begin{aligned}
& a_{V, W, U} \circ c_{U, V \otimes W} \circ a_{U, V, W}=\left(V \otimes c_{U, W}\right) \circ a_{V, U, W} \circ\left(c_{U, V} \otimes W\right), \\
& a_{W, U, V}^{-1} \circ c_{U \otimes V, W} \circ a_{U, V, W}^{-1}=\left(c_{U, W} \otimes V\right) \circ a_{U, W, V}^{-1} \circ\left(U \otimes c_{V, W}\right) .
\end{aligned}
$$

From now on we will omit the associativity and unity constraints unless needed to clarify the context.

A braided monoidal category is called symmetric if we further have $c_{V, U} \circ c_{U, V}=\operatorname{Id}_{U \otimes V}$ for every $U, V \in \mathcal{M}$.

A (symmetric) braided monoidal functor is a monoidal functor $F: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ such that

$$
F\left(c_{U, V}\right) \circ \phi_{2}(U, V)=\phi_{2}(V, U) \circ c_{F(U), F(V)}^{\prime}
$$

More details on these topics can be found in Ka, Chapter XIII].
REMARK 4.2. Given a braided monoidal category $(\mathcal{M}, \otimes, \mathbf{1}, c)$ the category $\operatorname{Alg}_{\mathcal{M}}$ becomes monoidal where, for every $A, B \in \operatorname{Alg}_{\mathcal{M}}$ the multiplication and unit of $A \otimes B$ are given by

$$
\begin{aligned}
m_{A \otimes B} & :=\left(m_{A} \otimes m_{B}\right) \circ\left(A \otimes c_{B, A} \otimes B\right):(A \otimes B) \otimes(A \otimes B) \rightarrow A \otimes B \\
u_{A \otimes B} & :=\left(u_{A} \otimes u_{B}\right) \circ l_{\mathbf{1}}^{-1}: \mathbf{1} \rightarrow A \otimes B
\end{aligned}
$$

Moreover the forgetful functor $\operatorname{Alg}_{\mathcal{M}} \rightarrow \mathcal{M}$ is a strict monoidal functor, cf. [JS, page 60].
Definition 4.3. A bialgebra in a braided monoidal category $(\mathcal{M}, \otimes, \mathbf{1}, c)$ is a coalgebra $(B, \Delta, \varepsilon)$ in the monoidal category $\operatorname{Alg}_{\mathcal{M}}$. Equivalently a bialgebra is a quintuple $(A, m, u, \Delta, \varepsilon)$ where $(A, m, u)$ is an algebra in $\mathcal{M},(A, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{M}$ such that $\Delta$ and $\varepsilon$ are morphisms of algebras where $A \otimes A$ is an algebra as in the previous remark. Denote by Bialg $\mathcal{M}_{\mathcal{M}}$ the category of bialgebras in $\mathcal{M}$ and their morphisms, defined in the expected way.

Proposition 4.4. Let $\mathcal{M}$ be a braided monoidal category. Consider the obvious functors

$$
J: \mathcal{M} \rightarrow \operatorname{Br}_{\mathcal{M}}, \quad J_{\mathrm{Alg}}: \operatorname{Alg}_{\mathcal{M}} \rightarrow \operatorname{BrAlg}_{\mathcal{M}} \quad \text { and } \quad J_{\text {Bialg }}: \operatorname{Bialg}_{\mathcal{M}} \rightarrow \operatorname{BrBialg}_{\mathcal{M}}
$$

which act as the identity on morphisms and defined on objects by

$$
\begin{aligned}
J V & =\left(V, c_{V, V}\right) \\
J_{\mathrm{Alg}}\left(A, m_{A}, u_{A}\right) & =\left(A, m_{A}, u_{A}, c_{A, A}\right) \\
J_{\text {Bialg }}\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) & =\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B, B}\right) .
\end{aligned}
$$

Then $J, J_{\mathrm{Alg}}$ and $J_{\text {Bialg }}$ are full, faithful and conservative. Moreover the following diagram commutes.


Proof. It is clear that $\left(V, c_{V, V}\right)$ is an object in $\mathrm{Br}_{\mathcal{M}}$. Moreover any morphism in $\mathcal{M}$ becomes a morphism in $\mathrm{Br}_{\mathcal{M}}$ with respect to the braiding of $\mathcal{M}$. Thus the functor $J$ is well-defined. Let us check that $J$ is full and faithful. For $V, V^{\prime} \in \mathcal{M}$,

$$
\operatorname{Br}_{\mathcal{M}}\left(J V, J V^{\prime}\right)=\operatorname{Br}_{\mathcal{M}}\left(\left(V, c_{V, V}\right),\left(V^{\prime}, c_{V^{\prime}, V^{\prime}}\right)\right)=\mathcal{M}\left(V, V^{\prime}\right)
$$

Using the naturality of the braiding in $\mathcal{M}$, one proves that $\left(A, m_{A}, u_{A}, c_{A, A}\right)$ is a braided algebra in $\mathcal{M}$ for every algebra $\left(A, m_{A}, u_{A}\right)$ in $\mathcal{M}$. Moreover any morphism of algebras becomes a morphism of braided algebras with respect to the braiding of $\mathcal{M}$. Thus the functor $J_{\mathrm{Alg}}$ is well-defined. Let us check that $J_{\mathrm{Alg}}$ is full and faithful. For $\left(A, m_{A}, u_{A}\right)$ and $\left(A^{\prime}, m_{A^{\prime}}, u_{A^{\prime}}\right)$ objects in $\operatorname{Alg}_{\mathcal{M}}$,

$$
\begin{aligned}
& \operatorname{BrAlg}_{\mathcal{M}}\left(J_{\mathrm{Alg}}\left(A, m_{A}, u_{A}\right), J_{\mathrm{Alg}}\left(A^{\prime}, m_{A^{\prime}}, u_{A^{\prime}}\right)\right) \\
= & \operatorname{BrAlg}_{\mathcal{M}}\left(\left(A, m_{A}, u_{A}, c_{A, A}\right),\left(A^{\prime}, m_{A^{\prime}}, u_{A^{\prime}}, c_{A^{\prime}, A^{\prime}}\right)\right) \\
= & \operatorname{Alg}_{\mathcal{M}}\left(\left(A, m_{A}, u_{A}\right),\left(A^{\prime}, m_{A^{\prime}}, u_{A^{\prime}}\right)\right)
\end{aligned}
$$

By Definition 4.3, a bialgebra in $\mathcal{M}$ is a quintuple $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)$ where $\left(B, m_{B}, u_{B}\right)$ is an algebra and $\left(B, \Delta_{B}, \varepsilon_{B}\right)$ a coalgebra in $\mathcal{M}$ such that $\Delta_{B}$ and $\varepsilon_{B}$ are morphisms of algebras where $B \otimes B$ is an algebra via the braiding of $\mathcal{M}$. Using the naturality of the braiding in $\mathcal{M}$, one proves that $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B, B}\right)$ is indeed a braided bialgebra in $\mathcal{M}$. Furthermore any morphism of bialgebras $f$ is indeed a morphism of braided bialgebras. Thus the functor $J_{\text {Bialg }}$ is well-defined. Let us check that $J_{\text {Bialg }}$ is full and faithful. For $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)$ and $\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right)$ objects in $\operatorname{Bialg}_{\mathcal{M}}$,

$$
\begin{aligned}
& \operatorname{BrBialg}_{\mathcal{M}}\left(J_{\mathrm{Bialg}}\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right), J_{\mathrm{Bialg}}\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right)\right) \\
= & \operatorname{BrBialg}_{\mathcal{M}}\left(\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B, B}\right),\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}, c_{B^{\prime}, B^{\prime}}\right)\right) \\
= & \operatorname{Bialg}_{\mathcal{M}}\left(\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right),\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right)\right)
\end{aligned}
$$

The commutativity of the diagrams and the fact that $J, J_{\text {Alg }}$ and $J_{\text {Bialg }}$ are conservative are clear.

Proposition 4.5. Take the hypotheses and notations of Proposition 3.1 and assume that $\mathcal{M}$ is braided. Then we have a commutative diagram

where $J$ and $J_{\mathrm{Alg}}$ are as in Proposition 4.4.
Proof. Set $\left(T, m_{T}, u_{T}\right):=T V$. We have

$$
\begin{aligned}
\left(J_{\mathrm{Alg}} \circ T\right)(V) & =J_{\mathrm{Alg}}\left(T, m_{T}, u_{T}\right)=\left(T, m_{T}, u_{T}, c_{T, T}\right) \\
\left(T_{\mathrm{Br}} \circ J\right)(V) & =T_{\mathrm{Br}}\left(V, c_{V, V}\right)=\left(T, m_{T}, u_{T}, c_{T}\right)
\end{aligned}
$$

where $c_{T}$ is uniquely determined by (42). Let us check that $c_{T}=c_{T, T}$. Since $\mathcal{M}$ is braided and $\alpha_{n} V: V^{\otimes n} \rightarrow T$ is a morphism in $\mathcal{M}$, we have that $c_{T, T} \circ\left(\alpha_{m} V \otimes \alpha_{n} V\right)=\left(\alpha_{n} V \otimes \alpha_{m} V\right) \circ$ $c_{V \otimes m, V \otimes n}$. Since the tensor functors preserves denumerable coproducts, $c_{T, T}$ is uniquely determined by this equality. Thus it will coincide with $c_{T}$ once proved that $c_{V \otimes m, V \otimes n}=c_{T}^{m, n}$ for every $m, n \in \mathbb{N}$. In view of Proposition 2.7, it suffices to check that $c_{V \otimes m, V \otimes n}$ fulfills the analogues of the equalities in that statement (which are fulfilled, by construction, by $c_{T}^{m, n}$ ). For objects $X, Y, Z \in \mathcal{M}$,

$$
\begin{aligned}
\left(c_{X, Z} \otimes Y\right) \circ\left(X \otimes c_{Y, Z}\right) & =c_{X \otimes Y, Z}, \quad\left(Y \otimes c_{X, Z}\right) \circ\left(c_{X, Y} \otimes Z\right)=c_{X, Y \otimes Z} \\
c_{\mathbf{1}, Z} \circ l_{Z}^{-1} & =r_{Z}^{-1}, \quad c_{Z, \mathbf{1}} \circ r_{Z}^{-1}=l_{Z}^{-1}
\end{aligned}
$$

If we take $X=V^{\otimes l}, Y=V^{\otimes m}$ and $Z=V^{\otimes n}$ we get

$$
\begin{aligned}
\left(c_{V^{\otimes l}, V^{\otimes n}} \otimes V^{\otimes m}\right) \circ\left(V^{\otimes l} \otimes c_{V^{\otimes m}, V^{\otimes n}}\right) & =c_{V^{\otimes l} \otimes V^{\otimes m}, V^{\otimes n}}=c_{V^{\otimes(l+m)}, V^{\otimes n}}, \\
\left(V^{\otimes m} \otimes c_{V^{\otimes l}, V^{\otimes n}}\right) \circ\left(c_{\left.V^{\otimes l}, V^{\otimes m} \otimes V^{\otimes n}\right)}\right. & =c_{V^{\otimes l}, V^{\otimes m} \otimes V^{\otimes n}}=c_{V^{\otimes l}, V^{\otimes(m+n)}}, \\
c_{\mathbf{1}, V^{\otimes n}} \circ l_{V^{\otimes n}}^{-1} & =r_{V^{\otimes n}}^{-1}, \quad c_{V^{\otimes n}, \mathbf{1}} \circ r_{V^{\otimes n}}^{-1}=l_{V^{\otimes n}}^{-1}
\end{aligned}
$$

Hence, $c_{V} \otimes m, V^{\otimes n}$ fulfills equalities of the same form of the ones defining $c_{T}^{m, n}$. Hence, in order to check that $c_{V \otimes m, V \otimes n}=c_{T}^{m, n}$ we have only to prove that it holds for $m, n \in\{0,1\}$. But this is true. Summing up we have proved that $c_{T, T}=c_{T}$ and hence $\left(J_{\mathrm{Alg}} \circ T\right)(V)=\left(T_{\mathrm{Br}} \circ J\right)(V)$. Moreover, for every morphism $f$ in $\mathcal{M}$ we have $\left(J_{\mathrm{Alg}} \circ T\right)(f)=T(f)=\left(T_{\mathrm{Br}} \circ J\right)(f)$. Hence $J_{\mathrm{Alg}} \circ T=T_{\mathrm{Br}} \circ J$.
4.6. Let $\mathcal{M}$ be a preadditive braided monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. Let $\operatorname{Bialg}_{\mathcal{M}}$ be the category of bialgebras in $\mathcal{M}$ and $\Theta: \operatorname{Bialg}_{\mathcal{M}} \rightarrow \mathcal{M}$ be the forgetful functor. Define the functor

$$
P:=H \circ P_{\mathrm{Br}} \circ J_{\mathrm{Bialg}}: \operatorname{Bialg}_{\mathcal{M}} \rightarrow \mathcal{M}
$$

For any $\mathbb{B}:=\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) \in \operatorname{Bialg}_{\mathcal{M}}$ we have

$$
\begin{aligned}
P(\mathbb{B}) & =\left(H \circ P_{\mathrm{Br}} \circ J_{\mathrm{Bialg}}\right)(\mathbb{B})=\left(H \circ P_{\mathrm{Br}}\right)\left(J_{\mathrm{Bialg}}(\mathbb{B})\right) \\
& =H\left(P\left(J_{\mathrm{Bialg}}(\mathbb{B})\right), c_{P\left(J_{\text {Bialg }}(\mathbb{B})\right)}\right)=P\left(J_{\text {Bialg }}(\mathbb{B})\right) \\
& =P\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B, B}\right)=P\left(B, \Delta_{B}, \varepsilon_{B}, u_{B}\right)
\end{aligned}
$$

The canonical inclusion $\xi P\left(B, \Delta_{B}, \varepsilon_{B}, u_{B}\right): P\left(B, \Delta_{B}, \varepsilon_{B}, u_{B}\right) \rightarrow B$ will be denoted by $\xi \mathbb{B}$. Thus we have the equalizer

$$
P(\mathbb{B}) \xrightarrow{\xi \mathbb{B}} B \underset{\left(B \otimes u_{B}\right) r_{B}^{-1}+\left(u_{B} \otimes B\right) l_{B}^{-1}}{\Delta_{B}} B \otimes B
$$

Proposition 4.7. Let $\mathcal{M}$ be a preadditive braided monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. Then we have a commutative diagram

where the horizontal arrows are the functors of Proposition 4.4. Furthermore

$$
\begin{equation*}
\xi J_{\mathrm{Bialg}}=J \xi \tag{72}
\end{equation*}
$$

Proof. Let $\mathbb{B}:=\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) \in \operatorname{Bialg}_{\mathcal{M}}$. Then

$$
J_{\text {Bialg }} \mathbb{B}=\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B, B}\right)
$$

where $c_{B, B}$ is the braiding of $B$ in $\mathcal{M}$. Looking at (4.6) and Lemma (3.3) we have $P_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}=$ $\left(P \mathbb{B}, c_{P}\right)$ where $(\xi \mathbb{B} \otimes \xi \mathbb{B}) c_{P}=c_{B, B}(\xi \mathbb{B} \otimes \xi \mathbb{B})$ and $\xi \mathbb{B}: P(\mathbb{B}) \rightarrow B$ is the morphism of definition of the equalizer. Since $\xi \mathbb{B}$ belongs to $\mathcal{M}$, it is compatible with the braiding so that $(\xi \mathbb{B} \otimes \xi \mathbb{B}) c_{P, P}=$ $c_{B, B}(\xi \mathbb{B} \otimes \xi \mathbb{B})$. Since the tensor functors preserve equalizers we have that $\xi \mathbb{B} \otimes \xi \mathbb{B}$ is a monomorphism and hence $c_{P}=c_{P, P}$. Thus $P_{\text {Br }} J_{\text {Bialg }} \mathbb{B}=\left(P \mathbb{B}, c_{P}\right)=\left(P \mathbb{B}, c_{P, P}\right)=J P \mathbb{B}$. Let $f: \mathbb{B} \rightarrow \mathbb{B}^{\prime}$ be a morphism in $\operatorname{Bialg}_{\mathcal{M}}$. Then $P_{\mathrm{Br}} J_{\mathrm{Bialg}} f \in \operatorname{Br}_{\mathcal{M}}\left(P_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}, P_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}^{\prime}\right)=\operatorname{Br}_{\mathcal{M}}\left(J P \mathbb{B}, J P \mathbb{B}^{\prime}\right)$. By Proposition 4.4, $J$ is full and faithful so that there is a unique morphism $g: P \mathbb{B} \rightarrow P \mathbb{B}^{\prime}$ such that $J g=P_{\mathrm{Br}} J_{\text {Bialg }} f$. By definition of $P$, we have $P f=H P_{\mathrm{Br}} J_{\mathrm{Bialg}} f=H J g=g$ so that we get $J P f=J g=P_{\mathrm{Br}} J_{\text {Bialg }} f$. This implies that $P_{\mathrm{Br}} \circ J_{\mathrm{Bialg}}=J \circ P$.

Note that $\xi J_{\mathrm{Bialg}} \mathbb{B}$ goes from $P_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}$ to $\Omega_{\mathrm{Br}} \mho_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}$. Now the first object is $J P \mathbb{B}$. The second object is $\Omega_{\mathrm{Br}} \mho_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}=\Omega_{\mathrm{Br}} J_{\mathrm{Alg}} \mho \mathbb{B}=J \Omega \mho \mathbb{B}$. Thus $\xi J_{\mathrm{Bialg}} \mathbb{B} \in \operatorname{Br}_{\mathcal{M}}(J P \mathbb{B}, J \Omega \mho \mathbb{B})$. Using again that $J$ is full and faithful, we get $\xi J_{\text {Bialg }} \mathbb{B}=J \alpha$ for a unique morphism $\alpha$. If we compose both sides of this equality by $H$ we get $\alpha=H \xi J_{\text {Bialg }} \mathbb{B}$. By 4.6 and Lemma 3.3, we have that $H \xi J_{\text {Bialg }} \mathbb{B}=H \xi \mathbb{B}=\xi \mathbb{B}$. Thus $\alpha=\xi \mathbb{B}$ and hence we get $\xi J_{\text {Bialg }}=J \xi$.
4.8. Let $\mathcal{M}$ be a preadditive braided monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. Assume further that $\mathcal{M}$ has denumerable coproducts and that the tensor functors preserve such coproducts. By Remark 1.3, the forgetful functor $\Omega: \operatorname{Alg}_{\mathcal{M}} \rightarrow \mathcal{M}$ has a left adjoint $T: \mathcal{M} \rightarrow \operatorname{Alg}_{\mathcal{M}}$. Let us check that $T(V)$ becomes an object in $\operatorname{Bialg}_{\mathcal{M}}$. Let $V \in \mathcal{M}$ and consider

$$
\left(\bar{T}_{\mathrm{Br}} \circ J\right)(V)=\bar{T}_{\mathrm{Br}}\left(V, c_{V, V}\right)
$$

Denote this braided bialgebra by $\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right)$. Note that

$$
\begin{aligned}
& T(V)=T H J V \stackrel{(44)}{=} H_{\mathrm{Alg}} T_{\mathrm{Br}} J(V) \\
& \stackrel{(55)}{=} H_{\mathrm{Alg}} \mho_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} J(V)=\left(A, m_{A}, u_{A}\right)
\end{aligned}
$$

Let us compute the braiding $c_{A}$. We have

$$
\begin{aligned}
\left(A, m_{A}, u_{A}, c_{A}\right) & =\mho_{\mathrm{Br}}\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right) \\
& =\mho_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} J(V) \stackrel{\sqrt{55}}{=} T_{\mathrm{Br}} J(V) \stackrel{(70)}{=} J_{\mathrm{Alg}} T(V) \\
& =\left(A, m_{A}, u_{A}, c_{A, A}\right)
\end{aligned}
$$

so that $c_{A}=c_{A, A}$. Since $\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A, A}\right)=\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}, c_{A}\right)$ which is a braided bialgebra, it is clear that $\left(A, m_{A}, u_{A}, \Delta_{A}, \varepsilon_{A}\right)$ is a bialgebra in $\mathcal{M}$ that will be denoted by $\bar{T}(V)$. By construction we have

$$
\left(\bar{T}_{\mathrm{Br}} \circ J\right)(V)=J_{\mathrm{Bialg}}(\bar{T}(V))
$$

Let $f: V \rightarrow W$ be a morphism in $\mathcal{M}$. Then

$$
\left(\bar{T}_{\mathrm{Br}} \circ J\right)(f) \in \operatorname{BrBialg}_{\mathcal{M}}\left(J_{\mathrm{Bialg}}(\bar{T}(V)), J_{\mathrm{Bialg}}(\bar{T}(W))\right)
$$

By Proposition 4.4, we have that $J_{\text {Bialg }}$ is full and faithful so that there is a unique morphism $\bar{T}(f) \in \operatorname{Bialg}_{\mathcal{M}}(\bar{T}(V), \bar{T}(W))$ such that $\left(\bar{T}_{\mathrm{Br}} \circ J\right)(f)=J_{\mathrm{Bialg}}(\bar{T}(f))$. In this way we have defined a functor

$$
\bar{T}: \mathcal{M} \rightarrow \operatorname{Bialg}_{\mathcal{M}}
$$

such that $\bar{T}_{\mathrm{Br}} \circ J=J_{\mathrm{Bialg}} \circ \bar{T}$. Thus we get the commutative diagram


Note that, by construction we have that (52) and (54) hold. We compute

$$
\begin{gathered}
\mho \bar{T}=H_{\mathrm{Alg}} J_{\mathrm{Alg}} \mho \bar{T} \stackrel{(69)}{=} H_{\mathrm{Alg}} \mho_{\mathrm{Br}} J_{\mathrm{Bialg}} \bar{T} \stackrel{73)}{=} H_{\mathrm{Alg}} \mho_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} J \\
\text { (止) } H_{\mathrm{Alg}} T_{\mathrm{Br}} J \stackrel{(44)}{=} T H J=T
\end{gathered}
$$

so that the following diagram commutes.


Let us check $\bar{T}$ is a left adjoint of the functor $P: \operatorname{Bialg}_{\mathcal{M}} \rightarrow \mathcal{M}$.
Theorem 4.9. Let $\mathcal{M}$ be a preadditive braided monoidal category with equalizers. Assume that the tensor functors are additive and preserve equalizers. Assume further that $\mathcal{M}$ has denumerable coproducts and that the tensor functors preserve such coproducts. Then we can consider the morphisms $\bar{\eta}_{\mathrm{Br}}$ and $\bar{\epsilon}_{\mathrm{Br}}$ of Theorem 3.5 and the functor $\bar{T}$ of (4.8).

1) There are unique natural transformations $\bar{\eta}: \operatorname{Id}_{\mathcal{M}} \rightarrow P \bar{T}$ and $\bar{\epsilon} \mathbb{B}: \bar{T} P \rightarrow \operatorname{Id}_{B_{B i a l g}^{\mathcal{M}}}$ such that

$$
\begin{align*}
\bar{\eta}_{\mathrm{Br}} J & =J \bar{\eta}  \tag{75}\\
\bar{\epsilon}_{\mathrm{Br}} J_{\mathrm{Bialg}} & =J_{\mathrm{Bialg}} \bar{\epsilon} . \tag{76}
\end{align*}
$$

2) The pair $(\bar{T}, P)$ is an adjunction with unit $\bar{\eta}$ and counit $\bar{\epsilon}$.
3) The unit $\bar{\eta}$ and the counit $\bar{\epsilon}$ are uniquely determined by the following equalities

$$
\begin{align*}
\xi \bar{T} \circ \bar{\eta} & =\eta  \tag{77}\\
\epsilon \mho \circ T \xi & =\mho \bar{\epsilon} \tag{78}
\end{align*}
$$

where $\eta$ and $\epsilon$ denote the unit and counit of the adjunction $(T, \Omega)$ respectively.
Proof. 1) Let $V \in \mathcal{M}$ and $\mathbb{B} \in \operatorname{Bialg}_{\mathcal{M}}$. Since $P_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} J V \stackrel{733}{=} P_{\mathrm{Br}} J_{\mathrm{Bialg}} \bar{T} V \stackrel{71}{=} J P \bar{T} V$, we have that $\bar{\eta}_{\mathrm{Br}} J V \in \operatorname{Br}_{\mathcal{M}}\left(J V, P_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} J V\right)=\operatorname{Br}_{\mathcal{M}}(J V, J P \bar{T} V)$. Since $\left.\bar{T}_{\mathrm{Br}} P_{\mathrm{Br}} J_{\mathrm{Bialg}} \stackrel{\mathbb{B}}{=} \bar{T}_{\mathrm{Br}} J P \mathbb{B} \xrightarrow{73}\right)$ $J_{\text {Bialg }} \bar{T} P \mathbb{B}$, we have that

$$
\bar{\epsilon}_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B} \in \operatorname{BrBialg}_{\mathcal{M}}\left(\bar{T}_{\mathrm{Br}} P_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}, J_{\mathrm{Bialg}} \mathbb{B}\right)=\operatorname{BrBialg}_{\mathcal{M}}\left(J_{\mathrm{Bialg}} \bar{T} P \mathbb{B}, J_{\mathrm{Bialg}} \mathbb{B}\right)
$$

Now, by Proposition 4.4, both $J$ and $J_{\text {Bialg }}$ are full and faithful. Thus there are unique morphisms $\bar{\eta} V: V \rightarrow P \bar{T} V$ and $\bar{\epsilon} \mathbb{B}: \bar{T} P \mathbb{B} \rightarrow \mathbb{B}$ such that $\bar{\eta}_{\mathrm{Br}} J V=J \bar{\eta} V$ and $\bar{\epsilon}_{\mathrm{Br}} J_{\mathrm{Bialg}} \mathbb{B}=J_{\mathrm{Bialg}} \bar{\epsilon} \mathbb{B}$. Note that $\bar{\eta} V=H J \bar{\eta} V=H \bar{\eta}_{\mathrm{Br}} J V$ so that $\bar{\eta} V$ is natural in $V$. Let us check that $\bar{\epsilon} \mathbb{B}$ is natural in $\mathbb{B}$. Given a morphism $f: \mathbb{B} \rightarrow \mathbb{B}^{\prime}$ we have

$$
\begin{aligned}
J_{\text {Bialg }}\left(\bar{\epsilon} \mathbb{B}^{\prime} \circ \bar{T} P f\right)= & J_{\text {Bialg }} \bar{\epsilon} \mathbb{B}^{\prime} \circ J_{\mathrm{Bialg}} \bar{T} P f \stackrel{\bar{\epsilon}_{\mathrm{Br}}}{ } J_{\mathrm{Bialg}} \mathbb{B}^{\prime} \circ \bar{T}_{\mathrm{Br}} J P f \\
& =J_{\text {Bialg }} f \circ \bar{\epsilon}_{\mathrm{Br}} J_{\mathrm{Bialg}} J_{\mathrm{B}}=J_{\mathrm{Bialg}} f \circ \mathbb{B}_{\mathrm{Bialg}} \bar{\epsilon} \overline{\mathbb{B}}^{\prime}=J_{\mathrm{Bialg}}(f \circ \bar{\epsilon} \mathbb{B})
\end{aligned}
$$


2) We compute

$$
\begin{gathered}
\left.J(P \bar{\epsilon} \circ \bar{\eta} P) \stackrel{\text { 71 }}{=} P_{\mathrm{Br}} J_{\mathrm{Bialg}} \bar{\epsilon} \circ J \bar{\eta} P\right), \sqrt{76} P_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}} J_{\mathrm{Bialg}} \circ \bar{\eta}_{\mathrm{Br}} J P \\
\stackrel{\text { 71 }}{=} P_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}} J_{\mathrm{Bialg}} \circ \bar{\eta}_{\mathrm{Br}} P_{\mathrm{Br}} J_{\mathrm{Bialg}}=P_{\mathrm{Br}} J_{\mathrm{Bialg}} \stackrel{71}{=} J P .
\end{gathered}
$$

Since $J$ is faithful, we obtain $P \bar{\epsilon} \circ \bar{\eta} P=P$. We also have

$$
\begin{aligned}
& J_{\text {Bialg }}(\bar{\epsilon} \bar{T} \circ \bar{T} \bar{\eta}) \stackrel{(73)}{=} J_{\text {Bialg }} \bar{\epsilon} \bar{T} \circ \bar{T}_{\mathrm{Br}} J \bar{\eta} \stackrel{\text { 75) , (76) }}{=} \bar{\epsilon}_{\mathrm{Br}} J_{\mathrm{Bialg}} \bar{T} \circ \bar{T}_{\mathrm{Br}} \bar{\eta}_{\mathrm{Br}} J \\
& \text { (73) } \bar{\epsilon}_{\mathrm{Br}} \bar{T}_{\mathrm{Br}} J \circ \bar{T}_{\mathrm{Br}} \bar{\eta}_{\mathrm{Br}} J=\bar{T}_{\mathrm{Br}} J \text { (73) } J_{\mathrm{Bialg}} \bar{T} \text {. }
\end{aligned}
$$

Since $J_{\text {Bialg }}$ is faithful, we get $\bar{\epsilon} \bar{T} \circ \bar{T} \bar{\eta}=\bar{T}$. We have so proved that $(\bar{T}, P)$ is an adjunction with unit $\bar{\eta}$ and counit $\bar{\epsilon}$.
3) We have

$$
\begin{gathered}
\xi \bar{T} \circ \bar{\eta}=H J(\xi \bar{T} \circ \bar{\eta})=H(J \xi \bar{T} \circ J \bar{\eta}) \stackrel{(72),(75)}{=} H\left(\xi J_{\mathrm{Bialg}} \bar{T} \circ \bar{\eta}_{\mathrm{Br}} J\right) \\
\stackrel{\text { (73) }}{=} H\left(\xi \bar{T}_{\mathrm{Br}} J \circ \bar{\eta}_{\mathrm{Br}} J\right) \stackrel{(6 \mathrm{~g})}{=} H \eta_{\mathrm{Br}} J \stackrel{(43)}{=} \eta H J=\eta
\end{gathered}
$$

and

$$
\begin{gathered}
\mho \bar{\epsilon}=H_{\mathrm{Alg}} \mho_{\mathrm{Br}} J_{\mathrm{Bialg}} \overline{=} H_{\mathrm{Alg}} \mho_{\mathrm{Br}} \overline{\mathrm{~B}}_{\mathrm{Br}} J_{\mathrm{Bialg}} H_{\mathrm{Alg}}\left(\epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \circ T_{\mathrm{Br}} \xi\right) J_{\mathrm{Bialg}} \\
=H_{\mathrm{Alg}} \epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} J_{\mathrm{Bialg}} \circ H_{\mathrm{Alg}} T_{\mathrm{Br}} \xi J_{\mathrm{Bialg}} \stackrel{43}{=} \epsilon H_{\mathrm{Alg}} \mho_{\mathrm{Br}} J_{\mathrm{Bialg}} \circ T H \xi J_{\mathrm{Bialg}} \\
\stackrel{72}{=} \epsilon \mho \circ T H J \xi=\epsilon \mho \circ T \xi .
\end{gathered}
$$

Since $\xi \bar{T}$ is a monomorphism and $\mho$ is faithful, we get that $\bar{\eta}$ and $\bar{\epsilon}$ are uniquely determined by (77) and (78) respectively.

Proposition 4.10. Let $\mathcal{M}$ and $\mathcal{M}^{\prime}$ be braided monoidal categories. Let $\left(F, \phi_{0}, \phi_{2}\right): \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ be a braided monoidal functor. Then $F$ induces a functor $\operatorname{Bialg} F: \operatorname{Bialg}_{\mathcal{M}} \rightarrow \operatorname{Bialg}_{\mathcal{M}^{\prime}}$ which acts as $F$ on morphisms and which is defined, on objects, by

$$
\operatorname{Bialg} F\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right):=\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}\right)
$$

where

$$
\begin{aligned}
m_{F B} & :=F m_{B} \circ \phi_{2}(B, B): F B \otimes F B \rightarrow F B \\
u_{F B} & :=F u_{B} \circ \phi_{0}: \mathbf{1} \rightarrow F B \\
\Delta_{F B} & :=\phi_{2}^{-1}(B, B) \circ F \Delta_{B}: F B \rightarrow F B \otimes F B \\
\varepsilon_{F B} & :=\phi_{0}^{-1} \circ F \varepsilon_{B}: F B \rightarrow \mathbf{1}
\end{aligned}
$$

and the following diagrams commute.


1) Bialg $F$ is an equivalence (resp. category isomorphism or conservative) whenever $F$ is.
2) If $F$ preserves equalizers, the following diagram commutes

and

$$
\begin{equation*}
\xi(\operatorname{Bialg} F)=F \xi \tag{79}
\end{equation*}
$$

Proof. Let us check that the first diagram commutes.

$$
\begin{aligned}
(\operatorname{Br} F \circ J)(M) & =\operatorname{Br} F\left(M, c_{M, M}\right)=\left(F M, \phi_{2}^{-1}(M, M) \circ F c_{M, M} \circ \phi_{2}(M, M)\right) \\
\stackrel{(*)}{=}\left(F M, c_{F M, F M}\right) & =\left(J^{\prime} \circ F\right)(M)
\end{aligned}
$$

where in $\left(^{*}\right)$ we used that $F$ is braided. The functors $\operatorname{Br} F \circ J$ and $J^{\prime} \circ F$ trivially coincide on morphisms. We have

$$
\begin{aligned}
& \left(\operatorname{BrBialg} F \circ J_{\mathrm{Bialg}}\right)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) \\
= & \operatorname{BrBialg} F\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B, B}\right) \\
= & \left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, \phi_{2}^{-1}(B, B) \circ F c_{B, B} \circ \phi_{2}(B, B)\right) \\
\stackrel{(*)}{=} & \left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, c_{F B, F B}\right)
\end{aligned}
$$

Now, since

$$
\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, c_{F B, F B}\right)=\left(\mathrm{BrBialg} F \circ J_{\mathrm{Bialg}}\right)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) \in \mathrm{BrBialg}_{\mathcal{M}^{\prime}}
$$

and $c_{F B, F B}$ is the braiding of $F B$ in $\mathcal{M}^{\prime}$ we conclude that $\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}\right) \in \operatorname{Bialg}_{\mathcal{M}^{\prime}}$. Moreover for every morphism $f:\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) \rightarrow\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right)$ in $\operatorname{Bialg}_{\mathcal{M}}$, we have $\left(\operatorname{BrBialg} F \circ J_{\text {Bialg }}\right)(f)=(\operatorname{BrBialg} F)(f)=F f$ so that $F f$ is a morphism with domain $\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, c_{F B, F B}\right)$ and codomain ( $\left.F B^{\prime}, m_{F B^{\prime}}, u_{F B^{\prime}}, \Delta_{F B^{\prime}}, \varepsilon_{F B^{\prime}}, c_{F B^{\prime}, F B^{\prime}}\right)$. Thus $F f$ is a morphism in $\operatorname{Bialg}_{\mathcal{M}^{\prime}}$. Hence $\operatorname{Bialg} F$ is a well-defined functor. Note also that

$$
\begin{gathered}
\left(J_{\mathrm{Bialg}}^{\prime} \circ \operatorname{Bialg} F\right)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)=J_{\mathrm{Bialg}}^{\prime}\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}\right) \\
=\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, c_{F B, F B}\right)
\end{gathered}
$$

so that the functors $J_{\text {Bialg }}^{\prime} \circ \operatorname{Bialg} F$ and $\operatorname{BrBialg} F \circ J_{\text {Bialg coincide on objects. They trivially }}$ coincide on morphisms too so that the second diagram commutes. The third diagram commutes by definition of Bialg $F$ and $\operatorname{Alg} F$.

1) Assume that $F$ preserves equalizers. By Proposition 3.6, we have

$$
\begin{array}{rl}
P^{\prime}(\operatorname{Bialg} F) & =H^{\prime} P_{\mathrm{Br}}^{\prime} J_{\mathrm{Bialg}}^{\prime}(\operatorname{Bialg} F)=H^{\prime} P_{\mathrm{Br}}^{\prime}(\operatorname{BrBialg} F) J_{\mathrm{Bialg}} H^{\prime}(\operatorname{Br} F) P_{\mathrm{Br}} J_{\mathrm{Bialg}} \\
710 & F H J P
\end{array}=F P .
$$

and

$$
\xi(\operatorname{Bialg} F)=\xi^{\prime} J_{\mathrm{Bialg}}^{\prime}(\operatorname{Bialg} F)=\xi^{\prime}(\mathrm{BrBialg} F) J_{\mathrm{Bialg}}(\mathrm{Br} F) \xi J_{\mathrm{Bialg}}=(\operatorname{Br} F) \xi=F \xi
$$

2) Assume that $F$ is an equivalence. By Proposition 4.4, $J_{\text {Bialg }}$ and $J_{\text {Bialg }}^{\prime}$ are both full and faithful. By Proposition 2.5, BrBialg $F$ is a category equivalence. Given $X$ and $Y$ objects in $\operatorname{Bialg}_{\mathcal{M}}$ we have

$$
\begin{aligned}
& \operatorname{Bialg}_{\mathcal{M}^{\prime}}((\operatorname{Bialg} F) X,(\operatorname{Bialg} F) Y) \cong \operatorname{BrBialg}_{\mathcal{M}^{\prime}}\left(\left(J_{\mathrm{Bialg}}^{\prime} \circ \operatorname{Bialg} F\right) X,\left(J_{\mathrm{Bialg}}^{\prime} \circ \operatorname{Bialg} F\right) Y\right) \\
= & \operatorname{BrBialg}_{\mathcal{M}^{\prime}}\left(\left(\operatorname{BrBialg} F \circ J_{\mathrm{Bialg}}\right) X,\left(\operatorname{BrBialg} F \circ J_{\mathrm{Bialg}}\right) Y\right) \cong \operatorname{Bialg}_{\mathcal{M}}(X, Y)
\end{aligned}
$$

The composition of these maps is the map assigning ( $\operatorname{Bialg} F)(f)$ to a morphism $f$ so that $\operatorname{Bialg} F$ is full and faithful. In order to prove it is an equivalence, it remains to check that it is essentially surjective i.e. that each object $\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right) \in \operatorname{Bialg}_{\mathcal{M}^{\prime}}$ is isomorphic to $(\operatorname{Bialg} F)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)$ for some object $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)$ in $\operatorname{Bialg}_{\mathcal{M}}$.

Let $\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right) \in \operatorname{Bialg}_{\mathcal{M}^{\prime}}$. Then

$$
\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}, c_{B^{\prime}, B^{\prime}}\right)=J_{\text {Bialg }}^{\prime}\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right) \in \operatorname{BrBialg}_{\mathcal{M}^{\prime}}
$$

Since $\operatorname{BrBialg} F$ is essentially surjective, there exists $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right) \in \operatorname{BrBialg}_{\mathcal{M}}$ and an isomorphism

$$
f:\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}, c_{B^{\prime}, B^{\prime}}\right) \rightarrow(\operatorname{BrBialg} F)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right)
$$

in $\operatorname{BrBialg}_{\mathcal{M}^{\prime}}$. Since
$(\operatorname{BrBialg} F)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right)=\left(F B, m_{F B}, u_{F B}, \Delta_{F B}, \varepsilon_{F B}, \phi_{2}^{-1}(B, B) \circ F c_{B} \circ \phi_{2}(B, B)\right)$,
we have

$$
\phi_{2}^{-1}(B, B) \circ F c_{B} \circ \phi_{2}(B, B) \circ(f \otimes f)=(f \otimes f) \circ c_{B^{\prime}, B^{\prime}}
$$

Since $f$ is ,in particular, a morphism in $\mathcal{M}^{\prime}$, by the naturality of the braiding, we get

$$
\begin{aligned}
F c_{B} & =\phi_{2}(B, B) \circ(f \otimes f) \circ c_{B^{\prime}, B^{\prime}} \circ\left(f^{-1} \otimes f^{-1}\right) \circ \phi_{2}^{-1}(B, B) \\
& =\phi_{2}(B, B) \circ c_{F B, F B}^{\prime} \circ(f \otimes f) \circ\left(f^{-1} \otimes f^{-1}\right) \circ \phi_{2}^{-1}(B, B) \\
& =\phi_{2}(B, B) \circ c_{F B, F B}^{\prime} \circ \phi_{2}^{-1}(B, B)=F c_{B, B}
\end{aligned}
$$

Since $F$ is faithful, we obtain $c_{B}=c_{B, B}$. Thus $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right) \in \operatorname{BrBialg}_{\mathcal{M}}$ means that $\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) \in \operatorname{Bialg}_{\mathcal{M}}$ so that

$$
\begin{aligned}
& (\operatorname{BrBialg} F)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}\right)=(\operatorname{BrBialg} F)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B, B}\right) \\
= & (\operatorname{BrBialg} F) J_{\mathrm{Bialg}}\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)=\left(J_{\mathrm{Bialg}}^{\prime} \circ \operatorname{Bialg} F\right)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right) .
\end{aligned}
$$

Thus $f \in \operatorname{BrBialg}_{\mathcal{M}^{\prime}}\left(J_{\text {Bialg }}^{\prime}\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right),\left(J_{\text {Bialg }}^{\prime} \circ \operatorname{Bialg} F\right)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)\right)$. Since $J_{\text {Bialg }}^{\prime}$ is full, there is a morphism $g:\left(B^{\prime}, m_{B^{\prime}}, u_{B^{\prime}}, \Delta_{B^{\prime}}, \varepsilon_{B^{\prime}}\right) \rightarrow(\operatorname{Bialg} F)\left(B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}\right)$ in such Bialg $\mathcal{M}^{\prime}$ that $f=J_{\text {Bialg }}^{\prime}(g)$. Since $J_{\text {Bialg }}^{\prime}$ is full and faithful, we get that $g$ is an isomorphism too. Therefore $\operatorname{Bialg} F$ is essentially surjective and hence an equivalence.

Assume that $F$ is a category isomorphism. By Proposition 2.5. BrBialg $F$ is a category isomorphism. Indeed the inverse is, by construction $\operatorname{BrBialg} G$ where $G$ is the inverse of $F$. We have

$$
J_{\text {Bialg }}^{\prime} \circ \operatorname{Bialg} F \circ \operatorname{Bialg} G=\operatorname{BrBialg} F \circ J_{\text {Bialg }} \circ \operatorname{Bialg} G=\operatorname{BrBialg} F \circ \operatorname{BrBialg} G \circ J_{\text {Bialg }}^{\prime}=J_{\text {Bialg }}^{\prime}
$$

and hence Bialg $F \circ \operatorname{Bialg} G=\operatorname{Id}_{\text {Bialg }_{\mathcal{M}^{\prime}}}$ (as $J_{\text {Bialg }}^{\prime}$ is faithful and trivially injective on objects). Similarly Bialg $G \circ \operatorname{Bialg} F=\operatorname{Id}_{\text {Bialg }_{\mathcal{M}}}$. Hence $\operatorname{Bialg} F$ is a category isomorphism.

If $F$ is conservative, then, by Proposition 2.5 and Proposition 4.4, we have that $\operatorname{BrBialg} F$ o $J_{\text {Bialg }}$ is conservative. Hence $J_{\text {Bialg }}^{\prime} \circ \operatorname{Bialg} F$ is conservative. From this we get that $\operatorname{Bialg} F$ is conservative.

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