Scalar field instability in de Sitter space-time

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Abstract

Starting from the equation of motion of the quantum operator of a real scalar field φ in de Sitter space-time, a simple differential equation is derived which describes the evolution of quantum fluctuations $\langle \varphi^2 \rangle$ of this field. Full de Sitter invariance is assumed and no *ad hoc* infrared cutoff is introduced. This equation is solved explicitly and in massive case our result agrees with the standard one. In massless case the large time behavior of our solution differs by sign from the expression found in earlier papers. A possible cause of discrepancy may be a spontaneous breaking of de Sitter invariance.

I. INTRODUCTION

Since de Sitter found his exact solution to Einstein's equations in 1917, this model of space-time has been intensively studied because of its maximal symmetry. Moreover, according to inflationary scenarios the universe was in (quasi) de Sitter state at some part of its history. Most probably inflation was induced by a scalar field (inflaton) and this fact amplifies the interest to the study of scalar fields in de Sitter space. Recent observations of Supernovae of type Ia [1], implying that present universe is in an accelerated stage, refreshed the interest to de Sitter space. A small observed value of vacuum (or dark) energy may be possibly explained by adjustment mechanism realized by a scalar field (for a recent review see ref. [2]), or by quantum instability of de Sitter space [3].

On the other hand, considerable progress in quantum field theory in curved space-time has been achieved in the well suited arena of de Sitter space-time, where calculations can be carried out explicitly with different techniques [4]. In particular, scalar fields were studied in detail because of their simple properties.

Massless minimally coupled scalar field has the same description of physical modes as gravitons in transverse-traceless gauge. This relation allows to model infrared problems of quantum gravity [5], which may generate space-time instability [6], in terms of scalar fields. Appearance of instability means that de Sitter invariance may be spontaneously broken, and thus the true vacuum is not invariant under the full symmetry group [7]; this in turn may lead to the adoption of vacua which are invariant only under a subgroup of the de Sitter group [8].

In this work we calculate vacuum expectation value of the operator of scalar field squared, $\langle \varphi^2 \rangle$, in de Sitter background, both in massive and massless case. Though this quantity has been calculated in several papers there is still some confusion about its infrared properties. Instead of relying on rather *ad hoc* infrared cut-off we have derived an equation governing evolution of $\langle \varphi^2 \rangle$ which is solely based on de Sitter invariance. We have found the standard expression for the massive case, while for massless field our result for large time differs by sign from the previously published ones.

This paper is organized as follows. In section II de Sitter geometry is reviewed. In section III field quantization in this space-time is discussed. In section IV a scalar field φ coupled to gravitation is considered. In section V an equation describing the quantum average of φ^2 is derived and its solutions are found both for massive and massless cases with an emphasis on the minimally coupled field. The conclusion is presented in section VI.

Below the following notations are used. An overdot means derivative with respect to time, the index "0" means time component of tensor or vector, a letter from the middle of the Latin alphabet, such as i, j, k, \ldots , means spatial components of tensor or vector. The system of units is $c = \hbar = k = 1$ and $m_{Pl}^2/8\pi = 1$.

II. DE SITTER SPACE-TIME

The equations of gravitational field (the Einstein's equations) with non-zero cosmological constant Λ , given the gravitational action functional $S_g = -\int d^4x \sqrt{-g}(R+2\Lambda)/2$, have the form:

$$R_{ab} - \frac{1}{2} R g_{ab} - \Lambda g_{ab} = T_{ab}, \qquad (1)$$

where R is the scalar curvature, g is the determinant of metric g_{ab} and T_{ab} is the energymomentum tensor of matter fields.

A very simple but non-trivial case is represented by space-times with a constant curvature, locally characterized by the condition $R_{abcd} = R(g_{ac} g_{bd} - g_{ad} g_{bc})/12$, which is equivalent to having a zero Weyl tensor C_{abcd} , indicating that the space-time is conformally flat.

These space-times can be viewed as empty spaces with $\Lambda = -R/4$ or as filled with a perfect fluid with the equation of state $\rho = -p$. The space with zero curvature R = 0is Minkowski space-time: it is flat four-dimensional hyperplane. The space with negative constant curvature is de Sitter space-time, whose topology is $\mathcal{R}^1 \times S^3$, while the one with R >0 is anti-de Sitter space-time. The former can be viewed as a four-dimensional hyperboloid embedded in five-dimensional Riemannian space: in cartesian five-dimensional coordinates x^a , with the metric $\eta_{ab} = \text{diag}(+1, -1, -1, -1, -1)$, its points must satisfy the relation:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -H^{-2}.$$
 (2)

Different coordinate systems may be used to describe de Sitter space-time, this freedom corresponds to different choices of Cauchy hyper-surfaces of constant time.

In flat coordinates the metric can be written in the Friedmann–Robertson–Walker form:

$$ds^{2} = dt^{2} - a^{2}(t) d\mathbf{r}^{2}, \quad \text{with} \quad a(t) = \exp(Ht), \quad H = \sqrt{\Lambda/3}.$$
 (3)

The isometry group of de Sitter space-time is SO(4, 1), that is the Lorentz group in a five-dimensional universe with four spatial dimensions.

Actually SO(4, 1) is only one of the four disconnected parts of the full symmetry group O(4, 1) [7], the other parts are obtained reflecting time, space, and both of them. The antipodal transformation which is an element of the second part sends a point x to its antipodal point \overline{x} , lying in the opposite side of the whole de Sitter space-time (for a discussion of antipodal points, topology and symmetry see [9, 10]). If $X^a(x)$ is a five-vector locating x in the metric η_{ab} , then \overline{x} is located by $-X^a(x)$.

Matter fields in general relativity are analyzed under some energy conditions, among which we mention the assumed weak energy condition, stating that at each space-time point x there must be $T_{ab} W^a W^b \ge 0$ for any time-like vector $W \in \mathcal{T}_x$ and by continuity for any null vector $W \in \mathcal{T}_x$ too (\mathcal{T}_x being the space of tangent vectors at x) [11].

In de Sitter space-time the Ricci tensor and the scalar curvature have a very simple form:

$$R_{ab} = -3H^2 g_{ab}, \qquad R = -12H^2. \tag{4}$$

From the Einstein equations follows the conservation law for the energy-momentum tensor that can be written, in the chosen metric, as:

$$\nabla_a T^a_{\ 0} = \dot{\varrho} + 3H(\varrho + p) = 0. \tag{5}$$

Thus in the exact de Sitter state, implying $\rho + p = 0$, energy density ρ must be constant.

III. QUANTUM FIELD THEORY IN DE SITTER SPACE.

The choice among different possible quantization procedures must both take into account their consequences for symmetry breaking, anomalies and zero point energy, and the representations of the symmetry group in question. Scalar representations of de Sitter group can be divided into the principal series $(m^2 \ge 4H^2)$, the complementary series $(0 < m^2 < 4H^2)$, and the discrete series $(m^2 = 0$ is the only interesting case) [12].

For massive scalar fields, corresponding to the first two cases, several different procedures are known which lead to a well defined Fock space. Massless case is more cumbersome, because, as it has been shown by several authors [7], there is no Fock space corresponding to a de Sitter invariant vacuum. This consideration leads to the Gupta–Bleuler quantization (naturally extensible to the massive case) in order to avoid such problem and keep intact the full de Sitter invariance [13].

A rigorous covariant quantization is achieved in the global approach to quantum field theory [14], where the starting point is the local symmetry and de Sitter group is considered as a charge group generated by the Killing vector fields, arising in the treatment of the gravitational field as an external field in the frameworks of the background field method.

An axiomatic approach, generalizing to curved space the axiomatic methods of Minkowskian quantum field theory [15], impose some basic requirements on the two-point functions which are necessary to obtain a well defined quantum theory [16]. Reasonable general principles seem to be covariance, locality, and positive definiteness of the two-point functions, as hold in Minkowskian case. The spectral condition, instead, cannot be literally translated [17], and one must adopt a weaker one, based on the property that the two-point function has to be a boundary value of an analytic function with the correct $i\epsilon$ prescription. The latter is also a generalization of the Hadamard condition [18]. Being formulated only for free field theory, this condition may be considered as a consequence of the Einstein's equivalence principle. It assures that Green functions have the same singularities as in flat space.

Causal problems arise from antipodal points, where Green functions are also singular, although singularities occurring at those points are unobservable, since such points are always separated by a horizon [19]. Instead of implying global condition on the propagators, one can avoid this problem in the flat coordinate system, where a half of the space, containing antipodal points, is excluded. Below the standard quantization procedure will be assumed, with the usual commutation relation between the scalar field and its conjugate momentum, which can be recast into the usual commutation relations among creation and annihilation operators, provided the Wronskian condition on the mode functions holds [20]. There is a close resemblance to the flat space-time case, the only difference being in the explicit form of the mode functions. In de Sitter space the latter are not plane waves, but a linear combination of the Hankel functions, with two constants coefficients, determined by the Wronskian condition plus the choice of the vacuum state.

Actually there is a one-parameter family of de Sitter invariant vacua [7], called α -vacua. The best known and often used one is determined by the Bunch–Davies prescription, leading to the usual Euclidean vacuum in the limiting case of flat space-time. Since the discrete PT (parity and time reversal) symmetry is a subgroup of de Sitter group, the condition of invariance with respect to the total group automatically makes the free vacuum propagator PT invariant. Supplementing this choice with the above mentioned Hadamard condition leads to the Euclidean or Bunch–Davies vacuum [7, 20].

It can be shown [21] that among α -vacua in planar coordinates the Euclidean vacuum is naturally interpreted as the state with no particles on the horizon at $t = \infty$ [22]. Another feature of the Euclidean vacuum is that α -vacua in an interacting theory are ill-defined [23], but can be regarded at least approximately as excited states in the Euclidean vacuum [24]. Renormalization considerations also lead to the conclusion that the Euclidean vacuum is the only physically acceptable state [25].

One more indication of the physical relevance of the Euclidean vacuum comes from the study of the behavior of the energy-momentum tensor. It has been proved [26] that the renormalized value of this tensor for all ultraviolet and infrared physically allowed initial states asymptotically approaches the one obtained in the Bunch–Davies state. The renormalized value has been calculated by the adiabatic regularization. Although this scheme is not covariant, in the Friedmann–Robertson–Walker space-times it is equivalent to the covariant point-splitting procedure.

Through back-reaction of the expectation value of the energy-momentum tensor on grav-

ity its quantum fluctuations could influence the metric. Since for massless free fields T_{ab} contains only derivatives of the field, it is not sensitive to long-wavelength modes. That is why in non-interacting theory it does not manifest breaking of the full de Sitter group. Purely gravitational back-reaction mechanism, instead, could drive changes in the metric, terminating the de Sitter phase: this might be a viable way to exit inflationary era (see [27, 28] and references therein).

It was proved [7] that in massless case there is no Fock space for a full de Sitter invariant propagator, because the calculation in the limit $m \to 0$ are performed through an infrared cutoff, which excludes zero modes from the integration over the phase space and hence the set of mode functions is not complete. The nature of those cutoffs is not clear. It may be understood through reasonable physical arguments, for example from the size of the initial wedge that inflated [29]. Infrared properties of massless self-interacting scalar field have been considered in [30].

Particle interpretation requires a Fock space, that is why some authors [8],[31] postulated less invariant vacua, for example compact O(4) invariant vacua or non-compact E(3) invariant vacua. In the former case the compactness allows to construct new zero modes in order to have a complete set of mode functions. In the E(3) invariant vacua this is not possible. For this reason such vacua are considered as idealizations of physical vacua.

Instability of de Sitter space-time in the presence of massless minimally coupled scalar field forces one to rely on the Allen–Folacci vacuum [31], which is not invariant under the full symmetry group. This may be seen, as in the massive case, through the energy-momentum tensor: its renormalized value in the Allen–Folacci vacuum is an attractor solution for any other vacuum choice in the massless case [26]. In the non-invariant vacua the expectation value of the energy-momentum tensor becomes different from the one obtained in the Euclidean vacuum. It can also depend on time [8], leading to possible changes in the metric through the back-reaction mechanism.

IV. SCALAR FIELD

A scalar field φ with mass m and coupling ξ to the curvature scalar, is described by the action:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[\nabla_a \varphi \nabla^a \varphi - m^2 \varphi^2 - \xi R \varphi^2 \right].$$
(6)

which leads to the following equation of motion

$$\nabla_a \nabla^a \varphi + m^2 \varphi + \xi R \varphi = 0.$$
⁽⁷⁾

In metric (3) the equation of motion reads:

$$\ddot{\varphi} - \left(\Delta/a^2\right)\varphi + 3H\dot{\varphi} + m^2\varphi + \xi R\varphi = 0, \tag{8}$$

where \triangle means the tridimensional Laplacian operator in flat space.

The energy-momentum tensor of the scalar field φ is defined as

$$T^{ab} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{ab}}.$$
(9)

Remembering that $g_{ab} g^{bc} = \delta_a{}^c$ and the matrix property tr $\ln g_{ab} = \ln \det g_{ab}$ and taking the infinitesimal variation of the former one can obtain the following useful identities:

$$\frac{\delta\sqrt{-g}}{\delta g_{ab}} = \frac{1}{2}\sqrt{-g} g^{ab}, \qquad \frac{\delta g^{cd}}{\delta g_{ab}} = -g^{ac} g^{bd}, \qquad g^{cd} \frac{\delta R_{cd}}{\delta g_{ab}} = \nabla^a \nabla^b - g^{ab} \nabla^e \nabla_e.$$

At this point the expression for the operator of the energy-momentum tensor can be easily derived:

$$T^{ab} = -\frac{1}{2} g^{ab} \left(\nabla_c \varphi \,\nabla^c \varphi - m^2 \varphi^2 \right) + \nabla^a \varphi \,\nabla^b \varphi - \xi \left(R^{ab} - \frac{1}{2} g^{ab} R \right) \,\varphi^2 - \xi \left(\nabla^a \,\nabla^b - g^{ab} \,\nabla_c \,\nabla^c \right) \varphi^2.$$
(10)

As expected this tensor is covariantly conserved:

$$\nabla_a T^a_{\ b} = 0. \tag{11}$$

Let us note that the conservation is realized only on solutions of the equation of motion (7).

It's straightforward to write the operator expression for the trace of T_{ab} :

$$T^{a}_{\ a} = -\nabla_{a} \varphi \nabla^{a} \varphi + 2 m^{2} \varphi^{2} + \xi R \varphi^{2} + 3 \xi g^{ab} \nabla_{a} \nabla_{b} \varphi^{2}.$$
(12)

In the special case of a massless conformally coupled scalar field $(m = 0, \xi = 1/6)$ this expression vanishes on solutions of the equation of motion (7). However, since the product of two operators in coinciding space-time points is ill-defined a regularization prescription is necessary. The latter may break the classical identities and lead to well known quantum anomalies. In particular, the vacuum expectation value of the trace of T_{ab} possesses such an anomaly called the *trace anomaly* or *conformal anomaly*. Independently on renormalization technique the resulting expression for the anomaly in four dimensions is [14, 20]:

$$\langle T^a_{\ a} \rangle_{ren} = -\frac{1}{2880 \,\pi^2} \left(R_{abcd} \, R^{abcd} - R_{ab} \, R^{ab} - \Box \, R \right).$$
 (13)

In de Sitter space it becomes:

$$\langle T^a_{\ a} \rangle_{ren} = \frac{H^4}{240\pi^2}.\tag{14}$$

The complete expression for the regularized vacuum expectation value of T_{ab} in de Sitter space for arbitrary m and ξ was calculated in ref. [32, 33] and reads:

$$\langle T_{ab} \rangle_{ren} = \frac{g_{ab}}{64\pi^2} \left\{ m^2 \left[m^2 - \left(\xi - \frac{1}{6}\right) R \right] \left[\psi \left(\frac{3}{2} + \nu\right) + \psi \left(\frac{3}{2} - \nu\right) - \ln \left(\frac{12m^2}{|R|}\right) \right] - \frac{1}{2} \left(\xi - \frac{1}{6}\right)^2 R^2 + \frac{R^2}{2160} - m^2 \left(\xi - \frac{1}{6}\right) R - \frac{m^2 R}{18} \right\},$$

$$(15)$$

where ψ is the logarithmic derivative of the Gamma-function and

$$\nu^2 = \frac{9}{4} + 12\left(\frac{m^2}{R} - \xi\right). \tag{16}$$

The result (15) is obtained with a covariant regularization procedure which respects conservation condition (11) and de Sitter symmetry according to which $T_{ab} \sim g_{ab}$.

In the conformal case, m = 0 and $\xi = 1/6$, the trace of (15) coincides, as expected, with (14). In the limit of $m \to 0$ and minimal coupling to gravity, $\xi = 0$, the energy-momentum tensor becomes:

$$T_{ab} = \frac{g_{ab}H^4}{\pi^2} \left(\frac{3}{32} + \frac{1}{960} - \frac{1}{32}\right).$$
 (17)

The first term in the brackets comes from $m^2 R \psi(3/2 - \nu)/6$ in eq. (15), simply using the definition $\psi(z) = d(\ln \Gamma(z))/dz$ and the identity $\Gamma(z+1) = z\Gamma(z)$ in the limit $z \to 0$. It is the standard non-anomalous term in the energy density at m = 0. The other two terms come from anomalous contributions: the first of them is purely anomalous one which survives in conformal limit, while the second disappears only for $\xi = 1/6$ and is non-vanishing for m = 0 and $\xi = 0$.

V. CALCULATION OF QUANTUM AVERAGE OF φ^2 IN DE SITTER SPACE.

The quantity $\langle \varphi^2 \rangle$, that is the quantum average value of the product of the scalar field operators $\hat{\varphi}$ in coincident space-time points, plays a primary role in quantum field theory. There are different methods to obtain its expression, which is divergent without an appropriate renormalization of the field operator $\hat{\varphi}$.

Such methods [34] comprise the use of zeta function techniques, point-splitting regularization, dimensional regularization, etc. giving rise in general to different renormalized value of $\langle \varphi^2 \rangle$. The usual expansion of the field operator $\hat{\varphi}$ through its mode functions, that is through the Hankel functions in de Sitter space, allows to find an expression for $\langle \varphi^2 \rangle$ in terms of Digamma and Hypergeometric functions, which gives the result [35] (originally obtained with point-splitting regularization):

$$\langle \varphi^2 \rangle = \frac{1}{16\pi^2} \left\{ m^2 \ln\left(\frac{\mu^2}{12m^2}\right) + \left[m^2 - \left(\xi - \frac{1}{6}\right)R\right] \left[\ln\left(-\frac{R}{\mu^2}\right) + \psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right)\right] \right\}$$
(18)

The renormalization mass μ is to be chosen in such a way that $\langle \varphi^2 \rangle$ vanishes in flat space limit i.e. when $R \to 0$. In massive case it can be achieved with $\mu^2 = 12m^2$. Indeed, when m/H is large then $\nu \approx im/H$ and

$$\psi(3/2 + \nu) + \psi(3/2 - \nu) \approx 2\ln(m/H).$$
 (19)

This term cancels down with $\ln(-R/\mu^2)$ if $\mu^2 = 12m^2$. In massless case and vanishing R $\langle \varphi^2 \rangle \to 0$ for arbitrary value of μ^2 and almost any ξ . However, for $\xi = 0$ the limit $m \to 0$ is singular and

$$\langle \varphi^2 \rangle \to \frac{3H^4}{8\pi^2 m^2}.$$
 (20)

In this case the transition to H = 0 is non-trivial, if first $m \to 0$.

In this work we will develop a different approach to calculation of the quantum average value of φ^2 in de Sitter space. We will derive an ordinary differential equation governing the evolution of $f(t) \equiv \langle \varphi^2 \rangle$. To this end we will use the equation of motion for the quantum operator $\hat{\varphi}$ (8) and the commutation of the quantum averaging and differentiation. This approach is similar to that indicated in ref. [36]. The scalar field $\hat{\varphi}$ is an operator-valued distribution, and the rule for derivatives of distributions allows to use the standard rule for derivatives of the product of functions; in our case it is $\hat{\varphi} \cdot \hat{\varphi}$. In this way we obtain:

$$g^{ab} \nabla_a \nabla_b f = g^{ab} \nabla_a \langle 2\varphi \nabla_b \varphi \rangle = \langle 2 g^{ab} \nabla_a \varphi \nabla_b \varphi + 2\varphi g^{ab} \nabla_a \nabla_b \varphi \rangle.$$
(21)

This identity permits to write the following equation for f:

$$(\nabla_a \nabla^a + m^2 + \xi R)f = 2 \langle \nabla_a \varphi \nabla^a \varphi \rangle - m^2 f - \xi R f.$$
(22)

To make this equation meaningful we need to calculate the first term in the r.h.s. of (22) in terms of known quantities. Expressing it through the trace of the energy-momentum tensor (12), and assuming unbroken de Sitter invariance, one can get a solvable closed equation for f. After straightforward algebra one obtains:

$$(1 - 6\xi)\nabla_a \nabla^a f - 2m^2 f = -2\langle T^a_a \rangle = -8\langle \varrho \rangle, \qquad (23)$$

where $\langle \varrho \rangle$ is the vacuum energy density of quantum fluctuations of φ . If de Sitter invariance is unbroken then $\langle \varrho \rangle = \langle T^a_{\ a} \rangle_{ren}/4$ and the trace can be trivially calculated from eq. (15) for arbitrary values of m and ξ .

In homogeneous background f should depend only on time and the equation can be rewritten as:

$$(1 - 6\xi)\left(\ddot{f} + 3H\dot{f}\right) - 2m^2 f = -8\langle \varrho \rangle.$$

$$(24)$$

Because of de Sitter invariance $\langle \varrho \rangle$ = constant, and this equation can be solved explicitly. In the case of minimal coupling to gravity, i.e. $\xi = 0$, the solution is:

$$f = C_1 \exp\left[\left(-\frac{3}{2}H - \sqrt{\frac{9}{4}H^2 + 2m^2}\right)t\right] + C_2 \exp\left[\left(-\frac{3}{2}H + \sqrt{\frac{9}{4}H^2 + 2m^2}\right)t\right] + \frac{4\langle\varrho\rangle}{m^2},$$
(25)

where C_1 and C_2 are some numerical constants. Surprisingly the solution is time dependent if any of $C_{1,2}$ is non-zero, moreover it contains an exponentially rising term, proportional to C_2 . The solution is constant, $f = 4\langle \rho \rangle / m^2$, independently of initial conditions only for conformal coupling, $\xi = 1/6$. It is unclear if time dependence or, at least, exponential rise can be killed by an appropriate choice of physically justified initial conditions. The limit of zero mass indicates the opposite, that C_2 should be non-zero (see the end of this Section). Possibly fast (exponential) evolution of $\langle \varphi^2 \rangle$ can be helpful for solution of the well known problem of vacuum energy.

In the case $9H^2/4 \gg 2m^2$ and $t < 3H/2m^2$ the solution is

$$f = C_1 \exp\left(-3\,H\,t\right) + C_2 \left(1 + \frac{2m^2}{3H}\,t\right) + \frac{4\langle\varrho\rangle}{m^2}.$$
(26)

If C_1 and C_2 are not singular at m = 0, the dominant term for $m \to 0$ is

$$\langle \varphi^2 \rangle = \frac{4\langle \varrho \rangle}{m^2}.$$
 (27)

It is formally the same as that found for $\xi = 1/6$ above but one should keep in mind that $\langle \rho \rangle$ depends both on m and ξ . Expression (27) would agree with the previously established one [35, 37] if we substitute for the energy density of quantum fluctuations of φ the smallmass limit of the result [32, 33]: $\langle \varrho \rangle = 3H^4/(32\pi^2)$, i.e. only the first term in eq. (17), while the anomalous contributions are disregarded. An account of the anomaly changes the numerical coefficient and $\langle \varphi^2 \rangle = 3H^4/8\pi^2m^2$ turns into $\langle \varphi^2 \rangle = 61H^4/240\pi^2m^2$.

However, it may be not as simple as that because, taken as it is, eq. (24) is not consistent in conformal limit, $m = 0, \xi = 1/6$. Indeed, in this limit the l.h.s. of this equation vanishes, while the r.h.s. is non-zero due to trace anomaly. A possible way out is a singularity in $\langle \varphi^2 \rangle$ at m = 0 or $\xi = 1/6$, as indicated by eqs. (25),(27) according to which $\langle \varphi^2 \rangle \sim \langle \varrho \rangle / m^2$. One can also solve eq. (23) directly for m = 0 (and $\xi = 0$):

$$f = C_1 + C_2 e^{-3Ht} - \frac{8}{3} \frac{\langle \varrho \rangle}{H} t,$$
 (28)

which gives the late time behavior

$$f \approx -\frac{8}{3} \frac{\langle \varrho \rangle}{H} t. \tag{29}$$

This solution almost coincides with the earlier found one [35, 37] - the absolute value is the same if anomaly is not included but, surprisingly, the sign is opposite. The same solution (28) can be obtained from expression (26) in the limit of zero mass under condition that the solution is not singular at m = 0. To realize that the coefficient C_2 should be non-vanishing and singular in m: $C_2 = C_{20} - 4\langle \varrho \rangle / m^2$, where C_{20} is a non-singular constant at m = 0. With m tending to zero the last term cancels out the singular in m part in eq. (26) and the remaining one coincides with (29).

We see that for small mass and large time the average value of φ^2 becomes negative. It looks strange because φ^2 is a positive definite operator. Still it may be true because the vacuum expectation value of this operator needs to be renormalized and after (infinite) renormalization can become negative.

VI. CONCLUSION

The case of spontaneous symmetry breaking in de Sitter space-time is similar to that of two-dimensional flat space-time with a broken Lorentz invariance [38]. Although it has been proposed in ref. [39], on the basis of functional methods, that in de Sitter space-time symmetries are always dynamically restored, this is probably not the case.

The occurrence of this phenomenon may be connected with the choice of the vacuum state [40] and related to this choice character of the infrared behavior in the massless limit. It also relies on the assumptions imposed on admissible physical states and theoretical ambiguities in quantum field theory in curved space. The properties of the admitted states, as e.g. behavior at large distances, zero modes, or possible interactions, determine the character of infrared divergences or directly the symmetry breaking [41]. In any case infrared divergences

could quite naturally lead to an instability. Possible manifestation of this phenomenon may be realized by quantum gravity effects in de Sitter space. They may lead to quantum instability of this state [42, 43]. In other words, due to quantum effects de Sitter solution could tend to some other solutions of the Einstein equations loosing de Sitter symmetry [44].

In this paper we have derived and solved a differential equation which describes the behavior of the vacuum expectation value of the quantum operator product φ^2 in de Sitter space. We have found that generally the solution is time dependent, though for non-zero mass a very special constant solution exits. However this solution cannot be continuously transformed to massless limit.

Our result is not based on any *ad hoc* infrared cutoff and it agrees with the standard one in the case of small but non-vanishing mass. This agreement is an indication for the validity of our approach. In the massless case, however, we obtain a surprising difference the magnitude of $\langle \varphi^2 \rangle$ is the same as found in the earlier papers but the sign is opposite. The large time behavior of this solution does not depend upon initial conditions.

We would like to stress that the result is obtained under assumption of de Sitter symmetry for quantum expectation values, which demands $\rho = \text{constant}$ and $\langle \rho \rangle + \langle p \rangle = 0$. The numerical coefficient in eq. (29) is determined by the expression [32, 33] for $\langle \rho \rangle$. Possibly time dependence of quantum fluctuations of scalar field is an indication of spontaneous symmetry breaking.

The minus sign in the massless case with respect to other calculations could be a signal of a quantum infrared anomaly in de Sitter space. On the other hand, this discrepancy could emerge as an indication of some problem in the limiting procedure to get zero mass from the massive case. This situation has a close resemblance to the appearance of the conformal anomaly in the massless limit of the renormalized value of the trace of the energy-momentum tensor (12). After the preparation of this work we became aware of a recent paper about gravitons in de Sitter space [45], where the renormalized value of the energy-momentum tensor has the sign opposite to that in the literature.

Further investigations may clarify the situation, in particular a step by step comparison with results which can be obtained in the Euclidean sector which, although not shown here, would present the same anomalous behavior in the massless case. An investigation of Euclidean case may be helpful in understanding where a possible breakdown of the standard behavior can occur.

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