# WEDGE PRODUCTS AND COTENSOR COALGEBRAS IN MONOIDAL CATEGORIES

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ABSTRACT. The construction of the cotensor coalgebra for an "abelian monoidal" category  $\mathcal{M}$  which is also cocomplete, complete and AB5, was performed in [A. Ardizzoni, C. Menini and D. Ştefan, *Cotensor Coalgebras in Monoidal Categories*, Comm. Algebra, to appear]. It was also proved that this coalgebra satisfies a meaningful universal property which resembles the classical one. Here the lack of the coradical filtration for a coalgebra E in  $\mathcal{M}$  is filled by considering a direct limit  $\tilde{D}$  of a filtration consisting of wedge products of a subcoalgebra D of E. The main aim of this paper is to characterize hereditary coalgebras  $\tilde{D}$ , where D is a coseparable coalgebra in  $\mathcal{M}$ , by means of a cotensor coalgebra: more precisely, we prove that, under suitable assumptions,  $\tilde{D}$  is hereditary if and only if it is formally smooth if and only if it is the cotensor coalgebra  $T_D^c(D \wedge_E D/D)$  if and only if it is a cotensor coalgebra  $T_D^c(N)$ , where N is a certain D-bicomodule in  $\mathcal{M}$ . Because of our choice, even when we apply our results in the category of vector spaces, new results are obtained.

## INTRODUCTION

Let C be a coalgebra over a field K and let M be a C-bicomodule. The cotensor coalgebra  $T_C^c(M)$  was introduced by Nichols in [Ni] and it appears as a main step in the classification of finite dimensional Hopf algebras problem (see, e.g., [AG] and [AS]). In [Ch] the relation between quiver coalgebras and hereditary coalgebras is investigated. In [JLMS], hereditary coalgebras with coseparable coradical are characterized by means of a suitable cotensor coalgebra: more precisely, the authors prove that a coalgebra C with coseparable coradical D is hereditary if and only if it is formally smooth if and only if it is a cotensor coalgebra  $T_D^c(N)$ , where N is a certain D-bicomodule. The main aim of this paper is to prove Theorem 3.23 which establishes an analogous result inside the framework of monoidal categories. This is pursued using the notion of formally smooth coalgebra for "abelian monoidal" categories performed in [AMS2] and using the construction of the cotensor coalgebra for "abelian monoidal" categories performed in [AMS1].

The basic point when dealing with coalgebras in monoidal categories is that there is no notion of coradical. The idea then is to take a subcoalgebra D of a coalgebra C and to consider the coalgebra  $\tilde{D}$  which is the direct limit of the iterated wedge powers of D in E. Then the coalgebra D acts, in a certain sense, as the coradical of  $\tilde{D}$ . Thus, because of our choice, even when we apply our results (e.g. Theorem 3.16) in the category of vector spaces, new results are obtained. It is also interesting to point out that, working in this wider context, we had to develop some properties of the wedge product that have an intrinsic interest. Due to the width of our setting, many technical results were needed. To enable an easier reading, we decided to postpone a number of them in two appendices that can be found at the end of the paper.

**Notations.** Let  $[(X, i_X)]$  be a subobject of an object E in an abelian category  $\mathcal{M}$ , where  $i_X = i_X^E : X \hookrightarrow E$  is a monomorphism and  $[(X, i_X)]$  is the associated equivalence class. By abuse of language, we will say that  $(X, i_X)$  is a subobject of E and we will write  $(X, i_X) = (Y, i_Y)$  to mean that  $(Y, i_Y) \in [(X, i_X)]$ . The same convention applies to cokernels. If  $(X, i_X)$  is a subobject

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of *E* then we will write  $(E/X, p_X) = \operatorname{Coker}(i_X)$ , where  $p_X = p_X^E : E \to E/X$ . Let  $(X_1, i_{X_1}^{Y_1})$  be a subobject of  $Y_1$  and let  $(X_2, i_{X_2}^{Y_2})$  be a subobject of  $Y_2$ . Let  $x : X_1 \to X_2$  and  $y : Y_1 \to Y_2$  be morphisms such that  $y \circ i_{X_1}^{Y_1} = i_{X_2}^{Y_2} \circ x$ . Then there exists a unique morphism, which we denote by  $y/x = \frac{y}{x} : Y_1/X_1 \to Y_2/X_2$ , such that  $\frac{y}{x} \circ p_{X_1}^{Y_1} = p_{X_2}^{Y_2} \circ y$ :



## 1. Monoidal Categories

1.1. A monoidal category means a category  $\mathcal{M}$  that is endowed with a functor  $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ , an object  $\mathbf{1} \in \mathcal{M}$  and functorial isomorphisms:  $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), l_X : \mathbf{1} \otimes X \to X$ and  $r_X : X \otimes \mathbf{1} \to X$ . The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom*, that is the following diagram



is commutative, for every U, V, W, X in  $\mathcal{M}$ . The morphisms l and r are called the *unit constraints* and they are assumed to satisfy the *Triangle Axiom*, i.e. the following diagram



is commutative. The object 1 is called the *unit* of  $\mathcal{M}$ . For details on monoidal categories we refer to [Ka, Chapter XI] and [Maj]. A monoidal category is called *strict* if the associativity constraint and unit constraints are the corresponding identity morphisms.

1.2. As it is noticed in [Maj, p. 420], the Pentagon Axiom solves the consistency problem that appears because there are two ways to go from  $((U \otimes V) \otimes W) \otimes X$  to  $U \otimes (V \otimes (W \otimes X))$ . The coherence theorem, due to S. Mac Lane, solves the similar problem for the tensor product of an arbitrary number of objects in  $\mathcal{M}$ . Accordingly with this theorem, we can always omit all brackets and simply write  $X_1 \otimes \cdots \otimes X_n$  for any object obtained from  $X_1, \ldots, X_n$  by using  $\otimes$  and brackets. Also as a consequence of the coherence theorem, the morphisms a, l, r take care of themselves, so they can be omitted in any computation involving morphisms in  $\mathcal{M}$ .

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. For more details, see [AMS2].

# 2. Wedge and cotensor products

We quote from [AMS1, 2.4]

DEFINITION 2.1. A monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$  will be called an *abelian monoidal category* if:

- (1)  $\mathcal{M}$  is an abelian category
- (2) both the functors  $X \otimes (-) : \mathcal{M} \to \mathcal{M}$  and  $(-) \otimes X : \mathcal{M} \to \mathcal{M}$  are additive and left exact, for every object  $X \in \mathcal{M}$ .

2.2. Let *E* be a coalgebra in an abelian monoidal category  $\mathcal{M}$ . Let us recall, (see [Mo, page 60]), the definition of wedge of two subobjects X, Y of *E* in  $\mathcal{M}$ :

$$X \wedge_E Y, i_{X \wedge_E Y}^E) := Ker[(p_X \otimes p_Y) \circ \triangle_E],$$

where  $p_X : E \to E/X$  and  $p_Y : E \to E/Y$  are the canonical quotient maps. In particular we have the following exact sequence:

$$0 \longrightarrow X \wedge_E Y \xrightarrow{i_{X \wedge_E Y}^E} E \xrightarrow{(p_X \otimes p_Y) \circ \triangle_E} E/X \otimes E/Y.$$

Consider the following commutative diagrams in  $\mathcal{M}$ 



where e is a coalgebra homomorphism. Then there is a unique morphism  $x \wedge_e y : X_1 \wedge_{E_1} Y_1 \to X_2 \wedge_{E_2} Y_2$  such that the following diagram

commutes. In fact we have

$$(p_{X_2}^{E_2} \otimes p_{Y_2}^{E_2}) \circ \Delta_{E_2} \circ e \circ i_{X_1 \wedge E_1}^{E_1} Y_1$$
  
=  $(p_{X_2}^{E_2} \otimes p_{Y_2}^{E_2}) \circ (e \otimes e) \circ \Delta_{E_1} \circ i_{X_1 \wedge E_1}^{E_1} Y_1$   
=  $(\frac{e}{x} \otimes \frac{e}{y}) \circ (p_{X_1}^{E_1} \otimes p_{Y_1}^{E_1}) \circ \Delta_{E_1} \circ i_{X_1 \wedge E_1}^{E_1} Y_1 = 0$ 

so that, since  $(X_2 \wedge_{E_2} Y_2, i_{X_2 \wedge_{E_2} Y_2}^{E_2})$  is the kernel of  $(p_{X_2}^{E_2} \otimes p_{Y_2}^{E_2}) \circ \Delta_{E_2}$ , we conclude. LEMMA 2.3. Consider the following commutative diagrams in  $\mathcal{M}$ 



where e and e' are coalgebra homomorphisms. Then we have

(1) 
$$(x' \wedge_{e'} y') \circ (x \wedge_{e} y) = (x'x \wedge_{e'e} y'y)$$

*Proof.* : straightforward.

2.4. Let  $\mathcal{M}$  be an abelian monoidal category and let E be a coalgebra in  $(\mathcal{M}, \otimes, \mathbf{1})$ . Given a right Ebicomodule  $(V, \rho_V^r)$  and a left E-comodule  $(W, \rho_W^l)$ , their cotensor product over E in  $\mathcal{M}$  is defined to be the equalizer  $(V \Box_E W, \chi(V, W) = \chi_E(V, W))$  of the couple of morphism  $(\rho_V^r \otimes W, V \otimes \rho_W^l)$ :

$$0 \longrightarrow V \square_E W \xrightarrow{\chi_E(V,W)} V \otimes W \xrightarrow{\rho_V^r \otimes W} V \otimes E \otimes W$$

Since the tensor functors are left exact, in view of [AMS1, Proposition 1.3], then  $V \Box_E W$  is also a *E*-bicomodule, namely it is *E*-sub-bicomodule of  $V \otimes W$ , whenever V and W are *E*-bicomodules. Furthermore, in this case, the category  $({}^E\mathcal{M}^E, \Box_E, E)$  is still an abelian monoidal category; the associative and unit constraints are induced by the ones in  $\mathcal{M}$  (the proof is dual to [AMS2, Theorem 1.11]). Therefore, also using  $\Box_E$ , one can forget about brackets. Moreover the functors  $M \Box_E(-) : {}^E\mathcal{M} \to \mathcal{M}$  and  $(-) \Box_E M : \mathcal{M}^E \to \mathcal{M}$  are left exact for any  $M \in \mathcal{M}$ .

We will write  $\Box$  instead of  $\Box_C$ , whenever there is no danger of misunderstanding. One has the following result.

2.5. Let  $e: E_1 \to E_2$  be a coalgebra homomorphism in an abelian monoidal category  $\mathcal{M}$ . Let  $(V_1, \rho_{V_1}^{E_1})$  be a right  $E_1$ -comodule, let  $(W_1, {}^{E_1}\rho_{W_1})$  be a left  $E_1$ -comodule, let  $(V_2, \rho_{V_2}^{E_2})$  be a right  $E_2$ -comodule and let  $(W_2, {}^{E_2}\rho_{W_2})$  be a left  $E_2$ -comodule. Let  $v: V_1 \to V_2$  and  $w: W_1 \to W_2$  be  $E_2$ -comodule homomorphisms (where  $V_1$  and  $W_1$  are regarded as  $E_2$ -comodules via e). Then there is a unique morphism  $v \Box_e w: V_1 \Box_{E_1} W_1 \to V_2 \Box_{E_2} W_2$  such that the following diagram

commutes. In fact we have

$$\begin{aligned} (\rho_{V_2}^{E_2} \otimes W_2) \circ (v \otimes w) \circ \chi_{E_1}(V_1, W_1) \\ &= (v \otimes E_2 \otimes w) \circ (\rho_{V_1}^{E_2} \otimes W_1) \circ \chi_{E_1}(V_1, W_1) \\ &= (v \otimes E_2 \otimes w) \circ [(V_1 \otimes e)\rho_{V_1}^{E_1} \otimes W_1] \circ \chi_{E_1}(V_1, W_1) \\ &= (v \otimes e \otimes w) \circ (\rho_{V_1}^{E_1} \otimes W_1) \circ \chi_{E_1}(V_1, W_1) \\ &= (v \otimes e \otimes w) \circ [V_1 \otimes ^{E_1} \rho_{W_1}] \circ \chi_{E_1}(V_1, W_1) \\ &= (v \otimes E_2 \otimes w) \circ [V_1 \otimes (e \otimes W_1) \circ ^{E_1} \rho_{W_1}] \circ \chi_{E_1}(V_1, W_1) \\ &= (v \otimes E_2 \otimes w) \circ (V_1 \otimes ^{E_2} \rho_{W_1}) \circ \chi_{E_1}(V_1, W_1) \\ &= (V_2 \otimes ^{E_2} \rho_{W_2}) \circ (v \otimes w) \circ \chi_{E_1}(V_1, W_1) \end{aligned}$$

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so that, since  $(V_2 \Box_{E_2} W_2, \chi_{E_2}(V_2, W_2))$  is the equalizer of  $\rho_{V_2}^{E_2} \otimes W_2$  and  $V_2 \otimes E_2 \rho_{W_2}$ , we conclude. Note that if  $E_1 = E_2 = E$  and  $e = \mathrm{Id}_E$ , one has

$$v\Box_e w = v\Box_E w.$$

LEMMA 2.6. Let  $e: E_1 \to E_2$  and  $e': E_2 \to E_3$  be coalgebra homomorphisms in  $\mathcal{M}$ . Let

$$(V_1, \rho_{V_1}^{E_1}) \in \mathcal{M}^{E_1}, \quad (V_2, \rho_{V_2}^{E_2}) \in \mathcal{M}^{E_2}, \quad (V_3, \rho_{V_3}^{E_3}) \in \mathcal{M}^{E_3}, (W_1, {}^{E_1}\rho_{W_1}) \in {}^{E_1}\mathcal{M}, \quad (W_2, {}^{E_2}\rho_{W_2}) \in {}^{E_2}\mathcal{M}, \quad (W_3, {}^{E_3}\rho_{W_3}) \in {}^{E_3}\mathcal{M}.$$

Let  $v: V_1 \to V_2$  and  $w: W_1 \to W_2$  be  $E_2$ -comodule homomorphisms (where  $V_1$  and  $W_1$  are regarded as  $E_2$ -comodules via e) and let  $v': V_2 \to V_3$  and  $w': W_2 \to W_3$  be  $E_3$ -comodule homomorphisms (where  $V_2$  and  $W_2$  are regarded as  $E_3$ -comodules via e'). Then

(2) 
$$(v'\Box_{e'}w') \circ (v\Box_{e}w) = (v'v\Box_{e'e}w'w).$$

*Proof.* : straightforward.

PROPOSITION 2.7. Let  $\mathcal{M}$  be an abelian monoidal category, let E be a coalgebra in  $\mathcal{M}$  and let  $(V, \rho_V^E)$  be a right E-comodule. Then the morphism  $\rho_V^E : V \to V \otimes E$  factorizes to a unique morphism  $\overline{\rho}_V^E : V \to V \Box_E E$  such that  $\chi(V, E) \circ \overline{\rho}_V^E = \rho_V^E$ . Moreover  $\overline{\rho}_V^E$  is an isomorphism whose inverse is given by

$$(\overline{\rho}_V^E)^{-1} = r_V \circ (V \otimes \varepsilon_E) \circ \chi(V, E).$$

An analogous statement holds for a left E-comodule  $(W, \rho_W^l)$ .

*Proof.* Since  $(V, \rho_V^E)$  is a right *E*-comodule, by universal property of the equalizer, we get the existence of  $\overline{\rho}_V^E$ . Set  $s_V := r_V \circ (V \otimes \varepsilon_E) \circ \chi(V, E)$ . We have

$$s_V \circ \overline{\rho}_V^E = r_V \circ (V \otimes \varepsilon_E) \circ \chi(V, E) \circ \overline{\rho}_V^E = r_V \circ (V \otimes \varepsilon_E) \circ \rho_V^E = r_V \circ r_V^{-1} = \mathrm{Id}_V.$$

Moreover we have

$$\begin{split} \chi(V,E) \circ \overline{\rho}_{V}^{E} \circ s_{V} &= \rho_{V}^{E} \circ s_{V} \\ &= \rho_{V}^{E} \circ r_{V} \circ (V \otimes \varepsilon_{E}) \circ \chi(V,E) \\ &= r_{V \otimes E} \circ (\rho_{V}^{E} \otimes \mathbf{1}) \circ (V \otimes \varepsilon_{E}) \circ \chi(V,E) \\ &= r_{V \otimes E} \circ (V \otimes E \otimes \varepsilon_{E}) \circ (\rho_{V}^{E} \otimes E) \circ \chi(V,E) \\ &= r_{V \otimes E} \circ (V \otimes E \otimes \varepsilon_{E}) \circ (V \otimes \Delta_{E}) \circ \chi(V,E) \\ &= (V \otimes r_{E}) \circ [V \otimes (E \otimes \varepsilon_{E}) \circ \Delta_{E}] \circ \chi(V,E) \\ &= (V \otimes r_{E}) \circ (V \otimes r_{E}^{-1}) \circ \chi(V,E) = \chi(V,E). \end{split}$$

Since  $\chi(V, E)$  is a monomorphism, we get  $\overline{\rho}_V^E \circ s_V = \mathrm{Id}_{V \otimes E}$ .

LEMMA 2.8. Let  $\alpha : A \to E$  be a monomorphism which is a coalgebra homomorphism in an abelian monoidal category  $\mathcal{M}$ . Let  $(W, {}^{E}\rho_{W})$  be a left E-comodule and let  ${}^{A}\rho_{W} : W \to A \otimes W$  be a morphism such that

$${}^{E}\rho_{W} = (\alpha \otimes W) \circ {}^{A}\rho_{W}.$$

Then  $(W, {}^{A}\rho_{W})$  is a left A-comodule.

Let  ${}^{A}\overline{\rho}_{W}$  be the unique morphism such that  ${}^{A}\rho_{W} = \chi(A, W) \circ {}^{A}\overline{\rho}_{W}$ . Then  ${}^{A}\overline{\rho}_{W} : W \to A \Box_{A} W$  is a morphism of left *E*-comodules.

*Proof.* We have

$$(\alpha \otimes \alpha \otimes W) \circ (A \otimes {}^{A}\rho_{W}) \circ {}^{A}\rho_{W}$$

$$= [E \otimes (\alpha \otimes W)^{A}\rho_{W}] \circ (\alpha \otimes W) \circ {}^{A}\rho_{W}$$

$$= (E \otimes {}^{E}\rho_{W}) \circ {}^{E}\rho_{W}$$

$$= (\Delta_{E} \otimes W) \circ {}^{E}\rho_{W}$$

$$= (\Delta_{E} \otimes W) \circ (\alpha \otimes W) \circ {}^{A}\rho_{W} = (\alpha \otimes \alpha \otimes W) \circ (\Delta_{A} \otimes W) \circ {}^{A}\rho_{W}.$$

Since  $\alpha \otimes \alpha \otimes W$  is a monomorphism, we get

$$(A \otimes {}^{A}\rho_{W}) \circ {}^{A}\rho_{W} = (\Delta_{A} \otimes W) \circ {}^{A}\rho_{W}$$

Moreover we have

$$(\varepsilon_A \otimes W) \circ {}^A \rho_W = (\varepsilon_E \otimes W) \circ (\alpha \otimes W) \circ {}^A \rho_W = (\varepsilon_E \otimes W) \circ {}^E \rho_W = l_W.$$

Therefore  $(W, {}^{A}\rho_{W})$  is a left A-comodule. Let us prove that  ${}^{A}\overline{\rho}_{W}$  is a morphism of left E-comodules. We have:

$$(E \otimes \alpha \otimes W) \circ [E \otimes \chi(A, W)] \circ (E \otimes {}^{A}\overline{\rho}_{W}) \circ {}^{E}\rho_{W}$$

$$= (E \otimes \alpha \otimes W) \circ (E \otimes {}^{A}\rho_{W}) \circ {}^{E}\rho_{W}$$

$$= (E \otimes {}^{E}\rho_{W}) \circ {}^{E}\rho_{W}$$

$$= (\Delta_{E} \otimes W) \circ (\alpha \otimes W) \circ {}^{A}\rho_{W}$$

$$= (\Delta_{E} \otimes W) \circ (\alpha \otimes W) \circ \chi(A, W) \circ {}^{A}\overline{\rho}_{W}$$

$$= (E \otimes \alpha \otimes W) \circ ({}^{E}\rho_{A} \otimes W) \circ \chi(A, W) \circ {}^{A}\overline{\rho}_{W}$$

$$= (E \otimes \alpha \otimes W) \circ [E \otimes \chi(A, W)] \circ ({}^{E}\rho_{A}\Box_{A}W) \circ {}^{A}\overline{\rho}_{W}.$$

Since  $E \otimes \alpha \otimes W$  and  $E \otimes \chi(A, W)$  are monomorphisms, we obtain:

$$(E \otimes {}^{A}\overline{\rho}_{W}) \circ {}^{E}\rho_{W} = {}^{E}\rho_{A\square_{A}W} \circ {}^{A}\overline{\rho}_{W},$$

i.e. that  ${}^{A}\overline{\rho}_{W}$  is a morphism of left *E*-comodules.

REMARK 2.9. Let E be a coalgebra in an abelian monoidal category  $\mathcal{M}$ , let X be a right coideal and let Y be a left coideal of E in  $\mathcal{M}$ . Then we have

$$\begin{aligned} (X \wedge_E Y, i_{X \wedge_E Y}^E) &= \operatorname{Ker}[(p_X \otimes p_Y) \circ \triangle_E] \\ &= \operatorname{Ker}[(p_X \otimes p_Y) \circ \chi_E(E, E) \circ \overline{\triangle}_E] \\ &= \operatorname{Ker}[\chi_E(E/X, E/Y) \circ (p_X \Box_E p_Y) \circ \overline{\triangle}_E] = \operatorname{Ker}[(p_X \Box_E p_Y) \circ \overline{\triangle}_E], \end{aligned}$$

where  $\overline{\bigtriangleup}_E : E \to E \square_E E$  denotes the canonical isomorphism. In particular we have the following exact sequence:

(3) 
$$0 \longrightarrow X \wedge_E Y \xrightarrow{i_{X \wedge_E Y}^E} E \xrightarrow{(p_X \square_E p_Y) \circ \overline{\Delta}_E} E/X \square_E E/Y.$$

DEFINITION 2.10. Let  $\mathcal{M}$  be an abelian monoidal category. Let  $i_F = i_F^E : F \hookrightarrow E, i_A^B : A \hookrightarrow B$ and  $i_B = i_B^E : B \hookrightarrow E$  be monomorphisms which are coalgebra homomorphisms. Consider in  $\mathcal{M}$  the following commutative diagram with exact rows and columns



In this particular case, we denote the connecting homomorphism by

$$\eta(F, B, A) = \eta^E(F, B, A) : F \wedge_E B \to \frac{E}{F} \Box_E \frac{B}{A}.$$

 $\mathbf{6}$ 

Note that, by Proposition A.2, the morphism  $\eta(F, B, A)$  is uniquely defined by the following relation:

(4) 
$$\left(\frac{E}{F}\Box_E i_{B/A}\right) \circ \eta(F, B, A) = \left(p_F \Box_E p_A\right) \circ \overline{\Delta}_E \circ i_{F \wedge_E B}^E$$

By the Snake Lemma we get the following exact sequence: (5)

$$0 \longrightarrow F \wedge_E A \xrightarrow{i_{F \wedge_E A}^{F \wedge_E B}} F \wedge_E B \xrightarrow{\eta(F,B,A)} \xrightarrow{E}_F \Box_E \xrightarrow{B} \longrightarrow \operatorname{Coker}[(p_F \Box_E p_A) \circ \overline{\Delta}_E].$$

Observe that, if A = 0, then  $p_A = \text{Id}_E$ , so that

$$(p_F \Box_E p_A) \circ \overline{\Delta}_E = (p_F \Box_E E) \circ \overline{\Delta}_E = \overline{\rho}_{E/F}^r \circ p_F$$

is an epimorphism and

$$F \wedge_E A = \operatorname{Ker}[(p_F \Box_E p_A) \circ \overline{\Delta}_E] = \operatorname{Ker}(\overline{\rho}_{E/F}^r \circ p_F) = \operatorname{Ker}(p_F) = F,$$

that is we have the following exact sequence:

(6) 
$$0 \longrightarrow F \xrightarrow{i_F^{F \wedge_E B}} F \wedge_E B \xrightarrow{\eta(F,B,0)} \xrightarrow{E}_F \Box_E B \longrightarrow 0.$$

PROPOSITION 2.11. Let  $\mathcal{M}$  be an abelian monoidal category. Let  $i_F = i_F^E : F \hookrightarrow E$ ,  $i_A^B : A \hookrightarrow B$ and  $i_B = i_B^E : B \hookrightarrow E$  be monomorphisms which are coalgebra homomorphisms. Then we have

(7) 
$$(\frac{E}{F} \Box_E \frac{i_B^{F \wedge_E B}}{A}) \circ \eta(F, B, A) = (p_F \circ i_{F \wedge_E B}^E \Box_E p_A^{F \wedge_E B}) \circ \overline{\Delta}_{F \wedge_E B}.$$

*Proof.* By (4) we have

$$\left(\frac{E}{F}\Box_E \frac{i_B^E}{A}\right) \circ \eta(F, B, A) = \left(p_F^E \Box_E p_A^E\right) \circ \overline{\Delta}_E \circ i_{F \wedge EB}^E$$

Thus we obtain

$$\begin{split} & (\frac{E}{F}\Box_{E}\frac{i_{F\wedge_{E}B}^{E}}{A})\circ(\frac{E}{F}\Box_{E}\frac{i_{B}^{F\wedge_{E}B}}{A})\circ\eta(F,B,A) \\ = & (\frac{E}{F}\Box_{E}\frac{i_{F\wedge_{E}B}^{E}}{A}\circ\frac{i_{B}^{F\wedge_{E}B}}{A})\circ\eta(F,B,A) \\ = & (\frac{E}{F}\Box_{E}\frac{i_{B}^{E}}{A})\circ\eta(F,B,A) \\ = & (p_{F}^{E}\Box_{E}p_{A}^{E})\circ\overline{\Delta}_{E}\circ i_{F\wedge_{E}B}^{E} \\ = & (p_{F}^{E}\circ i_{F\wedge_{E}B}^{E}\Box_{E}p_{A}^{E}\circ i_{F\wedge_{E}B})\circ\overline{\Delta}_{F\wedge_{E}B} \\ = & (p_{F}^{E}\circ i_{F\wedge_{E}B}^{E}\Box_{E}\frac{i_{F\wedge_{E}B}^{E}}{A}\circ p_{A}^{F\wedge_{E}B})\circ\overline{\Delta}_{F\wedge_{E}B} \\ = & (\frac{E}{F}\Box_{E}\frac{i_{F\wedge_{E}B}^{E}}{A})\circ(p_{F}^{E}\circ i_{F\wedge_{E}B}^{E}\Box_{E}p_{A}^{F\wedge_{E}B})\circ\overline{\Delta}_{F\wedge_{E}B}. \end{split}$$

Now, since  $\frac{E}{F} \Box_E \frac{i_{F \wedge E}^E B}{A}$  is a monomorphism, we conclude.

**PROPOSITION 2.12.** Let  $\mathcal{M}$  be an abelian monoidal category. Consider the following commutative diagrams in  $\mathcal{M}$ :

where the morphisms are coalgebra homomorphisms. Then the following diagram

$$\begin{array}{c|c} F_1 \wedge_{E_1} B_1 & \xrightarrow{\eta^{E_1}(F_1, B_1, A_1)} & \xrightarrow{E_1} \Box_{E_1} \frac{B_1}{A_1} \\ f \wedge_e b \\ F_2 \wedge_{E_2} B_2 & \xrightarrow{\eta^{E_2}(F_2, B_2, A_2)} & \xrightarrow{E_2} \Box_{E_2} \frac{B_2}{A_2} \end{array}$$

is commutative.

*Proof.* We have:

$$\begin{split} & (\frac{E_2}{F_2} \Box_{E_2} \frac{i_{B_2}^{E_2}}{A_2}) \circ \eta^{E_2}(F_2, B_2, A_2) \circ (f \wedge_e b) \\ = & (p_{F_2}^{E_2} \Box_{E_2} p_{A_2}^{E_2}) \circ \overline{\Delta}_{E_2}^{E_2} \circ i_{F_2 \wedge_{E_2} B_2}^{E_2} \circ (f \wedge_e b) \\ = & (p_{F_2}^{E_2} \Box_{E_2} p_{A_2}^{E_2}) \circ \overline{\Delta}_{E_2}^{E_2} \circ e \circ i_{F_1 \wedge_{E_1} B_1}^{E_1} \\ = & (p_{F_2}^{E_2} \circ e \Box_{E_2} p_{A_2}^{E_2} \circ e) \circ \overline{\Delta}_{E_1}^{E_2} \circ i_{F_1 \wedge_{E_1} B_1}^{E_1} \\ = & (\frac{e}{f} \circ p_{F_1}^{E_1} \Box_{E_2} \frac{e}{a} \circ p_{A_1}^{E_1}) \circ \overline{\Delta}_{E_1}^{E_2} \circ i_{F_1 \wedge_{E_1} B_1}^{E_1} \\ = & (\frac{e}{f} \Box_{E_2} \frac{e}{a}) \circ (p_{F_1}^{E_1} \Box_{E_2} p_{A_1}^{E_1}) \circ \overline{\Delta}_{E_1}^{E_2} \circ i_{F_1 \wedge_{E_1} B_1}^{E_1} \\ = & (\frac{e}{f} \Box_{E_2} \frac{e}{a}) \circ (p_{F_1}^{E_1} \Box_{E_2} p_{A_1}^{E_1}) \circ (E_1 \Box_e E_1) \circ \overline{\Delta}_{E_1}^{E_1} \circ i_{F_1 \wedge_{E_1} B_1}^{E_1} \\ = & (\frac{e}{f} \Box_{E_2} \frac{e}{a}) \circ (\frac{E_1}{F_1} \Box_e \frac{E_1}{A_1}) \circ (p_{F_1}^{E_1} \Box_{E_1} p_{A_1}^{E_1}) \circ \overline{\Delta}_{E_1}^{E_1} \circ i_{F_1 \wedge_{E_1} B_1} \\ = & (\frac{e}{f} \Box_{E_2} \frac{e}{a}) \circ (\frac{E_1}{F_1} \Box_e \frac{E_1}{A_1}) \circ (\frac{E_1}{F_1} \Box_{E_1} \frac{i_{B_1}^{E_1}}{A_1}) \circ \eta^{E_1}(F_1, B_1, A_1) \\ = & (\frac{e}{f} \Box_e \frac{e}{a} \circ \frac{i_{B_1}^{E_1}}{A_1}) \circ \eta^{E_1}(F_1, B_1, A_1) \\ = & (\frac{e}{f} \Box_e \frac{i_{B_2}}{A_2} \circ \frac{b}{a}) \circ \eta^{E_1}(F_1, B_1, A_1) \\ = & (\frac{E_2}{F_2} \Box_{E_2} \frac{i_{B_2}^{E_2}}{A_2}) \circ (\frac{e}{f} \Box_e \frac{b}{a}) \circ \eta^{E_1}(F_1, B_1, A_1). \end{split}$$

Since  $\frac{E_2}{F_2} \Box_{E_2} \frac{i_{B_2}^{E_2}}{A_2}$  is a monomorphism, we finally obtain:

$$\eta^{E_2}(F_2, B_2, A_2) \circ (f \wedge_e b) = \left(\frac{e}{f} \Box_e \frac{b}{a}\right) \circ \eta^{E_1}(F_1, B_1, A_1).$$

PROPOSITION 2.13. Let  $\mathcal{M}$  be an abelian monoidal category. Let  $i_F = i_F^E : F \hookrightarrow E$ ,  $i_A^B : A \hookrightarrow B$ and  $i_B = i_B^E : B \hookrightarrow E$  be monomorphisms which are coalgebra homomorphisms. Then the following diagram

(8) 
$$F \wedge B \xrightarrow{\eta(F,B,0)} \xrightarrow{E}_{F} \Box_{E} B$$
$$\bigvee_{\eta(F,B,A)} \xrightarrow{E}_{F} \Box_{E} p_{A}^{B}$$
$$\underset{E}{\longrightarrow} E_{F} \Box_{E} B$$

is commutative. Furthermore, if the morphism  $E/F \Box_E p_A^B : {}^E\mathcal{M} \to \mathcal{M}$  is an epimorphism we get the following exact sequence:

(9) 
$$0 \longrightarrow F \wedge_E A \xrightarrow{i_{F \wedge_E B}^{F \wedge_E B}} F \wedge_E B \xrightarrow{\eta(F,B,A)} \xrightarrow{E}_F \Box_E \frac{B}{A} \longrightarrow 0.$$

*Proof.* We apply Proposition 2.12 in the case when

$$E_1 = E_2 = E, \ F_1 = F_2 = F, \ B_1 = B_2 = B, \ A_1 = 0, \ A_2 = A$$
  
 $e = \operatorname{Id}_E, \ a = 0, \ f = \operatorname{Id}_F, \ b = \operatorname{Id}_B.$ 

If the morphism  $E/F \square_E p_A^B$  is an epimorphism, then  $\eta(F, B, A)$  is an epimorphism as a composition of epimorphisms. Thus, in view of (5) we obtain (9). 

LEMMA 2.14. Let  $i_F: F \to E$  and  $i_B: B \to E$  be monomorphisms which are coalgebra homomorphisms phisms in an abelian monoidal category  $\mathcal{M}$ . Let

$$(L,p) := Coker(i_B^{F \wedge_E B}) = \frac{F \wedge_E B}{B}.$$

Then there is a unique morphism  ${}^{F}\rho_{L}: L \to F \otimes L$  such that

$${}^{E}\rho_{L} = (i_{F} \otimes L) \circ {}^{F}\rho_{L}.$$

Moreover  $(L, {}^{F}\rho_{L})$  is a left F-comodule and  ${}^{F}\overline{\rho}_{L}: L \to F \Box_{F}L$  is a morphism of left E-comodules.

*Proof.* Tensorize the following exact sequence

$$0 \to F \xrightarrow{i_F} E \xrightarrow{p_F} \frac{E}{F} \to 0$$

by L to get the exact sequence

(10) 
$$0 \longrightarrow F \otimes L \xrightarrow{i_F \otimes L} E \otimes L \xrightarrow{p_F \otimes L} \frac{E}{F} \otimes L$$

We have

$$(E/F \otimes i_{L}^{E/B}) \circ (p_{F} \otimes L) \circ {}^{E}\rho_{L} \circ p$$

$$= (E/F \otimes i_{L}^{E/B}) \circ (p_{F} \otimes L) \circ (E \otimes p) \circ {}^{E}\rho_{F \wedge EB}$$

$$= (p_{F} \otimes i_{L}^{E/B} \circ p) \circ {}^{E}\rho_{F \wedge EB}$$

$$= (p_{F} \otimes i_{L}^{E/B} \circ p) \circ [i_{F \wedge EB}^{E} \otimes (F \wedge EB)] \circ \Delta_{F \wedge EB}$$

$$= (p_{F} \circ i_{F \wedge EB}^{E} \otimes p_{B} \circ i_{F \wedge EB}^{E}) \circ \Delta_{F \wedge EB}$$

$$= (p_{F} \otimes p_{B}) \circ \Delta_{E} \circ i_{F \wedge EB}^{E} = 0.$$

As  $E/F \otimes i_L^{E/B}$  is a monomorphism and p is an epimorphism, we obtain  $(p_F \otimes L) \circ {}^E \rho_L = 0$ . Since, by 10, we have

 $(F \otimes L, i_F \otimes L) = \operatorname{Ker}(p_F \otimes L),$ 

there exists a unique morphisms  ${}^{F}\rho_{L}: L \to F \otimes L$  such that  ${}^{E}\rho_{L} = (i_{F} \otimes L) \circ {}^{F}\rho_{L}$ . Moreover, by Lemma 2.8  $(L, {}^{F}\rho_{L})$  is a left *F*-comodule and  ${}^{F}\overline{\rho}_{L}: L \to F \Box_{F}L$  is a morphism of left *E*comodules.  $\square$ 

LEMMA 2.15. Let  $\alpha : F \to A$  be a homomorphism and let  $i_A^E : A \to E$  be a monomorphism which is a coalgebra homomorphism in an abelian monoidal category  $\mathcal{M}$ . Let  $(W, {}^A \rho_W)$  be a left A-comodule. For a morphism  ${}^F \rho_W : W \to F \otimes W$  the following statement are equivalent. (1)  ${}^A \rho_W = (\alpha \otimes W) \circ {}^F \rho_W$ 

(1) 
$${}^{E}\rho_{W} = (\alpha \otimes W) \circ {}^{E}\rho_{W}.$$
  
(2)  ${}^{E}\rho_{W} = (i_{A}^{E}\alpha \otimes W) \circ {}^{F}\rho_{W}, where {}^{E}\rho_{W} = (i_{A}^{E}\otimes W) \circ {}^{A}\rho_{W}.$ 

*Proof.* (1)  $\Rightarrow$  (2) is trivial. (2)  $\Rightarrow$  (1) We have that  $(i_A^E \otimes W) \circ {}^A \rho_W = {}^E \rho_W = (i_A^E \alpha \otimes W) \circ {}^F \rho_W = (i_A^E \otimes W) \circ (\alpha \otimes W) \circ {}^F \rho_W.$ Since  $i_A^E \otimes W$  is a monomorphism, we have  ${}^A \rho_W = (\alpha \otimes W) \circ {}^F \rho_W.$ 

PROPOSITION 2.16. Let  $\alpha : F \to A$  be a coalgebra homomorphism, let  $i_B^A : B \to A$  and  $i_A^E : A \to E$  be monomorphisms which are coalgebra homomorphisms in an abelian monoidal category  $\mathcal{M}$ . Assume there is a homomorphism  $\pi : A \to F$  such that  $\pi \alpha = \operatorname{Id}_F$ . Assume that there exists a morphism  ${}^F\rho_{A/B} : A/B \to F \otimes A/B$  such that  ${}^E\rho_{A/B} = (i_A^E \alpha \otimes A/B) \circ {}^F\rho_{A/B}$ , where  ${}^E\rho_{A/B} = (i_A^E \otimes A/B) \circ {}^A\rho_{A/B}$ . Then we have

$${}^{F}\rho_{A/B} = (\pi \otimes A/B) \circ {}^{A}\rho_{A/B}.$$

Furthermore  ${}^{F}\rho_{A/B}$  is uniquely defined by the following relation

$${}^{F}\rho_{A/B} \circ p_{B}^{A} = (\pi \otimes p_{B}^{A}) \circ \Delta_{A}$$

and we have

$$(A \otimes p_B^A) \circ \Delta_A = (\alpha \pi \otimes p_B^A) \circ \Delta_A.$$

*Proof.* By Lemma 2.15 one has  ${}^{A}\rho_{A/B} = (\alpha \otimes A/B) \circ {}^{F}\rho_{A/B}$ , so that we have

$${}^{F}\rho_{A/B} = (\pi \otimes A/B) \circ (\alpha \otimes A/B) \circ {}^{F}\rho_{A/B} = (\pi \otimes A/B) \circ {}^{A}\rho_{A/B}.$$

Furthermore we get:

$${}^{F}\rho_{A/B} \circ p_{B}^{A} = (\pi \otimes A/B) \circ {}^{A}\rho_{A/B} \circ p_{B}^{A} = (\pi \otimes A/B) \circ (A \otimes p_{B}^{A}) \circ \Delta_{A} = (\pi \otimes p_{B}^{A}) \circ \Delta_{A}.$$

Finally we obtain:

$$\begin{aligned} (\alpha \pi \otimes p_B^A) \circ \Delta_A &= (\alpha \otimes A/B) \circ (\pi \otimes p_B^A) \circ \Delta_A \\ &= (\alpha \otimes A/B) \circ {}^F \rho_{A/B} \circ p_B^A \\ &= {}^A \rho_{A/B} \circ p_B^A = (A \otimes p_B^A) \circ \Delta_A. \end{aligned}$$

COROLLARY 2.17. Let  $i_F : F \to E$  and  $i_B : B \to E$  be monomorphisms which are coalgebra homomorphisms in an abelian monoidal category  $\mathcal{M}$ . Let

$$(L,p) := Coker(i_B^{F \wedge_E B}) = \frac{F \wedge_E B}{B}.$$

Assume that  $\pi: F \wedge_E B \to F$  is a morphism such that  $\pi \circ i_F^{F \wedge_E B} = \mathrm{Id}_F$ . Then the morphism  ${}^F \rho_L: L \to F \otimes L$  defined in Lemma 2.14 is uniquely defined by the following relation

(11) 
$${}^{F}\rho_{L} \circ p_{B}^{F \wedge_{E} B} = (\pi \otimes p_{B}^{F \wedge_{E} B}) \circ \Delta_{F \wedge_{E} B}$$

(which means that  $p_B^{F \wedge_E B}$  is a morphism of left F-comodules whenever  $\pi$  is a coalgebra homomorphism) and we have:

(12) 
$$[(F \wedge_E B) \otimes p_B^{F \wedge_E B}] \circ \Delta_{F \wedge_E B} = (i_F^{F \wedge_E B} \pi \otimes p_B^{F \wedge_E B}) \circ \Delta_{F \wedge_E B}.$$

*Proof.* Since, by Lemma 2.14, one has

$${}^{E}\rho_{L} = (i_{F} \otimes L) \circ {}^{F}\rho_{L} = (i_{F \wedge_{E} B}^{E} \circ i_{F}^{F \wedge_{E} B} \otimes L) \circ {}^{F}\rho_{L}$$

so that, we can apply Proposition 2.16 in the case when  $A = F \wedge_E B, B = B, E = E, F = F, \alpha = i_F^{F \wedge_E B}, i_A^E = i_{F \wedge_E B}^E$ .

2.18. Let  $\alpha: F \to A$  be a coalgebra homomorphism and let  $i_A^E: A \to E$  be a monomorphism which is a coalgebra homomorphism in an abelian monoidal category  $\mathcal{M}$ . Let  $(L, p) = \operatorname{Coker}(\alpha)$ . Since F and A are left E-comodules via  $i_A^E \alpha$  and  $i_A^E$  respectively, so is L. Its left E-comodule structure is uniquely defined by a morphism  ${}^E \rho_L: L \to E \otimes L$  such that

$${}^{E}\rho_{L}\circ p = (E\otimes p)\circ {}^{E}\rho_{A} = (i_{A}^{E}\otimes p)\circ \Delta_{A}.$$

We also point out that, since F and A are left A-comodules via  $\alpha$  and Id<sub>A</sub> respectively, so is L. Its left A-comodule structure is uniquely defined by a morphism  ${}^{A}\rho_{L}: L \to A \otimes L$  such that

$${}^{A}\rho_{L}\circ p = (A\otimes p)\circ {}^{A}\rho_{A} = (A\otimes p)\circ \Delta_{A}$$

Assume that there exists a morphism

$$^{F}\rho_{L}:L\rightarrow F\otimes L$$

such that  ${}^{E}\rho_{L} = (i_{A}^{E}\alpha \otimes L) \circ {}^{F}\rho_{L}$  (this happens, for example, in the case  $A = F \wedge_{E} F$  and  $\alpha = i_{F}^{F \wedge_{E} F}$ ). Then, by Lemma 2.8,  $(L, {}^{F}\rho_{L})$  is a left *F*-comodule. Hence one can endow *L* with a left *A*-comodule structure via  ${}^{A}\rho'_{L} = (\alpha \otimes L) \circ^{F} \rho_{L}$ . One has

$${}^{A}\rho_{L}' = {}^{A}\rho_{L}.$$

In fact we have

$$(i_A^E \otimes L) \circ {}^A \rho'_L \circ p = (i_A^E \alpha \otimes L) \circ {}^F \rho_L \circ p = {}^E \rho_L \circ p = (i_A^E \otimes p) \circ \Delta_A = (i_A^E \otimes L) \circ (A \otimes p) \circ \Delta_A = (i_A^E \otimes L) \circ {}^A \rho_L \circ p.$$

Since  $i_A^E \otimes L$  is a monomorphism and p is an epimorphism, we conclude. Assume there is a homomorphism  $\pi : A \to F$  such that  $\pi \alpha = \mathrm{Id}_F$ . Then we have

$$(F \otimes p) \circ (\pi \otimes A) \circ \Delta_A \circ \alpha = (\pi \otimes p) \circ (\alpha \otimes \alpha) \circ \Delta_F = 0$$

Since  $(L, p) = \operatorname{Coker}(\alpha)$ , there exists a morphism  ${}^{F}\rho_{L}'' : L \to F \otimes L$  such that

$${}^{F}\rho_{L}^{\prime\prime}\circ p = (\pi\otimes p)\circ\Delta_{A}$$

(when  $\pi$  is a coalgebra morphism,  ${}^{F}\rho_{L}''$  defines the left *F*-comodule structure that *L* has via  $\pi$ ). By Lemma 2.16, we have that  ${}^{F}\rho_{L}$  is uniquely defined by the following relation

$${}^{F}\rho_{L}\circ p=(\pi\otimes p)\circ\Delta_{A}.$$

Therefore

$${}^{F}\rho_{L}^{\prime\prime} = {}^{F}\rho_{L}.$$

PROPOSITION 2.19. Let  $\alpha : F \to A$  be a coalgebra homomorphism and let  $i_A^E : A \to E$  be a monomorphism which is a coalgebra homomorphism in an abelian monoidal category  $\mathcal{M}$ . For any left F-comodule  $(W, F \rho_W)$ , let

$${}^{A}\rho_{W} = (\alpha \otimes W)^{F}\rho_{W}, \qquad {}^{E}\rho_{W} = (i_{A}^{E}\alpha \otimes W)^{F}\rho_{W}$$

Let  $f: W_1 \to W_2$  be a morphism of left F-comodules. Then

$$(i_A^E \Box_F f) \circ^A \overline{\rho}_{W_1} =^E \overline{\rho}_{W_2} \circ f.$$

*Proof.* We have

$${}^{E}\rho_{W_{2}}\circ f = (i_{A}^{E}\alpha\otimes W_{2})\circ {}^{F}\rho_{W_{2}}\circ f = (i_{A}^{E}\alpha\otimes W_{2})\circ (F\otimes f)\circ {}^{F}\rho_{W_{1}} = (i_{A}^{E}\otimes f)\circ {}^{A}\rho_{W_{1}}$$

so that

$$\chi_F(E, W_2) \circ {}^E \overline{\rho}_{W_2} \circ f$$

$$= {}^E \rho_{W_2} \circ f$$

$$= (i_A^E \otimes f) \circ {}^A \rho_{W_1} = (i_A^E \otimes f) \circ \chi_F(A, W_1) \circ {}^A \overline{\rho}_{W_1} = \chi_F(E, W_2) \circ (i_A^E \Box_F f) \circ {}^A \overline{\rho}_{W_1}.$$

Since  $\chi_F(E, W_2)$  is a monomorphism, we conclude.

## 3. MAIN RESULTS

We now recall some definitions and results established in [AMS1].

3.1. Let X be an object in an abelian monoidal category  $(\mathcal{M}, \otimes, \mathbf{1})$ . Set

$$X^{\otimes 0} = \mathbf{1}, \qquad X^{\otimes 1} = X \qquad \text{and} \qquad X^{\otimes n} = X^{\otimes n-1} \otimes X, \text{ for every } n > 1$$

and for every morphism  $f: X \to Y$  in  $\mathcal{M}$ , set

$$f^{\otimes 0} = \mathrm{Id}_{1}, \qquad f^{\otimes 1} = f \qquad \text{and} \qquad f^{\otimes n} = f^{\otimes n-1} \otimes f, \text{ for every } n > 1.$$

Let  $(C, \Delta_C, \varepsilon_C)$  be a coalgebra in  $\mathcal{M}$  and for every  $n \in \mathbb{N}$ , define the  $n^{\text{th}}$  iterated comultiplication of  $C, \Delta_C^n : C \to C^{\otimes n+1}$ , by

$$\Delta_C^0 = \mathrm{Id}_C, \qquad \Delta_C^1 = \Delta_C \qquad \text{and} \qquad \Delta_C^n = \left(\Delta_C^{\otimes n-1} \otimes C\right) \Delta_C, \text{ for every } n > 1.$$

Let  $\delta : D \to E$  be a monomorphism which is a homomorphism of coalgebras in  $\mathcal{M}$ . Denote by (L, p) the cokernel of  $\delta$  in  $\mathcal{M}$ . Regard D as a E-bicomodule via  $\delta$  and observe that L is a E-bicomodule and p is a morphism of bicomodules. Let

$$(D^{\wedge_E^n}, \delta_n) := \ker(p^{\otimes n} \Delta_E^{n-1})$$

for any  $n \in \mathbb{N} \setminus \{0\}$ . Note that  $(D^{\wedge_E^1}, \delta_1) = (D, \delta)$  and  $(D^{\wedge_E^2}, \delta_2) = D \wedge_E D$ . In order to simplify the notations we set  $(D^{\wedge_E^0}, \delta_0) = (0, 0)$ .

Now, since  $\mathcal{M}$  has left exact tensor functors and since  $p^{\otimes n} \Delta_E^{n-1}$  is a morphism of *E*-bicomodules (as a composition of morphisms of *E*-bicomodules), we get that  $D^{\wedge_E^n}$  is a coalgebra and  $\delta_n : D^{\wedge_E^n} \to E$  is a coalgebra homomorphism for any n > 0 and hence for any  $n \in \mathbb{N}$ .

PROPOSITION 3.2. [AMS1, Proposition 1.10]Let  $\delta : D \to E$  be a monomorphism which is a morphism of coalgebras in an abelian monoidal category  $\mathcal{M}$ . Then for any  $i \leq j$  in  $\mathbb{N}$  there is a (unique) morphism  $\xi_i^j : D^{\wedge_E^i} \to D^{\wedge_E^j}$  such that

(13) 
$$\delta_i \xi_i^j = \delta_i$$

Moreover  $\xi_i^j$  is a coalgebra homomorphism and  $((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$  is a direct system in  $\mathcal{M}$  whose direct limit, if it exists, carries a natural coalgebra structure that makes it the direct limit of  $((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$  as a direct system of coalgebras.

NOTATION 3.3. Let  $\delta : D \to E$  be a morphism of coalgebras in an abelian monoidal category  $\mathcal{M}$  cocomplete and with left exact tensor functors. By Proposition 3.2  $((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$  is a direct system in  $\mathcal{M}$  whose direct limit carries a natural coalgebra structure that makes it the direct limit of  $((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$  as a direct system of coalgebras.

From now on we set:  $(\widetilde{D}_E, (\xi_i)_{i \in \mathbb{N}}) = \varinjlim (D^{\wedge_E^i})_{i \in \mathbb{N}}$ , where  $\xi_i : D^{\wedge_E^i} \to \widetilde{D}_E$  denotes the structural morphism of the direct limit. We simply write  $\widetilde{D}$  if there is no danger of confusion. We note that, since  $\widetilde{D}$  is a direct limit of coalgebras, the canonical (coalgebra) homomorphisms  $(\delta_i : D^{\wedge_E^i} \to E)_{i \in \mathbb{N}}$ , which are compatible by (13), factorize to a unique coalgebra homomorphism  $\widetilde{\delta} : \widetilde{D} \to E$  such that  $\widetilde{\delta}\xi_i = \delta_i$  for any  $i \in \mathbb{N}$ .

3.4. Let  $(C, \Delta, \varepsilon)$  be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  and let  $(M, \rho_M^r, \rho_M^l)$  be a C-bicomodule. Set

 $M^{\square 0} = C, M^{\square 1} = M \quad \text{and} \quad M^{\square n} = M^{\square n - 1} \square M \text{ for any } n > 1$ 

and define  $(C^n(M))_{n \in \mathbb{N}}$  by

$$C^{0}(M) = 0, C^{1}(M) = C$$
 and  $C^{n}(M) = C^{n-1}(M) \oplus M^{\Box n-1}$  for any  $n > 1$ .

Let  $\sigma_i^{i+1}: C^i(M) \to C^{i+1}(M)$  be the canonical inclusion and for any j > i, define:

$$\sigma_{i}^{j} = \sigma_{j-1}^{j} \sigma_{j-2}^{j-1} \cdots \sigma_{i+1}^{i+2} \sigma_{i}^{i+1} : C^{i}(M) \to C^{j}(M).$$

Then  $((C^i(M))_{i\in\mathbb{N}}, (\sigma^j_i)_{i,j\in\mathbb{N}})$  is a direct system in  $\mathcal{M}$ . We set

$$T_C^c(M) = \bigoplus_{n \in \mathbb{N}} M^{\Box n} = C \oplus M \oplus M^{\Box 2} \oplus M^{\Box 3} \oplus \cdots$$

and we denote by  $\sigma_i : C^i(M) \to T^c_C(M)$  the canonical inclusion.

Throughout let

$$\begin{aligned} \pi_n^m &: \quad C^n(M) \to C^m(M) \ (m \le n), \qquad \pi_n : T_C^c(M) \to C^n(M), \\ p_n^m &: \quad C^n(M) \to M^{\Box m} \ (m < n), \qquad p_n : T_C^c(M) \to M^{\Box n}, \end{aligned}$$

be the canonical projections and let

$$\begin{split} \sigma^n_m &: \quad C^m(M) \to C^n(M) \ (m \leq n), \qquad \sigma_n : C^n(M) \to T^c_C(M), \\ i^n_m &: \quad M^{\Box m} \to C^n(M) \ (m < n), \qquad i_m : M^{\Box m} \to T^c_C(M), \end{split}$$

be the canonical injection for any  $m, n \in \mathbb{N}$ .

For technical reasons we set  $\pi_n^m = 0$ ,  $\sigma_m^n = 0$  for any n < m and  $p_n^m = 0$ ,  $i_m^n = 0$  for any  $n \le m$ . Then, we have the following relations:

$$p_n \sigma_k = p_k^n, \qquad p_n i_k = \delta_{n,k} \mathrm{Id}_{M^{\Box k}}, \qquad \pi_n i_k = i_k^n.$$

Moreover, we have:

$$\begin{aligned} \pi_n^m \sigma_k^n &= \sigma_k^m, \text{ if } k \leq m \leq n, \\ p_n^m \pi_k^n &= p_k^m, \text{ if } m < n \leq k, \\ p_n^m \sigma_k^n &= p_k^m, \text{ if } m < n \leq k, \\ p_n^m \sigma_k^n &= p_k^m, \text{ if } m < k \leq n, \\ p_n^m \sigma_k^n &= p_m, \text{ if } m < n, \\ \pi_n \sigma_k &= \sigma_k^n, \text{ if } k \leq n, \\ \pi_n \sigma_k &= \sigma_k^n, \text{ if } k \leq n, \\ p_n^m n_m^n &= \text{Id}_{M^{\Box m}}, \text{ if } m < n. \end{aligned} \qquad \begin{aligned} & \text{and} \qquad & \pi_n^m \sigma_k^n &= \pi_k^m, \text{ if } m \leq k \leq n, \\ \sigma_n n_m^n n_k^n &= n, \\ \sigma_n n_m^n n_k^n &= n, \\ \sigma_n n_m^n n_k^n &= n, \\ \pi_n \sigma_k &= \pi_k^n, \text{ if } n \leq k, \end{aligned}$$

In the other cases, these compositions are zero.

Let  $(C, \Delta, \varepsilon)$  be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  with left exact tensor functors and let  $(M, \rho_M^r, \rho_M^l)$  be a C-bicomodule. Then  $(T_C^c(M), (\sigma_n)_{n \in \mathbb{N}}) = \lim_{n \in \mathbb{N}} C^i(M)$ .

THEOREM 3.5. [AMS1, Theorem 2.9] Let  $(C, \Delta, \varepsilon)$  be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  and let  $(M, \rho_M^r, \rho_M^l)$  be a C-bicomodule.  $(T_C^c(M), (\sigma_i)_{i \in \mathbb{N}})$  carries a natural coalgebra structure that makes it the direct limit of  $((C^i(M))_{i \in \mathbb{N}}, (\sigma_i^j)_{i,j \in \mathbb{N}})$  as a direct system of coalgebras.

THEOREM 3.6. [AMS1, Theorem 2.13] Let  $(C, \Delta, \varepsilon)$  be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  and let  $(\mathcal{M}, \rho_{\mathcal{M}}^r, \rho_{\mathcal{M}}^l)$  be a C-bicomodule. Let  $\delta : D \to E$  be a monomorphism which is a morphism of coalgebras such that the canonical morphism  $\widetilde{\delta} : \widetilde{D} \to E$  of Notation 3.3 is a monomorphism. Let  $f_C : \widetilde{D} \to C$  be a coalgebra homomorphism and let  $f_{\mathcal{M}} : \widetilde{D} \to \mathcal{M}$  be a morphism of C-bicomodules such that  $f_{\mathcal{M}}\xi_1 = 0$ , where  $\widetilde{D}$  is a C-bicomodule via  $f_C$ . Then there is a unique morphism  $f : \widetilde{D} \to T_C^c(\mathcal{M})$  such that

$$f\xi_n = \sigma_n f_n$$
, for any  $n \in \mathbb{N}$ ,

where

(14) 
$$f_n = \sum_{t=0}^n i_t^n f_M^{\Box t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n$$

and  $\overline{\Delta}_{\widetilde{D}}^{n}: \widetilde{D} \to \widetilde{D}^{\Box n+1}$  is the  $n^{th}$  iteration of  $\overline{\Delta}_{\widetilde{D}}$   $(\overline{\Delta}_{\widetilde{D}}^{-1} = f_{C}, \overline{\Delta}_{\widetilde{D}}^{0} = Id_{\widetilde{D}}, \overline{\Delta}_{\widetilde{D}}^{1} = \overline{\Delta}_{\widetilde{D}}: \widetilde{D} \to \widetilde{D} \Box \widetilde{D}).$ Moreover:

1) f is a coalgebra homomorphism;

2)  $p_0 f = f_C$  and  $p_1 f = f_M$ , where  $p_n : T_C^c(M) \to M^{\Box n}$  denotes the canonical projection.

Furthermore, any coalgebra homomorphism  $f: \widetilde{D} \to T_C^c(M)$  that fulfils 2) satisfies the following relation:

(15) 
$$p_k f = f_M^{\Box k} \overline{\Delta}_{\widetilde{D}}^{k-1} \text{ for any } k \in \mathbb{N}.$$

THEOREM 3.7. [AMS1, Theorem 2.15] Let  $(C, \Delta, \varepsilon)$  be a coalgebra in a cocomplete and complete abelian monoidal category  $\mathcal{M}$  satisfying AB5. Let  $(\mathcal{M}, \rho_M^r, \rho_M^l)$  be a C-bicomodule. Let  $\delta : D \to E$ be a monomorphism which is a homomorphism of coalgebras. Let  $f_C : \widetilde{D} \to C$  be a coalgebra homomorphism and let  $f_M : \widetilde{D} \to M$  be a morphism of C-bicomodules such that  $f_M \xi_1 = 0$ , where  $\widetilde{D}$  is a bicomodule via  $f_C$ . Then there is a unique coalgebra homomorphism  $f : \widetilde{D} \to T_C^c(M)$  such that  $p_0 f = f_C$  and  $p_1 f = f_M$ , where  $p_n : T_C^c(M) \to M^{\Box n}$  denotes the canonical projection.



3.8. Let  $\mathcal{M}$  be an abelian category and let  $\mathcal{H}$  be a class of monomorphisms in  $\mathcal{M}$ . We recall that an object I in  $\mathcal{M}$  is called injective rel  $\lambda$ , where  $\lambda : X \to Y$  is a monomorphism in  $\mathcal{H}$ , if  $\mathcal{M}(\lambda, I) : \mathcal{M}(Y, I) \to \mathcal{M}(X, I)$  is surjective. I is called  $\mathcal{H}$ -injective if it is injective rel  $\lambda$  for every  $\lambda$  in  $\mathcal{H}$ . The *closure* of  $\mathcal{H}$  is the class  $\mathcal{C}(\mathcal{H})$  containing all monomorphisms  $\lambda$  in  $\mathcal{M}$  such that every  $\mathcal{H}$ -injective object is also injective rel  $\lambda$ . The class  $\mathcal{H}$  is called *closed* if  $\mathcal{H}$  is  $\mathcal{C}(\mathcal{H})$ . A closed class  $\mathcal{H}$  is called *injective* if for any object M in  $\mathcal{M}$  there is an monomorphism  $\lambda : M \to I$  in  $\mathcal{H}$  such that I is  $\mathcal{H}$ -injective.

3.9. We fix a coalgebra C in a monoidal category  $\mathcal{M}$ . Let  $\mathbb{U} : {}^{C}\mathcal{M}^{C} \to \mathcal{M}$  be the forgetful functor. Then

(16) 
$$\mathcal{I} := \{ f \in {}^{C}\mathcal{M}^{C} \mid \mathbb{U}(f) \text{ cosplits in } \mathcal{M} \}.$$

is an injective class of monomorphisms.

Now, for any C-bicomodule  $M \in {}^{C}\mathcal{M}^{C}$ , we define the Hochschild cohomology of C with coefficients in M by:

$$\mathbf{H}^{\bullet}(M,C) = \mathbf{Ext}^{\bullet}_{\mathcal{T}}(M,C),$$

where  $\mathbf{Ext}^{\bullet}_{\mathcal{I}}(M, -)$  are the relative left derived functors of  ${}^{C}\mathcal{M}^{C}(M, -)$ . The notion of Hochschild cohomology for algebras and coalgebras in monoidal categories has been deeply investigated in [AMS2]. Here we quote some results that will be needed afterwards.

THEOREM 3.10. [AMS2, Theorem 4.22] Let  $(D, \Delta, \varepsilon)$  be a coalgebra in an abelian monoidal category  $\mathcal{M}$ . Then the following conditions are equivalent:

(a) D is formally smooth (i.e.  $\mathrm{H}^{2}(M,D) = 0$ , for any  $M \in {}^{D}\mathcal{M}^{D}$ .).

(b) The canonical morphism  $\xi_1: D \to \widetilde{D}_E$  has a coalgebra homomorphism retraction, whenever:

- E is a coalgebra endowed with a coalgebra homomorphism  $\delta: D \to E$ ;
- $\delta$  is a monomorphism;
- $\tilde{D}_E$  exists;
- for any  $r \in \mathbb{N}$ , the canonical injection  $\xi_r^{r+1} : D^r \to D^{r+1}$  cosplits in  $\mathcal{M}$ .

THEOREM 3.11. [AMS1, Theorem 4.15] Let  $(C, \Delta, \varepsilon)$  be a formally smooth coalgebra in a cocomplete and complete abelian monoidal category  $\mathcal{M}$  satisfying AB5, with left and right exact tensor functors. Assume that denumerable coproducts commute with  $\otimes$ . Let  $(M, \rho_M^r, \rho_M^l)$  be a  $\mathcal{I}$ -injective C-bicomodule. Then the cotensor coalgebra  $T_C^c(M)$  is formally smooth.

From now on we will use the following notation

$$D^n := D^{\wedge_E n}$$
, for every  $n \in \mathbb{N}$ .

LEMMA 3.12. Let  $\delta: D \to E$  be a monomorphism which is a morphism of coalgebras in an abelian monoidal category  $\mathcal{M}$  and assume that for any  $r \in \mathbb{N}$ , the canonical injection  $\xi_r^{r+1}: D^r \to D^{r+1}$ cosplits in  $\mathcal{M}$  i.e. there exists  $\lambda_{r+1}^r: D^{r+1} \to D^r$  such that  $\lambda_{r+1}^r \circ \xi_r^{r+1} = \mathrm{Id}_{D^r}$ . Then, for any  $r \in \mathbb{N}$ , the canonical injection  $\xi_r: D^r \to \widetilde{D}$  cosplits in  $\mathcal{M}$ . *Proof.* For every  $i \in \mathbb{N}$  let us define  $\lambda_i : D^i \to D^r$  by setting

$$\begin{cases} \lambda_i = \lambda_{r+1}^r \circ \dots \circ \lambda_i^{i-1}, \text{ if } i > r \\ \lambda_r = \mathrm{Id}_{D^r} \\ \lambda_i = \xi_i^r \text{ if } i < r. \end{cases}$$

Let us prove that  $(\lambda_i : D^i \to D^r)_{i \in \mathbb{N}}$  is a compatible family of morphisms in  $\mathcal{M}$  i.e. that  $\lambda_{i+1} \circ$  $\begin{aligned} \xi_i^{i+1} &= \hat{\lambda}_i. \\ \text{If } i+1 < r, \text{ we have} \end{aligned}$ 

$$\lambda_{i+1} \circ \xi_i^{i+1} = \xi_{i+1}^r \circ \xi_i^{i+1} = \xi_i^r = \lambda_i.$$

If i + 1 = r, we have

$$\lambda_{i+1} \circ \xi_i^{i+1} = \mathrm{Id}_{D^r} \circ \xi_i^{i+1} = \xi_i^r = \lambda_i$$

If i + 1 > r, we have

$$\lambda_{i+1} \circ \xi_i^{i+1} = \lambda_{r+1}^r \circ \cdots \circ \lambda_i^{i-1} \circ \lambda_{i+1}^i \circ \xi_i^{i+1} = \lambda_{r+1}^r \circ \cdots \circ \lambda_i^{i-1} = \lambda_i.$$

Now, since  $(\lambda_i : D^i \to D^r)_{i \in \mathbb{N}}$  is a compatible family of morphisms in  $\mathcal{M}$  there exists a morphism  $\lambda:\widetilde{D}\to D^r$  such that

 $\lambda \circ \xi_i = \lambda_i$ , for every  $i \in \mathbb{N}$ .

In particular  $\lambda \circ \xi_r = \lambda_r = \mathrm{Id}_{D^r}$ .

THEOREM 3.13. Let  $(D, \Delta, \varepsilon)$  be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  and let  $(M, \rho_M^r, \rho_M^l)$  be a D-bicomodule. Let  $\delta: D \to E$  be a monomorphism which is a morphism of coalgebras in  $\mathcal{M}$ . Let  $M := E/D\Box_E D \simeq D^2/D$ . Assume that

i) D is a formally smooth coalgebra in  $\mathcal{M}$ .

ii) M is  $\mathcal{I}$ -injective where  $\mathcal{I} := \{ f \in {}^{D}\mathcal{M}^{D} \mid f \text{ cosplits in } \mathcal{M} \}.$ iii) For any  $r \in \mathbb{N}$ , the canonical injection  $\xi_{r}^{r+1} : D^{r} \to D^{r+1}$  cosplits in  $\mathcal{M}$ .

Then there is a coalgebra homomorphism  $f_D: \widetilde{D} \to D$  such that

$$f_D \circ \xi_1 = \mathrm{Id}_D$$

and a D-bicomodule homomorphism  $f_M: \widetilde{D} \to M$  such that

$$f_M \circ \xi_2 = \eta(D, D, 0).$$

Moreover there is a unique morphism  $f: \widetilde{D} \to T^c_D(M)$  such that

(17) 
$$f\xi_n = \sigma_n f_n, \text{ for any } n \in \mathbb{N},$$

where

$$f_n = \sum_{t=0}^n i_t^n f_M^{\Box_D t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n.$$

Also:

1) f is a coalgebra homomorphism;

2)  $p_0 f = f_D$  and  $p_1 f = f_M$ , where  $p_n : T_D^c(M) \to M^{\Box n}$  denotes the canonical projection. Furthermore, any coalgebra homomorphism  $f: \widetilde{D} \to T^c_D(M)$  that fulfils 2) satisfies the following relation:

(18) 
$$p_k f = f_M^{\Box k} \overline{\Delta}_{\widetilde{D}}^{k-1} \text{ for any } k \in \mathbb{N}.$$



*Proof.* By Theorem 3.10, the morphism  $\xi_1 : D \to \widetilde{D}$  has a retraction  $f_D : \widetilde{D} \to D$  which is a coalgebra homomorphism:  $f_D \circ \xi_1 = \operatorname{Id}_D$ . Thus  $\widetilde{D}$  becomes a *D*-bicomodule via  $f_D$ . Now we point out that  $\xi_2 \in \mathcal{I}$ . In fact by Lemma 3.12  $\xi_2$  cosplits in  $\mathcal{M}$ . Moreover,  $D^2$  is a *D*-bicomodule via  $f_D$  and  $\xi_2$  a morphism of *D*-bicomodules. As explained in (2.18)  $(M, \eta(D, D, 0)) = \operatorname{Coker}(\xi_1^2)$  is a *D*-bicomodule and  $\eta(D, D, 0)$  is a morphism of *D*-bicomodules. Since  $\xi_2 \in \mathcal{I}$  and *M* is  $\mathcal{I}$ -injective, there exists a *D*-bicomodule homomorphism  $f_M : \widetilde{D} \to M$  such that

$$f_M \circ \xi_2 = \eta(D, D, 0).$$

Finally  $f_M \circ \xi_1 = f_M \circ \xi_2 \circ \xi_1^2 = \eta(D, D, 0) \circ \xi_1^2 = 0$ , so that, by Theorem 3.6, there exists a coalgebra homomorphism  $f: \widetilde{D} \to T_D^c(M)$  which fulfills the required conditions.

DEFINITION 3.14. Keep the hypothesis and notations of Theorem 3.13. Then we have:

$$f_M^{\square_D n} \circ \overline{\Delta}_{\widetilde{D}}^{n-1} \circ \xi_n \stackrel{(18)}{=} p_n \circ f \circ \xi_n \stackrel{(17)}{=} p_n \circ \sigma_n \circ f_n = p_n^n \circ f_n = 0.$$

Therefore, given any  $k \ge 0$ , we have

$$f_M^{\Box_D n} \circ \overline{\Delta}_{\widetilde{D}}^{n-1} \circ \xi_{n+k} \circ \xi_n^{n+k} = f_M^{\Box_D n} \circ \overline{\Delta}_{\widetilde{D}}^{n-1} \circ \xi_n = 0.$$

Since  $[E/D^n \Box_E D^k, \eta(D^n, D^k, 0)] = \operatorname{Coker}(\xi_n^{n+k})$ , there exists a unique morphism

$$\mathcal{D}(D^n, D^k, 0) : E/D^n \square_E D^k \to (E/D \square_E D)^{\square_D n}$$

such that

(19) 
$$\Phi(D^n, D^k, 0) \circ \eta(D^n, D^k, 0) = f_M^{\Box_D n} \circ \overline{\Delta}_{\widetilde{D}}^{n-1} \circ \xi_{n+k}.$$

Note that, since  $\eta(D^n, D^k, 0)$ ,  $f_M^{\Box_D n} \circ \overline{\Delta}_{\widetilde{D}}^{n-1} \circ \xi_{n+k}$  and  $\xi_n^{n+k}$  are morphisms of *D*-bicomodules, so is  $\Phi(D^n, D^k, 0)$ .

Moreover observe that

$$\Phi(D,D,0)\circ\eta(D,D,0)=f_M^{\Box_D 1}\circ\overline{\Delta}_{\widetilde{D}}^0\circ\xi_2=f_M\circ\xi_2=\eta(D,D,0).$$

Since  $\eta(D, D, 0)$  is an epimorphism we get

(20) 
$$\Phi(D, D, 0) = \mathrm{Id}_{E/D\square_E D}.$$

The proof of the following theorem requires some technicalities that, for an easier reeding, were included in Appendix B.

THEOREM 3.15. Let  $\mathcal{M}$  be a cocomplete abelian monoidal category. Keep the hypothesis and notations of Theorem 3.13. Then  $\Phi(D^n, D, 0)$  is a monomorphism for any  $n \in \mathbb{N}$ . Moreover, if we assume that the morphism  $E/D^n \Box_E p_D^E$  is an epimorphism for every  $n \in \mathbb{N}$ , we get that  $\Phi(D^n, D, 0)$ is an isomorphism for any  $n \in \mathbb{N}$ .

*Proof.* By Theorem B.8, we have (44):

$$\Phi(D^n, D, 0) = [\Phi(D^{n-1}, D, 0) \Box_D(E/D \Box_E D)] \circ [E/D^{n-1} \Box_E^{\ D} \overline{\rho}_{E/D \Box_E D}] \circ (\gamma_n \Box_E D).$$

We point out that, by Proposition 2.7, the morphism  ${}^D\overline{\rho}_{E/D\square_E D}: E/D\square_E D \to D\square_D(E/D\square_E D)$  is always an isomorphism.

As explained in Definition B.6 the morphism  $\gamma_n: E/D^n \to E/D^{n-1} \square_E E/D$  defined by relation

$$\gamma_n \circ p_{D^n}^E = (p_{D^{n-1}}^E \Box_E p_D^E) \circ \overline{\Delta}_E.$$

is always a monomorphism. By Proposition B.9,  $(p_{D^{n-1}}^E \Box_E p_D^E) \circ \overline{\Delta}_E$  is an epimorphism whenever  $E/D^{n-1} \Box_E p_D^E$  is an epimorphism. In this case  $\gamma_n$  is an epimorphism too and hence it is an isomorphism. Since  $\Phi(D, D, 0) = Id_{E/D \Box_E D}$ , by induction, we conclude.

THEOREM 3.16. Let  $(D, \Delta, \varepsilon)$  be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  satisfying AB5. Let  $\delta: D \to E$  be a monomorphism which is a morphism of coalgebras in  $\mathcal{M}$ . Let  $M := E/D\Box_E D \simeq D^2/D$ . Assume that

1) D is a formally smooth coalgebra in  $\mathcal{M}$ .

2) M is  $\mathcal{I}$ -injective where  $\mathcal{I} := \{ f \in {}^{D}\mathcal{M}^{D} \mid f \text{ cosplits in } \mathcal{M} \}.$ 

3) For any  $r \in \mathbb{N}$ , the canonical injection  $\xi_r^{r+1} : D^r \to D^{r+1}$  cosplits in  $\mathcal{M}$ . Let

$$f: \widetilde{D} \to T^c_D(M)$$

be the unique morphism, constructed in Theorem 3.13, such that

$$f\xi_n = \sigma_n f_n$$
, for any  $n \in \mathbb{N}$ ,

where

$$f_n = \sum_{t=0}^n i_t^n f_M^{\Box_D t} \overline{\Delta}_{\widetilde{D}}^{t-1} \xi_n.$$

Then f is a monomorphism in  $\mathcal{M}$ . Moreover, if we assume that

4) the morphism  $E/D^n \Box_E p_D^E$  is an epimorphism for every  $n \in \mathbb{N}$ ,

then f is an isomorphism and, with the further assumptions that

5)  $\mathcal{M}$  is complete with right exact tensor functors;

6) denumerable coproducts commute with  $\otimes$ ;

then  $\widetilde{D}$  is a formally smooth coalgebra in  $\mathcal{M}$ .

*Proof.* Since  $\mathcal{M}$  fulfills AB5 condition, it is enough to prove that  $f_n$  is a monomorphism (resp. isomorphism) for any  $n \ge 0$ . We proceed by induction on  $n \ge 0$ .

For n = 0, since  $i_0^0 = 0$ , we have  $f_n = f_0 = i_0^0 f_M^{\Box_D 0} \overline{\Delta}_{\widetilde{D}}^{-1} \xi_0 = 0$ . As  $f_0 : D^0 = 0 \to C^0(M) = 0$ , we get that  $f_0$  is an isomorphism.

For n = 1 we have  $f_n = f_1 = i_0^1 f_M^{\Box_D 0} \overline{\Delta}_{\tilde{D}}^{-1} \xi_1 + i_1^1 f_M^{\Box_D 1} \overline{\Delta}_{\tilde{D}}^0 \xi_1 = i_0^1 f_M^{\Box_D 0} \overline{\Delta}_{\tilde{D}}^{-1} \xi_1 = i_0^1 f_D \xi_1 = i_0^1 \circ \mathrm{Id}_D = i_0^1 = \mathrm{Id}_D$ , as  $i_1^1 = 0$ . Thus  $f_1$  is an isomorphism.

Let n > 1. Assume that  $f_{n-1}$  is a monomorphism (resp. isomorphism) and let us prove that  $f_n$  is a monomorphism (resp. isomorphism) too.

We have

$$\Phi(D^{n-1}, D, 0) \circ \eta(D^{n-1}, D, 0) \stackrel{(19)}{=} f_M^{\Box_D n-1} \circ \overline{\Delta}_{\tilde{D}}^{n-2} \circ \xi_n \stackrel{(18)}{=} p_{n-1} \circ f \circ \xi_n \stackrel{(17)}{=} p_{n-1} \circ \sigma_n \circ f_n = p_n^{n-1} \circ f_n,$$
  
so that the following diagram

 $\rightarrow 0$ 

commutes and hence  $\Phi(D^{n-1}, D, 0) = \frac{f_n}{f_{n-1}}$ . By Theorem 3.15, the morphism  $\Phi(D^{n-1}, D, 0)$  is a monomorphism (resp. an isomorphism if we assume also that the morphism  $E/D^n \square_E p_D^E$  is an epimorphism for every  $n \in \mathbb{N}$ ). Hence, by applying 5-Lemma to the diagram above, we get that  $f_n$  is a monomorphism (resp. isomorphism) too.

We conclude by observing that, if 5) and 6) hold true, by Theorem 3.11, we get that  $T_{C}^{c}(M)$  is formally smooth.  $\square$ 

Let E be a coalgebra in  $\mathcal{M}$  and set

$$\begin{aligned} \mathcal{I}^E &= \left\{ f \in \mathcal{M}^E \mid f \text{ cosplits in } \mathcal{M} \right\}, \\ ^E \mathcal{I} &= \left\{ f \in ^E \mathcal{M} \mid f \text{ cosplits in } \mathcal{M} \right\}, \end{aligned}$$

so that

$$\mathcal{I} = {}^{E} \mathcal{I} \cap \mathcal{I}^{E}$$

LEMMA 3.17. Let  $\mathcal{M}$  be an abelian monoidal category, let N be an  $\mathcal{I}^E$ -injective right E-comodule and let  $i_L^M : L \to M$  be a morphism in  ${}^E\mathcal{I}$ . Then  $N \Box_E p_L^M$  is an epimorphism.

*Proof.* Since the tensor products are left exact, the sequence

$$0 \to N \square_E L \stackrel{N \square_E i_L^M}{\to} N \square_E M \stackrel{N \square_E p_L^M}{\to} N \square_E M / L$$

is exact. We have to prove that  $N \square_E p_L^M$  is an epimorphism. Since  $i_L^M \in^E \mathcal{I}$ , the sequence

$$0 \to L \stackrel{i_L^M}{\to} M \stackrel{p_L^M}{\to} M/L \to 0$$

splits in  $\mathcal{M}$  and hence the sequence

$$0 \to N \otimes L \stackrel{N \otimes i_L^M}{\to} N \otimes M \stackrel{N \otimes p_L^M}{\to} N \otimes M/L \to 0$$

is exact so that the morphism

$$(N \otimes E) \square_E p_L^M : (N \otimes E) \square_E M \to (N \otimes E) \square_E M/L$$

is an epimorphism since  $(N \otimes E) \square_E p_L^M \cong N \otimes p_L^M$ . Now, as N is an  $\mathcal{I}^E$ -injective right E-comodule, the monomorphism

$$\rho_N: N \to N \otimes E$$

which belongs to  $\mathcal{I}^E$ , has a retraction  $\beta_N \in \mathcal{M}^E$  and we have

$$\left(N \Box_E p_L^M\right) \left(\beta_N \Box_E M\right) = \beta_N \Box_E p_L^M = \left(\beta_N \Box_E M/L\right) \left[\left(N \otimes E\right) \Box_E p_L^M\right].$$

Now  $\beta_N \Box_E M/L$  is an epimorphism since it has a section  $\rho_N \Box_E M/L$ , and  $(N \otimes E) \Box_E p_L^M$  is an epimorphism so that we deduce that  $N \Box_E p_L^M$  is an epimorphism.  $\Box$ 

LEMMA 3.18. Let  $(D, \Delta, \varepsilon)$  be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  satisfying AB5. Let  $\delta: D \to E$  be a monomorphism which is a morphism of coalgebras in  $\mathcal{M}$ . Then

$$D^{\wedge_E n} = D^{\wedge_{\widetilde{D}} n}$$

for every  $n \in \mathbb{N}$ .

*Proof.* We proceed by induction on  $n \ge 1$  being the cases n = 0, 1 trivial. Assume that  $D^{\wedge_E n} = D^{\wedge_{\widetilde{D}} n}$ . Note that

$$\ker\left[\left(p_{D^{\wedge_E n}}^{\tilde{D}} \otimes p_{D}^{\tilde{D}}\right) \Delta_{\tilde{D}}\right] = D^{\wedge_E n} \wedge_{\tilde{D}} D = D^{\wedge_{\widetilde{D}} n} \wedge_{\tilde{D}} D = D^{\wedge_{\widetilde{D}} n+1}.$$

Let us prove that

$$\left(D^{\wedge_E n+1}, \xi_{n+1}\right) = \ker \left[\left(p_{D^{\wedge_E n}}^{\widetilde{D}} \otimes p_D^{\widetilde{D}}\right) \Delta_{\widetilde{D}}\right].$$

Let  $f: X \to \widetilde{D}$  be a morphism such that

$$\left(p_{D^{\wedge_E n}}^{\widetilde{D}} \otimes p_D^{\widetilde{D}}\right) \circ \Delta_{\widetilde{D}} \circ f = 0.$$

We have to prove that there exists a unique morphism  $\overline{f}: X \to D^{\wedge_E n+1}$  such that

$$\xi_{n+1} \circ f = f$$

We have

$$(p_{D^{\wedge_{E^{n}}}}^{E} \otimes p_{D}^{E}) \circ \Delta_{E} \circ \widetilde{\delta} \circ f = (p_{D^{\wedge_{E^{n}}}}^{E} \otimes p_{D}^{E}) \circ (\widetilde{\delta} \otimes \widetilde{\delta}) \circ \Delta_{\widetilde{D}} \circ f$$

$$= \left(\frac{\widetilde{\delta}}{D^{\wedge_{E^{n}}}} \otimes \frac{\widetilde{\delta}}{D}\right) \circ \left(p_{D^{\wedge_{E^{n}}}}^{\widetilde{D}} \otimes p_{D}^{\widetilde{D}}\right) \circ \Delta_{\widetilde{D}} \circ f = 0.$$

Since

$$\left(D^{\wedge_E n+1}, \delta_{n+1}\right) = \ker\left[\left(p_{D^{\wedge_E n}}^E \otimes p_D^E\right) \circ \Delta_E\right]$$

by the universal property of the kernel, there exists a unique morphism  $\overline{f}: X \to D^{\wedge_E n+1}$  such that

$$\delta_{n+1} \circ \overline{f} = \widetilde{\delta} \circ f.$$

Since  $\delta_{n+1} = \widetilde{\delta} \circ \xi_{n+1}$  and since, by AB5,  $\widetilde{\delta}$  is a monomorphism, we conclude.

DEFINITIONS 3.19. Let C be a coalgebra in an abelian monoidal category  $\mathcal{M}$ . We say that a quotient M/L of a right C-comodule M is a  $\mathcal{I}^C$ -quotient of M, whenever the canonical injection  $L \to M$  is in  $\mathcal{I}^C$  that is the sequence

$$0 \to L \to M \to \frac{M}{L} \to 0$$

is  $\mathcal{I}^C$ -exact.

A coalgebra C is called (right) hereditary whenever every  $\mathcal{I}^C$ -quotient of an  $\mathcal{I}^C$ -injective comodule is  $\mathcal{I}^C$ -injective.

DEFINITION 3.20. A coalgebra C is called *coseparable* whenever  $\mathbf{H}^1(M, C) = 0$  for every  $M \in {}^{C}\mathcal{M}^{C}$ .

The following theorem was proved for the abelian monoidal category of vector spaces in [JLMS].

THEOREM 3.21. Let D be a coalgebra in a cocomplete abelian monoidal category  $\mathcal{M}$  satisfying AB5. Let  $\delta: D \to E$  be a monomorphism which is a morphism of coalgebras in  $\mathcal{M}$ . Assume that

1) D is a coseparable coalgebra in  $\mathcal{M}$ .

2) For any  $r \in \mathbb{N}$ , the canonical injection  $\xi_r^{r+1} : D^{\wedge_E r} \to D^{\wedge_E r+1}$  cosplits in  $\mathcal{M}$  (e.g.  $\mathcal{M}$  is a semisimple category).

3) D is a right hereditary coalgebra in  $\mathcal{M}$ . Let

$$f: \widetilde{D} \to T_D^c(\frac{D \wedge_E D}{D})$$

be the unique morphism, constructed in Theorem 3.13.

Then f is an isomorphism in  $\mathcal{M}$ . Moreover, if we assume that

4)  $\mathcal{M}$  is complete with right exact tensor functors;

5) denumerable coproducts commute with  $\otimes$ ;

then D is a formally smooth coalgebra in  $\mathcal{M}$ .

Proof. First of all let us point out that, by Lemma 3.18 one has

$$D^{\wedge_E n} = D^{\wedge_{\widetilde{D}} n}.$$

Set  $D^n = D^{\wedge_E n} = D^{\wedge_{\widetilde{D}} n}$ .

We apply Theorem 3.16 to the case  $D = D, E = \tilde{D}, \delta = \xi_1$ . In fact, since D is coseparable, it is in particular a formally smooth coalgebra in  $\mathcal{M}$  [AMS2, Corollary 4.18]. Moreover every D-bicomodule and in particular  $D^2/D$  is  $\mathcal{I}$ -injective [AMS2, Theorem 4.4]. It remains to prove that the morphism

$$\frac{\widetilde{D}}{D^n} \Box_{\widetilde{D}} p_D^{\widetilde{D}} : \frac{\widetilde{D}}{D^n} \Box_{\widetilde{D}} \widetilde{D} \to \frac{\widetilde{D}}{D^n} \Box_{\widetilde{D}} \frac{\widetilde{D}}{D}$$

is an epimorphism. Since, by Lemma 3.12, the canonical injection  $\xi_n : D^n \to \widetilde{D}$  cosplits in  $\mathcal{M}$  for every  $n \in \mathbb{N}$ , we get that  $\frac{\widetilde{D}}{D^n}$  is an  $\mathcal{I}^{\widetilde{D}}$ -quotient of  $\widetilde{D}$ . Since, by assumption,  $\widetilde{D}$  is a right hereditary, then  $\frac{\widetilde{D}}{D^n}$  is  $\mathcal{I}^{\widetilde{D}}$ -injective for every  $n \in \mathbb{N}$ . Now,  $i_D^{\widetilde{D}} = \xi_1 : D \to \widetilde{D}$  has a retraction  $f_D : \widetilde{D} \to D$  in  $\mathcal{M}$  and hence  $i_D^{\widetilde{D}} \in \widetilde{D} \mathcal{I}$ . By Lemma 3.17, we conclude.

THEOREM 3.22. Let  $(C, \Delta, \varepsilon)$  be a formally smooth coalgebra in an abelian monoidal category  $\mathcal{M}$ . Then C is a right hereditary coalgebra.

*Proof.* Let  $M \in \mathcal{M}^C$  and  $(\mathcal{O}_1 C, \pi) = \operatorname{Coker}(\Delta)$ . Let us consider the following exact sequence in  ${}^C\mathcal{M}^C$ :

$$0 \to C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi} \mho_1 C \to 0.$$

Clearly  $r_C(C \otimes \varepsilon) \in {}^C \mathcal{M}$  is a retraction of  $\Delta$  so that the sequence above splits in  ${}^C \mathcal{M}$  and hence

$$0 \to M \square_C C \xrightarrow{M \square_C \Delta} M \square_C C \otimes C \xrightarrow{M \square_C \pi} M \square_C \mho_1 C \to 0$$

is an exact sequence in  $\mathcal{M}^C$ . Since  $M \square_C C \cong M$ , we get the exact sequence

$$0 \xrightarrow{\rho_{M}^{r}} M \otimes C \xrightarrow{(M \square_{C} \pi)[\overline{\rho}_{M}^{C} \otimes C]} M \square_{C} \mho_{1} C \xrightarrow{\rho_{M}} 0$$

in  $\mathcal{M}^C$ . Since C is formally smooth, by [AMS2, Corollary 4.21],  $\mathfrak{V}_1C$  is  $\mathcal{I}$ -injective, so that , by [Ar, Theorem 2.3], there is a morphism of C-bicomodules  $j : \mathfrak{V}_1C \to C \otimes X \otimes C$  that cosplits in  ${}^C\mathcal{M}^C$ . Thus

$$M \square_C j : M \square_C \mathcal{O}_1 C \to M \square_C C \otimes X \otimes C \cong M \otimes X \otimes C$$

is a morphism in  $\mathcal{M}^C$  that cosplits in  $\mathcal{M}^C$ . Therefore, by [Ar, Theorem 2.3],  $M \Box_C \mho_1 C$  is  $\mathcal{I}^C$ injective and hence the previous sequence is an  $\mathcal{I}^C$ -injective resolution of length 1. Since this is true for every  $M \in \mathcal{M}^C$ , we conclude that  $\operatorname{Ext}^n_{\mathcal{I}^C}(M, N) = 0$ , for every  $M, N \in \mathcal{M}^C, n \geq 2$  i.e. that C is a right hereditary coalgebra.

THEOREM 3.23. Let D be a coalgebra in a cocomplete and complete abelian monoidal category  $\mathcal{M}$  satisfying AB5.

Let  $\delta: D \to E$  be a monomorphism which is a morphism of coalgebras in  $\mathcal{M}$ . Assume that

- 1) D is a coseparable coalgebra in  $\mathcal{M}$ .
- 2) For every  $r \in \mathbb{N}$ , the canonical injection  $\xi_r^{r+1} : D^{\wedge_E r} \to D^{\wedge_E r+1}$  cosplits in  $\mathcal{M}$  (e.g.  $\mathcal{M}$  is a semisimple category).
- 3)  $\mathcal{M}$  has right exact tensor functors.

4) Denumerable coproducts commute with  $\otimes$ .

Then the following assertions are equivalent.

(i) D is a formally smooth coalgebra.

(ii) D is a right hereditary coalgebra.

(ii')  $\widetilde{D}$  is a left hereditary coalgebra.

(iii)  $\widetilde{D} \simeq T_D^c(\frac{D \wedge_E D}{D})$  as coalgebras.

(iv)  $D \simeq T_D^c(N)$  as coalgebras, for some D-bicomodule N.

*Proof.*  $(i) \Rightarrow (ii)$  follows by Theorem 3.22.

 $(ii) \Rightarrow (iii)$  follows by Theorem 3.21.

 $(iii) \Rightarrow (iv)$  is trivial.

 $(iv) \Rightarrow (i)$ . Since *D* is coseparable, it is in particular formally smooth as a coalgebra in  $\mathcal{M}$  [AMS2, Corollary 4.18]. Moreover every *D*-bicomodule and in particular *N* is  $\mathcal{I}$ -injective (see [AMS2, Theorem 4.4]). We conclude by applying Theorem 3.11.

COROLLARY 3.24. [JLMS] Let K be a vector space. Let E be a K-coalgebra with coseparable coradical D. Then the following assertions are equivalent.

(i) E is a formally smooth coalgebra.

(ii) E is a right hereditary coalgebra.

(ii') E is a left hereditary coalgebra.

(iii)  $E \simeq T_D^c(\frac{D \wedge_E D}{D})$  as coalgebras.

(iv)  $E \simeq T_D^c(N)$  as coalgebras, for some D-bicomodule N.

*Proof.* Since D is the coradical of E is well known that E = D (see e.g. [Sw, Corollary 9.0.4, page 185]). The conclusion follows by Theorem 3.23 applied in the case when  $\mathcal{M}$  is the category of vector spaces over K. We point out that, since, in this case,  $\mathcal{M}$  is a semisimple category, the notions of formally smooth and right hereditary coalgebras reduce to the classical ones.

EXAMPLES 3.25. We now provide a number of examples of abelian monoidal categories for which our results apply. These categories are all Grothendieck categories and hence cocomplete and complete abelian categories satisfying AB5.

Let B be a bialgebra over a field K.

• The category  ${}_B\mathfrak{M} = ({}_B\mathfrak{M}, \otimes_K, K)$ , of all left modules over B. The tensor  $V \otimes W$  of two left B-modules is an object in  ${}_B\mathfrak{M}$  via the diagonal action; the unit is K regarded as a left B-module via  $\varepsilon_B$ .

• The category  ${}_B\mathfrak{M}_B = ({}_B\mathfrak{M}_B, \otimes_K, K)$ , of all two-sided modules over B. The tensor  $V \otimes W$  of two B-bimodules carries, on both sides, the diagonal action; the unit is K regarded as a B-bimodule via  $\varepsilon_B$ .

• The category  ${}^{B}\mathfrak{M} = ({}^{B}\mathfrak{M}, \otimes_{K}, K)$ , of all left comodules over B. The tensor product  $V \otimes W$  of two left B-comodules is an object in  ${}^{B}\mathfrak{M}$  via the diagonal coaction; the unit is K regarded as a left B-comodule via the map  $k \mapsto 1_{B} \otimes k$ .

• The category  ${}^{B}\mathfrak{M}^{B} = ({}^{B}\mathfrak{M}^{B}, \otimes_{K}, K)$  of all two-sided comodules over B. The tensor  $V \otimes W$  of two B-bicomodules carries, on both sides, the diagonal coaction; the unit is K regarded as a B-bicomodule via the maps  $k \mapsto 1_{B} \otimes k$  and  $k \mapsto k \otimes 1_{B}$ .

• Let H be a Hopf algebra over a field K with bijective antipode.

The category  ${}^{H}_{H}\mathcal{YD} = ({}^{H}_{H}\mathcal{YD}, \otimes_{K}, K)$  of left Yetter-Drinfeld modules over H. Recall that an object V in  ${}^{H}_{H}\mathcal{YD}$  is a left H-module and a left H-comodule satisfying, for any  $h \in H, v \in V$ , the compatibility condition:

$$\sum (h_{(1)}v)_{<-1>}h_{(2)} \otimes (h_{(1)}v)_{<0>} = \sum h_{(1)}v_{<-1>} \otimes h_{(2)}v_{<0>}$$

where  $\Delta_H(h) = \sum h_{(1)} \otimes h_{(2)}$  and  $\rho(v) = \sum v_{<-1>} \otimes v_{<0>}$  denote the comultiplication of H and the left H-comodule structure of V respectively (we used Sweedler notation).

The tensor product  $V \otimes W$  of two Yetter-Drinfeld modules is an object in  ${}^{H}_{H}\mathcal{YD}$  via the diagonal action and the codiagonal coaction; the unit in  ${}^{H}_{H}\mathcal{YD}$  is K regarded as a left H-comodule via the map  $x \mapsto 1_{H} \otimes x$  and as a left H-module via the counit  $\varepsilon_{H}$ .

• The category  $_{Q}\mathfrak{M} = (_{Q}\mathfrak{M}, \otimes_{K}, K)$ , of all left modules over a quasi-bialgebra Q over a field K (see [Ka, Definition XV.1.1, page 368]).

# APPENDIX A. THE SNAKE LEMMA

We collect here some technical results that are used in the paper. Some of them might be found in the literature, nevertheless we decided to include them here for the reader's sake.

First of all, we need to recall some results related to the so called "Snake Lemma". Thus let  $\mathcal{M}$  be an abelian category and consider in  $\mathcal{M}$  the following commutative diagram with exact rows



where  $P = P(\alpha_2^3, k_3)$  denotes the pullback of  $(\alpha_2^3, k_3)$ . We have

$$\beta_2^3 f_2 k_3' = f_3 \alpha_2^3 k_3' = f_3 k_3 \alpha_2^{3'} = 0.$$

Since  $(B_1, \beta_1^2) = \ker(\beta_2^3)$ , there exists a unique morphism  $\omega : P \to B_1$  such that

(21) 
$$\beta_1^2 \omega = f_2 k_3'$$

Let  $\alpha_1^2 = ib$  be the canonical factorization of  $\alpha_1^2$  as the composition of a monomorphism  $i : \operatorname{Im}(\alpha_1^2) \to A_2$  and an epimorphism  $b : A_1 \to \operatorname{Im}(\alpha_1^2)$ . As  $\alpha_2^3 ib = \alpha_2^3 \alpha_1^2 = 0$  and b is an epimorphism, we get  $\alpha_2^3 \circ i = 0 = k_3 \circ 0$ . By the universal property of P, there exists a unique morphism  $\xi : \operatorname{Im}(\alpha_1^2) \to P$  such that

(22) 
$$k'_3\xi = i$$
 and  $\alpha_2^{3'}\xi = 0.$ 

It is straightforward to check that  $(\text{Im}(\alpha_1^2), \xi) = \text{ker}(\alpha_2^{3'})$ . We point out that, since  $\alpha_2^3$  is an epimorphism, also  $\alpha_2^{3'}$  is an epimorphism and hence  $(\text{ker}(f_3), \alpha_2^{3'}) = \text{Coker}(\xi)$ . Now, we have:

$$\beta_1^2 \omega \xi b = f_2 k'_3 \xi b = f_2 i b = f_2 \alpha_1^2 = \beta_1^2 f_1 f_1$$

 $\beta_1 \omega \xi \theta = f_2 \kappa_3 \xi \theta$ Since  $\beta_1^2$  is a monomorphism, we deduce that

 $\omega \xi b = f_1$ 

and hence  $c_1 \omega \xi b = c_1 f_1 = 0$ . As b is an epimorphism, we conclude that  $c_1 \omega \xi = 0$ . By the universal property of  $\operatorname{Coker}(\xi)$ , there exists a unique morphism  $\overline{\omega} : \operatorname{ker}(f_3) \to \operatorname{Coker}(f_1)$  such that

(23) 
$$\overline{\omega}\alpha_2^{3'} = c_1\omega.$$

The morphism  $\overline{\omega}$  is usually called *connecting homomorphism*.

In fact it is easy to prove the existence of morphisms  $k_1^2, k_2^3, c_1^2, c_2^3$  such that the following diagram commutes



and one has the following well known result.

THEOREM A.1 (Snake Lemma). The following sequence is exact:

(24) 
$$Ker(f_1) \xrightarrow{k_1^2} Ker(f_2) \xrightarrow{k_2^3} Ker(f_3) \xrightarrow{\overline{\omega}} Coker(f_1) \xrightarrow{c_1^2} Coker(f_2) \xrightarrow{c_2^3} Coker(f_3)$$

*Proof.* It is easy to check that the proof of [Mac, Lemma 5, page 206] works also in this more general setting (where  $\alpha_1^2$  is not assumed to be a monomorphism and  $\beta_2^3$  is not necessarily an epimorphism).

PROPOSITION A.2. Let  $\mathcal{M}$  be an abelian category and consider in  $\mathcal{M}$  the following commutative diagram with exact rows



where  $P = P(\alpha_2^3, k_3)$  denotes the pullback of  $(\alpha_2^3, k_3)$ . Then the connecting homomorphism  $\overline{\omega}$  is uniquely defined by the following relation:

(25) 
$$\beta_1^2 \overline{\omega} = f_2(\alpha_2^3)^{-1} k_3$$

*Proof.* By (23) we have  $\overline{\omega}\alpha_2^{3'} = c_1\omega = \omega$ . Therefore we have

$$\beta_1^2 \overline{\omega} \alpha_2^{3'} = \beta_1^2 \omega \stackrel{(21)}{=} f_2 k_3' = f_2(\alpha_2^3)^{-1} \alpha_2^3 k_3' = f_2(\alpha_2^3)^{-1} k_3 \alpha_2^{3'}$$

Since  $\alpha_2^{3'}$  is an epimorphism, we conclude. Note that, as  $\beta_1^2$  is a monomorphism, we have that  $\overline{\omega}$  is the unique morphism satisfying the required relation.

# Appendix B. Technical results to prove Theorem 3.15

PROPOSITION B.1. Let  $\mathcal{M}$  be a cocomplete abelian monoidal category. Then, with the hypothesis and notations of Theorem 3.13, the following relations hold true:

(26) 
$${}^{D}\overline{\rho}_{E/D\square_{E}D} \circ \eta(D,D,0) = [D\square_{D}\eta(D,D,0)] \circ {}^{D}\overline{\rho}_{D^{2}} = [f_{D}\xi_{2}\square_{D}\eta(D,D,0)] \circ \overline{\Delta}_{D^{2}}$$

(27) 
$$[\xi_1^2 f_D \xi_2 \Box_D \eta(D, D, 0)] \circ \overline{\Delta}_{D^2} = [D^2 \Box_D \eta(D, D, 0)] \circ \overline{\Delta}_{D^2}$$

(28) 
$$(E/D\square_E \xi_1^n) \circ \eta(D, D, 0) = \eta(D, D^n, 0) \circ \xi_2^{n+1}$$

(29) 
$$[\xi_1^2 \Box_D(E/D \Box_E D)] \circ {}^D \overline{\rho}_{E/D \Box_E D} \circ \eta(D, D, 0) = [D^2 \Box_D \eta(D, D, 0)] \circ \overline{\Delta}_{D^2}$$

$$(30) \qquad [\xi_1^2 \square_D(E/D \square_E \xi_1^n)] \circ {}^D \overline{\rho}_{E/D \square_E D} \circ \eta(D, D, 0) = [D^2 \square_D \eta(D, D^n, 0) \circ \xi_2^{n+1}] \circ \overline{\Delta}_{D^2}$$

(31) 
$$\eta(D^{n-1}, D^2, D) = [E/D^{n-1} \Box_E \eta(D, D, 0)] \circ \eta(D^{n-1}, D^2, 0)$$

(32) 
$$(E/D^{n-1}\Box_E\xi_2^{n+1}) \circ \eta(D^{n-1}, D^2, 0) = (p_{D^{n-1}} \circ \delta_{n+1}\Box_E D^{n+1}) \circ \overline{\Delta}_{D^{n+1}}$$

*Proof.* 1) By (11) applied to the case

$$F = B = D, E = E$$
 and  $\pi = f_D \xi_2$ 

we get

$$L = \frac{D \wedge_E D}{D} \quad \text{and} \quad {}^D \rho_{\frac{D \wedge_E D}{D}} \circ p_D^{D^2} = (f_D \otimes p_D^{D^2}) \circ \Delta_{D \wedge_E D}$$

from which, by identifying  $(\frac{D^2}{D}, p_D^{D^2})$  with  $(E/D\Box_E D, \eta(D, D, 0))$ , we obtain

$${}^{D}\rho_{E/D\square_{E}D}\circ\eta(D,D,0)=[D\otimes\eta(D,D,0)]\circ{}^{D}\rho_{D^{2}}$$

which means that  $\eta(D, D, 0) : D^2 \to E/D \square_E D$  is a morphism of left *D*-comodules. Thus we can apply Proposition 2.19 in the case

$$W_1 = D^2, W_2 = E/D\square_E D, f = \eta(D, D, 0), A = F = E = D, \alpha = i_A^E = \mathrm{Id}_D,$$

in order to obtain

$${}^{D}\overline{\rho}_{E/D\square_{E}D}\circ\eta(D,D,0)=[D\square_{D}\eta(D,D,0)]\circ{}^{D}\overline{\rho}_{D^{2}}$$

Note that, in view of 2.16 applied to the case

$$F = D, A = D^2, B = 0, \alpha = \xi_1^2, \pi = f_D \xi_2,$$

we have  ${}^D\rho_{D^2} = [f_D\xi_2 \otimes D^2] \circ \triangle_{D^2}$  so that  ${}^D\overline{\rho}_{D^2} = [f_D\xi_2\Box_D D^2] \circ \overline{\triangle}_{D^2}$  and hence we get (26).

2) By (12), applied to the case

$$F = B = D, E = E, \pi = f_D \xi_2,$$

we get

$$\left(D^2 \otimes p_D^{D^2}\right) \circ \Delta_{D^2} = \left(\xi_1^2 f_D \xi_2 \otimes p_D^{D^2}\right) \circ \Delta_{D^2}$$

from which, by identifying  $(\frac{D^2}{D}, p_D^{D^2})$  with  $(E/D \Box_E D, \eta(D, D, 0))$ , we obtain

$$[\xi_1^2 f_D \xi_2 \otimes \eta(D, D, 0)] \circ \triangle_{D^2} = [D^2 \otimes \eta(D, D, 0)] \circ \triangle_{D^2}$$

Therefore, since  $\eta(D, D, 0): D^2 \to E/D \square_E D$  is a morphism of left D-comodules, one has

$$\begin{split} \chi_D(D^2, E/D\Box_E D) &\circ [\xi_1^2 f_D \xi_2 \Box_D \eta(D, D, 0)] \circ \triangle_{D^2} \\ = & [\xi_1^2 f_D \xi_2 \otimes \eta(D, D, 0)] \circ \chi_D(D^2, D^2) \circ \overline{\triangle}_{D^2} \\ = & [\xi_1^2 f_D \xi_2 \otimes \eta(D, D, 0)] \circ \triangle_{D^2} \\ = & [D^2 \otimes \eta(D, D, 0)] \circ \triangle_{D^2} \\ = & [D^2 \otimes \eta(D, D, 0)] \circ \chi_D(D^2, D^2) \circ \overline{\triangle}_{D^2} \\ = & \chi_D(D^2, E/D\Box_E D) \circ [D^2 \Box_D \eta(D, D, 0)] \circ \overline{\triangle}_{D^2}. \end{split}$$

Since  $\chi_D(D^2, E/D\square_E D)$  is a monomorphism we obtain (27). 3) By applying Proposition 2.12 in the case

$$E_1 = E_2 = E, F_1 = F_2 = B_1 = D, B_2 = D^n, A_1 = A_2 = 0, e = \mathrm{Id}_E, f = \mathrm{Id}_D, b = \xi_1^n, a = 0,$$

we obtain(28). 4) We have

$$\begin{aligned} [\xi_1^2 \Box_D(E/D \Box_E D)] \circ {}^D \overline{\rho}_{E/D \Box_E D} \circ \eta(D, D, 0) \\ \stackrel{(26)}{=} & [\xi_1^2 \Box_D(E/D \Box_E D)] \circ [f_D \xi_2 \Box_D \eta(D, D, 0)] \circ \overline{\Delta}_{D^2} \\ = & [\xi_1^2 f_D \xi_2 \Box_D \eta(D, D, 0)] \circ \overline{\Delta}_{D^2} \\ \stackrel{(27)}{=} & [D^2 \Box_D \eta(D, D, 0)] \circ \overline{\Delta}_{D^2}. \end{aligned}$$

Hence we get (29). 5) We have

$$\begin{split} & [\xi_1^2 \Box_D(E/D \Box_E \xi_1^n)] \circ {}^D \overline{\rho}_{E/D \Box_E D} \circ \eta(D, D, 0) \\ &= [D^2 \Box_D(E/D \Box_E \xi_1^n)] \circ [\xi_1^2 \Box_D(E/D \Box_E D)] \circ {}^D \overline{\rho}_{E/D \Box_E D} \circ \eta(D, D, 0) \\ \end{split} \\ \begin{split} & (29) \\ &= [D^2 \Box_D(E/D \Box_E \xi_1^n)] \circ [D^2 \Box_D \eta(D, D, 0)] \circ \overline{\Delta}_{D^2} \\ &= [D^2 \Box_D(E/D \Box_E \xi_1^n) \circ \eta(D, D, 0)] \circ \overline{\Delta}_{D^2} \\ \end{split} \\ \end{split} \\ \end{split}$$

Hence we obtain (30).

6) By applying Proposition 2.13 in the case

$$E = E, F = D^{n-1}, B = D^2, A = D,$$

we obtain 31:

$$\eta(D^{n-1}, D^2, D) = [E/D^{n-1} \Box_E \eta(D, D, 0)] \circ \eta(D^{n-1}, D^2, 0)$$

7) By applying (7) in the case

$$E = E, F = D^{n-1}, B = D^2, A = 0, i_B^{F \wedge_E B} = \xi_2^{n+1}, i_{F \wedge_E B}^E = \delta_{n+1}, p_A^{F \wedge_E B} = \mathrm{Id}_{D^{n+1}},$$

we obtain 32.

**PROPOSITION B.2.** Let  $\mathcal{M}$  be an abelian monoidal category. Then we have:

$$(33) \quad [(E/D^{n-1}\Box_E\xi_1^2)\Box_D(E/D\Box_E\xi_1^n)] \circ \varphi(D^{n-1}, D^2, D) = [\eta(D^{n-1}, D^2, 0)\Box_D\eta(D, D^n, 0)] \circ \overline{\Delta}_{D^{n+1}},$$

where

(34) 
$$\varphi(D^{n-1}, D^2, D) := (E/D^{n-1} \Box_E{}^D \overline{\rho}_{E/D} \Box_E D) \circ \eta(D^{n-1}, D^2, D).$$

*Proof.* We have:

$$\begin{split} & (E/D^{n-1}\Box_{E}\xi_{2}^{n+1}\Box_{D}E/D\Box_{E}D^{n})\circ[(E/D^{n-1}\Box_{E}\xi_{1}^{n})]\cup (C/D\Box_{E}\xi_{1}^{n})]\circ\varphi(D^{n-1},D^{2},D) \\ & = (E/D^{n-1}\Box_{E}\xi_{2}^{n+1}\Box_{D}E/D\Box_{E}D^{n})\circ[(E/D^{n-1}\Box_{E}\xi_{1}^{2})\Box_{D}(E/D\Box_{E}\xi_{1}^{n})]\circ \\ & \circ(E/D^{n-1}\Box_{E}\xi_{2}^{n+1}\Box_{D}E/D\Box_{E}D^{n})\circ[E/D^{n-1}\Box_{E}(\xi_{1}^{2}\Box_{D}E/D\Box_{E}\xi_{1}^{n})^{D}\overline{\rho}_{E/D\Box_{E}D}]\circ \\ & \circ[E/D^{n-1}\Box_{E}\eta(D,D,0)]\circ\eta(D^{n-1},D^{2},0) \\ & = (E/D^{n-1}\Box_{E}\xi_{2}^{n+1}\Box_{D}E/D\Box_{E}D^{n})\circ[E/D^{n-1}\Box_{E}(\xi_{1}^{2}\Box_{D}E/D\Box_{E}\xi_{1}^{n})^{D}\overline{\rho}_{E/D\Box_{E}D}\eta(D,D,0)]\circ \\ & \circ\eta(D^{n-1},D^{2},0) \\ & (30) & (E/D^{n-1}\Box_{E}\xi_{2}^{n+1}\Box_{D}E/D\Box_{E}D^{n})\circ[E/D^{n-1}\Box_{E}(D^{2}\Box_{D}\eta(D,D^{n},0)\circ\xi_{2}^{n+1})\circ\overline{\Delta}_{D^{2}}]\circ\eta(D^{n-1},D^{2},0) \\ & = [E/D^{n-1}\Box_{E}[\xi_{2}^{n+1}\Box_{D}\mu(D,D^{n},0)\circ\xi_{2}^{n+1}]\circ\overline{\Delta}_{D^{2}}]\circ\eta(D^{n-1},D^{2},0) \\ & \text{since } \xi_{2}^{n+1} \coloneqq a \text{ coalgebra homomorphism, we get} \\ & = [E/D^{n-1}\Box_{E}[D^{n+1}\Box_{D}\eta(D,D^{n},0)]\circ\overline{\Delta}_{D^{n+1}}\circ\xi_{2}^{n+1}]\circ\eta(D^{n-1},D^{2},0) \\ & (32) & [E/D^{n-1}\Box_{E}[D^{n+1}\Box_{D}\eta(D,D^{n},0)]\circ\overline{\Delta}_{D^{n+1}}\circh_{n+1}\Box_{E}D^{n+1})\circ\overline{\Delta}_{D^{n+1}} \\ & = [p_{D^{n-1}}\circ\delta_{n+1}\Box_{E}[D^{n+1}\Box_{D}\eta(D,D^{n},0)]\circ(\overline{\Delta}_{D^{n+1}})\circ\delta_{D^{n+1}} \\ & = [p_{D^{n-1}}\circ\delta_{n+1}\Box_{E}D^{n+1}\Box_{D}\eta(D,D^{n},0)]\circ(\overline{\Delta}_{D^{n+1}}\Box_{D}n^{n+1})\circ\overline{\Delta}_{D^{n+1}} \\ & = [p_{D^{n-1}}\circ\delta_{n+1}\Box_{E}D^{n+1}\Box_{D}\eta(D,D^{n},0)]\circ(\overline{\Delta}_{D^{n+1}}\Box_{D}n^{n+1})\circ\overline{\Delta}_{D^{n+1}} \\ & = [(E/D^{n-1}\Box_{E}\xi_{2}^{n+1})\circ\eta(D,D^{n},0)]\circ(\overline{\Delta}_{D^{n+1}}\Box_{D}n^{n+1})\circ\overline{\Delta}_{D^{n+1}} \\ & = [(E/D^{n-1}\Box_{E}\xi_{2}^{n+1})\circ\eta(D,D^{n},0)]\circ(\overline{\Delta}_{D^{n+1}}\Box_{D}n^{n+1})\circ\overline{\Delta}_{D^{n+1}} \\ & = [(E/D^{n-1}\Box_{E}\xi_{2}^{n+1})\circ\eta(D,D^{n},0)]\circ(\overline{\Delta}_{D^{n+1}}\Box_{D}n^{n+1})\circ\overline{\Delta}_{D^{n+1}} \\ & = (E/D^{n-1}\Box_{E}\xi_{2}^{n+1})\circ\eta(D^{n-1},D^{2},0)\Box_{D}\eta(D,D^{n},0)]\circ\overline{\Delta}_{D^{n+1}} \\ & = (E/D^{n-1}\Box_{E}\xi_{2}^{n+1})\circ\mu(D^{n-1},D^{2},0)\Box_{D}\eta(D,D^{n},0)]\circ\overline{\Delta}_{D^{n+1}} \\ & = (E/D^{n-1}\Box_{E}\xi_{2}^{n+1}\Box_{D}E/D\Box_{E}D^{n})\circ[\eta(D^{n-1},D^{2},0)\Box_{D}\eta(D,D^{n},0)]\circ\overline{\Delta}_{D^{n+1}} \\ & = (E/D^{n-1}\Box_{E}\xi_{2}^{n+1}\Box_{D}E/D\Box_{E}D^{n})\circ[\eta(D^{n-1},D^{2},0)\Box_{D}\eta(D,D^{n},0)]\circ\overline{\Delta}_{D^{n+1}}. \end{aligned}$$

$$[(E/D^{n-1}\Box_E\xi_1^2)\Box_D(E/D\Box_E\xi_1^n)] \circ \varphi(D^{n-1}, D^2, D) = [\eta(D^{n-1}, D^2, 0)\Box_D\eta(D, D^n, 0)] \circ \overline{\Delta}_{D^{n+1}}.$$

LEMMA B.3. For any  $s \ge k \ge 0$ , we have

(35) 
$$\Phi(D^n, D^s, 0) \circ (E/D^n \Box_E \xi_k^s) = \Phi(D^n, D^k, 0).$$

*Proof.* We apply Proposition 2.12 in the case

 $E_1 = E_2 = E, F_1 = F_2 = D^n, B_1 = D^k, B_2 = D^s, A_1 = A_2 = 0, e = \mathrm{Id}_E, f = \mathrm{Id}_{D^n}, b = \xi_k^s, a = 0.$ in order to obtain:

$$(E/D^n \Box_E \xi^s_k) \circ \eta(D^n, D^k, 0) = \eta(D^n, D^s, 0) \circ \xi^{n+s}_{n+k}$$

Therefore we get

$$\Phi(D^n, D^s, 0) \circ (E/D^n \Box_E \xi^s_k) \circ \eta(D^n, D^k, 0)$$

$$= \Phi(D^n, D^s, 0) \circ \eta(D^n, D^s, 0) \circ \xi^{n+s}_{n+k}$$

$$= f^{\Box_D n}_M \circ \overline{\Delta}^{n-1}_{\tilde{D}} \circ \xi_{n+s} \circ \xi^{n+s}_{n+k}$$

$$= f^{\Box_D n}_M \circ \overline{\Delta}^{n-1}_{\tilde{D}} \circ \xi_{n+k} = \Phi(D^n, D^k, 0) \circ \eta(D^n, D^k, 0)$$

Since  $\eta(D^n, D^k, 0)$  is an epimorphism, we conclude.

PROPOSITION B.4. Let  $\mathcal{M}$  be a cocomplete abelian monoidal category. Then, with the hypothesis and notations of Theorem 3.13, we have:

(36) 
$$[\Phi(D^{n-1}, D, 0)\Box_D(E/D\Box_E D)] \circ \varphi(D^{n-1}, D^2, D) = f_M^{\Box_D n} \circ \overline{\Delta}_{\widetilde{D}}^{n-1} \circ \xi_{n+1}.$$

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*Proof.* By (19) we get

(37) 
$$\Phi(D^{n-1}, D^2, 0) \circ \eta(D^{n-1}, D^2, 0) = f_M^{\Box_D n-1} \circ \overline{\Delta}_{\tilde{D}}^{n-2} \circ \xi_{n+1}.$$

(38) 
$$\Phi(D, D^n, 0) \circ \eta(D, D^n, 0) = f_M^{\square_D 1} \circ \overline{\Delta}_{\widetilde{D}}^0 \circ \xi_{1+n} = f_M \circ \xi_{n+1}.$$

By (35) we obtain

(39) 
$$\Phi(D^{n-1}, D^2, 0) \circ (E/D^{n-1} \Box_E \xi_1^2) = \Phi(D^{n-1}, D, 0).$$
  
(40) 
$$\Phi(D, D^n, 0) \circ (E/D \Box_E \xi_1^n) = \Phi(D, D, 0).$$

Finally we compute:

$$\begin{split} f_{M}^{\Box_{D}n} \circ \overline{\Delta}_{D}^{n-1} \circ \xi_{n+1} \\ &= [f_{M}^{\Box_{D}n-1} \Box_{D}f_{M}] \circ (\overline{\Delta}_{D}^{n-2} \Box_{D}\widetilde{D}) \circ \overline{\Delta}_{D} \circ \xi_{n+1} \\ &= [f_{M}^{\Box_{D}n-1} \circ \overline{\Delta}_{D}^{n-2} \Box_{D}f_{M}] \circ \overline{\Delta}_{D} \circ \xi_{n+1} \\ &= [f_{M}^{\Box_{D}n-1} \circ \overline{\Delta}_{D}^{n-2} \circ \xi_{n+1} \Box_{D}f_{M} \circ \xi_{n+1}] \circ \overline{\Delta}_{D^{n+1}} \\ (37),(38) &= [\Phi(D^{n-1}, D^{2}, 0)\eta(D^{n-1}, D^{2}, 0)\Box_{D}\Phi(D, D^{n}, 0)\eta(D, D^{n}, 0)] \circ \overline{\Delta}_{D^{n+1}} \\ &= [\Phi(D^{n-1}, D^{2}, 0)\Box_{D}\Phi(D, D^{n}, 0)] \circ [\eta(D^{n-1}, D^{2}, 0)\Box_{D}\eta(D, D^{n}, 0)] \circ \overline{\Delta}_{D^{n+1}} \\ (\frac{33}{=} &= [\Phi(D^{n-1}, D^{2}, 0)\Box_{D}\Phi(D, D^{n}, 0)] \circ [(E/D^{n-1}\Box_{E}\xi_{1}^{2})\Box_{D}(E/D\Box_{E}\xi_{1}^{n})] \circ \varphi(D^{n-1}, D^{2}, D) \\ &= & [\Phi(D^{n-1}, D^{2}, 0)(E/D^{n-1}\Box_{E}\xi_{1}^{2})\Box_{D}\Phi(D, D^{n}, 0)(E/D\Box_{E}\xi_{1}^{n})] \circ \varphi(D^{n-1}, D^{2}, D) \\ (\frac{39}{=}, (40) &= & [\Phi(D^{n-1}, D, 0)\Box_{D}\Phi(D, D, 0)] \circ \varphi(D^{n-1}, D^{2}, D) \\ (\frac{20}{=} &= & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D, 0)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1}, D^{2}, D). \\ \Box = & [\Phi(D^{n-1}, D^{2}, D)\Box_{D}(E/D\Box_{E}D)] \circ \varphi(D^{n-1$$

LEMMA B.5. Let  $\delta: D \to E$  be a monomorphism which is a coalgebra homomorphism in an abelian monoidal category  $\mathcal{M}$ . Then we have

$$(D^{m+n}, \delta_{m+n}) = \ker[(p_{D^m}^E \Box_E p_{D^n}^E) \circ \overline{\Delta}_E].$$

*Proof.* By Remark 2.9, we have the following exact sequence:

$$0 \longrightarrow D^m \wedge_E D^n = D^{m+n} \xrightarrow{i_{D^m \wedge_E D^n}^E = \delta_{m+n}} E \xrightarrow{(p_{D^m}^E \square_E p_{D^n}^E) \circ \overline{\Delta}_E} \xrightarrow{E} \overline{D^m} \square_E \overline{E} \overline{D^n}.$$

Then we conclude.

DEFINITION B.6. By Lemma B.5, we have

$$(D^n, \delta_n) = \ker[(p_{D^{n-1}}^E \Box_E p_D^E) \circ \overline{\Delta}_E]$$

so that

$$(E/D^n, p_{D^n}^E) = \operatorname{Coker}(\delta_n) = \operatorname{Coker}[\operatorname{Ker}((p_{D^{n-1}}^E \Box_E p_D^E) \circ \overline{\Delta}_E)] = Im[(p_{D^{n-1}}^E \Box_E p_D^E) \circ \overline{\Delta}_E].$$

Thus there exists a unique morphism  $\gamma_n: E/D^n \to E/D^{n-1} \square_E E/D$  such that

(41) 
$$\gamma_n \circ p_{D^n}^E = (p_{D^{n-1}}^E \Box_E p_D^E) \circ \overline{\Delta}_E.$$

Obviously  $\gamma_n$  is a monomorphism and is a morphism of *E*-bicomodules.

LEMMA B.7. Let  $\mathcal{M}$  be an abelian monoidal category. We have

(42) 
$$(\gamma_n \Box_E D) \circ \eta(D^n, D, 0) = \eta(D^{n-1}, D^2, D).$$

*Proof.* We apply (7) in the case

$$\begin{split} E &= E, F = D^n, B = D, A = 0, i_B^{F \wedge E B} = \xi_1^{n+1}, i_{F \wedge E B}^E = \delta_{n+1}, p_A^{F \wedge E B} = \mathrm{Id}_{D^{n+1}}. \\ \text{in order to obtain:} \\ (43) \qquad (E/D^n \Box_E \xi_1^{n+1}) \circ \eta(D^n, D, 0) = (p_{D^n}^E \circ \delta_{n+1} \Box_E D^{n+1}) \circ \overline{\Delta}_{D^{n+1}}. \\ \text{Therefore we have} \\ \qquad (E/D^{n-1} \Box_E E/D \Box_E \xi_1^{n+1}) \circ (\gamma_n \Box_E D) \circ \eta(D^n, D, 0) \\ &= (\gamma_n \Box_E D^{n+1}) \circ (E/D^n \Box_E \xi_1^{n+1}) \circ \eta(D^n, D, 0) \\ (\overset{(43)}{3} \qquad (\gamma_n \Box_E D^{n+1}) \circ (p_{D^n}^E \circ \delta_{n+1} \Box_E D^{n+1}) \circ \overline{\Delta}_{D^{n+1}} \\ &= (\gamma_n \circ p_D^E \circ \delta_{n+1} \Box_E D^{n+1}) \circ \overline{\Delta}_{D^{n+1}} \\ &= (\gamma_n \circ p_D^E \circ \delta_{n+1} \Box_E D^{n+1}) \circ \overline{\Delta}_{D^{n+1}} \\ (\overset{(41)}{4} = [(p_{D^{n-1}}^E \Box_E p_D^E \circ \delta_{n+1} \Box_E D^{n+1}] \circ \overline{\Delta}_{D^{n+1}} \\ &= (p_{D^{n-1}}^E \circ \delta_{n+1} \Box_E p_D^E \circ \delta_{n+1} \odot (D^{n+1} \Box_E D^{n+1}) \circ \overline{\Delta}_{D^{n+1}} \\ &= (p_{D^{n-1}}^E \circ \delta_{n+1} \Box_E p_D^E \circ \delta_{n+1} \Box_E D^{n+1}) \circ (D^{n-1} \Box_E D_{D^{n+1}}) \circ \overline{\Delta}_{D^{n+1}} \\ &= (p_{D^{n-1}}^E \circ \delta_{n+1} \Box_E p_D^{n+1}) (p_{D^{n-1}}^{n+1}) \circ (D^{n-1} \Box_E D_{D^{n+1}}) \circ \overline{\Delta}_{D^{n+1}} \\ &= (E/D^{n-1} \Box_E (p_D^E \circ \delta_{n+1} \Box_E D^{n+1}) \overline{\Delta}_{D^{n+1}}] \circ (E/D^{n-1} \Box_E \xi_2^{n+1}) \circ \eta(D^{n-1}, D^2, 0) \\ &= [E/D^{n-1} \Box_E (p_D^E \circ \delta_{n+1} \Box_E D^{n+1}) \overline{\Delta}_{D^n} \circ (D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E (p_D^E \circ \delta_{n+1} (\Box_E D^{n-1} \Box_E (p_D^E \circ \delta_2 \Box_E D^2) \overline{\Delta}_D^2] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_2^{n+1}) \circ [E/D^{n-1} \Box_E (p_D (D^n \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_2^{n+1}) \circ (E/D^{n-1} \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_2^{n+1}) \circ (E/D^{n-1} \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_2^{n+1}) \circ (E/D^{n-1} \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_1^{n+1}) \circ (E/D^{n-1} \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_1^{n+1}) \circ (E/D^{n-1} \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_1^{n+1}) \circ (E/D^{n-1} \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_1^{n+1}) \circ (E/D^{n-1} \Box_E \eta(D, D, 0)] \eta(D^{n-1}, D^2, 0) \\ &= (E/D^{n-1} \Box_E E/D \Box_E \xi_1^{n+1}) \circ \eta(D$$

Since  $E/D^{n-1}\Box_E E/D\Box_E \xi_1^{n+1}$  is a monomorphism, we conclude.

THEOREM B.8. Let  $\mathcal{M}$  be a cocomplete abelian monoidal category. Then, with the hypothesis and notations of Theorem 3.13, we have:

(44) 
$$\Phi(D^n, D, 0) = [\Phi(D^{n-1}, D, 0) \Box_D(E/D \Box_E D)] \circ [E/D^{n-1} \Box_E^{-D} \overline{\rho}_{E/D \Box_E D}] \circ (\gamma_n \Box_E D).$$

*Proof.* We have

$$\begin{split} & [\Phi(D^{n-1}, D, 0) \Box_D(E/D \Box_E D)] \circ [E/D^{n-1} \Box_E{}^D \overline{\rho}_{E/D \Box_E D}] \circ (\gamma_n \Box_E D) \circ \eta(D^n, D, 0) \\ & \stackrel{(42)}{=} [\Phi(D^{n-1}, D, 0) \Box_D(E/D \Box_E D)] \circ [E/D^{n-1} \Box_E{}^D \overline{\rho}_{E/D \Box_E D}] \circ \eta(D^{n-1}, D^2, D) \\ & \stackrel{(34)}{=} [\Phi(D^{n-1}, D, 0) \Box_D(E/D \Box_E D)] \circ \varphi(D^{n-1}, D^2, D) \\ & \stackrel{(36)}{=} f_M^{\Box_D n} \circ \overline{\Delta}_{\widetilde{D}}^{n-1} \circ \xi_{n+1} \\ & \stackrel{(19)}{=} \Phi(D^n, D, 0) \circ \eta(D^n, D, 0). \end{split}$$

Since  $\eta(D^n, D, 0)$  is an epimorphism, we conclude.

PROPOSITION B.9. Let E be a coalgebra in an abelian monoidal category  $\mathcal{M}$ , let X be a right coideal and let Y be a left coideal of E in  $\mathcal{M}$ . Assume that the morphism  $E/X \Box_{EPY}$  is an epimorphism. Then we have that the following sequence is exact in  $\mathcal{M}$ :

$$0 \longrightarrow X \wedge_E Y \xrightarrow{i_{X \wedge_E Y}^E} E \xrightarrow{(p_X \square_E p_Y) \circ \overline{\Delta}_E} \xrightarrow{E} \square_E \xrightarrow{E} 0$$

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*Proof.* We have:

$$(E/X\square_E p_Y) \circ \overline{\rho}_{E/X}^r \circ p_X = (E/X\square_E p_Y) \circ (p_X\square_E E) \circ \overline{\Delta}_E = (p_X\square_E p_Y) \circ \overline{\Delta}_E,$$

so that  $(p_X \Box_E p_Y) \circ \overline{\Delta}_E$  is an epimorphism as a composition of epimorphisms. The conclusion follows by (3). 

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