# A REMARK ON THE UENO-CAMPANA'S THREEFOLD 

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#### Abstract

We show that the Ueno-Campana's threefold cannot be obtained as the blow-up of any smooth threefold along a smooth centre, answering negatively a question raised by Oguiso and Truong.


## 1. Introduction

Let $E_{\tau}=\mathbb{C} /(\mathbb{Z}+\mathbb{Z} \tau)$ be the complex elliptic curve of period $\tau$. There exist exactly two elliptic curves with automorphism group bigger than $\{ \pm 1\}$ : these are defined respectively by the periods $\sqrt{-1}$ and the cubic root of unity $\omega:=(-1+\sqrt{-3}) / 2$.

We consider the diagonal action of the cyclic group generated by $\sqrt{-1}$ (resp. $-\omega$ ) on the product

$$
E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} \quad\left(\text { resp. } E_{\omega} \times E_{\omega} \times E_{\omega}\right)
$$

and we denote by $X_{4}$ (resp. $X_{6}$ ) the minimal resolution of their quotients:

$$
E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} /\langle\sqrt{-1}\rangle \quad\left(\text { resp. } E_{\omega} \times E_{\omega} \times E_{\omega} /\langle-\omega\rangle\right) .
$$

The minimal resolutions are obtained by a single blow-up at the maximal ideal of each singular point of the quotients above.

The threefolds $X_{4}$ and $X_{6}$ have been extensively studied in the past. In particular, they admit an automorphism of positive entropy (e.g. see Ogu15 for more details). The variety $X_{4}$ is now referred as the

[^0]Ueno-Campana's threefold. It has been recently shown that $X_{4}$ and $X_{6}$ are rational. Indeed, Oguiso and Truong OT15 showed the rationality of $X_{6}$, and Colliot-Théléne CT15 showed the rationality of $X_{4}$, after the work of Catanese, Oguiso and Truong [COT14]. The unirationality of $X_{4}$ was conjectured by Ueno [Uen75], whilst Campana asked about the rationality of $X_{4}$ in Cam11].

The aim of this note is to give a negative answer to the following question raised by Oguiso and Truong (see Ogu15[Question 5.11] and [Tru15][Question 2]).
Question 1.1. Can $X_{4}$ or $X_{6}$ be obtained as the blow-up of $\mathbb{P}^{3}, \mathbb{P}^{2} \times \mathbb{P}^{1}$ or $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{1}$ along smooth centres?

Our main result is the following:
Theorem 1.2. Let $A$ be an abelian variety of dimension three and let $G$ be a finite group acting on $A$ such that the quotient map

$$
\rho: A \rightarrow Z=A / G
$$

is étale in codimension 2.
Assume that there exists a resolution $f: X \rightarrow Z$ given by the blow-up of the singular points of $Z$ and such that the exceptional divisor at each singular point of $Z$ is irreducible.

Then $X$ cannot be obtained as the blow-up of a smooth threefold along a smooth centre.

Note that Theorem 1.2 provides a negative answer to Question 1.1. Very recently, Lesieutre Les15 announced that Question 1.1 admits a negative answer, using different methods.

## 2. Preliminary results

We use some of the methods introduced in CT14. Let $X$ be a normal projective threefold with isolated quotient singularities. Given a basis $\gamma_{1}, \ldots, \gamma_{m}$ of $H^{2}(X, \mathbb{C})$, the cubic form associated to $X$ is the homogeneous polynomial of degree 3 defined by:

$$
F_{X}\left(x_{1}, \ldots, x_{m}\right)=\left(x_{1} \gamma_{1}+\cdots+x_{m} \gamma_{m}\right)^{3} \in \mathbb{C}\left[x_{1}, \ldots, x_{m}\right] .
$$

Note that, modulo the natural action of $\mathrm{GL}(m, \mathbb{C})$, the cubic $F_{X}$ does not depend on the choice of the base and it is a topological invariant of the underlying manifold $X$ (see OVdV95 for more details). In particular, if

$$
\mathcal{H}_{F_{X}}=\left(\partial_{x_{i}} \partial_{x_{j}} F_{X}\right)_{i, j=1, \ldots, m}
$$

denotes the Hessian matrix associated to $F_{X}$ and $p \in H^{2}(X, \mathbb{C})$, then the rank of $\mathcal{H}_{F_{X}}$ at $p$ is well-defined.

The following basic tool was used in CT14 in a more general context. We provide a proof for the reader's convenience.

Lemma 2.1. Let $Y$ be a normal projective threefold with isolated quotient singularities and let $f: X \rightarrow Y$ be the blow-up of $Y$ along a point $q \in Y$ (resp. a curve $C \subseteq Y$ ). Assume that the exceptional divisor of $f$ is irreducible and let $E$ be its class in $H^{2}(X, \mathbb{C})$.

Then the rank of the Hessian matrix $\mathcal{H}_{F_{X}}$ of $F_{X}$ at $E$ is one (resp. at most two).

Note that by [CT14][Lemma 2.7 and Lemma 2.12] the rank of $\mathcal{H}_{F_{X}}$ is never zero.

Proof. We have $H^{2}(X, \mathbb{C})=\left\langle E, f^{*}\left(\gamma_{1}\right), \ldots, f^{*}\left(\gamma_{m}\right)\right\rangle$ where $\gamma_{1}, \ldots, \gamma_{m}$ is a basis of $H^{2}(Y, \mathbb{C})$.

Consider the cubic form $F_{X}$ associated to $X$ with respect to this basis:

$$
F_{X}\left(x_{0}, \ldots, x_{m}\right)=\left(x_{0} E+\sum_{i=1}^{m} x_{i} f^{*}\left(\gamma_{i}\right)\right)^{3}
$$

Since $f^{*}\left(\gamma_{i}\right) \cdot f^{*}\left(\gamma_{j}\right) \cdot E=0$ for all $i, j=1, \ldots m$, we have

$$
F_{X}\left(x_{0}, \ldots, x_{m}\right)=x_{0}^{3} E^{3}+3 \sum_{i=1}^{m} x_{0}^{2} x_{i} E^{2} f^{*}\left(\gamma_{i}\right)+\left(\sum_{i=1}^{m} x_{i} f^{*}\left(\gamma_{i}\right)\right)^{3} .
$$

Let $a=E^{3}$ and let $b_{i}=E^{2} f^{*}\left(\gamma_{i}\right)$ for $i=1, \ldots, m$. Note that if $f$ is the blow-up of a point $q \in Y$ then $b_{1}=\ldots=b_{m}=0$.

Thus, we have

$$
F_{X}\left(x_{0}, \ldots, x_{m}\right)=a x_{0}^{3}+3 \sum_{i=1}^{m} b_{i} x_{0}^{2} x_{i}+G\left(x_{1}, \ldots, x_{m}\right),
$$

where $G$ is a homogeneous cubic polynomial in the variables $x_{1}, \ldots, x_{m}$, i.e. it does not depend on $x_{0}$. Let $p=y_{0} E+\sum_{i=1}^{m} y_{i} f^{*} \gamma_{i} \in H^{2}(X, \mathbb{C})$, for some $y_{0}, \ldots, y_{m} \in \mathbb{C}$ and let $p^{\prime}=\left(y_{1}, \ldots, y_{m}\right)$. After removing the first row and the first column, the Hessian matrix $\mathcal{H}_{F_{X}}(p)$ of $F_{X}$ at $p$, coincides with the Hessian matrix $\mathcal{H}_{G}\left(p^{\prime}\right)$ of $G$ at $p^{\prime}$.

In particular, if $p=E$, then $p^{\prime}=(0, \ldots, 0)$ and $\mathcal{H}_{G}\left(p^{\prime}\right)$ is the zero matrix. Thus, the rank of the Hessian of $F_{X}$ at $p$ is at most two. In addition, if $b_{1}=\ldots=b_{m}=0$, then the rank of $\mathcal{H}_{F}$ at $p$ is exactly one.

## 3. Proofs

Lemma 3.1. Let $A$ be an abelian variety of dimension 3 and let $G$ be a finite group acting on $A$ such that the quotient map $\rho: A \rightarrow Z=A / G$
is étale in codimension 2. Let $F_{Z}$ be the cubic form associated to $Z$ and let $p \in H^{2}(Z, \mathbb{C})$ such that $\operatorname{rk} \mathcal{H}_{F_{Z}}(p) \leq 1$.

Then $p=0$.
Proof. The morphism $\rho$ induces an immersion of vector spaces

$$
\rho^{*}: H^{2}(Z, \mathbb{C}) \rightarrow H^{2}(A, \mathbb{C})
$$

Thus, there exists a basis of $H^{2}(A, \mathbb{C})$ such that if $F_{A}$ is the cubic associated to $A$ with respect to this basis and $d$ is the degree of $\rho$, then

$$
F_{Z}\left(x_{1}, \ldots, x_{m}\right)=d \cdot F_{A}\left(x_{1}, \ldots, x_{m}, 0, \ldots, 0\right) .
$$

It is enough to show that if $q \in H^{2}(A, \mathbb{C})$ is such that the rank of $\mathcal{H}_{F_{A}}$ at $q$ is not greater than one, then $q=0$.

Write $A=\mathbb{C}^{3} / \Gamma$ and consider $z_{1}, z_{2}, z_{3}$ coordinates on $\mathbb{C}^{3}$. Then a basis of $H^{2}(A, \mathbb{C})$ is given by

$$
\begin{array}{ll}
z_{i j}=d z_{i} \wedge d z_{j} & 1 \leq i<j \leq 3 \\
z_{i \bar{j}}=d z_{i} \wedge d \bar{z}_{j} & i, j \in\{1,2,3\} \\
z_{\overline{i j}}=d \bar{z}_{i} \wedge d \bar{z}_{j} & 1 \leq i<j \leq 3
\end{array}
$$

For any $x \in H^{2}(A, \mathbb{C})$, let $x_{i j}, x_{i \bar{j}}$ and $x_{\bar{i} \bar{j}}$ be the coordinates of $x$ with respect to the basis above and let $F_{A}^{\prime}$ be the cubic associated to this basis. It is enough to show that if $q \in H^{2}(A, \mathbb{C})$ is such that the rank of $\mathcal{H}_{F_{A}^{\prime}}$ at $q$ is not greater than one, then $q=0$. Let $q_{i j}, q_{i \bar{j}}$ and $q_{\overline{i j}}$ be the coordinates of $q$.

The $(2 \times 2)$-minor of $\mathcal{H}_{F_{A}^{\prime}}$ at $x$ defined by the rows corresponding to $x_{12}$ and $x_{13}$ and the columns corresponding to $x_{2 \overline{1}}$ and $x_{3 \overline{1}}$ is given by

$$
\left(\begin{array}{cc}
0 & 6 x_{\overline{2} \overline{3}} \\
6 x_{\overline{2} \overline{3}} & 0
\end{array}\right) .
$$

It follows that $q_{\overline{2} \overline{3}}=0$. By choosing suitable $(2 \times 2)$-minors, it follows easily that each coordinate of $q$ is zero. Thus, the claim follows.

Proof of Theorem 1.2. Suppose not. Then there exists a smooth projective threefold $Y$ such that $X$ can be obtained as the blow-up $g: X \rightarrow$ $Y$ at a smooth centre. Let $E$ be the exceptional divisor of $g$. Let $k$ be the number of singular points of $Z$ and let $E_{1}, \ldots, E_{k}$ be the exceptional divisors on $X$ corresponding to the singular points of $Z$.

We want to prove that $E=E_{i}$ for some $i=1, \ldots, k$. Denote by $p$ the class of $E$ in $H^{2}(X, \mathbb{C})$. Lemma 2.1 implies that the rank of $\mathcal{H}_{F_{X}}$ at $p$ is not greater than two.

Let $\gamma_{1}, \ldots, \gamma_{m} \in H^{2}(Z, \mathbb{C})$ be a basis and let $F_{Z}$ be the associated cubic form. Then $f^{*} \gamma_{1}, \ldots, f^{*} \gamma_{m},\left[E_{1}\right], \ldots,\left[E_{k}\right]$ is a basis of $H^{2}(X, \mathbb{C})$ and if $F_{X}$ denotes the associated cubic form, we have

$$
F_{X}\left(x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{k}\right)=F_{Z}\left(x_{1}, \ldots, x_{m}\right)+\sum_{i=1}^{k} a_{i} y_{i}^{3}
$$

where $a_{i}=E_{i}^{3}$ is a non-zero integer, for $i=1, \ldots, k$.
Thus, the Hessian matrix of $F_{X}$ is composed by two blocks: one is the Hessian matrix of $F_{Z}$ and the other one is a diagonal matrix, whose only non-zero entries are $6 a_{i}$ for $i=1, \ldots, k$. We may write $p=\left(p^{0}, p^{1}\right)=\left(p_{1}^{0}, \ldots, p_{m}^{0}, p_{1}^{1}, \ldots, p_{k}^{1}\right)$. We have rk $\mathcal{H}_{F_{Z}}\left(p^{0}\right) \leq 2$.

We distinguish two cases. If $\operatorname{rk} \mathcal{H}_{F_{Z}}\left(p^{0}\right)=2$, then $p^{1}=(0, \ldots, 0)$ and in particular $E$ is numerically equivalent to $f^{*} D$, for some pseudoeffective Cartier divisor $D$ on $Z$. Since $A$ is abelian, it follows that $\rho^{*} D$ is a nef divisor. Thus $E$ is nef, a contradiction.

If rk $\mathcal{H}_{F_{Z}}\left(p^{0}\right) \leq 1$, then Lemma 3.1 implies that $p^{0}=0$. Thus,

$$
E \equiv c_{s} E_{s}+c_{t} E_{t}
$$

for some distinct $s, t \in\{1, \ldots, k\}$ and $c_{s}, c_{t}$ rational numbers. Since $E$ is effective non-trivial, at least one of the $c_{i}$ is positive. By symmetry, we may assume $c_{s}>0$. By the negativity lemma, the divisor $E_{s}$ is covered by rational curves $C$ such that $E_{s} \cdot C<0$. Since $E_{s}$ and $E_{t}$ are disjoint, it follows that $E \cdot C<0$, which implies that $C$ is contained in $E$. Thus $E_{s}$ is contained in $E$. Since $E$ is prime, it follows that $E=E_{s}$ and $c_{t}=0$.

Finally, note that $g$ contracts $E=E_{s}$ to a point, as otherwise there exists a small contraction $\eta: Y \rightarrow Z$ and in particular $Z$ is not $\mathbb{Q}$ factorial, a contradiction. Thus, $g: X \rightarrow Y$ is the contraction of $E_{s}$ to the corresponding singular point on $Z$, which is again a contradiction. The claim follows.

Remark 3.2. As K. Oguiso kindly pointed out to us, the same proof shows that if $f: X \rightarrow Z$ is as in Theorem 1.2 and $g$ is an automorphism on $X$ then the set of exceptional divisors of $f$ is invariant with respect to $g$. Thus, there exists a positive integer $m$ such that the power $g^{m}$ descends to an automorphism on $Z$.

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