FINITE PLASTICITY IN $\boldsymbol{P}^\top\boldsymbol{P}$. PART II: QUASISTATIC EVOLUTION AND LINEARIZATION

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Abstract. We address a finite-plasticity model based on the symmetric tensor $\boldsymbol{P}^\top\boldsymbol{P}$ instead of the classical plastic strain $\boldsymbol{P}$. Such a structure arises by assuming that the material behavior is invariant with respect to frame transformations of the intermediate configuration. The resulting variational model is lower-dimensional, symmetric, and based solely on the reference configuration. The constitutive model has been introduced in Part I [24] where the corresponding analysis at the material-point level has also been developed. Here, we combine the constitutive relation with the equilibrium system and prove the existence of quasistatic evolutions and convergence of time-discretizations. Secondly, we investigate the linearization limit for small deformations via a rigorous evolution Γ-convergence argument.

1. Introduction

Elastoplastic materials at finite strains are classically described in terms of their *deformation* $y : \Omega \to \mathbb{R}^3$ with respect to the reference configuration $\Omega \subset \mathbb{R}^3$ and of the *plastic deformation tensor* $\boldsymbol{P} : \Omega \to \mathrm{SL} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3} \,|\, \det \boldsymbol{A} = 1\}$, encoding indeed all the information on previous plastic transformations [26]. The evolution of the material is driven by the competition between energy-storage and plastic-dissipation mechanisms [52, 61]. The former calls for the specification of the *local energy density* of the medium which is assumed to be additively decomposed as $W_{\rm e}(\nabla y\, \boldsymbol{P}^{-1}) + W_{\rm p}(\boldsymbol{P})$. Here, $\nabla y\, \boldsymbol{P}^{-1}$ stands for the *elastic deformation tensor*. Its specific multiplicative form is classical [32, 33] and has recently received a novel justification from the kinematic viewpoint [57, 58]. The functions $W_{\rm e}$ and $W_{\rm p}$ are the elastic and plastic energy density, respectively.

A relevant class of finite plasticity models can be formulated in terms of the symmetric tensor $\boldsymbol{C}_{\rm p} := \boldsymbol{P}^\top\boldsymbol{P}$ only [36, 38, 60, 66]. This possibility hinges upon the invariance of such models with respect to frame tranformations of the so-called *intermediate* configuration. This requires the *isotropy* of the elastic energy density $W_{\rm e}$ and the *plastic-rotation indifference* of the plastic energy density $W_{\rm p}$, see (2.3). Note that the elastic isotropy assumption is common for polycrystals under moderate plastic deformations [15, 29, 55, 56, 67], possibly also in combination with additional mechanical effects [14, 34, 63], martensitic phase

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change [16, 17, 22], and structures [30, 48]. On the other hand, the plastic-rotation indifference of $W_{\rm p}$ is already advocated in [8, 27] as a natural requirement in relation with the multiplicative structure of the elastic deformation tensor $\nabla y\, \boldsymbol{P}^{-1}$, see also [50, Formula (4.5)], and allows to avoid introducing the intermediate configuration, a commonly controversial issue [50].

Finite-plasticity formulations in $\boldsymbol{C}_{\rm p}$ are fully Lagrangian and advantageous from the computational viewpoint [13, 60]. The symmetricity of $\boldsymbol{C}_{\rm p}$ makes the model lower-dimensional with respect to formulations in $\boldsymbol{P}$ and allows for more efficient matrix computations. In addition $\boldsymbol{C}_{\rm p}$ is a true tensor, defined indeed in the fixed reference configuration and the model requires no intermediate configurations. The reader is referred to the recent [51] for a comparative discussion of the many finite-plasticity models based on $\boldsymbol{C}_{\rm p}$ available in the literature. The main result of [51] consists in proving in the isotropic case that all these constitutive relations coincide with the one studied in this work.

Our interest in finite-plasticity models in $\boldsymbol{C}_{\rm p}$ is motivated by the possibility of phrasing them in variational terms, which in turn allows a rather complete mathematical treatment. Such analysis has been initiated in Part I of this work [24], where we focus on the constitutive model by identifying structural assumptions which make a description in $\boldsymbol{C}_{\rm p}$ amenable. The above mentioned invariances of the energy densities allow the local energy density of the medium to be rewritten in terms of ∇y and $\boldsymbol{C}_{\rm p}$ as $W_{\rm e}(\nabla y\, \boldsymbol{C}_{\rm p}^{-1/2}) + \widehat{W_{\rm p}}(\boldsymbol{C}_{\rm p})$. In addition, a corresponding plastic-rotation indifference of the dissipation mechanism has to be enforced. This modeling aspects are detailed in Part I [24] and combined with the analysis of the constitutive problem at the material point, i.e. by assuming the stress of the material to be given.

The aim of the present paper is to extend the results of Part I [24] to quasistatic evolution in three dimensions. This results from the combination of the material relation with the quasistatic equilibrium system, so that the stress of the material is itself an unknown, evolving indeed under the effect of external forces. The stored energy of the system takes the form

$$\int_\Omega \Big(W_{\rm e}(\nabla y\, \boldsymbol{C}_{\rm p}^{-1/2}) + \widehat{W_{\rm p}}(\boldsymbol{C}_{\rm p}) + \frac{\mu}{r} |\nabla \boldsymbol{C}_{\rm p}|^r \Big)\, {\rm d}x.$$

Our specific assumptions on the elastic energy density $W_{\rm e}$ are modeled on polycrystalline isotropic metallic materials. Following the classical setup in elasticity [4], $W_{\rm e}$ is here assumed to be *polyconvex*. In addition, the stored energy is augmented by a gradient term with $\mu > 0$ and $r > 1$. Such a term describes nonlocal plastic effects and is inspired to the by-now classical *gradient plasticity* theory [19, 20, 49]. In particular, its occurrence turns out to be crucial in order to prevent the formation of plastic microstructures and ultimately ensures the necessary compactness for the analysis. Note that in the finite-plasticity context, the introduction of suitable regularizing terms on the plastic variables seems at the moment unavoidable. Even at the level of *incremental* problems, gradient-like regularization are crucial [41] and their absence calls for strong modeling restrictions [39] .

A first result of this paper is the solvability of the quasistatic evolution problem within the variational frame of *energetic* solutions [40, 47]. The corresponding existence result at the material-point level is presented in Part I of this work [24]. Energetic solution of the quasistatic evolution problem are obtained as time-continuous limits of time discretizations. The existence proof follows the by now well-established general theory of energetic solvability of rate-independent systems [44]. This analysis can be compared with the first quasistatic-evolution existence result for finite plasticity from [37], where a regularizing term in $\nabla \boldsymbol{P}$ is considered. Our current regularizing term in $\nabla \boldsymbol{C}_{\rm p}$ is clearly different from the latter for it vanishes on all rotations.

The second main result of the paper is the rigorous justification of the classical linearization approach for small deformations. Within the small-deformation regime it is indeed customary to leave the nonlinear finite-strain frame and resort to linearized theories [28]. This model reduction is classically justified by heuristic Taylor-expansion arguments. Here, we aim instead at providing a rigorous linearization proof by means of an *evolution* Γ-*convergence* analysis in the spirit of the general abstract theory of [45]. This again extends to the quasistatic evolution setting the material-point results of Part I [24].

Note that a rigorous convergence result in case of the $\boldsymbol{P}$-based formulation was provided in [46] while we focus here on $\boldsymbol{C}_{\rm p}$-based plasticity instead. In contrast with [46], we discuss here the convergence of solutions for which existence is known. This involves the additional difficulty of discussing the convergence of the gradient terms. On the other hand, within the small-deformation limit we can handle the situation of a vanishing parameter μ. In other words, both local and gradient linearized elastoplasticity can be obtained as limits of our model by letting $\mu \to 0$ along the limit.

While referring the reader to Part I [24] for a thorough discussion, we briefly recall the constitutive model and its coupling with quasistatic equilibrium in Section 2. Section 3 is then devoted to the proof of the existence of energetic solutions to the quasistatic evolution problem. Eventually, the linearization limit for small deformations is detailed in Section 4 and two technical tools are presented in Appendices A and B.

2. Mechanical model

Aim of this preliminary section is to introduce the mechanical model. As the constitutive relation has been thoroughly discussed in Part I of this work [24], we limit ourselves here at recalling its basic features in order to make the exposition self-contained.

All notation is obviously borrowed from [24]. Boldface symbols indicate 2-tensors in $\mathbb{R}^3$, $\boldsymbol{A}{:}\boldsymbol{B} := \operatorname{tr}(\mathbf{A}\mathbf{B}^\top) = A_{ij}B_{ij}$ is the classical tensor contraction product (summation convention) where $\top$ denotes transposition, and $|\boldsymbol{A}|^2 = \boldsymbol{A}{:}\boldsymbol{A}$ is the Frobenius norm. The deviatoric part of a tensor $\boldsymbol{A}$ is defined as $\operatorname{dev}\boldsymbol{A} = \boldsymbol{A} - (\operatorname{tr}\boldsymbol{A})\boldsymbol{I}/3$, where $\boldsymbol{I}$ is the identity

2-tensor. The following tensor sets will turn out relevant

$$\mathbb{R}^{3\times 3}_{\mathrm{sym}} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{A} = \boldsymbol{A}^\top\}, \quad \mathbb{R}^{3\times 3}_{\mathrm{dev}} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \mid \mathrm{tr}\,\boldsymbol{A} = 0\}$$
$$\mathrm{SL} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3} \mid \det \boldsymbol{A} = 1\}, \quad \mathrm{SO} := \{\boldsymbol{A} \in \mathrm{SL} \mid \boldsymbol{A}^{-1} = \boldsymbol{A}^\top\},$$
$$\mathrm{GL}^+ := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3} \mid \det \boldsymbol{A} > 0\}, \quad \mathrm{SL}^+_{\mathrm{sym}} := \{\boldsymbol{A} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \cap \mathrm{SL} \mid \boldsymbol{A} > \boldsymbol{0}\}.$$

We consider an elastoplastic body occupying the reference configuration $\Omega \subset \mathbb{R}^3$, which is assumed to be nonempty, open, connected, and bounded with Lipschitz boundary $\partial\Omega$ and outward normal ν. The boundary $\partial\Omega$ is partitioned as $\partial\Omega = \overline{(\Gamma_{\mathrm{tr}} \cup \Gamma_{\mathrm{D}})}$ where the two sets Γ_{tr}, $\Gamma_{\mathrm{D}} \subset \partial\Omega$ are measurable and relatively open in $\partial\Omega$, $\Gamma_{\mathrm{D}} \subset \partial\Omega$ has positive surface measure, and $\Gamma_{\mathrm{tr}} \cap \Gamma_{\mathrm{D}} = \emptyset$. The set Γ_{tr} represents the portion of the boundary where traction is exerted. On the other hand, Γ_{D} corresponds to the part of the boundary where the body is fixed.

The deformation of the body $y : \Omega \to \mathbb{R}^3$ is assumed to be almost everywhere differentiable and the deformation gradient $\boldsymbol{F} := \nabla y$ is asked to have positive determinant, namely $\boldsymbol{F} \in \mathrm{GL}^+$. $\boldsymbol{F}$ is classically decomposed as [32, 33]

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}}\boldsymbol{P} \tag{2.1}$$

where $\boldsymbol{F}_{\mathrm{e}}$ denotes the *elastic* part of $\boldsymbol{F}$ and $\boldsymbol{P}$ its *plastic* part.

We assume that the plastic deformation is *isochoric*, which is typically the case for metals [61], and we impose accordingly the constraint $\det \boldsymbol{P} = 1$.

The *(right) Cauchy-Green* symmetric tensors associated to the three deformation gradients are defined by

$$\boldsymbol{C} := \boldsymbol{F}^\top\boldsymbol{F}, \quad \boldsymbol{C}_{\mathrm{e}} := \boldsymbol{F}_{\mathrm{e}}^\top\boldsymbol{F}_{\mathrm{e}}, \quad \boldsymbol{C}_{\mathrm{p}} := \boldsymbol{P}^\top\boldsymbol{P}.$$

Since $\det \boldsymbol{C}_{\mathrm{p}} = (\det \boldsymbol{P})^2 = 1$, we have that the plastic variable $\boldsymbol{C}_{\mathrm{p}}$ belongs to the five-dimensional manifold of symmetric, positive-definite, and unit-determinant matrices $\mathrm{SL}^+_{\mathrm{sym}}$.

The quasistatic evolution of the medium is governed by the interplay of energy-storage and dissipation mechanisms. The local energy density of the body is additively decomposed as

$$W_{\mathrm{e}}(\boldsymbol{F}_{\mathrm{e}}) + W_{\mathrm{p}}(\boldsymbol{P}) \tag{2.2}$$

in a *hyperelastic* and a *local plastic* contribution. The elastic-energy density $W_{\mathrm{e}} : \mathrm{GL}^+ \to [0, \infty]$ is assumed to be *frame indifferent* [65], namely $W_{\mathrm{e}}(\boldsymbol{R}\boldsymbol{F}_{\mathrm{e}}) = W_{\mathrm{e}}(\boldsymbol{F}_{\mathrm{e}})$ for all $\boldsymbol{R} \in \mathrm{SO}$. The *local plastic-energy density* $W_{\mathrm{p}} : \mathrm{SL} \to [0, \infty]$ describes purely kinematic hardening effects and is assumed to be smooth on its domain. By introducing additional internal variables, different hardening dynamics (e.g. isotropic) could be considered as well (see for example [37, 41] for a detailed account).

The main assumption of the model is *plastic-rotation indifference* [24], namely that the material behavior is invariant by frame transformations of the so-called *intermediate* configuration, that is

$$W_{\rm e}(\boldsymbol{F}_{\rm e}\boldsymbol{Q}) = W_{\rm e}(\boldsymbol{F}_{\rm e}), \quad W_{\rm p}(\boldsymbol{Q}\boldsymbol{P}) = W_{\rm p}(\boldsymbol{P}) \quad \forall \boldsymbol{Q} \in {\rm SO} \tag{2.3}$$

for all $\boldsymbol{F}_{\rm e} \in {\rm GL}^+$ and $\boldsymbol{P} \in {\rm SL}$. The condition on $W_{\rm e}$ states that elastic energy is *isotropic* while the one on $W_{\rm p}$ is the frame indifference with respect to the intermediate configuration. Both conditions are considered to be well-suited for describing finite-plastic phenomena in polycrystalline materials, see among others [15, 29, 55, 66] and [8, 27, 50], respectively.

Plastic-rotation indifference (2.3) implies that the energy density (2.2) can be expressed solely in terms of the tensors ∇y and $\boldsymbol{C}_{\rm p}$ as

$$W_{\rm e}(\nabla y\, \boldsymbol{C}_{\rm p}^{-1/2}) + \widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}) \tag{2.4}$$

where we have used the multiplicative decomposition (2.1) and we have introduced the function $\widehat{W}_{\rm p} : {\rm SL}^+_{\rm sym} \to [0, \infty]$ in terms of $W_{\rm p}$ via $\widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}) = W_{\rm p}(\boldsymbol{C}_{\rm p}^{1/2})$.

Assume now to be given the *body force* $b : \Omega \times (0, T) \to \mathbb{R}^3$ and the *traction* $g : \Gamma_{\rm tr} \times (0, T) \to \mathbb{R}^3$. By neglecting inertial effects we focus on the quasistatic approximation of the equilibrium system given by

$$\nabla\cdot\boldsymbol{\sigma} + b(t) = 0 \quad \text{in} \ \ \Omega \times (0, T) \tag{2.5}$$

where the *total stress* $\boldsymbol{\sigma}$ fulfills

$$\boldsymbol{\sigma} := \partial_{\nabla y} W_{\rm e}(\nabla y\, \boldsymbol{C}_{\rm p}^{-1/2}) = \partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\nabla y\, \boldsymbol{C}_{\rm p}^{-1/2})\boldsymbol{C}_{\rm p}^{-1/2}.$$

The equilibrium system (2.5) is complemented with the boundary conditions

$$y = {\rm id} \quad \text{in} \ \ \Gamma_{\rm D} \times (0, T), \tag{2.6}$$

$$\boldsymbol{\sigma}\nu = g(t) \quad \text{in} \ \ \Gamma_{\rm tr} \times (0, T) \tag{2.7}$$

Let us comment that our choice of boundary conditions is dictated by simplicity. Nonconstant imposed boundary deformations could be considered as well [18].

The evolution of the plastic variable $\boldsymbol{C}_{\rm p}$ will be described in terms of a balance of dissipative and internal forces as

$$\partial_{\dot{\boldsymbol{C}}_{\rm p}} \widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p}) + \partial_{\boldsymbol{C}_{\rm p}} W_{\rm e}(\nabla y \boldsymbol{C}_{\rm p}^{-1/2}) + \partial_{\boldsymbol{C}_{\rm p}} \widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}) - \mu\nabla\cdot(|\nabla \boldsymbol{C}_{\rm p}|^{r-2}\nabla \boldsymbol{C}_{\rm p}) \ni 0. \tag{2.8}$$

Here $\widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p})$ is the local *local dissipation density*, defined by

$$\widehat{R}(\boldsymbol{C}_{\rm p}, \dot{\boldsymbol{C}}_{\rm p}) := \widetilde{R}(\boldsymbol{C}_{\rm p}^{-1/2}\dot{\boldsymbol{C}}_{\rm p}\boldsymbol{C}_{\rm p}^{-1/2})$$

where we have set

$$\widetilde{R}(\boldsymbol{A}) := \begin{cases} \dfrac{\rho}{2}|\boldsymbol{A}| & \text{if } {\rm tr}\,(\boldsymbol{A}) = 0, \\ +\infty & \text{else} \end{cases}$$

for some given yield threshold $\rho > 0$, see [24]. Note that this corresponds to the standard *normality rule* along with a *von Mises*-type yield criterion whenever a plastic-rotation-invariance assumption analogous to (2.3) is imposed on the dissipation [24, Formula (2.12)]. The role of the coefficient 1/2 will be revealed by passing to the small-strain limit, see (4.7).

The symbol ∂ in (2.8) denotes the (partial) subdifferential in the sense of Convex Analysis [7]. We recall that, letting $\varphi : E \to (-\infty, \infty]$ denote a smooth or convex, proper, and lower semicontinuous function on the normed space E with duality pairing $\langle \cdot, \cdot \rangle$, we have $y^* \in \partial\varphi(x)$ iff $\varphi(x) < \infty$ and $\langle y^*, w - x\rangle \leq \varphi(w) - \varphi(x)$ for all $w \in E$. The second and third term in (2.8) are hence the subdifferential of the local energy density from (2.4) with respect to $\boldsymbol{C}_{\rm p}$.

The gradient term in (2.8) for $r > 1$ places the model within the by now classical frame of *gradient plasticity* theory [19, 20, 49] and introduces a length scale for oscillations in $\boldsymbol{C}_{\rm p}$ modulated by $\mu > 0$. Here, the gradient $\nabla\boldsymbol{C}_{\rm p}$ is a 3-tensor, namely $(\nabla\boldsymbol{C}_{\rm p})_{ijk} = (\boldsymbol{C}_{\rm p})_{ij,k}$, and its divergence is a 2-tensor reading $(\nabla\cdot\nabla\boldsymbol{C}_{\rm p})_{ij} = (\boldsymbol{C}_{\rm p})_{ij,kk}$. Note that a gradient term $\nabla\boldsymbol{P}$ is considered in the quasistatic evolution analysis in [37] as well. Although the compactifying effects of the two terms $\nabla\boldsymbol{C}_{\rm p}$ and $\nabla\boldsymbol{P}$ are comparable, such terms deliver different contributions. By defining for all 3-tensors $\boldsymbol{D}$ the norm $|\boldsymbol{D}|^2 := D^2_{ijk}$, the partial transposition $(\boldsymbol{D}^\top)_{ijk} = D_{jik}$, and the products $(\boldsymbol{D}\boldsymbol{A})_{ijk} = D_{ij\ell}A_{\ell k}$ and $(\boldsymbol{A}\boldsymbol{D})_{ijk} = A_{i\ell}D_{\ell jk}$ so that $|\boldsymbol{D}\boldsymbol{A}|, |\boldsymbol{A}\boldsymbol{D}| \leq |\boldsymbol{A}|\,|\boldsymbol{D}|$, one can compute

$$\begin{aligned}|\nabla\boldsymbol{C}_{\rm p}| &= |\nabla(\boldsymbol{P}^\top\boldsymbol{P})| = |(\boldsymbol{P}^\top\nabla\boldsymbol{P}^\top)^\top + \boldsymbol{P}^\top\nabla\boldsymbol{P}| \\ &\leq |\boldsymbol{P}^\top\nabla\boldsymbol{P}^\top| + |\boldsymbol{P}^\top\nabla\boldsymbol{P}| \leq 2|\boldsymbol{P}|\,|\nabla\boldsymbol{P}|.\end{aligned}$$

In particular, the term $|\nabla\boldsymbol{P}|$ controls $|\nabla\boldsymbol{C}_{\rm p}|$ for bounded $\boldsymbol{P}$. On the other hand, the term $|\nabla\boldsymbol{C}_{\rm p}|$ vanishes on SO. In particular, it does not control $|\nabla\boldsymbol{P}|$.

Eventually, relation (2.8) is complemented with boundary and initial conditions

$$|\nabla\boldsymbol{C}_{\rm p}|^{r-2}\nabla\boldsymbol{C}_{\rm p}\nu = \boldsymbol{0} \quad \text{in} \;\; \partial\Omega \times (0, T), \tag{2.9}$$

$$\boldsymbol{C}_{\rm p} = \boldsymbol{C}_{{\rm p},0} \quad \text{in} \;\; \Omega \tag{2.10}$$

where the initial datum $\boldsymbol{C}_{{\rm p},0} : \Omega \to {\rm SL}^+_{\rm sym}$ is given.

Given suitable data b, g, and $\boldsymbol{C}_{{\rm p},0}$, the *quasistatic evolution problem* consists in finding trajectories $t \mapsto (y(t), \boldsymbol{C}_{\rm p}(t))$ solving the equilibrium system (2.5)-(2.6) and the material constitutive relation (2.8), with the boundary condition (2.9) and the initial condition (2.10).

3. Energetic solvability of the quasistatic-evolution problem

This section aims at presenting an existence result for the quasistatic evolution problem (2.5)-(2.10). As the strong formulation of such problem seems at present inaccessible due to its highly nonlinear character, we follow [37] and resort to the by-now classical weak variational setting of *energetic formulations* [18, 40, 47, 44].

Let us start by introducing suitable functional spaces for the state variables. Assume to be given the coefficients $q_y, q_{\rm p}, r > 1$, specific requirements will be listed in (3.8). We will ask the deformation y to belong to

$$\mathcal{Y} := \{y \in W^{1,q_y}(\Omega, \mathbb{R}^3) \mid y = \mathrm{id} \text{ on } \Gamma_{\rm D}\},$$

whereas the state space for the internal variable $\boldsymbol{C}_{\rm p}$ is chosen to be

$$\mathcal{C} = \{\boldsymbol{C}_{\rm p} \in L^{q_{\rm p}}(\Omega, \mathbb{R}^{3\times3}) \cap W^{1,r}(\Omega, \mathbb{R}^{3\times3}) \mid \boldsymbol{C}_{\rm p} \in \mathrm{SL}^+_{\rm sym} \ \text{ a.e. in } \ \Omega\}.$$

Note that both $\mathcal{Y}$ and $\mathcal{C}$ are weakly closed subsets of separable and reflexive Banach spaces.

The *stored* energy of the system $\mathcal{W} : \mathcal{Y} \times \mathcal{C} \to [0, \infty]$ is defined by integrating on the reference configuration the sum of the local and the nonlocal energy densities as

$$\mathcal{W}(y, \boldsymbol{C}_{\rm p}) = \int_\Omega \Big(W_{\rm e}(\nabla y\, \boldsymbol{C}_{\rm p}^{-1/2}) + \widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}) + \frac{\mu}{r}|\nabla \boldsymbol{C}_{\rm p}|^r \Big)\, \mathrm{d}x$$

and the *generalized work* of external forces reads

$$\langle \ell(t), y\rangle := \int_\Omega b(x,t)\cdot y(x)\ \mathrm{d}x + \int_{\Gamma_{\rm tr}} g(x,t)\cdot y(x)\, \mathrm{d}S \tag{3.1}$$

where $\mathrm{d}S$ is the surface measure on $\partial\Omega$. Eventually, the *total energy* $\mathcal{E} : \mathcal{Y}\times\mathcal{C} \to (-\infty, \infty]$ of the body is given by

$$\mathcal{E}(y, \boldsymbol{C}_{\rm p}, t) = \mathcal{W}(y, \boldsymbol{C}_{\rm p}) - \langle \ell(t), y\rangle. \tag{3.2}$$

As regards the dissipation we let $\mathcal{D} : \mathcal{C} \times \mathcal{C} \to [0, \infty]$ be defined by

$$\mathcal{D}(\boldsymbol{C}_{\rm p}, \widehat{\boldsymbol{C}}_{\rm p}) := \int_\Omega D(\boldsymbol{C}_{\rm p}(x), \widehat{\boldsymbol{C}}_{\rm p}(x))\, \mathrm{d}x$$

where the local dissipation $D : \mathrm{SL}^+_{\rm sym} \times \mathrm{SL}^+_{\rm sym} \to [0, \infty]$ is given as

$$D(\boldsymbol{C}_{\rm p}, \widehat{\boldsymbol{C}}_{\rm p}) := \inf \Big\{ \int_0^1 \widehat{R}(\boldsymbol{C}_{\rm p}(t), \dot{\boldsymbol{C}}_{\rm p}(t))\mathrm{d}t \mid \boldsymbol{C}_{\rm p} \in C^1(0,1; \mathrm{SL}^+_{\rm sym}),$$
$$\boldsymbol{C}_{\rm p}(0) = \boldsymbol{C}_{\rm p},\ \boldsymbol{C}_{\rm p}(1) = \widehat{\boldsymbol{C}}_{\rm p} \Big\}.$$

We refer the reader to Part I of this work [24] and [22, 37] for a justification of the latter choice and limit ourselves in noting that D results in a *Finsler metric* on $\mathrm{SL}^+_{\rm sym}$ [43] as the function $R(\boldsymbol{C}_{\rm p}, \cdot)$ is smooth for $\dot{\boldsymbol{C}}_{\rm p} \neq 0$, positively 1-homogeneous, and has a strictly convex square power. For the sake of completeness we collect here some relevant properties of D from [24] to be used below.

Lemma 3.1 (Properties of D)**.** *The following hold.*

i) *D is symmetric and satisfies the triangle inequality*

$$D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) \le D(\boldsymbol{C}_\mathrm{p},\widetilde{\boldsymbol{C}}_\mathrm{p}) + D(\widetilde{\boldsymbol{C}}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) \quad \forall \boldsymbol{C}_\mathrm{p},\, \widehat{\boldsymbol{C}}_\mathrm{p},\, \widetilde{\boldsymbol{C}}_\mathrm{p} \in \mathrm{SL}^+_\mathrm{sym}.$$

ii) *The map D fulfills*

$$D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) \le \widetilde{R}(\log \boldsymbol{C}_\mathrm{p} - \log \widehat{\boldsymbol{C}}_\mathrm{p}) \quad \forall \boldsymbol{C}_\mathrm{p},\, \widehat{\boldsymbol{C}}_\mathrm{p} \in \mathrm{SL}^+_\mathrm{sym}. \tag{3.3}$$

In particular, D is locally Lipschitz continuous.

iii) *D satisfies the bound*

$$D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) \le 2\rho(|\boldsymbol{C}_\mathrm{p}|+|\widehat{\boldsymbol{C}}_\mathrm{p}|+6) \quad \forall \boldsymbol{C}_\mathrm{p},\, \widehat{\boldsymbol{C}}_\mathrm{p} \in \mathrm{SL}^+_\mathrm{sym}.$$

iv) *D can be explicitly computed by*

$$D(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) = \frac{\rho}{2}\big|\log\big(\boldsymbol{C}_\mathrm{p}^{-1/2}\widehat{\boldsymbol{C}}_\mathrm{p}\boldsymbol{C}_\mathrm{p}^{-1/2}\big)\big| \quad \forall \boldsymbol{C}_\mathrm{p},\, \widehat{\boldsymbol{C}}_\mathrm{p} \in \mathrm{SL}^+_\mathrm{sym}.$$

Note that properties (i)–(iii) directly translate to corresponding properties for $\mathcal{D}$. In particular, $\mathcal{D}$ is locally Lipschitz continuous, therefore weakly continuous in $\mathcal{C}\times\mathcal{C}$.

Given an initial datum $(y_0,\boldsymbol{C}_{\mathrm{p},0}) \in \mathcal{Y}\times\mathcal{C}$, we say that a trajectory $(y,\boldsymbol{C}_\mathrm{p}) : [0,T] \to \mathcal{Y}\times\mathcal{C}$ is an *energetic solution of the quasistatic evolution problem starting from* $(y_0,\boldsymbol{C}_{\mathrm{p},0})$ if $(y(0),\boldsymbol{C}_\mathrm{p}(0)) = (y_0,\boldsymbol{C}_{\mathrm{p},0})$ and, for all $t\in[0,T]$, the following two conditions are satisfied

$$\begin{aligned}(y(t),\boldsymbol{C}_\mathrm{p}(t)) \in \mathcal{S}(t) := \Big\{&(y,\boldsymbol{C}_\mathrm{p}) \in \mathcal{Y}\times\mathcal{C} \mid \mathcal{E}(y,\boldsymbol{C}_\mathrm{p},t) < \infty \ \text{ and} \\ &\mathcal{E}(y,\boldsymbol{C}_\mathrm{p},t) \le \mathcal{E}(\widehat{y},\widehat{\boldsymbol{C}}_\mathrm{p},t) + \mathcal{D}(\boldsymbol{C}_\mathrm{p},\widehat{\boldsymbol{C}}_\mathrm{p}) \quad \forall(\widehat{y},\widehat{\boldsymbol{C}}_\mathrm{p}) \in \mathcal{Y}\times\mathcal{C}\Big\},\end{aligned} \tag{3.4}$$

$$\mathcal{E}(y(t),\boldsymbol{C}_\mathrm{p}(t),t) + \mathrm{Diss}_{\mathcal{D},[0,t]}(\boldsymbol{C}_\mathrm{p}) = \mathcal{E}(y_0,\boldsymbol{C}_{\mathrm{p},0},0) - \int_0^t \langle\dot{\ell}(\tau),y(\tau)\rangle\,\mathrm{d}\tau. \tag{3.5}$$

Relation (3.4) is the so-called *global stability* condition and we call $\mathcal{S}(t)$ the set of *stable states* at time $t\in[0,T]$. Equality (3.5) is the *energy balance* where the *total dissipation* $\mathrm{Diss}_{\mathcal{D},[0,t]}(\boldsymbol{C}_\mathrm{p})$ on $[0,t]$ is given by

$$\mathrm{Diss}_{\mathcal{D},[0,t]}(\boldsymbol{C}_\mathrm{p}) := \sup\left\{\sum_{i=1}^N \mathcal{D}(\boldsymbol{C}_\mathrm{p}(t_{i-1}),\boldsymbol{C}_\mathrm{p}(t_i))\right\}$$

where the supremum is taken over all partitions $\{0=t_0\le t_1\le \ldots\ldots t_N=t\}$ of $[0,t]$.

Now we are in a position to state the main result of this section.

Theorem 3.2 (Energetic solvability of the quasistatic system)**.** *Assume polyconvexity of* W_e *and coercivity of* W_e *and* $\widehat{W}_\mathrm{p}$, *namely*

$$\begin{aligned}&W_\mathrm{e}(\boldsymbol{F}_\mathrm{e}) = \mathbb{W}(\boldsymbol{F}_\mathrm{e},\mathrm{cof}\boldsymbol{F}_\mathrm{e},\det\boldsymbol{F}_\mathrm{e}) \quad \forall\boldsymbol{F}_\mathrm{e}\in\mathrm{GL}^+ \ \textit{for some} \\ &\mathbb{W}:\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3}\times\mathbb{R}_+ \to (-\infty,\infty] \ \textit{convex and lower semicontinuous},\end{aligned} \tag{3.6}$$

$$W_\mathrm{e}(\boldsymbol{F}_\mathrm{e}) \ge c_1|\boldsymbol{F}_\mathrm{e}|^{q_\mathrm{e}} - \frac{1}{c_1}, \qquad \widehat{W}_\mathrm{p}(\boldsymbol{C}_\mathrm{p}) \ge c_1|\boldsymbol{C}_\mathrm{p}|^{q_\mathrm{p}} - \frac{1}{c_1} \tag{3.7}$$

for some positive constant c_1*, and* $q_{\rm e}$*,* $q_{\rm p} > 1$*. Moreover, assume that*

$$\frac{1}{q_y} = \frac{1}{q_{\rm e}} + \frac{1}{2q_{\rm p}}, \quad q_y > 3, \qquad r > 1, \tag{3.8}$$

$\ell \in W^{1,1}([0,T];\mathcal{Y}^*)$*, and* $(y_0, \boldsymbol{C}_{\rm p,0}) \in \mathcal{S}(0)$*. Then, there exists an energetic solution of the quasistatic evolution problem starting from* $(y_0, \boldsymbol{C}_{\rm p,0})$*.*

More precisely, for all partitions $\{0 = t_0^k < t_1^k < \ldots < t_{N^k}^k = T\}$ *with time step* $\tau^k = \max_{i=1}^{N^k}(t_i^k - t_{i-1}^k)$ *the incremental minimization problems*

$$\begin{aligned}(y_i, \boldsymbol{C}_{{\rm p},i}) = \operatorname{Argmin}\big\{\mathcal{E}(y,\boldsymbol{C}_{\rm p},t_i^k) + \mathcal{D}(\boldsymbol{C}_{{\rm p},i-1},\boldsymbol{C}_{{\rm p},i}) \mid (y,\boldsymbol{C}_{\rm p}) \in \mathcal{Y}\times\mathcal{C}\big\}\\ \text{for } i = 1,\ldots,N^k\end{aligned} \tag{3.9}$$

admit a solution $\{(y_0, \boldsymbol{C}_{\rm p,0}), (y_1^k, \boldsymbol{C}_{\rm p,1}^k), \ldots, (y_{N^k}^k, \boldsymbol{C}_{{\rm p},N^k}^k)\}$ *and, as* $\tau^k \to 0$*, the corresponding piecewise backward-constant interpolants* $t \mapsto (\overline{y}^k(t), \overline{\boldsymbol{C}}_{\rm p}^k(t))$ *on the partition admit a not relabeled subsequence such that, for all* $t \in [0,T]$*,*

$$\begin{aligned}&(\overline{y}^k(t), \overline{\boldsymbol{C}}_{\rm p}^k(t)) \to (y(t), \boldsymbol{C}_{\rm p}(t)), \quad {\rm Diss}_{\mathcal{D},[0,t]}(\overline{\boldsymbol{C}}_{\rm p}^k) \to {\rm Diss}_{\mathcal{D},[0,t]}(\boldsymbol{C}_{\rm p}),\\ &\mathcal{E}(\overline{y}^k(t), \overline{\boldsymbol{C}}_{\rm p}^k(t), t) \to \mathcal{E}(y(t), \boldsymbol{C}_{\rm p}(t), t),\end{aligned}$$

and $\partial_t\mathcal{E}(\overline{y}^k(\cdot), \overline{\boldsymbol{C}}_{\rm p}^k(\cdot), \cdot) \to \partial_t\mathcal{E}(y(t), \boldsymbol{C}_{\rm p}(\cdot), \cdot)$ *in* $L^1(0,T)$ *where* $(y, \boldsymbol{C}_{\rm p})$ *in an energetic solution of the quasistatic evolution problem starting from* $(y_0, \boldsymbol{C}_{\rm p,0})$*.*

Before moving on let us comment that the assumptions on the elastic energy density (3.6)-(3.7) are indeed met by the actually fairly general class of *Ogden materials* [9, Sec. 4.9] corresponding indeed to the choice

$$\begin{aligned}&W_{\rm e}(\boldsymbol{F}_{\rm e}) := \widehat{W}_{\rm e}(\boldsymbol{C}_{\rm e}) := \sum_{i=1}^n a_i {\rm tr}\, \boldsymbol{C}_{\rm e}^{\gamma_i/2} + \sum_{j=1}^m b_j {\rm tr}\,({\rm cof}\, \boldsymbol{C}_{\rm e})^{\delta_j/2} + \Gamma(\det \boldsymbol{C}_{\rm e}^{1/2}),\\ &n, m \geq 1, \quad a_i, b_j > 0, \quad \gamma_i, \delta_j \geq 1, \quad s \mapsto \Gamma(s) \text{ convex on } (0,\infty), \quad \lim_{s\to 0^+} \Gamma(s) = \infty.\end{aligned}$$

Specifically, the function $W_{\rm e}(\boldsymbol{F}_{\rm e})$ is polyconvex whenever $\gamma_i, \delta_j \geq 1$. These energies are designed for describing materials undergoing large deformations, including rubbers. In the case of metal plasticity, which is the reference setting here, only the behavior of these energies at moderate strains is relevant, for large strains are the result of the plastic flow rather than of large elastic deformations. As their small-strain limit is the classical Saint-Venant-Kirchhoff potential of isotropic elasticity [9, Sec. 4.10], Ogden energies appear to be well-suited for polycrystalline isotropic metallic materials as well.

The proof of Theorem 3.2 follows the standard scheme of energetic solvability of rate-independent systems [44]. In particular, it follows closely the theory in [37], by adapting the proofs to the occurrence of the metric tensor $\boldsymbol{P}^\top \boldsymbol{P}$ instead of $\boldsymbol{P}$. A *caveat* on notation: in the following we use the same symbol c in order to indicate a generic constant, possibly depending on data and varying from line to line.

The first step is to prove that the incremental-minimization problems (3.9) admit solutions. The existence of minimizers ensues from the Direct Method as soon as coercivity and lower semincontinuity is established. This is indeed the focus of the next three Lemmas. Let us start by checking that the energy is coercive.

Lemma 3.3 (Coercivity of the energy)**.** *Under the assumptions of Theorem* 3.2*, the energy* $\mathcal{E}$ *is coercive in the following sense*

$$\mathcal{E}(y, \boldsymbol{C}_{\rm p}, t) \geq c_2 \|y\|^{q_y}_{W^{1,q_y}} + c_2 \|\boldsymbol{C}_{\rm p}\|^{q_{\rm p}}_{L^{q_{\rm p}}} + c_2 \|\nabla \boldsymbol{C}_{\rm p}\|^{r}_{L^r} - \frac{1}{c_2} \tag{3.10}$$

where c_2 *is a positive constant.*

Proof. From the coercivity assumption (3.7), an application of Young's inequality gives

$$\frac{1}{c_1} W_{\rm e}(\boldsymbol{F}_{\rm e}) + \frac{1}{c_1^2} \geq |\nabla y \boldsymbol{C}_{\rm p}^{-1/2}|^{q_{\rm e}} \geq \left(|\nabla y| / |\boldsymbol{C}_{\rm p}^{1/2}|\right)^{q_{\rm e}} \geq 3^{-1/2} (|\nabla y|^{q_{\rm e}} / |\boldsymbol{C}_{\rm p}|^{q_{\rm e}/2}) \geq$$
$$3^{-1/2} \left((1+t)\delta^{t/(t+1)} |\nabla y|^{q_{\rm e}/(t+1)} - t\delta |\boldsymbol{C}_{\rm p}|^{q_{\rm e}/2t}\right)$$

for any δ and t positive. Moreover, $\widehat{W_{\rm p}}(\boldsymbol{C}_{\rm p}) \geq c_1 |\boldsymbol{C}_{\rm p}|^{q_{\rm p}} - 1/c_1$. By taking δ sufficiently small and t such that $q_{\rm e}/2t = q_{\rm p}$, we obtain

$$W_{\rm e}(\boldsymbol{F}_{\rm e}) + \widehat{W_{\rm p}}(\boldsymbol{C}_{\rm p}) \geq c |\nabla y|^{2q_{\rm e} q_{\rm p}/(2q_{\rm p}+q_{\rm e})} + c|\boldsymbol{C}_{\rm p}|^{q_{\rm p}} - c,$$

therefore

$$\mathcal{W}(y, \boldsymbol{C}_{\rm p}) \geq c\|\nabla y\|^{q_y}_{L^{q_y}} + \|\boldsymbol{C}_{\rm p}\|^{q_{\rm p}}_{L^{q_{\rm p}}} - c.$$

As $|\langle \ell, y\rangle| \leq \|\ell\| \|y\|_{W^{1,q_y}}$, by virtue of Korn's inequality the statement follows. □

Note that indeed the coercivity lower bound (3.10) holds under the weaker condition $1/q_y = 1/q_{\rm e} + 1/(2q_{\rm p}) < 1$ as well.

We shall now proceed to prove that the energy is weakly lower semicontinuous. For all $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ we indicate by $\operatorname{cof} \boldsymbol{A}$ its *cofactor tensor* ($\operatorname{cof} \boldsymbol{A} = (\det \boldsymbol{A})\, \boldsymbol{A}^{-\top}$ for $\boldsymbol{A}$ invertible) and by $\mathbb{M}_1(\boldsymbol{A}) = \boldsymbol{A} \in \mathbb{R}^{3\times 3}$, $\mathbb{M}_2(\boldsymbol{A}) = \operatorname{cof}\boldsymbol{A}$, and $\mathbb{M}_3(\boldsymbol{A}) = \det\boldsymbol{A}$ its *minors*. Using the shorthand notation $\mathbb{M}(\boldsymbol{A}) = (\mathbb{M}_1(\boldsymbol{A}), \mathbb{M}_2(\boldsymbol{A}), \mathbb{M}_3(\boldsymbol{A}))$ we can state the following.

Lemma 3.4 (Convergence of minors, [37, Prop. 5.1])**.** *Let* $y_k \rightharpoonup y$ *in* $W^{1,q_y}(\Omega; \mathbb{R}^3)$ *and* $\boldsymbol{P}_k \to \boldsymbol{P}$ *in* $L^p(\Omega; \mathrm{SL})$ *and*

$$q_y > 3, \quad \frac{1}{q_y} + \frac{2}{p} \leq 1. \tag{3.11}$$

Then,

$$\mathbb{M}(\nabla y_k \boldsymbol{P}_k^{-1}) \rightharpoonup \mathbb{M}(\nabla y \boldsymbol{P}^{-1}) \quad in \quad L^1(\Omega; \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R}).$$

Note that conditions on the exponents in (3.11) are weaker than those in (3.8).

We can now establish the weak lower semicontinuity of the energy functional, see also [37, Thm. 5.2] for the analogous result in terms of $\boldsymbol{P}$.

Lemma 3.5 (Lower semicontinuity of the energy)**.** *Under the assumptions of Theorem 3.2 the energy $\mathcal{E}$ is weakly lower semicontinuous.*

Proof. Let $(y_k, \boldsymbol{C}_{\mathrm{p}k}) \rightharpoonup (y, \boldsymbol{C}_{\mathrm{p}})$ in $\mathcal{Y} \times \mathcal{C}$. The compact embedding of $W^{1,r} \subset\subset L^r$ and the weak convergence of $\boldsymbol{C}_{\mathrm{p}k} \rightharpoonup \boldsymbol{C}_{\mathrm{p}}$ in $L^{q_{\mathrm{p}}} \cap W^{1,r}$ entail strong convergence in L^s, for all $s \in [1, q_{\mathrm{p}})$ if $r < q_{\mathrm{p}}$ and all $s \in [1, r]$ if $r \geq q_{\mathrm{p}}$. The term

$$\boldsymbol{C}_{\mathrm{p}} \mapsto \int_\Omega \left(\widehat{W}_{\mathrm{p}}(\boldsymbol{C}_{\mathrm{p}}) + \frac{\mu}{r} |\nabla \boldsymbol{C}_{\mathrm{p}}|^r \right) \mathrm{d}x$$

is hence lower semicontinuous (see, e.g., [64, Thm. 1.6, p. 9]).

The convergence in measure of $\boldsymbol{C}_{\mathrm{p}k}$ and the local Lipschitz continuity of the square root map on $\mathrm{SL}^+_{\mathrm{sym}}$ [24, Lemma B.1] implies the convergence in measure of $\boldsymbol{C}_{\mathrm{p}k}^{1/2}$. As $\boldsymbol{C}_{\mathrm{p}k}^{1/2}$ are bounded in $L^{2q_{\mathrm{p}}}$ we obtain the strong convergence $\boldsymbol{C}_{\mathrm{p}k}^{1/2} \to \boldsymbol{C}_{\mathrm{p}}^{1/2}$ in L^p for $p \in [1, 2q_{\mathrm{p}})$. Thanks to Lemma 3.4, we have

$$\mathbb{M}(\nabla y_k \boldsymbol{C}_{\mathrm{p}k}^{-1/2}) \rightharpoonup \mathbb{M}(\nabla y \boldsymbol{C}_{\mathrm{p}}^{-1/2}) \ \text{ in } \ L^1.$$

In fact, condition (3.11) is easily verified by checking that

$$\frac{1}{q_y} + \frac{2}{2q_{\mathrm{p}}} \overset{(3.8)}{=} \frac{1}{q_{\mathrm{e}}} + \frac{3}{2q_{\mathrm{p}}} \overset{(3.8)}{<} 1 - \frac{2}{q_{\mathrm{e}}},$$

which implies $1/q_y + 2/p \leq 1$ for $p = 2q_{\mathrm{p}} - \varepsilon \in [1, 2q_{\mathrm{p}})$ and $\varepsilon > 0$ sufficiently small. As $W_{\mathrm{e}}(\nabla y \boldsymbol{C}_{\mathrm{p}}^{-1/2}) = \mathbb{W}(\mathbb{M}(\nabla y \boldsymbol{C}_{\mathrm{p}}^{-1/2}))$, with $\mathbb{W}$ convex and lower semicontinuous, the lower semicontinuity of the elastic energy term follows, see e.g. Lemma B.1. Eventually, the time-dependent linear term is weakly continuous. $\square$

From here on the proof of Theorem 3.2 is fairly classical [44]. The only slight difference with respect to the standard setting is that the power term $t \mapsto \int_0^t \langle \dot{\ell}, y \rangle \, \mathrm{d}\tau$ is here just absolutely continuous. This lack of smoothness in time is however compensated by linearity in y and we can argue exactly as in [2].

Once we have established that the incremental minimization problems (3.9) have a solution $(y_i^k, \boldsymbol{C}_{\mathrm{p},i}^k)$ for $i = 1, \ldots, N^k$, an energetic solution of the quasistatic evolution problem can be found by passing to the limit in the corresponding backward piecewise-constant interpolants $(\overline{y}^k, \overline{\boldsymbol{C}}_{\mathrm{p}}^k) : [0, T] \to \mathcal{Y} \times \mathcal{C}$ as the diameter τ^k of the partition goes to 0. Indeed, a nonrelabeled subsequence $(\overline{y}^k, \overline{\boldsymbol{C}}_{\mathrm{p}}^k)$ pointwise converging to a limit $(y, \boldsymbol{C}_{\mathrm{p}})$ with respect to the weak topology of $\mathcal{Y} \times \mathcal{C}$ can be found by compactness due to the coercivity of the energy (Lemma 3.3), the nondegeneracy of $\mathcal{D}$ given by $\mathcal{D}(\boldsymbol{C}_{\mathrm{p}}, \boldsymbol{C}_{\mathrm{p},k}) \to 0 \Rightarrow \boldsymbol{C}_{\mathrm{p},k} \rightharpoonup \boldsymbol{C}_{\mathrm{p}}$, and a generalized version of Helly's convergence theorem [40, Thm. 5.1].

The global stability (3.4) of the trajectory $(y, \boldsymbol{C}_{\mathrm{p}})$ and the energy balance (3.5) follow from the lower semicontinuity of the energy (Lemma 3.5), the weak continuity of $\mathcal{D}$, and [40, Prop. 5.7]. In particular, the separate convergence of energy, dissipation, and power can be proved as in [40, Thm. 5.1] and the assertion of Theorem 3.2 follows.

4. Small-deformation limit for quasistatic evolution

We turn now our attention to the analysis of the small-strain limit. The aim of this section is to prove that, under suitable rescaling, the finite-strain model reduces to classical linearized hardening elastoplasticity for small strains. Both gradient and local plasticity can be recovered in the linearization limit by suitably choosing the rescaling of the gradient term.

Our result consists in a variational convergence argument. In the static case, the seminal contribution in this direction is [10] where a variational justification of linearization in elasticity is provided. Successive refinements [1] and extensions [53, 54, 59] of the argument have been presented. Here the proof is adapted for rate-independent evolution situation by following the general abstract strategy of [45]. In the finite-plasticity context the first contribution in this direction is [46] where the linearization of hardening finite plasticity in $\boldsymbol{P}$ is ascertained. Then, linearized plate models have been derived from finite plasticity in [11, 12] and perfect plasticity in one dimension is considered in [23].

The small-strain assumption corresponds to consider both ∇y and $\boldsymbol{C}_{\rm p}$ close to the identity. By letting $\varepsilon > 0$ measure such small distance we write

$$\nabla y = I + \varepsilon \nabla u, \quad \boldsymbol{C}_{\rm p} = \exp(2\varepsilon \boldsymbol{z}) = 1 + 2\varepsilon \boldsymbol{z} + {\rm o}(\varepsilon \boldsymbol{z}).$$

For all $(y, \boldsymbol{C}_{\rm p}) \in \mathcal{Y} \times \mathcal{C}$ we hence aim at considering the equivalent variables

$$u = \frac{1}{\varepsilon}(y - {\rm id}) \in \mathcal{Y}_0 := \{u \in H^1(\Omega;\mathbb{R}^3) \, : \, u = 0 \text{ on } \Gamma_{\rm D}\}, \tag{4.1}$$

$$\boldsymbol{z} = \frac{1}{2\varepsilon}\log \boldsymbol{C}_{\rm p} \in \mathcal{C}_0 := \begin{cases} L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev}) & \text{if } \ \mu_0 = 0, \\ H^1(\Omega;\mathbb{R}^{3\times 3}_{\rm dev}) & \text{if } \ \mu_0 > 0. \end{cases} \tag{4.2}$$

The parameter μ_0 identifies the limiting model: linearized gradient plasticity corresponds to $\mu_0 > 0$ whereas classical linearized local plasticity to $\mu_0 = 0$ (see (4.5) below). Note that the matrix logarithm is uniquely defined on ${\rm SL}^+_{\rm sym}$. The position for $\boldsymbol{z}$ is nonlinear and has the effect of mapping the nonconvex set $\mathcal{C}$ to the linear space $\mathcal{C}_0$. We can in particular rewrite the problem in terms of $(u, \boldsymbol{z})$ instead of $(y, \boldsymbol{C}_{\rm p})$.

Let us deal with energy and dissipation densities first. Given ∇u and $\varepsilon > 0$ we define the rescaled Green-Saint Venant strain as

$$\boldsymbol{e} := \frac{1}{2\varepsilon}((\boldsymbol{I}+\varepsilon\nabla u)^\top(\boldsymbol{I}+\varepsilon\nabla u) - \boldsymbol{I}) = \nabla u^{\rm sym} + \frac{\varepsilon}{2}\nabla u^\top \nabla u$$

so that $\boldsymbol{e} = \nabla u^{\rm sym} := (\nabla u + \nabla u^\top)/2$ to first order. We assume the potentials $W_{\rm e}$, $W_{\rm p}$ to have a quadratic behavior in a neighborhood of the identity, the precise assumption being given in (4.10) below. In particular, we let $\widehat{W}_{\rm e}(\boldsymbol{I}) = \widehat{W}_{\rm p}(\boldsymbol{I}) = 0$ (which is an inconsequential normalization), and $\partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{I}) = \partial_{\boldsymbol{C}_{\rm p}} \widehat{W}_{\rm p}(\boldsymbol{I}) = \boldsymbol{0}$. Moreover the potentials $\widehat{W}_{\rm e}$ and $\widehat{W}_{\rm p}$ are supposed to be twice differentiable at $\boldsymbol{I}$, with

$$\mathbb{C} := 4\partial^2_{\boldsymbol{C}_{\rm e}} \widehat{W}_{\rm e}(\boldsymbol{I}) = \partial^2_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{I}), \qquad \mathbb{H} := 4\partial^2_{\boldsymbol{C}_{\rm p}} \widehat{W}_{\rm p}(\boldsymbol{I}),$$

where the *elasticity* $\mathbb{C}$ and *hardening* tensors $\mathbb{H}$ have been introduced. These fourth-order tensors are clearly symmetric, for they are Hessians. In addition, due to frame- and plastic-rotation indifference the tensors $\mathbb{C}$ and $\mathbb{H}$ present also the so-called *minor symmetries*, namely $\mathbb{C}_{ij\ell k} = \mathbb{C}_{ijk\ell}$ and $\mathbb{H}_{ij\ell k} = \mathbb{H}_{ijk\ell}$. Given any symmetric, positive-definite 4-tensor $\mathbb{B}$, henceforth we denote by $|\boldsymbol{A}|^2_{\mathbb{B}} := \boldsymbol{A}{:}\mathbb{B}\boldsymbol{A}$ the corresponding squared norm on $\mathbb{R}^{3\times3}_{\rm sym}$, where $\mathbb{B}\boldsymbol{A} \in \mathbb{R}^{3\times3}_{\rm sym}$ is defined as $(\mathbb{B}\boldsymbol{A})_{ij} := \mathbb{B}_{ij\ell k}A_{\ell k}$.

Let now the rescaled local energy density be given as

$$W_\varepsilon(\nabla u, \boldsymbol{z}) := \frac{1}{\varepsilon^2}\widehat{W}_{\rm e}\big(\exp(-\varepsilon\boldsymbol{z})(\boldsymbol{I}{+}2\varepsilon\boldsymbol{e})\exp(-\varepsilon\boldsymbol{z})\big) + \frac{1}{\varepsilon^2}\widehat{W}_{\rm p}\big(\exp(2\varepsilon\boldsymbol{z})\big).$$

The quadratic scaling in W_ε can be justified by computing the Taylor expansion

$$W_\varepsilon(\nabla u, \boldsymbol{z}) = \frac{1}{2}|\nabla u^{\rm sym}{-}\boldsymbol{z}|^2_{\mathbb{C}} + \frac{1}{2}|\boldsymbol{z}|^2_{\mathbb{H}} + {\rm o}(1)$$

where we have used the fact that $\boldsymbol{e} = \nabla u^{\rm sym} + {\rm o}(\varepsilon)$ and $\exp(-\varepsilon\boldsymbol{z}) = \boldsymbol{I} - \varepsilon\boldsymbol{z} + {\rm o}(\varepsilon)$. This in particular proves that the rescaled local energy density pointwise converges to its linearized counterpart

$$W_0(\nabla u, \boldsymbol{z}) := \frac{1}{2}|\nabla u^{\rm sym}{-}\boldsymbol{z}|^2_{\mathbb{C}} + \frac{1}{2}|\boldsymbol{z}|^2_{\mathbb{H}}.$$

Note that W_0 is function of $\nabla u^{\rm sym}$ and $\boldsymbol{z}$ only. Based on W_ε we define the rescaled total energy functional as

$$\mathcal{E}_\varepsilon(u, \boldsymbol{z}, t) = \int_\Omega W_\varepsilon(\nabla u, \boldsymbol{z})\,{\rm d}x + \frac{\mu_\varepsilon}{2\varepsilon^2}\int_\Omega |\nabla\exp(2\varepsilon\boldsymbol{z})|^2\,{\rm d}x - \langle \ell_0(t), u\rangle. \tag{4.3}$$

and, correspondingly, the rescaled stored energy as

$$\mathcal{W}_\varepsilon(u, \boldsymbol{z}) = \mathcal{E}_\varepsilon(u, \boldsymbol{z}, t) + \langle \ell_0(t), u\rangle. \tag{4.4}$$

This corresponds indeed to (3.2) by letting $r = 2$ and the generalized load ℓ be $\ell = \ell_0/\varepsilon$. Let us stress, that although existence of energetic solutions can be proved for all $r > 1$, the linearization result deals with the quadratic case $r = 2$ only.

Note that we have allowed in $\mathcal{E}_\varepsilon$ an independent scaling in μ_ε in order to account for two different linearization limits in a unified way. By assuming $\mu_\varepsilon \to \mu_0 \geq 0$ strictly monotonically, the linearized total energy takes the form

$$\mathcal{E}_0(u, \boldsymbol{z}, t) = \int_\Omega W_0(\nabla u, \boldsymbol{z})\,{\rm d}x + 2\mu_0\int_\Omega |\nabla\boldsymbol{z}|^2{\rm d}x - \langle \ell_0(t), u\rangle. \tag{4.5}$$

We define the rescaled local dissipation as

$$D_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2) := \frac{1}{\varepsilon}D(\boldsymbol{C}_{\rm p1}, \boldsymbol{C}_{\rm p2}) = \frac{1}{\varepsilon}D\big(\exp(2\varepsilon\boldsymbol{z}_1), \exp(2\varepsilon\boldsymbol{z}_2)\big). \tag{4.6}$$

The linear scaling is motivated by the explicit form of D from Lemma 3.1.iv. In fact, we have that

$$D_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2) = \frac{\rho}{2\varepsilon}\big|\log\big(\exp(-\varepsilon\boldsymbol{z}_1)\exp(-2\varepsilon\boldsymbol{z}_2)\exp(-\varepsilon\boldsymbol{z}_1)\big)\big| = \rho|\boldsymbol{z}_1 - \boldsymbol{z}_2| + {\rm o}(1). \tag{4.7}$$

The linearized dissipation distance ensues by taking the pointwise limit of D_ε, namely

$$D_0(\boldsymbol{z}_1, \boldsymbol{z}_2) := \rho|\boldsymbol{z}_1 - \boldsymbol{z}_2|.$$

The rescaled dissipation distance is hence defined as

$$\mathcal{D}_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2) = \int_\Omega D_\varepsilon(\boldsymbol{z}_1, \boldsymbol{z}_2)\,\mathrm{d}x$$

and the corresponding linearized dissipation distance is

$$\mathcal{D}_0(\boldsymbol{z}_1, \boldsymbol{z}_2) = \int_\Omega D_0(\boldsymbol{z}_1, \boldsymbol{z}_2)\,\mathrm{d}x$$

Assume now to be given $\ell_0 \in W^{1,1}(0,T;(W^{1,q_y}_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3)))^*)$ and initial values $\boldsymbol{z}_{0\varepsilon}$ such that $\exp(2\varepsilon\boldsymbol{z}_{0\varepsilon}) \in \mathcal{S}(0)$ where $\mathcal{S}(t)$ denotes stable states at time $t \in [0,T]$ corresponding to the rescaled energy

$$(y, \boldsymbol{C}_\mathrm{p}, t) \mapsto \frac{1}{\varepsilon^2}\mathcal{W}(y, \boldsymbol{C}_\mathrm{p}) - \frac{1}{\varepsilon}\langle \ell(t), y\rangle$$

and the rescaled dissipation $\mathcal{D}/(2\varepsilon)$. By arguing as in Theorem 3.2 one can check that there exists an energetic solution $(y_\varepsilon, \boldsymbol{C}_{\mathrm{p}\varepsilon})$ corresponding to such rescaled potentials. Correspondingly, by defining $(u_\varepsilon, \boldsymbol{z}_\varepsilon)$ from $(y_\varepsilon, \boldsymbol{C}_{\mathrm{p}\varepsilon})$ via (4.1)-(4.2) we find that $(u_\varepsilon, \boldsymbol{z}_\varepsilon)$ is an energetic solution with respect to the rescaled energy $\mathcal{E}_\varepsilon$ and the rescaled dissipation $\mathcal{D}_\varepsilon$ in the state space $\mathcal{Y}_0 \times \mathcal{C}_0$. We shall refer to such trajectories $(u_\varepsilon, \boldsymbol{z}_\varepsilon)$ as *finite-plasticity quasistatic evolutions* and denote the corresponding set of stable states at time $t \in [0,T]$ by $\mathcal{S}_\varepsilon(t)$.

The main result of this section is the convergence of finite-plasticity quasistatic evolutions $(u_\varepsilon, \boldsymbol{z}_\varepsilon)$ to the unique strong solution $(u, \boldsymbol{z})$ of the linearized elastoplasticity system corresponding to $\mathcal{E}_0$ and $\mathcal{D}_0$, namely the system

$$\begin{aligned}
&\nabla\cdot\boldsymbol{\sigma} + b_0(t) = 0 \quad \text{in } \ \Omega \times (0,T),\\
&\boldsymbol{\sigma} = \mathbb{C}(\nabla u^{\mathrm{sym}} - \boldsymbol{z}) \quad \text{in } \ \Omega \times (0,T),\\
&u = 0 \quad \text{in } \ \Gamma_\mathrm{D} \times (0,T),\\
&\boldsymbol{\sigma}\nu = \tau_0(t) \quad \text{in } \ \Gamma_\mathrm{tr} \times (0,T),\\
&\rho\partial|\dot{\boldsymbol{z}}| + \mathbb{H}\boldsymbol{z} - 4\mu_0\Delta\boldsymbol{z} \ni \boldsymbol{\sigma} \quad \text{in } \ \Omega \times (0,T),\\
&\mu_0\nabla\boldsymbol{z}\,\nu = \boldsymbol{0} \quad \text{in } \ \Gamma \times (0,T),\\
&\boldsymbol{z}(0) = \boldsymbol{z}_0 \quad \text{in } \ \Omega
\end{aligned}$$

where the body force b_0 and the traction τ_0 define the generalized load ℓ_0 as in (3.1). We refer to $(u, \boldsymbol{z})$ as *linearized-plasticity quasistatic evolution.* Note that such strong solution uniquely exists both in the gradient ($\mu_0 > 0$) and the local ($\mu_0 = 0$) case [28].

We now state our main convergence result.

Theorem 4.1 (Small-deformation limit of the quasistatic evolution)**.** *Let the compact set*

$$K := \{\boldsymbol{C} \in \mathrm{SL}^+_{\mathrm{sym}} : \ |\boldsymbol{C}| \le \kappa\}, \quad \kappa^2 > 3 \tag{4.8}$$

be given and assume $\widehat{W}_{\rm p}$ to be coercive in the following sense

$$\widehat{W}_{\rm p}(\boldsymbol{C}_{\rm p}) < \infty \ \Leftrightarrow \ \boldsymbol{C}_{\rm p},\, \boldsymbol{C}_{\rm p}^{-1} \in K. \tag{4.9}$$

Let $\widehat{W}_{\rm e}$ and $\widehat{W}_{\rm p}$ have quadratic behavior at identity, namely

$$\forall \delta > 0\, \exists c_\delta > 0 :$$
$$\left|\widehat{W}_{\rm e}(\boldsymbol{I}{+}2\boldsymbol{A}){-}\frac{1}{2}|\boldsymbol{A}|^2_{\mathbb{C}}\right| + \left|\widehat{W}_{\rm p}\big(\exp(2\boldsymbol{A})\big){-}\frac{1}{2}|\boldsymbol{A}|^2_{\mathbb{H}}\right| \leq \delta|\boldsymbol{A}|^2 \quad \forall |\boldsymbol{A}| \leq c_\delta. \tag{4.10}$$

Moreover, assume the control

$$|\boldsymbol{F}_{\rm e}^\top \partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{F}_{\rm e})| \leq c_3(1{+}W_{\rm e}(\boldsymbol{F}_{\rm e})) \quad \forall \boldsymbol{F}_{\rm e} \in {\rm GL}^+ \tag{4.11}$$

and let

$$W_{\rm e}(\boldsymbol{F}) \geq c_4\, {\rm dist}^2(\boldsymbol{F}, {\rm SO}) \quad \forall \boldsymbol{F} \in {\rm GL}^+, \tag{4.12}$$
$$\widehat{W}_{\rm p}(\exp(\boldsymbol{A})) \geq c_4|\boldsymbol{A}|^2 \quad \forall \boldsymbol{A} \in \mathbb{R}^{3\times 3}, \tag{4.13}$$

for some positive constants c_3, $c_4 > 0$. Let $(u_\varepsilon, \boldsymbol{z}_\varepsilon)$ be finite-plasticity quasistatic evolutions starting from well-prepared initial data $(u_{0\varepsilon}, \boldsymbol{z}_{0\varepsilon}) \in \mathcal{S}_\varepsilon(0)$, namely,

$$(u_{0\varepsilon}, \boldsymbol{z}_{0\varepsilon}) \to (u_0, \boldsymbol{z}_0) \quad \textit{and} \quad \mathcal{E}_\varepsilon(u_{0\varepsilon}, \boldsymbol{z}_{0\varepsilon}, 0) \to \mathcal{E}_0(u_0, \boldsymbol{z}_0, 0).$$

Then, for all $t \in [0, T]$,

$$(u_\varepsilon(t), \boldsymbol{z}_\varepsilon(t)) \to (u(t), \boldsymbol{z}(t)),$$
$${\rm Diss}_{\mathcal{D}_\varepsilon,[0,t]}(\boldsymbol{z}_\varepsilon) \to {\rm Diss}_{\mathcal{D}_0,[0,t]}(\boldsymbol{z}),$$
$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \boldsymbol{z}_\varepsilon(t), t) \to \mathcal{E}_0(u(t), \boldsymbol{z}(t), t)$$

where (u, z) is the unique linearized-plasticity quasistatic evolution starting from $(u_0, \boldsymbol{z}_0)$.

Note that the coercivity assumption (4.9) on $\widehat{W}_{\rm p}$ is stronger than the former (3.7). The assumption on the shape of K from (4.8) is of technical nature and could probably be relaxed. On the other hand, the choice for the shape of K is immaterial with respect to the linearization limit as all deformations concentrate around the identity for $\varepsilon \to 0$. Note additionally that it would be sufficient to assume $\boldsymbol{C}_{\rm p} \in K$ in order to deduce from $\det \boldsymbol{C}_{\rm p} = 1$ that $\boldsymbol{C}_{\rm p}^{-1} \in K'$ for a possibly larger compact set K'.

Assumption (4.11) expresses the controllability of the so called *Kirchhoff stress tensor* $\boldsymbol{F}_{\rm e}^\top \partial_{\boldsymbol{F}_{\rm e}} W_{\rm e}(\boldsymbol{F}_{\rm e})$ by means of the energy. In particular, it implies that $W_{\rm e}$ has polynomial growth [6, Prop. 2.7]. This condition is compatible with polyconvexity and plays an important role in finite-deformation theories [5, 6].

The proof of Theorem 4.1 follows the general path of [46] where the linearization limit for finite plasticity model in $\boldsymbol{P}$ was ascertained. Here we deal with a formulation in $\boldsymbol{C}_{\rm p}$ instead. This calls for adapting the argument of [46] in many technical points. In particular, the choice of rescaled variables and functionals is here different from [46] and especially tailored to cope with the nonlinear and symmetric structure of ${\rm SL}^+_{\rm sym}$. In addition, we allow here for the gradient-plasticity case by including a quadratic term

$|\nabla \boldsymbol{C}_\mathrm{p}|^2$ in the energy. Correspondingly, the construction of the recovery sequence is more involved. The advantage of including such gradient term is apparent as it delivers an existence theory for finite-plasticity trajectories, see Section 3. Note that such existence theory was not available in the framework of [46].

The rest of this section is devoted to the proof of Theorem 4.1 which is based on the general theory of the evolution Γ-convergence of rate-independent processes from [45]. The application of this general theory will follow upon proving uniform coercivity of the rescaled energy, two separate Γ-lim inf inequalities for energy and dissipation, namely,

$$\begin{aligned}\mathcal{E}_0(u,\boldsymbol{z},t) \leq \inf \Big\{ \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon,\boldsymbol{z}_\varepsilon,t_\varepsilon) \ \Big| \\ (u_\varepsilon,\boldsymbol{z}_\varepsilon,t_\varepsilon) \rightharpoonup (u,\boldsymbol{z},t) \ \text{ in } \ \mathcal{Y}_0\times\mathcal{C}_0\times[0,T]\Big\}, \qquad (4.14)\end{aligned}$$

$$\mathcal{D}_0(\boldsymbol{z},\hat{\boldsymbol{z}}) \leq \inf \Big\{ \liminf_{\varepsilon\to 0} \mathcal{D}_\varepsilon(\boldsymbol{z}_\varepsilon,\hat{\boldsymbol{z}}_\varepsilon) \ \Big| \ (\boldsymbol{z}_\varepsilon,\hat{\boldsymbol{z}}_\varepsilon) \rightharpoonup (\boldsymbol{z},\hat{\boldsymbol{z}}) \ \text{ in } \ L^2(\Omega;(\mathbb{R}^{3\times3}_\mathrm{dev})^2)\Big\} \qquad (4.15)$$

and, for any given $(\hat{u}_0,\hat{\boldsymbol{z}}_0,t_0)$ and $(u_\varepsilon,\boldsymbol{z}_\varepsilon,t_\varepsilon) \rightharpoonup (u_0,\boldsymbol{z}_0,t_0)$ in $\mathcal{Y}_0\times\mathcal{C}_0\times[0,T]$ with uniformly bounded energies, the existence of a *mutual recovery sequence* $(\hat{u}_\varepsilon,\hat{\boldsymbol{z}}_\varepsilon)$ such that

$$\limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(\boldsymbol{z}_\varepsilon,\hat{\boldsymbol{z}}_\varepsilon) \leq \mathcal{D}_0(\boldsymbol{z}_0,\hat{\boldsymbol{z}}_0) \qquad (4.16)$$

$$\limsup_{\varepsilon\to 0} \big(\mathcal{E}_\varepsilon(t,\hat{u}_\varepsilon,\hat{\boldsymbol{z}}_\varepsilon,t_\varepsilon) - \mathcal{E}_\varepsilon(t,u_\varepsilon,\boldsymbol{z}_\varepsilon,t_\varepsilon)\big) \leq \mathcal{E}_0(t,\hat{u}_0,\hat{\boldsymbol{z}}_0,t_0) - \mathcal{E}_0(t,u_0,\boldsymbol{z}_0,t_0) \qquad (4.17)$$

Let us start by presenting the coercivity result.

Lemma 4.2 (Coercivity of the rescaled energy)**.** *Under the assumptions of Theorem* 4.1 *we have that*

$$\|\nabla u\|^2_{L^2} + \|\boldsymbol{z}\|^2_{L^2} + \mu_\varepsilon\|\nabla\exp(2\varepsilon\boldsymbol{z})/\varepsilon\|^2_{L^2} + \varepsilon\|\boldsymbol{z}\|_{L^\infty} \leq c\big(1+\mathcal{W}_\varepsilon(u,\boldsymbol{z})\big) \qquad (4.18)$$

for all $(u,\boldsymbol{z}) \in \mathcal{Y}_0\times\mathcal{C}_0$.

Proof. Assume $\mathcal{W}_\varepsilon(u,z)$ to be finite. Then, by (4.9), $\exp(2\varepsilon\boldsymbol{z}) \in K$ so that $\varepsilon\|\boldsymbol{z}\|_{L^\infty} \leq c$. Indeed, by denoting by λ_i the eigenvalues of $\boldsymbol{z}$, condition $\mathrm{tr}\,\boldsymbol{z} = 0$ entails $\max_i \lambda_i \geq \|\boldsymbol{z}\|_\infty/6$, hence

$$\mathrm{e}^{\varepsilon\|\boldsymbol{z}\|_{L^\infty}/3} \leq \mathrm{e}^{2\varepsilon\,\max_i\lambda_i} \leq \|\exp(2\varepsilon\boldsymbol{z})\|_\infty < c.$$

From the coercivity (4.13) of $\widehat{W}_\mathrm{p}$ we have that $\|\boldsymbol{z}\|_{L^2}$ is bounded by $c\mathcal{W}_\varepsilon$ and the same holds for $\mu_\varepsilon\|\nabla\exp(2\varepsilon\boldsymbol{z})/\varepsilon\|_{L^2}$ from (4.3)-(4.4). Note that, for all $\alpha \in \mathbb{R}$

$$\exp(\alpha\varepsilon\boldsymbol{z}) = \boldsymbol{I} + \alpha\varepsilon\boldsymbol{z} + \varepsilon^2\boldsymbol{z}^2\boldsymbol{B}_\varepsilon, \quad \boldsymbol{B}_\varepsilon := \left(\sum_{k=2}^{\infty}\frac{\alpha^k(\varepsilon\boldsymbol{z})^{k-2}}{k!}\right). \qquad (4.19)$$

As $\|\varepsilon\boldsymbol{z}\|_{L^\infty} \leq c$, we have that $\|\boldsymbol{B}_\varepsilon\|_{L^\infty} \leq c$ as well.

The coercivity estimate on ∇u is based on a geometric rigidity argument as in [10]. The first step is to obtain an estimate of the distance of ∇y from SO. By recalling that

$\boldsymbol{F}_{\rm e} = \nabla y\, \boldsymbol{C}_{\rm p}^{-1/2}$ and that $\boldsymbol{C}_{\rm p} \in K$ is bounded, one obtains

$$|\nabla y - \boldsymbol{Q}|^2 = |(\boldsymbol{F}_{\rm e} - \boldsymbol{Q})\boldsymbol{C}_{\rm p}^{1/2} + \boldsymbol{Q}(\boldsymbol{C}_{\rm p}^{1/2} - \boldsymbol{I})|^2 \leq c|\boldsymbol{F}_{\rm e} - \boldsymbol{Q}|^2 + |\boldsymbol{C}_{\rm p}^{1/2} - \boldsymbol{I}|^2$$

almost everywhere. We now use the development (4.19) with $\alpha = 1$ in order to get that

$$\boldsymbol{C}_{\rm p}^{1/2} = \boldsymbol{I} + \varepsilon \boldsymbol{z} + \varepsilon^2 \boldsymbol{z}^2 \boldsymbol{B}_\varepsilon = \boldsymbol{I} + \varepsilon \boldsymbol{z} \boldsymbol{B}'_\varepsilon,$$

where $\boldsymbol{B}'_\varepsilon = \boldsymbol{I} + (\varepsilon \boldsymbol{z})\boldsymbol{B}_\varepsilon$. Therefore

$$|\boldsymbol{C}_{\rm p}^{1/2} - \boldsymbol{I}|^2 \leq \varepsilon^2 |\boldsymbol{z}|^2 |\boldsymbol{B}'_\varepsilon|^2$$

and we can conclude that

$$|\nabla y - \boldsymbol{Q}|^2 = |(\boldsymbol{F}_{\rm e} - \boldsymbol{Q})\boldsymbol{C}_{\rm p}^{1/2} + \boldsymbol{Q}(\boldsymbol{C}_{\rm p}^{1/2} - \boldsymbol{I})|^2 \leq c(|\boldsymbol{F}_{\rm e} - \boldsymbol{Q}|^2 + \varepsilon^2 |\boldsymbol{z}|^2 |\boldsymbol{B}'_\varepsilon|^2)$$

where $\|\boldsymbol{B}'_\varepsilon\|_{L^\infty} \leq c$. We now proceed as in [46, Lemma 3.1]. The last inequality combined with the nondegeneracy condition (4.12) yields

$$\int_\Omega {\rm dist}^2(\nabla y, {\rm SO})\, {\rm d}x \leq c\varepsilon^2 \big(1 + \mathcal{W}_\varepsilon(u, \boldsymbol{z})\big).$$

Then, the Rigidity Lemma [21, Thm. 3.1] entails

$$\exists \widehat{\boldsymbol{Q}} \in {\rm SO}: \quad \|\nabla y - \widehat{\boldsymbol{Q}}\|_{L^2}^2 \leq c\varepsilon^2 \big(1 + \mathcal{W}_\varepsilon(u, \boldsymbol{z})\big).$$

As the rotation $\widehat{\boldsymbol{Q}}$ satisfies the estimate $|\widehat{\boldsymbol{Q}} - \boldsymbol{I}|^2 \leq c\varepsilon^2(1 + \mathcal{W}_\varepsilon(u, \boldsymbol{z}))$ as a result of the boundary conditions, see [10, Prop. 3.4], we have

$$\varepsilon^2 \|\nabla u\|_{L^2}^2 = \|\nabla y - \boldsymbol{I}\|_{L^2}^2 \leq 2\|\nabla y - \widehat{\boldsymbol{Q}}\|_{L^2}^2 + 2\|\widehat{\boldsymbol{Q}} - \boldsymbol{I}\|_{L^2}^2 \leq c\varepsilon^2 \big(1 + \mathcal{W}_\varepsilon(u, \boldsymbol{z})\big)$$

and the assertion follows. □

We now move to the proof of the Γ-$\liminf$ inequalities (4.14)-(4.15). This is done in the next two lemmas.

Lemma 4.3 (Γ-$\liminf$ inequality for $\mathcal{E}_\varepsilon$)**.** *Under the assumptions of Theorem* 4.1 *the* Γ-$\liminf$ *inequality* (4.14) *holds.*

Proof. As convergence for the linear external energy $\langle \ell_0(t_\varepsilon), u_\varepsilon \rangle$ is trivial, we concentrate on the terms

$$\begin{aligned}
I_\varepsilon^1 &:= \frac{1}{\varepsilon^2} \int_\Omega \widehat{W}_{\rm e}\big(\exp(-\varepsilon \boldsymbol{z}_\varepsilon)(\boldsymbol{I} + 2\varepsilon \boldsymbol{e}_\varepsilon) \exp(-\varepsilon \boldsymbol{z}_\varepsilon)\big)\, {\rm d}x, \\
&= \frac{1}{\varepsilon^2} \int_\Omega W_{\rm e}\big((\boldsymbol{I} + \varepsilon \nabla u_\varepsilon) \exp(-\varepsilon \boldsymbol{z}_\varepsilon)\big)\, {\rm d}x \\
I_\varepsilon^2 &:= \frac{1}{\varepsilon^2} \int_\Omega \widehat{W}_{\rm p}\big(\exp(2\varepsilon \boldsymbol{z}_\varepsilon)\big)\, {\rm d}x, \\
I_\varepsilon^3 &:= \frac{\mu_\varepsilon}{2\varepsilon^2} \int_\Omega |\nabla \exp(2\varepsilon \boldsymbol{z}_\varepsilon)|^2\, {\rm d}x = 2\mu_\varepsilon \int_\Omega |\nabla \exp(2\varepsilon \boldsymbol{z}_\varepsilon)/(2\varepsilon)|^2\, {\rm d}x.
\end{aligned}$$

Let $(u_\varepsilon, \boldsymbol{z}_\varepsilon) \rightharpoonup (u, \boldsymbol{z})$ in $\mathcal{Y}_0 \times \mathcal{C}_0$. We can assume with no loss of generality the energies $\mathcal{E}_\varepsilon(u_\varepsilon, \boldsymbol{z}_\varepsilon, t_\varepsilon)$ to be equibounded, so that the bound (4.18) holds for all $(u_\varepsilon, \boldsymbol{z}_\varepsilon)$ independently of ε. In particular, as noticed at the beginning of Lemma 4.2, $\exp(2\varepsilon \boldsymbol{z}_\varepsilon) \in K$ and $\|\varepsilon \boldsymbol{z}_\varepsilon\|_{L^\infty} \leq c$.

Relation (4.19) for $\alpha = 2$ entails that $\exp(2\varepsilon \boldsymbol{z}_\varepsilon) = \boldsymbol{I} + 2\varepsilon \boldsymbol{z}_\varepsilon + \varepsilon^2 \boldsymbol{z}_\varepsilon^2 \boldsymbol{B}_\varepsilon$ with $\|\boldsymbol{B}_\varepsilon\|_{L^\infty} \leq c$. As $\|\boldsymbol{z}_\varepsilon\|_{L^2} \leq c$, we readily check that

$$\frac{1}{2\varepsilon}\Big(\exp(2\varepsilon \boldsymbol{z}_\varepsilon) - \boldsymbol{I}\Big) - \boldsymbol{z}_\varepsilon = \frac{1}{2}\varepsilon \boldsymbol{z}_\varepsilon^2 \boldsymbol{B}_\varepsilon \stackrel{L^1}{\to} 0$$

and, taking into account the L^2-boundedness of the same sequence, we deduce that $(\exp(2\varepsilon \boldsymbol{z}_\varepsilon) - \boldsymbol{I})/(2\varepsilon) \rightharpoonup \boldsymbol{z}$ in L^2. On the other hand, by (4.18) we also have the gradient bound $\mu_\varepsilon \|\nabla(\exp(2\varepsilon \boldsymbol{z}_\varepsilon) - \boldsymbol{I})/(2\varepsilon)\|_{L^2} \leq c$. If $\mu_0 > 0$ the convergence

$$\frac{1}{2\varepsilon}\Big(\exp(2\varepsilon \boldsymbol{z}_\varepsilon) - \boldsymbol{I}\Big) \stackrel{H^1}{\rightharpoonup} \boldsymbol{z} \tag{4.20}$$

follows. In particular,

$$2\mu_0 \int_\Omega |\nabla \boldsymbol{z}|^2 \,\mathrm{d}x \leq \liminf_{\varepsilon \to 0} \left(2\mu_\varepsilon \int_\Omega |\nabla \exp(2\varepsilon \boldsymbol{z}_\varepsilon)/(2\varepsilon)|^2 \,\mathrm{d}x\right) = \liminf_{\varepsilon \to 0} I_\varepsilon^3.$$

In case $\mu_0 = 0$, the relevant inequality is $0 \leq \liminf_{\varepsilon \to 0} I_\varepsilon^3$ which is trivially satisfied.

As a consequence of the quadratic behavior (4.10) one can prove the continuous convergence [24, Lemma 4.2]

$$\boldsymbol{\zeta}_\varepsilon \to \boldsymbol{\zeta} \ \text{ in } \ \mathbb{R}^{3\times 3}_{\mathrm{dev}} \ \Rightarrow \ \frac{1}{\varepsilon^2}\widehat{W}_{\mathrm{p}}(\exp(2\varepsilon \boldsymbol{\zeta}_\varepsilon)) \ \to \ \frac{1}{2}|\boldsymbol{\zeta}|^2_{\mathbb{H}}.$$

so that Lemma B.1 directly yields

$$\frac{1}{2}\int_\Omega |\boldsymbol{z}|^2_{\mathbb{H}} \,\mathrm{d}x \leq \liminf_{\varepsilon \to 0} I_\varepsilon^2.$$

Let us now turn to the term I_ε^1. We introduce, also for future reference, the shorthand notation

$$\boldsymbol{A}_\varepsilon = \frac{1}{\varepsilon}\Big((\boldsymbol{I} + \varepsilon \nabla u_\varepsilon)\exp(-\varepsilon \boldsymbol{z}_\varepsilon) - \boldsymbol{I}\Big), \tag{4.21}$$

which allows to rewrite I_ε^1 as

$$I_\varepsilon^1 = \frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{e}}(1 + \varepsilon \boldsymbol{A}_\varepsilon) \,\mathrm{d}x.$$

Again the quadratic behavior (4.10) ensures the continuous convergence [24, Lemma 4.2]

$$\boldsymbol{a}_\varepsilon \to \boldsymbol{a} \ \text{ in } \ \mathbb{R}^{3\times 3} \ \Rightarrow \ \frac{1}{\varepsilon^2}W_{\mathrm{e}}(\boldsymbol{I} + \varepsilon \boldsymbol{a}_\varepsilon) \ \to \ \frac{1}{2}|\boldsymbol{a}|^2_{\mathbb{C}}.$$

Therefore, by Lemma B.1, in order to check the limiting behavior of I_ε^1 it suffices to prove that

$$\boldsymbol{A}_\varepsilon \stackrel{L^2}{\rightharpoonup} \nabla u - \boldsymbol{z}. \tag{4.22}$$

By expanding $\exp(-\varepsilon \boldsymbol{z}_\varepsilon) = \boldsymbol{I} - \varepsilon \boldsymbol{z}_\varepsilon + \varepsilon^2 \boldsymbol{z}_\varepsilon^2 \boldsymbol{B}_\varepsilon$ according to (4.19) with $\alpha = -1$ we compute

$$\boldsymbol{A}_\varepsilon = (\nabla u_\varepsilon - \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{z}_\varepsilon) + \varepsilon \boldsymbol{z}_\varepsilon^2 \boldsymbol{B}_\varepsilon + \varepsilon \nabla u_\varepsilon \boldsymbol{z}_\varepsilon (\varepsilon \boldsymbol{z}_\varepsilon \boldsymbol{B}_\varepsilon - \boldsymbol{I}).$$

As $\nabla u_\varepsilon \rightharpoonup \nabla u$ in L^2, $\boldsymbol{z}_\varepsilon \rightharpoonup \boldsymbol{z}$ in L^2, and $\|\boldsymbol{B}_\varepsilon\|_{L^\infty} + \|\varepsilon \boldsymbol{z}_\varepsilon\|_{L^\infty} \leq c$, the convergence (4.22) follows. □

Lemma 4.4 (Γ-lim inf inequality for $\mathcal{D}_\varepsilon$)**.** *Under the assumptions of Theorem* 4.1 *the* Γ-lim inf *inequality* (4.15) *holds.*

Proof. By Lemma 3.1.iv and the expansion (4.7) we have that D_ε continuously converge to D_0, namely [24, Lemma 4.3]

$$(\boldsymbol{\zeta}_\varepsilon, \widehat{\boldsymbol{\zeta}}_\varepsilon) \to (\boldsymbol{\zeta}, \widehat{\boldsymbol{\zeta}}) \ \text{ in } \mathbb{R}^{3\times3}_{\text{dev}} \times \mathbb{R}^{3\times3}_{\text{dev}} \ \Rightarrow \ D_\varepsilon(\boldsymbol{\zeta}_\varepsilon, \widehat{\boldsymbol{\zeta}}_\varepsilon) \to D_0(\boldsymbol{\zeta}, \widehat{\boldsymbol{\zeta}}).$$

The assertion follows by applying the lower semicontinuity tool of Lemma B.1. □

Having established the above Γ-lim inf inequalities, the next ingredient of the evolutive-Γ-convergence argument is the specification of a *mutual recovery sequence*. This is done in the following lemma.

Lemma 4.5 (Mutual recovery sequence)**.** *Under the assumptions of Theorem* 4.1 *let* $(u_\varepsilon, \boldsymbol{z}_\varepsilon) \rightharpoonup (u_0, \boldsymbol{z}_0)$ *in* $\mathcal{Y}_0 \times \mathcal{C}_0$ *be given with* $\sup_\varepsilon \mathcal{W}_\varepsilon(u_\varepsilon, \boldsymbol{z}_\varepsilon) < \infty$*. Moreover, let*

$$(\widehat{u}_0, \widehat{\boldsymbol{z}}_0) = (u_0, \boldsymbol{z}_0) + (\widetilde{u}, \widetilde{\boldsymbol{z}})$$

where $(\widetilde{u}, \widetilde{\boldsymbol{z}}) \in C^\infty_{\text{c}}(\Omega; \mathbb{R}^3) \times C^\infty_{\text{c}}(\Omega; \mathbb{R}^{3\times3}_{\text{dev}})$*. Then, there exists* $(\widehat{u}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) \in \mathcal{Y}_0 \times \mathcal{C}_0$ *such that the* lim sup *relations* (4.16)-(4.17) *hold.*

Proof. We divide the proof into several steps.

Step 1: Definition of the recovery sequence. We define

$$\widehat{u}_\varepsilon := u_\varepsilon + \widetilde{u} \circ (\text{id} + \varepsilon u_\varepsilon) \tag{4.23}$$

$$\widehat{\boldsymbol{z}}_\varepsilon := \frac{1}{2\varepsilon} \log \Big(\boldsymbol{\Pi} \left(\exp(2\varepsilon(\boldsymbol{z}_\varepsilon + \widetilde{\boldsymbol{z}})) \right) \Big) \tag{4.24}$$

where $\boldsymbol{\Pi} : \text{SL}^+_{\text{sym}} \to K$ is a contraction mapping onto the compact set K defined by (4.8). More precisely, we ask $\boldsymbol{\Pi}$ to have the following properties

$$\boldsymbol{\Pi}\,|_K = \text{id}_K \quad \text{ and } \quad |\boldsymbol{\Pi}(\boldsymbol{C}_{\text{p1}}) - \boldsymbol{\Pi}(\boldsymbol{C}_{\text{p2}})| \leq |\boldsymbol{C}_{\text{p1}} - \boldsymbol{C}_{\text{p2}}| \quad \forall \boldsymbol{C}_{\text{p1}}, \boldsymbol{C}_{\text{p2}} \in \text{SL}^+_{\text{sym}}. \tag{4.25}$$

An explicit construction of a map $\boldsymbol{\Pi}$ fulfilling these properties is provided in Appendix A.

The definition of $\widehat{u}_\varepsilon$ can be rewritten in terms of $\widehat{y}_\varepsilon = \text{id} + \varepsilon \widehat{u}_\varepsilon$ as

$$\widehat{y}_\varepsilon = \text{id} + \varepsilon u_\varepsilon + \varepsilon \widetilde{u} \circ (\text{id} + \varepsilon u_\varepsilon) = \widetilde{y} \circ y_\varepsilon, \tag{4.26}$$

where now $\widetilde{y} = \text{id} + \varepsilon \widetilde{u}$. This choice for the recovery sequence $\widehat{u}_\varepsilon$ corresponds to the one used in [46]. Note in particular that

$$\det \nabla \widehat{y}_\varepsilon = \det(\nabla \widetilde{y}(y_\varepsilon) \nabla y_\varepsilon) = \det \nabla \widetilde{y}(y_\varepsilon) \ \det \nabla y_\varepsilon > 0$$

for small ε, as $\det \nabla \widetilde{y}(y_\varepsilon) \to 1$ uniformly. That is $\boldsymbol{I} + \varepsilon \nabla \widehat{u}_\varepsilon \in \mathrm{GL}^+$ almost everywhere for all ε sufficiently small. Moreover, we immediately check that

$$\widehat{u}_\varepsilon \stackrel{L^2}{\to} \widehat{u}_0. \tag{4.27}$$

The recovery sequence $\widehat{\boldsymbol{z}}_\varepsilon$ is different from the one used in [46] in two respects. At first, the choice is tailored to have a recovery sequence made of *symmetric* tensors whereas no symmetry of the recovery sequence needs to be imposed in [46]. Secondly, we address here the additional intricacy of keeping the gradient of the recovery sequence bounded in L^2 while gradient terms were not discussed in [46]. Note that by neglecting log, $\boldsymbol{\Pi}$, and exp in the definition of $\widehat{\boldsymbol{z}}_\varepsilon$ we would retrieve the classical choice $\widehat{\boldsymbol{z}}_\varepsilon = \boldsymbol{z}_\varepsilon + \tilde{\boldsymbol{z}}$ which is well-suited for quadratic energies in the linear-space setting [40]. The actual definition of $\widehat{\boldsymbol{z}}_\varepsilon$ can hence be seen as an adaptation of the latter to the nonlinear structure of $\mathrm{SL}^+_{\mathrm{sym}}$. On the other, in case $\mu_0 > 0$ the choice of the recovery sequence $\widehat{\boldsymbol{z}}_\varepsilon$ can be much simplified and, under minor adjustments (see [25] for an analogous case) one would be entitled to take the constant sequence $\widehat{\boldsymbol{z}}_\varepsilon = \widehat{\boldsymbol{z}}_0$.

In the following we will use the shorthand notation

$$\boldsymbol{C}_{\mathrm{p}\varepsilon} := \exp(2\varepsilon \boldsymbol{z}_\varepsilon), \quad \widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon} := \exp(2\varepsilon(\boldsymbol{z}_\varepsilon + \widetilde{\boldsymbol{z}})), \quad \widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon} := \boldsymbol{\Pi}(\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}) = \exp(2\varepsilon \widehat{\boldsymbol{z}}_\varepsilon).$$

Step 2: Preliminary results. Owing to the coercivity Lemma 4.2 we have the bound

$$\|\boldsymbol{z}_\varepsilon\|^2_{L^2} + \frac{\mu_\varepsilon}{\varepsilon^2} \|\nabla \exp(2\varepsilon \boldsymbol{z}_\varepsilon)\|^2_{L^2} + \varepsilon \|\boldsymbol{z}_\varepsilon\|_{L^\infty} \le c. \tag{4.28}$$

The uniform Lipschitz continuity of the logarithm on the set K [24, Lemma B.1] entails that

$$2\varepsilon |\nabla \boldsymbol{z}_\varepsilon| = |\nabla \log \boldsymbol{C}_{\mathrm{p}\varepsilon}| \le c |\nabla \boldsymbol{C}_{\mathrm{p}\varepsilon}|.$$

From (4.28) and the bound $\|\exp(-2\varepsilon \boldsymbol{z}_\varepsilon)\|_{L^\infty} \le c$ (see (4.9)), we obtain

$$\sqrt{\mu_\varepsilon}\, \|\nabla \boldsymbol{z}_\varepsilon\|_{L^2} \le c \quad \text{and} \quad \|\nabla \boldsymbol{z}_\varepsilon\|_{L^2} \le c \ \text{ for } \mu_0 > 0. \tag{4.29}$$

For any $\alpha \in \mathbb{R}$, by expanding the exponential $\widetilde{\boldsymbol{C}}^\alpha_{\mathrm{p}\varepsilon} = \exp(2\alpha\varepsilon(\boldsymbol{z}_\varepsilon + \widetilde{\boldsymbol{z}}))$, we obtain the useful expression

$$\widetilde{\boldsymbol{C}}^\alpha_{\mathrm{p}\varepsilon} = 2\alpha\varepsilon \widetilde{\boldsymbol{z}} + \boldsymbol{C}^\alpha_{\mathrm{p}\varepsilon} + \varepsilon^2 \boldsymbol{L}_\varepsilon \tag{4.30}$$

where the tensor

$$\boldsymbol{L}_\varepsilon = \sum_{n=2}^\infty \frac{(2\alpha)^n \varepsilon^{n-2}}{n!} \big((\boldsymbol{z}_\varepsilon + \widetilde{\boldsymbol{z}})^n - \boldsymbol{z}_\varepsilon^n \big)$$

satisfies the bound

$$\|\varepsilon \boldsymbol{L}_\varepsilon\|_\infty + \|\boldsymbol{L}_\varepsilon\|_{L^2} \le c. \tag{4.31}$$

In fact, for all $n \ge 2$, the highest power of $\boldsymbol{z}_\varepsilon$ in $\boldsymbol{L}_\varepsilon$ is controlled by $c\varepsilon^{n-2}|\boldsymbol{z}_\varepsilon|^{n-1}$ and $\|\varepsilon \boldsymbol{z}_\varepsilon\|_\infty + \|\boldsymbol{z}_\varepsilon\|_{L^2} \le c$. Taking the gradient, for $\alpha = 1$, we have

$$\nabla \widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon} = 2\varepsilon \nabla \widetilde{\boldsymbol{z}} + \nabla \boldsymbol{C}_{\mathrm{p}\varepsilon} + \varepsilon^2 \nabla \boldsymbol{L}_\varepsilon. \tag{4.32}$$

In order to estimate $\nabla \boldsymbol{L}_\varepsilon$, we inspect the $\boldsymbol{z}_\varepsilon$-dependent terms in the expansion of $\boldsymbol{L}_\varepsilon$, which behave as $\varepsilon^{n-2}\boldsymbol{z}_\varepsilon^k$, with $1 \le k \le n-1$ for any $n \ge 2$. Correspondingly, the gradient terms fulfill

$$\begin{aligned}\|\varepsilon^{n-2}\nabla \boldsymbol{z}_\varepsilon^k\|_{L^2} &\le k\varepsilon^{n-2}\|(\nabla \boldsymbol{z}_\varepsilon)\boldsymbol{z}_\varepsilon^{k-1}\|_{L^2} = k\varepsilon^{n-k-1}\|(\nabla \boldsymbol{z}_\varepsilon)(\varepsilon \boldsymbol{z}_\varepsilon)^{k-1}\|_{L^2} \\ &\le kc^{k-1}\varepsilon^{n-k-1}\|\nabla \boldsymbol{z}_\varepsilon\|_{L^2}.\end{aligned}$$

In view of (4.29) this entails

$$\sqrt{\mu_\varepsilon}\,\|\nabla \boldsymbol{L}_\varepsilon\|_{L^2} \le c \quad \text{and} \quad \|\nabla \boldsymbol{L}_\varepsilon\|_{L^2} \le c \;\text{ for } \mu_0 > 0. \tag{4.33}$$

We now define the sets

$$K_\varepsilon := \{x \in \Omega \mid \widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}(x) \in K\} = \{x \in \Omega \mid \widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon}(x) = \widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}(x)\}.$$

In particular, note that

$$\widehat{\boldsymbol{z}}_\varepsilon - \widetilde{\boldsymbol{z}} - \boldsymbol{z}_\varepsilon = 0 \;\text{ on } K_\varepsilon.$$

The complement of K_ε has small measure. Indeed, from (4.19) and (4.30), it follows that $\|\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}-\boldsymbol{I}\|_{L^2}^2 \le c\varepsilon^2$. Moreover, one has that $|\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}(x)-\boldsymbol{I}| \ge \kappa/\sqrt{3}-1$ for $\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}(x) \in \mathrm{SL}^+_{\mathrm{sym}}\backslash K$, that is for $x \in \Omega\backslash K_\varepsilon$. Hence,

$$|\Omega\backslash K_\varepsilon| = \int_{\Omega\backslash K_\varepsilon} \mathrm{d}x \le \frac{1}{(\kappa/\sqrt{3}-1)^2}\int_{\Omega\backslash K_\varepsilon} |\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}-\boldsymbol{I}|^2\,\mathrm{d}x \le \frac{1}{(\kappa/\sqrt{3}-1)^2}\|\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}-\boldsymbol{I}\|_{L^2}^2 \le c\varepsilon^2.$$

The following convergences will be used in the estimate of the lim sup of the hardening terms

$$\widehat{\boldsymbol{z}}_\varepsilon - \boldsymbol{z}_\varepsilon \overset{L^2}{\to} \widetilde{\boldsymbol{z}}, \tag{4.34}$$

$$\widehat{\boldsymbol{z}}_\varepsilon + \boldsymbol{z}_\varepsilon \overset{L^2}{\rightharpoonup} \widehat{\boldsymbol{z}}_0 + \boldsymbol{z}_0. \tag{4.35}$$

Indeed, on K_ε we have $\widehat{\boldsymbol{z}}_\varepsilon - \boldsymbol{z}_\varepsilon = \widetilde{\boldsymbol{z}}$ and $\widehat{\boldsymbol{z}}_\varepsilon + \boldsymbol{z}_\varepsilon = \widetilde{\boldsymbol{z}} + 2\boldsymbol{z}_\varepsilon$ with $\boldsymbol{z}_\varepsilon \overset{L^2}{\rightharpoonup} \boldsymbol{z}_0$. Hence, convergences (4.34)-(4.35) follow from $|\Omega\backslash K_\varepsilon| < C\varepsilon^2$ and the L^2-boundedness of $\widehat{\boldsymbol{z}}_\varepsilon$ and $\boldsymbol{z}_\varepsilon$. In particular, by taking the sum of (4.34) and (4.35) we conclude that

$$\widehat{\boldsymbol{z}}_\varepsilon \overset{L^2}{\rightharpoonup} \widehat{\boldsymbol{z}}_0. \tag{4.36}$$

Step 3: The lim sup *inequality for the dissipation.* Let us decompose

$$\mathcal{D}_\varepsilon(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) = \frac{1}{2\varepsilon}\int_{\Omega\backslash K_\varepsilon} D(\boldsymbol{C}_{\mathrm{p}\varepsilon}, \widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon})\,\mathrm{d}x + \frac{1}{2\varepsilon}\int_{K_\varepsilon} D\big(\exp(2\varepsilon \boldsymbol{z}_\varepsilon), \exp(2\varepsilon\widehat{\boldsymbol{z}}_\varepsilon)\big)\,\mathrm{d}x.$$

Taking into account the uniform Lipschitz continuity of D on K, we have

$$\begin{aligned}\frac{1}{\varepsilon}D(\boldsymbol{C}_{\mathrm{p}\varepsilon}, \widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon}) &= \frac{1}{\varepsilon}D\big(\boldsymbol{\Pi}(\boldsymbol{C}_{\mathrm{p}\varepsilon}), \boldsymbol{\Pi}(\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon})\big) \le \frac{c}{\varepsilon}|\boldsymbol{\Pi}(\boldsymbol{C}_{\mathrm{p}\varepsilon}) - \boldsymbol{\Pi}(\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon})| \\ &\le \frac{c}{\varepsilon}|\boldsymbol{C}_{\mathrm{p}\varepsilon} - \widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}| \overset{(4.30)}{=} c|2\widetilde{\boldsymbol{z}} + \varepsilon \boldsymbol{L}_\varepsilon|\end{aligned}$$

and the right-hand side is uniformly bounded in L^∞. Since $|\Omega\backslash K_\varepsilon| < c\varepsilon^2$, it follows that

$$\limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) \le \limsup_{\varepsilon\to 0} \int_{K_\varepsilon} D_\varepsilon(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon)\,\mathrm{d}x.$$

On the other hand, by recalling (3.3), on the set K_ε we have that

$$\frac{1}{2\varepsilon} D\big(\exp(2\varepsilon\boldsymbol{z}_\varepsilon), \exp(2\varepsilon\widehat{\boldsymbol{z}}_\varepsilon)\big) \le \widetilde{R}(\widehat{\boldsymbol{z}}_\varepsilon - \boldsymbol{z}_\varepsilon) = \widetilde{R}(\widetilde{\boldsymbol{z}}),$$

hence

$$\limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(\boldsymbol{z}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) \le \int_\Omega R(\widetilde{\boldsymbol{z}})\,\mathrm{d}x = \mathcal{D}_0(\boldsymbol{z}_0, \widehat{\boldsymbol{z}}_0).$$

Step 4: The lim sup *inequality for the gradient term.* In case $\mu_0 > 0$, we aim at showing that

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \frac{\mu_\varepsilon}{2\varepsilon^2}\left(\int_\Omega |\nabla \exp(2\varepsilon\widehat{\boldsymbol{z}}_\varepsilon)|^2\mathrm{d}x - \int_\Omega |\nabla \exp(2\varepsilon\boldsymbol{z}_\varepsilon)|^2\,\mathrm{d}x\right)\\ &\le 2\mu_0\int_\Omega |\nabla\widehat{\boldsymbol{z}}_0|^2\mathrm{d}x - 2\mu_0\int_\Omega |\nabla\boldsymbol{z}_0|^2\,\mathrm{d}x.\end{aligned} \tag{4.37}$$

The contractive character of $\boldsymbol{\Pi}$, see (4.25), ensures that

$$|\nabla\exp(2\varepsilon\widehat{\boldsymbol{z}}_\varepsilon)| = |\nabla(\boldsymbol{\Pi}(\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}))| \le |\nabla\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}|.$$

By the decomposition (4.32) and the bound $\|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|_{L^2} \le c\varepsilon$, from the coercivity condition (4.18) we compute

$$\begin{aligned}&\frac{1}{\varepsilon^2}(\|\nabla\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}\|^2_{L^2} - \|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|^2_{L^2}) = \|2\nabla\widetilde{\boldsymbol{z}} + \nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}/\varepsilon + \varepsilon\nabla\boldsymbol{L}_\varepsilon\|^2_{L^2} - \|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}/\varepsilon\|^2_{L^2}\\ &\le 4\|\nabla\widetilde{\boldsymbol{z}}\|^2_{L^2} + 2\varepsilon^{-1}\|\nabla\widetilde{\boldsymbol{z}}\,\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon} + \nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\nabla\widetilde{\boldsymbol{z}}\|_{L^1} + c\varepsilon.\end{aligned}$$

Owing to convergence (4.20) we have that $(2\varepsilon)^{-1}\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon} \stackrel{L^2}{\rightharpoonup} \nabla\boldsymbol{z}_0$, so that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left(\frac{\mu_\varepsilon}{2\varepsilon^2}\left(\|\nabla\widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon}\|^2_{L^2} - \|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|^2_{L^2}\right)\right)\\ &\le 2\mu_0\|\nabla\widetilde{\boldsymbol{z}}\|^2_{L^2} + 2\mu_0\|\nabla\widetilde{\boldsymbol{z}}\,\nabla\boldsymbol{z}_0 + \nabla\boldsymbol{z}_0\nabla\widetilde{\boldsymbol{z}}\|_{L^1} =\\ &= 2\mu_0\|\nabla(\widetilde{\boldsymbol{z}} + \boldsymbol{z}_0)\|^2_{L^2} - 2\mu_0\|\nabla\boldsymbol{z}_0\|^2_{L^2} = 2\mu_0\|\nabla\widehat{\boldsymbol{z}}\|^2_{L^2} - 2\mu_0\|\nabla\boldsymbol{z}_0\|^2_{L^2}\end{aligned}$$

which corresponds to (4.37) for $\mu_0 > 0$.

Let us start by considering the case $\mu_0 = 0$. By using $|\nabla\widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon}| \le |\nabla\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}|$ and (4.32) we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\frac{\mu_\varepsilon}{2\varepsilon^2}\left(\|\nabla\widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon}\|^2_{L^2} - \|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|^2_{L^2}\right) \le \limsup_{\varepsilon\to 0}\frac{\mu_\varepsilon}{2\varepsilon^2}\left(\|\nabla\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}\|^2_{L^2} - \|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|^2_{L^2}\right)\\ &= \limsup_{\varepsilon\to 0}\frac{\mu_\varepsilon}{2\varepsilon^2}\left(\|2\varepsilon\nabla\widetilde{\boldsymbol{z}} + \nabla\boldsymbol{C}_{\mathrm{p}\varepsilon} + \varepsilon^2\nabla\boldsymbol{L}_\varepsilon\|^2_{L^2} - \|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|^2_{L^2}\right).\end{aligned}$$

The Schwarz inequality gives

$$\frac{\mu_\varepsilon}{2\varepsilon^2}\Big(\|2\varepsilon\nabla\widetilde{\boldsymbol{z}}+\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}+\varepsilon^2\nabla\boldsymbol{L}_\varepsilon\|^2_{L^2}-\|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|^2_{L^2}\Big)$$
$$\leq 2\mu_\varepsilon\|\nabla\widetilde{\boldsymbol{z}}\|^2_{L^2}+\frac{\varepsilon^2\mu_\varepsilon}{2}\|\nabla\boldsymbol{L}_\varepsilon\|^2_{L^2}+2\sqrt{\mu_\varepsilon}\|\nabla\widetilde{\boldsymbol{z}}\|_{L^2}\frac{\sqrt{\mu_\varepsilon}}{\varepsilon}\|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|_{L^2}$$
$$+\,2\varepsilon\sqrt{\mu_\varepsilon}\|\nabla\widetilde{\boldsymbol{z}}\|_{L^2}\sqrt{\mu_\varepsilon}\|\nabla\boldsymbol{L}_\varepsilon\|_{L^2}+\varepsilon\sqrt{\mu_\varepsilon}\|\nabla\boldsymbol{L}_\varepsilon\|_{L^2}\frac{\sqrt{\mu_\varepsilon}}{\varepsilon}\|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|_{L^2}.$$

The coercivity condition (4.18) and (4.33) ensure that

$$\frac{\sqrt{\mu_\varepsilon}}{\varepsilon}\|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|_{L^2}\leq c,\qquad \sqrt{\mu_\varepsilon}\|\nabla\boldsymbol{L}_\varepsilon\|_{L^2}\leq c \tag{4.38}$$

and, of course, $\|\nabla\widetilde{\boldsymbol{z}}\|_{L^2}\leq c$. Therefore

$$\limsup_{\varepsilon\to 0}\frac{\mu_\varepsilon}{2\varepsilon^2}\Big(\|\nabla\widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon}\|^2_{L^2}-\|\nabla\boldsymbol{C}_{\mathrm{p}\varepsilon}\|^2_{L^2}\Big)\leq$$
$$\leq\limsup_{\varepsilon\to 0}\left(2\mu_\varepsilon c^2+\frac{\varepsilon^2c^2}{2}+2\sqrt{\mu_\varepsilon}\,c^2+2\varepsilon\sqrt{\mu_\varepsilon}c^2+\varepsilon c^2\right)=0,$$

namely relation (4.37) for $\mu_0=0$.

Step 5: The lim sup *inequality for the elastic energies.* Let $\boldsymbol{A}_\varepsilon$ be defined by (4.21) and $\widehat{\boldsymbol{A}}_\varepsilon$ have an analogous expression in terms of $\widehat{u}_\varepsilon$ and $\widehat{\boldsymbol{z}}_\varepsilon$. We aim at proving that

$$\limsup_{\varepsilon\to 0}\left(\int_\Omega W^\varepsilon_{\mathrm{e}}(\widehat{\boldsymbol{A}}_\varepsilon)\,\mathrm{d}x-\int_\Omega W^\varepsilon_{\mathrm{e}}(\boldsymbol{A}_\varepsilon)\,\mathrm{d}x\right)$$
$$\leq\frac{1}{2}\int_\Omega|\nabla\widehat{u}^{\mathrm{sym}}_0-\widehat{\boldsymbol{z}}_0|^2_{\mathbb{C}}\,\mathrm{d}x-\frac{1}{2}\int_\Omega|\nabla u^{\mathrm{sym}}_0-\boldsymbol{z}_0|^2_{\mathbb{C}}\,\mathrm{d}x \tag{4.39}$$

where we have used the shorthand notation $W^\varepsilon_{\mathrm{e}}(\boldsymbol{A}):=\varepsilon^{-2}W_{\mathrm{e}}(\boldsymbol{I}+\varepsilon\boldsymbol{A})$. We preliminarily observe that

$$\|\boldsymbol{A}_\varepsilon\|_{L^2}+\|\widehat{\boldsymbol{A}}_\varepsilon\|_{L^2}\leq c. \tag{4.40}$$

Indeed, by using the decomposition (4.19) we have

$$\boldsymbol{C}^{-1/2}_{\mathrm{p}\varepsilon}=\boldsymbol{I}+\varepsilon\boldsymbol{L}_\varepsilon$$

with $\boldsymbol{L}_\varepsilon$ satisfying the bound (4.31). In particular, one has

$$\boldsymbol{A}_\varepsilon=\nabla u_\varepsilon+\boldsymbol{L}_\varepsilon+\nabla u_\varepsilon(\varepsilon\boldsymbol{L}_\varepsilon)$$

so that the bound for $\|\boldsymbol{A}_\varepsilon\|_{L^2}\leq c$ follows. The control of $\widehat{\boldsymbol{A}}_\varepsilon$ is analogous. On the set K_ε we have that

$$\widehat{\boldsymbol{A}}_\varepsilon=\nabla\widehat{u}_\varepsilon+\widehat{\boldsymbol{L}}_\varepsilon+\nabla\widehat{u}_\varepsilon(\varepsilon\widehat{\boldsymbol{L}}_\varepsilon)$$

for some $\widehat{\boldsymbol{L}}_\varepsilon$ fulfilling (4.31).

We next remark that

$$\nabla\widehat{u}_\varepsilon-\nabla u_\varepsilon\stackrel{L^2}{\to}\nabla\widetilde{u}. \tag{4.41}$$

Indeed, by computing

$$\nabla\widehat{u}_\varepsilon = \frac{1}{\varepsilon}\big(\nabla\tilde{y}(y_\varepsilon)\nabla y_\varepsilon - \boldsymbol{I}\big) = \frac{1}{\varepsilon}\big((\boldsymbol{I}+\varepsilon\nabla\tilde{u})(y_\varepsilon)\nabla y_\varepsilon - \boldsymbol{I}\big)$$
$$= \frac{1}{\varepsilon}\big(\nabla y_\varepsilon + \varepsilon\nabla\tilde{u}(y_\varepsilon)\nabla y_\varepsilon - \boldsymbol{I}\big) = \nabla u_\varepsilon + \nabla\tilde{u}(y_\varepsilon) + \varepsilon\nabla\tilde{u}(y_\varepsilon)\nabla u_\varepsilon$$

we obtain that

$$\|(\nabla\widehat{u}_\varepsilon - \nabla u_\varepsilon) - \nabla\tilde{u}\|_{L^2} \le \|\nabla\tilde{u}(y_\varepsilon) - \nabla\tilde{u}\|_{L^2} + \|\varepsilon\nabla\tilde{u}(y_\varepsilon)\nabla u_\varepsilon\|_{L^2}$$
$$\le c\varepsilon + c\varepsilon\|\nabla u_\varepsilon\|_{L^2} \le c\varepsilon.$$

On the other hand, we readily check that

$$\nabla\widehat{u}_\varepsilon + \nabla u_\varepsilon = 2\nabla u_\varepsilon + \nabla\tilde{u}(y_\varepsilon) + \varepsilon\nabla\tilde{u}(y_\varepsilon)\nabla u_\varepsilon$$

so that the convergence

$$\nabla\widehat{u}_\varepsilon + \nabla u_\varepsilon \overset{L^2}{\rightharpoonup} \nabla\widehat{u}_0 + \nabla u_0 \tag{4.42}$$

follows. By combining (4.27) and (4.41)-(4.42) we obtain that

$$\widehat{u}_\varepsilon \overset{H^1}{\rightharpoonup} \widehat{u}_0. \tag{4.43}$$

As $\|\nabla\widehat{\boldsymbol{z}}_\varepsilon\|_{L^2}$ is bounded, convergences (4.36) and (4.43) entail that

$$(\widehat{u}_\varepsilon, \widehat{\boldsymbol{z}}_\varepsilon) \rightharpoonup (\widehat{u}_0, \widehat{\boldsymbol{z}}_0) \ \text{ in } \ \mathcal{Y}_0 \times \mathcal{C}_0$$

as required.

We now turn to the proof of the two convergences

$$\widehat{\boldsymbol{A}}_\varepsilon - \boldsymbol{A}_\varepsilon \overset{L^2}{\to} \nabla\widetilde{u} - \widetilde{\boldsymbol{z}} = (\nabla\widehat{u}_0 - \widehat{\boldsymbol{z}}_0) - (\nabla u_0 - \boldsymbol{z}_0), \tag{4.44}$$
$$\widehat{\boldsymbol{A}}_\varepsilon + \boldsymbol{A}_\varepsilon \overset{L^2}{\rightharpoonup} (\nabla\widehat{u}_0 - \widehat{\boldsymbol{z}}_0) + (\nabla u_0 - \boldsymbol{z}_0). \tag{4.45}$$

On the set K_ε we use (4.30) for $\alpha = -1/2$ in order to get that

$$\widehat{\boldsymbol{A}}_\varepsilon - \boldsymbol{A}_\varepsilon = \frac{1}{\varepsilon}\left((\boldsymbol{I}+\varepsilon\nabla\widehat{u}_\varepsilon)\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}^{-1/2} - (\boldsymbol{I}+\varepsilon\nabla u_\varepsilon)\boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2}\right)$$
$$= \frac{1}{\varepsilon}\left(\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}^{-1/2} - \boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2}\right) + \nabla\widetilde{u}\boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2} + (\nabla\widehat{u}_\varepsilon - \nabla u_\varepsilon - \nabla\widetilde{u})\boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2} + \nabla\widehat{u}_\varepsilon\left(\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}^{-1/2} - \boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2}\right)$$
$$= (-\widetilde{\boldsymbol{z}} + \varepsilon\boldsymbol{L}_\varepsilon) + \nabla\widetilde{u}\boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2} + (\nabla\widehat{u}_\varepsilon - \nabla u_\varepsilon - \nabla\widetilde{u})\boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2} + \varepsilon\nabla\widehat{u}_\varepsilon(-\widetilde{\boldsymbol{z}} + \varepsilon\boldsymbol{L}_\varepsilon)$$

with $\|\boldsymbol{L}_\varepsilon\|_{L^2} \le c$. The first term in the above right-hand side converges L^2-strongly to $-\widetilde{\boldsymbol{z}}$. Since $\boldsymbol{C}_{\mathrm{p}\varepsilon}^{-1/2} \overset{L^2}{\longrightarrow} \boldsymbol{I}$ by (4.19), the second term strongly converges in L^2 to $\nabla\widetilde{u}$. The last two terms are easily seen to be strongly L^2 convergent to zero. Since $|\Omega\backslash K_\varepsilon| \le c\varepsilon^2$, the bound (4.40) yields convergence (4.44).

The proof of the weak convergence (4.45) results as a combination of the same argument for (4.22) on K_ε and the L^2-boundedness of $\boldsymbol{A}_\varepsilon$ and $\widehat{\boldsymbol{A}}_\varepsilon$.

Note that the quadratic behavior (4.10) of $\widehat{W}_{\rm e}$ is equivalent to

$$\forall\delta>0\,\exists\tilde c_\delta>0: \qquad \left|W_{\rm e}(\boldsymbol{I}{+}\boldsymbol{A}){-}\frac{1}{2}|\boldsymbol{A}|^2_{\mathbb{C}}\right|\le\delta|\boldsymbol{A}|^2 \quad \forall|\boldsymbol{A}|\le\tilde c_\delta.$$

We define for all $\delta>0$ the sets

$$U^\delta_\varepsilon:=\{x\in\Omega\mid|\varepsilon\boldsymbol{A}_\varepsilon(x)|+|\varepsilon\widehat{\boldsymbol{A}}_\varepsilon(x)|\le\tilde c_\delta\}$$

On these sets, also by using the bound (4.40), we have

$$\begin{aligned} W^\varepsilon_{\rm e}(\widehat{\boldsymbol{A}}_\varepsilon)-W^\varepsilon_{\rm e}(\boldsymbol{A}_\varepsilon) &\le (1{+}\delta)|\widehat{\boldsymbol{A}}_\varepsilon|^2_{\mathbb{C}}-(1{-}\delta)|\boldsymbol{A}_\varepsilon|^2_{\mathbb{C}}\le|\widehat{\boldsymbol{A}}_\varepsilon|^2_{\mathbb{C}}-|\boldsymbol{A}_\varepsilon|^2_{\mathbb{C}}+2c\delta\tilde c_\delta^2=\\ &= \frac{1}{2}(\widehat{\boldsymbol{A}}_\varepsilon{-}\boldsymbol{A}_\varepsilon){:}\mathbb{C}(\widehat{\boldsymbol{A}}_\varepsilon{+}\boldsymbol{A}_\varepsilon)+2c\delta\tilde c_\delta^2 \quad \text{on} \quad U^\delta_\varepsilon. \end{aligned} \tag{4.46}$$

We can easily estimate the measure of the sets U^δ_ε as follows

$$|\Omega\backslash U^\delta_\varepsilon|=\int_{\Omega\backslash U^\delta_\varepsilon}{\rm d}x\le\frac{1}{\tilde c_\delta^2}\int_{\Omega\backslash U^\delta_\varepsilon}(|\varepsilon\boldsymbol{A}_\varepsilon(x)|+|\varepsilon\widehat{\boldsymbol{A}}_\varepsilon(x)|)\,{\rm d}x\le\frac{c\varepsilon^2}{\tilde c_\delta^2}. \tag{4.47}$$

Thanks convergences (4.44) and(4.45), estimates (4.46) and (4.47) entail

$$\begin{aligned} &\limsup_{\varepsilon\to0}\int_{U^\delta_\varepsilon}\Big(W^\varepsilon_{\rm e}(\widehat{\boldsymbol{A}}_\varepsilon)-W^\varepsilon_{\rm e}(\boldsymbol{A}_\varepsilon)\Big)\,{\rm d}x\\ &\le\limsup_{\varepsilon\to0}\left(2c\delta\tilde c_\delta^2|\Omega|+\frac{1}{2}\int_{U^\delta_\varepsilon}(\widehat{\boldsymbol{A}}_\varepsilon{-}\boldsymbol{A}_\varepsilon){:}\mathbb{C}(\widehat{\boldsymbol{A}}_\varepsilon{+}\boldsymbol{A}_\varepsilon)\,{\rm d}x\right)\\ &\le 2c\delta\tilde c_\delta^2|\Omega|+\limsup_{\varepsilon\to0}\left(\frac{1}{2}\int_{U^\delta_\varepsilon}(\nabla\widetilde u{-}\widetilde{\boldsymbol{z}}){:}\mathbb{C}(\nabla\widehat u_0{-}\widehat{\boldsymbol{z}}_0{+}\nabla u_0{-}\boldsymbol{z}_0)\,{\rm d}x\right)\\ &\le 2c\delta\tilde c_\delta^2|\Omega|+\frac{1}{2}\int_\Omega\big(|\nabla\widehat u_0^{\rm sym}{-}\widehat{\boldsymbol{z}}_0|^2_{\mathbb{C}}-|\nabla u_0^{\rm sym}{-}\boldsymbol{z}_0|^2_{\mathbb{C}}\big)\,{\rm d}x \end{aligned}$$

where the minor-symmetry property $\mathbb{C}\boldsymbol{A}=\mathbb{C}\boldsymbol{A}^{\rm sym}$ has been used.

We shall now discuss the contribution of the sets $\Omega\setminus U^\delta_\varepsilon$. Our aim is to show that the corresponding elastic-energies terms are uniformly bounded by $c\varepsilon$. Consider relation

$$\nabla\widehat y_\varepsilon\widehat{\boldsymbol{C}}^{-1/2}_{{\rm p}\varepsilon}=(\nabla\widehat y_\varepsilon\nabla y_\varepsilon^{-1})(\nabla y_\varepsilon\boldsymbol{C}^{-1/2}_{{\rm p}\varepsilon})(\boldsymbol{C}^{1/2}_{{\rm p}\varepsilon}\widehat{\boldsymbol{C}}^{-1/2}_{{\rm p}\varepsilon}).$$

This can be rewritten as

$$1+\varepsilon\widehat{\boldsymbol{A}}_\varepsilon=\boldsymbol{G}_{1,\varepsilon}(1{+}\varepsilon\boldsymbol{A}_\varepsilon)\boldsymbol{G}_{2,\varepsilon}$$

with

$$\boldsymbol{G}_{1,\varepsilon}:=\nabla\widehat y_\varepsilon\nabla y_\varepsilon^{-1},\qquad \boldsymbol{G}_{2,\varepsilon}:=\boldsymbol{C}^{1/2}_{{\rm p}\varepsilon}\widehat{\boldsymbol{C}}^{-1/2}_{{\rm p}\varepsilon}.$$

Recalling the choice (4.26), we have

$$\boldsymbol{G}_{1,\varepsilon}{-}\boldsymbol{I}=\nabla(\widetilde y\circ y_\varepsilon)\nabla y_\varepsilon^{-1}-\boldsymbol{I}=\nabla\widetilde y-\boldsymbol{I}=\varepsilon(\nabla\widetilde u)\circ y_\varepsilon.$$

Now we consider

$$\boldsymbol{G}_{2,\varepsilon}{-}\boldsymbol{I}=\boldsymbol{C}^{1/2}_{{\rm p}\varepsilon}\left((\boldsymbol{\Pi}(\widetilde{\boldsymbol{C}}_{{\rm p}\varepsilon}))^{-1/2}-\boldsymbol{C}^{-1/2}_{{\rm p}\varepsilon}\right).$$

By using the Lipschitz-continuity of the matrix square root [24, Lemma B.1], we have

$$|\boldsymbol{G}_{2,\varepsilon}-\boldsymbol{I}| \le c|\boldsymbol{\Pi}(\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon})-\boldsymbol{C}_{\mathrm{p}\varepsilon}| \le c|\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}-\boldsymbol{C}_{\mathrm{p}\varepsilon}| \le c\varepsilon|2\widetilde{\boldsymbol{z}}+\varepsilon\boldsymbol{L}_\varepsilon|.$$

The uniform bounds

$$\|\boldsymbol{G}_{1,\varepsilon}-\boldsymbol{I}\|_{L^\infty} \le c\varepsilon, \qquad \|\boldsymbol{G}_{2,\varepsilon}-\boldsymbol{I}\|_{L^\infty} \le c\varepsilon \tag{4.48}$$

then follow. These bounds allow us to use the following estimate [46, Lemma 4.1]

$$|W_\mathrm{e}(\boldsymbol{G}_1\boldsymbol{F}\boldsymbol{G}_2)-W_\mathrm{e}(\boldsymbol{F})| \le c(1+W_\mathrm{e}(\boldsymbol{F}))(|\boldsymbol{G}_1-\boldsymbol{I}|+|\boldsymbol{G}_2-\boldsymbol{I}|) \quad \forall |\boldsymbol{G}_1|,\, |\boldsymbol{G}_2| \le \delta \tag{4.49}$$

for some constants c, $\delta > 0$. By combining this with the bounds (4.48) one has that

$$\begin{aligned}
\int_{\Omega\setminus U_\varepsilon^\delta} (W_\mathrm{e}^\varepsilon(\widehat{\boldsymbol{A}}_\varepsilon) - W_\mathrm{e}^\varepsilon(\boldsymbol{A}_\varepsilon))\,\mathrm{d}x &= \frac{1}{\varepsilon^2}\int_{\Omega\setminus U_\varepsilon^\delta}(W_\mathrm{e}(\boldsymbol{G}_{1,\varepsilon}\boldsymbol{F}_\varepsilon\boldsymbol{G}_{1,\varepsilon}) - W_\mathrm{e}(\boldsymbol{F}_\varepsilon))\,\mathrm{d}x \\
&\overset{(4.49)}{\le} \frac{c}{\varepsilon^2}\int_{\Omega\setminus U_\varepsilon^\delta}(1+W_\mathrm{e}(\boldsymbol{F}_\varepsilon))(|\boldsymbol{G}_{1,\varepsilon}-\boldsymbol{I}|+|\boldsymbol{G}_{2,\varepsilon}-\boldsymbol{I}|)\,\mathrm{d}x \\
&\overset{(4.48)}{\le} \frac{c}{\varepsilon}\int_{\Omega\setminus U_\varepsilon^\delta}(1+W_\mathrm{e}(\boldsymbol{F}_\varepsilon))\,\mathrm{d}x \overset{(4.47)}{\le} c\varepsilon,
\end{aligned}$$

which completes the proof of the $\limsup$ relation (4.39).

Step 6: The $\limsup$ *inequality for the plastic energy.* We aim at showing that

$$\limsup_{\varepsilon\to 0}\left(\int_\Omega W_\mathrm{p}^\varepsilon(\widehat{\boldsymbol{z}}_\varepsilon)\,\mathrm{d}x - \int_\Omega W_\mathrm{p}^\varepsilon(\boldsymbol{z}_\varepsilon)\,\mathrm{d}x\right) \le \frac{1}{2}\int_\Omega |\widehat{\boldsymbol{z}}_0|^2_{\mathbb{H}}\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\boldsymbol{z}_0|^2_{\mathbb{H}}\,\mathrm{d}x, \tag{4.50}$$

where we have used the short-hand notation $W_\mathrm{p}^\varepsilon(\boldsymbol{z}) := \varepsilon^{-2}W_\mathrm{p}(\exp(2\varepsilon\boldsymbol{z}))$. The strategy is similar to the one used for the elastic energy. First, we define the sets

$$Z_\varepsilon^\delta := \{x \in \Omega \mid |\varepsilon\boldsymbol{z}_\varepsilon(x)| + |\varepsilon\widehat{\boldsymbol{z}}_\varepsilon(x)| \le c_\delta\},$$

with c_δ from (4.10), so that

$$W_\mathrm{p}^\varepsilon(\widehat{\boldsymbol{z}}_\varepsilon) - W_\mathrm{p}^\varepsilon(\boldsymbol{z}_\varepsilon) \le \frac{1}{2}(\widehat{\boldsymbol{z}}_\varepsilon-\boldsymbol{z}_\varepsilon){:}\mathbb{H}(\widehat{\boldsymbol{z}}_\varepsilon+\boldsymbol{z}_\varepsilon) + 2c\delta c_\delta^2 \quad \text{on} \quad Z_\varepsilon^\delta. \tag{4.51}$$

Arguing exactly as in Step 5, we can prove that the complementary sets $\Omega\setminus Z_\varepsilon^\delta$ fulfill

$$|\Omega\setminus Z_\varepsilon^\delta| \le \frac{c\varepsilon^2}{c_\delta^2}.$$

Owing to the Lipschitz-continuity of $\widehat{W}_\mathrm{p}$ on K, the contraction property of $\boldsymbol{\Pi}$ and (4.30)

$$\begin{aligned}
&\int_{\Omega\setminus Z_\varepsilon^\delta}\big(W_\mathrm{p}^\varepsilon(\widehat{\boldsymbol{z}}_\varepsilon)-W_\mathrm{p}^\varepsilon(\boldsymbol{z}_\varepsilon)\big)\,\mathrm{d}x = \frac{1}{\varepsilon^2}\int_{\Omega\setminus Z_\varepsilon^\delta}\Big(\widehat{W}_\mathrm{p}(\widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon})-\widehat{W}_\mathrm{p}(\boldsymbol{C}_{\mathrm{p}\varepsilon})\Big)\,\mathrm{d}x \le \\
&\frac{c}{\varepsilon^2}\int_{\Omega\setminus Z_\varepsilon^\delta}|\widehat{\boldsymbol{C}}_{\mathrm{p}\varepsilon}-\boldsymbol{C}_{\mathrm{p}\varepsilon}|\mathrm{d}x \le \frac{c}{\varepsilon^2}\int_{\Omega\setminus Z_\varepsilon^\delta}|\widetilde{\boldsymbol{C}}_{\mathrm{p}\varepsilon}-\boldsymbol{C}_{\mathrm{p}\varepsilon}|\mathrm{d}x \le \frac{c}{\varepsilon^2}|\Omega\setminus Z_\varepsilon^\delta|\varepsilon = c\varepsilon.
\end{aligned} \tag{4.52}$$

By combining (4.51)-(4.52) and (4.34)-(4.35), the $\limsup$ condition (4.50) follows from δ being arbitrary. $\square$

Proof of Theorem 4.1. Lemmas 4.3, 4.4, and 4.5, allow us to apply the abstract convergence theorem in [45, Thm. 3.1]. Although Lemma 4.5 deals with smooth and compactly supported competitors only, the full strength of the recovery condition in (4.16)-(4.17) can be easily deduced by density. The pointwise strong convergence of $(u_\varepsilon, \boldsymbol{z}_\varepsilon)$ and the convergence of energies and dissipation follow at once from the uniform convexity of the linearized energy $\mathcal{E}_0$ along the same lines as [46, Cor. 3.8 and Cor. 3.9]. □

Appendix A. The map $\boldsymbol{\Pi}$

We collect here some comment on the existence of a map

$$\boldsymbol{\Pi} : \mathrm{SL}^+_{\mathrm{sym}} \to K$$

having properties (4.25). Note that this has been used in the definition of the recovery sequence (4.24).

Recall that

$$K = \{\boldsymbol{C} \in \mathrm{SL}^+_{\mathrm{sym}} \mid |\boldsymbol{C}| \leq \kappa\}, \qquad \kappa > |\boldsymbol{I}| = \sqrt{3}$$

and let the flux Φ_t, $t \geq 0$, be associated to the following differential equation on GL^+

$$\dot{\boldsymbol{C}} = -\left(\boldsymbol{C} - 3|\boldsymbol{C}^{-1}|^{-2}\boldsymbol{C}^{-1}\right). \tag{A.1}$$

In particular, $t \mapsto \Phi_t(\boldsymbol{C})$ is the solution of the differential equation (A.1) with initial datum $\boldsymbol{C}$. Note that the manifold $\mathrm{SL}^+_{\mathrm{sym}}$ is invariant under the flux Φ_t. In fact, symmetry and determinant constraint are preserved along solutions $\boldsymbol{C}(t)$ of the equation (A.1), as the symmetry of $\boldsymbol{C}$ induces that of $\dot{\boldsymbol{C}}$ and

$$\mathrm{tr}\,(\boldsymbol{C}(t)^{-1}\dot{\boldsymbol{C}}(t)) = -\mathrm{tr}\,(\boldsymbol{I} - 3|\boldsymbol{C}^{-1}|^{-2}\boldsymbol{C}^{-2}) = -(3 - 3|\boldsymbol{C}^{-1}|^{-2}\boldsymbol{C}^{-2}{:}\boldsymbol{I}) = 0.$$

Moreover, the flux Φ_t is norm-contractive for we readily check that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{C}(t)|^2 &= \mathrm{tr}\,(\boldsymbol{C}\dot{\boldsymbol{C}}) = -\mathrm{tr}\,(\boldsymbol{C}^2 - 3|\boldsymbol{C}^{-1}|^{-2}\boldsymbol{I}) \\ &= -(|\boldsymbol{C}|^2 - 9|\boldsymbol{C}^{-1}|^{-2}) \leq 3 - |\boldsymbol{C}|^2 \leq 0.\end{aligned}$$

We have here used the fact that $\boldsymbol{C}^{-1} \in \mathrm{SL}^+_{\mathrm{sym}}$ and $|\boldsymbol{C}^{-1}|^2 \geq 3$. More precisely, as $|\boldsymbol{C}|^2 \geq 3$ on $\mathrm{SL}^+_{\mathrm{sym}}$ (with equality corresponding to $\boldsymbol{C} = \boldsymbol{I}$), we have checked that

$$|\boldsymbol{C}| > \sqrt{3} \ \Rightarrow\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{C}(t)|^2 < 0. \tag{A.2}$$

Let us record some additional properties of the flux Φ_t in the following lemma.

Lemma A.1. *The flux Φ_t satisfies the following properties*

i) *Let $\boldsymbol{C}$, $\boldsymbol{C}_0 \in \mathrm{SL}^+_{\mathrm{sym}}$. Then*

$$|\Phi_t(\boldsymbol{C})| \geq |\boldsymbol{C}_0| \ \Rightarrow\ \frac{\mathrm{d}}{\mathrm{d}t}|\Phi_t(\boldsymbol{C}) - \boldsymbol{C}_0| \leq 0. \tag{A.3}$$

ii) *For all* $t \geq 0$, Φ_t *is a contraction on* $\mathrm{SL}^+_{\mathrm{sym}}$, *namely*

$$|\Phi_t(\boldsymbol{C}_1) - \Phi_t(\boldsymbol{C}_2)| \leq |\boldsymbol{C}_1 - \boldsymbol{C}_2| \quad \forall \boldsymbol{C}_1, \boldsymbol{C}_2 \in \mathrm{SL}^+_{\mathrm{sym}}. \tag{A.4}$$

Proof. Ad i). Let $\boldsymbol{C}(t) = \Phi_t(\boldsymbol{C})$ for some $\boldsymbol{C} \in \mathrm{SL}^+_{\mathrm{sym}}$. The differential equation (A.1) entails that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{C}(t) - \boldsymbol{C}_0|^2 = \mathrm{tr}\,\left(\dot{\boldsymbol{C}}(\boldsymbol{C} - \boldsymbol{C}_0)\right) \\
&= -\mathrm{tr}\,\left(\boldsymbol{C}(\boldsymbol{C} - \boldsymbol{C}_0)\right) + \frac{3}{|\boldsymbol{C}^{-1}|^2}\left(3 - \mathrm{tr}\,(\boldsymbol{C}^{-1}\boldsymbol{C}_0)\right)
\end{aligned}$$

By the invariance of the trace by cyclic permutation of the factors we have that

$$\mathrm{tr}\,(\boldsymbol{C}^{-1}\boldsymbol{C}_0) = \mathrm{tr}\,(\boldsymbol{C}^{-1/2}\boldsymbol{C}_0\boldsymbol{C}^{-1/2}) \geq 3$$

since $\boldsymbol{C}^{-1/2}\boldsymbol{C}_0\boldsymbol{C}^{-1/2} \in \mathrm{SL}^+_{\mathrm{sym}}$. Moreover,

$$|\boldsymbol{C}| \geq |\boldsymbol{C}_0| \Rightarrow |\boldsymbol{C}|^2 = \mathrm{tr}\,(\boldsymbol{C}^2) \geq \mathrm{tr}\,(\boldsymbol{C}\boldsymbol{C}_0),$$

which proves the statement.

Ad ii). Let $\boldsymbol{C}_i(t) = \Phi_t(\boldsymbol{C}_i)$ for $\boldsymbol{C}_1, \boldsymbol{C}_2 \in \mathrm{SL}^+_{\mathrm{sym}}$. By using again the differential equation we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{C}_1(t) - \boldsymbol{C}_2(t)|^2 = \mathrm{tr}\,\left[(\dot{\boldsymbol{C}}_1 - \dot{\boldsymbol{C}}_2)(\boldsymbol{C}_1 - \boldsymbol{C}_2)\right] = \\
&= -\mathrm{tr}\,\left[\left(\boldsymbol{C}_1 - \frac{3}{|\boldsymbol{C}_1^{-1}|^2}\boldsymbol{C}_1^{-1} - \boldsymbol{C}_2 + \frac{3}{|\boldsymbol{C}_2^{-1}|^2}\boldsymbol{C}_2^{-1}\right)(\boldsymbol{C}_1 - \boldsymbol{C}_2)\right] = \\
&= -|\boldsymbol{C}_1 - \boldsymbol{C}_2|^2 + 3\left(\frac{3 - \mathrm{tr}\,(\boldsymbol{C}_1^{-1}\boldsymbol{C}_2)}{|\boldsymbol{C}_1^{-1}|^2} + \frac{3 - \mathrm{tr}\,(\boldsymbol{C}_2^{-1}\boldsymbol{C}_1)}{|\boldsymbol{C}_2^{-1}|^2}\right) \leq 0
\end{aligned}$$

proving the assertion. □

For any initial datum $\boldsymbol{C} \in \mathrm{SL}^+_{\mathrm{sym}}$, by (A.2) there exists a minimum time $t_0 \geq 0$ for which $|\Phi_{t_0}(\boldsymbol{C})| = \kappa > \sqrt{3}$, namely

$$t_0(\boldsymbol{C}) = \min\{t \geq 0 \mid \Phi_{t_0}(\boldsymbol{C}) \in K\} < \infty.$$

We define the map $\boldsymbol{\Pi} : \mathrm{SL}^+_{\mathrm{sym}} \to K$ as follows

$$\boldsymbol{\Pi}(\boldsymbol{C}) := \Phi_{t_0(\boldsymbol{C})}(\boldsymbol{C}).$$

Of course, $\boldsymbol{\Pi}|_K = \mathrm{id}$, so that the first condition in (4.25) is satisfied.

Let now $\boldsymbol{C}_1, \boldsymbol{C}_2 \in \mathrm{SL}^+_{\mathrm{sym}}$ be given and assume with no loss of generality that $t_1 := t_0(\boldsymbol{C}_1) \leq t_0(\boldsymbol{C}_2) =: t_1 + \delta$. We can write

$$\begin{aligned}
|\boldsymbol{\Pi}(\boldsymbol{C}_1) - \boldsymbol{\Pi}(\boldsymbol{C}_2)| &= |\Phi_{t_1}(\boldsymbol{C}_1) - \Phi_{t_2}(\boldsymbol{C}_2)| = |\Phi_{t_1}(\boldsymbol{C}_1) - \Phi_\delta(\Phi_{t_1}(\boldsymbol{C}_2))| \\
&\overset{(A.3)}{\leq} |\Phi_{t_1}(\boldsymbol{C}_1) - \Phi_{t_1}(\boldsymbol{C}_2)| \overset{(A.4)}{\leq} |\boldsymbol{C}_1 - \boldsymbol{C}_2|.
\end{aligned}$$

The map $\boldsymbol{\Pi}$ is hence contractive in $\mathrm{SL}^+_{\mathrm{sym}}$, which is the second condition in (4.25).

Appendix B. Lower semincontinuity tool

For the sake of completeness, we report here the lower-semicontinuity tool which has been repeatedly used above. The lemma is in the spirit of [3, Thm. 1] and [31]. A proof can be found in [62, Thm 4.3, Cor. 4.4], in [42, Lemma 3.1] in one dimension, and in [35] in case of local uniform convergence.

Lemma B.1 (Lower-semicontinuity tool)**.** *Let* f_0, $f_\varepsilon : \mathbb{R}^n \to [0,\infty]$ *be lower semicontinuous,*

$$f_0(v_0) \leq \inf \left\{ \liminf_{\varepsilon\to 0} f_\varepsilon(v_\varepsilon) \mid v_\varepsilon \to v_0 \right\} \quad \forall v_0 \in \mathbb{R}^n$$

and $w_\varepsilon \rightharpoonup w_0$ *in* $L^1(\Omega;\mathbb{R}^n)$*. By denoting by* ζ *the Young measure generated by* w_ε *we have that*

$$\int_\Omega \left(\int_{\mathbb{R}^n} f_0(w) \mathrm{d}\zeta_x(w) \right) \mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_\Omega f_\varepsilon(w_\varepsilon)\, \mathrm{d}x.$$

In particular, if f_0 *is convex we have*

$$\int_\Omega f_0(w_0)\, \mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_\Omega f_\varepsilon(w_\varepsilon)\, \mathrm{d}x.$$

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