# EXPLICIT LOG FANO STRUCTURES ON BLOW-UPS OF PROJECTIVE SPACES 

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#### Abstract

In this paper we determine which blow-ups $X$ of $\mathbb{P}^{n}$ at general points are log Fano, that is, when there exists an effective $\mathbb{Q}$-divisor $\Delta$ such that $-\left(K_{X}+\Delta\right)$ is ample and the pair $(X, \Delta)$ is klt. For these blow-ups, we produce explicit boundary divisors $\Delta$ making $X \log$ Fano.


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## 1. Introduction

In this paper we investigate special properties of blow-ups of complex projective spaces at general points. These varieties appear frequently in algebraic geometry, for example in moduli problem. In general, the more points are blown-up, the more complicated the resulting variety is. For positive integers $n$ and $k$, denote by $X_{k}^{n}$ a blow-up of $\mathbb{P}^{n}$ at $k$ points in general position. The 2-dimensional case has been understood classically. For $k \leq 8$, $S=X_{k}^{2}$ is a del Pezzo surface (i.e. $-K_{S}$ is ample), and its geometry can be completely described in terms of some finite data. For $k \geq 9$, the situation changes drastically. The anti-canonical class of $S$ is no longer big, and $S$ contains infinitely many ( -1 )-curves.

For $n \geq 3, X_{1}^{n}$ is a Fano manifold (i.e., $-K_{X_{1}^{n}}$ is ample), but as soon as $k \geq 2, X_{k}^{n}$ is no longer Fano. However, for small values of $k$, the blow-up $X_{k}^{n}$ behaves like a Fano manifold. A more appropriate notion here is that of a log Fano variety.

Definition 1.1. Let $X$ be a normal projective $\mathbb{Q}$-factorial variety. We say that $X$ is $\log$ Fano if there exists an effective $\mathbb{Q}$-divisor $\Delta$ such that $-\left(K_{X}+\Delta\right)$ is ample, and the pair ( $X, \Delta$ ) is klt. (See for instance [Ko, Definition 3.5] for the notion of klt singularities).

[^0]Log Fano varieties play an important role in the classification of algebraic varieties. It was proved in [BCHM, Corollary 1.3.2] that they are special instances of Mori dream spaces. We refer to Section 2 and references therein for the definition and special properties of Mori dream spaces. Here we just vaguely remark that a Mori dream space $X$ behaves very well with respect to the minimal model program. Moreover, the birational geometry of $X$ can be encoded in some finite data, namely its cone of effective divisors $\operatorname{Eff}(X)$ together with a chamber decomposition on it.

In this paper we address the following problems:
For which values of $n$ and $k$ is $X_{k}^{n}$ a log Fano variety?
In those cases, can we find an explicit $\mathbb{Q}$-divisor $\Delta$ making $X_{k}^{n} \log$ Fano?
In the second question, the expected properties of $\Delta$ depend on the context. In some cases, one would like $\Delta$ to be irreducible. On the other hand, when $X_{k}^{n}$ appears as a compactification of some moduli space, it is often desirable that $\Delta$ is supported on the boundary divisor.
Example 1.2. Let us consider the case $n=3$. By Proposition 1.4 below, if $k \leq 8$, then the Mori cone of $X=X_{k}^{3}$ is generated by classes of lines $R_{i}$ in the exceptional divisors, $1 \leq i \leq k$, and strict transforms $L_{i, j}$ of the lines through two of the blown-up points. Using this result, one checks easily that $-K_{X}$ is nef for $k \leq 8$. Moreover, by computing the top intersection $\left(-K_{X}\right)^{3}$, one concludes that $-K_{X}$ is big if and only if $k \leq 7$. Projective manifolds with nef and big anti-canonical class are called weak Fano. The fact that $X_{k}^{3}$ is weak Fano if and only if $k \leq 7$ has been proven, with slightly different techniques, in BL, Proposition 2.9]. By Lemma [2.5, weak Fano manifolds are log Fano. On the other hand, $\log$ Fano varieties have big anti-canonical class. So we conclude that $X$ is $\log$ Fano if and only if $k \leq 7$.

When $k \leq 4, X_{k}^{3}$ is a toric variety and one can take $\Delta$ to be a suitable combination of toric invariant divisors. Alternatively, we may choose $\Delta=\epsilon D$ to be irreducible. We describe such irreducible $\Delta$ when $k=4$. We may assume that the blown-up points are the fundamental points of $\mathbb{P}^{3}$. Let $D \subset X_{4}^{3}$ be the strict transform of the Cayley nodal cubic surface

$$
S=\left\{x_{0} x_{1} x_{2}+x_{0} x_{1} x_{3}+x_{0} x_{2} x_{3}+x_{1} x_{2} x_{3}=0\right\} \subset \mathbb{P}^{3} .
$$

The surface $S$ has ordinary double points at the fundamental points of $\mathbb{P}^{3}$, and is smooth elsewhere. Thus $D$ is smooth and $D \sim 3 H-2\left(E_{1}+\ldots+E_{4}\right)$. One computes that

$$
\left(-K_{X_{4}^{3}}-\epsilon D\right) \cdot R_{i}=2-2 \epsilon \text { and }\left(-K_{X_{4}^{3}}-\epsilon D\right) \cdot L_{i, j}=4 \epsilon .
$$

Thus $(X, \epsilon D)$ is klt and $-\left(K_{X_{4}^{3}}+\epsilon D\right)$ is ample for any $0<\epsilon<1$.
When $k=5$, let $H_{i, j, k} \subset \mathbb{P}^{3}$ be the plane spanned by three of the blown-up points $p_{i}, p_{j}$ and $p_{k}$, and take $D$ to be the strict transform of $\sum_{i, j, k} H_{i, j, k}$. Then $D \sim 10 H-6\left(E_{1}+\ldots+\right.$ $E_{5}$ ), and one computes

$$
-\left(K_{X_{5}^{3}}-\epsilon D\right) \cdot R_{i}=2-6 \epsilon \text { and }-\left(K_{X_{5}^{3}}-\epsilon D\right) \cdot L_{i, j}=2 \epsilon .
$$

So $-\left(K_{X_{5}^{3}}+\epsilon D\right)$ is ample for any $0<\epsilon<\frac{1}{3}$. Furthermore, we can take $\epsilon>0$ sufficiently small so that the pair $\left(X_{5}^{3}, \epsilon D\right)$ is klt.

When $k=6$, let $Q_{i} \subset \mathbb{P}^{3}$ be the unique irreducible quadric cone through the 6 blown-up points, with vertex at one of the points $p_{i}$. Let $D$ be the strict transform of $Q_{1}+Q_{2}+Q_{3}+$
$H_{4,5,6}$ in $X_{6}^{3}$. (As before, $H_{4,5,6} \subset \mathbb{P}^{3}$ denotes the plane spanned by $p_{4}, p_{5}$ and $p_{6}$.) Then $D \sim 7 H-4\left(E_{1}+\ldots+E_{6}\right)$, and one computes

$$
-\left(K_{X_{6}^{3}}+\epsilon D\right) \cdot R_{i}=2-4 \epsilon \text { and }-\left(K_{X_{6}^{3}}+\epsilon D\right) \cdot R_{i}=\epsilon .
$$

So $-\left(K_{X_{6}^{3}}+\epsilon D\right)$ is ample for any $0<\epsilon<\frac{1}{2}$. Furthermore we can take $\epsilon>0$ sufficiently small so that the pair $\left(X_{6}^{3}, \epsilon D\right)$ is klt.

Suppose now that $k=7$. For each triple $1 \leq i, j, k \leq 7$, consider the linear system of cubics defining the standard Cremona transformation of $\mathbb{P}^{3}$ centered at the four points $p_{h}$, $h \neq i, j, k$. There exists an unique irreducible cubic surface $S_{i, j, k} \subset \mathbb{P}^{3}$ in this linear system passing through $p_{i}, p_{j}$ and $p_{k}$. It is smooth at $p_{i}, p_{j}$ and $p_{k}$, and has a double point at $p_{h}$ for $h \neq i, j, k$. Its strict transform in $X_{7}^{3}$ is a rigid surface. Let $D$ be the strict transform of $\sum_{i, j, k} S_{i, j, k}$ in $X_{7}^{3}$. Then $D \sim 105 H-55\left(E_{1}+\ldots+E_{7}\right)$, and one computes

$$
-\left(K_{X_{7}^{3}}+\epsilon D\right) \cdot R_{i}=2-55 \epsilon \text { and }\left(K_{X_{7}^{3}}+\epsilon D\right) \cdot L_{i, j}=5 \epsilon
$$

So $-\left(K_{X_{5}^{3}}+\epsilon D\right)$ is ample for any $0<\epsilon<\frac{2}{55}$. Furthermore we can take $\epsilon>0$ sufficiently small so that the pair $\left(X_{7}^{3}, \epsilon D\right)$ is klt.

For $n \geq 4$, we can approach the first question using Mori dream spaces. The main results of Muk01 and CT06] put together show that $X_{k}^{n}$ is a Mori dream space if and only if one of the following holds:
$-n=4$ and $k \leq 8$.

- $n>4$ and $k \leq n+3$.

Using Mukai's description of the geometry of the boundary cases $X_{8}^{4}$ and $X_{n+3}^{n}$, see Section [3.2, we answer the first question.

Theorem 1.3. Let $X_{k}^{n}$ be a blow-up of $\mathbb{P}^{n}$ at $k$ points in general position, with $n \geq 2$ and $k \geq 0$. Then $X_{k}^{n}$ is log Fano if and only if one of the following holds:

- $n=2$ and $k \leq 8$,
- $n=3$ and $k \leq 7$,
- $n=4$ and $k \leq 8$,
- $n>4$ and $k \leq n+3$.

The proof of Theorem 1.3, which may already have been known to experts, does not give any hint on which $\mathbb{Q}$-divisor $\Delta$ makes $X_{k}^{n} \log$ Fano. So we proceed to find such explicit $\mathbb{Q}$-divisor $\Delta$. The first step is to determine the Mori cone of $X_{k}^{n}$.
Proposition 1.4. Let $X_{k}^{n}$ be the blow-up of $\mathbb{P}^{n}$ at points in general position $p_{1}, \ldots, p_{k}$, $n \geq 2$. Denote by $R_{i}$ the class of a line in the exceptional divisors over $p_{i}$, and by $L_{i, j}$ the class of the strict transforms of the line through two distinct points $p_{i}$ and $p_{j}$. Suppose that either of the following holds:

- $k \leq 2 n$.
- $n=3$ and $k \leq 8$.

Then the Mori cone $\overline{\mathrm{NE}}\left(X_{k}^{n}\right)$ is generated by the $R_{i}$ 's and $L_{i, j}$ 's.
Using Proposition 1.4, it is not hard to find a $\mathbb{Q}$-divisor $\Delta$ such that $-\left(K_{X_{k}^{n}}+\Delta\right)$ is ample. We often choose $\Delta$ as linear combinations of extremal divisors in $X_{k}^{n}$. The hard part is to show that for such divisors $\left(X_{k}^{n}, \Delta\right)$ is klt. We do so by providing explicit log resolutions for
these pairs, and computing discrepancies. Explicit $\mathbb{Q}$-divisors $\Delta$ making $X_{k}^{n} \log$ Fano are given in Theorems 4.3, 4.5, 5.7 and 5.9. In particular, they provide a new proof that these varieties are Mori dream spaces.

Some blow-ups of projective spaces at points (and, more generally, linear spaces) appear as moduli spaces $\bar{M}_{g, A[n]}$ of weighted pointed stable curves. These spaces were introduced and investigated by Hassett in [Ha. In [Ha, Problem 7.1], Hassett asks whether there is an effective $\mathbb{Q}$-divisor $\Delta$ on $\bar{M}_{g, A[n]}$, supported on the boundary, such that $\left(\bar{M}_{g, A[n]}, \Delta\right)$ is $\log$ canonical, and $K_{\bar{M}_{g, A[n]}}+\Delta$ is ample. We end the paper by addressing this question.

The paper is organized as follows. In Section 2, we recall the definition and some special properties of Mori dream spaces. In Section 3] we review the description from [CT06] of the cone of effective divisors of $X_{k}^{n}$, and make explicit the description of its Mori chamber decomposition proposed in Muk05. We end this section by proving Theorem 1.3. In Section 4, we exhibit an integral divisor $D \subset X_{k}^{n}$ and rational number $\epsilon>0$ such that $\Delta=\epsilon D$ makes $X_{k}^{n} \log$ Fano for $k \leq n+2$. This task is relatively easy, and serves as warm up for the next case $n=k+3$, treated in Section 5. For $X_{n+3}^{n}$, we construct $D$ from joins of suitable linear spaces and higher secant varieties of the unique rational normal curve through the blown-up points. In order to construct an explicit log resolution for the resulting pair $\left(X_{n+3}^{n}, \Delta\right)$, we need a good understanding of the intersections of such joins. Subsection 5.1 is devoted to this. The description of $D$ is given separately when $n$ is odd (Subsection 5.2) and even (Subsection 5.3). Finally, in Section 6, we address a question of Hassett about some moduli spaces $\bar{M}_{g, A[n]}$ of weighted pointed stable curves.

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## 2. Mori Dream Spaces

Let $X$ be a normal projective variety. We denote by $N^{1}(X)$ the real vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. The nef cone of $X$ is the (closed) convex cone $\operatorname{Nef}(X) \subset N^{1}(X)$ generated by classes of nef divisors. The movable cone of $X$ is the convex cone $\operatorname{Mov}(X) \subset N^{1}(X)$ generated by classes of movable divisors. These are Cartier divisors whose stable base locus has codimension at least two in $X$. The effective cone of $X$ is the convex cone $\operatorname{Eff}(X) \subset N^{1}(X)$ generated by classes of effective divisors. We have inclusions:

$$
\operatorname{Nef}(X) \subset \overline{\operatorname{Mov}(X)} \subset \overline{\operatorname{Eff}(X)}
$$

We say that a birational map $f: X \rightarrow X^{\prime}$ into a normal projective variety $X^{\prime}$ is a birational contraction if its inverse does not contract any divisor. We say that it is a small $\mathbb{Q}$-factorial modification if $X^{\prime}$ is $\mathbb{Q}$-factorial and $f$ is an isomorphism in codimension one. If $f: X \rightarrow X^{\prime}$ is a small $\mathbb{Q}$-factorial modification, then the natural pullback map $f^{*}$ : $N^{1}\left(X^{\prime}\right) \rightarrow N^{1}(X)$ sends $\operatorname{Mov}\left(X^{\prime}\right)$ and $\operatorname{Eff}\left(X^{\prime}\right)$ isomorphically onto $\operatorname{Mov}(X)$ and $\operatorname{Eff}(X)$, respectively. In particular, we have $f^{*}\left(\operatorname{Nef}\left(X^{\prime}\right)\right) \subset \overline{\operatorname{Mov}(X)}$.

Definition 2.1. A normal projective $\mathbb{Q}$-factorial variety $X$ is called a Mori dream space if the following conditions hold:
$-\operatorname{Pic}(X)$ is finitely generated,

- $\operatorname{Nef}(X)$ is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_{i}: X \rightarrow X_{i}$, such that each $X_{i}$ satisfies the second condition above, and

$$
\operatorname{Mov}(X)=\bigcup_{i} f_{i}^{*}\left(\operatorname{Nef}\left(X_{i}\right)\right)
$$

The collection of all faces of all cones $f_{i}^{*}\left(\operatorname{Nef}\left(X_{i}\right)\right)$ 's above forms a fan supported on $\operatorname{Mov}(X)$. If two maximal cones of this fan, say $f_{i}^{*}\left(\operatorname{Nef}\left(X_{i}\right)\right)$ and $f_{j}^{*}\left(\operatorname{Nef}\left(X_{j}\right)\right)$, meet along a facet, then there exists a commutative diagram:

where $Y$ is a normal projective variety, and $h_{i}$ and $h_{j}$ are small birational morphisms. The fan structure on $\operatorname{Mov}(X)$ can be extended to a fan supported on $\operatorname{Eff}(X)$ as follows.

Definition 2.2. Let $X$ be a Mori dream space. We describe a fan structure on the effective cone $\operatorname{Eff}(X)$, called the Mori chamber decomposition. We refer to HK00, Proposition $1.11(2)$ ] and [Oka11, Section 2.2] for details. There are finitely many birational contractions from $X$ to Mori dream spaces, denoted by $g_{i}: X \rightarrow Y_{i}$. The set $\operatorname{Exc}\left(g_{i}\right)$ of exceptional prime divisors of $g_{i}$ has cardinality $\rho\left(X / Y_{i}\right)=\rho(X)-\rho\left(Y_{i}\right)$. The maximal cones $\mathcal{C}_{i}$ of the Mori chamber decomposition of $\operatorname{Eff}(X)$ are of the form:

$$
\mathcal{C}_{i}=\operatorname{Cone}\left(g_{i}^{*}\left(\operatorname{Nef}\left(Y_{i}\right)\right), \operatorname{Exc}\left(g_{i}\right)\right)
$$

We call $\mathcal{C}_{i}$ or its interior $\mathcal{C}_{i}^{\circ}$ a maximal chamber of $\operatorname{Eff}(X)$.
By BCHM, Corollary 1.3.2], a log Fano variety is a Mori dream space. The converse does not hold in general, and there are several criteria for a Mori dream space to be log Fano [GOST]. We will use the following.

Proposition 2.3. Let $X$ be a log Fano variety. Then any small $\mathbb{Q}$-factorial modification of $X$ is also log Fano.

Proposition 2.3 follows from the properties of Mori dream spaces and Lemma 2.4 below. In what follows, a normal projective variety $X$ is said to be of Fano type if there exists an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $-\left(K_{X}+D\right)$ is $\mathbb{Q}$-Cartier and ample, and the pair $(X, D)$ is klt. This is weaker than our current notion of $\log$ Fano because it does not require that $X$ be $\mathbb{Q}$-factorial.

Lemma 2.4 ([GOST, Lemma 3.1]). Let $h: X \rightarrow Y$ be a small birational morphism between normal projective varieties. Then $X$ is of Fano type if and only if so is $Y$.

Lemma 2.5. Let $X$ be a normal $\mathbb{Q}$-factorial projective variety with at worst klt singularities. Suppose that $-K_{X}$ is nef and big. Then $X$ is log Fano.

Proof. Since $-K_{X}$ is big, by [La, Corollary 2.2.6], there exist an ample divisor $A$, an effective divisor $D$, and a positive integer $m$ such that $-m K_{X} \equiv A+D$. For $h>m$, we can write
$-h K_{X} \equiv-(h-m) K_{X}+A+D$. The divisor $D^{\prime}=-(h-m) K_{X}+A$ is a sum of a nef and an ample divisor, and so it is ample. Setting $\epsilon=\frac{1}{h}$, we get that $-\left(K_{X}+\epsilon D\right) \equiv \epsilon D^{\prime}$ is ample. Since $X$ has at worst klt singularities, by taking $h$ large enough, we get that the pair $(X, \epsilon D)$ is klt.

Remark 2.6. In the next section, we will use Proposition 2.3 and Lemma 2.5 to prove that certain blow-ups $X$ of $\mathbb{P}^{n}$ at general points are Mori dream spaces. To do so, we will use the fact that $X$ admits a small $\mathbb{Q}$-factorial modification $X^{\prime}$ which is smooth and has $-K_{X^{\prime}}$ nef and big. Notice that smoothness of $X^{\prime}$ is essential. In fact, there are examples of Mori dream spaces $X$ which are not log Fano, but admit a (very singular) small $\mathbb{Q}$-factorial modification $X^{\prime}$ with $-K_{X^{\prime}}$ ample, see for instance [CG, Example 5.1].

## 3. Cones of curves and divisors on blow-ups of $\mathbb{P}^{n}$ at general points

Let $p_{1}, \ldots, p_{k} \in \mathbb{P}^{n}$ be general points, and let $X_{k}^{n}$ be the blow-up of $\mathbb{P}^{n}$ at $p_{1}, \ldots, p_{k}$. In this section we describe several cones of curves and divisors on $X_{k}^{n}$.

Notation 3.1. We denote by $H \in N^{1}\left(X_{k}^{n}\right)$ the class of the pullback of the hyperplane section of $\mathbb{P}^{n}$. By abuse of notation, we denote by $E_{i}$ both the exceptional divisor over $p_{i}$ and its class in $N^{1}\left(X_{k}^{n}\right)$. Then $\left\{H, E_{1}, \ldots, E_{k}\right\}$ is a basis of $N^{1}\left(X_{k}^{n}\right)$, and we have

$$
-K_{X_{k}^{n}}=(n+1) H-(n-1) E_{1}-\ldots-(n-1) E_{k} .
$$

We denote by $L \in N_{1}\left(X_{k}^{n}\right)$ the class of the strict transform of a general line on $\mathbb{P}^{n}$. For each $i \in\{1, \ldots, k\}$, we denote by $R_{i} \in N_{1}\left(X_{k}^{n}\right)$ the class of a line on $E_{i} \cong \mathbb{P}^{n-1}$, and by $L_{i} \in N_{1}\left(X_{k}^{n}\right)$ the class of the strict transform of a general line on $\mathbb{P}^{n}$ passing through $p_{i}$. For $i \neq j$, we denote by $L_{i, j}$ the class of the strict transform of the line on $\mathbb{P}^{n}$ joining $p_{i}$ and $p_{j}$. Then $\left\{L, R_{1}, \ldots, R_{k}\right\}$ is a basis of $N_{1}\left(X_{k}^{n}\right)$, and we have

$$
\begin{equation*}
L \equiv L_{i, j}+R_{i}+R_{j} \text { and } L_{i} \equiv L-R_{i} \equiv L_{i, j}+R_{j} . \tag{3.1}
\end{equation*}
$$

3.1. The Mori cone of $X_{k}^{n}$. In this section we prove Proposition 1.4,

Lemma 3.2. Let $p_{1}, \ldots, p_{8} \in \mathbb{P}^{3}$ be general points, and $C \subset \mathbb{P}^{3}$ an irreducible curve of degree $d$ having multiplicity $m_{i}=\operatorname{mult}_{p_{i}}(C)$ at $p_{i}, 1 \leq i \leq 8$. Then $m_{1}+\ldots+m_{8} \leq 2 d$.

Proof. If $C$ is degenerate, then $m_{i} \neq 0$ for at most three points $p_{i}$, and the conclusion follows easily from Bézout's Theorem. So from now on we assume that $C$ is non degenerate. Let $\Lambda$ be the pencil of irreducible quadric surfaces passing through $p_{1}, \ldots, p_{8}$. Suppose that $m_{1}+\ldots+m_{8}>2 d$. It follows from Bézout's Theorem that $C$ is contained in every member of $\Lambda$. In particular, $C$ is a non degenerate irreducible curve contained in the intersection of two irreducible quadric surfaces. So $d \in\{3,4\}$. Suppose that $d=3$. Then $C$ must be a twisted cubic through at most 6 of the $p_{i}$ 's, and thus $m_{1}+\ldots+m_{8} \leq 6=2 d$, contradicting our assumptions. We conclude that $d=4, m_{i} \geq 1$ for every $i$, and $m_{j} \geq 2$ for some $j$. If follows from Bézout's Theorem that $m_{j}=2$, and $m_{i}=1$ for $i \neq j$. Consider the projection from $p_{j}$

$$
\pi_{p_{j}}: C \rightarrow \mathbb{P}^{2} .
$$

The image $\overline{\pi_{p_{1}}(C)}$ is a conic though the seven general points $\pi_{p_{j}}\left(p_{i}\right), i \neq j$, which is impossible. This shows that $m_{1}+\ldots+m_{8} \leq 2 d$.

Proof of Proposition 1.4. Let $X_{k}^{n}$ be the blow-up of $\mathbb{P}^{n}, n \geq 2$, at points in general position $p_{1}, \ldots, p_{k}$. We follow Notation 3.1, Let $\widetilde{C} \subset X_{k}^{n}$ be an irreducible curve not contained in any exceptional divisor $E_{i}$, and denote by $C$ the image of $\widetilde{C}$ in $\mathbb{P}^{n}$. It is an irreducible curve of degree $d>0$ and multiplicity $m_{i}=\operatorname{mult}_{p_{i}} C \geq 0$ at $p_{i}, \widetilde{C}$ is the strict transform of $C$, and

$$
\begin{equation*}
\widetilde{C} \equiv d L-m_{1} R_{1}-\ldots-m_{k} R_{k} \tag{3.2}
\end{equation*}
$$

We must show that the class of $\widetilde{C}$ in $N_{1}\left(X_{k}^{n}\right)$ lies in the cone generated by the $R_{i}$ 's and $L_{i, j}$ 's. We may assume that $m_{1} \leq m_{2} \leq \cdots \leq m_{k}$.

Suppose first that $k \leq 2 n$.
First let us assume that $k$ is even. We write

$$
\begin{align*}
\widetilde{C} \equiv & d L-m_{1}\left(R_{1}+R_{2}\right)-\left(m_{2}-m_{1}\right) R_{2}-m_{3}\left(R_{3}+R_{4}\right)-\left(m_{4}-m_{3}\right) R_{4}-  \tag{3.3}\\
& \ldots-m_{k-1}\left(R_{k-1}+R_{k}\right)-\left(m_{k}-m_{k-1}\right) R_{k} .
\end{align*}
$$

Note that $m_{1}+\left(m_{2}-m_{1}\right)+m_{3}+\left(m_{4}-m_{3}\right)+\ldots+m_{k-1}+\left(m_{k}-m_{k-1}\right)=m_{2}+m_{4}+\ldots+m_{k}$. We claim that $m_{2}+m_{4}+\ldots+m_{k} \leq d$. Indeed, since $k \leq 2 n$, the set $\left\{p_{2}, p_{4}, \ldots, p_{k}\right\}$ has cardinality at most $n$. Consider the linear space $P=\left\langle p_{2}, p_{4}, \ldots, p_{k}\right\rangle \varsubsetneqq \mathbb{P}^{n}$. If $\left.m_{2}+m_{4}+\ldots+m_{k}\right\rangle d$, then $C \subset P$ by Bézout's Theorem. Since the $p_{i}$ 's are general, $p_{1}, p_{3}, \ldots, p_{k-1} \notin P$, and so $m_{1}=m_{3}=\ldots=m_{k-1}=0$. But this implies that $m_{i}=0$ for $i \leq k-1$ and $m_{k}>d$, which is impossible. This proves the claim. So we can rewrite (3.3) as

$$
\begin{aligned}
\widetilde{C} \equiv & m_{1} L_{1,2}+\left(m_{2}-m_{1}\right) L_{2}+m_{3} L_{3,4}+\left(m_{4}-m_{3}\right) L_{4}- \\
& \ldots+m_{k-1} L_{k-1, k}+\left(m_{k}-m_{k-1}\right) L_{k}+\left(d-m_{2}-m_{4}-\ldots-m_{k}\right) L
\end{aligned}
$$

It follows from (3.1) that the class of $\widetilde{C}$ in $N_{1}\left(X_{k}^{n}\right)$ lies in the cone generated by the $R_{i}$ 's and $L_{i, j}$ 's.

Now suppose that $k$ is odd, and write

$$
\begin{align*}
\widetilde{C} \equiv & d L-m_{1}\left(R_{1}+R_{2}\right)-\left(m_{2}-m_{1}\right) R_{2}-m_{3}\left(R_{3}+R_{4}\right)-\left(m_{4}-m_{3}\right) R_{4}-  \tag{3.4}\\
& \ldots-m_{k-2}\left(R_{k-2}+R_{k-1}\right)-\left(m_{k-1}-m_{k-2}\right) R_{k-1}-m_{k} R_{k} .
\end{align*}
$$

In this case $m_{1}+\left(m_{2}-m_{1}\right)+m_{3}+\left(m_{4}-m_{3}\right)+\ldots+m_{k-2}+\left(m_{k-1}-m_{k-2}\right)+m_{k}=$ $m_{2}+m_{4}+\ldots+m_{k-1}+m_{k}$. Like in the even case, one shows that $m_{2}+m_{4}+\ldots+m_{k-1}+m_{k} \leq d$ and rewrite (3.4) as an effective linear combination of the $R_{i}$ 's and $L_{i, j}$ 's.

From now on we suppose that $n=3$ and $k \leq 8$. Then $m_{i} \leq d$ and $m_{1}+\ldots+m_{k} \leq 2 d$ by Lemma 3.2. If $m_{k-1}=0$, then $\widetilde{C} \equiv m_{k} L_{k}+\left(d-m_{k}\right) L$. It follows from (3.1) that the class of $\widetilde{C}$ in $N_{1}\left(X_{k}^{n}\right)$ lies in the cone generated by the $R_{i}$ 's and $L_{i, j}$ 's. If $m_{k-1} \neq 0$, then rewrite (3.2) as

$$
\widetilde{C} \equiv\left(L_{k-1, k}\right)+d^{\prime} L-m_{1}^{\prime} R_{1}-\ldots-m_{k}^{\prime} R_{k},
$$

where $d^{\prime}=d-1, m_{i}^{\prime}=m_{i}$ for $i \leq k-2$, and $m_{i}^{\prime}=m_{i}-1$ for $i=k-1$ or $k$. Note that $m_{i}^{\prime} \leq d^{\prime}$. This is clear for $i=k-1$ or $k$. For $i \leq k-2$ it follows from the assumptions that $m_{1} \leq m_{2} \leq \cdots \leq m_{k} \leq d$ and $m_{1}+\ldots+m_{k} \leq 2 d$. We also have $m_{1}^{\prime}+\ldots+m_{k}^{\prime} \leq 2 d^{\prime}$. So we can repeat the process and conclude by induction that the class of $\widetilde{C}$ in $N_{1}\left(X_{k}^{n}\right)$ lies in the cone generated by the $R_{i}$ 's and $L_{i, j}$ 's.
3.2. The effective cone of $X_{n+3}^{n}$. In this section we describe the effective cone of the blowup of $\mathbb{P}^{n}$ at $n+3$ points in general position, as well as its Mori chamber decomposition. The main references are CT06, Muk05 and Bau91. See also BDP15 for a recent new proof.
3.3 (The effective cone of the blow-up of $\mathbb{P}^{n}$ at $n+3$ points). Let $X=X_{n+3}^{n}$ be the blow-up of $\mathbb{P}^{n}$ at $n+3$ points $p_{i}$ in general position. We follow Notation 3.1] By [CT06, Theorems 1.3], $X$ is a Mori dream space. Next we describe the 1 -dimensional faces of $\operatorname{Eff}(X)$ (CT06, Theorem 1.2]). For each subset $I \subset\{1, \cdots, n+3\}$ whose complement has odd cardinality $\left|I^{c}\right|=2 k+1$, consider the divisor class

$$
E_{I}:=k H-k \sum_{i \in I} E_{i}-(k-1) \sum_{i \in I^{c}} E_{i} .
$$

There is a unique divisor in the linear system $\left|E_{I}\right|$, which we also denote by $E_{I}$. When $k=0$ we have $E_{\{i\}^{c}}=E_{i}$ When $k \geq 1, E_{I}$ can be described as follows. Let $\pi_{I}: \mathbb{P}^{n} \rightarrow \mathbb{P}^{2 k-2}$ be the projection from the linear space $\left\langle p_{i}\right\rangle_{i \in I}$. Let $C_{I} \subset \mathbb{P}^{2 k-2}$ be the image of the unique rational normal curve through all the $p_{i}^{\prime}$ s. The divisor $E_{I}$ is the cone with vertex $\left\langle p_{i}\right\rangle_{i \in I}$ over $\operatorname{Sec}_{k-1} C_{I}$. Each $E_{I}$ generates a 1-dimensional face of $\operatorname{Eff}(X)$, and all 1-dimensional faces are of this form.

Let $X$ be the blow-up of $\mathbb{P}^{n}$ at $n+3$ points in general position, and follow the notation of Paragraph 3.3 above. In order to describe the Mori chamber decomposition of Eff $(X)$, we make explicit the map to the weight space proposed by Mukai in Muk05. Write ( $y, x_{1}, \ldots, x_{n+3}$ ) for coordinates in $\mathbb{R}^{n+4}$, and $\left(\alpha_{1}, \ldots, \alpha_{n+3}\right)$ for coordinates in $\mathbb{R}^{n+3}$. We identify $\mathbb{R}^{n+4}$ with $N^{1}(X)$ by associating to a point $\bar{x}=\left(y, x_{1}, \ldots, x_{n+3}\right) \in \mathbb{R}^{n+4}$ the divisor class of $D_{\bar{x}}=y H+\sum x_{i} E_{i}$. Note that all the $E_{I}$ 's defined in Paragraph 3.3 lie in the hyperplane

$$
(n+1) y+\sum x_{i}=1
$$

Consider the projection from the origin

$$
\begin{gather*}
\varphi=\left(\varphi_{1}, \cdots, \varphi_{n+3}\right): \operatorname{Eff}(X) \rightarrow \mathbb{R}^{n+3}, \\
\varphi_{i}=\frac{y+x_{i}}{(n+1) y+\sum x_{i}} . \tag{3.5}
\end{gather*}
$$

We shall describe the image of $\operatorname{Eff}(X)$ under $\varphi$, along with the decomposition induced by the Mori chamber decomposition of $\operatorname{Eff}(X)$. Before we do so, let us introduce some notation. The vertices of the hypercube $[0,1]^{n+3} \subset \mathbb{R}^{n+3}$ are the points of the form $\xi_{I}=$ $\left(\left(\xi_{I}\right)_{1}, \ldots,\left(\xi_{I}\right)_{n+3}\right)$, where $I \subset\{1, \ldots, n+3\},\left(\xi_{I}\right)_{i}=1$ if $i \in I$, and $\left(\xi_{I}\right)_{i}=0$ otherwise. The parity of the vertex $\xi_{I}$ is the parity of $|I|$. For each subset $I \subset\{1, \ldots, n+3\}$, define the degree one polynomial in the $\alpha_{i}$ 's:

$$
\begin{equation*}
H_{I}:=\sum_{j \notin I} \alpha_{j}+\sum_{i \in I}\left(1-\alpha_{i}\right) . \tag{3.6}
\end{equation*}
$$

For any subset $J \subset\{1, \ldots, n+3\}$, we have:

$$
\begin{equation*}
H_{I}\left(\xi_{J}\right)=\#\left(I^{c} \cap J\right)+\#\left(J^{c} \cap I\right) \tag{3.7}
\end{equation*}
$$

Given $J \subset\{1, \ldots, n+3\}$ and $i_{0} \notin J$, set $I:=J \cup\left\{i_{0}\right\}$. Then

$$
\begin{equation*}
H_{I}=H_{J}+1-2 \alpha_{i_{0}} . \tag{3.8}
\end{equation*}
$$

One computes that $\varphi\left(E_{I}\right)=\xi_{I^{c}}$. Therefore, the image of $\operatorname{Eff}(X)$ under $\varphi$ is the polytope $\Delta \subset \mathbb{R}^{n+3}$ generated by the odd vertices of the hypercube. Using (3.7) above, one can easily check that the polytope $\Delta \subset \mathbb{R}^{n+3}$ is defined by the following set of inequalities:

$$
\Delta=\varphi(\operatorname{Eff}(X))=\left\{\begin{array}{lr}
0 \leq \alpha_{i} \leq 1, & i \in\{1, \ldots, n+3\}  \tag{3.9}\\
H_{I} \geq 1, & |I| \text { even }
\end{array}\right.
$$

Next we describe the chamber decomposition in $\Delta$ induced by the Mori chamber decomposition of $\operatorname{Eff}(X)$. For each subset $I \subset\{1, \ldots, n+3\}$, and each integer $k$ satisfying $2 \leq k \leq \frac{n+3}{2}$ and $|I| \not \equiv k \bmod 2$, consider the hyperplane $\left(H_{I}=k\right)$. Now take the complement in the interior of $\Delta$ of the hyperplane arrangement

$$
\begin{equation*}
\left(H_{I}=k\right)_{2 \leq k \leq \frac{n+3}{2},|I| \not \equiv k \bmod 2 .} \tag{3.10}
\end{equation*}
$$

and consider its decomposition into connected components. Each connected component is called a chamber of $\Delta$.

The following theorem summarizes the results of Muk05] and Bau91. The proof follows from the proof of the main theorem in [Muk05, Page 6] and the description of wall crosses in [Muk05, Propositions 2 and 3]. Mukai's proof relies on interpreting $X$ as a moduli space of parabolic vector bundles on $\mathbb{P}^{1}$, and the description of these spaces in Bau91, Section 2].

Theorem 3.4. Let $X$ be the blow-up of $\mathbb{P}^{n}$ at $n+3$ points in general position, and consider the projection

$$
\varphi: \operatorname{Eff}(X) \rightarrow \Delta
$$

defined in (3.5) above.

- The chamber decomposition of $\Delta$ defined by the hyperplane arrangement (3.10) coincides with that induced by the Mori chamber decomposition of $\operatorname{Eff}(X)$ via $\varphi$.
- The image of $\operatorname{Mov}(X)$ under $\varphi$ is given by

$$
\Pi=\varphi(\operatorname{Mov}(X))=\left\{\begin{array}{lr}
0 \leq \alpha_{i} \leq 1, & i \in\{1, \ldots, n+3\} \\
H_{I} \geq 2, & |I| \text { odd }
\end{array}\right.
$$

- All small $\mathbb{Q}$-factorial modifications of $X$ are smooth. Let $\mathcal{C}$ and $\mathcal{C}^{\prime}$ be two adjacent chambers of $\operatorname{Mov}(X)$, corresponding to small $\mathbb{Q}$-factorial modifications of $X, f$ : $X \rightarrow \widetilde{X}$ and $f^{\prime}: X \rightarrow \widetilde{X}^{\prime}$, respectively. The images of these chambers in $\Delta$ are separated by a hyperplane of the form $\left(H_{I}=k\right)$, with $3 \leq k \leq \frac{n+3}{2}$ and $|I| \not \equiv k$ $\bmod 2$. Suppose that $\varphi(\mathcal{C}) \subset\left(H_{I} \leq k\right)$ and $\varphi\left(\mathcal{C}^{\prime}\right) \subset\left(H_{I} \geq k\right)$. Then the birational map $f^{\prime} \circ f^{-1}: \widetilde{X} \rightarrow \widetilde{X}^{\prime}$ flips a $\mathbb{P}^{k-2}$ into a $\mathbb{P}^{n+1-k}$.
- Let $\mathcal{C}$ be a chamber of $\operatorname{Mov}(X)$, corresponding to small $\mathbb{Q}$-factorial modification $\tilde{X}$ of $X$. Let $\sigma \subset \partial \mathcal{C}$ be a wall such that $\sigma \subset \partial \operatorname{Mov}(X)$, and let $f: \tilde{X} \rightarrow Y$ be the corresponding elementary contraction. The image of $\sigma$ in $\Pi$ is supported on a hyperplane of one of the following forms:
(a) $\left(\alpha_{i}=0\right)$ or $\left(\alpha_{i}=1\right)$.
(b) $\left(H_{I}=2\right)$, with $|I|$ odd.

In case (a), $f: \widetilde{X} \rightarrow Y$ is a $\mathbb{P}^{1}$-bundle. In case (b), $f: \widetilde{X} \rightarrow Y$ is the blow-up of a smooth point, and the exceptional divisor of $f$ is the image in $\widetilde{X}$ of the divisor $E_{I^{c}}$.

Remark 3.5. In Theorem 3.4, note that

$$
\Delta \cap\left(H_{I} \geq 2\right)_{|I| \text { odd }}=[0,1]^{n+3} \cap\left(H_{I} \geq 2\right)_{|I| \text { odd }} .
$$

This can be checked using (3.8).
Remark 3.6. The image of $\operatorname{Nef}(X)$ under $\varphi$ is given by

$$
\Sigma=\varphi(\operatorname{Nef}(X))=\left\{\begin{array}{lr}
H_{\{i\}} \geq 2, & i \in\{1, \ldots, n+3\} \\
H_{\{i, j\}} \leq 3, & i, j \in\{1, \ldots, n+3\}, i \neq j .
\end{array}\right.
$$

Remark 3.7. Formula (3.5), together with the equations for the walls in the chamber decomposition of $\Delta$ defined by the hyperplane arrangement (3.10), allow us to find explicit inequalities defining the cones $\operatorname{Eff}(X), \operatorname{Mov}(X)$, and $\operatorname{Nef}(\widetilde{X})$, for any small $\mathbb{Q}$-factorial modification $\widetilde{X}$ of $X$.
3.3. Proof of Theorem 1.3. Let $X_{k}^{n}$ be a blow-up of $\mathbb{P}^{n}$ at $k$ points in general position. By [Muk01] and [CT06], $X_{k}^{n}$ is a Mori dream space if and only if one of the following holds:

- $n=2$ and $k \leq 8$.
- $n=3$ and $k \leq 7$.
- $n=4$ and $k \leq 8$.
- $n>4$ and $k \leq n+3$.

We will show that in each of these cases $X_{k}^{n}$ is $\log$ Fano. In view of the classification of del Pezzo surfaces and Example 1.2, we may assume that $n \geq 4$.

Suppose that $k=n+3$, set $X:=X_{n+3}^{n}$, and follow the notation of the previous subsection. The center of the polytopes $\Pi$ and $\Delta$ is the point $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)=\varphi\left(-K_{X}\right)$.

When $n$ is even, this point is in the interior of a chamber of $\Pi$, namely the chamber defined by:

$$
\Sigma^{\prime}=\left(H_{I} \geq \frac{n+2}{2}\right)_{|I| \neq \frac{n+2}{2}} \bmod 2 .
$$

Let $X^{\prime}$ be the small $\mathbb{Q}$-factorial modification of $X$ whose nef cone is the inverse image of the chamber $\Sigma^{\prime}$. Then $X^{\prime}$ is a smooth Fano manifold with very interesting geometry and symmetries. See Cas14 and references therein for several descriptions of $X^{\prime}$. By Proposition 2.3, $X$ is log Fano.

When $n$ is odd, the point $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)=\varphi\left(-K_{X}\right)$ lies in the intersection of the hyperplanes:

$$
\left(H_{I}=\frac{n+3}{2}\right)_{|I| \neq \frac{n+3}{2} \bmod 2}
$$

Let $X^{\prime}$ be the small $\mathbb{Q}$-factorial modification of $X$ whose nef cone is the inverse image of some chamber $\Sigma^{\prime}$ containing $\varphi\left(-K_{X}\right)$ in its boundary. Then $X^{\prime}$ is a smooth projective variety with $-K_{X^{\prime}}$ nef and big. By Lemma 2.5 and Proposition 2.3, $X$ is $\log$ Fano.

It follows from [GOST, Corollary 1.3] that $X_{k}^{n}$ is $\log$ Fano for any $k \leq n+3$.
The case $n=4$ and $k=8$ can be treated in a similar way. In [Muk05, Section 2], Mukai describes the Mori chamber decomposition of $\operatorname{Mov}\left(X_{8}^{4}\right)$. It follows from his description that $X_{8}^{4}$ admits a small $\mathbb{Q}$-factorial modification $X^{\prime}$ which is a Fano manifold. Again we conclude that $X_{8}^{4}$ is $\log$ Fano by Proposition 2.3.
4. Finding explicit divisors making $X_{k}^{n}$ LOG Fano for $k \leq n+2$

Throughout this section, we let $p_{1}, \ldots, p_{k} \in \mathbb{P}^{n}$ be general points, $k \leq n+2$, and let $X_{k}^{n}$ be the blow-up of $\mathbb{P}^{n}$ at $p_{1}, \ldots, p_{k}$. We shall exhibit an integral divisor $D \subset X_{k}^{n}$ and rational numbers $\epsilon>0$ such that $\Delta=\epsilon D$ makes $X_{k}^{n} \log$ Fano. In order to show that $-\left(K_{X_{k}^{n}}+\Delta\right)$ is ample, we will use Proposition 1.4 To show that $\left(X_{k}^{n}, \Delta\right)$ is klt, we will need an explicit log resolutions for this pair, which we introduce next.

Notation 4.1. For each $0 \leq h \leq k-1$ and each subset $I=\left\{i_{1}<\cdots<i_{h+1}\right\} \subset\{1, \ldots, k\}$, consider the $h$-dimensional linear subspace $H_{I}^{h}=\left\langle p_{i_{1}}, \ldots, p_{i_{h+1}}\right\rangle \subset \mathbb{P}^{n}$. Denote by $\mathcal{H}^{h}$ the collection of all such $h$-dimensional linear subspaces, and by $\rho_{h}=\binom{k}{h+1}$ its cardinality.

Let $\pi: Y \rightarrow X_{k}^{n}$ be the blow-up of the strict transforms of the lines in $\mathcal{H}^{1}$, followed by the blow-up of the strict transforms of the planes in $\mathcal{H}^{2}$, and so on, in order of increasing dimension, up to the blow-up of the strict transforms of the $(n-2)$-planes in $\mathcal{H}^{n-2}$. For each $1 \leq h \leq n-2$, denote by $E_{1}^{h}, \ldots, E_{\rho_{h}}^{h} \subset Y$ the exceptional divisors over the $\rho_{h} h$-planes in $\mathcal{H}^{h}$. We have

$$
\begin{equation*}
K_{Y}=\pi^{*} K_{X_{k}^{n}}+\sum_{h=1}^{n-2}(n-h-1)\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right) \tag{4.1}
\end{equation*}
$$

Remark 4.2. For $k=n+1$, the variety $Y$ constructed above is the Losev-Manin moduli space, introduced in LM as a toric compactification of $M_{0, n+3}$. For $k=n+2$, the construction above gives Kapranov's description of $Y=\bar{M}_{0, n+3}$ as an iterated blow-up of $\mathbb{P}^{n}$ (Ka]).
4.1. Blow-ups of $\mathbb{P}^{n}$ in at most $n+1$ points. For $k \leq n+1$, the variety $X_{k}^{n}$ is toric, and one can take $\Delta$ making $X_{k}^{n} \log$ Fano to be a suitable combination of toric invariant divisors. Alternatively, we show that the divisor $\Delta$ can be chosen irreducible. We work out the case $k=n+1$. When $k<n+1$, the boundary divisor can be taken to be the image of $\Delta$ under the natural morphism $X_{n+1}^{n} \rightarrow X_{k}^{n}$.
Theorem 4.3. Let $D \subset X_{n+1}^{n}$ be the strict transform of a general member of the linear system $\Gamma \subset\left|\mathcal{O}_{\mathbb{P}^{n}}(n)\right|$ of the standard Cremona transformation of $\mathbb{P}^{n}$, centered at $p_{1}, \ldots, p_{n+1}$. For any $\frac{n-3}{n-2}<\epsilon<1$ the divisor $-\left(K_{X_{n+1}^{n}}+\epsilon D\right)$ is ample, and the pair $\left(X_{n+1}^{n}, \epsilon D\right)$ is klt.

For the proof of Theorem 4.3, we will need the following.
Lemma 4.4. Let $\Gamma \subset\left|\mathcal{O}_{\mathbb{P}^{n}}(n)\right|$ be the linear system of the standard Cremona transformation of $\mathbb{P}^{n}$, centered at $p_{1}, \ldots, p_{n+1}$, and $D \in \Gamma$ be a general member. Let $H_{I}^{h}$ and $\pi: Y \rightarrow \mathbb{P}^{n}$ be as in Notation 4.1. Then the strict transform $\widetilde{D}$ of $D$ in $Y$ is smooth and transversal to all the exceptional divisors of $\pi$. Furthermore

$$
\operatorname{mult}_{H_{I}^{h}} D=n-h+1
$$

for any $h=0, \ldots, n-2$.
Proof. By MM, Theorem 1] the Cremona transformation induced by $\Gamma$ lifts to an automorphism of $Y$. This implies that $\widetilde{D}$ is smooth and transversal to all the exceptional divisors of $\pi$. In particular, $D$ is smooth away from the union of the codimension two linear subspaces $H_{I}^{n-2}$, .

We may assume that the $p_{i}$ 's are the fundamental points of $\mathbb{P}^{n}$, and consider the element of the linear system $\Gamma$ given by:

$$
D_{0}:=\left\{x_{0} x_{1} \ldots x_{n-1}+x_{0} x_{1} \ldots x_{n-2} x_{n}+\ldots+x_{1} x_{2} \ldots x_{n}=0\right\} .
$$

Let $p \in H_{I}^{h}$ be a general point. Then one checks easily that $\operatorname{mult}_{p} D_{0}=\operatorname{mult}_{H_{I}^{h}} D_{0}=$ $n-h+1$. To conclude, note that mult $H_{I}^{h} D \geq n-h+1$ for any $D \in \Gamma$.
Proof of Theorem 4.3. With Notation 3.1, we have

$$
D \sim n H-(n-1)\left(E_{1}+\ldots+E_{n-1}\right) .
$$

Recall from Proposition 1.4 that the Mori cone of $X_{n+1}^{n}$ is generated by the classes $R_{i}$ 's and $L_{i, j}$ 's. One computes

$$
-\left(K_{X_{n+1}^{n}}+\epsilon D\right) \cdot R_{i}=n-1-\epsilon(n-1) \text { and }-\left(K_{X_{n+1}^{n}}+\epsilon D\right) \cdot L_{i, j}=(n-1) \epsilon-n+3 .
$$

Therefore $-K_{X_{n+1}^{n}}-\epsilon D$ is ample provided that $\frac{n-3}{n-2}<\epsilon<1$.
Next we check when the pair $\left(X_{n+1}^{n}, \epsilon D\right)$ is klt. Let $\pi: Y \rightarrow X_{n+1}^{n}$ be the morphism introduced in Notation 4.1. By Lemma 4.4 $\pi: Y \rightarrow X_{n+1}^{n}$ is a $\log$ resolution of $\left(X_{n+1}^{n}, \epsilon D\right)$, and

$$
\pi^{*}(D)=\widetilde{D}+\sum_{h=1}^{n-2}(n-h-1)\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right)
$$

Together with (4.1), this gives

$$
K_{Y}+\epsilon \widetilde{D}=\pi^{*}\left(K_{n+1}^{n}+\epsilon D\right)+\sum_{h=1}^{n-2}(n-h-1)(1-\epsilon)\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right)
$$

Therefore the pair ( $X_{n+1}^{n}, \epsilon D$ ) is klt for any $0 \leq \epsilon<1$.
4.2. Blow-ups of $\mathbb{P}^{n}$ in $n+2$ points. In this subsection we construct divisors $\Delta$ making $X_{n+2}^{n} \log$ Fano. Note that $X_{n+2}^{n}$ is not toric. We follow Notation 4.1, and denote by $H_{1}, \ldots, H_{\rho_{n-1}} \subset \mathbb{P}^{n}$ the $\rho_{n-1}$ hyperplanes through $n$ of the $p_{i}$ 's.
Theorem 4.5. Let $D \subset X_{n+2}^{n}$ be the strict transform of the divisor $H_{1}+\cdots+H_{\rho_{n-1}}$. For any $\frac{2(n-3)}{(n+1)(n-2)}<\epsilon<\frac{2(n-1)}{n(n+1)}$ the divisor $-\left(K_{X_{n+2}^{n}}+\epsilon D\right)$ is ample, and the pair $\left(X_{n+2}^{n}, \epsilon D\right)$ is klt.

For the proof of Theorem 4.5, we will need the following.
Lemma 4.6. Let $D \subset X_{n+2}^{n}$ be the strict transform of the divisor $H_{1}+\cdots+H_{\rho_{n-1}}$. Let $\pi: Y \rightarrow X_{n+2}^{n}$ be the morphism introduced in Notation 4.1. Then $\pi: Y \rightarrow X_{n+2}^{n}$ is a $\log$ resolution of $\left(X_{n+2}^{n}, D\right)$, and

$$
\pi^{*}(D)=\widetilde{D}+\sum_{h=1}^{n-2}\binom{n-h+1}{n-h-1}\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right)
$$

Proof. Note that at each step in the description of $Y$ as an iterated blow-up, the center of the blow-up is a disjoint union of smooth subvarieties. Moreover, the divisor $\operatorname{Exc}(\pi) \cup \widetilde{D}$ is simple normal crossing, and so $\pi: Y \rightarrow X_{n+2}^{n}$ is a log resolution of $\left(X_{n+2}^{n}, D\right)$.

Any $H_{I}^{h} \in \mathcal{H}^{h}$ is contained in exactly $\binom{n-h+1}{n-h-1}$ of the $\rho_{n-1}$ hyperplanes $H_{i}$ 's. Thus mult $_{H_{I}^{h}} D=\binom{n-h+1}{n-h-1}$, and the formula for $\pi^{*}(D)$ follows.

Proof of Theorem 4.5. Each point $p_{i}$ lies in exactly $\binom{n+1}{n-1}=\frac{1}{2}(n+1) n$ hyperplanes among the $H_{i}$ 's, $1 \leq i \leq \rho_{n-1}=\frac{1}{2}(n+2)(n+1)$. So we have, with Notation 3.1,

$$
D \sim \frac{1}{2}(n+2)(n+1) H-\frac{1}{2}(n+1) n E_{1}-\ldots-\frac{1}{2}(n+1) n E_{n+2} .
$$

Recall from Proposition 1.4 that the Mori cone of $X_{n+2}^{n}$ is generated by the classes $R_{i}$ 's and $L_{i, j}$ 's. One computes
$-\left(K_{X_{n+2}^{n}}+\epsilon D\right) \cdot R_{i}=\left(n-1-\frac{\epsilon}{2}(n+1) n\right)$ and $-\left(K_{X_{n+2}^{n}}+\epsilon D\right) \cdot L_{i, j}=-n+3+\frac{\epsilon}{2}(n+1)(n-2)$.
Therefore $-K_{X_{n+2}^{n}}-\epsilon D$ is ample provided that $\frac{2(n-3)}{(n+1)(n-2)}<\epsilon<\frac{2(n-1)}{n(n+1)}$.
Next we check when the pair $\left(X_{n+2}^{n}, \epsilon D\right)$ is klt. Let $\pi: Y \rightarrow X_{n+2}^{n}$ be the morphism introduced in Notation 4.1. By Lemma 4.6 $\pi: Y \rightarrow X_{n+2}^{n}$ is a $\log$ resolution of $\left(X_{n+2}^{n}, \epsilon D\right)$, and

$$
\pi^{*}(D)=\widetilde{D}+\sum_{h=1}^{n-2}\binom{n-h+1}{n-h-1}\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right)
$$

Together with (4.1), this gives

$$
K_{Y}+\epsilon \widetilde{D}=\pi^{*}\left(K_{n+2}^{n}+\epsilon D\right)+\sum_{h=1}^{n-2}\left((n-h-1)-\epsilon\binom{n-h+1}{n-h-1}\right)\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right) .
$$

Therefore the pair ( $X_{n+2}^{n}, \epsilon D$ ) is klt for any $0 \leq \epsilon<\frac{2}{n}$.

## 5. Finding explicit divisors making $X_{n+3}^{n}$ Log Fano

Throughout this section, let $p_{1}, \ldots, p_{n+3} \in \mathbb{P}^{n}$ be general points, and let $X_{n+3}^{n}$ be the blow-up of $\mathbb{P}^{n}$ at $p_{1}, \ldots, p_{n+3}$. We shall exhibit integral divisors $D \subset X_{n+3}^{n}$ and rational numbers $\epsilon>0$ such that $\Delta=\epsilon D$ makes $X_{n+3}^{n} \log$ Fano. In the previous cases, $D$ was taken as sum of strict transforms of hyperplanes through $n$ of the $n+3$ points. For $X_{n+3}^{n}$, we will also need other extremal divisors $E_{I} \subset X_{n+3}^{n}$ introduced in Paragraph 3.3. This will make the $\log$ resolution of $(X, \Delta)$ more complicated, and we will need to understand well how the divisors $E_{I}$ 's intersect. For this purpose, we start this section with some preliminaries on secant varieties of rational normal curves. Then we will consider separately the cases $n=2 h+1$ odd, and $n=2 h$ even.
5.1. Preliminaries on secant varieties of rational normal curves. Given an irreducible and reduced non-degenerate variety $X \subset \mathbb{P}^{n}$, and a positive integer $k \leq n$ we denote by $\operatorname{Sec}_{k}(X)$ the $k$-secant variety of $X$. This is the subvariety of $\mathbb{P}^{n}$ obtained as the closure of the union of all $(k-1)$-planes $\left\langle x_{1}, \ldots, x_{k}\right\rangle$ spanned by $k$ general points of $X$. We will be concerned with the case when $X=C$ is a rational normal curve of degree $n$ in $\mathbb{P}^{n}$. The following proposition gathers some of the basic properties of the secant varieties $\operatorname{Sec}_{k}(C)$ in this case.

Proposition 5.1. Let $C \subset \mathbb{P}^{n}$ be a rational normal curve of degree $n$, and let $k$ be an integer such that $1 \leq k \leq \frac{n}{2}$. Then the following statements hold.
(1) $\operatorname{dim}\left(\operatorname{Sec}_{k}(C)\right)=2 k-1$ (see for instance [Har, Proposition 11.32]).
(2) $\operatorname{deg}\left(\operatorname{Sec}_{k}(C)\right)=\binom{n-k+1}{k}$ (see for instance [EH, Theorem 12.16]).
(3) $\operatorname{Sec}_{k}(C)$ is normal and $\operatorname{Sing}\left(\operatorname{Sec}_{k}(C)\right)=\operatorname{Sec}_{k-1}(C)$ (see for instance Ve1, Theorem 1.1]).
(4) If $n=2 h$ is even, then for any $1 \leq t<h$ we have

$$
\operatorname{mult}_{\operatorname{Sec}_{h-t}(C)} \operatorname{Sec}_{h}(C)=t+1
$$

Proof of (4). Suppose that $n=2 h$ is even, and consider the $(h+1) \times(h+1)$ matrix

$$
M_{h}=\left(\begin{array}{cccc}
x_{0} & x_{1} & \ldots & x_{h}  \tag{5.1}\\
x_{1} & x_{2} & \ldots & x_{h+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_{h} & x_{h+1} & \ldots & x_{2 h}
\end{array}\right)
$$

For any $1 \leq k \leq h$, the secant variety $\operatorname{Sec}_{k}(C)$ can be described as the determinantal variety:

$$
\operatorname{Sec}_{k}(C)=\left\{\operatorname{rank}\left(M_{h}\right) \leq k\right\}
$$

(See for instance Har, Proposition 9.7]). In particular, $\operatorname{Sec}_{h}(C) \subset \mathbb{P}^{2 h}$ is the degree $h+1$ hypersurface defined by the polynomial $F:=\operatorname{det}\left(M_{h}\right)$. For each $j \in\{0, \ldots, 2 h\}$, let $\left\{M_{i}^{j}\right\}$ be the set of $h \times h$ minors of $M_{h}$ produced by erasing in $M_{h}$ a row and a column meeting in an entry of type $x_{j}$ Denote by $\rho_{j}$ be the number of such minors. Then

$$
\frac{\partial F}{\partial x_{j}}=\sum_{i=1}^{\rho_{j}} \alpha_{i}^{j} \operatorname{det}\left(M_{i}^{j}\right)
$$

for suitable $\alpha_{i}^{j} \neq 0$. Inductively, we see that for any $1 \leq t<h$ the partial derivatives of order $t$ of $F$ are linear combinations of determinants of $(h+1-t) \times(h+1-t)$ minors of $M_{h}$. The vanishing of such determinants defines $\operatorname{Sec}_{h-t}(C)$, while the vanishing of the of determinants of the $(h-t) \times(h-t)$ minors of $M_{h}$ defines $\operatorname{Sec}_{h-t-1}(C) \subsetneq \operatorname{Sec}_{h-t}(C)$. Therefore, there is at least one partial derivative of order $t+1$ of $F$ not vanishing on $\operatorname{Sec}_{h-t}(C)$. This means that mult $\operatorname{Sec}_{h-t}(C) \operatorname{Sec}_{h}(C)=t+1$ for any $1 \leq t<h$.

The following proposition is just a particular instance of [Be, Theorem 1]. The general statement for smooth curves embedded via a $2 h$-very ample line bundle can be found in Ve, Theorem 3.1] as well.
Proposition 5.2. Let $C \subset \mathbb{P}^{n}$ be a rational normal curve of degree $n$, and set $h:=\left\lfloor\frac{n}{2}\right\rfloor$. Consider the following sequence of blow-ups:

$$
\begin{aligned}
& -\pi_{1}: X_{1} \rightarrow \mathbb{P}^{n} \text { the blow-up of } C \text {, } \\
& -\pi_{2}: X_{2} \rightarrow X_{1} \text { the blow-up of the strict transform of } \operatorname{Sec}_{2}(C) \text {, } \\
& \vdots \\
& -\pi_{h}: X_{h} \rightarrow X_{h-1} \text { the blow-up of the strict transform of } \operatorname{Sec}_{h}(C) \text {. }
\end{aligned}
$$

Let $\pi: X \rightarrow \mathbb{P}^{n}$ be the composition of these blow-ups. Then, for any $k \leq h$ the strict transform of $\operatorname{Sec}_{k}(C)$ in $X_{k-1}$ is smooth and transverse to all exceptional divisors. In particular $X$ is smooth and the exceptional locus of $\pi$ is a simple normal crossing divisor.

Notation 5.3. Let $p_{1}, \ldots, p_{n+3} \in \mathbb{P}^{n}$ be general points, and let $C \subset \mathbb{P}^{n}$ be the unique rational normal curve of degree $n$ through these points. Given $1 \leq m \leq n, I=\left\{i_{1}<\cdots<\right.$
$\left.i_{m}\right\} \subset\{1, \ldots, n+3\}$, and a positive integer $k$ such that $0 \leq k \leq \frac{n-m}{2}$, we consider the following variety of dimension $d=2 k-1+m$ :

$$
Y_{I}^{d}:=\operatorname{Join}\left(\left\langle p_{i_{1}}, \ldots, p_{i_{m}}\right\rangle, \operatorname{Sec}_{k}(C)\right) .
$$

Alternatively, $Y_{I}^{d}$ can be defined as follows. Let $\pi_{I}: \mathbb{P}^{n} \longrightarrow \mathbb{P}^{n-m}$ be the projection from the linear space $\left\langle p_{i_{1}}, \ldots, p_{i_{m}}\right\rangle$. Let $C_{I} \subset \mathbb{P}^{n-m}$ be the image of $C$ under $\pi_{I}$. It is the the unique rational normal curve of degree $n-m$ through the points $\pi\left(p_{j}\right), j \notin I$. Then $Y_{I}^{d}$ is the cone with vertex $\left\langle p_{i_{1}}, \ldots, p_{i_{m}}\right\rangle$ over $\operatorname{Sec}_{k}\left(C_{I}\right)$.

By convention, when $k=0$, we set $Y_{I}^{m-1}:=\left\langle p_{i_{1}}, \ldots, p_{i_{m}}\right\rangle$.
Fix $I=\left\{i_{1}<\cdots<i_{m}\right\} \subset\{1, \ldots, n+3\}$, with $m \leq n$. Given $k$ such that $0 \leq k \leq \frac{n-m}{2}$, set $d:=2 k-1+m$. By Proposition 5.1, we have

$$
\begin{equation*}
\operatorname{deg}\left(Y_{I}^{d}\right)=\binom{n-m-k+1}{k} \text { and } \operatorname{Sing}\left(Y_{I}^{d}\right)=Y_{I}^{d-2} \tag{5.2}
\end{equation*}
$$

Moreover, if $n-m$ is even and $d_{1}=2 k_{1}-1+m>2 k_{2}-1+m=d_{2}$, then $Y_{I}^{d_{2}} \subset Y_{I}^{d_{1}}$ and

$$
\begin{equation*}
\operatorname{mult}_{Y_{I}^{d_{2}}} Y_{I}^{d_{1}}=\frac{d_{1}-d_{2}}{2}+1 \tag{5.3}
\end{equation*}
$$

We also have analogs of Proposition 5.2 for sequences of blow-ups of $Y_{I}^{d}$, for $|I|-1 \leq$ $d \leq n-1$. More precisely:

Proposition 5.4. Let $C \subset \mathbb{P}^{n}$ be a rational normal curve of degree $n$, $p_{1}, \ldots, p_{m} \in C$ distinct points, with $1 \leq m \leq n$, and set $h:=\left\lfloor\frac{n-m}{2}\right\rfloor$. Consider the following sequence of blow-ups:

- $\pi_{1}: X_{1} \rightarrow \mathbb{P}^{n}$ the blow-up of $Y_{I}^{m-1}:=\left\langle p_{1}, \ldots, p_{m}\right\rangle$,
- $\pi_{2}: X_{2} \rightarrow X_{1}$ the blow-up of the strict transform of $Y_{I}^{m+1}$,
$\vdots$
- $\pi_{h}: X_{h} \rightarrow X_{h-1}$ the blow-up of the strict transform of $Y_{I}^{m+2 h-1}$.

Let $\pi: X \rightarrow \mathbb{P}^{n}$ be the composition of these blow-ups. Then, for any $k \leq h$ the strict transform of $Y_{I}^{m+2 k-1}$ in $X_{k-1}$ is smooth and transverse to all exceptional divisors.

Proposition 5.4 follows easily from Proposition 5.2, In the next sections, we will blowup varieties of type $Y_{I}^{d}$ for several subsets $I \subset\{1, \ldots, n+3\}$, in a suitable order. In order to show the smoothness and transversality of the strict transforms of the $Y_{I}^{d}$ 's in the intermediate blow-ups, we will need the following result.

Proposition 5.5. Let $W \subsetneq Z \subsetneq X$ be smooth projective varieties, and let $Y \subset X$ be $a$ projective variety such that $\operatorname{Sing}(Y)=Z$ and $Y$ has ordinary singularities along $Z$. Let $\pi_{W}: X_{W} \rightarrow X$ be the blow-up of $W$, and denote by $Z_{W}$ and $Y_{W}$ the strict transforms of $Z$ and $Y$, respectively. Then $\operatorname{Sing}\left(Y_{W}\right)=Z_{W}$ and $Y_{W}$ has ordinary singularities along $Z_{W}$.
Proof. Denote by $E_{W}$ the exceptional divisor of $\pi_{W}$. Then $\pi_{W}^{-1}(Z)=Z_{W} \cup E_{W}$. Let $\pi_{Z_{W}}: X_{Z_{W}} \rightarrow X_{W}$ be the blow-up of $X_{W}$ along $Z_{W}$, with exceptional divisor $E_{Z_{W}}$.

We claim that the composite morphism $\pi_{W} \circ \pi_{Z_{W}}: X_{Z_{W}} \rightarrow X$ is isomorphic to the blow-up $\pi_{Z}: X_{Z} \rightarrow X$ of $X$ along $Z$, followed by the blow-up of $X_{Z}$ along $\pi_{Z}^{-1}(W)$. Indeed,
by the universal property of the blow-up ([Hart, Proposition 7.14]), there exits a unique morphism $f: X_{Z_{W}} \rightarrow X_{Z}$ making the following diagram commute.


Note that all varieties in this diagram are smooth. Since $Z$ and $W$ are smooth, the intersection $Z_{W} \cap E_{W} \subset X_{W}$ is smooth. Thus, any normal direction of $Z_{W}$ in $X_{W}$ at a point $p \in Z_{W} \cap E_{W}$ is the image of a normal direction at $p$ of $Z_{W} \cap E_{W}$ in $E_{W}$. In other words, the inverse image of $W$ in $X_{Z_{W}}$ consists of the strict transform $\widetilde{E}_{W}$ of $E_{W}$ in $X_{Z_{W}}$. Therefore, the inverse image of the smooth variety $\pi_{Z}^{-1}(W)$ in $X_{W}$ is precisely $\widetilde{E}_{W}$. Using the the universal property of the blow-up, and comparing the Picard number of these smooth varieties, we conclude that $f: X_{Z_{W}} \rightarrow X_{Z}$ is the blow-up of $X_{Z}$ along $\pi_{Z}^{-1}(W)$, proving the claim.

Next we prove that $\operatorname{Sing}\left(Y_{W}\right)=Z_{W}$. Clearly $Z_{W} \subset \operatorname{Sing}\left(Y_{W}\right)$. Suppose that this inclusion is strict. Then the strict transform $Y_{Z_{W}}$ of $Y_{W}$ in $X_{Z_{W}}$ is singular. Since $f$ : $X_{Z_{W}} \rightarrow X_{Z}$ is a smooth blow-up, $f\left(Y_{Z_{W}}\right) \subset X_{Z}$ is singular as well. But notice that $f\left(Y_{Z_{W}}\right) \subset X_{Z}$ is the strict transform of $Y \subset X$ via $\pi_{Z}$. Since $\operatorname{Sing}(Y)=Z$ and $Y$ has ordinary singularities along $Z$, the blow-up $\pi_{Z}$ resolves the singularities of $Y$. This contradiction shows that $\operatorname{Sing}\left(Y_{W}\right)=Z_{W}$. Moreover, since $Y$ has ordinary singularities along $Z$, the intersection of its strict transform $Y_{Z}$ with the exceptional divisor $E_{Z}$ of $\pi_{Z}$ is transverse. This implies that the intersection $Y_{Z_{W}} \cap E_{Z_{W}}$ is also transverse, i.e., $Y_{W}$ has ordinary singularities along $Z_{W}$.

We end this section by describing the intersection of some of the $Y_{I}^{d}$ 's. This can be computed using elementary projective geometry. In what follows we adopt the following notation. Given two finite sets $I$ and $J$, we define their distance to be

$$
d(I, J):=|(I \cup J) \backslash(I \cap J)|
$$

We start by intersecting varieties $Y_{I}^{d}$ 's of the same dimension.
Proposition 5.6. Let the assumptions and notation be as in Notation 5.3. Let $I_{1}, I_{2} \subset$ $\{1, \ldots, n+3\}$ be subsets with cardinality $m_{1}$ and $m_{2}$, respectively, and suppose that $I_{1} \cap I_{2}=$ Ø. Let $k_{1}$ and $k_{2}$ be integers such that $0 \leq k_{i} \leq \frac{n-m_{i}}{2}, i=1,2$, and $m_{1}+2 k_{1}-1=$ $m_{2}+2 k_{2}-1=: d$. Set $s=\frac{m_{1}+m_{2}}{2}$ and suppose that $d \leq n-s$. Then

$$
Y_{I_{1}}^{d} \cap Y_{I_{2}}^{d}=\bigcup_{J} Y_{J}^{d-s}
$$

where the union is taken over all subsets $J \subset I_{1} \cup I_{2}$ satisfying $d\left(I_{i}, J\right)=s$ for $i=1,2$.
Moreover, for a general point in any irreducible component of the above intersections, the intersection is transverse.

Proof. We note that the assumptions of the theorem imply that $d=k_{1}+k_{2}+s-1$ and $m_{1}-m_{2}=2\left(k_{2}-k_{1}\right)$.

Let $J \subset I_{1} \cup I_{2}$ be such that $d\left(I_{i}, J\right)=s$ for $i=1,2$. We shall prove that $Y_{J}^{d-s} \subset Y_{I_{1}}^{d} \cap Y_{I_{2}}^{d}$. Write $J=J_{1} \cup J_{2}$, where $J_{i} \subset I_{i}, i=1,2$, set $\ell_{i}:=\left|J_{i}\right|, i=1,2$, and $\ell=|J|=\ell_{1}+\ell_{2}$.

The assumption that $d\left(I_{i}, J\right)=s$ for $i=1,2$ implies that $k_{2}-k_{1}=\ell_{1}-\ell_{2}$. We set $k:=k_{2}-\ell_{1}=k_{1}-\ell_{2}$, and note that $d-s=\ell+2 k-1$.

Let $x \in Y_{J}^{d-s}$. Then there exists a point $q \in \operatorname{Sec}_{k}(C)$ such that $x \in\left\langle q, p_{i} \mid i \in J\right\rangle \cong \mathbb{P}^{\ell}$. The following two linear subspaces of this $\mathbb{P}^{\ell}$

$$
\left\langle x, p_{i} \mid i \in I_{1}\right\rangle \cong \mathbb{P}^{\ell_{1}} \text { and }\left\langle q, p_{i} \mid i \in I_{2}\right\rangle \cong \mathbb{P}^{\ell_{2}}
$$

have complementary dimensions. Hence there exists a point

$$
z \in\left\langle x, p_{i} \mid i \in J_{1}\right\rangle \cap\left\langle q, p_{i} \mid i \in J_{2}\right\rangle .
$$

In particular, $z \in \operatorname{Sec}_{k+\ell_{2}}(C)$. Since $k+\ell_{2}=k_{1}$, we conclude that $x \in Y_{I_{1}}^{d}$. Similarly we show that $x \in Y_{I_{2}}^{d}$.

Now assume that $x$ is a general point of $Y_{J}^{d-s}$. Keeping the same notation as above, we will prove now that $Y_{I_{1}}^{d}$ and $Y_{I_{2}}^{d}$ intersect transversely at $x$. This amounts to proving that $T_{x}\left(Y_{I_{1}}^{d}\right) \cap T_{x}\left(Y_{I_{2}}^{d}\right)=T_{x}\left(Y_{J}^{d-s}\right)$. By Terracini's Lemma Te, we have

$$
\begin{aligned}
T_{x}\left(Y_{I_{1}}^{d}\right) & =\left\langle\left\langle p_{i} \mid i \in I_{1}\right\rangle,\left\langle T_{q_{i}} C \mid 1 \leq i \leq k\right\rangle,\left\langle T_{p_{i}} C \mid i \in J_{2}\right\rangle\right\rangle, \\
T_{x}\left(Y_{I_{2}}^{d}\right) & =\left\langle\left\langle p_{i} \mid i \in I_{2}\right\rangle,\left\langle T_{q_{i}} C \mid 1 \leq i \leq k\right\rangle,\left\langle T_{p_{i}} C \mid i \in J_{1}\right\rangle\right\rangle, \\
T_{x}\left(Y_{J}^{d-s}\right) & =\left\langle\left\langle p_{i} \mid i \in J\right\rangle,\left\langle T_{q_{i}} C \mid 1 \leq i \leq k\right\rangle\right\rangle,
\end{aligned}
$$

where $q_{1}, \ldots, q_{k} \in C$ are such that $q \in\left\langle q_{i} \mid 1 \leq i \leq k\right\rangle$.
Consider the linear subspaces:

$$
\begin{aligned}
L_{1} & :=\left\langle\left\langle p_{i} \mid i \in I_{1}\right\rangle,\left\langle T_{p_{i}} C \mid i \in J_{2}\right\rangle\right\rangle, \\
L_{2} & :=\left\langle\left\langle p_{i} \mid i \in I_{2}\right\rangle,\left\langle T_{p_{i}} C \mid i \in J_{1}\right\rangle\right\rangle, \\
L & :=\left\langle\left\langle p_{i} \mid i \in J\right\rangle\right\rangle \subset L_{1} \cap L_{2} .
\end{aligned}
$$

We have that $\operatorname{dim}\left(\left\langle L_{1}, L_{2}\right\rangle\right) \leq m_{1}+m_{2}+\ell-1$, and equality holds if and only if $L_{1} \cap L_{2}=L$. On the other hand, note that $L$ intersects $C$ in at least $m_{1}+m_{2}+\ell$ points, counted with multiplicity. Therefore we must have $\operatorname{dim}\left(\left\langle L_{1}, L_{2}\right\rangle\right)=m_{1}+m_{2}+\ell-1$, and $L_{1} \cap L_{2}=L$. It follows from the description of the tangent spaces above that $T_{x}\left(Y_{I_{1}}^{d}\right) \cap T_{x}\left(Y_{I_{2}}^{d}\right)=T_{x}\left(Y_{J}^{d-s}\right)$.

It remains to prove that $Y_{I_{1}}^{d} \cap Y_{I_{2}}^{d} \subset \bigcup_{J} Y_{J}^{d-s}$. Write $\left\{p_{i} \mid i \in I_{1}\right\}=\left\{x_{1}, \ldots, x_{m_{1}}\right\}$ and $\left\{p_{i} \mid i \in I_{2}\right\}=\left\{y_{1}, \ldots, y_{m_{2}}\right\}$. Suppose that $x \in Y_{I_{1}}^{d} \cap Y_{I_{2}}^{d}$. This means that there exist points $z_{1}, \ldots, z_{k_{1}}, w_{1}, \ldots, w_{k_{2}} \in C$ such that:

$$
\begin{gathered}
\left\langle x_{1}, \ldots, x_{m_{1}}\right\rangle \cap\left\langle z_{1}, \ldots, z_{k_{1}}\right\rangle=\emptyset=\left\langle y_{1}, \ldots, y_{m_{2}}\right\rangle \cap\left\langle w_{1}, \ldots, w_{k_{2}}\right\rangle, \text { and } \\
x \in\left\langle x_{1}, \ldots, x_{m_{1}}, z_{1}, \ldots, z_{k_{1}}\right\rangle \cap\left\langle y_{1}, \ldots, y_{m_{2}}, w_{1}, \ldots, w_{k_{2}}\right\rangle .
\end{gathered}
$$

The assumption that $d \leq n-s$ implies that $m_{1}+m_{2}+k_{1}+k_{2} \leq n+1$, and thus

$$
\begin{array}{r}
\left\langle x_{1}, \ldots, x_{m_{1}}, z_{1}, \ldots, z_{k_{1}}\right\rangle \cap\left\langle y_{1}, \ldots, y_{m_{2}}, w_{1}, \ldots, w_{k_{2}}\right\rangle= \\
\left\langle\left\{x_{1}, \ldots, x_{m_{1}}, z_{1}, \ldots, z_{k_{1}}\right\} \cap\left\{y_{1}, \ldots, y_{m_{2}}, w_{1}, \ldots, w_{k_{2}}\right\}\right\rangle .
\end{array}
$$

By relabeling the points if necessary, we may write, for suitable integers $s_{1}, s_{2}$ and $r$ :

$$
\begin{aligned}
\left\{x_{1}, \ldots, x_{s_{1}}\right\} & =\left\{x_{1}, \ldots, x_{m_{1}}\right\} \cap\left\{w_{1}, \ldots, w_{k_{2}}\right\} \\
\left\{y_{1}, \ldots, y_{s_{2}}\right\} & =\left\{y_{1}, \ldots, y_{m_{2}}\right\} \cap\left\{z_{1}, \ldots, z_{k_{1}}\right\} \\
\left\{z_{1}=w_{1}, \ldots, z_{r}=w_{r}\right\} & =\left\{z_{1}, \ldots, z_{k_{1}}\right\} \cap\left\{w_{1}, \ldots, w_{k_{2}}\right\} .
\end{aligned}
$$

Note that $s_{i}+r \leq k_{j},\{i, j\}=\{1,2\}$, and we have

$$
\begin{equation*}
x \in\left\langle x_{1}, \ldots, x_{s_{1}}, y_{1}, \ldots, y_{s_{2}}, z_{1}, \ldots, z_{r}\right\rangle \tag{5.4}
\end{equation*}
$$

Let $J_{0} \subset I_{1} \cup I_{2}$ be the subset of indices corresponding to the subset $\left\{x_{1}, \ldots, x_{s_{1}}, y_{1}, \ldots, y_{s_{2}}\right\} \subset$ $\left\{p_{1}, \ldots, p_{n+3}\right\}$. Note that $d\left(J_{0}, I_{i}\right)=m_{i}-s_{i}+s_{j}$, for $\{i, j\}=\{1,2\}$. In particular we have

$$
d\left(J_{0}, I_{1}\right)+d\left(J_{0}, I_{2}\right)=2 s
$$

Suppose first that $d\left(J_{0}, I_{1}\right)=d\left(J_{0}, I_{2}\right)=s$. It follows from (5.4) that

$$
x \in \operatorname{Join}\left(\left\langle p_{i} \mid i \in J_{0}\right\rangle, \operatorname{Sec}_{r}(C)\right) .
$$

Since $s_{i}+r \leq k_{j},\{i, j\}=\{1,2\}$, we get that

$$
\left|J_{0}\right|+2 r-1=s_{1}+s_{2}+2 r-1 \leq k_{1}+k_{2}-1=d-s .
$$

Hence $x \in Y_{J_{0}}^{d-s}$.
From now on we consider the case when $d\left(J_{0}, I_{1}\right) \neq d\left(J_{0}, I_{2}\right)$. Without lost of generality, we assume that

$$
d\left(J_{0}, I_{1}\right)-d\left(J_{0}, I_{2}\right)=m_{1}-m_{2}+2 s_{2}-2 s_{1}>0 .
$$

We will modify the subset $J_{0} \subset I_{1} \cup I_{2}$ by adding points of $I_{1} \backslash J_{0}$ or removing points of $I_{2} \cap J_{0}$ to obtain another subset $J \subset I_{1} \cup I_{2}$ satisfying $d\left(I_{i}, J\right)=s$ for $i=1,2$. Note that if $i \in I_{1} \backslash J_{0}$, then $d\left(J_{0} \cup\{i\}, I_{1}\right)=d\left(J_{0}, I_{1}\right)-1$ and $d\left(J_{0} \cup\{i\}, I_{2}\right)=d\left(J_{0}, I_{2}\right)+1$. Similarly, if $i \in I_{2} \cap J_{0}$, then $d\left(J_{0} \backslash\{i\}, I_{1}\right)=d\left(J_{0}, I_{1}\right)-1$ and $d\left(J_{0} \backslash\{i\}, I_{2}\right)=d\left(J_{0}, I_{2}\right)+1$. So we have to modify $J_{0}$ by adding or removing exactly $\frac{m_{1}-m_{2}}{2}+s_{2}-s_{1}$ points of the appropriate $I_{i}$.

Suppose first that $\left|I_{1} \backslash J_{0}\right|=m_{1}-s_{1} \geq \frac{m_{1}-m_{2}}{2}+s_{2}-s_{1}$. This is equivalent to the inequality $s \geq s_{2}$. We construct $J_{1} \subset I_{1} \cup I_{2}$ by adding to $J_{0}$ exactly $\frac{m_{1}-m_{2}}{2}+s_{2}-s_{1}$ points of $I_{1} \backslash J_{0}$. Then $d\left(I_{i}, J_{1}\right)=s$ for $i=1,2$, and it follows from (5.4) that

$$
x \in \operatorname{Join}\left(\left\langle p_{i} \mid i \in J_{1}\right\rangle, \operatorname{Sec}_{r}(C)\right) .
$$

Since $s_{2}+r \leq k_{1}$, we get that

$$
\left|J_{1}\right|+2 r-1=\left(k_{2}-k_{1}+2 s_{2}\right)+2 r-1 \leq k_{1}+k_{2}-1=d-s .
$$

Hence $x \in Y_{J_{1}}^{d-s}$.
Next we suppose that $s<s_{2}$. Let $I_{2}^{\prime} \subset I_{2}$ be the subset of indices corresponding to the subset $\left\{y_{1}, \ldots, y_{s}\right\}$, and set $J_{2}:=I_{1} \cup I_{2}^{\prime}$. Then $d\left(I_{i}, J_{2}\right)=s$ for $i=1,2$, and it follows from (5.4) that

$$
x \in \operatorname{Join}\left(\left\langle p_{i} \mid i \in J_{2}\right\rangle, \operatorname{Sec}_{r+s_{2}-s}(C)\right) .
$$

Since $s_{2}+r \leq k_{1}$, we get that

$$
\left|J_{2}\right|+2\left(r+s_{2}-s\right)-1=m_{1}+2\left(r+s_{2}\right)-s-1 \leq m_{1}+2 k_{1}-1-s=d-s .
$$

Hence $x \in Y_{J_{2}}^{d-s}$.
5.2. The odd case $n=2 h+1$. In this subsection we construct divisors $\Delta$ making $X_{n+3}^{n} \log$ Fano when $n=2 h+1$ is odd. We follow Notation 5.3) For each $1 \leq i \leq 3$, let $\Delta_{i} \subset X_{n+3}^{n}$ be the strict transform of the divisor $Y_{i}^{2 h} \subset \mathbb{P}^{n}$, and denote by $H_{4, \ldots, n+3} \subset X_{n+3}^{n}$ the strict transform of the hyperplane $\left\langle p_{4}, \ldots, p_{n+3}\right\rangle \subset \mathbb{P}^{n+3}$.

Theorem 5.7. Let $n=2 h+1 \geq 5$ be an odd integer. Set

$$
D:=\Delta_{1} \cup \Delta_{2} \cup \Delta_{3} \cup H_{4, \ldots, n+3} \subset X_{n+3}^{n} .
$$

For any $\frac{2 h-2}{3 h-2}<\epsilon<\frac{2 h}{3 h+1}$ the divisor $-\left(K_{X_{n+3}^{n}}+\epsilon D\right)$ is ample, and the pair $\left(X_{n+3}^{n}, \epsilon D\right)$ is klt.

For the proof of Theorem 5.7, we will need the following.
Proposition 5.8. Let the assumptions be as in Theorem 5.7, and follow Notation 5.3. For $0 \leq m \leq n-3$, we define a modification $X_{m}$ of $X_{n+3}^{n}$ recursively as follows:

- $X_{0}=X_{n+3}^{n}$,
- $X_{2 k+1}$ is the blow-up of $X_{2 k}$ along the strict transforms of $\operatorname{Sec}_{k+1}(C)$, and of the $Y_{i, j}^{2 k+1}$ 's $(0 \leq k \leq h-2)$,
- $X_{2 k}$ is the blow-up of $X_{2 k-1}$ along the strict transforms of the $Y_{i}^{2 k}$ 's, and of $Y_{1,2,3}^{2 k}$ ( $1 \leq k \leq h-1$ ),
- $X_{n-2}$ is the blow-up of $X_{n-3}$ along the strict transform of $\operatorname{Sec}_{h}(C)$.

Then, for any $0 \leq m \leq n-3$, the center of the blow-up $X_{m+1} \rightarrow X_{m}$ is a disjoint union of smooth subvarieties, all transverse to the exceptional divisors of $X_{m} \rightarrow X_{0}$. Moreover, the composition $\pi: X_{n-2} \rightarrow X_{n+3}^{n}$ of these blow-ups is a log resolution of the pair $\left(X_{n+3}^{n}, D\right)$.
Proof. We will prove the result by induction on $m$. The statement is clearly try for $m=0$. For simplicity of notation we will denote by $\widetilde{Z}$ the strict transform of a subvariety $Z \subset X_{n+3}^{n}$ in any $X_{m}$.

Suppose that the statement is true $m=2 k$. We will show that it holds for $X_{2 k+1}$ and $X_{2 k+2}$. We start with the following observation. Let $I \subset\{1, \ldots, n+3\}$ be such that either $|I| \in\{0,1,2\}$ or $I=\{1,2,3\}, 0 \leq k \leq \frac{n-|I|}{2}$, and $d=2 k-1+|I|$. Then, for any $m<d-1$, each component of the center of the blow-up $X_{m+1} \rightarrow X_{m}$ is either contained in $\operatorname{Sing}\left(Y_{I}^{d}\right)$, or is disjoint from it. This allows us to apply Proposition 5.5, together with Propositions 5.2 and 5.4 and conclude by induction that the following holds.

- The subvarieties $\widetilde{Y}_{i}^{2 k+2} \subset X_{2 k+1}, 1 \leq i \leq 3$, are smooth and transverse to the exceptional divisors over $X_{0}$.
- The subvarieties $\mathbb{S e c} \widetilde{c_{k+2}}(C), \widetilde{Y}_{i, j}^{2 k+3} \subset X_{2 k+2}, 1 \leq i<j \leq 3$, are smooth and transverse to the exceptional divisors over $X_{0}$.
Next we show that the $\widetilde{Y}_{i}^{2 k+2}$, s and $\widetilde{Y}_{1,2,3}^{2 k+2}$ are pairwise disjoint in $X_{2 k+1}$, and similarly for $\widetilde{\operatorname{Sec} c_{k+2}(C)}$ and the $\widetilde{Y}_{i, j}^{2 k+3}$ 's in $X_{2 k+2}$.

Consider the blow-up $X_{2 k+1} \rightarrow X_{2 k}$. By Proposition 5.6, on $X_{2 k}$ we have

$$
\widetilde{Y}_{i}^{2 k+2} \cap \widetilde{Y}_{j}^{2 k+2}=\widetilde{\operatorname{Sec}_{k+1}(C)} \cup \widetilde{Y}_{i, j}^{2 k+1}, \quad \widetilde{Y}_{i}^{2 k+2} \cap \widetilde{Y}_{i, r, s}^{2 k+2}=\widetilde{Y}_{i, r}^{2 k+1} \cup \widetilde{Y}_{i, s}^{2 k+1}
$$

By the induction hypothesis, $\widetilde{\operatorname{Sec}\left(\widetilde{c_{k+1}( } C\right)}$ and $\widetilde{Y}_{i, j}^{2 k+1}$ are smooth and disjoint. So the intersections are everywhere transverse. We conclude that on $X_{2 k+1}$, which is obtained from $X_{2 k}$ by blowing-up $\widetilde{\operatorname{Sec}(C)}(C)$ and $\widetilde{Y}_{i, j}^{2 k+1}$, the $\widetilde{Y}_{i}^{2 k+2}$,s and $\widetilde{Y}_{1,2,3}^{2 k+2}$ are pairwise disjoint.

Now consider the blow-up $X_{2 k+2} \rightarrow X_{2 k+1}$. By Proposition 5.6, on $X_{2 k+1}$ we have

$$
\widetilde{\operatorname{Sec}_{k+2}(C)} \cap \widetilde{Y}_{i, j}^{2 k+3}=\widetilde{Y}_{i}^{2 k+2} \cup \widetilde{Y}_{j}^{2 k+2}, \quad \widetilde{Y}_{i, j}^{2 k+3} \cap \widetilde{Y}_{i, r}^{2 k+3}=\widetilde{Y}_{i}^{2 k+2} \cup \widetilde{Y}_{i, j, r}^{2 k+2}
$$

By the induction hypothesis, the $\widetilde{Y}_{i}^{2 k+2}$,s and $\widetilde{Y}_{1,2,3}^{2 k+2}$ are smooth and pairwise disjoint. So the intersections are everywhere transverse. We conclude that on $X_{2 k+2}$, which is obtained from $X_{2 k+1}$ by blowing-up the $\widetilde{Y}_{i}^{2 k+2}$,s and $\widetilde{Y}_{1,2,3}^{2 k+2}$, the varieties $\mathbb{S e c} \widetilde{c_{k+2}(C)}$ and the $\widetilde{Y}_{i, j}^{2 k+3}$ 's are pairwise disjoint.

As before, we have that the divisors $\widetilde{H}_{4, \ldots, n+3}, \widetilde{\Delta}_{1}, \widetilde{\Delta}_{2}$ and $\widetilde{\Delta}_{3}$ on $X_{n-3}$ are smooth and transverse to the exceptional divisors over $X_{0}$, and their intersection are pairwise smooth and everywhere transverse. By Proposition 5.6 we have

$$
\widetilde{\Delta}_{1} \cap \widetilde{\Delta}_{2} \cap \widetilde{\Delta}_{3}=\widetilde{\sec _{h}(C)} .
$$

So, after the blow-up $X_{n-2} \rightarrow X_{n-3}$ of $\widetilde{\operatorname{Sec}_{h}(C)}$, we get a log resolution of $\left(X_{n+3}^{n}, D\right)$.
Proof of Theorem 5.7. With Notation 3.1, we have

$$
D=\Delta_{1}+\Delta_{2}+\Delta_{3}+H_{4, \ldots, n+3} \sim(3 h+4) H-(3 h+1)\left(E_{1}+\ldots+E_{n+3}\right) .
$$

Recall from Proposition 1.4 that the Mori cone of $X_{n+3}^{n}$ is generated by the classes $R_{i}$ 's and $L_{i, j}$ 's. One computes

$$
-\left(K_{X_{n+3}^{n}}+\epsilon D\right) \cdot R_{i}=2 h-\epsilon(3 h+1) \text { and }-\left(K_{X_{n+3}^{n}}+\epsilon D\right) \cdot L_{i, j}=\epsilon(3 h-2)-2 h+2 .
$$

Therefore $-K_{X_{n+3}^{n}}-\epsilon D$ is ample provided that $\frac{2 h-2}{3 h-2}<\epsilon<\frac{2 h}{3 h+1}$.
Next we check when the pair $\left(X_{n+3}^{n}, \epsilon D\right)$ is klt. Let $\pi: \widetilde{X}:=X_{n-2} \rightarrow X_{n+3}^{n}$ be the log resolution of $\left(X_{n+3}^{n}, \epsilon D\right)$ introduced in Proposition 5.8 above. We have

$$
K_{\tilde{X}}=\pi^{*} K_{X_{n+3}^{n}}+\sum_{k=1}^{h}(n-2 k) E_{\operatorname{Sec}_{k}(C)}+\sum_{k=1}^{h-1}(n-2 k) \sum_{i, j} E_{Y_{i, j}^{2 k-1}}+\sum_{k=1}^{h-1}(n-2 k-1)\left(\sum_{i} E_{Y_{i}^{2 k}}+E_{Y_{1,2,3}^{2 k}}\right) .
$$

Here we denote by $E_{Y}$ the exceptional divisor with center $Y \subset \mathbb{P}^{n}$. In order to compute discrepancies, we will compute the the multiplicities of the $Y_{i}^{2 h}$, along the images in $\mathbb{P}^{n}$ of the subvarieties blown-up by $\pi$. By Proposition 5.1 we have mult $_{\operatorname{Sec}_{k}(C)} \operatorname{Sec}_{h}(C)=h-k+1$. Moreover, $\operatorname{mult}_{\text {Sec }}^{k}(C) Y_{r}^{2 h}=h-k+1$,

$$
\begin{gathered}
\operatorname{mult}_{Y_{i, j}^{2 k-1}} Y_{r}^{2 h}= \begin{cases}\operatorname{mult}_{\operatorname{Sec}_{k}(C)} \operatorname{Sec}_{h}(C)=h-k+1 & \text { if } r \in\{i, j\}, \\
\operatorname{mult}_{\operatorname{Sec}}^{k_{k+1}(C)} \\
\operatorname{Sec} & (C)=h-k \\
\text { if } r \notin\{i, j\},\end{cases} \\
\operatorname{mult}_{Y_{i}^{2 k}} Y_{r}^{2 h}= \begin{cases}\operatorname{mult}_{\operatorname{Sec} c_{k}(C)} \operatorname{Sec}_{h}(C)=h-k+1 & \text { if } r=i, \\
\operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h}(C)=h-k & \text { if } r \neq i,\end{cases}
\end{gathered}
$$

and $\operatorname{mult}_{Y_{1,2,3}^{2 k}} Y_{r}^{2 h}=\operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h}(C)=h-k$ for for $k=1, \ldots, h-1$. Let $\Delta \subset \mathbb{P}^{n}$ be the divisor whose strict transform is $D$. We have

$$
\begin{align*}
& \operatorname{mult}_{\sec _{k}(C)} \Delta=3(h-k+1) \\
& \operatorname{mult}_{Y_{i, j}^{2 k-1}}^{2 k} \Delta=2(h-k+1)+h-k=3(h-k)+2,  \tag{5.5}\\
& \operatorname{mult}_{Y_{i, j, r}^{2 k}} \Delta=3(h-k) \\
& \operatorname{mult}_{Y_{i}^{2 k}} \Delta=h-k+1+2(h-k)=3 h-3 k+1
\end{align*}
$$

Equalities 5.5 yield:

$$
\begin{aligned}
\pi^{*}(D)= & \widetilde{D}+\sum_{k=1}^{h} 3(h-k+1) E_{\mathbb{S e c}_{k}(C)}+\sum_{k=1}^{h-1}(3(h-k)+2) \sum_{i, j} E_{Y_{i, j}^{2 k-1}} \\
& +\sum_{k=1}^{h-1}(3 h-3 k+1) \sum_{i} E_{Y_{i}^{2 k}}+\sum_{k=1}^{h-1} 3(h-k) E_{Y_{1,2,3}^{2 k}}
\end{aligned}
$$

and hence

$$
\begin{aligned}
K_{\tilde{X}}=\pi^{*}\left(K_{X_{n+3}^{n}}+\epsilon D\right) & +\sum_{k=1}^{h}(2 h-2 k+1-3 \epsilon(h-k+1)) E_{\text {Sec }_{k}(C)} \\
& +\sum_{k=1}^{h-1}(2 h-2 k+1-\epsilon(3(h-k)+2)) \sum_{i, j} E_{Y_{i, j}^{2 k-1}} \\
& +\sum_{k=1}^{h-1}(2(h-k)-\epsilon(3 h-3 k+1)) \sum_{i} E_{Y_{2}^{2 k}} \\
& +\sum_{k=1}^{h-1}(2(h-k)-\epsilon(3 h-3 k)) E_{Y_{1,2,3}^{22}}-\epsilon D .
\end{aligned}
$$

Therefore the pair $\left(X_{n+3}^{n}, \epsilon D\right)$ is klt for any $0 \leq \epsilon<\frac{2}{3}$.
5.3. The even case $n=2 h$. In this subsection we construct divisors $\Delta$ making $X_{n+3}^{n} \log$ Fano when $n=2 h$ is even. We follow Notation [5.3] For each $1 \leq i<j \leq n+3$, let $\Delta_{i, j} \subset$ $X_{n+3}^{n}$ be the strict transform of the divisor $Y_{i, j}^{2 h-1} \subset \mathbb{P}^{n}$, and denote by $H_{5, \ldots, n+3} \subset X_{n+3}^{n}$ the strict transform of a general hyperplane in $\mathbb{P}^{n+3}$ through $p_{5}, \ldots, p_{n+3}$.
Theorem 5.9. Let $n=2 h \geq 4$ be an even integer. Set

$$
D:=\Delta_{1,2} \cup \Delta_{3,4} \cup \widetilde{\operatorname{Sec}_{h}(C)} \cup H_{5, \ldots, n+3} \subset X_{n+3}^{n}
$$

For any $\frac{2 h-3}{3 h-4}<\epsilon<\frac{2 h-1}{3 h-1}$ the divisor $-\left(K_{X_{n+3}^{n}}+\epsilon D\right)$ is ample, and the pair $\left(X_{n+3}^{n}, \epsilon D\right)$ is klt.

To provide a log resolution of the pair $\left(X_{n+3}^{n}, D\right)$ in Proposition 5.11 below, we will need the following result.
Lemma 5.10. Any point of $Y_{1,2}^{n-1} \cap Y_{3,4}^{n-1} \subset \mathbb{P}^{n}$ which is smooth for both divisors $Y_{1,2}^{n-1}$ and $Y_{3,4}^{n-1}$ is a smooth point of $Y_{1,2}^{n-1} \cap Y_{3,4}^{n-1}$.
Proof. Let $x \in\left(Y_{1,2}^{n-1} \cap Y_{3,4}^{n-1}\right) \backslash\left(\operatorname{Sing}\left(Y_{1,2}^{n-1}\right) \cup \operatorname{Sing}\left(Y_{3,4}^{n-1}\right)\right)$. We shall prove that the intersection of $Y_{1,2}^{n-1}$ and $Y_{3,4}^{n-1}$ is transverse at $x$, that is, $T_{x} Y_{1,2}^{n-1} \neq T_{x} Y_{3,4}^{n-1}$.

Suppose otherwise, and set $P=T_{x} Y_{1,2}^{n-1}=T_{x} Y_{3,4}^{n-1}$. By Terracini's Lemma Te] we have

$$
P=\left\langle p_{1}, p_{2}, T_{z_{1}} C, \ldots, T_{z_{h-1}} C\right\rangle=\left\langle p_{2}, p_{3}, T_{w_{1}} C, \ldots, T_{w_{h-1}} C\right\rangle
$$

for suitable $z_{i}, w_{i} \in C$, with $z_{i} \notin\left\{p_{1}, p_{2}\right\}$ and $w_{i} \notin\left\{p_{3}, p_{4}\right\}$. Set $s=\mid\left\{z_{1}, \ldots, z_{h-1}\right\} \cap$ $\left\{w_{1}, \ldots, w_{h-1}\right\}\left|, r=\left|\left\{z_{1}, \ldots, z_{h-1}\right\} \cap\left\{p_{3}, p_{4}\right\}\right|\right.$, and $t=\left|\left\{w_{1}, \ldots, w_{h-1}\right\} \cap\left\{p_{1}, p_{2}\right\}\right|$. We may assume that $r \geq t$. Then $s \leq h-1-r$, and the number of intersection points in $P \cap C$, counted with multiplicity, is at least

$$
2(2(h-1)-s)+2-r+2-t \geq n+2+r-t \geq n+2 .
$$

This is impossible since $C$ has degree $n$.
Proposition 5.11. Let the assumptions be as in Theorem 5.9, and follow Notation 5.3 . For $0 \leq m \leq n-3$, we define a modification $X_{m}$ of $X_{n+3}^{n}$ recursively as follows:

- $X_{0}=X_{n+3}^{n}$,
- $X_{2 k+1}$ is the blow-up of $X_{2 k}$ along the strict transforms of $\operatorname{Sec}_{k+1}(C)$ and the $Y_{i, j}^{2 k+1}$,s ( $0 \leq k \leq h-3$ ), and also of $Y_{1,2,3,4}^{2 k+1}$ if $0<k \leq h-3$,
- $X_{2 k}$ is the blow-up of $X_{2 k-1}$ along the strict transforms of the $Y_{i}^{2 k}$,s, and of the $Y_{i, j, r}^{2 k}$ 's $(1 \leq k \leq h-2)$.
- $X_{n-3}$ is the blow-up of $X_{n-4}$ along the strict transforms of $\operatorname{Sec}_{h-1}(C)$, of $Y_{1,2}^{2 h-3}$ and of $Y_{3,4}^{2 h-3}$.
Then, for any $0 \leq m \leq n-4$, the center of the blow-up $X_{m+1} \rightarrow X_{m}$ is a disjoint union of smooth subvarieties, all transverse to the exceptional divisors of $X_{m} \rightarrow X_{0}$. Moreover, the composition $\pi: X_{n-3} \rightarrow X_{n+3}^{n}$ of these blow-ups is a log resolution of the pair $\left(X_{n+3}^{n}, D\right)$.

Proof. Using the same arguments as in the proof of Proposition 5.8, we can prove that, for any $0 \leq m \leq n-4$, the center of the blow-up $X_{m+1} \rightarrow X_{m}$ is a disjoint union of smooth subvarieties, all transverse to the exceptional divisors of $X_{m} \rightarrow X_{0}$. Moreover, the strict transforms of $\Delta_{1,2}, \Delta_{3,4}, \widetilde{\operatorname{Sec}_{h}(C)}$ and $H_{5, \ldots, 2 h+3}$ in $X_{n-3}$ are smooth and transverse to the exceptional divisors over $X_{0}$, and the intersection $\widetilde{\operatorname{Sec}_{h}(C)} \cap \widetilde{Y}_{i, j}^{2 h-1}$ is transverse. Clearly the strict transform of $H_{5, \ldots, 2 h+3}$ is transverse to $\widetilde{\operatorname{Sec}_{h}(C)}, \Delta_{1,2}, \Delta_{3,4}$ and to all exceptional divisors. To show that the strict transform of $D$ in $X_{n-3}$ is simple normal crossing, it remains to compute $\Delta_{1,2} \cap \Delta_{3,4}$. Note that we cannot use Proposition 5.6 in this case. To compute $\Delta_{1,2} \cap \Delta_{3,4}$, we first describe the intersection of $Y_{i, j}^{n-1}$ and $\operatorname{Sing}\left(Y_{r, s}^{n-1}\right)=Y_{r, s}^{n-3}$.
Claim 5.12. We have

$$
Y_{i, j}^{n-1} \cap Y_{r, s}^{n-3}=Y_{r}^{n-4} \cup Y_{s}^{n-4} \cup Y_{i, r, s}^{n-4} \cup Y_{j, r, s}^{n-4}
$$

Moreover, at a general point in any irreducible component of this intersection, the intersection is transverse.

Proof. Note that $Y_{i, j}^{n-1} \cap Y_{r, s}^{n-3}=\left(Y_{i, j}^{n-1} \cap Y_{i, j, r, s}^{n-1}\right) \cap Y_{r, s}^{n-3}$. By Proposition 5.6, $Y_{i, j}^{n-1} \cap Y_{i, j, r, s}^{n-1}=$ $Y_{i, j, r}^{n-2} \cup Y_{i, j, s}^{n-2}$. Applying Proposition 5.6 repeatedly, we have that
$Y_{i, j, r}^{n-2} \cap Y_{r, s}^{n-3}=\left(Y_{i, j, r}^{n-2} \cap Y_{i, r, s}^{n-2}\right) \cap Y_{r, s}^{2 h-3}=\left(Y_{i, r}^{n-3} \cup Y_{i, j, r, s}^{n-3}\right) \cap Y_{r, s}^{n-3}=Y_{r}^{n-4} \cup Y_{i, r, s}^{n-4} \cup Y_{j, r, s}^{n-4}$.
Similarly we show that $Y_{i, j, s}^{n-2} \cap Y_{r, s}^{n-3}=Y_{s}^{n-4} \cup Y_{i, r, s}^{2 h-4} \cup Y_{j, r, s}^{n-4}$.
The strict transforms $\widetilde{Y}_{1,2}^{n-1}$ and $\widetilde{Y}_{3,4}^{n-1}$ in $X_{n-4}$ are still singular along $\widetilde{Y}_{1,2}^{n-3}$ and $\widetilde{Y}_{3,4}^{n-3}$, respectively. However, by Claim 5.12 we have

$$
\widetilde{Y}_{1,2}^{n-1} \cap \operatorname{Sing}\left(\widetilde{Y}_{3,4}^{n-1}\right)=\widetilde{Y}_{3,4}^{n-1} \cap \operatorname{Sing}\left(\widetilde{Y}_{1,2}^{n-1}\right)=\emptyset
$$

Hence, by Lemma 5.10, in $X_{n-3}, \widetilde{Y}_{1,2}^{n-1} \cap \widetilde{Y}_{3,4}^{n-1}$ is smooth, and so the intersection $\widetilde{Y}_{1,2}^{n-1} \cap \widetilde{Y}_{3,4}^{n-1}$ is transverse.

Proof of Theorem 5.9. With Notation 3.1, we have

$$
D=\Delta_{1,2} \cup \Delta_{3,4} \cup \widetilde{\sec _{h}(C)} \cup H_{5, \ldots, 2 h+3} \sim(3 h+2) H-(3 h-1)\left(E_{1}+\ldots+E_{n+3}\right)
$$

and

$$
-K_{X_{n+3}^{n}}-\epsilon D \sim(2 h+1-\epsilon(3 h+2)) H-(2 h-1-\epsilon(3 h-1))\left(E_{1}+\ldots+E_{n+3}\right) .
$$

Recall from Proposition 1.4 that the Mori cone of $X_{n+3}^{n}$ is generated by the classes $R_{i}$ 's and $L_{i, j}$ 's. One computes

$$
\left(-K_{X_{n+3}^{n}}-\epsilon D\right) \cdot R_{i}=2 h-1-\epsilon(3 h-1) \text { and }\left(-K_{X_{n+3}^{n}}-\epsilon D\right) \cdot L_{i, j}=\epsilon(3 h-4)-2 h+3 .
$$

Therefore $-K_{X_{n+3}^{n}}-\epsilon D$ is ample provided that $\frac{2 h-3}{3 h-4}<\epsilon<\frac{2 h-1}{3 h-1}$.
Next we check when the pair $\left(X_{n+3}^{n}, \epsilon D\right)$ is klt. Let $\pi: \widetilde{X}:=X_{n-3} \rightarrow X_{n+3}^{n}$ be the log resolution of $\left(X_{n+3}^{n}, \epsilon D\right)$ introduced in Proposition 5.11 above. We have

$$
\begin{aligned}
K_{\tilde{X}}= & \pi^{*} K_{X_{n+3}^{n}}+\sum_{k=1}^{h-1}(n-2 k) E_{\operatorname{Sec}_{k}(C)}+\sum_{k=1}^{h-1}(n-2 k) \sum_{i, j} E_{Y_{i, j}^{2 k-1}} \\
& +\sum_{k=1}^{h-2}(n-2 k-1)\left(\sum_{i} E_{Y_{i}^{2 k}}+\sum_{i, j, r} E_{Y_{i, j, r}^{2 k}}\right)+\sum_{k=2}^{h-2}(n-2 k) E_{Y_{1,2,3,4}^{2 k-1}} .
\end{aligned}
$$

Here we denote by $E_{Y}$ the exceptional divisor with center $Y \subset \mathbb{P}^{n}$.
In order to compute discrepancies, we will compute the the multiplicities of $\operatorname{Sec}_{h}(C)$, $Y_{1,2}^{2 h-1}$, and $Y_{3,4}^{2 h-1}$ along the images in $\mathbb{P}^{n}$ of the subvarieties blown-up by $\pi$.

We start with the divisor $Y_{i, j}^{2 h-1}$. For $1 \leq k \leq h-1$, we have:

$$
\left.\begin{array}{c}
\operatorname{mult}_{Y_{r, s}^{2 k-1}}^{2 k-1} Y_{i, j}^{2 h-1}= \begin{cases}\operatorname{mult}_{\operatorname{Sec}_{k-1}(C)} \operatorname{Sec}_{h-1}(C)=h-k+1 & \text { if }\{i, j\}=\{r, s\}, \\
\operatorname{mult}_{\operatorname{Sec}_{k}(C)} \operatorname{Sec}_{h-1}(C)=h-k & \text { if }|\{i, j\} \cap\{r, s\}|=1, \\
\operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h-1}(C)=h-k-1 & \text { if }\{i, j\} \cap\{r, s\}=\emptyset .\end{cases} \\
\operatorname{mult}_{Y_{r, s, t}^{2 k}} Y_{i, j}^{2 h-1}= \begin{cases}\operatorname{mult}_{\operatorname{Sec}_{k+2}(C)} \operatorname{Sec}_{h-1}(C)=h-k-2 & \text { if } i,\{i, j\} \cap\{r, s, t\}=\emptyset ., \\
\operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h-1}(C)=h-k-1 & \text { if }|\{i, j\} \cap\{r, s, t\}|=1, \\
\operatorname{mult}_{\operatorname{Sec}_{k}(C)} \operatorname{Sec}_{h-1}(C)=h-k & \text { if }\{i, j\} \subset\{r, s, t\} .\end{cases} \\
\operatorname{mult}_{Y_{r}^{2 k}} Y_{i, j}^{2 h-1}= \begin{cases}\operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h-1}(C)=h-k-1 & \text { if } r \notin\{i, j\}, \\
\operatorname{mult}_{\operatorname{Sec}}^{k}(C) & \operatorname{Sec} c_{h-1}(C)=h-k\end{cases} \\
\text { if } r \in\{i, j\} .
\end{array}\right\} \begin{aligned}
& \operatorname{mult}_{Y_{1,2,3,4}^{2 k-1} Y_{i, j}^{2 h-1}=\operatorname{mult}_{\operatorname{Sec}_{k}(C)} \operatorname{Sec}_{h-1}(C)=h-k .}
\end{aligned}
$$

Next we consider the divisor $\operatorname{Sec}_{h}(C)$. For $1 \leq k \leq h-1$, we have:

$$
\begin{array}{ll}
\operatorname{mult}_{\operatorname{Sec}_{k}(C)} \operatorname{Sec}_{h}(C) & =h-k+1, \\
\operatorname{mult}_{Y_{i, j}^{2 k-1}} \operatorname{Sec}_{h}(C) & =\operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h}(C)=h-k, \\
\operatorname{mult}_{Y_{i, j, r}^{2 k}} \operatorname{Sec}_{h}(C) & =\operatorname{mult}_{\operatorname{Sec}_{k+2}(C)} \operatorname{Sec}_{h}(C)=h-k-1, \\
\operatorname{mult}_{Y_{i}} \operatorname{Sec}_{h}(C) & =\operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h}(C)=h-k, \\
\operatorname{mult}_{Y_{1,2,3,4}^{2 k-1}} \operatorname{Sec}_{h}(C) & =\operatorname{mult}_{\operatorname{Sec}_{k+2}(C)} \operatorname{Sec}_{h}(C)=h-k-1 .
\end{array}
$$

Now let $\bar{D} \subset \mathbb{P}^{n}$ be the divisor whose strict transform is $D$. The above formulas yield:

$$
\begin{array}{ll}
\operatorname{mult}_{\operatorname{Sec}_{k}(C)} \bar{D} & =2(h-k)+(h-k+1)=3 h-3 k+1 \\
\operatorname{mult}_{Y_{i, j}^{2 k-1}} \bar{D} & =2(h-k)+(h-k)=(h-k+1)+(h-k-1)+(h-k)=3 h-3 k \\
\operatorname{mult}_{Y_{i, j, r}}^{2 k} & \bar{D} \\
\operatorname{mult}_{Y_{i}^{2 k}} & =(h-k-1)+(h-k)+(h-k-2)=3 h-3 k-3 \\
\operatorname{mult}_{Y_{1,2,3,4}^{2 k-1}} \bar{D} & =(h-k-1)+(h-k)+(h-k)=3 h-3 k-1 \\
& =2(h-k)+h-k-1=3 h-3 k-1
\end{array}
$$

Thus

$$
\begin{aligned}
\pi^{*}(D)= & \widetilde{D}+\sum_{k=1}^{h-1}(3 h-3 k+1) E_{\operatorname{Sec}_{k}(C)}+\sum_{k=1}^{h-1}(3 h-3 k) \sum_{i, j} E_{Y_{i, j}^{2 k-1}} \\
& +\sum_{k=1}^{h-2}(3 h-3 k-1) \sum_{i} E_{Y_{i}^{2 k}}+\sum_{k=1}^{h-2}(3 h-3 k-3) \sum_{i, j, r} E_{Y_{i, j, r}^{2 k}} \\
& +\sum_{k=2}^{h-2}(3 h-3 k-1) E_{Y_{1,2,3,4}^{2 k-1}}
\end{aligned}
$$

and hence

$$
\begin{aligned}
K_{\widetilde{X}}=\pi^{*}\left(K_{X_{n+3}^{n}}+\epsilon D\right) & +\sum_{k=1}^{h-1}(2 h-2 k-\epsilon(3 h-3 k+1)) E_{\mathbb{S e c}_{k}(C)} \\
& +\sum_{k=1}^{h-1}(2 h-2 k-\epsilon(3 h-3 k)) \sum_{i, j} E_{Y_{i, j}^{2 k-1}} \\
& +\sum_{k=1}^{h-2}(2 h-2 k-1-\epsilon(3 h-3 k-1)) \sum_{i} E_{Y_{i}^{2 k}} \\
& +\sum_{k=1}^{h-2}(2 h-2 k-1-\epsilon(3 h-3 k-3)) \sum_{i, j, r} E_{Y_{i, j, r}^{2 k}} \\
& +\sum_{k=2}^{h-2}(2 h-2 k-\epsilon(3 h-3 k-1)) E_{Y_{1,2,3,4}^{2 k-1}}-\epsilon \widetilde{D}
\end{aligned}
$$

For $\epsilon<\frac{2 h-1}{3 h-2}$ all the discrepancies are greater than -1 . Therefore, for $\frac{2 h-3}{3 h-4}<\epsilon<\frac{2 h-1}{3 h-1}$ the divisor $-K_{X_{n+3}^{n}}-\epsilon D$ is ample and the pair $\left(X_{n+3}^{n}, \epsilon D\right)$ is klt.

## 6. On a question of Hassett

In [Ha], Hassett introduced moduli spaces of weighted pointed curves. Given $g \geq 0$ and rational weight data $A[n]=\left(a_{1}, \ldots, a_{n}\right), 0<a_{i} \leq 1$, satisfying $2 g-2+\sum_{i=1}^{n} a_{i}>0$, the moduli space $\bar{M}_{g, A[n]}$ parametrizes genus $g$ nodal $n$-pointed curves $\left\{C,\left(x_{1}, \ldots, x_{n}\right)\right\}$ subject to the following stability conditions:

- Each $x_{i}$ is a smooth point of $C$, and the points $x_{i_{1}}, \ldots, x_{i_{k}}$ are allowed to coincide only if $\sum_{j=1}^{k} a_{i_{j}} \leq 1$.
- The twisted dualizing sheaf $\omega_{C}\left(a_{1} x_{1}+\cdots+a_{n} x_{n}\right)$ is ample.

In particular, $\bar{M}_{g, A[n]}$ is a compactification of the moduli space $M_{g, n}$ of genus $g$ smooth $n$-pointed curves. The irreducible components of the boundary divisor $\bar{M}_{g, A[n]} \backslash M_{g, n}$ are well understood. In the special case when $g=0$, they are described as follows. Consider a partition $I \cup J=\{1, \ldots, n\}$ such that one of the following holds.
$-I=\left\{i_{1}, \ldots, i_{r}\right\}, J=\left\{j_{1}, \ldots, j_{n-r}\right\}$, with $r, n-r \geq 2, a_{i_{1}}+\ldots+a_{i_{r}}>1$ and $a_{j_{1}}+\ldots+$
$a_{j_{n-r}}>1$.

- $I=\left\{i_{1}, i_{2}\right\}$ and $i_{1}+i_{2} \leq 1$.

In the first case, there is a prime divisor $D_{I, J}(A)$ in $\bar{M}_{0, A[n]}$ whose general point corresponds to a nodal curve with two irreducible components, having marked points $x_{i_{1}}, \ldots, x_{i_{r}}$ on one component, and $x_{j_{1}}, \ldots, x_{j_{n-r}}$ on the other component. In the latter case, there is a
prime divisor $D_{I, J}(A)$ in $\bar{M}_{0, A[n]}$ parametrizing curves where the marked points $x_{i_{1}}$ and $x_{i_{2}}$ coincide. These are precisely the boundary divisors of $\bar{M}_{g, A[n]}$.
6.1 ([Ha, Section 4]). For fixed $g$ and $n$, given two collections of rational weight data $A[n]$ and $B[n]$ such that $a_{i} \geq b_{i}$ for any $i=1, \ldots, n$, there exists a birational reduction morphism

$$
\rho_{B[n], A[n]}: \bar{M}_{g, A[n]} \rightarrow \bar{M}_{g, B[n]} .
$$

This morphism associates to a curve $\left[C, x_{1}, \ldots, x_{n}\right] \in \bar{M}_{g, A[n]}$ the pointed curve obtained by collapsing components of $C$ along which $\omega_{C}\left(b_{1} x_{1}+\ldots+b_{n} x_{n}\right)$ fails to be ample.
Example 6.2 ( Ha , Sections 6.1 and 6.2]). Consider the weight data

$$
\begin{aligned}
& A_{0}[n]=(1 /(n-2), \ldots, 1 /(n-2), 1), \\
& A_{1}[n]=(1 /(n-3), \ldots, 1 /(n-3), 1) \\
& A_{1,2}[n]=(1 /(n-2), \ldots, 1 /(n-2), 2 /(n-2), 1) .
\end{aligned}
$$

Then we have $\bar{M}_{0, A_{0}[n]} \cong \mathbb{P}^{n-3}, \bar{M}_{0, A_{1}[n]} \cong X_{n-1}^{n-3}=B l_{p_{1}, \ldots, p_{n-1}} \mathbb{P}^{n-3}$ and $\bar{M}_{0, A_{1,2}[n]} \cong$ $X_{n-2}^{n-3}=B l_{p_{1}, \ldots, p_{n-2}} \mathbb{P}^{n-3}$. The reduction morphisms $\rho_{A_{1,2}[n], A_{1}[n]}: X_{n-1}^{n-3} \rightarrow X_{n-2}^{n-3}$ and $\rho_{A_{0}[n], A_{1}[n]}: X_{n-1}^{n-3} \rightarrow \mathbb{P}^{n-3}$ are the natural blow-up morphisms.

Let us describe some of the boundary divisors of $\bar{M}_{0, A_{1}[n]}$ under the blowup morphism $\rho: X_{n-1}^{n-3} \rightarrow \mathbb{P}^{n-3}$. There are $(n-1)$ partitions of type $I=\{\hat{\imath}, n\}, J=\{1, \ldots, \hat{\imath}, \ldots, n-1\}$. The corresponding $(n-1)$ divisors $D_{I, J}$ are the $(n-1)$ exceptional divisors of the blowup. There are $\binom{n-1}{2}$ partitions of type $I=\left\{\hat{\imath}_{1}, \hat{\imath}_{2}\right\}, J=\left\{1, \ldots, \hat{\imath}_{1}, \ldots, \hat{\imath}_{2}, \ldots, n-1\right\} \cup\{n\}$. The corresponding $\binom{n-1}{2}$ divisors $D_{I, J}$ are the strict transforms of the $\binom{n-1}{2}$ hyperplanes spanned subsets of cardinality $n-3$ of $\left\{p_{1}, \ldots, p_{n-1}\right\}$.

In Ha Hassett proposed the following problem.
Problem 6.3 ([Ha, Problem 7.1]). Let $A[n]$ be a vector of weights and consider the moduli space $\bar{M}_{0, A[n]}$. Do there exist rational numbers $\alpha_{I, J}$ such that

$$
K_{\bar{M}_{0, A[n]}}+\sum_{I, J} \alpha_{I, J} D_{I, J}(A)
$$

is ample and the pair $\left(\bar{M}_{0, A[n]}, \sum_{I, J} \alpha_{I, J} D_{I, J}(A)\right)$ is log canonical?
In Ha, Sections 7.1, 7.2, 7.3, Remark 8.5] Hassett gives examples in which Problem 6.3 admits a positive answer. The techniques developed in this paper allow us to give some more examples.
Proposition 6.4. For the moduli space $\bar{M}_{0, A_{1}[n]}$, Problem 6.3 admits a positive answer.
Proof. Consider the blow-up $\rho: \bar{M}_{0, A_{1}[n]} \cong X_{n-1}^{n-3} \rightarrow \mathbb{P}^{n-3}$ described in Example 6.2, We denote by $H$ the pullback of the hyperplane class of $\mathbb{P}^{n-3}$. Let $E_{1}, \ldots, E_{n-1}$ be the exceptional divisors, and $H_{i_{1}, \ldots, i_{n-3}}$ be the strict transform of the hyperplane $\left\langle p_{i_{1}}, \ldots, p_{i_{n-3}}\right\rangle$, where $1 \leq i_{j} \leq n-1$. Then

$$
K_{\bar{M}_{0, A_{1}[n]}}=-(n-2) H+(n-4)\left(E_{1}+\ldots+E_{n-1}\right)
$$

and

$$
H_{i_{1}, \ldots, i_{n-3}} \sim H-E_{i_{1}}-\ldots-E_{i_{n-3}} .
$$

Recall from Example 6.2 that the $E_{i}$ 's and the $H_{i_{1}, \ldots, i_{n-3}}$ 's are boundary divisors of $\bar{M}_{0, A_{1}[n]}$. So we set

$$
\Delta=\alpha\left(H_{1, \ldots, n-3}+\ldots+H_{3, \ldots, n-1}\right)+\beta\left(E_{1}+\ldots+E_{n-1}\right)
$$

where $\alpha$ and $\beta$ are positive numbers to be chosen. Then

$$
K_{\bar{M}_{0, A_{1}[n]}}+\Delta=\left(\alpha\binom{n-1}{2}-n+2\right) H-\left(\alpha\binom{n-2}{2}-n-\beta+4\right) \sum_{i=1}^{n-1} E_{i} .
$$

Recall from Proposition 1.4 that the Mori cone of $X_{n-1}^{n-3} \cong \bar{M}_{0, A_{1}[n]}$ is generated by the classes $R_{i}$ 's and $L_{i, j}$ 's introduced in Section 3.1. One computes:
$\left(K_{\bar{M}_{0, A_{1}[n]}}+\Delta\right) \cdot R_{i}=\frac{\alpha}{2}(n-2)(n-3)-n-\beta+4$ and $\left(K_{\bar{M}_{0, A_{1}[n]}}+\Delta\right) \cdot L_{i, j}=\frac{\alpha}{2}(n-2)(5-n)+2 \beta+n-6$.
Therefore $K_{\bar{M}_{0, A_{1}[n]}}+\Delta$ is ample for $\alpha=\frac{2}{n-2}$ and $\beta=\frac{2}{3}$.
Next we check that the pair $\left(\bar{M}_{0, A_{1}[n]}, \Delta\right)$ is $\log$ canonical. Let $\bar{\rho}: Y=\bar{M}_{0, n} \rightarrow \bar{M}_{0, A_{1}[n]}$ be the composition of blow-ups introduced in Notation 4.1. It is also a reduction morphism (see [Ha, Section 6.1]). By Proposition 4.6, the morphism $\bar{\rho}$ is a $\log$ resolution of the pair $\left(\bar{M}_{0, A_{1}[n]}, \Delta\right)$.

There are $\rho_{h}=\binom{n-1}{h+1} h$-planes spanned by subsets of cardinality $h+1$ of $\left\{p_{1}, \ldots, p_{n-1}\right\}$. Each such $h$-plane is contained in $\binom{n-h-2}{n-h-4}$ of the $H_{i_{1}, \ldots, i_{n-3}}$ 's. Denote by $E_{j}^{h} \subset, j=1, \ldots, \rho_{h}$, the exceptional divisors over the $h$-planes. Then we have

$$
K_{Y}=\bar{\rho}^{*} K_{\bar{M}_{0, A_{1}[n]}}+\sum_{h=1}^{n-5}(n-h-4)\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right)
$$

and

$$
\bar{\rho}^{*}(\Delta) \sim \sum_{h=1}^{n-5} \alpha\binom{n-h-2}{2}\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right)+\alpha \sum_{i_{1}, \ldots, i_{n-3}} \widetilde{H}_{i_{1}, \ldots, i_{n-3}}+\beta \sum_{i} \widetilde{E}_{i} .
$$

Thus

$$
K_{Y}+\widetilde{\Delta}=\bar{\rho}^{*}\left(K_{\bar{M}_{0, A_{1}[n]}}+\Delta\right)+\sum_{h=1}^{n-5}\left(n-h-4-\alpha\binom{n-h-2}{2}\right)\left(E_{1}^{h}+\ldots+E_{\rho_{h}}^{h}\right) .
$$

For $\alpha=\frac{2}{n-2}$ and $\beta=\frac{2}{3}$ all the discrepancies are greater than -1 . Therefore the pair $\left(\bar{M}_{0, A_{1}[n]}, \Delta\right)$ is log canonical.

Proposition 6.5. For the moduli space $\bar{M}_{0, A_{1,2}[n]}$, Problem 6.3 admits a positive answer.
Proof. Consider the blow-up $\rho: \bar{M}_{0, A_{1,2}[n]} \cong X_{n-2}^{n-3} \rightarrow \mathbb{P}^{n-3}$ described in Example 6.2, We denote by $H$ the pullback of the hyperplane class of $\mathbb{P}^{n-3}$. The prime divisors $D_{I, J}$ appearing in $\Delta$ will be the following:

- the ( $n-2$ ) exceptional divisors $E_{1}, \ldots, E_{n-2}$,
- the strict transforms $H_{i_{1}, \ldots, i_{n-3}}$ of the ( $n-2$ ) hyperplanes spanned by subsets of cardinality $(n-3)$ of $\left\{p_{1}, \ldots, p_{n-2}\right\}\left(H_{i_{1}, \ldots, i_{n-3}} \sim H-E_{i_{1}}-\ldots-E_{i_{n-3}}\right)$,
- the strict transforms $\Lambda_{j_{1}, \ldots, j_{n-4}}$ of the $\binom{n-2}{2}$ hyperplanes spanned by subsets of cardinality $(n-4)$ of $\left\{p_{1}, \ldots, p_{n-2}\right\}$ and $p_{n-1}\left(\Lambda_{j_{1}, \ldots, j_{n-4}} \sim H-E_{j_{1}}-\ldots-E_{j_{n-4}}\right)$.

Set

$$
\Delta=\frac{2}{n-2} \sum_{i_{1}, \ldots, i_{n-3}} H_{i_{1}, \ldots, i_{n-3}}+\frac{2}{n-2} \sum_{j_{1}, \ldots, j_{n-4}} \Lambda_{j_{1}, \ldots, j_{n-4}}+\frac{2}{3} \sum_{i=1}^{n-2} E_{i}
$$

Each $p_{i}, i=1, \ldots, n-2$, lies in exactly $(n-3)$ of the $H_{i_{1}, \ldots, i_{n-3}}$ 's, and $\binom{n-3}{2}$ of the $\Lambda_{i_{1}, \ldots, i_{n-3}}$ 's. So we have

$$
\Delta \sim(n-1) H+\left(\frac{2}{3}-\frac{2(n-3)}{n-2}-\frac{2}{n-2}\binom{n-3}{2}\right) \sum_{i=1}^{n-2} E_{i}=(n-1) H-\frac{3 n-11}{3} \sum_{i=1}^{n-2} E_{i}
$$

and

$$
K_{\bar{M}_{0, A_{1,2}[n]}}+\Delta=(-n+2+n-1) H+\left(n-4+\frac{11-3 n}{3}\right) \sum_{i=1}^{n-2} E_{i}=H-\frac{1}{3} \sum_{i=1}^{n-2} E_{i} .
$$

Recall from Proposition 1.4 that the Mori cone of $X_{n-2}^{n-3} \cong \bar{M}_{0, A_{1,2}[n]}$ is generated by the classes $R_{i}$ 's and $L_{i, j}$ 's introduced in Section 3.1. One computes:

$$
\left(K_{\bar{M}_{0, A_{1,2}[n]}}+\Delta\right) \cdot R_{i}=\left(K_{\bar{M}_{0, A_{1,2}[n]}}+\Delta\right) \cdot L_{i, j}=\frac{1}{3} .
$$

Therefore $K_{\bar{M}_{0, A_{1,2}[n]}}+\Delta$ is ample.
Next we check that the pair $\left(\bar{M}_{0, A_{1,2}[n]}, \Delta\right)$ is log canonical. Let $\pi_{n-1}: X_{n-1}^{n-3} \rightarrow X_{n-2}^{n-3}$ be the blow-up of $p_{n-1}$ and consider the composition

$$
Y \xlongequal[\tilde{\rho}]{\bar{\rho}} X_{n-1}^{n-3}=\bar{M}_{0, A_{1}[n]} \xrightarrow{\pi_{n-1}} X_{n-2}^{n-3}=\bar{M}_{0, A_{1,2}[n]},
$$

where $\bar{\rho}$ is the $\log$ resolution used in the proof of Proposition 6.4. Then $\widetilde{\rho}$ is a $\log$ resolution of the pair $\left(\bar{M}_{0, A_{1,2}[n]}, D\right)$. Let $E_{n-1}$ be the exceptional divisor over $p_{n-1}$. There are $\gamma_{h}=\binom{n-2}{h+1} h$-planes spanned by subsets of cardinality $h+1$ of $\left\{p_{1}, \ldots, p_{n-2}\right\}$. We denote by $E_{j}^{h}, 1 \leq j \leq \gamma_{h}$, the exceptional divisors over these $h$-planes. Similarly, there are $\bar{\gamma}_{h}=\binom{n-2}{h}$ $h$-planes spanned by $p_{n-1}$ and subsets of cardinality $h$ of $\left\{p_{1}, \ldots, p_{n-2}\right\}$. We denote by $\bar{E}_{j}^{h}$, $1 \leq j \leq \bar{\gamma}_{h}$, the exceptional divisors over these $h$-planes. Note that

- the point $p_{n-1}$ is contained all of the $\binom{n-2}{2} \Lambda_{j_{1}, \ldots, j_{n-4}}$ 's,
- any $h$-plane spanned by subsets of cardinality $h+1$ of $\left\{p_{1}, \ldots, p_{n-2}\right\}$ is contained in $n-h-3$ of the $H_{i_{1}, \ldots, i_{n-3}}$ 's and in $\binom{n-h-3}{2}$ of the $\Lambda_{j_{1}, \ldots, j_{n-4}}$ 's,
- any $h$-plane spanned by $p_{n-1}$ and subsets of cardinality $h$ of $\left\{p_{1}, \ldots, p_{n-2}\right\}$ is contained in $\binom{n-h-2}{2}$ of the $\Lambda_{j_{1}, \ldots, j_{n-4}}$ 's.
Therefore, we have

$$
\begin{aligned}
\widetilde{\rho}^{*}(\Delta)= & \frac{2}{n-2}\binom{n-2}{2} E_{n-1}+\frac{2}{n-2} \sum_{h=1}^{n-5}\left(n-h-3+\binom{n-h-3}{2}\right)\left(E_{1}^{h}+\ldots+E_{\gamma_{h}}^{h}\right)+ \\
& \frac{2}{n-2} \sum_{h=1}^{n-5}\binom{n-h-2}{2}\left(\bar{E}_{1}^{h}+\ldots+\bar{E}_{\bar{\gamma}_{h}}^{h}\right)+\widetilde{\Delta} .
\end{aligned}
$$

Since

$$
K_{Y}=\widetilde{\rho}^{*} K_{\bar{M}_{0, A_{1,2}[n]}}+(n-4) E_{n-1}+\sum_{h=1}^{n-5}(n-h-4)\left(E_{1}^{h}+\ldots+E_{\gamma_{h}}^{h}+\bar{E}_{1}^{h}+\ldots+\bar{E}_{\gamma_{h}}^{h}\right)
$$

we have

$$
\begin{aligned}
K_{Y}+\widetilde{\Delta}= & \widetilde{\rho}^{*}\left(K_{\bar{M}_{0, A_{1,2}[n]}}+\Delta\right)+\left(n-4-\frac{2}{n-2}\binom{n-2}{2}\right) E_{n-1}+ \\
& \sum_{h=1}^{n-5}\left(n-h-4-\frac{2}{n-2}\left(n-h-3+\binom{n-h-3}{2}\right)\right)\left(E_{1}^{h}+\ldots+E_{\gamma_{h}}^{h}\right)+ \\
& \sum_{h=1}^{n-5}\left(n-h-4-\frac{2}{n-2}\binom{n-h-2}{2}\right)\left(\bar{E}_{1}^{h}+\ldots+\bar{E}_{\bar{\gamma}_{h}}^{h}\right) .
\end{aligned}
$$

The discrepancies are all $\geq-1$, and hence the pair $\left(\bar{M}_{0, A_{1,2}[n]}, \Delta\right)$ is log canonical.
Remark 6.6. Any 3-dimensional Hassett's space $\bar{M}_{0, A[6]}$ admits a reduction morphism $\rho: \bar{M}_{0,6} \rightarrow \bar{M}_{0, A[6]}$ (Ha, Theorem 4.1]). The moduli space $\bar{M}_{0,6}$ is log Fano by [HK00]. So, by GOST, Corollary 1.3], $\bar{M}_{0, A[6]}$ is also log Fano. Examples of 3-dimensional Hassett's spaces are the following.

- The blow-up of $\mathbb{P}^{3}$ in four general points, along the strict transforms of the lines spanned by them, and in a fifth general point. This variety corresponds to $A[6]=$ ( $1 / 3,1 / 3,1 / 3,1 / 3,1,1$ ).
- The blow-up of $\mathbb{P}^{3}$ in five general points, and along the strict transforms of the lines spanned by them. This is $\bar{M}_{0,6}$ itself.
- The blow-up $X_{1}$ of $\mathbb{P}_{1}^{1} \times \mathbb{P}_{2}^{1} \times \mathbb{P}_{3}^{1}$ in $p_{1}=([0: 1],[0: 1],[0: 1]), p_{2}=([1: 0],[1:$ $0],[1: 0])$, and $p_{3}=([1: 1],[1: 1],[1: 1])$. This variety corresponds to $A_{1}[6]=$ (2/3,2/3, 2/3, 1/6, 1/6, 1/6) (see [Ha, Section 6.3]).
- Consider the projections $\pi_{i}: \mathbb{P}_{1}^{1} \times \mathbb{P}_{2}^{1} \times \mathbb{P}_{3}^{1} \rightarrow \mathbb{P}_{i}^{1}$, and set $F_{0}=\bigcup_{i=1}^{3} \pi_{i}^{-1}([0: 1]), F_{1}=$ $\bigcup_{i=1}^{3} \pi_{i}^{-1}([1: 0]), F_{\infty}=\bigcup_{i=1}^{3} \pi_{i}^{-1}([1: 1])$. Let $\Delta_{2}$ be the union of the 2-dimensional diagonals of $\mathbb{P}_{1}^{1} \times \mathbb{P}_{2}^{1} \times \mathbb{P}_{3}^{1}$. Let $X_{2}$ be the blow-up of $X_{1}$ along the strict transform of $\Delta_{2} \cap\left(F_{0} \cup F_{1} \cup F_{\infty}\right)$. This variety corresponds to $A_{2}[6]=(2 / 3,2 / 3,2 / 3,1 / 3,1 / 3,1 / 3)$ (see [Ha, Section 6.3]).
- The blow-up $X_{3}$ of $X_{2}$ along the strict transform of the 1-dimension diagonal $\Delta_{1}$ of $\mathbb{P}_{1}^{1} \times \mathbb{P}_{2}^{1} \times \mathbb{P}_{3}^{1}$. This is $\bar{M}_{0,6}$ (see [Ha, Section 6.3]).
Let $q_{1}=([1: 0], \ldots,[1: 0]), q_{2}=([0: 1], \ldots,[0: 1]), q_{3}=([1: 1], \ldots,[1: 1]) \in$ $\left(\mathbb{P}^{1}\right)^{n-3}$, and set $Y_{3}^{n-3}=B l_{q_{1}, q_{2}, q_{3}}\left(\mathbb{P}^{1}\right)^{n-3}$. By Ha, Section 6.3], $Y_{3}^{n-3} \cong \bar{M}_{0, A[n]}$ for $A[n]=\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3(n-4)}, \ldots, \frac{1}{3(n-4)}\right)$.
Proposition 6.7. Then there exists a small birational modification

$$
X_{n-1}^{n-3}-\rightarrow Y_{3}^{n-3} .
$$

In particular, $Y_{3}^{n-3}$ is log Fano.
Proof. Note that the Picard numbers satisfy $\rho\left(X_{n-1}^{n-3}\right)=\rho\left(Y_{3}^{n-3}\right)=n$. Without lost of generality, we may assume that $X_{n-1}^{n-3}$ is the blow-up of $\mathbb{P}^{n-3}$ at the points $p_{1}=[1: 0:$ $\ldots: 0], p_{2}=[0: 1: \ldots: 0], \ldots, p_{n-2}=[0: \ldots: 0: 1]$ and $p_{n-1}=[1: 1: \ldots: 1]$. Set $X_{n-2}^{n-3}=B l_{p_{1}, \ldots, p_{n-2}} \mathbb{P}^{n-3}, X_{n-3}^{n-3}=B l_{p_{1}, \ldots, p_{n-3}} \mathbb{P}^{n-3}, Y_{2}^{n-3}=B l_{q_{1}, q_{2}}\left(\mathbb{P}^{1}\right)^{n-3}$ and and $Y_{1}^{n-3}=B l_{q_{1}}\left(\mathbb{P}^{1}\right)^{n-3}$. These are all toric varieties. Let $e_{1}, \ldots, e_{n-3}$ be the standard basis
vectors of the co-character lattice of $\left(k^{*}\right)^{n-3}$. The rays of the fan of $\mathbb{P}^{n-3}$ are $e_{1}, \ldots, e_{n-3}$ and $-e_{1}-\ldots-e_{n-3}$. By blowing-up $p_{1}, \ldots, p_{n-2}$ we add the rays $-e_{1}, \ldots,-e_{n-3}$ and $e_{1}+\ldots+e_{n-3}$. On the other hand the rays of $\left(\mathbb{P}^{1}\right)^{n-3}$ are $e_{1}, \ldots, e_{n-3},-e_{1}, \ldots,-e_{n-3}$, and blowing-up $q_{1}, q_{2}$ corresponds to introducing the two rays $e_{1}+\ldots+e_{n-3}$ and $-e_{1}-\ldots-e_{n-3}$. So the fans of $X_{n-2}^{n-3}$ and $Y_{2}^{n-3}$ have the same 1-dimensional rays. Therefore, $X_{n-2}^{n-3}$ and $Y_{2}^{n-3}$ are isomorphic in codimension one.

Given $1 \leq i_{1}<\ldots<i_{n-4} \leq n-3$, set $H_{i_{1}, \ldots, i_{n-4}}^{n-5}=\left\langle p_{i_{1}}, \ldots, p_{i_{n-4}}\right\rangle$, and $\left\{j_{1}, j_{2}\right\}=$ $\{0, \ldots, n-3\} \backslash\left\{i_{1}-1, \ldots, i_{n-4}-1\right\}$. The projection from $H_{i_{1}, \ldots, i_{n-4}}^{n-5}$ is the rational map

$$
\begin{array}{cccc}
\pi_{i_{1}, \ldots, i_{n-4}}: & \mathbb{P}^{n-3} & \rightarrow & \mathbb{P}^{1} \\
{\left[x_{0}: \ldots: x_{n-3}\right]} & \mapsto & {\left[x_{j_{1}}: x_{j_{2}}\right] .}
\end{array}
$$

There are $(n-3)$ of those, inducing a rational map

$$
\begin{array}{cccc}
g: & \mathbb{P}^{n-3} & -- & \left(\mathbb{P}^{1}\right)^{n-3} \\
& x=\left[x_{0}: \ldots: x_{n-3}\right] & \mapsto & \left(\pi_{1, \ldots, n-4}(x), \ldots, \pi_{2, \ldots, n-3}(x)\right)
\end{array}
$$

The hyperplane $W=\left\langle p_{1}, \ldots, p_{n-3}\right\rangle=\left\{x_{n-3}=0\right\}$ is mapped to the point $q_{1} \in\left(\mathbb{P}^{1}\right)^{n-3}$ by $g$. This is the only divisor contracted by $g$. Therefore, by blowing-up $q_{1} \in\left(\mathbb{P}^{1}\right)^{n-3}$ we obtain a small transformation $g_{1}: X_{n-3}^{n-3} \longrightarrow Y_{1}^{n-3}$ fitting in the following diagram:


Note that $g_{1}$ maps the strict transform $\widetilde{W}$ of $W$ to the exceptional divisor $E_{q_{1}}$, while the exceptional divisors $E_{p_{1}}, \ldots, E_{p_{n-3}}$ are mapped to the strict transforms of the $(n-3)$ divisors in $\left(\mathbb{P}^{1}\right)^{n-3}$ obtained by fixing one of the factors. Note also that $g([0: \ldots: 0: 1])=([0:$ $1], \ldots,[0: 1])$ and $g([1: \ldots: 1])=([1: 1], \ldots,[1: 1])$. It follows from the universal property of the blow-up that $g_{1}$ lifts to a small modification $f: X_{n-1}^{n-3} \rightarrow Y_{3}^{n-3}$ mapping $E_{p_{n-2}}$ to $E_{q_{2}}$, and $E_{p_{n-3}}$ to $E_{q_{3}}$.

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