EXPLICIT LOG FANO STRUCTURES ON BLOW-UPS OF PROJECTIVE SPACES

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ABSTRACT. In this paper we determine which blow-ups X of \mathbb{P}^n at general points are log Fano, that is, when there exists an effective \mathbb{Q} -divisor Δ such that $-(K_X + \Delta)$ is ample and the pair (X, Δ) is klt. For these blow-ups, we produce explicit boundary divisors Δ making X log Fano.

CONTENTS

1.	Introduction	1
2.	Mori Dream Spaces	4
3.	Cones of curves and divisors on blow-ups of \mathbb{P}^n at general points	6
4.	Finding explicit divisors making X_k^n log Fano for $k \le n+2$	11
5.	Finding explicit divisors making X_{n+3}^n log Fano	13
6.	On a question of Hassett	24
References		29

1. INTRODUCTION

In this paper we investigate special properties of blow-ups of complex projective spaces at general points. These varieties appear frequently in algebraic geometry, for example in moduli problem. In general, the more points are blown-up, the more complicated the resulting variety is. For positive integers n and k, denote by X_k^n a blow-up of \mathbb{P}^n at k points in general position. The 2-dimensional case has been understood classically. For $k \leq 8$, $S = X_k^2$ is a del Pezzo surface (i.e. $-K_S$ is ample), and its geometry can be completely described in terms of some finite data. For $k \geq 9$, the situation changes drastically. The anti-canonical class of S is no longer big, and S contains infinitely many (-1)-curves.

For $n \geq 3$, X_1^n is a Fano manifold (i.e., $-K_{X_1^n}$ is ample), but as soon as $k \geq 2$, X_k^n is no longer Fano. However, for small values of k, the blow-up X_k^n behaves like a Fano manifold. A more appropriate notion here is that of a log Fano variety.

Definition 1.1. Let X be a normal projective Q-factorial variety. We say that X is log Fano if there exists an effective Q-divisor Δ such that $-(K_X + \Delta)$ is ample, and the pair (X, Δ) is klt. (See for instance [Ko, Definition 3.5] for the notion of klt singularities).

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Log Fano varieties play an important role in the classification of algebraic varieties. It was proved in [BCHM, Corollary 1.3.2] that they are special instances of *Mori dream spaces*. We refer to Section 2 and references therein for the definition and special properties of Mori dream spaces. Here we just vaguely remark that a Mori dream space X behaves very well with respect to the minimal model program. Moreover, the birational geometry of X can be encoded in some finite data, namely its cone of effective divisors Eff(X) together with a chamber decomposition on it.

In this paper we address the following problems:

For which values of n and k is X_k^n a log Fano variety? In those cases, can we find an explicit \mathbb{Q} -divisor Δ making X_k^n log Fano?

In the second question, the expected properties of Δ depend on the context. In some cases, one would like Δ to be irreducible. On the other hand, when X_k^n appears as a compactification of some moduli space, it is often desirable that Δ is supported on the boundary divisor.

Example 1.2. Let us consider the case n = 3. By Proposition 1.4 below, if $k \leq 8$, then the Mori cone of $X = X_k^3$ is generated by classes of lines R_i in the exceptional divisors, $1 \leq i \leq k$, and strict transforms $L_{i,j}$ of the lines through two of the blown-up points. Using this result, one checks easily that $-K_X$ is nef for $k \leq 8$. Moreover, by computing the top intersection $(-K_X)^3$, one concludes that $-K_X$ is big if and only if $k \leq 7$. Projective manifolds with nef and big anti-canonical class are called *weak Fano*. The fact that X_k^3 is weak Fano if and only if $k \leq 7$ has been proven, with slightly different techniques, in [BL, Proposition 2.9]. By Lemma 2.5, weak Fano manifolds are log Fano. On the other hand, log Fano varieties have big anti-canonical class. So we conclude that X is log Fano if and only if $k \leq 7$.

When $k \leq 4$, X_k^3 is a toric variety and one can take Δ to be a suitable combination of toric invariant divisors. Alternatively, we may choose $\Delta = \epsilon D$ to be irreducible. We describe such irreducible Δ when k = 4. We may assume that the blown-up points are the fundamental points of \mathbb{P}^3 . Let $D \subset X_4^3$ be the strict transform of the Cayley nodal cubic surface

$$S = \{x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0\} \subset \mathbb{P}^3.$$

The surface S has ordinary double points at the fundamental points of \mathbb{P}^3 , and is smooth elsewhere. Thus D is smooth and $D \sim 3H - 2(E_1 + ... + E_4)$. One computes that

$$(-K_{X_4^3} - \epsilon D) \cdot R_i = 2 - 2\epsilon$$
 and $(-K_{X_4^3} - \epsilon D) \cdot L_{i,j} = 4\epsilon.$

Thus $(X, \epsilon D)$ is klt and $-(K_{X_4^3} + \epsilon D)$ is ample for any $0 < \epsilon < 1$.

When k = 5, let $H_{i,j,k} \subset \mathbb{P}^3$ be the plane spanned by three of the blown-up points p_i, p_j and p_k , and take D to be the strict transform of $\sum_{i,j,k} H_{i,j,k}$. Then $D \sim 10H - 6(E_1 + ... + E_5)$, and one computes

$$-(K_{X_5^3} - \epsilon D) \cdot R_i = 2 - 6\epsilon \text{ and } -(K_{X_5^3} - \epsilon D) \cdot L_{i,j} = 2\epsilon.$$

So $-(K_{X_5^3} + \epsilon D)$ is ample for any $0 < \epsilon < \frac{1}{3}$. Furthermore, we can take $\epsilon > 0$ sufficiently small so that the pair $(X_5^3, \epsilon D)$ is klt.

When k = 6, let $Q_i \subset \mathbb{P}^3$ be the unique irreducible quadric cone through the 6 blown-up points, with vertex at one of the points p_i . Let D be the strict transform of $Q_1 + Q_2 + Q_3 + Q_3 + Q_4$

 $H_{4,5,6}$ in X_6^3 . (As before, $H_{4,5,6} \subset \mathbb{P}^3$ denotes the plane spanned by p_4 , p_5 and p_6 .) Then $D \sim 7H - 4(E_1 + \ldots + E_6)$, and one computes

$$-(K_{X_{\epsilon}^3} + \epsilon D) \cdot R_i = 2 - 4\epsilon$$
 and $-(K_{X_{\epsilon}^3} + \epsilon D) \cdot R_i = \epsilon.$

So $-(K_{X_6^3} + \epsilon D)$ is ample for any $0 < \epsilon < \frac{1}{2}$. Furthermore we can take $\epsilon > 0$ sufficiently small so that the pair $(X_6^3, \epsilon D)$ is klt.

Suppose now that k = 7. For each triple $1 \le i, j, k \le 7$, consider the linear system of cubics defining the standard Cremona transformation of \mathbb{P}^3 centered at the four points p_h , $h \ne i, j, k$. There exists an unique irreducible cubic surface $S_{i,j,k} \subset \mathbb{P}^3$ in this linear system passing through p_i, p_j and p_k . It is smooth at p_i, p_j and p_k , and has a double point at p_h for $h \ne i, j, k$. Its strict transform in X_7^3 is a rigid surface. Let D be the strict transform of $\sum_{i,j,k} S_{i,j,k}$ in X_7^3 . Then $D \sim 105H - 55(E_1 + ... + E_7)$, and one computes

$$-(K_{X_{\tau}^3} + \epsilon D) \cdot R_i = 2 - 55\epsilon$$
 and $(K_{X_{\tau}^3} + \epsilon D) \cdot L_{i,j} = 5\epsilon.$

So $-(K_{X_5^3} + \epsilon D)$ is ample for any $0 < \epsilon < \frac{2}{55}$. Furthermore we can take $\epsilon > 0$ sufficiently small so that the pair $(X_7^3, \epsilon D)$ is klt.

For $n \ge 4$, we can approach the first question using Mori dream spaces. The main results of [Muk01] and [CT06] put together show that X_k^n is a Mori dream space if and only if one of the following holds:

- n = 4 and $k \leq 8$.
- n > 4 and $k \le n + 3$.

Using Mukai's description of the geometry of the boundary cases X_8^4 and X_{n+3}^n , see Section 3.2, we answer the first question.

Theorem 1.3. Let X_k^n be a blow-up of \mathbb{P}^n at k points in general position, with $n \ge 2$ and $k \ge 0$. Then X_k^n is log Fano if and only if one of the following holds:

-
$$n = 2$$
 and $k \le 8$,
- $n = 3$ and $k \le 7$,
- $n = 4$ and $k \le 8$,
- $n > 4$ and $k \le n + 3$.

The proof of Theorem 1.3, which may already have been known to experts, does not give any hint on which \mathbb{Q} -divisor Δ makes X_k^n log Fano. So we proceed to find such explicit \mathbb{Q} -divisor Δ . The first step is to determine the Mori cone of X_k^n .

Proposition 1.4. Let X_k^n be the blow-up of \mathbb{P}^n at points in general position p_1, \ldots, p_k , $n \geq 2$. Denote by R_i the class of a line in the exceptional divisors over p_i , and by $L_{i,j}$ the class of the strict transforms of the line through two distinct points p_i and p_j . Suppose that either of the following holds:

- $k \le 2n$. - n = 3 and $k \le 8$.

Then the Mori cone $\overline{NE}(X_k^n)$ is generated by the R_i 's and $L_{i,j}$'s.

Using Proposition 1.4, it is not hard to find a \mathbb{Q} -divisor Δ such that $-(K_{X_k^n} + \Delta)$ is ample. We often choose Δ as linear combinations of extremal divisors in X_k^n . The hard part is to show that for such divisors (X_k^n, Δ) is klt. We do so by providing explicit log resolutions for these pairs, and computing discrepancies. Explicit \mathbb{Q} -divisors Δ making X_k^n log Fano are given in Theorems 4.3, 4.5, 5.7 and 5.9. In particular, they provide a new proof that these varieties are Mori dream spaces.

Some blow-ups of projective spaces at points (and, more generally, linear spaces) appear as moduli spaces $\overline{M}_{g,A[n]}$ of weighted pointed stable curves. These spaces were introduced and investigated by Hassett in [Ha]. In [Ha, Problem 7.1], Hassett asks whether there is an effective Q-divisor Δ on $\overline{M}_{g,A[n]}$, supported on the boundary, such that $(\overline{M}_{g,A[n]}, \Delta)$ is log canonical, and $K_{\overline{M}_{g,A[n]}} + \Delta$ is ample. We end the paper by addressing this question.

The paper is organized as follows. In Section 2, we recall the definition and some special properties of Mori dream spaces. In Section 3 we review the description from [CT06] of the cone of effective divisors of X_k^n , and make explicit the description of its Mori chamber decomposition proposed in [Muk05]. We end this section by proving Theorem 1.3. In Section 4, we exhibit an integral divisor $D \subset X_k^n$ and rational number $\epsilon > 0$ such that $\Delta = \epsilon D$ makes X_k^n log Fano for $k \leq n+2$. This task is relatively easy, and serves as warm up for the next case n = k + 3, treated in Section 5. For X_{n+3}^n , we construct D from joins of suitable linear spaces and higher secant varieties of the unique rational normal curve through the blown-up points. In order to construct an explicit log resolution for the resulting pair (X_{n+3}^n, Δ) , we need a good understanding of the intersections of such joins. Subsection 5.1 is devoted to this. The description of D is given separately when n is odd (Subsection 5.2) and even (Subsection 5.3). Finally, in Section 6, we address a question of Hassett about some moduli spaces $\overline{M}_{g,A[n]}$ of weighted pointed stable curves.

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2. Mori Dream Spaces

Let X be a normal projective variety. We denote by $N^1(X)$ the real vector space of \mathbb{R} -Cartier divisors modulo numerical equivalence. The *nef cone* of X is the (closed) convex cone Nef $(X) \subset N^1(X)$ generated by classes of nef divisors. The *movable cone* of X is the convex cone Mov $(X) \subset N^1(X)$ generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in X. The *effective cone* of X is the convex cone Eff $(X) \subset N^1(X)$ generated by classes of *effective divisors*. We have inclusions:

$$\operatorname{Nef}(X) \subset \overline{\operatorname{Mov}(X)} \subset \overline{\operatorname{Eff}(X)}.$$

We say that a birational map $f : X \dashrightarrow X'$ into a normal projective variety X' is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small* \mathbb{Q} -factorial modification if X' is \mathbb{Q} -factorial and f is an isomorphism in codimension one. If $f : X \dashrightarrow X'$ is a small \mathbb{Q} -factorial modification, then the natural pullback map $f^* : N^1(X') \to N^1(X)$ sends $\operatorname{Mov}(X')$ and $\operatorname{Eff}(X')$ isomorphically onto $\operatorname{Mov}(X)$ and $\operatorname{Eff}(X)$, respectively. In particular, we have $f^*(\operatorname{Nef}(X')) \subset \overline{\operatorname{Mov}(X)}$.

Definition 2.1. A normal projective \mathbb{Q} -factorial variety X is called a *Mori dream space* if the following conditions hold:

- $\operatorname{Pic}(X)$ is finitely generated,
- Nef (X) is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small \mathbb{Q} -factorial modifications $f_i: X \dashrightarrow X_i$, such that each X_i satisfies the second condition above, and

$$Mov(X) = \bigcup_{i} f_{i}^{*}(Nef(X_{i})).$$

The collection of all faces of all cones $f_i^*(\text{Nef}(X_i))$'s above forms a fan supported on Mov(X). If two maximal cones of this fan, say $f_i^*(\text{Nef}(X_i))$ and $f_j^*(\text{Nef}(X_j))$, meet along a facet, then there exists a commutative diagram:



where Y is a normal projective variety, and h_i and h_j are small birational morphisms. The fan structure on Mov(X) can be extended to a fan supported on Eff(X) as follows.

Definition 2.2. Let X be a Mori dream space. We describe a fan structure on the effective cone Eff(X), called the *Mori chamber decomposition*. We refer to [HK00, Proposition 1.11(2)] and [Oka11, Section 2.2] for details. There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i : X \to Y_i$. The set $\text{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones C_i of the Mori chamber decomposition of Eff(X) are of the form:

$$C_i = \operatorname{Cone} \left(g_i^* \left(\operatorname{Nef}(Y_i) \right), \operatorname{Exc}(g_i) \right).$$

We call \mathcal{C}_i or its interior \mathcal{C}_i° a maximal chamber of $\mathrm{Eff}(X)$.

By [BCHM, Corollary 1.3.2], a log Fano variety is a Mori dream space. The converse does not hold in general, and there are several criteria for a Mori dream space to be log Fano [GOST]. We will use the following.

Proposition 2.3. Let X be a log Fano variety. Then any small \mathbb{Q} -factorial modification of X is also log Fano.

Proposition 2.3 follows from the properties of Mori dream spaces and Lemma 2.4 below. In what follows, a normal projective variety X is said to be of Fano type if there exists an effective \mathbb{Q} -divisor D on X such that $-(K_X + D)$ is \mathbb{Q} -Cartier and ample, and the pair (X, D) is klt. This is weaker than our current notion of log Fano because it does not require that X be \mathbb{Q} -factorial.

Lemma 2.4 ([GOST, Lemma 3.1]). Let $h : X \to Y$ be a small birational morphism between normal projective varieties. Then X is of Fano type if and only if so is Y.

Lemma 2.5. Let X be a normal \mathbb{Q} -factorial projective variety with at worst klt singularities. Suppose that $-K_X$ is nef and big. Then X is log Fano.

Proof. Since $-K_X$ is big, by [La, Corollary 2.2.6], there exist an ample divisor A, an effective divisor D, and a positive integer m such that $-mK_X \equiv A + D$. For h > m, we can write

 $-hK_X \equiv -(h-m)K_X + A + D$. The divisor $D' = -(h-m)K_X + A$ is a sum of a nef and an ample divisor, and so it is ample. Setting $\epsilon = \frac{1}{h}$, we get that $-(K_X + \epsilon D) \equiv \epsilon D'$ is ample. Since X has at worst klt singularities, by taking h large enough, we get that the pair $(X, \epsilon D)$ is klt.

Remark 2.6. In the next section, we will use Proposition 2.3 and Lemma 2.5 to prove that certain blow-ups X of \mathbb{P}^n at general points are Mori dream spaces. To do so, we will use the fact that X admits a small \mathbb{Q} -factorial modification X' which is smooth and has $-K_{X'}$ nef and big. Notice that smoothness of X' is essential. In fact, there are examples of Mori dream spaces X which are not log Fano, but admit a (very singular) small \mathbb{Q} -factorial modification X' with $-K_{X'}$ ample, see for instance [CG, Example 5.1].

3. Cones of curves and divisors on blow-ups of \mathbb{P}^n at general points

Let $p_1, ..., p_k \in \mathbb{P}^n$ be general points, and let X_k^n be the blow-up of \mathbb{P}^n at $p_1, ..., p_k$. In this section we describe several cones of curves and divisors on X_k^n .

Notation 3.1. We denote by $H \in N^1(X_k^n)$ the class of the pullback of the hyperplane section of \mathbb{P}^n . By abuse of notation, we denote by E_i both the exceptional divisor over p_i and its class in $N^1(X_k^n)$. Then $\{H, E_1, \ldots, E_k\}$ is a basis of $N^1(X_k^n)$, and we have

$$-K_{X_k^n} = (n+1)H - (n-1)E_1 - \dots - (n-1)E_k.$$

We denote by $L \in N_1(X_k^n)$ the class of the strict transform of a general line on \mathbb{P}^n . For each $i \in \{1, \ldots, k\}$, we denote by $R_i \in N_1(X_k^n)$ the class of a line on $E_i \cong \mathbb{P}^{n-1}$, and by $L_i \in N_1(X_k^n)$ the class of the strict transform of a general line on \mathbb{P}^n passing through p_i . For $i \neq j$, we denote by $L_{i,j}$ the class of the strict transform of the line on \mathbb{P}^n joining p_i and p_j . Then $\{L, R_1, \ldots, R_k\}$ is a basis of $N_1(X_k^n)$, and we have

(3.1)
$$L \equiv L_{i,j} + R_i + R_j \text{ and } L_i \equiv L - R_i \equiv L_{i,j} + R_j.$$

3.1. The Mori cone of X_k^n . In this section we prove Proposition 1.4.

Lemma 3.2. Let $p_1, ..., p_8 \in \mathbb{P}^3$ be general points, and $C \subset \mathbb{P}^3$ an irreducible curve of degree d having multiplicity $m_i = \text{mult}_{p_i}(C)$ at $p_i, 1 \leq i \leq 8$. Then $m_1 + ... + m_8 \leq 2d$.

Proof. If C is degenerate, then $m_i \neq 0$ for at most three points p_i , and the conclusion follows easily from Bézout's Theorem. So from now on we assume that C is non degenerate. Let Λ be the pencil of irreducible quadric surfaces passing through p_1, \ldots, p_8 . Suppose that $m_1 + \ldots + m_8 > 2d$. It follows from Bézout's Theorem that C is contained in every member of Λ . In particular, C is a non degenerate irreducible curve contained in the intersection of two irreducible quadric surfaces. So $d \in \{3, 4\}$. Suppose that d = 3. Then C must be a twisted cubic through at most 6 of the p_i 's, and thus $m_1 + \ldots + m_8 \leq 6 = 2d$, contradicting our assumptions. We conclude that d = 4, $m_i \geq 1$ for every i, and $m_j \geq 2$ for some j. If follows from Bézout's Theorem that $m_j = 2$, and $m_i = 1$ for $i \neq j$. Consider the projection from p_j

$$\pi_{p_i}: C \dashrightarrow \mathbb{P}^2.$$

The image $\overline{\pi_{p_1}(C)}$ is a conic though the seven general points $\pi_{p_j}(p_i)$, $i \neq j$, which is impossible. This shows that $m_1 + \ldots + m_8 \leq 2d$.

Proof of Proposition 1.4. Let X_k^n be the blow-up of \mathbb{P}^n , $n \geq 2$, at points in general position p_1, \ldots, p_k . We follow Notation 3.1. Let $\widetilde{C} \subset X_k^n$ be an irreducible curve not contained in any exceptional divisor E_i , and denote by C the image of \widetilde{C} in \mathbb{P}^n . It is an irreducible curve of degree d > 0 and multiplicity $m_i = \text{mult}_{p_i} C \geq 0$ at p_i , \widetilde{C} is the strict transform of C, and

$$(3.2) \qquad \qquad \widetilde{C} \equiv dL - m_1 R_1 - \dots - m_k R_k.$$

We must show that the class of \widetilde{C} in $N_1(X_k^n)$ lies in the cone generated by the R_i 's and $L_{i,j}$'s. We may assume that $m_1 \leq m_2 \leq \cdots \leq m_k$.

Suppose first that $k \leq 2n$.

First let us assume that k is even. We write

(3.3)
$$\widetilde{C} \equiv dL - m_1(R_1 + R_2) - (m_2 - m_1)R_2 - m_3(R_3 + R_4) - (m_4 - m_3)R_4 - \dots - m_{k-1}(R_{k-1} + R_k) - (m_k - m_{k-1})R_k.$$

Note that $m_1+(m_2-m_1)+m_3+(m_4-m_3)+\ldots+m_{k-1}+(m_k-m_{k-1})=m_2+m_4+\ldots+m_k$. We claim that $m_2+m_4+\ldots+m_k \leq d$. Indeed, since $k \leq 2n$, the set $\{p_2, p_4, \ldots, p_k\}$ has cardinality at most n. Consider the linear space $P = \langle p_2, p_4, \ldots, p_k \rangle \subsetneq \mathbb{P}^n$. If $m_2 + m_4 + \ldots + m_k > d$, then $C \subset P$ by Bézout's Theorem . Since the p_i 's are general, $p_1, p_3, \ldots, p_{k-1} \notin P$, and so $m_1 = m_3 = \ldots = m_{k-1} = 0$. But this implies that $m_i = 0$ for $i \leq k-1$ and $m_k > d$, which is impossible. This proves the claim. So we can rewrite (3.3) as

$$C \equiv m_1 L_{1,2} + (m_2 - m_1)L_2 + m_3 L_{3,4} + (m_4 - m_3)L_4 - \dots + m_{k-1}L_{k-1,k} + (m_k - m_{k-1})L_k + (d - m_2 - m_4 - \dots - m_k)L.$$

It follows from (3.1) that the class of \widetilde{C} in $N_1(X_k^n)$ lies in the cone generated by the R_i 's and $L_{i,j}$'s.

Now suppose that k is odd, and write

(3.4)
$$\widetilde{C} \equiv dL - m_1(R_1 + R_2) - (m_2 - m_1)R_2 - m_3(R_3 + R_4) - (m_4 - m_3)R_4 - \dots - m_{k-2}(R_{k-2} + R_{k-1}) - (m_{k-1} - m_{k-2})R_{k-1} - m_k R_k.$$

In this case $m_1 + (m_2 - m_1) + m_3 + (m_4 - m_3) + ... + m_{k-2} + (m_{k-1} - m_{k-2}) + m_k = m_2 + m_4 + ... + m_{k-1} + m_k$. Like in the even case, one shows that $m_2 + m_4 + ... + m_{k-1} + m_k \le d$ and rewrite (3.4) as an effective linear combination of the R_i 's and $L_{i,j}$'s.

From now on we suppose that n = 3 and $k \le 8$. Then $m_i \le d$ and $m_1 + \ldots + m_k \le 2d$ by Lemma 3.2. If $m_{k-1} = 0$, then $\widetilde{C} \equiv m_k L_k + (d - m_k)L$. It follows from (3.1) that the class of \widetilde{C} in $N_1(X_k^n)$ lies in the cone generated by the R_i 's and $L_{i,j}$'s. If $m_{k-1} \ne 0$, then rewrite (3.2) as

$$\widetilde{C} \equiv (L_{k-1,k}) + d'L - m'_1 R_1 - \dots - m'_k R_k,$$

where d' = d - 1, $m'_i = m_i$ for $i \le k - 2$, and $m'_i = m_i - 1$ for i = k - 1 or k. Note that $m'_i \le d'$. This is clear for i = k - 1 or k. For $i \le k - 2$ it follows from the assumptions that $m_1 \le m_2 \le \cdots \le m_k \le d$ and $m_1 + \ldots + m_k \le 2d$. We also have $m'_1 + \ldots + m'_k \le 2d'$. So we can repeat the process and conclude by induction that the class of \widetilde{C} in $N_1(X^n_k)$ lies in the cone generated by the R_i 's and $L_{i,j}$'s.

3.2. The effective cone of X_{n+3}^n . In this section we describe the effective cone of the blowup of \mathbb{P}^n at n+3 points in general position, as well as its Mori chamber decomposition. The main references are [CT06], [Muk05] and [Bau91]. See also [BDP15] for a recent new proof.

3.3 (The effective cone of the blow-up of \mathbb{P}^n at n+3 points). Let $X = X_{n+3}^n$ be the blow-up of \mathbb{P}^n at n+3 points p_i in general position. We follow Notation 3.1. By [CT06, Theorems 1.3], X is a Mori dream space. Next we describe the 1-dimensional faces of Eff(X) ([CT06, Theorem 1.2]). For each subset $I \subset \{1, \dots, n+3\}$ whose complement has odd cardinality $|I^c| = 2k + 1$, consider the divisor class

$$E_I := kH - k \sum_{i \in I} E_i - (k-1) \sum_{i \in I^c} E_i.$$

There is a unique divisor in the linear system $|E_I|$, which we also denote by E_I . When k = 0we have $E_{\{i\}^c} = E_i$ When $k \ge 1$, E_I can be described as follows. Let $\pi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{2k-2}$ be the projection from the linear space $\langle p_i \rangle_{i \in I}$. Let $C_I \subset \mathbb{P}^{2k-2}$ be the image of the unique rational normal curve through all the p'_i s. The divisor E_I is the cone with vertex $\langle p_i \rangle_{i \in I}$ over $\mathbb{S}ec_{k-1}C_I$. Each E_I generates a 1-dimensional face of $\mathrm{Eff}(X)$, and all 1-dimensional faces are of this form.

Let X be the blow-up of \mathbb{P}^n at n + 3 points in general position, and follow the notation of Paragraph 3.3 above. In order to describe the Mori chamber decomposition of Eff(X), we make explicit the map to the weight space proposed by Mukai in [Muk05]. Write $(y, x_1, \ldots, x_{n+3})$ for coordinates in \mathbb{R}^{n+4} , and $(\alpha_1, \ldots, \alpha_{n+3})$ for coordinates in \mathbb{R}^{n+3} . We identify \mathbb{R}^{n+4} with $N^1(X)$ by associating to a point $\bar{x} = (y, x_1, \ldots, x_{n+3}) \in \mathbb{R}^{n+4}$ the divisor class of $D_{\bar{x}} = yH + \sum x_i E_i$. Note that all the E_I 's defined in Paragraph 3.3 lie in the hyperplane

$$(n+1)y + \sum x_i = 1.$$

Consider the projection from the origin

(3.5)
$$\varphi = (\varphi_1, \cdots, \varphi_{n+3}) : \operatorname{Eff}(X) \to \mathbb{R}^{n+3},$$
$$\varphi_i = \frac{y + x_i}{(n+1)y + \sum x_i}.$$

We shall describe the image of Eff(X) under φ , along with the decomposition induced by the Mori chamber decomposition of Eff(X). Before we do so, let us introduce some notation. The vertices of the hypercube $[0,1]^{n+3} \subset \mathbb{R}^{n+3}$ are the points of the form $\xi_I =$ $((\xi_I)_1, \ldots, (\xi_I)_{n+3})$, where $I \subset \{1, \ldots, n+3\}$, $(\xi_I)_i = 1$ if $i \in I$, and $(\xi_I)_i = 0$ otherwise. The parity of the vertex ξ_I is the parity of |I|. For each subset $I \subset \{1, \ldots, n+3\}$, define the degree one polynomial in the α_i 's:

(3.6)
$$H_I := \sum_{j \notin I} \alpha_j + \sum_{i \in I} (1 - \alpha_i).$$

For any subset $J \subset \{1, \ldots, n+3\}$, we have:

(3.7)
$$H_I(\xi_J) = \#(I^{c} \cap J) + \#(J^{c} \cap I).$$

Given $J \subset \{1, \ldots, n+3\}$ and $i_0 \notin J$, set $I := J \cup \{i_0\}$. Then

(3.8)
$$H_I = H_J + 1 - 2\alpha_{i_0}$$

One computes that $\varphi(E_I) = \xi_{I^c}$. Therefore, the image of Eff(X) under φ is the polytope $\Delta \subset \mathbb{R}^{n+3}$ generated by the odd vertices of the hypercube. Using (3.7) above, one can easily check that the polytope $\Delta \subset \mathbb{R}^{n+3}$ is defined by the following set of inequalities:

(3.9)
$$\Delta = \varphi \big(\operatorname{Eff}(X) \big) = \begin{cases} 0 \le \alpha_i \le 1, & i \in \{1, \dots, n+3\} \\ H_I \ge 1, & |I| \text{ even.} \end{cases}$$

Next we describe the chamber decomposition in Δ induced by the Mori chamber decomposition of Eff(X). For each subset $I \subset \{1, \ldots, n+3\}$, and each integer k satisfying $2 \le k \le \frac{n+3}{2}$ and $|I| \ne k \mod 2$, consider the hyperplane $(H_I = k)$. Now take the complement in the interior of Δ of the hyperplane arrangement

(3.10)
$$\left(\begin{array}{cc} H_I &=& k \end{array}\right)_{2 \leq k \leq \frac{n+3}{2}, \ |I| \not\equiv k \mod 2}$$

and consider its decomposition into connected components. Each connected component is called a *chamber* of Δ .

The following theorem summarizes the results of [Muk05] and [Bau91]. The proof follows from the proof of the main theorem in [Muk05, Page 6] and the description of wall crosses in [Muk05, Propositions 2 and 3]. Mukai's proof relies on interpreting X as a moduli space of parabolic vector bundles on \mathbb{P}^1 , and the description of these spaces in [Bau91, Section 2].

Theorem 3.4. Let X be the blow-up of \mathbb{P}^n at n+3 points in general position, and consider the projection

$$\varphi$$
 : Eff(X) $\rightarrow \Delta$

defined in (3.5) above.

- The chamber decomposition of Δ defined by the hyperplane arrangement (3.10) coincides with that induced by the Mori chamber decomposition of Eff(X) via φ .
- The image of Mov(X) under φ is given by

$$\Pi = \varphi(\operatorname{Mov}(X)) = \begin{cases} 0 \le \alpha_i \le 1, & i \in \{1, \dots, n+3\} \\ H_I \ge 2, & |I| \text{ odd.} \end{cases}$$

- All small Q-factorial modifications of X are smooth. Let C and C' be two adjacent chambers of Mov(X), corresponding to small Q-factorial modifications of X, f : $X \to \widetilde{X}$ and $f' : X \to \widetilde{X}'$, respectively. The images of these chambers in Δ are separated by a hyperplane of the form $(H_I = k)$, with $3 \le k \le \frac{n+3}{2}$ and $|I| \ne k$ mod 2. Suppose that $\varphi(C) \subset (H_I \le k)$ and $\varphi(C') \subset (H_I \ge k)$. Then the birational map $f' \circ f^{-1} : \widetilde{X} \to \widetilde{X}'$ flips a \mathbb{P}^{k-2} into a \mathbb{P}^{n+1-k} .
- Let \mathcal{C} be a chamber of Mov(X), corresponding to small \mathbb{Q} -factorial modification \widetilde{X} of X. Let $\sigma \subset \partial \mathcal{C}$ be a wall such that $\sigma \subset \partial Mov(X)$, and let $f : \widetilde{X} \to Y$ be the corresponding elementary contraction. The image of σ in Π is supported on a hyperplane of one of the following forms:
 - (a) $(\alpha_i = 0)$ or $(\alpha_i = 1)$.
 - (b) $(H_I = 2)$, with |I| odd.

In case (a), $f: \widetilde{X} \to Y$ is a \mathbb{P}^1 -bundle. In case (b), $f: \widetilde{X} \to Y$ is the blow-up of a smooth point, and the exceptional divisor of f is the image in \widetilde{X} of the divisor E_{I^c} .

Remark 3.5. In Theorem 3.4, note that

$$\Delta \cap (H_I \ge 2)_{|I| \text{ odd }} = [0,1]^{n+3} \cap (H_I \ge 2)_{|I| \text{ odd }}.$$

This can be checked using (3.8).

Remark 3.6. The image of Nef(X) under φ is given by

$$\Sigma = \varphi \big(\operatorname{Nef}(X) \big) = \begin{cases} H_{\{i\}} \ge 2, & i \in \{1, \dots, n+3\} \\ H_{\{i,j\}} \le 3, & i, j \in \{1, \dots, n+3\}, i \neq j. \end{cases}$$

Remark 3.7. Formula (3.5), together with the equations for the walls in the chamber decomposition of Δ defined by the hyperplane arrangement (3.10), allow us to find explicit inequalities defining the cones Eff(X), Mov(X), and $\text{Nef}(\widetilde{X})$, for any small \mathbb{Q} -factorial modification \widetilde{X} of X.

3.3. **Proof of Theorem 1.3.** Let X_k^n be a blow-up of \mathbb{P}^n at k points in general position. By [Muk01] and [CT06], X_k^n is a Mori dream space if and only if one of the following holds:

-
$$n = 2$$
 and $k \le 8$.
- $n = 3$ and $k \le 7$.
- $n = 4$ and $k \le 8$.
- $n > 4$ and $k \le n + 3$.

We will show that in each of these cases X_k^n is log Fano. In view of the classification of del Pezzo surfaces and Example 1.2, we may assume that $n \ge 4$.

Suppose that k = n+3, set $X := X_{n+3}^n$, and follow the notation of the previous subsection. The center of the polytopes Π and Δ is the point $(\frac{1}{2}, \ldots, \frac{1}{2}) = \varphi(-K_X)$.

When n is even, this point is in the interior of a chamber of Π , namely the chamber defined by:

$$\Sigma' = \left(H_I \geq \frac{n+2}{2} \right)_{|I| \not\equiv \frac{n+2}{2} \mod 2}$$

Let X' be the small Q-factorial modification of X whose nef cone is the inverse image of the chamber Σ' . Then X' is a smooth Fano manifold with very interesting geometry and symmetries. See [Cas14] and references therein for several descriptions of X'. By Proposition 2.3, X is log Fano.

When n is odd, the point $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) = \varphi(-K_X)$ lies in the intersection of the hyperplanes:

$$\left(H_I = \frac{n+3}{2} \right)_{|I| \not\equiv \frac{n+3}{2} \mod 2}.$$

Let X' be the small Q-factorial modification of X whose nef cone is the inverse image of some chamber Σ' containing $\varphi(-K_X)$ in its boundary. Then X' is a smooth projective variety with $-K_{X'}$ nef and big. By Lemma 2.5 and Proposition 2.3, X is log Fano.

It follows from [GOST, Corollary 1.3] that X_k^n is log Fano for any $k \le n+3$.

The case n = 4 and k = 8 can be treated in a similar way. In [Muk05, Section 2], Mukai describes the Mori chamber decomposition of $Mov(X_8^4)$. It follows from his description that X_8^4 admits a small Q-factorial modification X' which is a Fano manifold. Again we conclude that X_8^4 is log Fano by Proposition 2.3.

4. Finding explicit divisors making X_k^n log fand for $k \leq n+2$

Throughout this section, we let $p_1, ..., p_k \in \mathbb{P}^n$ be general points, $k \leq n+2$, and let X_k^n be the blow-up of \mathbb{P}^n at $p_1, ..., p_k$. We shall exhibit an integral divisor $D \subset X_k^n$ and rational numbers $\epsilon > 0$ such that $\Delta = \epsilon D$ makes X_k^n log Fano. In order to show that $-(K_{X_k^n} + \Delta)$ is ample, we will use Proposition 1.4. To show that (X_k^n, Δ) is klt, we will need an explicit log resolutions for this pair, which we introduce next.

Notation 4.1. For each $0 \le h \le k-1$ and each subset $I = \{i_1 < \cdots < i_{h+1}\} \subset \{1, \ldots, k\}$, consider the *h*-dimensional linear subspace $H_I^h = \langle p_{i_1}, \ldots, p_{i_{h+1}} \rangle \subset \mathbb{P}^n$. Denote by \mathcal{H}^h the collection of all such *h*-dimensional linear subspaces, and by $\rho_h = \binom{k}{h+1}$ its cardinality.

Let $\pi: Y \to X_k^n$ be the blow-up of the strict transforms of the lines in \mathcal{H}^1 , followed by the blow-up of the strict transforms of the planes in \mathcal{H}^2 , and so on, in order of increasing dimension, up to the blow-up of the strict transforms of the (n-2)-planes in \mathcal{H}^{n-2} . For each $1 \leq h \leq n-2$, denote by $E_1^h, \ldots, E_{\rho_h}^h \subset Y$ the exceptional divisors over the ρ_h h-planes in \mathcal{H}^h . We have

(4.1)
$$K_Y = \pi^* K_{X_k^n} + \sum_{h=1}^{n-2} (n-h-1)(E_1^h + \dots + E_{\rho_h}^h).$$

Remark 4.2. For k = n + 1, the variety Y constructed above is the Losev-Manin moduli space, introduced in [LM] as a toric compactification of $M_{0,n+3}$. For k = n + 2, the construction above gives Kapranov's description of $Y = \overline{M}_{0,n+3}$ as an iterated blow-up of \mathbb{P}^n ([Ka]).

4.1. Blow-ups of \mathbb{P}^n in at most n + 1 points. For $k \leq n + 1$, the variety X_k^n is toric, and one can take Δ making X_k^n log Fano to be a suitable combination of toric invariant divisors. Alternatively, we show that the divisor Δ can be chosen irreducible. We work out the case k = n + 1. When k < n + 1, the boundary divisor can be taken to be the image of Δ under the natural morphism $X_{n+1}^n \to X_k^n$.

Theorem 4.3. Let $D \subset X_{n+1}^n$ be the strict transform of a general member of the linear system $\Gamma \subset |\mathcal{O}_{\mathbb{P}^n}(n)|$ of the standard Cremona transformation of \mathbb{P}^n , centered at $p_1, ..., p_{n+1}$. For any $\frac{n-3}{n-2} < \epsilon < 1$ the divisor $-(K_{X_{n+1}^n} + \epsilon D)$ is ample, and the pair $(X_{n+1}^n, \epsilon D)$ is klt.

For the proof of Theorem 4.3, we will need the following.

Lemma 4.4. Let $\Gamma \subset |\mathcal{O}_{\mathbb{P}^n}(n)|$ be the linear system of the standard Cremona transformation of \mathbb{P}^n , centered at $p_1, ..., p_{n+1}$, and $D \in \Gamma$ be a general member. Let H_I^h and $\pi : Y \to \mathbb{P}^n$ be as in Notation 4.1. Then the strict transform \widetilde{D} of D in Y is smooth and transversal to all the exceptional divisors of π . Furthermore

$$\operatorname{mult}_{H_I^h} D = n - h + 1$$

for any h = 0, ..., n - 2.

Proof. By [MM, Theorem 1] the Cremona transformation induced by Γ lifts to an automorphism of Y. This implies that \widetilde{D} is smooth and transversal to all the exceptional divisors of π . In particular, D is smooth away from the union of the codimension two linear subspaces H_I^{n-2} 's.

We may assume that the p_i 's are the fundamental points of \mathbb{P}^n , and consider the element of the linear system Γ given by:

$$D_0 := \{x_0 x_1 \dots x_{n-1} + x_0 x_1 \dots x_{n-2} x_n + \dots + x_1 x_2 \dots x_n = 0\}.$$

Let $p \in H_I^h$ be a general point. Then one checks easily that $\operatorname{mult}_p D_0 = \operatorname{mult}_{H_I^h} D_0 = n - h + 1$. To conclude, note that $\operatorname{mult}_{H_I^h} D \ge n - h + 1$ for any $D \in \Gamma$.

Proof of Theorem 4.3. With Notation 3.1, we have

$$D \sim nH - (n-1)(E_1 + \dots + E_{n-1}).$$

Recall from Proposition 1.4 that the Mori cone of X_{n+1}^n is generated by the classes R_i 's and $L_{i,j}$'s. One computes

$$-(K_{X_{n+1}^n} + \epsilon D) \cdot R_i = n - 1 - \epsilon(n-1) \text{ and } -(K_{X_{n+1}^n} + \epsilon D) \cdot L_{i,j} = (n-1)\epsilon - n + 3.$$

Therefore $-K_{X_{n+1}^n} - \epsilon D$ is ample provided that $\frac{n-3}{n-2} < \epsilon < 1$.

Next we check when the pair $(X_{n+1}^n, \epsilon D)$ is klt. Let $\pi : Y \to X_{n+1}^n$ be the morphism introduced in Notation 4.1. By Lemma 4.4 $\pi : Y \to X_{n+1}^n$ is a log resolution of $(X_{n+1}^n, \epsilon D)$, and

$$\pi^*(D) = \widetilde{D} + \sum_{h=1}^{n-2} (n-h-1)(E_1^h + \dots + E_{\rho_h}^h)$$

Together with (4.1), this gives

$$K_Y + \epsilon \widetilde{D} = \pi^* (K_{n+1}^n + \epsilon D) + \sum_{h=1}^{n-2} (n-h-1)(1-\epsilon)(E_1^h + \dots + E_{\rho_h}^h).$$

Therefore the pair $(X_{n+1}^n, \epsilon D)$ is klt for any $0 \le \epsilon < 1$.

4.2. Blow-ups of \mathbb{P}^n in n+2 points. In this subsection we construct divisors Δ making X_{n+2}^n log Fano. Note that X_{n+2}^n is not toric. We follow Notation 4.1, and denote by $H_1, \ldots, H_{\rho_{n-1}} \subset \mathbb{P}^n$ the ρ_{n-1} hyperplanes through n of the p_i 's.

Theorem 4.5. Let $D \subset X_{n+2}^n$ be the strict transform of the divisor $H_1 + \cdots + H_{\rho_{n-1}}$. For any $\frac{2(n-3)}{(n+1)(n-2)} < \epsilon < \frac{2(n-1)}{n(n+1)}$ the divisor $-(K_{X_{n+2}^n} + \epsilon D)$ is ample, and the pair $(X_{n+2}^n, \epsilon D)$ is klt.

For the proof of Theorem 4.5, we will need the following.

Lemma 4.6. Let $D \subset X_{n+2}^n$ be the strict transform of the divisor $H_1 + \cdots + H_{\rho_{n-1}}$. Let $\pi: Y \to X_{n+2}^n$ be the morphism introduced in Notation 4.1. Then $\pi: Y \to X_{n+2}^n$ is a log resolution of (X_{n+2}^n, D) , and

$$\pi^*(D) = \widetilde{D} + \sum_{h=1}^{n-2} \binom{n-h+1}{n-h-1} (E_1^h + \dots + E_{\rho_h}^h).$$

Proof. Note that at each step in the description of Y as an iterated blow-up, the center of the blow-up is a disjoint union of smooth subvarieties. Moreover, the divisor $\text{Exc}(\pi) \cup \widetilde{D}$ is simple normal crossing, and so $\pi: Y \to X_{n+2}^n$ is a log resolution of (X_{n+2}^n, D) .

Any $H_I^h \in \mathcal{H}^h$ is contained in exactly $\binom{n-h+1}{n-h-1}$ of the ρ_{n-1} hyperplanes H_i 's. Thus $\operatorname{mult}_{H_I^h} D = \binom{n-h+1}{n-h-1}$, and the formula for $\pi^*(D)$ follows.

Proof of Theorem 4.5. Each point p_i lies in exactly $\binom{n+1}{n-1} = \frac{1}{2}(n+1)n$ hyperplanes among the H_i 's, $1 \le i \le \rho_{n-1} = \frac{1}{2}(n+2)(n+1)$. So we have, with Notation 3.1,

$$D \sim \frac{1}{2}(n+2)(n+1)H - \frac{1}{2}(n+1)nE_1 - \dots - \frac{1}{2}(n+1)nE_{n+2}.$$

Recall from Proposition 1.4 that the Mori cone of X_{n+2}^n is generated by the classes R_i 's and $L_{i,j}$'s. One computes

$$-(K_{X_{n+2}^n} + \epsilon D) \cdot R_i = (n - 1 - \frac{\epsilon}{2}(n+1)n) \text{ and } -(K_{X_{n+2}^n} + \epsilon D) \cdot L_{i,j} = -n + 3 + \frac{\epsilon}{2}(n+1)(n-2).$$

Therefore $-K_{X_{n+2}^n} - \epsilon D$ is ample provided that $\frac{2(n-3)}{(n+1)(n-2)} < \epsilon < \frac{2(n-1)}{n(n+1)}$. Next we check when the pair $(X_{n+2}^n, \epsilon D)$ is klt. Let $\pi : Y \to X_{n+2}^n$ be the morphism

Next we check when the pair $(X_{n+2}^n, \epsilon D)$ is klt. Let $\pi : Y \to X_{n+2}^n$ be the morphism introduced in Notation 4.1. By Lemma 4.6 $\pi : Y \to X_{n+2}^n$ is a log resolution of $(X_{n+2}^n, \epsilon D)$, and

$$\pi^*(D) = \widetilde{D} + \sum_{h=1}^{n-2} \binom{n-h+1}{n-h-1} (E_1^h + \dots + E_{\rho_h}^h).$$

Together with (4.1), this gives

$$K_Y + \epsilon \widetilde{D} = \pi^* (K_{n+2}^n + \epsilon D) + \sum_{h=1}^{n-2} \left((n-h-1) - \epsilon \binom{n-h+1}{n-h-1} \right) (E_1^h + \dots + E_{\rho_h}^h).$$

Therefore the pair $(X_{n+2}^n, \epsilon D)$ is klt for any $0 \le \epsilon < \frac{2}{n}$.

5. Finding explicit divisors making X_{n+3}^n log Fano

Throughout this section, let $p_1, ..., p_{n+3} \in \mathbb{P}^n$ be general points, and let X_{n+3}^n be the blow-up of \mathbb{P}^n at $p_1, ..., p_{n+3}$. We shall exhibit integral divisors $D \subset X_{n+3}^n$ and rational numbers $\epsilon > 0$ such that $\Delta = \epsilon D$ makes X_{n+3}^n log Fano. In the previous cases, D was taken as sum of strict transforms of hyperplanes through n of the n+3 points. For X_{n+3}^n , we will also need other extremal divisors $E_I \subset X_{n+3}^n$ introduced in Paragraph 3.3. This will make the log resolution of (X, Δ) more complicated, and we will need to understand well how the divisors E_I 's intersect. For this purpose, we start this section with some preliminaries on secant varieties of rational normal curves. Then we will consider separately the cases n = 2h + 1 odd, and n = 2h even.

5.1. Preliminaries on secant varieties of rational normal curves. Given an irreducible and reduced non-degenerate variety $X \subset \mathbb{P}^n$, and a positive integer $k \leq n$ we denote by $\mathbb{S}ec_k(X)$ the *k*-secant variety of X. This is the subvariety of \mathbb{P}^n obtained as the closure of the union of all (k-1)-planes $\langle x_1, ..., x_k \rangle$ spanned by k general points of X. We will be concerned with the case when X = C is a rational normal curve of degree n in \mathbb{P}^n . The following proposition gathers some of the basic properties of the secant varieties $\mathbb{S}ec_k(C)$ in this case.

Proposition 5.1. Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n, and let k be an integer such that $1 \leq k \leq \frac{n}{2}$. Then the following statements hold.

- (1) $\dim(\mathbb{S}ec_k(C)) = 2k 1$ (see for instance [Har, Proposition 11.32]).
- (2) $\deg(\operatorname{Sec}_k(C)) = \binom{n-k+1}{k}$ (see for instance [EH, Theorem 12.16]).

- (3) $\operatorname{Sec}_k(C)$ is normal and $\operatorname{Sing}(\operatorname{Sec}_k(C)) = \operatorname{Sec}_{k-1}(C)$ (see for instance [Ve1, Theorem 1.1]).
- (4) If n = 2h is even, then for any $1 \le t < h$ we have

$$\operatorname{nult}_{\operatorname{Sec}_{h-t}(C)} \operatorname{Sec}_h(C) = t+1.$$

Proof of (4). Suppose that n = 2h is even, and consider the $(h + 1) \times (h + 1)$ matrix

(5.1)
$$M_{h} = \begin{pmatrix} x_{0} & x_{1} & \dots & x_{h} \\ x_{1} & x_{2} & \dots & x_{h+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{h} & x_{h+1} & \dots & x_{2h} \end{pmatrix}$$

For any $1 \le k \le h$, the secant variety $\mathbb{S}ec_k(C)$ can be described as the determinantal variety:

$$\mathbb{S}ec_k(C) = \{ \operatorname{rank}(M_h) \le k \}.$$

(See for instance [Har, Proposition 9.7]). In particular, $\mathbb{S}ec_h(C) \subset \mathbb{P}^{2h}$ is the degree h + 1 hypersurface defined by the polynomial $F := \det(M_h)$. For each $j \in \{0, ..., 2h\}$, let $\{M_i^j\}$ be the set of $h \times h$ minors of M_h produced by erasing in M_h a row and a column meeting in an entry of type x_j Denote by ρ_j be the number of such minors. Then

$$\frac{\partial F}{\partial x_j} = \sum_{i=1}^{\rho_j} \alpha_i^j \det(M_i^j),$$

for suitable $\alpha_i^j \neq 0$. Inductively, we see that for any $1 \leq t < h$ the partial derivatives of order t of F are linear combinations of determinants of $(h+1-t) \times (h+1-t)$ minors of M_h . The vanishing of such determinants defines $\mathbb{S}ec_{h-t}(C)$, while the vanishing of the of determinants of the $(h-t) \times (h-t)$ minors of M_h defines $\mathbb{S}ec_{h-t-1}(C) \subsetneq \mathbb{S}ec_{h-t}(C)$. Therefore, there is at least one partial derivative of order t+1 of F not vanishing on $\mathbb{S}ec_{h-t}(C)$. This means that $\operatorname{mult}_{\mathbb{S}ec_{h-t}(C)} \mathbb{S}ec_h(C) = t+1$ for any $1 \leq t < h$.

The following proposition is just a particular instance of [Be, Theorem 1]. The general statement for smooth curves embedded via a 2h-very ample line bundle can be found in [Ve, Theorem 3.1] as well.

Proposition 5.2. Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n, and set $h := \lfloor \frac{n}{2} \rfloor$. Consider the following sequence of blow-ups:

- $\pi_1 : X_1 \to \mathbb{P}^n$ the blow-up of C, - $\pi_2 : X_2 \to X_1$ the blow-up of the strict transform of $\mathbb{S}ec_2(C)$, :

- $\pi_h: X_h \to X_{h-1}$ the blow-up of the strict transform of $Sec_h(C)$.

Let $\pi : X \to \mathbb{P}^n$ be the composition of these blow-ups. Then, for any $k \leq h$ the strict transform of $\operatorname{Sec}_k(C)$ in X_{k-1} is smooth and transverse to all exceptional divisors. In particular X is smooth and the exceptional locus of π is a simple normal crossing divisor.

Notation 5.3. Let $p_1, ..., p_{n+3} \in \mathbb{P}^n$ be general points, and let $C \subset \mathbb{P}^n$ be the unique rational normal curve of degree *n* through these points. Given $1 \le m \le n$, $I = \{i_1 < \cdots < n\}$

 $i_m \} \subset \{1, \ldots, n+3\}$, and a positive integer k such that $0 \le k \le \frac{n-m}{2}$, we consider the following variety of dimension d = 2k - 1 + m:

$$Y_I^d := \operatorname{Join} \left(\langle p_{i_1}, \dots, p_{i_m} \rangle, \operatorname{Sec}_k(C) \right).$$

Alternatively, Y_I^d can be defined as follows. Let $\pi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-m}$ be the projection from the linear space $\langle p_{i_1}, \ldots, p_{i_m} \rangle$. Let $C_I \subset \mathbb{P}^{n-m}$ be the image of C under π_I . It is the the unique rational normal curve of degree n-m through the points $\pi(p_j), j \notin I$. Then Y_I^d is the cone with vertex $\langle p_{i_1}, \ldots, p_{i_m} \rangle$ over $\mathbb{S}ec_k(C_I)$.

By convention, when k = 0, we set $Y_I^{m-1} := \langle p_{i_1}, \dots, p_{i_m} \rangle$.

Fix $I = \{i_1 < \cdots < i_m\} \subset \{1, \ldots, n+3\}$, with $m \le n$. Given k such that $0 \le k \le \frac{n-m}{2}$, set d := 2k - 1 + m. By Proposition 5.1, we have

(5.2)
$$\deg(Y_I^d) = \binom{n-m-k+1}{k} \text{ and } \operatorname{Sing}(Y_I^d) = Y_I^{d-2}$$

Moreover, if n-m is even and $d_1 = 2k_1 - 1 + m > 2k_2 - 1 + m = d_2$, then $Y_I^{d_2} \subset Y_I^{d_1}$ and

(5.3)
$$\operatorname{mult}_{Y_I^{d_2}} Y_I^{d_1} = \frac{d_1 - d_2}{2} + 1.$$

We also have analogs of Proposition 5.2 for sequences of blow-ups of Y_I^d , for $|I| - 1 \le d \le n - 1$. More precisely:

Proposition 5.4. Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree $n, p_1, \ldots, p_m \in C$ distinct points, with $1 \leq m \leq n$, and set $h := \lfloor \frac{n-m}{2} \rfloor$. Consider the following sequence of blow-ups:

 $\begin{array}{l} -\pi_1: X_1 \to \mathbb{P}^n \ the \ blow-up \ of \ Y_I^{m-1} := \langle p_1, \dots, p_m \rangle, \\ -\pi_2: X_2 \to X_1 \ the \ blow-up \ of \ the \ strict \ transform \ of \ Y_I^{m+1}, \\ \vdots \\ -\pi_h: X_h \to X_{h-1} \ the \ blow-up \ of \ the \ strict \ transform \ of \ Y_I^{m+2h-1}. \end{array}$

Let $\pi : X \to \mathbb{P}^n$ be the composition of these blow-ups. Then, for any $k \leq h$ the strict transform of Y_I^{m+2k-1} in X_{k-1} is smooth and transverse to all exceptional divisors.

Proposition 5.4 follows easily from Proposition 5.2. In the next sections, we will blowup varieties of type Y_I^d for several subsets $I \subset \{1, \ldots, n+3\}$, in a suitable order. In order to show the smoothness and transversality of the strict transforms of the Y_I^d 's in the intermediate blow-ups, we will need the following result.

Proposition 5.5. Let $W \subsetneq Z \subsetneq X$ be smooth projective varieties, and let $Y \subset X$ be a projective variety such that $\operatorname{Sing}(Y) = Z$ and Y has ordinary singularities along Z. Let $\pi_W : X_W \to X$ be the blow-up of W, and denote by Z_W and Y_W the strict transforms of Z and Y, respectively. Then $\operatorname{Sing}(Y_W) = Z_W$ and Y_W has ordinary singularities along Z_W .

Proof. Denote by E_W the exceptional divisor of π_W . Then $\pi_W^{-1}(Z) = Z_W \cup E_W$. Let $\pi_{Z_W} : X_{Z_W} \to X_W$ be the blow-up of X_W along Z_W , with exceptional divisor E_{Z_W} .

We claim that the composite morphism $\pi_W \circ \pi_{Z_W} : X_{Z_W} \to X$ is isomorphic to the blow-up $\pi_Z : X_Z \to X$ of X along Z, followed by the blow-up of X_Z along $\pi_Z^{-1}(W)$. Indeed,

by the universal property of the blow-up ([Hart, Proposition 7.14]), there exits a unique morphism $f: X_{Z_W} \to X_Z$ making the following diagram commute.



Note that all varieties in this diagram are smooth. Since Z and W are smooth, the intersection $Z_W \cap E_W \subset X_W$ is smooth. Thus, any normal direction of Z_W in X_W at a point $p \in Z_W \cap E_W$ is the image of a normal direction at p of $Z_W \cap E_W$ in E_W . In other words, the inverse image of W in X_{Z_W} consists of the strict transform \tilde{E}_W of E_W in X_{Z_W} . Therefore, the inverse image of the smooth variety $\pi_Z^{-1}(W)$ in X_W is precisely \tilde{E}_W . Using the the universal property of the blow-up, and comparing the Picard number of these smooth varieties, we conclude that $f: X_{Z_W} \to X_Z$ is the blow-up of X_Z along $\pi_Z^{-1}(W)$, proving the claim.

Next we prove that $\operatorname{Sing}(Y_W) = Z_W$. Clearly $Z_W \subset \operatorname{Sing}(Y_W)$. Suppose that this inclusion is strict. Then the strict transform Y_{Z_W} of Y_W in X_{Z_W} is singular. Since $f : X_{Z_W} \to X_Z$ is a smooth blow-up, $f(Y_{Z_W}) \subset X_Z$ is singular as well. But notice that $f(Y_{Z_W}) \subset X_Z$ is the strict transform of $Y \subset X$ via π_Z . Since $\operatorname{Sing}(Y) = Z$ and Yhas ordinary singularities along Z, the blow-up π_Z resolves the singularities of Y. This contradiction shows that $\operatorname{Sing}(Y_W) = Z_W$. Moreover, since Y has ordinary singularities along Z, the intersection of its strict transform Y_Z with the exceptional divisor E_Z of π_Z is transverse. This implies that the intersection $Y_{Z_W} \cap E_{Z_W}$ is also transverse, i.e., Y_W has ordinary singularities along Z_W .

We end this section by describing the intersection of some of the Y_I^{d} 's. This can be computed using elementary projective geometry. In what follows we adopt the following notation. Given two finite sets I and J, we define their distance to be

$$d(I,J) := |(I \cup J) \setminus (I \cap J)|.$$

We start by intersecting varieties Y_I^{d} 's of the same dimension.

Proposition 5.6. Let the assumptions and notation be as in Notation 5.3. Let $I_1, I_2 \subset \{1, \ldots, n+3\}$ be subsets with cardinality m_1 and m_2 , respectively, and suppose that $I_1 \cap I_2 = \emptyset$. Let k_1 and k_2 be integers such that $0 \leq k_i \leq \frac{n-m_i}{2}$, i = 1, 2, and $m_1 + 2k_1 - 1 = m_2 + 2k_2 - 1 =: d$. Set $s = \frac{m_1+m_2}{2}$ and suppose that $d \leq n - s$. Then

$$Y_{I_1}^d \cap Y_{I_2}^d = \bigcup_J Y_J^{d-s},$$

where the union is taken over all subsets $J \subset I_1 \cup I_2$ satisfying $d(I_i, J) = s$ for i = 1, 2.

Moreover, for a general point in any irreducible component of the above intersections, the intersection is transverse.

Proof. We note that the assumptions of the theorem imply that $d = k_1 + k_2 + s - 1$ and $m_1 - m_2 = 2(k_2 - k_1)$.

Let $J \subset I_1 \cup I_2$ be such that $d(I_i, J) = s$ for i = 1, 2. We shall prove that $Y_J^{d-s} \subset Y_{I_1}^d \cap Y_{I_2}^d$. Write $J = J_1 \cup J_2$, where $J_i \subset I_i$, i = 1, 2, set $\ell_i := |J_i|$, i = 1, 2, and $\ell = |J| = \ell_1 + \ell_2$. The assumption that $d(I_i, J) = s$ for i = 1, 2 implies that $k_2 - k_1 = \ell_1 - \ell_2$. We set $k := k_2 - \ell_1 = k_1 - \ell_2$, and note that $d - s = \ell + 2k - 1$.

Let $x \in Y_J^{d-s}$. Then there exists a point $q \in Sec_k(C)$ such that $x \in \langle q, p_i | i \in J \rangle \cong \mathbb{P}^{\ell}$. The following two linear subspaces of this \mathbb{P}^{ℓ}

$$\langle x, p_i \mid i \in I_1 \rangle \cong \mathbb{P}^{\ell_1} \text{ and } \langle q, p_i \mid i \in I_2 \rangle \cong \mathbb{P}^{\ell_2}$$

have complementary dimensions. Hence there exists a point

 $z \in \langle x, p_i \mid i \in J_1 \rangle \cap \langle q, p_i \mid i \in J_2 \rangle.$

In particular, $z \in \mathbb{S}ec_{k+\ell_2}(C)$. Since $k + \ell_2 = k_1$, we conclude that $x \in Y_{I_1}^d$. Similarly we show that $x \in Y_{I_2}^d$.

Now assume that x is a general point of Y_J^{d-s} . Keeping the same notation as above, we will prove now that $Y_{I_1}^d$ and $Y_{I_2}^d$ intersect transversely at x. This amounts to proving that $T_x(Y_{I_1}^d) \cap T_x(Y_{I_2}^d) = T_x(Y_J^{d-s})$. By Terracini's Lemma [Te], we have

$$T_x(Y_{I_1}^d) = \langle \langle p_i \mid i \in I_1 \rangle, \langle T_{q_i}C \mid 1 \le i \le k \rangle, \langle T_{p_i}C \mid i \in J_2 \rangle \rangle,$$

$$T_x(Y_{I_2}^d) = \langle \langle p_i \mid i \in I_2 \rangle, \langle T_{q_i}C \mid 1 \le i \le k \rangle, \langle T_{p_i}C \mid i \in J_1 \rangle \rangle,$$

$$T_x(Y_J^{d-s}) = \langle \langle p_i \mid i \in J \rangle, \langle T_{q_i}C \mid 1 \le i \le k \rangle \rangle,$$

where $q_1, \ldots, q_k \in C$ are such that $q \in \langle q_i | 1 \le i \le k \rangle$. Consider the linear subspaces:

$$\begin{split} L_1 &:= \langle \langle p_i \mid i \in I_1 \rangle, \ \langle \ T_{p_i}C \mid i \in J_2 \rangle \rangle, \\ L_2 &:= \langle \langle p_i \mid i \in I_2 \rangle, \ \langle \ T_{p_i}C \mid i \in J_1 \rangle \rangle, \\ L &:= \langle \langle p_i \mid i \in J \rangle \rangle \subset L_1 \cap L_2. \end{split}$$

We have that $\dim(\langle L_1, L_2 \rangle) \leq m_1 + m_2 + \ell - 1$, and equality holds if and only if $L_1 \cap L_2 = L$. On the other hand, note that L intersects C in at least $m_1 + m_2 + \ell$ points, counted with multiplicity. Therefore we must have $\dim(\langle L_1, L_2 \rangle) = m_1 + m_2 + \ell - 1$, and $L_1 \cap L_2 = L$. It follows from the description of the tangent spaces above that $T_x(Y_{I_1}^d) \cap T_x(Y_{I_2}^d) = T_x(Y_J^{d-s})$.

It remains to prove that $Y_{I_1}^d \cap Y_{I_2}^d \subset \bigcup_J Y_J^{d-s}$. Write $\{p_i \mid i \in I_1\} = \{x_1, \ldots, x_{m_1}\}$ and $\{p_i \mid i \in I_2\} = \{y_1, \ldots, y_{m_2}\}$. Suppose that $x \in Y_{I_1}^d \cap Y_{I_2}^d$. This means that there exist points $z_1, \ldots, z_{k_1}, w_1, \ldots, w_{k_2} \in C$ such that:

$$\langle x_1, \dots, x_{m_1} \rangle \cap \langle z_1, \dots, z_{k_1} \rangle = \emptyset = \langle y_1, \dots, y_{m_2} \rangle \cap \langle w_1, \dots, w_{k_2} \rangle, \text{ and }$$
$$x \in \langle x_1, \dots, x_{m_1}, z_1, \dots, z_{k_1} \rangle \cap \langle y_1, \dots, y_{m_2}, w_1, \dots, w_{k_2} \rangle.$$

The assumption that $d \leq n - s$ implies that $m_1 + m_2 + k_1 + k_2 \leq n + 1$, and thus

$$\langle x_1,\ldots,x_{m_1},z_1,\ldots,z_{k_1}\rangle \cap \langle y_1,\ldots,y_{m_2},w_1,\ldots,w_{k_2}\rangle$$

 $\langle \{x_1, \ldots, x_{m_1}, z_1, \ldots, z_{k_1}\} \cap \{y_1, \ldots, y_{m_2}, w_1, \ldots, w_{k_2}\} \rangle.$

By relabeling the points if necessary, we may write, for suitable integers s_1 , s_2 and r:

$$\{x_1, \dots, x_{s_1}\} = \{x_1, \dots, x_{m_1}\} \cap \{w_1, \dots, w_{k_2}\}$$

$$\{y_1, \dots, y_{s_2}\} = \{y_1, \dots, y_{m_2}\} \cap \{z_1, \dots, z_{k_1}\}$$

$$\{z_1 = w_1, \dots, z_r = w_r\} = \{z_1, \dots, z_{k_1}\} \cap \{w_1, \dots, w_{k_2}\}.$$

Note that $s_i + r \leq k_j$, $\{i, j\} = \{1, 2\}$, and we have

$$(5.4) x \in \langle x_1, \ldots, x_{s_1}, y_1, \ldots, y_{s_2}, z_1, \ldots, z_r \rangle.$$

Let $J_0 \subset I_1 \cup I_2$ be the subset of indices corresponding to the subset $\{x_1, \ldots, x_{s_1}, y_1, \ldots, y_{s_2}\} \subset \{p_1, \ldots, p_{n+3}\}$. Note that $d(J_0, I_i) = m_i - s_i + s_j$, for $\{i, j\} = \{1, 2\}$. In particular we have

$$d(J_0, I_1) + d(J_0, I_2) = 2s.$$

Suppose first that $d(J_0, I_1) = d(J_0, I_2) = s$. It follows from (5.4) that

 $x \in \text{Join} (\langle p_i \mid i \in J_0 \rangle, \mathbb{S}ec_r(C)).$

Since $s_i + r \le k_j$, $\{i, j\} = \{1, 2\}$, we get that

$$|J_0| + 2r - 1 = s_1 + s_2 + 2r - 1 \le k_1 + k_2 - 1 = d - s.$$

Hence $x \in Y_{J_0}^{d-s}$.

From now on we consider the case when $d(J_0, I_1) \neq d(J_0, I_2)$. Without lost of generality, we assume that

$$d(J_0, I_1) - d(J_0, I_2) = m_1 - m_2 + 2s_2 - 2s_1 > 0.$$

We will modify the subset $J_0 \subset I_1 \cup I_2$ by adding points of $I_1 \setminus J_0$ or removing points of $I_2 \cap J_0$ to obtain another subset $J \subset I_1 \cup I_2$ satisfying $d(I_i, J) = s$ for i = 1, 2. Note that if $i \in I_1 \setminus J_0$, then $d(J_0 \cup \{i\}, I_1) = d(J_0, I_1) - 1$ and $d(J_0 \cup \{i\}, I_2) = d(J_0, I_2) + 1$. Similarly, if $i \in I_2 \cap J_0$, then $d(J_0 \setminus \{i\}, I_1) = d(J_0, I_1) - 1$ and $d(J_0 \setminus \{i\}, I_2) = d(J_0, I_2) + 1$. So we have to modify J_0 by adding or removing exactly $\frac{m_1 - m_2}{2} + s_2 - s_1$ points of the appropriate I_i .

Suppose first that $|I_1 \setminus J_0| = m_1 - s_1 \ge \frac{m_1 - m_2}{2} + s_2 - s_1$. This is equivalent to the inequality $s \ge s_2$. We construct $J_1 \subset I_1 \cup I_2$ by adding to J_0 exactly $\frac{m_1 - m_2}{2} + s_2 - s_1$ points of $I_1 \setminus J_0$. Then $d(I_i, J_1) = s$ for i = 1, 2, and it follows from (5.4) that

$$x \in \text{Join} (\langle p_i \mid i \in J_1 \rangle, \mathbb{S}ec_r(C)).$$

Since $s_2 + r \leq k_1$, we get that

$$|J_1| + 2r - 1 = (k_2 - k_1 + 2s_2) + 2r - 1 \le k_1 + k_2 - 1 = d - s_1$$

Hence $x \in Y_{J_1}^{d-s}$.

Next we suppose that $s < s_2$. Let $I'_2 \subset I_2$ be the subset of indices corresponding to the subset $\{y_1, \ldots, y_s\}$, and set $J_2 := I_1 \cup I'_2$. Then $d(I_i, J_2) = s$ for i = 1, 2, and it follows from (5.4) that

 $x \in \text{Join} (\langle p_i | i \in J_2 \rangle, \mathbb{S}ec_{r+s_2-s}(C)).$

Since $s_2 + r \leq k_1$, we get that

$$|J_2| + 2(r+s_2-s) - 1 = m_1 + 2(r+s_2) - s - 1 \le m_1 + 2k_1 - 1 - s = d - s.$$

Hence $x \in Y_{J_2}^{d-s}$.

5.2. The odd case n = 2h+1. In this subsection we construct divisors Δ making $X_{n+3}^n \log$ Fano when n = 2h + 1 is odd. We follow Notation 5.3. For each $1 \le i \le 3$, let $\Delta_i \subset X_{n+3}^n$ be the strict transform of the divisor $Y_i^{2h} \subset \mathbb{P}^n$, and denote by $H_{4,\dots,n+3} \subset X_{n+3}^n$ the strict transform of the hyperplane $\langle p_4, ..., p_{n+3} \rangle \subset \mathbb{P}^{n+3}$.

Theorem 5.7. Let $n = 2h + 1 \ge 5$ be an odd integer. Set

$$D := \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup H_{4,\dots,n+3} \subset X_{n+3}^n.$$

For any $\frac{2h-2}{3h-2} < \epsilon < \frac{2h}{3h+1}$ the divisor $-(K_{X_{n+3}^n} + \epsilon D)$ is ample, and the pair $(X_{n+3}^n, \epsilon D)$ is klt.

For the proof of Theorem 5.7, we will need the following.

Proposition 5.8. Let the assumptions be as in Theorem 5.7, and follow Notation 5.3. For $0 \le m \le n-3$, we define a modification X_m of X_{n+3}^n recursively as follows:

- $X_0 = X_{n+3}^n$, X_{2k+1} is the blow-up of X_{2k} along the strict transforms of $\mathbb{S}ec_{k+1}(C)$, and of the $Y_{i,j}^{2k+1}$'s $(0 \le k \le h-2)$,
- X_{2k} is the blow-up of X_{2k-1} along the strict transforms of the Y_i^{2k} 's, and of $Y_{1,2,3}^{2k}$ $(1 \le k \le h - 1),$
- X_{n-2} is the blow-up of X_{n-3} along the strict transform of $Sec_h(C)$.

Then, for any $0 \le m \le n-3$, the center of the blow-up $X_{m+1} \to X_m$ is a disjoint union of smooth subvarieties, all transverse to the exceptional divisors of $X_m \to X_0$. Moreover, the composition $\pi: X_{n-2} \to X_{n+3}^n$ of these blow-ups is a log resolution of the pair (X_{n+3}^n, D) .

Proof. We will prove the result by induction on m. The statement is clearly try for m = 0. For simplicity of notation we will denote by \widetilde{Z} the strict transform of a subvariety $Z \subset X_{n+3}^n$ in any X_m .

Suppose that the statement is true m = 2k. We will show that it holds for X_{2k+1} and X_{2k+2} . We start with the following observation. Let $I \subset \{1, \ldots, n+3\}$ be such that either $|I| \in \{0, 1, 2\}$ or $I = \{1, 2, 3\}, 0 \le k \le \frac{n - |I|}{2}$, and d = 2k - 1 + |I|. Then, for any m < d - 1, each component of the center of the blow-up $X_{m+1} \to X_m$ is either contained in $\operatorname{Sing}(Y_I^d)$, or is disjoint from it. This allows us to apply Proposition 5.5, together with Propositions 5.2 and 5.4, and conclude by induction that the following holds.

- The subvarieties $\widetilde{Y}_i^{2k+2} \subset X_{2k+1}, 1 \leq i \leq 3$, are smooth and transverse to the exceptional divisors over X_0 .
- The subvarieties $\widetilde{\mathbb{S}ec_{k+2}(C)}$, $\widetilde{Y}_{i,j}^{2k+3} \subset X_{2k+2}$, $1 \leq i < j \leq 3$, are smooth and transverse to the exceptional divisors over X_0 .

Next we show that the \widetilde{Y}_i^{2k+2} 's and $\widetilde{Y}_{1,2,3}^{2k+2}$ are pairwise disjoint in X_{2k+1} , and similarly for $\widetilde{Sec_{k+2}(C)}$ and the $\widetilde{Y}_{i,j}^{2k+3}$'s in X_{2k+2} . Consider the blow-up $X_{2k+1} \to X_{2k}$. By Proposition 5.6, on X_{2k} we have

$$\widetilde{Y}_{i}^{2k+2} \ \cap \ \widetilde{Y}_{j}^{2k+2} \ = \ \widetilde{\mathbb{S}ec_{k+1}(C)} \ \cup \ \widetilde{Y}_{i,j}^{2k+1}, \quad \widetilde{Y}_{i}^{2k+2} \cap \widetilde{Y}_{i,r,s}^{2k+2} = \widetilde{Y}_{i,r}^{2k+1} \cup \widetilde{Y}_{i,s}^{2k+1}.$$

By the induction hypothesis, $\widetilde{Sec_{k+1}(C)}$ and $\widetilde{Y}_{i,j}^{2k+1}$ are smooth and disjoint. So the inter-sections are everywhere transverse. We conclude that on X_{2k+1} , which is obtained from X_{2k} by blowing-up $\widetilde{\mathbb{Sec}_{k+1}(C)}$ and $\widetilde{Y}_{i,j}^{2k+1}$, the \widetilde{Y}_i^{2k+2} 's and $\widetilde{Y}_{1,2,3}^{2k+2}$ are pairwise disjoint. Now consider the blow-up $X_{2k+2} \to X_{2k+1}$. By Proposition 5.6, on X_{2k+1} we have

$$\widetilde{\mathbb{S}ec_{k+2}(C)} \cap \widetilde{Y}_{i,j}^{2k+3} = \widetilde{Y}_i^{2k+2} \cup \widetilde{Y}_j^{2k+2}, \quad \widetilde{Y}_{i,j}^{2k+3} \cap \widetilde{Y}_{i,r}^{2k+3} = \widetilde{Y}_i^{2k+2} \cup \widetilde{Y}_{i,j,r}^{2k+2}$$

By the induction hypothesis, the \tilde{Y}_i^{2k+2} 's and $\tilde{Y}_{1,2,3}^{2k+2}$ are smooth and pairwise disjoint. So the intersections are everywhere transverse. We conclude that on X_{2k+2} , which is obtained from X_{2k+1} by blowing-up the \widetilde{Y}_i^{2k+2} 's and $\widetilde{Y}_{1,2,3}^{2k+2}$, the varieties $\widetilde{\mathbb{Sec}_{k+2}(C)}$ and the $\widetilde{Y}_{i,i}^{2k+3}$'s are pairwise disjoint.

As before, we have that the divisors $\widetilde{H}_{4,\dots,n+3}$, $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$ and $\widetilde{\Delta}_3$ on X_{n-3} are smooth and transverse to the exceptional divisors over X_0 , and their intersection are pairwise smooth and everywhere transverse. By Proposition 5.6 we have

$$\widetilde{\Delta}_1 \cap \widetilde{\Delta}_2 \cap \widetilde{\Delta}_3 = \widetilde{\mathbb{Sec}_h(C)}.$$

So, after the blow-up $X_{n-2} \to X_{n-3}$ of $Sec_h(C)$, we get a log resolution of (X_{n+3}^n, D) .

Proof of Theorem 5.7. With Notation 3.1, we have

$$D = \Delta_1 + \Delta_2 + \Delta_3 + H_{4,\dots,n+3} \sim (3h+4)H - (3h+1)(E_1 + \dots + E_{n+3}).$$

Recall from Proposition 1.4 that the Mori cone of X_{n+3}^n is generated by the classes R_i 's and $L_{i,j}$'s. One computes

$$-(K_{X_{n+3}^n} + \epsilon D) \cdot R_i = 2h - \epsilon(3h+1) \text{ and } -(K_{X_{n+3}^n} + \epsilon D) \cdot L_{i,j} = \epsilon(3h-2) - 2h + 2.$$

Therefore $-K_{X_{n+3}^n} - \epsilon D$ is ample provided that $\frac{2h-2}{3h-2} < \epsilon < \frac{2h}{3h+1}$. Next we check when the pair $(X_{n+3}^n, \epsilon D)$ is klt. Let $\pi : \widetilde{X} := X_{n-2} \to X_{n+3}^n$ be the log

resolution of $(X_{n+3}^n, \epsilon D)$ introduced in Proposition 5.8 above. We have

$$K_{\widetilde{X}} = \pi^* K_{X_{n+3}^n} + \sum_{k=1}^h (n-2k) E_{\mathbb{S}ec_k(C)} + \sum_{k=1}^{h-1} (n-2k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} + \sum_{k=1}^{h-1} (n-2k-1) (\sum_i E_{Y_i^{2k}} + E_{Y_{1,2,3}^{2k}}) + \sum_{k=1}^{h-1} (n-2k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} + \sum_{k=1}^{h-1} (n-2k-1) (\sum_i E_{Y_i^{2k}} + E_{Y_{1,2,3}^{2k}}) + \sum_{k=1}^{h-1} (n-2k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} + \sum_{k=1}^{h-1} (n-2k-1) (\sum_i E_{Y_i^{2k-1}} + E_{Y_{1,2,3}^{2k-1}}) + \sum_{k=1}^{h-1} (n-2k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} + \sum_{k=1}^{h-1} (n-2k-1) (\sum_i E_{Y_i^{2k-1}} + E_{Y_{1,2,3}^{2k-1}}) + \sum_{k=1}^{h-1} (n-2k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} + \sum_{i,j} E_{Y_{i,j}^{2$$

Here we denote by E_Y the exceptional divisor with center $Y \subset \mathbb{P}^n$. In order to compute discrepancies, we will compute the the multiplicities of the Y_i^{2h} 's along the images in \mathbb{P}^n of the subvarieties blown-up by π . By Proposition 5.1 we have $\operatorname{mult}_{\operatorname{Sec}_k(C)} \operatorname{Sec}_h(C) = h - k + 1$. Moreover, $\operatorname{mult}_{\operatorname{Sec}_k(C)} Y_r^{2h} = h - k + 1,$

$$\operatorname{mult}_{Y_{i,j}^{2k-1}} Y_r^{2h} = \begin{cases} \operatorname{mult}_{\operatorname{Sec}_k(C)} \operatorname{Sec}_h(C) = h - k + 1 & \text{if } r \in \{i,j\},\\ \operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_h(C) = h - k & \text{if } r \notin \{i,j\}, \end{cases}$$

$$\operatorname{mult}_{Y_i^{2k}} Y_r^{2h} = \begin{cases} \operatorname{mult}_{\operatorname{Sec}_k(C)} \operatorname{Sec}_h(C) = h - k + 1 & \text{if } r = i, \\ \operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_h(C) = h - k & \text{if } r \neq i, \end{cases}$$

and $\operatorname{mult}_{Y_{1,2,3}^{2k}} Y_r^{2h} = \operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_h(C) = h - k$ for for k = 1, ..., h - 1. Let $\Delta \subset \mathbb{P}^n$ be the divisor whose strict transform is D. We have

(5.5)
$$\begin{array}{rcl} \operatorname{mult}_{Sec_k(C)}\Delta &=& 3(h-k+1),\\ \operatorname{mult}_{Y_{i,j}^{2k-1}}\Delta &=& 2(h-k+1)+h-k=3(h-k)+2,\\ \operatorname{mult}_{Y_{i,j,r}^{2k}}\Delta &=& 3(h-k),\\ \operatorname{mult}_{Y_{i,j,r}^{2k}}\Delta &=& h-k+1+2(h-k)=3h-3k+1. \end{array}$$

Equalities 5.5 yield:

$$\begin{aligned} \pi^*(D) &= \quad \widetilde{D} + \sum_{k=1}^h 3(h-k+1) E_{\mathbb{S}ec_k(C)} + \sum_{k=1}^{h-1} (3(h-k)+2) \sum_{i,j} E_{Y_{i,j}^{2k-1}} \\ &+ \sum_{k=1}^{h-1} (3h-3k+1) \sum_i E_{Y_i^{2k}} + \sum_{k=1}^{h-1} 3(h-k) E_{Y_{1,2,3}^{2k}}, \end{aligned}$$

and hence

$$\begin{split} K_{\widetilde{X}} &= \pi^* (K_{X_{n+3}^n} + \epsilon D) &+ \sum_{k=1}^h (2h - 2k + 1 - 3\epsilon(h - k + 1)) E_{\mathbb{S}ec_k(C)} \\ &+ \sum_{k=1}^{h-1} (2h - 2k + 1 - \epsilon(3(h - k) + 2)) \sum_{i,j} E_{Y_{i,j}^{2k-1}} \\ &+ \sum_{k=1}^{h-1} (2(h - k) - \epsilon(3h - 3k + 1)) \sum_i E_{Y_{i,2}^{2k}} \\ &+ \sum_{k=1}^{h-1} (2(h - k) - \epsilon(3h - 3k)) E_{Y_{1,2,3}^{2k}} - \epsilon D. \end{split}$$

Therefore the pair $(X_{n+3}^n, \epsilon D)$ is klt for any $0 \le \epsilon < \frac{2}{3}$.

5.3. The even case n = 2h. In this subsection we construct divisors Δ making $X_{n+3}^n \log$ Fano when n = 2h is even. We follow Notation 5.3. For each $1 \leq i < j \leq n+3$, let $\Delta_{i,j} \subset X_{n+3}^n$ be the strict transform of the divisor $Y_{i,j}^{2h-1} \subset \mathbb{P}^n$, and denote by $H_{5,\dots,n+3} \subset X_{n+3}^n$ the strict transform of a general hyperplane in \mathbb{P}^{n+3} through p_5, \dots, p_{n+3} .

Theorem 5.9. Let $n = 2h \ge 4$ be an even integer. Set

$$D := \Delta_{1,2} \cup \Delta_{3,4} \cup \widetilde{Sec_h(C)} \cup H_{5,\dots,n+3} \subset X_{n+3}^n.$$

For any $\frac{2h-3}{3h-4} < \epsilon < \frac{2h-1}{3h-1}$ the divisor $-(K_{X_{n+3}^n} + \epsilon D)$ is ample, and the pair $(X_{n+3}^n, \epsilon D)$ is klt.

To provide a log resolution of the pair (X_{n+3}^n, D) in Proposition 5.11 below, we will need the following result.

Lemma 5.10. Any point of $Y_{1,2}^{n-1} \cap Y_{3,4}^{n-1} \subset \mathbb{P}^n$ which is smooth for both divisors $Y_{1,2}^{n-1}$ and $Y_{3,4}^{n-1}$ is a smooth point of $Y_{1,2}^{n-1} \cap Y_{3,4}^{n-1}$.

Proof. Let $x \in (Y_{1,2}^{n-1} \cap Y_{3,4}^{n-1}) \setminus (\operatorname{Sing}(Y_{1,2}^{n-1}) \cup \operatorname{Sing}(Y_{3,4}^{n-1}))$. We shall prove that the intersection of $Y_{1,2}^{n-1}$ and $Y_{3,4}^{n-1}$ is transverse at x, that is, $T_x Y_{1,2}^{n-1} \neq T_x Y_{3,4}^{n-1}$.

Suppose otherwise, and set $P = T_x Y_{1,2}^{n-1} = T_x Y_{3,4}^{n-1}$. By Terracini's Lemma [Te] we have

$$P = \langle p_1, p_2, T_{z_1}C, ..., T_{z_{h-1}}C \rangle = \langle p_2, p_3, T_{w_1}C, ..., T_{w_{h-1}}C \rangle$$

for suitable $z_i, w_i \in C$, with $z_i \notin \{p_1, p_2\}$ and $w_i \notin \{p_3, p_4\}$. Set $s = |\{z_1, ..., z_{h-1}\} \cap \{w_1, ..., w_{h-1}\}|$, $r = |\{z_1, ..., z_{h-1}\} \cap \{p_3, p_4\}|$, and $t = |\{w_1, ..., w_{h-1}\} \cap \{p_1, p_2\}|$. We may assume that $r \geq t$. Then $s \leq h - 1 - r$, and the number of intersection points in $P \cap C$, counted with multiplicity, is at least

$$2(2(h-1)-s) + 2 - r + 2 - t \ge n + 2 + r - t \ge n + 2.$$

This is impossible since C has degree n.

Proposition 5.11. Let the assumptions be as in Theorem 5.9, and follow Notation 5.3. For $0 \le m \le n-3$, we define a modification X_m of X_{n+3}^n recursively as follows:

- $X_0 = X_{n+3}^n$,
- X_{2k+1} is the blow-up of X_{2k} along the strict transforms of $\mathbb{S}ec_{k+1}(C)$ and the $Y_{i,j}^{2k+1}$'s $(0 \le k \le h-3)$, and also of $Y_{1,2,3,4}^{2k+1}$ if $0 < k \le h-3$,
- X_{2k} is the blow-up of X_{2k-1} along the strict transforms of the Y_i^{2k} 's, and of the $Y_{i,j,r}^{2k}$'s $(1 \le k \le h-2)$.
- X_{n-3} is the blow-up of X_{n-4} along the strict transforms of $\mathbb{S}ec_{h-1}(C)$, of $Y_{1,2}^{2h-3}$ and of $Y_{3,4}^{2h-3}$.

Then, for any $0 \le m \le n-4$, the center of the blow-up $X_{m+1} \to X_m$ is a disjoint union of smooth subvarieties, all transverse to the exceptional divisors of $X_m \to X_0$. Moreover, the composition $\pi: X_{n-3} \to X_{n+3}^n$ of these blow-ups is a log resolution of the pair (X_{n+3}^n, D) .

Proof. Using the same arguments as in the proof of Proposition 5.8, we can prove that, for any $0 \le m \le n-4$, the center of the blow-up $X_{m+1} \to X_m$ is a disjoint union of smooth subvarieties, all transverse to the exceptional divisors of $X_m \to X_0$. Moreover, the strict transforms of $\Delta_{1,2}$, $\Delta_{3,4}$, $\widehat{Sec_h(C)}$ and $H_{5,\dots,2h+3}$ in X_{n-3} are smooth and transverse to the exceptional divisors over X_0 , and the intersection $\widehat{Sec_h(C)} \cap \widetilde{Y}_{i,j}^{2h-1}$ is transverse. Clearly the strict transform of $H_{5,\dots,2h+3}$ is transverse to $\widehat{Sec_h(C)}$, $\Delta_{1,2}$, $\Delta_{3,4}$ and to all exceptional

divisors. To show that the strict transform of D in X_{n-3} is simple normal crossing, it remains to compute $\Delta_{1,2} \cap \Delta_{3,4}$. Note that we cannot use Proposition 5.6 in this case. To compute $\Delta_{1,2} \cap \Delta_{3,4}$, we first describe the intersection of $Y_{i,j}^{n-1}$ and $\operatorname{Sing}(Y_{r,s}^{n-1}) = Y_{r,s}^{n-3}$.

Claim 5.12. We have

$$Y_{i,j}^{n-1} \cap Y_{r,s}^{n-3} = Y_r^{n-4} \cup Y_s^{n-4} \cup Y_{i,r,s}^{n-4} \cup Y_{j,r,s}^{n-4}.$$

Moreover, at a general point in any irreducible component of this intersection, the intersection is transverse.

Proof. Note that $Y_{i,j}^{n-1} \cap Y_{r,s}^{n-3} = (Y_{i,j}^{n-1} \cap Y_{i,j,r,s}^{n-1}) \cap Y_{r,s}^{n-3}$. By Proposition 5.6, $Y_{i,j}^{n-1} \cap Y_{i,j,r,s}^{n-1} = Y_{i,j,r}^{n-2} \cup Y_{i,j,s}^{n-2}$. Applying Proposition 5.6 repeatedly, we have that

$$Y_{i,j,r}^{n-2} \cap Y_{r,s}^{n-3} = (Y_{i,j,r}^{n-2} \cap Y_{i,r,s}^{n-2}) \cap Y_{r,s}^{2h-3} = (Y_{i,r}^{n-3} \cup Y_{i,j,r,s}^{n-3}) \cap Y_{r,s}^{n-3} = Y_r^{n-4} \cup Y_{i,r,s}^{n-4} \cup Y_{j,r,s}^{n-4}.$$

Similarly we show that $Y_{i,j,s}^{n-2} \cap Y_{r,s}^{n-3} = Y_s^{n-4} \cup Y_{i,r,s}^{2h-4} \cup Y_{j,r,s}^{n-4}.$

The strict transforms $\widetilde{Y}_{1,2}^{n-1}$ and $\widetilde{Y}_{3,4}^{n-1}$ in X_{n-4} are still singular along $\widetilde{Y}_{1,2}^{n-3}$ and $\widetilde{Y}_{3,4}^{n-3}$, respectively. However, by Claim 5.12 we have

$$\widetilde{Y}_{1,2}^{n-1} \cap \operatorname{Sing}(\widetilde{Y}_{3,4}^{n-1}) = \widetilde{Y}_{3,4}^{n-1} \cap \operatorname{Sing}(\widetilde{Y}_{1,2}^{n-1}) = \emptyset.$$

Hence, by Lemma 5.10, in X_{n-3} , $\widetilde{Y}_{1,2}^{n-1} \cap \widetilde{Y}_{3,4}^{n-1}$ is smooth, and so the intersection $\widetilde{Y}_{1,2}^{n-1} \cap \widetilde{Y}_{3,4}^{n-1}$ is transverse.

22

Proof of Theorem 5.9. With Notation 3.1, we have

$$D = \Delta_{1,2} \cup \Delta_{3,4} \cup \mathbb{S}ec_h(C) \cup H_{5,\dots,2h+3} \sim (3h+2)H - (3h-1)(E_1 + \dots + E_{n+3})$$

and

$$-K_{X_{n+3}^n} - \epsilon D \sim (2h+1-\epsilon(3h+2))H - (2h-1-\epsilon(3h-1))(E_1 + \dots + E_{n+3}).$$

Recall from Proposition 1.4 that the Mori cone of X_{n+3}^n is generated by the classes R_i 's and $L_{i,j}$'s. One computes

$$(-K_{X_{n+3}^n} - \epsilon D) \cdot R_i = 2h - 1 - \epsilon(3h - 1)$$
 and $(-K_{X_{n+3}^n} - \epsilon D) \cdot L_{i,j} = \epsilon(3h - 4) - 2h + 3.$

Therefore $-K_{X_{n+3}^n} - \epsilon D$ is ample provided that $\frac{2h-3}{3h-4} < \epsilon < \frac{2h-1}{3h-1}$.

Next we check when the pair $(X_{n+3}^n, \epsilon D)$ is klt. Let $\pi : \widetilde{X} := X_{n-3} \to X_{n+3}^n$ be the log resolution of $(X_{n+3}^n, \epsilon D)$ introduced in Proposition 5.11 above. We have

$$\begin{split} K_{\widetilde{X}} &= \pi^* K_{X_{n+3}^n} + \sum_{k=1}^{h-1} (n-2k) E_{\mathbb{S}ec_k(C)} + \sum_{k=1}^{h-1} (n-2k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} \\ &+ \sum_{k=1}^{h-2} (n-2k-1) (\sum_i E_{Y_i^{2k}} + \sum_{i,j,r} E_{Y_{i,j,r}^{2k}}) + \sum_{k=2}^{h-2} (n-2k) E_{Y_{1,2,3,4}^{2k-1}} . \end{split}$$

Here we denote by E_Y the exceptional divisor with center $Y \subset \mathbb{P}^n$.

In order to compute discrepancies, we will compute the the multiplicities of $Sec_h(C)$, $Y_{1,2}^{2h-1}$, and $Y_{3,4}^{2h-1}$ along the images in \mathbb{P}^n of the subvarieties blown-up by π . We start with the divisor $Y_{i,j}^{2h-1}$. For $1 \le k \le h-1$, we have:

$$\operatorname{mult}_{Y_{r,s}^{2k-1}} Y_{i,j}^{2h-1} = \begin{cases} \operatorname{mult}_{\mathbb{S}ec_{k-1}(C)} \mathbb{S}ec_{h-1}(C) = h - k + 1 & \text{if } \{i,j\} = \{r,s\}, \\ \operatorname{mult}_{\mathbb{S}ec_{k}(C)} \mathbb{S}ec_{h-1}(C) = h - k & \text{if } |\{i,j\} \cap \{r,s\}| = 1, \\ \operatorname{mult}_{\mathbb{S}ec_{k+1}(C)} \mathbb{S}ec_{h-1}(C) = h - k - 1 & \text{if } \{i,j\} \cap \{r,s\} = \emptyset. \end{cases}$$

$$\operatorname{mult}_{Y_{r,s,t}^{2k}} Y_{i,j}^{2h-1} = \begin{cases} \operatorname{mult}_{\mathbb{S}ec_{k+2}(C)} \mathbb{S}ec_{h-1}(C) = h - k - 2 & \text{if } i, \{i,j\} \cap \{r,s,t\} = \emptyset, \\ \operatorname{mult}_{\mathbb{S}ec_{k+1}(C)} \mathbb{S}ec_{h-1}(C) = h - k - 1 & \text{if } |\{i,j\} \cap \{r,s,t\}| = 1, \\ \operatorname{mult}_{\mathbb{S}ec_{k}(C)} \mathbb{S}ec_{h-1}(C) = h - k & \text{if } \{i,j\} \subset \{r,s,t\}. \end{cases}$$

$$\operatorname{mult}_{Y_r^{2k}} Y_{i,j}^{2h-1} = \begin{cases} \operatorname{mult}_{\operatorname{Sec}_{k+1}(C)} \operatorname{Sec}_{h-1}(C) = h-k-1 & \text{if } r \notin \{i,j\},\\ \operatorname{mult}_{\operatorname{Sec}_k(C)} \operatorname{Sec}_{h-1}(C) = h-k & \text{if } r \in \{i,j\}. \end{cases}$$

$$\operatorname{mult}_{Y_{1,2,3,4}^{2k-1}} Y_{i,j}^{2h-1} = \operatorname{mult}_{\operatorname{Sec}_k(C)} \operatorname{Sec}_{h-1}(C) = h - k.$$

Next we consider the divisor $\mathbb{S}ec_h(C)$. For $1 \leq k \leq h-1$, we have:

$$\begin{aligned} & \operatorname{mult}_{\operatorname{Sec}_{k}(C)}\operatorname{Sec}_{h}(C) &= h - k + 1, \\ & \operatorname{mult}_{Y_{i,j}^{2k-1}}\operatorname{Sec}_{h}(C) &= \operatorname{mult}_{\operatorname{Sec}_{k+1}(C)}\operatorname{Sec}_{h}(C) = h - k, \\ & \operatorname{mult}_{Y_{i,j,r}^{2k}}\operatorname{Sec}_{h}(C) &= \operatorname{mult}_{\operatorname{Sec}_{k+2}(C)}\operatorname{Sec}_{h}(C) = h - k - 1, \\ & \operatorname{mult}_{Y_{i}^{2k}}\operatorname{Sec}_{h}(C) &= \operatorname{mult}_{\operatorname{Sec}_{k+1}(C)}\operatorname{Sec}_{h}(C) = h - k, \\ & \operatorname{mult}_{Y_{i,2,3,4}^{2k-1}}\operatorname{Sec}_{h}(C) &= \operatorname{mult}_{\operatorname{Sec}_{k+2}(C)}\operatorname{Sec}_{h}(C) = h - k. \end{aligned}$$

Now let $\overline{D} \subset \mathbb{P}^n$ be the divisor whose strict transform is D. The above formulas yield:

$$\begin{split} & \operatorname{mult}_{\operatorname{Sec}_k(C)} D &= 2(h-k) + (h-k+1) = 3h - 3k + 1, \\ & \operatorname{mult}_{Y_{i,j}^{2k-1}} \bar{D} &= 2(h-k) + (h-k) = (h-k+1) + (h-k-1) + (h-k) = 3h - 3k, \\ & \operatorname{mult}_{Y_{i,j,r}^{2k}} \bar{D} &= (h-k-1) + (h-k) + (h-k-2) = 3h - 3k - 3, \\ & \operatorname{mult}_{Y_i^{2k}} \bar{D} &= (h-k-1) + (h-k) + (h-k) = 3h - 3k - 1, \\ & \operatorname{mult}_{Y_{i,2,3,4}^{2k-1}} \bar{D} &= 2(h-k) + h - k - 1 = 3h - 3k - 1. \end{split}$$

Thus

$$\begin{aligned} \pi^*(D) &= \quad \widetilde{D} + \sum_{k=1}^{h-1} (3h - 3k + 1) E_{\mathbb{S}ec_k(C)} + \sum_{k=1}^{h-1} (3h - 3k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} \\ &+ \sum_{k=1}^{h-2} (3h - 3k - 1) \sum_i E_{Y_i^{2k}} + \sum_{k=1}^{h-2} (3h - 3k - 3) \sum_{i,j,r} E_{Y_{i,j,r}^{2k}} \\ &+ \sum_{k=2}^{h-2} (3h - 3k - 1) E_{Y_{1,2,3,4}^{2k-1}}, \end{aligned}$$

and hence

$$\begin{split} K_{\widetilde{X}} &= \pi^* (K_{X_{n+3}^n} + \epsilon D) &+ \sum_{k=1}^{h-1} (2h - 2k - \epsilon (3h - 3k + 1)) E_{\mathbb{S}ec_k(C)} \\ &+ \sum_{k=1}^{h-1} (2h - 2k - \epsilon (3h - 3k)) \sum_{i,j} E_{Y_{i,j}^{2k-1}} \\ &+ \sum_{k=1}^{h-2} (2h - 2k - 1 - \epsilon (3h - 3k - 1)) \sum_i E_{Y_i^{2k}} \\ &+ \sum_{k=1}^{h-2} (2h - 2k - 1 - \epsilon (3h - 3k - 3)) \sum_{i,j,r} E_{Y_{i,j,r}^{2k}} \\ &+ \sum_{k=2}^{h-2} (2h - 2k - \epsilon (3h - 3k - 1)) E_{Y_{1,2,3,4}^{2k-1}} - \epsilon \widetilde{D}. \end{split}$$

For $\epsilon < \frac{2h-1}{3h-2}$ all the discrepancies are greater than -1. Therefore, for $\frac{2h-3}{3h-4} < \epsilon < \frac{2h-1}{3h-1}$ the divisor $-K_{X_{n+3}^n} - \epsilon D$ is ample and the pair $(X_{n+3}^n, \epsilon D)$ is klt.

6. On a question of Hassett

In [Ha], Hassett introduced moduli spaces of weighted pointed curves. Given $g \ge 0$ and rational weight data $A[n] = (a_1, ..., a_n), 0 < a_i \leq 1$, satisfying $2g - 2 + \sum_{i=1}^n a_i > 0$, the moduli space $\overline{M}_{g,A[n]}$ parametrizes genus g nodal n-pointed curves $\{C, (x_1, ..., x_n)\}$ subject to the following stability conditions:

- Each x_i is a smooth point of C, and the points x_{i_1}, \ldots, x_{i_k} are allowed to coincide only if $\sum_{j=1}^{k} a_{i_j} \leq 1$. - The twisted dualizing sheaf $\omega_C(a_1x_1 + \dots + a_nx_n)$ is ample.

In particular, $\overline{M}_{q,A[n]}$ is a compactification of the moduli space $M_{q,n}$ of genus g smooth *n*-pointed curves. The irreducible components of the boundary divisor $M_{q,A[n]} \setminus M_{g,n}$ are well understood. In the special case when q = 0, they are described as follows. Consider a partition $I \cup J = \{1, ..., n\}$ such that one of the following holds.

- $I = \{i_1, ..., i_r\}, J = \{j_1, ..., j_{n-r}\}, \text{ with } r, n-r \ge 2, a_{i_1} + ... + a_{i_r} > 1 \text{ and } a_{j_1} + ... + a_{j_r} > 1$ $a_{j_{n-r}} > 1.$ - $I = \{i_1, i_2\}$ and $i_1 + i_2 \le 1.$

In the first case, there is a prime divisor $D_{I,J}(A)$ in $\overline{M}_{0,A[n]}$ whose general point corresponds to a nodal curve with two irreducible components, having marked points $x_{i_1}, ..., x_{i_r}$ on one component, and $x_{j_1}, ..., x_{j_{n-r}}$ on the other component. In the latter case, there is a

prime divisor $D_{I,J}(A)$ in $\overline{M}_{0,A[n]}$ parametrizing curves where the marked points x_{i_1} and x_{i_2} coincide. These are precisely the boundary divisors of $\overline{M}_{g,A[n]}$.

6.1 ([Ha, Section 4]). For fixed g and n, given two collections of rational weight data A[n] and B[n] such that $a_i \ge b_i$ for any i = 1, ..., n, there exists a birational reduction morphism

$$\rho_{B[n],A[n]}: \overline{M}_{g,A[n]} \to \overline{M}_{g,B[n]}.$$

This morphism associates to a curve $[C, x_1, ..., x_n] \in \overline{M}_{g,A[n]}$ the pointed curve obtained by collapsing components of C along which $\omega_C(b_1x_1 + ... + b_nx_n)$ fails to be ample.

Example 6.2 ([Ha, Sections 6.1 and 6.2]). Consider the weight data

$$\begin{split} A_0[n] &= (1/(n-2), ..., 1/(n-2), 1), \\ A_1[n] &= (1/(n-3), ..., 1/(n-3), 1), \\ A_{1,2}[n] &= (1/(n-2), ..., 1/(n-2), 2/(n-2), 1) \end{split}$$

Then we have $\overline{M}_{0,A_0[n]} \cong \mathbb{P}^{n-3}$, $\overline{M}_{0,A_1[n]} \cong X_{n-1}^{n-3} = Bl_{p_1,\dots,p_{n-1}}\mathbb{P}^{n-3}$ and $\overline{M}_{0,A_{1,2}[n]} \cong X_{n-2}^{n-3} = Bl_{p_1,\dots,p_{n-2}}\mathbb{P}^{n-3}$. The reduction morphisms $\rho_{A_{1,2}[n],A_1[n]} : X_{n-1}^{n-3} \to X_{n-2}^{n-3}$ and $\rho_{A_0[n],A_1[n]} : X_{n-1}^{n-3} \to \mathbb{P}^{n-3}$ are the natural blow-up morphisms.

Let us describe some of the boundary divisors of $\overline{M}_{0,A_1[n]}$ under the blowup morphism $\rho: X_{n-1}^{n-3} \to \mathbb{P}^{n-3}$. There are (n-1) partitions of type $I = \{\hat{i}, n\}, J = \{1, ..., \hat{i}, ..., n-1\}$. The corresponding (n-1) divisors $D_{I,J}$ are the (n-1) exceptional divisors of the blow-up. There are $\binom{n-1}{2}$ partitions of type $I = \{\hat{i}_1, \hat{i}_2\}, J = \{1, ..., \hat{i}_1, ..., \hat{i}_2, ..., n-1\} \cup \{n\}$. The corresponding $\binom{n-1}{2}$ divisors $D_{I,J}$ are the strict transforms of the $\binom{n-1}{2}$ hyperplanes spanned subsets of cardinality n-3 of $\{p_1, ..., p_{n-1}\}$.

In [Ha] Hassett proposed the following problem.

Problem 6.3 ([Ha, Problem 7.1]). Let A[n] be a vector of weights and consider the moduli space $\overline{M}_{0,A[n]}$. Do there exist rational numbers $\alpha_{I,J}$ such that

$$K_{\overline{M}_{0,A[n]}} + \sum_{I,J} \alpha_{I,J} D_{I,J}(A)$$

is ample and the pair $(\overline{M}_{0,A[n]}, \sum_{I,J} \alpha_{I,J} D_{I,J}(A))$ is log canonical?

In [Ha, Sections 7.1, 7.2, 7.3, Remark 8.5] Hassett gives examples in which Problem 6.3 admits a positive answer. The techniques developed in this paper allow us to give some more examples.

Proposition 6.4. For the moduli space $\overline{M}_{0,A_1[n]}$, Problem 6.3 admits a positive answer.

Proof. Consider the blow-up $\rho : \overline{M}_{0,A_1[n]} \cong X_{n-1}^{n-3} \to \mathbb{P}^{n-3}$ described in Example 6.2. We denote by H the pullback of the hyperplane class of \mathbb{P}^{n-3} . Let $E_1, ..., E_{n-1}$ be the exceptional divisors, and $H_{i_1,...,i_{n-3}}$ be the strict transform of the hyperplane $\langle p_{i_1}, ..., p_{i_{n-3}} \rangle$, where $1 \leq i_j \leq n-1$. Then

$$K_{\overline{M}_{0,A_1[n]}} = -(n-2)H + (n-4)(E_1 + \dots + E_{n-1})$$

and

$$H_{i_1,\ldots,i_{n-3}} \sim H - E_{i_1} - \ldots - E_{i_{n-3}}.$$

Recall from Example 6.2 that the E_i 's and the $H_{i_1,\ldots,i_{n-3}}$'s are boundary divisors of $\overline{M}_{0,A_1[n]}$. So we set

$$\Delta = \alpha(H_{1,\dots,n-3} + \dots + H_{3,\dots,n-1}) + \beta(E_1 + \dots + E_{n-1}),$$

where α and β are positive numbers to be chosen. Then

$$K_{\overline{M}_{0,A_{1}[n]}} + \Delta = \left(\alpha \binom{n-1}{2} - n + 2\right) H - \left(\alpha \binom{n-2}{2} - n - \beta + 4\right) \sum_{i=1}^{n-1} E_{i}.$$

Recall from Proposition 1.4 that the Mori cone of $X_{n-1}^{n-3} \cong \overline{M}_{0,A_1[n]}$ is generated by the classes R_i 's and $L_{i,j}$'s introduced in Section 3.1. One computes:

$$(K_{\overline{M}_{0,A_{1}[n]}}+\Delta)\cdot R_{i} = \frac{\alpha}{2}(n-2)(n-3)-n-\beta+4 \text{ and } (K_{\overline{M}_{0,A_{1}[n]}}+\Delta)\cdot L_{i,j} = \frac{\alpha}{2}(n-2)(5-n)+2\beta+n-6$$

Therefore $K_{\overline{M}_{0,A_{1}[n]}} + \Delta$ is ample for $\alpha = \frac{2}{n-2}$ and $\beta = \frac{2}{3}$.

Next we check that the pair $(\overline{M}_{0,A_1[n]}, \Delta)$ is log canonical. Let $\overline{\rho} : Y = \overline{M}_{0,n} \to \overline{M}_{0,A_1[n]}$ be the composition of blow-ups introduced in Notation 4.1. It is also a reduction morphism (see [Ha, Section 6.1]). By Proposition 4.6, the morphism $\overline{\rho}$ is a log resolution of the pair $(\overline{M}_{0,A_1[n]}, \Delta)$.

There are $\rho_h = \binom{n-1}{h+1}$ *h*-planes spanned by subsets of cardinality h + 1 of $\{p_1, ..., p_{n-1}\}$. Each such *h*-plane is contained in $\binom{n-h-2}{n-h-4}$ of the $H_{i_1,...,i_{n-3}}$'s. Denote by $E_j^h \subset, j = 1, ..., \rho_h$, the exceptional divisors over the *h*-planes. Then we have

$$K_Y = \overline{\rho}^* K_{\overline{M}_{0,A_1[n]}} + \sum_{h=1}^{n-5} (n-h-4) (E_1^h + \dots + E_{\rho_h}^h)$$

and

$$\overline{\rho}^*(\Delta) \sim \sum_{h=1}^{n-5} \alpha \binom{n-h-2}{2} (E_1^h + \dots + E_{\rho_h}^h) + \alpha \sum_{i_1,\dots,i_{n-3}} \widetilde{H}_{i_1,\dots,i_{n-3}} + \beta \sum_i \widetilde{E}_i.$$

Thus

$$K_Y + \widetilde{\Delta} = \overline{\rho}^* (K_{\overline{M}_{0,A_1[n]}} + \Delta) + \sum_{h=1}^{n-5} \left(n - h - 4 - \alpha \binom{n-h-2}{2} \right) (E_1^h + \dots + E_{\rho_h}^h).$$

For $\alpha = \frac{2}{n-2}$ and $\beta = \frac{2}{3}$ all the discrepancies are greater than -1. Therefore the pair $(\overline{M}_{0,A_1[n]}, \Delta)$ is log canonical.

Proposition 6.5. For the moduli space $\overline{M}_{0,A_{1,2}[n]}$, Problem 6.3 admits a positive answer.

Proof. Consider the blow-up $\rho : \overline{M}_{0,A_{1,2}[n]} \cong X_{n-2}^{n-3} \to \mathbb{P}^{n-3}$ described in Example 6.2. We denote by H the pullback of the hyperplane class of \mathbb{P}^{n-3} . The prime divisors $D_{I,J}$ appearing in Δ will be the following:

- the (n-2) exceptional divisors $E_1, ..., E_{n-2}$,
- the strict transforms $H_{i_1,\ldots,i_{n-3}}$ of the (n-2) hyperplanes spanned by subsets of cardinality (n-3) of $\{p_1,\ldots,p_{n-2}\}$ $(H_{i_1,\ldots,i_{n-3}} \sim H E_{i_1} \ldots E_{i_{n-3}})$,
- the strict transforms $\Lambda_{j_1,...,j_{n-4}}$ of the $\binom{n-2}{2}$ hyperplanes spanned by subsets of cardinality (n-4) of $\{p_1,...,p_{n-2}\}$ and p_{n-1} $(\Lambda_{j_1,...,j_{n-4}} \sim H - E_{j_1} - ... - E_{j_{n-4}})$.

Set

$$\Delta = \frac{2}{n-2} \sum_{i_1,\dots,i_{n-3}} H_{i_1,\dots,i_{n-3}} + \frac{2}{n-2} \sum_{j_1,\dots,j_{n-4}} \Lambda_{j_1,\dots,j_{n-4}} + \frac{2}{3} \sum_{i=1}^{n-2} E_i$$

Each p_i , i = 1, ..., n-2, lies in exactly (n-3) of the $H_{i_1,...,i_{n-3}}$'s, and $\binom{n-3}{2}$ of the $\Lambda_{i_1,...,i_{n-3}}$'s. So we have

$$\Delta \sim (n-1)H + \left(\frac{2}{3} - \frac{2(n-3)}{n-2} - \frac{2}{n-2}\binom{n-3}{2}\right)\sum_{i=1}^{n-2} E_i = (n-1)H - \frac{3n-11}{3}\sum_{i=1}^{n-2} E_i$$

and

$$K_{\overline{M}_{0,A_{1,2}[n]}} + \Delta = (-n+2+n-1)H + \left(n-4+\frac{11-3n}{3}\right)\sum_{i=1}^{n-2} E_i = H - \frac{1}{3}\sum_{i=1}^{n-2} E_i.$$

Recall from Proposition 1.4 that the Mori cone of $X_{n-2}^{n-3} \cong \overline{M}_{0,A_{1,2}[n]}$ is generated by the classes R_i 's and $L_{i,j}$'s introduced in Section 3.1. One computes:

$$(K_{\overline{M}_{0,A_{1,2}[n]}} + \Delta) \cdot R_i = (K_{\overline{M}_{0,A_{1,2}[n]}} + \Delta) \cdot L_{i,j} = \frac{1}{3}$$

Therefore $K_{\overline{M}_{0,A_{1,2}[n]}} + \Delta$ is ample.

Next we check that the pair $(\overline{M}_{0,A_{1,2}[n]}, \Delta)$ is log canonical. Let $\pi_{n-1} : X_{n-1}^{n-3} \to X_{n-2}^{n-3}$ be the blow-up of p_{n-1} and consider the composition

$$Y \xrightarrow{\overline{\rho}} X_{n-1}^{n-3} = \overline{M}_{0,A_1[n]} \xrightarrow{\pi_{n-1}} X_{n-2}^{n-3} = \overline{M}_{0,A_{1,2}[n]},$$

$$\widetilde{\rho}$$

where $\overline{\rho}$ is the log resolution used in the proof of Proposition 6.4. Then $\tilde{\rho}$ is a log resolution of the pair $(\overline{M}_{0,A_{1,2}[n]}, D)$. Let E_{n-1} be the exceptional divisor over p_{n-1} . There are $\gamma_h = \binom{n-2}{h+1}$ h-planes spanned by subsets of cardinality h+1 of $\{p_1, ..., p_{n-2}\}$. We denote by $E_j^h, 1 \leq j \leq \gamma_h$, the exceptional divisors over these *h*-planes. Similarly, there are $\overline{\gamma}_h = \binom{n-2}{h}$ *h*-planes spanned by p_{n-1} and subsets of cardinality *h* of $\{p_1, ..., p_{n-2}\}$. We denote by \overline{E}_i^h , $1 \leq j \leq \overline{\gamma}_h$, the exceptional divisors over these *h*-planes. Note that

- the point p_{n-1} is contained all of the (ⁿ⁻²₂) Λ_{j1,...,jn-4}'s,
 any h-plane spanned by subsets of cardinality h + 1 of {p₁, ..., p_{n-2}} is contained in n h 3 of the H_{i1,...,in-3}'s and in (^{n-h-3}₂) of the Λ_{j1,...,jn-4}'s,
 any h-plane spanned by p_{n-1} and subsets of cardinality h of {p₁, ..., p_{n-2}} is con-
- tained in $\binom{n-h-2}{2}$ of the $\Lambda_{j_1,\ldots,j_{n-4}}$'s.

Therefore, we have

$$\widetilde{\rho}^{*}(\Delta) = \frac{2}{n-2} \binom{n-2}{2} E_{n-1} + \frac{2}{n-2} \sum_{h=1}^{n-5} \left(n-h-3 + \binom{n-h-3}{2} \right) (E_{1}^{h} + \dots + E_{\gamma_{h}}^{h}) + \frac{2}{n-2} \sum_{h=1}^{n-5} \binom{n-h-2}{2} (\overline{E}_{1}^{h} + \dots + \overline{E}_{\overline{\gamma_{h}}}^{h}) + \widetilde{\Delta}.$$

Since

$$K_Y = \tilde{\rho}^* K_{\overline{M}_{0,A_{1,2}[n]}} + (n-4)E_{n-1} + \sum_{h=1}^{n-5} (n-h-4)(E_1^h + \dots + E_{\gamma_h}^h + \overline{E}_1^h + \dots + \overline{E}_{\gamma_h}^h),$$

we have

$$K_{Y} + \widetilde{\Delta} = \widetilde{\rho}^{*} (K_{\overline{M}_{0,A_{1,2}[n]}} + \Delta) + \left(n - 4 - \frac{2}{n-2} \binom{n-2}{2}\right) E_{n-1} + \sum_{h=1}^{n-5} \left(n - h - 4 - \frac{2}{n-2} \left(n - h - 3 + \binom{n-h-3}{2}\right)\right) (E_{1}^{h} + \dots + E_{\gamma_{h}}^{h}) + \sum_{h=1}^{n-5} \left(n - h - 4 - \frac{2}{n-2} \binom{n-h-2}{2}\right) (\overline{E}_{1}^{h} + \dots + \overline{E}_{\gamma_{h}}^{h}).$$

The discrepancies are all ≥ -1 , and hence the pair $(\overline{M}_{0,A_{1,2}[n]}, \Delta)$ is log canonical.

Remark 6.6. Any 3-dimensional Hassett's space $\overline{M}_{0,A[6]}$ admits a reduction morphism $\rho: \overline{M}_{0,6} \to \overline{M}_{0,A[6]}$ ([Ha, Theorem 4.1]). The moduli space $\overline{M}_{0,6}$ is log Fano by [HK00]. So, by [GOST, Corollary 1.3], $\overline{M}_{0,A[6]}$ is also log Fano. Examples of 3-dimensional Hassett's spaces are the following.

- The blow-up of \mathbb{P}^3 in four general points, along the strict transforms of the lines spanned by them, and in a fifth general point. This variety corresponds to A[6] = (1/3, 1/3, 1/3, 1/3, 1, 1).
- The blow-up of \mathbb{P}^3 in five general points, and along the strict transforms of the lines spanned by them. This is $\overline{M}_{0,6}$ itself.
- The blow-up X_1 of $\mathbb{P}_1^1 \times \mathbb{P}_2^1 \times \mathbb{P}_3^1$ in $p_1 = ([0:1], [0:1], [0:1]), p_2 = ([1:0], [1:0], [1:0]), and <math>p_3 = ([1:1], [1:1], [1:1])$. This variety corresponds to $A_1[6] = (2/3, 2/3, 2/3, 1/6, 1/6, 1/6)$ (see [Ha, Section 6.3]).
- Consider the projections $\pi_i : \mathbb{P}_1^1 \times \mathbb{P}_2^1 \times \mathbb{P}_3^1 \to \mathbb{P}_i^1$, and set $F_0 = \bigcup_{i=1}^3 \pi_i^{-1}([0:1]), F_1 = \bigcup_{i=1}^3 \pi_i^{-1}([1:0]), F_\infty = \bigcup_{i=1}^3 \pi_i^{-1}([1:1])$. Let Δ_2 be the union of the 2-dimensional diagonals of $\mathbb{P}_1^1 \times \mathbb{P}_2^1 \times \mathbb{P}_3^1$. Let X_2 be the blow-up of X_1 along the strict transform of $\Delta_2 \cap (F_0 \cup F_1 \cup F_\infty)$. This variety corresponds to $A_2[6] = (2/3, 2/3, 2/3, 1/3, 1/3, 1/3)$ (see [Ha, Section 6.3]).
- The blow-up X_3 of X_2 along the strict transform of the 1-dimension diagonal Δ_1 of $\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^1_3$. This is $\overline{M}_{0,6}$ (see [Ha, Section 6.3]).

Let $q_1 = ([1 : 0], ..., [1 : 0]), q_2 = ([0 : 1], ..., [0 : 1]), q_3 = ([1 : 1], ..., [1 : 1]) \in (\mathbb{P}^1)^{n-3}$, and set $Y_3^{n-3} = Bl_{q_1,q_2,q_3}(\mathbb{P}^1)^{n-3}$. By [Ha, Section 6.3], $Y_3^{n-3} \cong \overline{M}_{0,A[n]}$ for $A[n] = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3(n-4)}, ..., \frac{1}{3(n-4)}).$

Proposition 6.7. Then there exists a small birational modification

$$X_{n-1}^{n-3} \dashrightarrow Y_3^{n-3}.$$

In particular, Y_3^{n-3} is log Fano.

Proof. Note that the Picard numbers satisfy $\rho(X_{n-1}^{n-3}) = \rho(Y_3^{n-3}) = n$. Without lost of generality, we may assume that X_{n-1}^{n-3} is the blow-up of \mathbb{P}^{n-3} at the points $p_1 = [1:0:\dots:0]$, $p_2 = [0:1:\dots:0], \dots, p_{n-2} = [0:\dots:0:1]$ and $p_{n-1} = [1:1:\dots:1]$. Set $X_{n-2}^{n-3} = Bl_{p_1,\dots,p_{n-2}}\mathbb{P}^{n-3}$, $X_{n-3}^{n-3} = Bl_{p_1,\dots,p_{n-3}}\mathbb{P}^{n-3}$, $Y_2^{n-3} = Bl_{q_1,q_2}(\mathbb{P}^1)^{n-3}$ and and $Y_1^{n-3} = Bl_{q_1}(\mathbb{P}^1)^{n-3}$. These are all toric varieties. Let e_1,\dots,e_{n-3} be the standard basis

vectors of the co-character lattice of $(k^*)^{n-3}$. The rays of the fan of \mathbb{P}^{n-3} are e_1, \ldots, e_{n-3} and $-e_1 - \dots - e_{n-3}$. By blowing-up p_1, \dots, p_{n-2} we add the rays $-e_1, \dots, -e_{n-3}$ and $e_1 + \dots + e_{n-3}$. On the other hand the rays of $(\mathbb{P}^1)^{n-3}$ are $e_1, \ldots, e_{n-3}, -e_1, \ldots, -e_{n-3}$, and blowing-up q_1, q_2 corresponds to introducing the two rays $e_1 + \ldots + e_{n-3}$ and $-e_1 - \ldots - e_{n-3}$. So the fans of X_{n-2}^{n-3} and Y_2^{n-3} have the same 1-dimensional rays. Therefore, X_{n-2}^{n-3} and Y_2^{n-3} are isomorphic in codimension one.

Given $1 \leq i_1 < ... < i_{n-4} \leq n-3$, set $H_{i_1,...,i_{n-4}}^{n-5} = \langle p_{i_1},...,p_{i_{n-4}} \rangle$, and $\{j_1, j_2\} = \{0, ..., n-3\} \setminus \{i_1-1, ..., i_{n-4}-1\}$. The projection from $H_{i_1,...,i_{n-4}}^{n-5}$ is the rational map

$$\begin{array}{cccc} \pi_{i_1,\ldots,i_{n-4}}: & \mathbb{P}^{n-3} & \dashrightarrow & \mathbb{P}^1 \\ & & [x_0:\ldots:x_{n-3}] & \mapsto & [x_{j_1}:x_{j_2}] \end{array}$$

There are (n-3) of those, inducing a rational map

The hyperplane $W = \langle p_1, ..., p_{n-3} \rangle = \{x_{n-3} = 0\}$ is mapped to the point $q_1 \in (\mathbb{P}^1)^{n-3}$ by g. This is the only divisor contracted by g. Therefore, by blowing-up $q_1 \in (\mathbb{P}^1)^{n-3}$ we obtain a small transformation $g_1: X_{n-3}^{n-3} \dashrightarrow Y_1^{n-3}$ fitting in the following diagram:

6

Note that g_1 maps the strict transform \widetilde{W} of W to the exceptional divisor E_{q_1} , while the exceptional divisors $E_{p_1}, ..., E_{p_{n-3}}$ are mapped to the strict transforms of the (n-3) divisors in $(\mathbb{P}^1)^{n-3}$ obtained by fixing one of the factors. Note also that g([0:...:0:1]) = ([0:...])1],..., [0:1]) and g([1:...:1]) = ([1:1],...,[1:1]). It follows from the universal property of the blow-up that g_1 lifts to a small modification $f: X_{n-1}^{n-3} \to Y_3^{n-3}$ mapping $E_{p_{n-2}}$ to E_{q_2} , and $E_{p_{n-3}}$ to E_{q_3} .

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