Existence and uniqueness of very weak solutions to the steady–state Navier–Stokes problem in Lipschitz domains*

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October 18, 2016

Abstract

It is proved that in a bounded Lipschitz domain of $\mathbb{R}^3$ the Navier–Stokes equations with boundary data in $L^2(\partial\Omega)$ have a very weak solution $\boldsymbol{u} \in L^3(\Omega)$, unique for large viscosity.

1 Introduction

In the last years several papers have been devoted to existence of solutions to the Navier Stokes equations[1]

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p + \boldsymbol{f} = \boldsymbol{0} \quad & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad & \text{on } \partial\Omega \end{aligned} \tag{1}$$

*2000 *Mathematical Subject classification*. Primary 76D05, 35Q30; Secondary 31B10, 76D03; *Key words*: stationary Navier Stokes equations, bounded Lipschitz domains, boundary–value problem.

[1]We use a standard notation as, *e.g.*, in [4], [11], [26]. In particular, italic light–face letters except o, x, y, ξ, ζ that denote points of $\mathbb{R}^3$, and small upper–case letters indicate scalars and vectors respectively; $o \in \mathbb{R}^3 \setminus \overline{\Omega}$ is the origin of the reference frame $(o, \{\boldsymbol{e}_i\}_{i=1,2,3})$; $\boldsymbol{x} = x - o$, $r = r(x) = |\boldsymbol{x}|$; $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is the vector with components $u_i\partial_i u_j$. S_R is a ball centered at o, containing $\complement\Omega$ and $\Omega_R = \Omega \cap S_R$. For $q \in (1, +\infty)$ $D^{1,q}(\Omega) = \{\varphi : \|\nabla\varphi\|_{L^q(\Omega)} < +\infty\}$, $D_0^{1,q}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla\varphi\|_{L^q(\Omega)}$ and $D^{-1,q'}(\Omega)$ is the dual space of $D_0^{1,q}(\Omega)$. A subscript σ in a vector function space donotes its free divergence fields and the symbol c is reversed to indicate a positive constant unessential to our purposes

in a bounded domain of $\mathbb{R}^3$ under weak assumptions on $\boldsymbol{f}$ and $\boldsymbol{a}$. The first results in this directions are due to D. Serre [21] and E. Marušić-Paloka [13]. In [21], assuming that $\partial\Omega$ is connected and of class C^2 , $\boldsymbol{f} \in W^{-1,q}(\Omega)$, and $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, $q > 3/2$, satisfies the necessary compatibility condition

$$\int\limits_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0, \tag{2}$$

where $\boldsymbol{n}$ is the outward (with respect to Ω) unit normal to $\partial\Omega$, it is proved that (1) admits a weak solution, *i.e.*, a field $\boldsymbol{u} \in W^{1,q}_\sigma(\Omega)$ such that

$$\nu\int\limits_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi} = \int\limits_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{u} + \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle, \quad \forall\boldsymbol{\varphi}\in C^\infty_{0,\sigma}(\Omega), \tag{3}$$

and whose trace in the sense of the Sobolev spaces coincides with $\boldsymbol{a}$. In [13], assuming that $\partial\Omega$ is connected and of class $C^{1,1}$, $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfies (2) and $\boldsymbol{f} \in W^{-1,3}(\Omega)$, and it is proved that (1) has a very weak solution, *i.e.*, a field $\boldsymbol{u} \in L^3(\Omega)$ such that

$$\begin{aligned} &\nu\int\limits_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{z} = -\int\limits_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{z}\cdot\boldsymbol{u} - \int\limits_{\partial\Omega}\boldsymbol{u}\cdot\partial_n\boldsymbol{z} + \langle\boldsymbol{f},\boldsymbol{z}\rangle,\\ &\int\limits_\Omega \boldsymbol{u}\cdot\nabla Q = \int\limits_{\partial\Omega} Q(\boldsymbol{a}\cdot\boldsymbol{n}), \end{aligned} \tag{4}$$

for every $\boldsymbol{z} \in W^{2,3/2}_\sigma(\Omega)\cap W^{1,3/2}_0(\Omega)$ and for every $Q \in W^{1,3/2}(\Omega)$. The results of [13] were extended to more general situations, like boundary data in $W^{-1/3,3}(\partial\Omega)$ and $\partial\Omega$ not connected for "small fluxes", in [1], [2], [3], [5], [7], [18], [19], [20], [25] (see also [4] and the reference therein)[2] [3]. In [17] a definition of very weak solution is given following a point of view of J. Nečas [15]. To to this we consider, as usual, the *Stokes problem*

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} = \boldsymbol{0} &\quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{a} &\quad \text{on } \partial\Omega. \end{aligned} \tag{5}$$

Let $\boldsymbol{f}$ belongs to a normal space of distribution, *i.e*, the dual of a Banach space which has $C^\infty_0(\Omega)$ as dense subspace. We say that a field $\boldsymbol{u} \in L^3(\Omega)$

[2]In some of these papers the more general problem $(1)_{1,3}$ with the condition $\operatorname{div}\boldsymbol{u} = \gamma$ is considered. In Section 4 we shall observe that our results retain their validity also in this case under weak assumptions on γ (see Theorem 4).

[3]Existence of a weak solution to (1) in 2–D and 3–D axially symmetric domains without assumptions on the magnitude of fluxes have recently established in [8], [9], [10].

is a *very weak solution* to (5) if

$$\nu \int\limits_{\Omega} \boldsymbol{u} \cdot \boldsymbol{\varphi} = -\int\limits_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{s}(\boldsymbol{z}, Q) + \langle \boldsymbol{f}, \boldsymbol{z} \rangle, \tag{6}$$

for every $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$, with $(\boldsymbol{z}, Q)$ solution to the equations

$$\begin{aligned} \nu\Delta \boldsymbol{z} - \nabla Q + \boldsymbol{\varphi} = \mathbf{0} \quad & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{z} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{z} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{7}$$

and

$$\boldsymbol{s}(\boldsymbol{z}, Q) = \nu(\nabla \boldsymbol{z} + \nabla \boldsymbol{z}^\top)\boldsymbol{n} - p\boldsymbol{n}. \tag{8}$$

Choosing $\boldsymbol{\varphi} = \nabla\zeta$ we see that $\boldsymbol{z} = \mathbf{0}$ and

$$\int_{\Omega} \boldsymbol{u} \cdot \nabla\zeta = 0, \quad \forall \zeta \in C_0^\infty(\Omega),$$

so that $\boldsymbol{z}$ is weakly divergence free. Moreover, for $\boldsymbol{\varphi} = \Delta\boldsymbol{\phi}$, with $\boldsymbol{\phi} \in C^\infty 0_\sigma(\Omega)$ and Q constant, (7) writes

$$\nu \int\limits_{\Omega} \boldsymbol{u} \cdot \Delta\boldsymbol{\phi} = \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle, \quad \boldsymbol{\phi} \in C^\infty 0_\sigma(\Omega).$$

Hence by a classical result of G. De Rham (see, *e.g.*, [26] p.14) there is a "pressure field" $p \in C_0^\infty(\Omega)$ such that

$$\langle \nu\Delta\boldsymbol{u} + \boldsymbol{f} - \nabla p, \boldsymbol{\varphi} \rangle = 0, \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega), \tag{9}$$

so that $(\boldsymbol{u}, p)$ is a solution to (5) in the sense of the distributions. By a standard theoretical functional approach it can be proved that a solution $\boldsymbol{u}$ to (10) takes the boundary values in a suitable weak sense. Nevertheless, we shall observe that $\boldsymbol{u}$ takes the value $\boldsymbol{a}$ is a well–defined sense which coincide with that of the nontangential convergence for $\boldsymbol{f} \in W^{-1,q}(\Omega)$, $(q > 3)$, say [17]. Now, a very weak weak solution to (1) is defined as a field which satisfies

$$\nu \int\limits_{\Omega} \boldsymbol{u} \cdot \boldsymbol{\varphi} = -\int\limits_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{s}(\boldsymbol{z}, Q) + \langle \boldsymbol{f}, \boldsymbol{z} \rangle - \int\limits_{\Omega} \boldsymbol{u} \cdot \nabla\boldsymbol{z} \cdot \boldsymbol{u}, \tag{10}$$

for every $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$, with $(\boldsymbol{z}, Q)$ solution to (7). Classical results assure that if $\boldsymbol{f}$ is more regular in Ω, then so does $\boldsymbol{u}$ and p, In particular, if $\boldsymbol{f} \in C^\infty(\Omega)$, then $(\boldsymbol{u}, p) \in C^\infty(\Omega) \times C^\infty(\Omega)$ (see, *e.g.*, Section IX.5 in [4]).

Owing on the results of [19], [16], [17] and the argument of [13], we proved existence of a very weak solution to the Navier–Stokes problem for Lipschitz domains under somewhat weak assumptions.

Let

$$\Omega = \Omega_0 \setminus \bigcup_{1=1}^{m} \overline{\Omega}_i \tag{11}$$

where Ω_j, $j = 0, 1, \dots, m$ are bounded domains with connected boundaries such that $\Omega_i \Subset \Omega_0$, $i = 1, \dots, m$ and $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j \neq 0$. Moreover, set

$$\mathcal{F} = \frac{1}{8\pi} \sum_{i=1}^{m} \left| \int\limits_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n} \right| \left(\max_{\partial\Omega} \frac{1}{|x - x_i|} - \min_{\partial\Omega} \frac{1}{|x - x_i|} \right),$$

where x_i is an assigned point in Ω_i. The solution $\boldsymbol{z}$ to (7) satisfies the estimates

$$\|\nabla z\|_{L^3(\Omega)} \leq c_\ell \|\boldsymbol{\varphi}\|_{L^{3/2}(\Omega)}. \tag{12}$$

The main result of the present paper is to generalize those of [17] by proving the following

Theorem 1. *Let the domain (11) be Lipschitz, let $\boldsymbol{f} \in W^{s-2,q}(\Omega)$, with $sq > 1$ and $q(1+s) = 3$, and let $\boldsymbol{a} \in L^2(\partial\Omega)$ satisfies (2). If*

$$\mathcal{F} < \nu, \tag{13}$$

then (10) has a very weak solution $\boldsymbol{u} \in L^3(\Omega)$. It is unique in the ball $\{\|\boldsymbol{u}\|_{L^3_\sigma(\Omega)} < \nu/(2c_\ell)\}$. If Ω is of class $C^{1,1}$, then we can assume $\boldsymbol{a} \in W^{-1/3,3}(\partial\Omega)$.

Theorem 1 is proved in Sections 3. In the next section we collect the main tools we shall need to get our results and that have some interest in themselves.

2 Preliminary results

The fundamental solution to

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \nabla p &= \boldsymbol{0}, \\ \operatorname{div} \boldsymbol{u} &= 0, \end{aligned} \tag{14}$$

in $\mathbb{R}^3$ writes

$$\begin{aligned} \mathcal{U}_{ij}(x-y) &= \frac{1}{8\pi\nu|x-y|} \left\{ \delta_{ij} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} \right\}, \\ \varpi_i(x-y) &= \frac{x_i - y_i}{4\pi|x-y|^3}. \end{aligned} \tag{15}$$

The Stokes simple layer potential with density $\boldsymbol{\psi} \in L^q(\partial\Omega)$ is defined by

$$
\begin{aligned}
v_i[\boldsymbol{\psi}](x) &= \int\limits_{\partial\Omega} \mathcal{U}_{ij}(x-\xi)\cdot\psi_j(\xi)\mathrm{da}_\xi,\\
P[\boldsymbol{\psi}](x) &= \int\limits_{\partial\Omega} \varpi_j(x-\xi)\psi_j(\xi)\mathrm{da}_\xi.
\end{aligned}
\tag{16}
$$

It is a solution to (14) in $\mathbb{R}^3 \setminus \partial\Omega$. The trace of $(48)_1$ on $\partial\Omega$ defines a map

$$
\mathcal{S} : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega). \tag{17}
$$

which is continuous for every $q \in (1, +\infty)$ and Fredholm with index zero for $q = 2$ [14], [17]. Moreover,

$$
\operatorname{Kern}\mathcal{S} = \operatorname{Kern}\mathcal{S}' = \operatorname{spn}\{\boldsymbol{n}_i\}_{i=1,\dots,m}, \tag{18}
$$

where

$$
\boldsymbol{n}_i(\xi) = \begin{cases} \boldsymbol{n}(\xi), & \xi \in \partial\Omega_i,\\ \mathbf{0}, & \xi \in \partial\Omega \setminus \partial\Omega_i,\end{cases}
$$

and

$$
\mathcal{S}' : W^{-1,2}(\partial\Omega) \to L^2(\partial\Omega) \tag{19}
$$

is the adjoint of $\mathcal{S}$[4]. Since $\partial\Omega$ is Lipschitz, there is a finite cone Γ such that a conguent cone Γ_ξ to Γ with vertex at ξ is contained in $\mathbb{R}^3 \setminus \Omega$ for every $\xi \in \partial\Omega$. The simple layer potential $\boldsymbol{v}[\boldsymbol{\psi}]$, with $\boldsymbol{\psi} \in W^{-1,q}(\partial\Omega)$ takes the value $\mathcal{S}'[\boldsymbol{\psi}]$ in the sense of *nontangential convergence*:

$$
\lim_{x(\in\Gamma_\xi)\to\xi} \boldsymbol{v}[\boldsymbol{\psi}](x) = \mathcal{S}'[\boldsymbol{\psi}](\xi) \tag{20}
$$

for almost all $\xi \in \partial\Omega$. Moreover, the following estimates hold

$$
\begin{aligned}
\|\boldsymbol{v}[\boldsymbol{\psi}]\|_{W^{1/2,2}(\Omega)} &\le \|\boldsymbol{\psi}\|_{W^{-1,2}(\partial\Omega)},\\
\|\boldsymbol{v}[\boldsymbol{\psi}]\|_{W^{3/2,2}(\Omega)} &\le \|\boldsymbol{\psi}\|_{L^2(\partial\Omega)}.
\end{aligned}
\tag{21}
$$

If Ω is of class $C^{1,1}(\partial\Omega)$, then

$$
\mathcal{S} : W^{1-1/q,q}(\partial\Omega) \to W^{2-1/q,q}(\partial\Omega)
$$

is is Fredholm with index zero and (18) holds, where $\mathcal{S}' : W^{-1-1/q,q}(\partial\Omega) \to W^{-1/q,q}(\partial\Omega)$, $q \in (1, +\infty)$. Also,

$$
\|\boldsymbol{v}[\boldsymbol{\psi}]\|_{L^q(\Omega)} \le \|\boldsymbol{\psi}\|_{W^{-1-1/q,q}(\partial\Omega)}. \tag{22}
$$

[4] In such a case (16) has to be understood as the value of the functional $\boldsymbol{\psi}$ at $\boldsymbol{\mathfrak{U}}$.

The Stokes volume potential with density $\boldsymbol{f} \in C_0^\infty(\Omega)$ is the pair

$$\begin{aligned}\boldsymbol{\mathcal{V}}[\boldsymbol{f}](x) &= \int_\Omega \boldsymbol{\mathcal{U}}(x-y)\cdot\boldsymbol{f}(y)dv_y,\\ p[\boldsymbol{f}](x) &= \int_\Omega \boldsymbol{\varpi}(x-y)\cdot\boldsymbol{f}(y)dv_y.\end{aligned} \tag{23}$$

By classical theorems about integral transforms (see, *e.g.*, [24]), $\boldsymbol{\mathcal{V}}$ maps boundedly $W^{s-2,q}(\Omega)$ into $W^{s,q}(\Omega)$. Hence, if $\boldsymbol{f} \in W^{s-2,q}(\Omega)$ and $sq > 1$, then by the trace theorem $\mathrm{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{f}] \in W^{s-1/q,q} \subset L^{2q/(3-sq)}(\partial\Omega)$.

From the above results and Fredholm theory it follows

Theorem 2. *Let the domain* (11) *be Lipschitz and let* $\mathrm{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{f}] \in L^2(\partial\Omega)$. *If* $\boldsymbol{a} \in L^2(\partial\Omega)$ *satisfies* (2), *then* (5) *has a unique very weak solution expressed by*

$$\begin{aligned}\boldsymbol{u}_s(x) &= \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{\sigma}(x) + \boldsymbol{\mathcal{V}}[\boldsymbol{f}](x),\\ p_s(x) &= P[\boldsymbol{\psi}](x) + p[\boldsymbol{f}],\end{aligned} \tag{24}$$

for some $\boldsymbol{\psi} \in W^{-1,2}(\partial\Omega)$, *where*

$$\boldsymbol{\sigma}(x) = \frac{1}{4\pi}\sum_{i=1}^m \frac{x_i - x}{|x-x_i|^3}\int_{\partial\Omega_i}\boldsymbol{a}\cdot\boldsymbol{n},$$

with x_i *fixed point in* Ω_i. *Moreover* [22],

- *(i)* *if* $\mathrm{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{f}]$ *and* $\boldsymbol{a} \in L^q(\partial\Omega)$, *then* $\boldsymbol{u} \in W^{1/q,q}_{\mathrm{loc}}(\overline{\Omega})$.
- *(ii)* *There is a positive* ϵ *such that if* $\mathrm{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{f}]$, $\boldsymbol{a} \in W^{1,q}(\partial\Omega)$, $q \in [2, 2+\epsilon)$, *then* $\boldsymbol{u} \in W^{1+1/q,q}_{\mathrm{loc}}(\overline{\Omega})$; *if* Ω *is of class* C^1, *then we can take* $q \in [2, +\infty)$.
- *(iii)* *There is a positive* μ_0 *such that if* $\mathrm{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{f}]$, $\boldsymbol{a} \in C^{0,\mu}(\partial\Omega)$, $\mu \in [0, \mu_0)$, *then* $\boldsymbol{u} \in C^{0,\mu}_{\mathrm{loc}}(\overline{\Omega})$; *if* Ω *is of class* C^1, *then we can take* $\mu_0 = 1$.

If Ω *is of class* $C^{1,1}$ *and* $\mathrm{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{f}]$, $\boldsymbol{a} \in W^{-1/q,q}(\partial\Omega)$, *then* $\boldsymbol{\psi} \in W^{-1-1/q,q}(\partial\Omega)$.

For $\boldsymbol{u} \in L^q_\sigma(\Omega)$, $q \geq 3$, denote by $\boldsymbol{\mathcal{L}}[\boldsymbol{u}]$ the solution to (5) with boundary datum $-\mathrm{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{u}\cdot\nabla\boldsymbol{u}]$. The field $\boldsymbol{\mathcal{N}}[\boldsymbol{u}] = \boldsymbol{\mathcal{V}}[\boldsymbol{u}\cdot\nabla\boldsymbol{u}] + \boldsymbol{\mathcal{L}}[\boldsymbol{u}] \in W^{1,q/2}_\sigma(\Omega)$ is the weak solution to the equations

$$\begin{aligned}\Delta\boldsymbol{v} - \nabla p - \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \boldsymbol{0} &\quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v} = 0 &\quad \text{in } \Omega,\\ \boldsymbol{v} = \boldsymbol{0} &\quad \text{on } \partial\Omega.\end{aligned} \tag{25}$$

for a suitable pressure $p \in L^{3/2}(\Omega)$. Taking into account that well–known estimates about integral transform and the trace theorem

$$\begin{aligned}
&\|\boldsymbol{\mathcal{V}}[\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1]-\boldsymbol{\mathcal{V}}[\boldsymbol{u}_2\cdot\nabla\boldsymbol{u}_2]\|_{L^q(\Omega)}\le\|\boldsymbol{\mathcal{V}}[(\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u}_1]\|_{L^q(\Omega)}\\
&\quad+\|\boldsymbol{\mathcal{V}}[\boldsymbol{u}_2\cdot\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)]\|_{L^q(\Omega)}\le c_q\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^q(\Omega)}\|\boldsymbol{u}_1\|_{L^q(\Omega)}\\
&\quad+c_q\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^q(\Omega)}\|\boldsymbol{u}_2\|_{L^q(\Omega)}\le c_q\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^q(\Omega)}(\|\boldsymbol{u}_1\|_{L^q(\Omega)}+\|\boldsymbol{u}_2\|_{L^q(\Omega)}),
\end{aligned}$$

and

$$\|\boldsymbol{\mathcal{L}}[\boldsymbol{u}]\|_{W^{1-2/q,q/2}(\partial\Omega)}\le c\|\boldsymbol{\mathcal{V}}[\boldsymbol{u}\cdot\nabla\boldsymbol{u}]\|_{L^q(\Omega)}\le c\|\boldsymbol{u}\|^2_{L^q(\Omega)},$$

we have that there is a positive constant c_q such that

$$\|\boldsymbol{\mathcal{N}}[\boldsymbol{u}_1]-\boldsymbol{\mathcal{N}}[\boldsymbol{u}_2]\|_{L^q(\Omega)}\le c_q\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^q(\Omega)}(\|\boldsymbol{u}_1\|_{L^q(\Omega)}+\|\boldsymbol{u}_2\|_{L^q(\Omega)}).$$

for every $\boldsymbol{u}_1,\boldsymbol{u}_2 \in L^q(\Omega)$. Therefore, if

$$\|\boldsymbol{u}'_s\|_{L^q(\Omega)}<\frac{1}{2c_q}, \tag{26}$$

then the map

$$\boldsymbol{u}'=\boldsymbol{u}_s+\boldsymbol{\mathcal{N}}[\boldsymbol{u}], \tag{27}$$

where $\boldsymbol{u}_s$ is a very weak solution to (5), is a contraction in $L^3_\sigma(\Omega)$ and its (unique) fixed point is a very weak solution to (1).

If $(\boldsymbol{u}', Q')$ is another very weak solution to (1), then

$$\nu\int\limits_\Omega \boldsymbol{w}\cdot\boldsymbol{\varphi}=\int\limits_\Omega[\boldsymbol{u}'\cdot\nabla\boldsymbol{z}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\nabla\boldsymbol{z}\cdot\boldsymbol{u}].$$

By Hölder's inequality we have

$$\begin{aligned}
\nu\left|\int\limits_\Omega \boldsymbol{w}\cdot\boldsymbol{\varphi}\right| &\le \left\{\|\boldsymbol{u}'\|_{L^q(\Omega)}+\|\boldsymbol{u}\|_{L^3(\Omega)}\right\}\|\boldsymbol{w}\|_{L^3(\Omega)}\|\nabla\boldsymbol{z}\|_{L^3(\Omega)}\\
&\le c_\ell\left\{\|\boldsymbol{u}'\|_{L^3(\Omega)}+\|\boldsymbol{u}\|_{L^3(\Omega)}\right\}\|\boldsymbol{w}\|_{L^3(\Omega)}\|\boldsymbol{\varphi}\|_{L^{3/2}(\Omega)}
\end{aligned}$$

for every $\boldsymbol{\varphi}\in C^\infty_0(\Omega)$, so that

$$\nu\|\boldsymbol{w}\|_{L^3(\Omega)}\le c_\ell\left\{\|\boldsymbol{u}'\|_{L^3(\Omega)}+\|\boldsymbol{u}\|_{L^3(\Omega)}\right\}\|\boldsymbol{w}\|_{L^3(\Omega)}.$$

Therefore, in the ball $\|\boldsymbol{v}\|_{L^3(\Omega)}<\nu/(2c_\ell)$ a very weak solution to (1) is unique. If $q\ge 6$, then writing $\boldsymbol{u}=\boldsymbol{u}_s+\boldsymbol{w}$, (10) a simple computation yields

$$\nu\int_\Omega|\nabla\boldsymbol{w}|^2=-\int_\Omega\boldsymbol{u}_s\cdot\nabla\boldsymbol{w}\cdot(\boldsymbol{u}_s+\boldsymbol{w}). \tag{28}$$

Since by Hölder's inequality

$$\left|\int_\Omega \boldsymbol{u}_s\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}\right| \le \|\boldsymbol{u}_s\|_{L^3(\Omega)}\|\boldsymbol{w}\|_{L^6(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \le c_s\|\boldsymbol{u}_s\|_{L^3(\Omega)}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)},$$

$$\left|\int_\Omega \boldsymbol{u}_s\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{u}_s\right| \le \|\boldsymbol{u}_s\|_{L^3(\Omega)}\|\boldsymbol{w}\|_{L^6(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \le \|\boldsymbol{u}_s\|^2_{L^4(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)},$$

(28) yields

$$(\nu - c_s\|\boldsymbol{u}_s\|_{L^3(\Omega)})\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \le \|\boldsymbol{u}_s\|^2_{L^4(\Omega)}. \tag{29}$$

Taking into account that

$$\|\boldsymbol{w}\|_{L^3(\Omega)} \le |\Omega|^{1/6}\|\boldsymbol{w}\|_{L^6(\Omega)} \le c_s|\Omega|^{1/6}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}$$

for $c_s\|\boldsymbol{u}_s\|_{L^3(\Omega)} < \nu$, (29) implies

$$\|\boldsymbol{u}\|_{L^3(\Omega)} \le \|\boldsymbol{u}_s\|_{L^3(\Omega)} + \|\boldsymbol{w}\|_{L^3(\Omega)} \le \|\boldsymbol{u}_s\|_{L^3(\Omega)} + \frac{c_s|\Omega|^{1/6}\|\boldsymbol{u}_s\|^2_{L^4(\Omega)}}{\nu - c_s\|\boldsymbol{u}_s\|_{L^3(\Omega)}}.$$

Therefore, we can then collect the above results in the following existence and uniqueness theorem for small data.

Theorem 3. *Let the domain (11) be Lipschitz. If (26) holds, then the Navier–Stokes problem (1) has a very weak solution in $L^q(\Omega)$. It is unique in the ball $\|\boldsymbol{u}\|_{L^q(\Omega)} < \nu/(2c_\ell)$ and, if $q \ge 6$, for*

$$c_s\|\boldsymbol{u}_s\|_{L^3(\Omega)} < \nu, \quad \|\boldsymbol{u}_s\|_{L^3(\Omega)} + \frac{c_s|\Omega|^{1/6}\|\boldsymbol{u}_s\|^2_{L^4(\Omega)}}{\nu - c_s\|\boldsymbol{u}_s\|_{L^3(\Omega)}} < \frac{1}{2c_\ell}. \tag{30}$$

3 Proof of Theorem 1

Following [13], for $\epsilon > 0$ denote by $\boldsymbol{a}_\epsilon$ and $\boldsymbol{f}_\epsilon$ regular fields on $\partial\Omega$ and Ω respectively such that

$$\|\boldsymbol{a} - \boldsymbol{a}_\epsilon\|_{L^2(\partial\Omega)} + \|\boldsymbol{f} - \boldsymbol{f}_\epsilon\|_{W^{s-2,q}(\Omega)} < \epsilon.$$

Clearly,

$$\sum_{i=1}^m \left|\int_{\partial\Omega_i} (\boldsymbol{a} - \boldsymbol{a}_\epsilon)\cdot\boldsymbol{n}\right| \le c\epsilon. \tag{31}$$

By Theorem 3 for ϵ sufficiently small the equations

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p + \boldsymbol{f} - \boldsymbol{f}_\epsilon &= 0 && \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{a} - \boldsymbol{a}_\epsilon && \text{on } \partial\Omega \end{aligned} \tag{32}$$

has a solution $\boldsymbol{u}_\epsilon \in L^3_\sigma(\Omega)$ such that

$$\|\boldsymbol{u}_\epsilon\|_{L^3(\Omega)} \le c\epsilon. \tag{33}$$

If we show that the equations

$$\begin{aligned} \nu\Delta\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\boldsymbol{u}_\epsilon - \boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{v} - \nabla p + \boldsymbol{f}_\epsilon = \boldsymbol{0} &\quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{a}_\epsilon &\quad \text{on } \partial\Omega \end{aligned} \tag{34}$$

has a solution $(\boldsymbol{v}, Q)$, then the pair $(\boldsymbol{u}_\epsilon + \boldsymbol{v}, p_\epsilon + Q)$ is a solution to (1). Write $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{v}_s$, where $\boldsymbol{v}_s$ is the solution to the Stokes equations

$$\begin{aligned} \nu\Delta\boldsymbol{v}_s - \nabla p_s + \boldsymbol{f}_\epsilon = \boldsymbol{0} &\quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v}_s = 0 &\quad \text{in } \Omega,\\ \boldsymbol{v} = \boldsymbol{a}_\epsilon &\quad \text{on } \partial\Omega, \end{aligned}$$

expressed by

$$\boldsymbol{v}_s = \bar{\boldsymbol{v}} + \boldsymbol{\sigma}_\epsilon, \tag{35}$$

with $\bar{\boldsymbol{v}} = \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{\mathcal{V}}[\operatorname{div}\boldsymbol{G}_\epsilon]$ and

$$\boldsymbol{\sigma}_\epsilon(x) = \frac{1}{4\pi}\sum_{i=1}^{m}\frac{x_i - x}{|x - x_i|^3}\int_{\partial\Omega}\boldsymbol{a}_\epsilon\cdot\boldsymbol{n}. \tag{36}$$

The field $\boldsymbol{w}$ is a solution to the equations

$$\begin{aligned} -\nu\Delta\boldsymbol{w} + \operatorname{div}\big[(\boldsymbol{v}_s + \boldsymbol{w})\otimes(\boldsymbol{v}_s + \boldsymbol{w} + \boldsymbol{u}_\epsilon) + \boldsymbol{u}_\epsilon\otimes(\boldsymbol{v}_s + \boldsymbol{w})\big] + \nabla p = \boldsymbol{0} &\quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} = 0 &\quad \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{a}_\epsilon &\quad \text{on } \partial\Omega. \end{aligned} \tag{37}$$

If we show that there is a positive constant c such that

$$\int_\Omega |\nabla\boldsymbol{w}|^2 \le c, \tag{38}$$

for every solution $\boldsymbol{w} \in W^{1,2}_0(\Omega)$, then classical methods (see, $e, g.$, [4], [11], [26]) ensure that (34) has a solution $\boldsymbol{w} \in W^{1,2}_0(\Omega)$. To get (38) we follow a classical argument of J. Leray [12] and O.A. Ladyzhenskaia [11]. If (38) were not true, then a sequence $\boldsymbol{w}_k \in W^{1,2}_{\sigma,0}(\Omega)$ should exist such that

$$\begin{aligned} \nu\int_\Omega \nabla\boldsymbol{w}_k\cdot\nabla\boldsymbol{\varphi} = &-\int_\Omega(\boldsymbol{v}_s + \boldsymbol{w}_k)\cdot\nabla\boldsymbol{\varphi}\cdot(\boldsymbol{v}_s + \boldsymbol{w}_k + \boldsymbol{u}_\epsilon)\\ &-\int_\Omega[\boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{\varphi}\cdot(\boldsymbol{v}_s + \boldsymbol{w}_k) \end{aligned} \tag{39}$$

and

$$\lim_{k\to+\infty} J_k^2 = \lim_{k\to+\infty}\int_\Omega |\nabla \boldsymbol{w}_k|^2 = +\infty.$$

Thus, the field

$$\boldsymbol{w}'_k = \frac{\boldsymbol{w}_k}{J_k} \tag{40}$$

satisfies

$$\begin{aligned}\frac{1}{J_k}\int_\Omega \nabla \boldsymbol{w}'_k\cdot\nabla\boldsymbol{\varphi} &= -\int_\Omega \boldsymbol{w}'_k\cdot\nabla\boldsymbol{w}'_k\cdot\boldsymbol{\varphi} + \frac{1}{J_k}\int_\Omega \boldsymbol{w}'_k\cdot\nabla\boldsymbol{\varphi}\cdot(\boldsymbol{v}_s+\boldsymbol{u}_\epsilon)\\ &+\frac{1}{J_k}\int_\Omega(\boldsymbol{v}_s+\boldsymbol{u}_\epsilon)\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{w}'_k + \frac{1}{J_k^2}\int_\Omega\big[\boldsymbol{v}_s\cdot\nabla\boldsymbol{\varphi}\cdot(\boldsymbol{v}_s+\boldsymbol{u}_\epsilon)+\boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{v}_s\big].\end{aligned} \tag{41}$$

Since the sequence $\{\boldsymbol{w}'_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega)$, we can extract a subsequence from it, denoted by the same symbol, such that $\boldsymbol{w}'_k$ converges weakly in $W^{1,2}(\Omega)$ and strongly in $L^q(\Omega)$, $q<6$, to a field $\boldsymbol{w}'\in W^{1,2}_{\sigma,0}(\Omega)$, with $\|\nabla\boldsymbol{w}'\|_{L^2(\Omega)}\le 1$. Therefore, choosing $\boldsymbol{\varphi}\in C^\infty 0_\sigma(\Omega)$ and letting $k\to+\infty$ in (41), we see that $\boldsymbol{w}'$ is a weak solution to the Euler equations

$$\begin{aligned}\boldsymbol{w}'\cdot\nabla\boldsymbol{w}'+\nabla Q' = \boldsymbol{0} &\quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{w}' = 0 &\quad \text{in } \Omega,\\ \boldsymbol{w}' = \boldsymbol{0} &\quad \text{on } \partial\Omega,\end{aligned} \tag{42}$$

for some $Q'\in W^{1,3/2}(\Omega)$ constant on each $\partial\Omega_i$ possibly with different values (say Q_i on $\partial\Omega_i$) [6]. Choosing $\boldsymbol{\varphi}=\boldsymbol{w}_k$ in (41), we have

$$\nu = \int_\Omega \boldsymbol{w}'_k\cdot\nabla{\boldsymbol{w}'}_k\cdot(\boldsymbol{v}_s+\boldsymbol{u}_\epsilon) + \frac{1}{J_k}\int_\Omega\big[\boldsymbol{v}_s\cdot\nabla\boldsymbol{w}'_k\cdot(\boldsymbol{v}_s+\boldsymbol{u}_\epsilon)+\boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{w}'_k\cdot\boldsymbol{v}_s\big]. \tag{43}$$

By Hölder's inequality, Sobolev's inequality and (33)

$$\begin{aligned}\left|\int_\Omega [\boldsymbol{w}'_k\cdot\nabla{\boldsymbol{w}'}_k\cdot\boldsymbol{u}_\epsilon\right| &\le \|\boldsymbol{u}_\epsilon\|_{L^3(\Omega)}\|\boldsymbol{w}'_k\|_{L^6(\Omega)}\|\nabla\boldsymbol{w}'_k\|_{L^2(\Omega)} \le c\epsilon\|\nabla\boldsymbol{w}'_k\|^2_{L^2(\Omega)}\\ \left|\int_\Omega \boldsymbol{v}_s\cdot\nabla\boldsymbol{w}'_k\cdot(\boldsymbol{v}_s+\boldsymbol{u}_\epsilon)\right| &\le \|\boldsymbol{v}_s\|_{L^6(\Omega)}\|\boldsymbol{v}_s+\boldsymbol{u}_\epsilon\|_{L^3(\Omega)}\|\nabla\boldsymbol{w}'_k\|_{L^2(\Omega)}\\ \left|\int_\Omega \boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{w}'_k\cdot\boldsymbol{v}_s\right| &\le \|\boldsymbol{v}_s\|_{L^6(\Omega)}\|\boldsymbol{u}_\epsilon\|_{L^3(\Omega)}\|\nabla\boldsymbol{w}'_k\|_{L^2(\Omega)}.\end{aligned}$$

$$
\left|\int_\Omega [\boldsymbol{w}'_k\cdot\nabla\boldsymbol{w}'_k - \boldsymbol{w}'\cdot\nabla\boldsymbol{w}']\cdot\boldsymbol{v}_s\right| \le \left|\int_\Omega (\boldsymbol{w}'_k-\boldsymbol{w}')\cdot\nabla\boldsymbol{w}'_k\cdot\boldsymbol{v}_s\right|
$$

$$
+\left|\int_\Omega \boldsymbol{w}'\cdot(\nabla\boldsymbol{w}'_k-\nabla\boldsymbol{w}')\cdot\boldsymbol{v}_s\right| \le \|\boldsymbol{v}_s\|_{L^6(\Omega)}\|\boldsymbol{w}'_k-\boldsymbol{w}'\|_{L^3(\Omega)}\|\nabla\boldsymbol{w}'_k\|_{L^2(\Omega)}
$$

$$
+\left|\int_\Omega \boldsymbol{w}'\cdot(\nabla\boldsymbol{w}'_k-\nabla\boldsymbol{w}')\cdot\boldsymbol{v}_s\right|
$$

Therefore, letting $k\to+\infty$ in (43) and taking into account that

$$
\int_\Omega \boldsymbol{w}'\cdot\nabla\boldsymbol{w}'\cdot\bar{\boldsymbol{v}} = -\int_\Omega \bar{\boldsymbol{v}}\cdot\nabla Q = -\sum_{i=0}^m Q_i\int_{\partial\Omega_i}\bar{\boldsymbol{v}}\cdot\boldsymbol{n} = 0
$$

and that by (31)

$$
\left|\int_\Omega \boldsymbol{w}'\cdot\nabla\boldsymbol{w}'\cdot(\boldsymbol{\sigma}_\epsilon-\boldsymbol{\sigma})\right| \le c\epsilon,
$$

we have

$$
(\nu-c\epsilon)\le\int_\Omega \boldsymbol{w}'\cdot\nabla\boldsymbol{w}'\cdot\boldsymbol{\sigma} = \frac{1}{4\pi}\left|\sum_{i=1}^m\left(\int_{\partial\Omega_i}\boldsymbol{a}\cdot\boldsymbol{n}\right)\int_\Omega\frac{\nabla\boldsymbol{w}'\cdot\nabla\boldsymbol{w}'^\top}{|x-x_i|}\right|. \tag{44}
$$

Taking into account that $|\nabla\boldsymbol{w}'|^2 = |\hat{\nabla}\boldsymbol{w}'|^2+|\tilde{\nabla}\boldsymbol{w}'|^2$, with $\hat{\nabla}\boldsymbol{w}'$ and $\tilde{\nabla}\boldsymbol{w}'$ symmetric and skew parts of $\nabla\boldsymbol{w}'$ respectively, and

$$
2\int_\Omega|\hat{\nabla}\boldsymbol{w}'|^2 = 2\int_\Omega|\tilde{\nabla}\boldsymbol{w}'|^2 = \int_\Omega|\nabla\boldsymbol{w}'|^2,
$$

(44) implies

$$
\nu - c\epsilon - \mathfrak{F}\le 0 \tag{45}
$$

Since for sufficiently small ϵ (45) contradicts assumption (13), we conclude that (38) holds and the theorem is proved. □

4 Some remarks on a more general system

The methods of the above sections can be used to deal with the more general systems considered, *e.g.*, in [1] and [5]

$$
\begin{aligned}
\nu\Delta\boldsymbol{u}-\nabla p+\boldsymbol{f}=\boldsymbol{0} &\quad \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u}+\gamma=0 &\quad \text{in } \Omega,\\
\boldsymbol{u}=\boldsymbol{a} &\quad \text{on } \partial\Omega
\end{aligned} \tag{46}
$$

and

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p + \boldsymbol{f} = \boldsymbol{0} \quad &\text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} + \gamma = 0 \quad &\text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{a} \quad &\text{on } \partial\Omega. \end{aligned} \tag{47}$$

Let

$$\mathcal{V}[\gamma](x) = \int\limits_{\Omega} \frac{\gamma(y)dv_y}{|x-y|} \tag{48}$$

be the harmonic volume potentials. Clearly, the field $\nabla\mathcal{V}[\gamma]$ is a solution to $(47)_2$. Therefore, if Ω is Lipschitz, $\boldsymbol{f},\gamma$ are such that $\operatorname{tr}_{|\partial\Omega}\boldsymbol{\mathcal{V}}[\boldsymbol{f}], \operatorname{tr}_{|\partial\Omega}\mathcal{V}[\gamma] \in L^2(\partial\Omega)$ and $\boldsymbol{a} \in L^2(\Omega)$ satisfies

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = -\langle\gamma, 1\rangle \tag{49}$$

a solution to (46) is expressed by

$$\begin{aligned} \boldsymbol{u}_s(x) &= \boldsymbol{v}[\boldsymbol{\psi}] + \boldsymbol{\sigma}(x) + \nabla\mathcal{V}[\gamma](x) + \boldsymbol{\mathcal{V}}[\boldsymbol{f}](x), \\ p_s(x) &= P[\boldsymbol{\psi}](x) + \mathcal{V}[\gamma](x) + p[\boldsymbol{f}](x), \end{aligned} \tag{50}$$

for some $\boldsymbol{\psi} \in W^{-1,2}(\partial\Omega)$ solution to the equation

$$\mathcal{S}[\boldsymbol{\psi}](\xi) = (\boldsymbol{a} - \boldsymbol{\sigma} - \nabla\mathcal{V}[\gamma](x) - \boldsymbol{\mathcal{V}}[\boldsymbol{f}])(\xi),$$

for almost all $\xi \in \partial\Omega$. Starting from these results and taking into account that

$$\left|\int\limits_{\Omega} \boldsymbol{w}\cdot\nabla\boldsymbol{w}\cdot\nabla\mathcal{V}[\gamma]\right| \le c_s\|\nabla\mathcal{V}[\gamma]\|_{L^3(\Omega)}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)}, \quad \forall\boldsymbol{w} \in W^{1,2}_0(\Omega),$$

and for $\gamma \in C(\overline{\Omega})$

$$\left|\int\limits_{\Omega} \gamma\nabla\boldsymbol{w}\cdot\nabla\boldsymbol{w}^\top\right| \le \tfrac{1}{2}\left(\max_{\Omega}\gamma - \min_{\Omega}\gamma\right)\int\limits_{\Omega}|\nabla\boldsymbol{w}|^2, \quad \forall\boldsymbol{w} \in W^{1,2}_0(\Omega),$$

we can repeat *ad litteram* the argument in the proof of Theorem 1 with obvious modification, to get the following theorem which improve results obtained in [1], [5].

Theorem 4. *Let the domain* (11) *be Lipschitz, let* $\boldsymbol{f} \in W^{s-2,q}(\Omega)$, $\gamma \in W^{t-1,r}(\Omega)$, *with* $sq > 1$, $rt >$ *and* $q(1+s) = r(1+t) = 3$, *and let* $\boldsymbol{a} \in L^2(\partial\Omega)$ *satisfies* (49). *If*

$$\mathcal{F} + c_s\|\nabla\mathcal{V}[\gamma]\|_{L^3(\Omega)} < \nu,$$

or for $\gamma \in C(\overline{\Omega})$

$$\mathcal{F} + \tfrac{1}{2}\left(\max_{\Omega}\gamma - \min_{\Omega}\gamma\right) < \nu,$$

then (47) *has a very weak solution* $\boldsymbol{u} \in L^3(\Omega)$. *If* Ω *is of class* $C^{1,1}$, *then we can assume* $\boldsymbol{a} \in W^{-1/3,3}(\partial\Omega)$ *and* $\gamma \in W^{-1/2,2}(\Omega)$.

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