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EQUILIBRIUM FOR MULTIPHASE SOLIDS WITH EULERIAN INTERFACES

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Abstract. We describe a general phase-field model for hyperelastic multiphase materials. The model features an elastic energy functional that depends on the phase-field variable and a surface energy term that depends in turn on the elastic deformation, as it measures interfaces in the deformed configuration. We prove existence of energy minimizing equilibrium states and Γ-convergence of diffuse-interface approximations to the sharp-interface limit.

1. Introduction

Mathematical models of multi-component (or multi-phase) materials have attracted the attention of researchers for decades. A prominent example of multi-phase materials is provided by shape memory alloys, i.e., intermetallic materials having a high-temperature phase called austenite and a low-temperature phase called martensite, existing in many symmetry-related variants, see [7, 9]. Mathematical analysis of elastostatic problems of such materials is involved because of the lack of suitable convexity properties. In fact, these materials exhibit complicated microstructures which are reflected in faster and faster oscillations of minimizing sequences driving the elastic energy functional to its infimum. Consequently, no minimizer generically exists and various methods have been developed to cope with this difficulty.

A possibility to overcome the nonexistence issue is to search for a lower semicontinuous envelope of the energy functional that describes macroscopic behaviour of the specimen [11]. This provides us with a solvable minimization problem and ensures that every minimizer is reachable by a minimizing sequence of the original problem. The downside of this method, called *relaxation*, is that such envelope is usually not known closed form.

A second option is to include a higher-order deformation gradient to the energy functional. In this case, we resort to nonsimple materials, see e.g. [5, 6, 8, 25, 31, 32] for various attempts in this direction. Here, a convex function of the second deformation gradient (strain gradient) penalizes spatial changes of the first gradient, which introduces a second length scale in the model and implies that oscillations in minimizing sequences have finite fineness. Besides, some models that are discussed in the above contributions

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include surface terms along the discontinuity set of the first deformation gradient, see also [14, 34].

A third option is the phase-field approach to multiphase materials, in which each phase of the material is identified by some value of a suitable phase indicator. A surface energy is generally assigned to each phase-separating interface, which prevents repeated phase jumps at small scale, see for instance the general theory by Šilhavý [36, 37]. In the gradient theory of phase transitions, the surface area penalization is relaxed by assuming that the change of phase takes place in a small but finite layer. This is the typical approach to the theory of Cahn-Hilliard fluids [3, 20, 29, 38], the fundamental convergence result to the sharp interface limit being established in [29] based on the Modica-Mortola Theorem [30].

In this paper, we consider an elastic model for multi-phase materials inspired by [36, 37]. We introduce an energy functional depending on the first deformation gradient and a phase indicator distinguishing particular material phases or variants. In particular, to each pair of continuous phases we associate an interfacial energy, where interfaces are measured in the deformed configuration. In fact, variational theories featuring Eulerian interfacial energy terms can be traced back at least to [19] and have been considered, for instance, in [24, 26, 27, 28], among others.

In the particular case of a two-component material, a diffuse-interface approximation to the Šilhavý's model was discussed in [18]. There, we proved that the approximations Γ-converge to the sharp-interface model. The aim of this paper is to extend that theory in several ways. We shall introduce a more general model, allowing for a finite number of material components. For the treatment of this model, we shall further develop the analysis of *interfacial measures* from [18], where a key role is played by the notion of mappings of finite distorsion [22]. We shall also consider the borderline case of deformations in $W^{1,p}(\Omega;\mathbb{R}^n)$ with $p = n$, hence without requiring their Hölder continuity.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open domain representing the reference configuration of a multi-component material. The composition of the material at each point is described by a *component vector* $z(x) = (z_1(x), \dots, z_h(x)) \in \mathbb{R}^h$. For instance, a mixture of h chemical species can be described by the relative mass fraction $z_i \in [0, 1]$, of the i^{th} component of the mixture for $i = 1, \dots, h$. If the components are immiscible, then at each point $x \in \Omega$ we have $z_i(x) \in \{0, 1\}$ and $z_i(x) = 1$ if and only if the material component i is present at x. As a second example, we can mention ferromagnetic materials, in which the spontaneous magnetization vector $z(x) \in \mathbb{R}^3$ can serve as the component descriptor of the phase.

We introduce the discrete set $P = \{p_\alpha \in \mathbb{R}^h \mid \alpha = 1, \dots, m\}$, $m \geq 2$, of *stable phases* characterized by the component vectors $z = p_\alpha$. The relation between the components number h and the number of stable phases m depends on the specific model. For instance, in an immiscible mixture with h components, we may have $m = h$ and $(p_\alpha)_i = \delta_{i\alpha}$, where $(p_\alpha)_i$ is the i^{th} component of p_α. On the other hand, if the component vector

$z \in \mathbb{R}^3$ represents the (saturated) magnetization vector of an anisotropic magnetic crystal, for instance with cubic anisotropy, one needs to consider $m = 2h = 6$ stable phases corresponding to the six magnetization directions $\pm(1,0,0)$, $\pm(0,1,0)$, $\pm(0,0,1)$.

Sharp interface model. In the sharp-interface setting, given a component-configuration field $z : \Omega \to \mathbb{R}^h$ taking values in P, we let $E_\alpha(z) := \{x \in \Omega : z(x) = p_\alpha\}$, $\alpha = 1,\dots,m$. The sets $(E_\alpha)_\alpha$ form a partition of Ω describing the spatial distribution of phases. For a given deformation $y : \Omega \to \Omega^y \subset \mathbb{R}^n$, we let $\zeta : \Omega^y \to \mathbb{R}^h$ denote the associated indicator function in the deformed configuration, i.e., $\zeta_i := z_i \circ y^{-1}$, $i = 1,\dots,h$. The set $E^y_\alpha := y(E_\alpha)$ is the region occupied by phase α in the deformed configuration.

We consider the stored energy functional for an elastic multiphase material

$$\mathcal{F}_0(\zeta, y) = \int_\Omega W(\nabla y(x), \zeta(y(x)))\, dx + \frac{1}{2} \sum_{\alpha,\beta=1}^{m} d_{\alpha,\beta} \mathcal{H}^{n-1}(E^y_{\alpha,\beta}) \tag{1.1}$$

where W is the stored bulk energy and

$$E^y_{\alpha,\beta} := \partial^* E^y_\alpha \cap \partial^* E^y_\beta \cap \Omega^y$$

is the interface between E_α and E_β in the deformed configuration. Here, ∂^* denotes the reduced boundary. The coefficients $d_{\alpha,\beta}$ are suitable surface-tension parameters such that: $d_{\alpha,\beta} = d_{\beta,\alpha} \geq 0$ and $d_{\alpha,\beta} = 0$ if and only if $\alpha = \beta$. The coefficients are assumed to satisfy the following inequalities

$$d_{\alpha,\beta} + d_{\beta,\gamma} \geq d_{\alpha,\gamma} \tag{1.2}$$

for any admissible triple of indexes α, β, γ. This condition is necessary for lower semicontinuity of $\mathcal{F}_0$, see [1]. Indeed, assume that $d_{\alpha,\gamma} > d_{\alpha,\beta} + d_{\beta,\gamma}$ for some triple of phases and consider a sequence of states where a layer of the phase β, of thickness tending to zero, is inserted between the layers α and γ. The bulk contribution in (1.1) tends to the value taken in absence of the phase β (the limit state); instead, the interfacial energy undergoes an increasing jump discontinuity in the limit process. The meaning of (1.2) also resides in its relation with the notion of *separability* of interfaces from [36], which would require the existence of coefficients g_α, $\alpha = 1,\dots,m$, such that $d_{\alpha,\beta} = g_\alpha + g_\beta$ for any α and any β between 1 and m. The separability assumption implies (1.2) and the two are equivalent if $m = 2, 3$.

This model, as in [3], features a standard sharp interface term for a multiphase material. On the other hand, the interface penalization is complemented by an elastic energy term that accounts for macroscopic deformation of the specimen, and the choice of taking the interface term in the deformed configuration is an example of interface polyconvex energy as described by [36, 37].

Diffuse-interface model. We are interested in providing a diffuse-interface approximation of the above energy. In a diffuse-interface model, the phase field z takes values in $\mathbb{R}^h$. The phase-field functional is defined as

$$\mathcal{F}_\varepsilon(\zeta, y) = \mathcal{F}^{\mathrm{bulk}}(\zeta, y) + \mathcal{F}^{\mathrm{int}}_\varepsilon(\zeta, y),$$

where

$$\mathcal{F}^{\rm bulk}(\zeta, y) := \int_\Omega W(\nabla y(x), \zeta(y(x)))\, dx, \quad \mathcal{F}^{\rm int}_\varepsilon(\zeta, y) := \int_{\Omega^y} \frac{\varepsilon}{2} |\nabla \zeta(\xi)|^2 + \frac{1}{\varepsilon}\, \Phi(\zeta(\xi))\, d\xi,$$

and where we denote by ξ (here and through the paper) the variable in deformed configuration, i.e., $\xi \in \Omega^y$. We have introduced a continuous multi-well potential $\Phi : \mathbb{R}^h \to \mathbb{R}^+$ with zeros only at $p_1, \dots p_m$. The relationship between the two models is established by letting

$$d_{\alpha,\beta} := d_\Phi(p_\alpha, p_\beta), \qquad \alpha = 1, \dots, m, \quad \beta = 1, \dots, m, \tag{1.3}$$

where d_Φ the Riemaniann distance in $\mathbb{R}^h$ induced by $\sqrt{2\Phi}$, i.e.,

$$d_\Phi(p_\alpha, p_\beta) := \inf \left\{ \int_0^1 \sqrt{2\Phi(\gamma(t))} |\gamma'(t)|\, dt : \gamma \in C^1([0,1]; \mathbb{R}^h),\ \gamma(0) = p_\alpha,\ \gamma(1) = p_\beta \right\}.$$

This guarantees symmetry, positivity, and the validity of the triangle inequality (1.2) for the coefficients $d_{\alpha,\beta}$.

Plan of the paper. We first state our main results in Section 2. In particular, we address the existence of minimizers for the diffuse-interface, as well as for the sharp-interface functionals $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ in Theorems 2.1 and 2.2, respectively. The approximation result is stated in Theorem 2.3. Properties of admissible deformations are reviewed in Section 3 whereas properties of interfacial measures, implicitly introduced in [36, 37], are detailed in Section 4. Our results mainly rest on proving a Γ-convergence statement. Indeed, a proof of the fact that $\mathcal{F}_0$ is a lower bound for $\mathcal{F}_\varepsilon$ is contained in Section 6. Eventually, proofs of the main theorems can be found in Section 7.

2. Main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open Lipschitz set representing the reference configuration. In this section, we introduce the set of admissible couples (y, ζ) (deformation and phase indicator) and we state the main results.

2.1. Admissible states.

Following [18], we introduce the functional spaces of the admissible states. For fixed $q > n-1$ and $p \geq n$ (not included in the notation for simplicity), we define the space of admissible deformations as

$$\mathbb{Y} := \left\{ y \in W^{1,p}(\Omega; \mathbb{R}^n) \mid \det \nabla y > 0 \text{ a.e.}, \int_\Omega \det \nabla y(x)\, \mathrm{d}x \leq |\Omega^y|,\ K_y \in L^q(\Omega) \right\} \tag{2.1}$$

Here, K_y denotes the optimal distorsion function associated to the deformation map y, see Definition 3.1 below. Any element of $\mathbb{Y}$ has a continuous representative which is a homeomorphism. This is a consequence of the Ciarlet-Nečas [10] condition appearing in (2.1) and of the L^q integrability of the distorsion function as shown in [18] for $n = 3$. The arguments therein straightforwardly apply for any dimension $n \geq 2$. Later in section 3 we shall derive more properties of the set of admissible deformations.

Recalling that $P \subset \mathbb{R}^h$ is the finite set of stable phases, we define the sets of the states, including the states for the sharp interface model

$$\mathbb{Q} := \{(y,\zeta) \mid y \in \mathbb{Y},\ \zeta \in BV(\Omega^y;\mathbb{R}^h),\ \zeta(\xi) \in P \text{ for a.e. } \xi \in \Omega^y\},$$

and for the diffuse interface model

$$\widetilde{\mathbb{Q}}^R := \{(y,\zeta) \mid y \in \mathbb{Y},\ \zeta \in W^{1,2}(\Omega^y;\mathbb{R}^h),\ |\zeta(\xi)| \le R \text{ for a.e. } \xi \in \Omega^y\},$$

where $R > 0$. A natural compatibility condition for the two models is $R > \max_{\alpha\in\{1,\dots,m\}} |p_\alpha|$, so that for a couple $(y,\zeta) \in \widetilde{\mathbb{Q}}^R$, ζ may take values in P.

Letting $\Gamma_0 \subset \partial\Omega$ be relatively open in $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma_0) > 0$, and letting $y_0 \in \mathbb{Y}$ be continuous up to $\partial\Omega$, we introduce the associated function spaces with Dirichlet boundary conditions

$$\mathbb{Q}_{(y_0,\Gamma_0)} := \{(y,\zeta) \in \mathbb{Q} \mid y = y_0 \text{ on } \Gamma_0\}, \qquad \widetilde{\mathbb{Q}}^R_{(y_0,\Gamma_0)} := \{(y,\zeta) \in \widetilde{\mathbb{Q}} \mid y = y_0 \text{ on } \Gamma_0\},$$

where the relation $y = y_0$ on Γ_0 is understood in the sense of traces. Moreover, y_0 is required to be nonconstant on Γ_0 (i.e. Γ_0 does not shrink to a point). We further define $\mathbb{Q}_{y_0} := \mathbb{Q}_{(y_0,\partial\Omega)}$ and $\widetilde{\mathbb{Q}}^R_{y_0} := \widetilde{\mathbb{Q}}^R_{(y_0,\partial\Omega)}$.

Given $y_0 \in \mathbb{Y}$, the compatibility between the boundary condition $y = y_0$ on Γ_0 and the choice of the energy functional is enforced by assuming that

$$\text{there exists } \ (y,\zeta) \in \mathbb{Q}_{(y_0,\Gamma_0)} \ \text{ such that } \ \mathcal{F}_0(y,\zeta) < \infty. \tag{2.2}$$

2.2. **The elastic energy.** The elastic energy, both in the diffuse- and in the sharp-interface case, is given by the bulk integral functional $\mathcal{F}^{\text{bulk}}(\zeta, y)$. The following assumptions are made for the energy density $W : \mathbb{R}^{n\times n} \times \mathbb{R} \to (-\infty, +\infty]$.

$$\begin{aligned}
&\text{The map } W(\cdot,\cdot) \text{ is lower semicontinuous in } \mathbb{R}^{n\times n} \times \mathbb{R}^h,\\
&\text{for any } z \in \mathbb{R}^h, \text{ the map } F \mapsto W(F,z) \text{ is } \textit{polyconvex},\\
&W(RF,z) = W(F,z) \quad \forall R \in \mathrm{SO}(n),\ \ \forall F \in \mathbb{R}^{n\times n},\ \ \forall z \in \mathbb{R}^h,
\end{aligned} \tag{2.3}$$

where $\mathrm{SO}(n)$ appearing in the standard frame-indifference property is the special orthogonal group, i.e., $\mathrm{SO}(n) = \{R \in \mathbb{R}^{n\times n} \mid RR^T = I,\ \det R = 1\}$. The notion of polyconvexity [4] requires that the map $F \mapsto W(F,z)$ can be written as a convex function of all of the minors (subdeterminants) of F. For instance, if $n = 3$,

$$W(F,z) := \begin{cases} w(F, \operatorname{cof} F, \det F, z) & \text{if } \det F > 0,\\ \infty \text{ otherwise} \end{cases}$$

for a convex function $w(\cdot, z) : \mathbb{R}^{19} \to \mathbb{R}$, at all $z \in \mathbb{R}^h$, where $\operatorname{cof} F$ denotes the cofactor matrix of F. We further assume that $W(\cdot, z)$ satisfies a suitable coercivity property. More precisely, we require that there exists $C > 0$, $p \ge n$, $r > 1$, and $q > n-1$ such that

$$W(F,z) \ge C\left(|F|^p + (\det F)^r + \frac{|F|^{nq}}{(\det F)^q}\right) - \frac{1}{C} \qquad \forall F \in \mathbb{R}^{n\times n},\ \ \forall z \in \mathbb{R}^h. \tag{2.4}$$

The third term on the right-hand side of (2.4) ensures that deformation gradients $F = \nabla y$ with finite energy will have a q-integrable distorsion function $F \mapsto |F|^n/\det F$. Notice that $F \mapsto |F|^n/\det F$ is polyconvex on the set of matrices with positive determinant. On the other hand, we mention that it is possible to drop the restriction $W(F,z) \geq C(\det F)^r$ in case $p > n$.

A typical example of a bulk energy functional W is

$$W(F,z) = \sum_{i=1}^{h} z_i^+ W_i(F) + \Big(1 - (z_1 + \cdots + z_h)\Big)^+ W_{h+1}(F) \tag{2.5}$$

where we assume that the listed properties (2.3)-(2.4) are uniformly satisfied by every elastic potential W_i at the place of W. The latter corresponds to a mixture ansatz, where notation is prepared for the general case $z \in \mathbb{R}^h$ of the phase-field approximation. In the sharp-interface case, we have that $z \in P$, where the set P of stable phases includes the origin and the standard orthonormal basis of $\mathbb{R}^h$ (thus, $m = h+1$) and the latter elastic energy takes the classical form

$$W(F,z) = \sum_{i=1}^{h} z_i W_i(F) + (1 - (z_1 + \cdots + z_h))\, W_{h+1}(F), \qquad z \in P.$$

We also note that the assumptions (2.3)-(2.4) on W could be imposed in the physical case $z \in \mathrm{Conv}(P)$ first, and then extended to the whole $\mathbb{R}^h$ by a suitable projection construction.

2.3. **Statement of the main results.** Owing to the above-introduced notation, we are now in the position of stating the main results of the paper. In the next three statements, the following underlying assumptions are understood to hold. Ω is a bounded open Lipschitz domain. The exponents p, q in the definition of the set of admissible deformations $\mathbb{Y}$ are of course given by assumption (2.4). As discussed in the introduction, the multiwell potential $\Phi : \mathbb{R}^h \to \mathbb{R}^+$ is continuous and vanishing only at points of P, and the coefficients $d_{\alpha,\beta}$ appearing in (1.1) are given by (1.3). About the Dirichlet datum, we require that $y_0 \in \mathbb{Y}$ is continuous up to the boundary of Ω and not constant on Γ_0. Here, $\Gamma_0 \subset \partial\Omega$ is relatively open in $\partial\Omega$ and $\mathcal{H}^{n-1}(\Gamma_0) > 0$.

Theorem 2.1 (Existence for the diffuse-interface model)**.** *Let $\varepsilon > 0$ and $R > 0$ be fixed. Suppose that $(y,\zeta) \in \widetilde{\mathbb{Q}}^R_{(y_0,\Gamma_0)}$ exists such that $\mathcal{F}_\varepsilon(y,\zeta) < \infty$. Let assumptions* (2.3), (2.4) *hold. Then, there is a minimizer of $\mathcal{F}_\varepsilon$ on $\widetilde{\mathbb{Q}}^R_{(y_0,\Gamma_0)}$.*

Theorem 2.2 (Existence for the sharp-interface model)**.** *Under assumptions* (2.2), (2.3), (2.4)*, the functional $\mathcal{F}_0$ admits a minimizer on $\mathbb{Q}_{(y_0,\Gamma_0)}$.*

The third main result states that the phase-field indeed approximates the sharp-interphase model, namely $\mathcal{F}_0$ is the Γ-limit [12] of the family $(\mathcal{F}_\varepsilon)_\varepsilon$. It requires an additional assumption on the boundary datum y_0. Namely, we ask for $\Gamma_0 = \partial\Omega$, i.e., Dirichlet conditions are imposed on the whole boundary, and Ω^{y_0} is assumed to be a Lipschitz domain. Moreover, assumptions on W have to be strengthened by additionally asking

$$\text{the map } W(F,\cdot):\mathbb{R}^h\to\mathbb{R} \text{ is continuous for any } F\in\mathbb{R}^{n\times n}. \tag{2.6}$$

We shall also require that for any $R>0$, given $y\in\mathbb{Y}$ and given $z\in L^1(\Omega;\mathbb{R}^h)$ such that $|z|\le R$ a.e. in Ω, there holds

$$\int_\Omega W(y(x),z(x))\,dx<\infty \quad\Rightarrow\quad \int_\Omega \sup_{\{z\in\mathbb{R}^h:|z|\le R\}} W(y(x),z)\,dx<\infty. \tag{2.7}$$

When considering the mixture example (2.5), the latter assumption is satisfied under the following comparability condition between the elastic potentials of the different phases: if $y\in\mathbb{Y}$ is such that $W_i(\nabla y)$ is integrable on Ω for some $i=1,\dots,m$, then $W_j(\nabla y)$ is integrable on Ω for any $j\neq i$.

Theorem 2.3 (Phase-field approximation)**.** *Let assumptions* (2.2), (2.3), (2.4), (2.6) *and* (2.7) *hold. Let $y_0\in\mathbb{Y}$ be such that Ω^{y_0} is a Lipschitz domain. There exists $R_0>0$ such that if $R>R_0$ the following holds. For every vanishing sequence $(\varepsilon_k)_k$ of positive numbers and every sequence $(y_k,\zeta_k)_k$ of minimizers of $\mathcal{F}_{\varepsilon_k}$ on $\widetilde{\mathbb{Q}}^R_{y_0}$, there exists $(y,\zeta)\in\mathbb{Q}_{y_0}$ such that, up to not relabeled subsequences,*

i) *$y_k\to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $k\to\infty$*
ii) *$\zeta_k\to\zeta$ strongly in $L^1(\Omega^y;\mathbb{R}^h)$ as $k\to\infty$*
iii) *(y,ζ) minimizes $\mathcal{F}_0$ on $\mathbb{Q}_{y_0}$.*

Remark 2.4 (Incompressibility)**.** The above results can be specialized to the case of an incompressible material. Indeed, one could impose the incompressibility constraint by letting $W(F,z)=+\infty$ if $\det F\neq 1$, which is compatible with the assumptions on W. For the model case (2.5) one might require $W_\alpha(F)=+\infty$ if $\det F\neq 1$ for any $\alpha=1,\dots,m$.

Remark 2.5 (Mass constraint)**.** Our analysis would allow additionally imposing the constraint

$$\int_{\Omega^y}\zeta_i(\xi)\,d\xi=\int_\Omega \zeta_i(y(x))\,\det\nabla y(x)\,dx=M_i,\qquad i=1,\dots,h$$

for given values M_i. By interpreting ζ_i as volume densities, the latter corresponds to constraining the mass of the single phases. In the incompressible case, see Remark 2.4, such constraints can be equivalently rewritten, for couples (y,ζ) with finite energy, in the more standard form

$$\int_\Omega z_i(x)\,dx=\int_\Omega \zeta_i(y(x))\,dx=M_i \qquad i=1,\dots,h.$$

3. Properties of admissible deformations

In this section we introduce the notion of mappings of finite distorsion and the distorsion function which appears in the definition (2.1) of the set $\mathbb{Y}$ of admissible deformations. Based on the properties of such mappings, for which we mostly refer to [22], we shall obtain a suitable closure property of $\mathbb{Y}$. Let us start by some basic definitions. In this section, Ω is an arbitrary open set of $\mathbb{R}^n$.

The set of finite Radon measures μ on Ω with value in $\mathbb{R}^n$ is denoted by $\mathcal{M}(\Omega;\mathbb{R}^n)$ and it is normed by the total variation

$$|\mu|(\Omega) := \sup\left\{\int_\Omega f\cdot\, d\mu \mid f\in C_c^0(\Omega;\mathbb{R}^n),\ \|f\|_\infty\le 1\right\}.$$

The weak convergence in $\mathcal{M}(\Omega;\mathbb{R}^n)$ of a sequence $(\mu_n)\subset\mathcal{M}(\Omega;\mathbb{R}^n)$ to $\mu\in\mathcal{M}(\Omega;\mathbb{R}^n)$ is defined by

$$\int_\Omega f\cdot d\mu_n \to \int_\Omega f\cdot\, d\mu \quad \text{for any } f\in C_c^0(\Omega;\mathbb{R}^n).$$

For a measurable set $E\subset\Omega$, we denote the n-dimensional Lebesgue measure by $|E|$ and the m-dimensional Hausdorff measure by $\mathcal{H}^m(E)$. By χ_E we denote the characteristic function of E. If $g\in L^1_{loc}(\Omega)$, we say that $g\in BV(\Omega)$ if

$$|\nabla g|(\Omega) := \sup\left\{\int_\Omega g\,\mathrm{div}\varphi\,\mathrm{d}x \mid \varphi\in C_\mathrm{c}^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty\le 1\right\} < +\infty,$$

and we say that a measurable set $E\subset\Omega$ is a set of finite perimeter in Ω if $\chi_E\in BV(\Omega)$. We use the notation $\mathrm{Per}(E,\Omega) := |\nabla\chi_E|(\Omega)$. For a set of finite perimeter E in Ω, there is a subset ∂^*E of ∂E (called *reduced boundary*) such that $\mathrm{Per}(E,\Omega) = \mathcal{H}^{n-1}(\partial^*E\cap\Omega)$, see [2]. Given $y:\Omega\to\mathbb{R}^n$, we will use the notations $\Omega^y := y(\Omega)$ and $E^y := y(E)$, and we recall that y is said to satisfy the Lusin condition N if $|E| = 0 \Rightarrow |E^y| = 0$.

Definition 3.1 (Finite distorsion)**.** Let $\Omega\subset\mathbb{R}^n$ for $n\ge 2$ be an open set. A Sobolev map $y\in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ with $\det\nabla y\ge 0$ almost everywhere in Ω is said to be of *finite distorsion* if $\det\nabla y\in L^1_{\mathrm{loc}}(\Omega)$ and there is a function $K:\Omega\to[1,+\infty]$ with $K<+\infty$ almost everywhere in Ω such that $|\nabla y|^n\le K\det\nabla y$. For a mapping y of finite distorsion, the *(optimal) distorsion function* $K_y:\Omega\to\mathbb{R}$ is defined as

$$K_y(x) := \begin{cases} |\nabla y(x)|^n/\det\nabla y(x) & \text{if } \det\nabla y(x)\neq 0,\\ 1 & \text{if } \det\nabla y(x) = 0.\end{cases}$$

The following result is a closure property of the set of admissible deformations.

Lemma 3.2 (Closure)**.** *Let $p\ge n$ and let $q>n-1$. Let $y\in W^{1,p}(\Omega;\mathbb{R}^n)$ and let $(y_k)_k\subset\mathbb{Y}$ be a sequence such that*

i) *y is not constant*
ii) *$y_k\to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $k\to\infty$,*
iii) *$C := \sup_{k\in\mathbb{N}}\|K_{y_k}\|_{L^q(\Omega)} < +\infty$.*

Then $y \in \mathbb{Y}$. In particular, y has a continuous representative which is a homeomorphism.

Proof. It is enough to consider the hardest case $p = n$. We recall from [18, Section 3] that any element of $\mathbb{Y}$ has a continuous representative which is a homeomorphisms of Ω onto Ω^y.

First of all, up to extraction of a not relabeled subsequence, there exists a function $K \in L^q(\Omega)$ such that $K_{y_k} \to K$ weakly in $L^q(\Omega)$ as $k \to \infty$. Then, a result by Gehring and Iwaniec [16], see also [15] ensures that y is a mapping of finite distorsion such that

$$\|K_y\|_{L^q(\Omega)} \leq \|K\|_{L^q(\Omega)} \leq \liminf_{k\to+\infty} \|K_{y_k}\|_{L^q(\Omega)} \leq C.$$

In particular, y has a continuous representative by [22, Theorem 2.3]. Moreover, since $y_k \to y$ weakly in $W^{1,n}(\Omega;\mathbb{R}^n)$, the higher integrability result by Müller [31] entails $\det \nabla y^k \to \det \nabla y$ weakly in $L^1(E)$ for any open set E compactly contained in Ω. Therefore, we may invoke the result in [17, Theorem 4.4] to infer that $|E^{y_k}| \to |E^y|$ as $k \to +\infty$, recalling that the measure-theoretic images from [17] are in this case reduced to the usual images through the continuous representatives of y_k and y. Moreover, by [22, Theorem 4.5], continuous representatives of $W^{1,n}(\Omega;\mathbb{R}^n)$ mappings of finite distorsion satisfy the Lusin condition N, and thus the area formula holds with equality, see [22, Theorem A.35]. In particular, since the y_k's are in fact homeomorphisms, the area formula yields

$$\int_E \det \nabla y \, dx = \lim_{k\to+\infty} \int_E \det \nabla y_k \, dx = \lim_{k\to+\infty} |E^{y_k}| = |E^y|.$$

By taking now an increasing sequence of open sets E_j, compactly contained in Ω, such that $\cup_{j=1}^{\infty} E_j = \Omega$, and by applying the monotone convergence theorem, we obtain the validity of the Ciarlet-Nečas condition (with equality) for y. We notice that since y is not constant, it is an open map by [22, Theorem 3.4], therefore Ω^y is open. The Ciarlet-Nečas condition entails that the multiplicity function $N(\Omega, y, \cdot)$ of y on Ω is a.e. equal to 1 in Ω^y: indeed, since y satisfies the Lusin condition N, the area formula and the Ciarlet-Nečas condition yield

$$|\Omega^y| \leq \int_{\Omega^y} N(\Omega, y, \xi)\, d\xi = \int_\Omega \det \nabla y \leq |\Omega^y|$$

so that $N(\Omega, y, \xi) = 1$ for a.e. $\xi \in \Omega^y$. By invoking [22, Lemma 4.13] we conclude that $\det \nabla y > 0$ a.e. in Ω. This proves that $y \in \mathbb{Y}$. $\square$

4. Interfacial measures

This section is devoted to introduce a fundamental notions of our theory, in particular we introduce *interfacial measures* and provide a generalization of [18, Theorem 2.2]. In this section, $\Omega \subset \mathbb{R}^n$ denotes a generic open set.

Definition 4.1 (Interfacial measure)**.** Let $p \geq n$. Given a homeomorphism of finite distorsion $y \in W^{1,p}_{loc}(\Omega;\mathbb{R}^n)$ and $g \in L^r_{loc}(\Omega)$ for some $r \in [\frac{p}{p-n}, +\infty]$, we say that $p_{y,g} \in \mathcal{M}(\Omega;\mathbb{R}^n)$ is an interfacial measure for the couple (y, g) if

$$\int_\Omega g \operatorname{cof}(\nabla y) : \nabla \psi \, \mathrm{d}x = \int_\Omega \psi \cdot \mathrm{d}p_{y,g} \qquad \text{for any } \psi \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^n). \tag{4.1}$$

The relevance of this notion comes from its role in the characterization of interface areas in the deformed configurations, in case g is a distance function from an energy well. It will be thoroughly discussed in Theorem 4.2 and in the rest of the paper. If y is the identity map on Ω, requiring the existence of an interfacial measure is equivalent to saying that $g \in BV(\Omega)$. If (4.1) holds, $p_{y,g}$ is the distributional divergence of $-g \operatorname{cof} \nabla y$ in Ω.

In the following, we give a characterization of those couples (g, y), where $g \in L^\infty_{loc}(\Omega)$ and y is a homeomorphism in $W^{1,n}_{loc}(\Omega)$, such that $g \circ y^{-1} \in BV(\Omega^y)$. We state the theorem after having introduced some preliminary notation.

For a homeomorphism $y : \Omega \to \mathbb{R}^n$ and a finite Radon measure $\mu \in \mathcal{M}(\Omega^y;\mathbb{R}^n)$, the pull-back measure of μ through y, denoted $y_\flat\mu$, is the measure in $\mathcal{M}(\Omega;\mathbb{R}^n)$ defined by

$$\int_\Omega \psi \cdot \, d(y_\flat\mu) = \int_{\Omega^y} \psi \circ y^{-1} \cdot \, d\mu \qquad \text{for any bounded Borel function } \psi : \Omega \to \mathbb{R}^n.$$

Clearly, $y_\flat\mu(\Omega) = \mu(\Omega^y)$. Moreover, $|y_\flat\mu|(\Omega) = |\mu|(\Omega^y)$, since is y is a homeomorphism.

Theorem 4.2 (Characterization)**.** *Let $p \geq n$ and let $y \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ be a homeomorphism of finite distorsion. Let $g \in L^\infty_{loc}(\Omega)$. Then, $g \circ y^{-1} \in BV(\Omega^y)$ if and only if a finite Radon measure $p_{y,g} \in \mathcal{M}(\Omega;\mathbb{R}^n)$ exists such that* (4.1) *holds. In such case,*

$$p_{y,g} = y_\flat(\nabla(g \circ y^{-1})) = -\operatorname{div}(g \operatorname{cof} \nabla y) \qquad \textit{in } \mathcal{M}(\Omega;\mathbb{R}^n). \tag{4.2}$$

Proof. We preliminarily observe that a homeomorphism in $W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ satisfies the Lusin's condition N [35, Theorem. 3], i.e., $|E| = 0 \Rightarrow |E^y| = 0$ for any measurable set $E \in \Omega$. As a consequence E^y is measurable for any measurable set $E \in \Omega$ and we may apply the area formula, see [22, Theorem A.35]: if $f \in L^r_{loc}(\Omega)$ for some $r \in [\frac{p}{p-n}, +\infty]$, for the measurable function $f \circ y^{-1}$ there holds

$$\int_{E^y} |f| \circ y^{-1} \, d\xi = \int_E |f| \det \nabla y \, dx,$$

for any measurable set $E \subset \Omega$. In particular we obtain $f \circ y^{-1} \in L^1_{loc}(\Omega^y)$, since we have by assumption $\det \nabla y \in L^{p/n}_{loc}(\Omega)$ and $f \in L^r_{loc}(\Omega)$. The Lusin condition N also implies that $\|g \circ y^{-1}\|_{L^\infty(E^y)} = \|g\|_{L^\infty(E)}$ for any measurable set $E \subset \Omega$ so that we obtain $g \circ y^{-1} \in L^\infty_{loc}(\Omega^y)$, since $g \in L^\infty_{loc}(\Omega)$.

Step 1. Let us assume $g \circ y^{-1} \in BV(\Omega^y)$. We shall verify that, by taking $p_{y,g} := y_\flat(\nabla(g \circ y^{-1}))$, (4.1) holds along with (4.2).

First, we observe that $y_\flat(\nabla(g\circ y^{-1}))\in\mathcal{M}(\Omega;\mathbb{R}^n)$ by definition of pull-back, since $\nabla(g\circ y^{-1})\in\mathcal{M}(\Omega^y;\mathbb{R}^n)$. Let $\psi\in C_c^\infty(\Omega;\mathbb{R}^n)$. Let $G_\varepsilon:=(g\circ y^{-1})*\rho_\varepsilon$, where $\rho_\varepsilon(x):=\varepsilon^{-n}\rho(x/\varepsilon)$, $x\in\mathbb{R}^n$, and ρ is the standard unit symmetric mollifier in $\mathbb{R}^n$, so that (up to passing to a vanishing sequence, which we do not include in the notation) $G_\varepsilon\to g\circ y^{-1}$ a.e. in Ω^y and $\nabla G_\varepsilon\rightharpoonup\nabla(g\circ y^{-1})$ weakly in $\mathcal{M}(\Omega^y;\mathbb{R}^n)$. Therefore,

$$\int_\Omega\psi\cdot d(y_\flat(\nabla(g\circ y^{-1})))=\int_{\Omega^y}(\psi\circ y^{-1})\cdot d(\nabla(g\circ y^{-1}))=\lim_{\varepsilon\to0}\int_{\Omega^y}(\psi\circ y^{-1})\cdot\nabla G_\varepsilon\,d\xi.\tag{4.3}$$

There holds $(\nabla y)^{-T}\nabla(G_\varepsilon\circ y)=(\nabla G_\varepsilon)\circ y$ a.e. in $D:=\{x\in\Omega:\det\nabla y(x)>0\}$. The cofactor matrix is divergence-free, implying $\operatorname{div}((\operatorname{cof}\nabla y)^T\psi)=\operatorname{cof}\nabla y:\nabla\psi$. Moreover, $\operatorname{cof}\nabla y=0$ holds a.e. on $\Omega\setminus D$ since y is a mapping of finite distorsion. Hence,

$$\begin{aligned}\int_{\Omega^y}(\psi\circ y^{-1})\cdot\nabla G_\varepsilon\,d\xi&=\int_D(\det\nabla y)\,\psi\cdot(\nabla G_\varepsilon)\circ y\,dx\\&=\int_D(\det\nabla y)\,\psi\cdot(\nabla y)^{-T}\nabla(G_\varepsilon\circ y)\,dx=\int_D(\det\nabla y)\,(\nabla y)^{-1}\psi\cdot\nabla(G_\varepsilon\circ y)\,dx\\&=-\int_\Omega(G_\varepsilon\circ y)\operatorname{div}((\operatorname{cof}\nabla y)^T\,\psi)\,dx=-\int_\Omega(G_\varepsilon\circ y)\operatorname{cof}\nabla y:\nabla\psi\,dx\end{aligned}\tag{4.4}$$

Since $G_\varepsilon\to g\circ y^{-1}$ pointwise a.e. in Ω_y, we obtain $G_\varepsilon\circ y\to g$ a.e. in D. Indeed, the area formula again implies that for a measurable set $E\subset D$ there holds $|E^y|=\int_E\det\nabla y$ so that $|E^y|=0$ implies $|E|=0$. In particular, if $E=D\cap\operatorname{supp}(\psi)$, then $\|G_\varepsilon\circ y\|_{L^\infty(E)}=\|G_\varepsilon\|_{L^\infty(E^y)}\le\|g\circ y^{-1}\|_{L^\infty(E^y)}=\|g\|_{L^\infty(E)}<\infty$. As $\operatorname{cof}\nabla y\in L^1_{loc}(\Omega)$ and $\operatorname{cof}\nabla y=0$ a.e. on $\Omega\setminus D$, by dominated convergence we obtain

$$\lim_{\varepsilon\to0}\int_\Omega(G_\varepsilon\circ y)\,\nabla\psi:\operatorname{cof}\nabla y\,dx=\int_\Omega g\,\nabla\psi:\operatorname{cof}\nabla y\,dx.\tag{4.5}$$

By combining (4.3), (4.4) and (4.5) we get

$$\int_\Omega\psi\cdot d(y_\flat^{-1}(\nabla(g\circ y^{-1})))=-\int_\Omega g\,\nabla\psi:\operatorname{cof}\nabla y\,dx$$

for any $\psi\in C_c^\infty(\Omega;\mathbb{R}^n)$. Hence, $p_{y,g}$ satisfies (4.1) and (4.2) holds.

Step 2. Let us now assume that $p_{y,g}\in\mathcal{M}(\Omega;\mathbb{R}^n)$ exists such that (4.1) holds and let us verify that $g\circ y^{-1}\in BV(\Omega^y)$.

The area formula gives

$$\begin{aligned}|\nabla(g\circ y^{-1})|(\Omega^y)&=\sup\left\{\int_{\Omega^y}g(y^{-1}(\xi))\operatorname{div}\varphi(\xi)\,\mathrm{d}\xi\mid\varphi\in C_\mathrm{c}^\infty(\Omega^y;\mathbb{R}^n),\ \|\varphi\|_\infty\le1\right\}\\&=\sup\left\{\int_\Omega g(x)\operatorname{div}\varphi(y(x))\det\nabla y(x)\,\mathrm{d}x\mid\varphi\in C_\mathrm{c}^\infty(\Omega^y;\mathbb{R}^n),\ \|\varphi\|_\infty\le1\right\}\\&=\sup\left\{\int_\Omega g\operatorname{cof}(\nabla y):\nabla(\varphi\circ y)\,\mathrm{d}x\mid\varphi\in C_\mathrm{c}^\infty(\Omega^y;\mathbb{R}^3),\ \|\varphi\|_\infty\le1\right\},\end{aligned}\tag{4.6}$$

where the second equality is due to the identity $(\operatorname{div}\varphi)\circ y\det\nabla y=\operatorname{cof}\nabla y:\nabla(\varphi\circ y)$ which holds a.e. in Ω, as a consequence of the chain rule and of the matrix identity

$(\operatorname{cof} A)A^T = I \det A$. As $y \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^n)$, we have $\operatorname{cof}\nabla y \in L^q_{\rm loc}(\Omega)$ with $q = p/(n-1)$. Since $g \in L^{p/(p-n)}_{loc}(\Omega)$ we get $g \operatorname{cof}\nabla y : \nabla(\varphi\circ y) \in L^1_{loc}(\Omega)$. The function $g \operatorname{cof}\nabla y : \nabla(\varphi\circ y)$ is compactly supported in Ω, as y is a homeomorphism and φ is compactly supported in Ω^y. As a consequence, the relation (4.1) can be extended by continuity to all test functions in the class $W^{1,p}(\Omega;\mathbb{R}^n)\cap C^0_{\rm c}(\Omega;\mathbb{R}^n)$ since $p_{y,g} \in \mathcal{M}(\Omega;\mathbb{R}^n)$. Therefore, $\varphi\circ y$ is an admissible test function for equality (4.1). From (4.6), from the validity (4.1) and from the fact that (4.1) holds with test functions in $W^{1,p}(\Omega;\mathbb{R}^n)\cap C^0_{\rm c}(\Omega;\mathbb{R}^n)$ we obtain

$$|\nabla(g\circ y^{-1})|(\Omega^y) = \sup\left\{ \int_\Omega (\varphi\circ y)\cdot {\rm d}p_{y,g} \mid \varphi \in C^\infty_{\rm c}(\Omega^y;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \right\}. \qquad (4.7)$$

The definition of total variation and (4.7) directly imply $|\nabla(g\circ y^{-1})|(\Omega^y) \le |p_{y,g}|(\Omega)$. $\square$

5. Convergence of the phases

From here and through the rest of the paper, Ω is a bounded open Lipschitz set. In this section, we prepare some tools which will later be used in the limit passages in Sections 6 and 7.

Lemma 5.1. *Let $p \ge n$. Let $(y_k)_k \subset W^{1,p}(\Omega;\mathbb{R}^n)$, $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ be homeomorphisms of finite distorsion such that $y_k \to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$.*

- i) *If $A \subset\subset \Omega^y$, then there exists $k_0 \in \mathbb{N}$ such that $A \subset \Omega^{y_k}$ for any $k > k_0$.*
- ii) *Assuming in addition that the sequence $(\det\nabla y_k)_k$ is equiintegrable on Ω, there holds $\lim_{k\to\infty} |\Omega^y \Delta \Omega^{y_k}| = 0$.*

Proof. i) First we prove that the sequence y_k is uniformly converging on any compact subset $K \subset\subset \Omega$. From [21, Theorem 1.3] we deduce that there exists a constant $C(K,n)$ such that, for any k,

$$\forall x_1, x_2 \in K \quad |y_k(x_1) - y_k(x_2)| \le C(K,n)\|\nabla y_k\|_{L^n(\Omega)}\,\theta(|x_1 - x_2|), \quad \theta(t) := |\ln(2/t)|^{-1/n}.$$

Since $\|\nabla y_k\|_{L^n(\Omega)}$ is bounded, we obtain the equicontinuity of the sequence (y_k) over K. Moreover, by combining equicontinuity on compact domains with the bound $\|y_k\|_{L^1(\Omega)} < C$, the uniform boundedness of y_k on K follows:

$$\sup\{|y_k(x)| : k \in \mathbb{N},\, x \in K\} < \infty.$$

In fact, suppose by contradiction that there exist sequences $(k_\ell)_\ell$ and $(x_\ell)_\ell \subset K$ such that $|y_{k_\ell}(x_\ell)| \ge \ell$ for any $\ell \in \mathbb{N}$. Fix a $\delta > 0$ such that $K + B_\delta(0) \subset K' \subset\subset \Omega$ for a compact set K'. By the equicontinuity on K', there exist $r \in (0,\delta)$ such that

$$\forall k \in \mathbb{N}\ \ \forall x \in K\ \ \forall x' \in B_r(x), \quad x' \in K' \quad \text{and} \quad |y_k(x') - y_k(x)| < 1.$$

Therefore, $\|y_{k_\ell}\|_{L^1} \ge \int_{B_r(x_\ell)} |y_{k_\ell}(x')|dx' \ge |B_r(0)|\,(\ell-1) \to \infty$ for $\ell \to \infty$, a contradiction. By Ascoli-Arzelà theorem, $y_k \to y$ uniformly on any compact subset of Ω.

Let S be such that $A \subset\subset S \subset\subset \Omega^y$. Since y, y_k are homeomorphisms and there is uniform convergence of y_k to y on compact subsets of Ω, it is easy to conclude. Indeed, let $\varepsilon := \mathrm{dist}(\overline{A}, \partial S)$ so that $\varepsilon > 0$. Let $U = y^{-1}(S)$ so that $U \subset\subset \Omega$ as y is a homeomorphism. Let $S_k = y_k(U)$. Since $y, y^k \in \mathbb{Y}$ are homeomorphisms on $\overline{U}$, we have $\partial S = y(\partial U)$ and $\partial S_k = y_k(\partial U)$. By the above result we have $y_k \to y$ uniformly on $\overline{U}$, thus fixing $\delta < \varepsilon/2$ we get $\sup_{x\in\overline{U}} |y(x) - y_k(x)| < \delta$ for k large enough. Hence, for any boundary point $\xi \in \partial S_k$, we have that $d(\xi, \partial S) < \delta$ for k large enough. Since $d(\overline{A}, \partial S) = \varepsilon > 2\delta$, we obtain $d(\overline{A}, \partial S_k) > \delta$, hence $\overline{A} \subset S_k \subset \Omega^{y_k}$ for any large enough k.

ii) We have $\det \nabla y_k \to \det \nabla y$ weakly in $L^1(\Omega)$ as $k \to +\infty$. This follows from the boundedness in $L^p(\Omega; \mathbb{R}^n)$ of the sequence $(\nabla y_k)_k$ if $p > n$ and from the additional equiintegrability assumption if $p = n$. Then, the property $\lim_{k\to\infty} |\Omega^y \Delta \Omega^{y_k}| = 0$ is a consequence of [17, Theorem 4.4]. Indeed, the measure-theoretic images appearing in [17] are the usual images for our mappings that have a continuous representative. □

Lemma 5.2. *Let $p \geq n$ and $q > n - 1$. Let $(y_k)_k \subset W^{1,p}(\Omega; \mathbb{R}^n)$, $y \in W^{1,p}(\Omega; \mathbb{R}^n)$ be homeomorphisms of finite distorsion such that $y_k \to y$ weakly in $W^{1,p}(\Omega; \mathbb{R}^n)$ and $\sup_{k\in\mathbb{N}} \|K_{y_k}\|_{L^q(\Omega)} < +\infty$. Suppose that the sequence $(\det \nabla y_k)_k$ is equiintegrable on Ω. Then $|y^{-1}(O_k)| \to |\Omega|$ and $|y_k^{-1}(O_k)| \to |\Omega|$ as $k \to \infty$, where $O^k := \Omega^y \cap \Omega^{y_k}$.*

Proof. We preliminarily observe that $\nabla y^{-1} \in L^n(\Omega^y; \mathbb{R}^n)$ and $\nabla y_k^{-1} \in L^n(\Omega^{y_k}; \mathbb{R}^n)$ for any $k \in \mathbb{N}$. This follows from the L^q-integrability of the distorsion, see [23]. In particular, $\det \nabla y^{-1} \in L^1(\Omega^y)$ and $\det \nabla y_k^{-1} \in L^1(\Omega^{y_k})$. Moreover, since y, y_k are homeomorphisms, we have $\det \nabla y > 0$ a.e. in Ω^y and $\det \nabla y_k^{-1} > 0$ a.e. in Ω^{y_k} for any $k \in \mathbb{N}$ and then y, y_k satisfy the Lusin condition N^{-1}, see [22, Theorem 4.13]. In particular, y^{-1} satisfies the Lusin condition N so that the area formula holds (with equality) and entails

$$|\Omega| = |y^{-1}(\Omega^y)| = \int_{\Omega^y} \det \nabla y^{-1} \, d\xi, \qquad |y^{-1}(O^k)| = \int_{O^k} \det \nabla y^{-1} \, d\xi, \quad k \in \mathbb{N}. \tag{5.1}$$

Since $|\Omega^y \setminus O^k| \to 0$ by Lemma 5.1, we get from (5.1) as $k \to \infty$

$$|y^{-1}(O_k)| = \int_{O^k} \det \nabla y^{-1} \, \mathrm{d}\xi \to \int_{\Omega^y} \det \nabla y^{-1} \, \mathrm{d}\xi = |\Omega|.$$

With the same change of variables for y_k^{-1} that satisfies the Lusin condition N we get

$$\begin{aligned} |y_k^{-1}(O_k)| &= \int_{O^k} \det \nabla y_k^{-1} \, \mathrm{d}\xi = \int_{\Omega^{y_k}} \det \nabla y_k^{-1} \, \mathrm{d}\xi - \int_{\Omega^{y_k} \setminus O^k} \det \nabla y_k^{-1} \, \mathrm{d}\xi \\ &= |\Omega| - \int_{\Omega^{y_k} \setminus \Omega^y} \det \nabla y_k^{-1} \, \mathrm{d}\xi. \end{aligned}$$

Thanks to the results in [33], the uniform bound on $\|K_{y_k}\|_{L^q(\Omega)}$ yields the equi-integrability of the family $(\det \nabla y_k^{-1})_k$, as proven in [18, Lemma 5.1]. Since $|\Omega^{y_k} \setminus \Omega^y| \to 0$ as $k \to \infty$ by Lemma 5.1, the statement follows. □

Lemma 5.3 (Convergence of the reference phases)**.** *Let $p \geq n$, $q > n - 1$. Suppose that*

i) $(y_k)_k \subset W^{1,p}(\Omega;\mathbb{R}^n)$, $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ *are homeomorphisms of finite distortion such that* $y_k \to y$ *weakly in* $W^{1,p}(\Omega;\mathbb{R}^n)$ *as* $k \to \infty$,
ii) $\sup_{k\in\mathbb{N}} \|K_{y_k}\|_{L^q(\Omega)} < +\infty$ *and the sequence* $(\det \nabla y_k)_k$ *is equiintegrable on* Ω,
iii) $(\zeta_k)_k \subset L^\infty(\Omega^{y_k};\mathbb{R}^h)$, $\zeta \in L^\infty(\Omega^y)$ *and* $\|\zeta_k - \zeta\|_{L^1(\Omega^y\cap\Omega^{y_k})} \to 0$ *as* $k \to \infty$,
iv) $|\zeta_k(\xi)| \le M$ *holds a.e. in* Ω^{y_k}, *for any* $k \in \mathbb{N}$.

Then $K_y \in L^q(\Omega)$, $|\zeta(\xi)| \le M$ *a.e. in* Ω^y *and* $\|\zeta_k \circ y_k - \zeta \circ y\|_{L^1(\Omega)} \to 0$ *as* $k \to \infty$.

Proof. We use the notations $O^k := \Omega^y \cap \Omega^{y_k}$ and $E_k := y^{-1}(O^k) \cap y_k^{-1}(O^k)$. Since $y^{-1}(O_k) \subset \Omega$ and $y_k^{-1}(O_k) \subset \Omega$, in order to prove that $|\Omega \setminus E_k| \to 0$ as $k \to \infty$ it is sufficient to show $|y^{-1}(O_k)| \to |\Omega|$ and $|y_k^{-1}(O_k)| \to |\Omega|$, which are in turn proven in Lemma 5.2.

Let us prove that $|\zeta| \le M$ a.e. in Ω^y. Indeed, suppose not and let $B := \{\xi \in \Omega^y : |\zeta(\xi)| > M\}$ so that $|B| > 0$. Then there exists $\varepsilon > 0$ and $B' \subset B$ such that $|B'| > |B|/2$ and $|\zeta| > M + \varepsilon$ a.e. in B'. By assumption iv), this implies $|\zeta_k(\xi) - \zeta(\xi)| > \varepsilon$ a.e. on B' for any k. Let $A \subset\subset \Omega^y$ be an open set such that $|\Omega^y \setminus A| < |B|/4$, so that $|A \cap B'| > |B|/4$. Therefore, $\|\zeta_k - \zeta\|_{L^1(A)} \ge \|\zeta_k - \zeta\|_{L^1(A\cap B')} \ge \varepsilon|B|/4$ for any k. On the other hand, for k large enough we have $A \subset\subset \Omega^{y_k}$ by Lemma 5.1, hence assumption iii) implies that $\|\zeta_k - \zeta\|_{L^1(A)}$ goes to zero as $k \to \infty$, a contradiction.

As seen in the proof of Lemma 5.2, we have $\det \nabla y > 0$ a.e. in Ω^y and $\det \nabla y_k^{-1} > 0$ a.e. in Ω^{y_k} for any $k \in \mathbb{N}$. Then, the property $K_y \in L^q(\Omega)$ follows by the polyconvexity of the optimal distortion function on the set of matrices of positive determinant.

Next we prove the convergence of reference phases $\|\zeta_k \circ y_k - \zeta \circ y\|_{L^1(\Omega)} \to 0$ as $k \to \infty$. We clearly bound such norm by $2M|\Omega \setminus E_k| + \|\zeta_k \circ y_k - \zeta \circ y\|_{L^1(E_k)}$, therefore we are reduced to prove that $\|\zeta_k \circ y_k - \zeta \circ y\|_{L^1(E_k)}$ goes to zero as $k \to \infty$. The argument is similar to the one of [18, Lemma 5.3]. Indeed, there holds $\|\zeta_k \circ y_k - \zeta \circ y\|_{L^1(E_k)} \le I_k + J_k$, where

$$I_k := \|\zeta_k \circ y_k - \zeta \circ y_k\|_{L^1(E_k)}, \qquad J_k := \|\zeta \circ y_k - \zeta \circ y\|_{L^1(E_k)}.$$

About I_k, since y_k^{-1} satisfies the Lusin condition N we may change variables as done in the proof of Lemma 5.2 and obtain

$$I_k = \int_{E_k^{y_k}} |\zeta_k(\xi) - \zeta(\xi)| \det \nabla y_k^{-1}(\xi)\, d\xi. \tag{5.2}$$

We fix a small value $\delta > 0$, and since $E_k^{y_k} \subset O_k$, by (5.2) we have

$$I_k \le \int_{O^k} \det \nabla y_k^{-1} |\zeta_k - \zeta|\, d\xi \le \delta \int_{O_k\setminus A_k(\delta)} \det \nabla y_k^{-1}\, d\xi + 2M \int_{A_k(\delta)} \det \nabla y_k^{-1}\, d\xi, \tag{5.3}$$

where $A_k(\delta) := \{\xi \in O^k : |\zeta_k(\xi) - \zeta(\xi) > \delta|\}$. Notice that

$$\delta |A_k(\delta)| \le \int_{A_k(\delta)} |\zeta_k - \zeta|\, d\xi \le \int_{O_k} |\zeta_k - \zeta|\, d\xi,$$

so that assumption iii) yields $|A_k(\delta)| \to 0$ as $k \to \infty$. We deduce that

$$\lim_{k\to\infty} \int_{A_k(\delta)} \det \nabla y_k^{-1}\, d\xi = 0,$$

thanks to the equi-integrability property of ∇y_k^{-1} from [18, Lemma 5.1]. Inserting this in (5.3) we get

$$\limsup_{k\to 0} I_k \le \limsup_{k\to 0}\, \delta \int_{O_k\setminus A_k(\delta)} \det \nabla y_k^{-1}\, d\xi \le \delta|\Omega|,$$

where we changed back variables and used $y_k^{-1}(O_k \setminus A_k(\delta)) \subset \Omega$.

Concerning J_k, Let $\bar\zeta_\delta$ be a continuous compactly supported function in Ω^y such that $|\bar\zeta_\delta| \le M$ and such that $|\bar A_\delta| < \delta$, where $\bar A_\delta := \{\xi \in \Omega^y : |\bar\zeta_\delta(\xi) - \zeta(\xi)| > \delta\}$. For instance, we may may take a mollification of the restriction of ζ to a large enough open set compactly contained in Ω^y. We write $J_k = J_k^{(1)} + J_k^{(2)} + J_k^{(3)}$, where

$$J_k^{(1)} = \|\zeta \circ y_k - \overline{\zeta}_\delta \circ y_k\|_{L^1(E_k)}, J_k^{(2)} = \|\overline{\zeta}_\delta \circ y_k - \overline{\zeta}_\delta \circ y\|_{L^1(E_k)}, J_k^{(3)} = \|\overline{\zeta}_\delta \circ y - \zeta \circ y\|_{L^1(E_k)}.$$

Here, $J_k^{(1)}$ and $J_k^{(3)}$ can be treated exactly as I_k by change of variables, and with the help of assumption iv) we have for any $k \in \mathbb{N}$

$$J_k^{(1)} \le \delta|\Omega| + 2M \int_{\bar A_\delta} \det \nabla y_k^{-1}\, d\xi, \quad J_k^{(3)} \le \delta|\Omega| + 2M \int_{\bar A_\delta} \det \nabla y^{-1}\, d\xi. \tag{5.4}$$

On the other hand, if $A \subset\subset \Omega$ is an open set such that $|\Omega \setminus A| < \delta$, we have

$$J_k^{(2)} \le 2M\delta + \int_A |\bar\zeta_\delta \circ y_k - \bar\zeta_\delta \circ y|\, dx \le 2M\delta + |\Omega| \sup_{x\in A} \omega_\delta(|y_k(x) - y(x)|) \tag{5.5}$$

where ω_δ is the modulus of continuity of $\bar\zeta_\delta$. Taking the limit as $k \to \infty$ in (5.5), since by Lemma 5.1 we have uniform convergence of y_k to y in A, we get

$$\limsup_{k\to\infty} J_k^{(2)} \le 2M\delta. \tag{5.6}$$

By (5.4), (5.6), the equi-integrability of $\det \nabla y_k^{-1}$, the integrability of $\det \nabla y^{-1}$, by $|\bar A_\delta| < \delta$ and the arbitrariness of δ, we conclude that $J_k \to 0$ as $k \to \infty$. □

6. LOWER BOUND

This section collects some lower semicontinuity arguments, leading to the proof of the Γ-lim inf inequality, namely Proposition 6.4. We start by a lower semicontinuity property of interfacial measures (see Definition 4.1).

Proposition 6.1 (Double lower semicontinuity of $p_{y,g}$). *Let $(y_k)_k \subset W^{1,p}(\Omega;\mathbb{R}^n)$, $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ be homeomorphisms of finite distorsion such that $y_k \to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$, for $p \ge n$. Let $(g_k)_k \subset L^r_{loc}(\Omega)$, $g \in L^r_{loc}(\Omega)$, $r \in [\frac{p}{p-n}, +\infty)$, be such that $g_k \to g$ strongly* *in* $L^r_{loc}(\Omega)$. *If* $\liminf_{k\to+\infty}|p_{y_k,g_k}|(\Omega)<\infty$, *then there exists* $p_{y,g}\in\mathcal{M}(\Omega;\mathbb{R}^n)$ *satisfying* (4.1) *and*

$$|p_{y,g}|(\Omega)\le\liminf_{k\to+\infty}|p_{y_k,g_k}|(\Omega). \tag{6.1}$$

Proof. Since $\nabla y_k\to\nabla y$ weakly in $L^p(\Omega)$, the convergence $\mathrm{cof}\,\nabla y_k\to\mathrm{cof}\,\nabla y$ holds weakly in $L^{p/(n-1)}(\Omega)$. Therefore, for any test function $\psi\in C^\infty_{\rm c}(\Omega;\mathbb{R}^3)$, as $k\to\infty$ we have,

$$\int_\Omega\psi\cdot\mathrm{d}p_{y_k,g_k}=\int_\Omega g_k\mathrm{cof}\,\nabla y_k:\nabla\psi\,\mathrm{d}x\to\int_\Omega g\,\mathrm{cof}\,\nabla y:\nabla\psi\,\mathrm{d}x=:p_{y,g}(\psi),$$

by weak-times-strong convergence; the last equality is a definition of the distribution on the right side. By the lower semicontinuity of the total variation, we have that $p_{y,g}\in\mathcal{M}(\Omega;\mathbb{R}^n)$ and (6.1) holds. ☐

Lemma 6.2 (Lower semicontinuity of bulk energy)**.** *Let assumptions* (2.3) *and* (2.4) *hold. Let* $R>0$, *let* $(y,\zeta)\in\widetilde{\mathbb{Q}}^R$ *and let the sequence* $(y_k,\zeta_k)_k\subset\widetilde{\mathbb{Q}}^R$ *be such that*

i) $y_k\to y$ *weakly in* $W^{1,p}(\Omega;\mathbb{R}^n)$,
ii) $\lim_{k\to+\infty}\|\zeta_k-\zeta\|_{L^1(O^k)}=0$, *with* $O^k:=\Omega^{y_k}\cap\Omega^y$.

Then, $\mathcal{F}^{\rm bulk}(y,\zeta)\le\liminf_{k\to\infty}\mathcal{F}^{\rm bulk}(y_k,\zeta_k)$.

Proof. We may assume that along a not relabeled subsequence $\sup_{k\in\mathbb{N}}\mathcal{F}^{\rm bulk}(y_k,\zeta_k)<+\infty$. Thanks to the coercivity assumption (2.4), the hypotheses of Lemma 5.3 are satisfied. Letting $z_k:=\zeta_k\circ y_k$, Lemma 5.3 entails $z_k\to z=\zeta\circ y$ in $L^1(\Omega;\mathbb{R}^h)$. Now, write the bulk energy functional as a function of z:

$$\widetilde{\mathcal{F}}^{\rm bulk}(y,z):=\mathcal{F}^{\rm bulk}(y,z\circ y^{-1})=\int_\Omega W(\nabla y(x),z(x))\,\mathrm{d}x,$$

Since $W(\cdot,\cdot)$ is lower semicontinuous in $\mathbb{R}^{n\times n}\times\mathbb{R}^h$ and is poly-convex in the first argument, we can apply the result [13, Corollary 7.9], getting $\liminf_{k\to\infty}\widetilde{\mathcal{F}}^{\rm bulk}(y_k,z_k)\ge\widetilde{\mathcal{F}}^{\rm bulk}(y,z)$, which proves the claim. ☐

In the following, we recall that $\Phi:\mathbb{R}^h\to\mathbb{R}^+$ is a continuous potential that vanishes only at the points of P, and that (1.3) holds. We take advantage of the following inequality, for a proof see [3, Proposition 2.1].

Proposition 6.3. *For any* $\alpha\in\{1,\dots,m\}$, *let* $\varphi_\alpha:\mathbb{R}^h\to\mathbb{R}$ *be defined by* $\varphi_\alpha(z):=d_\Phi(p_\alpha,z)$, *where the* p_α*'s are the zeros of* Φ. *Let* $u\in W^{1,2}(\Omega;\mathbb{R}^h)\cap L^\infty(\Omega;\mathbb{R}^h)$. *Then* $\varphi_\alpha\circ u\in W^{1,2}(\Omega)$ *and for any open set* $A\subseteq\Omega$ *there holds*

$$\int_A|\nabla(\varphi_\alpha\circ u)|\le\int_A\sqrt{\Phi}\circ u\,|\nabla u|.$$

Before stating the liminf inequality, we recall that for a collection $\{\mu_\alpha\}_{\alpha=1,\ldots,m}$ of positive Borel measures on Ω, the supremum measure is defined on open sets $A \subseteq \Omega$ as

$$\left(\bigvee_{\alpha=1}^{m} \mu\right)(A) := \sup\left\{\sum_{\alpha=1}^{k} \mu_\alpha(A_\alpha) : (A_\alpha) \text{ pairw. disjoint open sets, } \bigcup_{\alpha=1}^{m} A_\alpha = A\right\} \quad (6.2)$$

Equivalently, the supremum measure is the smallest positive Borel measure ν such that $\nu(A) \geq \mu_\alpha(A)$ for any $\alpha \in \{1,\ldots,m\}$ and any open set $A \subseteq \Omega$.

The theory that we developed in Section 4 shall play a crucial role in the liminf inequality. Indeed, as we will see through the next proof, as soon as $(y,\zeta) \in \widetilde{\mathbb{Q}}^R$ is a state with finite energy, i.e., $\mathcal{F}_\varepsilon(y,\zeta) < +\infty$, an interfacial measure exists for the couple $(\varphi_\alpha \circ \zeta \circ y, y)$, for any $\alpha = 1,\ldots,m$. This is reminiscent of the notion of admissible states from [36, 37], which are indeed defined as those couples of deformations and phase indicators that admit a suitable interfacial measure.

Proposition 6.4 (Γ-$\liminf$ inequality)**.** *Let $p \geq n$, $q > n-1$. Let $R > \max_{\alpha\in\{1,\ldots,m\}} |p_\alpha|$, where $p_1,\ldots,p_m$ are the zeroes of Φ. Let $(y,\zeta) \in \widetilde{\mathbb{Q}}^R$ and let $(y_k,\zeta_k)_k, \subset \widetilde{\mathbb{Q}}^R$ be a sequence such that*

i) $\liminf_{k\to+\infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k,\zeta_k) < \infty$ *for some vanishing sequence* $(\varepsilon_k)_k \subset (0,+\infty)$,
ii) $y_k \to y$ *weakly in* $W^{1,p}(\Omega;\mathbb{R}^n)$,
iii) $\lim_{k\to+\infty} \|\zeta_k - \zeta\|_{L^1(O^k)} = 0$, *with* $O^k := \Omega^{y_k} \cap \Omega^y$.

Then, there exist sets of finite perimeter $E^y_\alpha \subset \Omega^y$, $\alpha = 1,\ldots,m$ such that

$$\zeta = \sum_{\alpha=1}^{m} p_\alpha \chi_{E^y_\alpha} \quad (6.3)$$

and

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{m} d_{\alpha,\beta}\mathcal{H}^{n-1}(E^y_{\alpha,\beta}) \leq \liminf_{k\to+\infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k,\zeta_k), \quad (6.4)$$

where $E^y_{\alpha,\beta} := \Omega^y \cap \partial^* E^y_\alpha \cap \partial^* E^y_\beta$. In particular, one has that $(y,\zeta) \in \mathbb{Q}$.

Proof. Let $F \subset\subset \Omega^y$ be open. By Lemma 5.1 we have $F \subset \Omega^{y_k}$ for any large enough k. Therefore, assumption i) and Fatou lemma imply

$$\int_F \Phi(\zeta)\,d\xi \leq \liminf_{k\to\infty} \int_F \Phi(\zeta_k)\,d\xi \leq \liminf_{k\to\infty} \varepsilon_k \int_{\Omega^{y_k}} \frac{1}{\varepsilon_k}\Phi(\zeta_k)\,d\xi \leq \liminf_{k\to\infty} \varepsilon_k\, \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k,\zeta_k) = 0.$$

The arbitrariness of F and $\Phi \geq 0$ show that $\Phi(\zeta) = 0$ a.e. in Ω^y.

For any $\alpha \in \{1,\ldots,m\}$ and any open set $A \subseteq \Omega$ (so that A^{y_k} is open as well, since y_k is a homeomorphism) we have by Proposition 6.3

$$\int_{A^{y_k}} \left(\frac{\varepsilon_k}{2}|\nabla\zeta_k|^2 + \frac{1}{\varepsilon_k}\Phi(\zeta_k)\right) \mathrm{d}\xi \geq \int_{A^{y_k}} \sqrt{2\Phi(\zeta_k)}\,|\nabla\zeta_k|\,\mathrm{d}\xi \geq \int_{A^{y_k}} |\nabla(\varphi_\alpha \circ \zeta_k)|\,d\xi.$$

Therefore,

$$\int_{\Omega^{y_k}} \left(\frac{\varepsilon_k}{2}|\nabla\zeta_k|^2 + \frac{1}{\varepsilon_k}\Phi(\zeta_k)\right) \mathrm{d}\xi \geq \int_{\Omega^{y_k}} \max_{\alpha=1,\dots,m} |\nabla(\varphi_\alpha \circ \zeta_k)|\, d\xi = \left(\bigvee_{\alpha=1}^{m} |\nabla(\varphi_\alpha \circ \zeta_k)|\right)(\Omega^{y_k}). \tag{6.5}$$

We have $|\zeta_k| \leq R$ and we let $z_k := \zeta_k \circ y_k$, thus $z_k \in L^\infty(\Omega;\mathbb{R}^h)$. We clearly have $g_k^\alpha := \varphi_\alpha \circ z_k \in L^\infty(\Omega)$, and since $\varphi_\alpha \circ \zeta_k = g_k^\alpha \circ y_k^{-1}$, by invoking Theorem 4.2 we see that

$$|\nabla(\varphi_\alpha \circ \zeta_k)|(A^{y_k}) = |(y_k)_\flat(\nabla(\varphi_\alpha \circ \zeta_k))|(A) = |p_{y_k,g_k^\alpha}|(A), \tag{6.6}$$

for any open set $A \subseteq \Omega$, where p_{y_k,g_k^α} is an interfacial measure. By Lemma 5.3 we have $z_k \to z$ strongly in $L^1(\Omega;\mathbb{R}^h)$, hence $g_k^\alpha \to g^\alpha$ strongly $L^1(\Omega)$. As in the proof of Proposition 6.1, we get the weak convergence of measures $p_{y_k,g_k^\alpha} \rightharpoonup p_{y,g^\alpha}$, which yields lower semicontinuity for any open set $A \subseteq \Omega$, i.e.

$$|p_{y,g^\alpha}|(A) \leq \liminf_{k\to\infty} |p_{y_k,g_k^\alpha}|(A). \tag{6.7}$$

By defining $g^\alpha := \varphi_\alpha \circ z$, still by Theorem 4.2 we have

$$|p_{y,g^\alpha}|(A) = |\nabla(g^\alpha \circ y^{-1})|(A^y) = |\nabla(\varphi_\alpha \circ \zeta)|(A^y). \tag{6.8}$$

From (6.6), (6.7) and (6.8) we get

$$|\nabla(\varphi_\alpha \circ \zeta)|(A^y) \leq \liminf_{k\to\infty} |\nabla(\varphi_\alpha \circ \zeta_k)|(A^{y_k})$$

for any open set $A \subseteq \Omega$ and any $\alpha \in \{1,\dots,m\}$. By the latter semicontinuity property and the definition (6.2) of supremum measure, we obtain

$$\left(\bigvee_{\alpha=1}^{m} |\nabla(\varphi_\alpha \circ \zeta)|\right)(\Omega^y) \leq \liminf_{k\to\infty} \left(\bigvee_{\alpha=1}^{m} |\nabla(\varphi_\alpha \circ \zeta_k)|\right)(\Omega^{y_k}). \tag{6.9}$$

In conclusion we obtain from (6.5) and (6.9)

$$\liminf_{k\to\infty} \mathcal{F}^{\mathrm{int}}_{\varepsilon_k}(y_k,\zeta_k) = \liminf_{k\to\infty} \int_{\Omega^{y_k}} \left(\frac{\varepsilon_k}{2}|\nabla\zeta_k|^2 + \frac{1}{\varepsilon_k}\Phi(\zeta_k)\right) \mathrm{d}\xi \geq \left(\bigvee_{\alpha=1}^{m} |\nabla(\varphi_\alpha \circ \zeta)|\right)(\Omega^y). \tag{6.10}$$

In particular, $\varphi_\alpha \circ \zeta \in BV(\Omega^y)$ for any $\alpha \in \{1,\dots,m\}$. Since $\Phi(\zeta) = 0$ a.e. in Ω^y, by invoking [3, Proposition 2.2] we get (6.3) and

$$\left(\bigvee_{\alpha=1}^{m} |\nabla(\varphi_\alpha \circ \zeta)|\right)(\Omega^y) = \frac{1}{2}\sum_{\alpha,\beta=1}^{m} d_{\alpha,\beta}\mathcal{H}^{n-1}(E^y_{\alpha,\beta}).$$

Together with (6.10), this proves (6.4). □

7. Proof of the main results

We are now in the position of providing a proof of our main results, Theorems 2.1, 2.2, and 2.3.

Proof of Theorem 2.1. Let $(y_k,\zeta_k)_k \subset \widetilde{\mathbb{Q}}^R_{(y_0,\Gamma_0)}$ be a minimizing sequence for functional $\mathcal{F}_\varepsilon$, which is bounded from below due to (2.4). The coercivity of the potential W from (2.4) and the generalized Friedrichs inequality imply that one can extract a not relabeled subsequence such that $y_k \to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$. The boundary condition is preserved in the limit. We conclude by Lemma 3.2 that $y \in \mathbb{Y}$ and $y = y_0$ on Γ_0, recalling that the assumption on y_0 (not constant on Γ_0) prevents y from being a constant map.

Denote by η_k and H_k the zero extensions on $\mathbb{R}^n$ of ζ_k and $\nabla\zeta_k$ respectively. The coercivity of $\mathcal{F}^{\rm int}_\varepsilon$ and the uniform bound $|\zeta_k| \le R$ imply that one can extract not relabeled subsequences such that $\eta_k \to \eta$ weakly* in $L^\infty(\mathbb{R}^n;\mathbb{R}^h)$ and $H_k \to H$ weakly in $L^2(\mathbb{R}^n;\mathbb{R}^{h\times h})$. Set now $\zeta := \eta|_{\Omega^y}$. For every $\delta > 0$, let $O_\delta := \{\xi \in \Omega^y |\, \operatorname{dist}(\xi,\partial\Omega^y) > \delta\} \subset\subset \Omega^y$. We have that $\Omega^y = \cup_\delta O_\delta$ and, by Lemma 5.1, $O_\delta \subset \Omega^{y_k}$ for k large. For every $\xi_0 \in O_\delta$ and $B(\xi_0,r) \subset O_\delta$ we have that $\eta_k \to \eta$ weakly in $W^{1,2}(B(\xi_0,r);\mathbb{R}^h)$. This implies that $H = \nabla\eta = \nabla\zeta$ almost everywhere in $B(\xi_0,r)$. Moreover, by possibly extracting again, one has that $\eta_k \to \eta$ strongly in $L^2(B(\xi_0,r);\mathbb{R}^h)$. As every $\xi \in \Omega^y$ belongs to some O_δ for δ small enough, we get that $H = \nabla\zeta$ almost everywhere in Ω^y. Now, by the weak lower semicontinuity of the L^2-norm

$$\liminf_{k\to\infty}\int_{\Omega^{y_k}}|\nabla\zeta_k|^2\,\mathrm{d}\xi = \liminf_{k\to\infty}\int_{\mathbb{R}^n}|H_k|^2\,\mathrm{d}\xi \ge \int_{\mathbb{R}^n}|H|^2\,\mathrm{d}\xi \ge \int_{\Omega^y}|\nabla\zeta|^2\,\mathrm{d}\xi. \tag{7.1}$$

The local strong convergence $\eta_k \to \eta$ in $L^2(B(\xi_0,r);\mathbb{R}^h)$ for any $B(\xi_0,r) \subset\subset \Omega^y$ and $|\eta - \eta_k| \le C$ imply the strong L^2-convergence on Ω^y, hence, up to extracting again, the pointwise convergence to ζ on Ω^y, and thus $|\zeta| \le R$. By the Fatou Lemma, we find

$$\liminf_{k\to\infty}\int_{\Omega^{y_k}}\Phi(\eta_k)\,\mathrm{d}\xi = \liminf_{k\to\infty}\int_{\mathbb{R}^n}\Phi(\eta_k)\,\mathrm{d}\xi \ge \liminf_{k\to\infty}\int_{\Omega^y}\Phi(\eta_k)\,\mathrm{d}\xi \ge \int_{\Omega^y}\Phi(\zeta)\,\mathrm{d}\xi.$$

Thus we have proven the weak lower semicontinuity of the interfacial energy $\mathcal{F}^{\rm int}_\varepsilon(y_k,\zeta_k)$.

As for the bulk contribution, because of the convergence $\|\zeta - \zeta_k\|_{L^1(\Omega^{y_k}\cap\Omega^y)} \to 0$, we can apply Lemma 6.2 and obtain the lower semicontinuity of $\mathcal{F}^{\rm bulk}(y,\zeta)$. Together with (7.1), this proves that (y,ζ) is a minimizer of $\mathcal{F}_\varepsilon$ on $\widetilde{\mathbb{Q}}^R_{(y_0,\Gamma_0)}$ by means of the direct method [11]. □

Proof of Theorem 2.2. Let $(y_k,\zeta_k) \in \mathbb{Q}_{(y_0,\Gamma_0)}$ be a minimizing sequence for $\mathcal{F}_0$. As in the proof of Theorem 2.1, we can assume, up to extraction of a not relabeled subsequence, that $y_k \to y$ weakly in $W^{1,p}$ for some $y \in \mathbb{Y}$, and the coercivity assumption (2.4) also implies that $\det\nabla y_k$ are equiintegrable functions on Ω.

Let $F_k = (F_k^1, \dots, F_k^m)$, with $F_k^\alpha = \{\zeta_k = p_\alpha\}$, $\alpha = 1, \dots, m$, be the partition of Ω^{y_k} corresponding to a phase configuration ζ_k; we can identify the sequence of states with the sequence $(y_k, F_k)_k$. Since the interface energy

$$\sum_{\alpha,\beta=1}^{m} d_{\alpha,\beta}\mathcal{H}^{n-1}((F_k)_{\alpha,\beta}),$$

where $(F_k)_{\alpha,\beta} := \partial^* F_k^\alpha \cap \partial^* F_k^\beta \cap \Omega^{y_k}$, is bounded along the sequence $(y_k, F_k)_k$, the sets F_k have uniformly bounded perimeters, namely, $\mathrm{Per}(F_k^\alpha, \Omega^{y_k}) \le c$.

For $\ell \in \mathbb{N}$, let $O^\ell := \{x \in \Omega^y | \operatorname{dist}(x, \partial\Omega^y) > 2^{-\ell}\} \subset\subset \Omega^y$. As $O^\ell \subset \Omega^{y_k}$ for k large enough due to Lemma 5.1, for any given $\ell \in \mathbb{N}$ we have that $\limsup_k \mathrm{Per}(F_k^\alpha, O^\ell) \le c$ for any α. We can hence find a measurable set $(G^\alpha)^\ell \subset O^\ell$ and a not relabeled subsequence F_h such that

$$|(F_h^\alpha \Delta (G^\alpha)^\ell) \cap O^\ell| \to 0 \quad \text{for} \quad h \to \infty.$$

For all $\ell' > \ell$ we can further extract a subsequence $F_{h'}$ from F_h above in such a way that $|(F_{h'}^\alpha \Delta (G^\alpha)^{\ell'}) \cap O^{\ell'}| \to 0$ and $(G^\alpha)^{\ell'} \cap O^\ell = (G^\alpha)^\ell$. From the nested family of subsequences corresponding to $\ell = 1, 2, \dots$ we extract by a diagonal argument a further subsequence $F_{k'}$. By setting $F^\alpha := \cup_\ell (G^\alpha)^\ell$ and, owing to $O^\ell \nearrow \Omega^y$, we get that

$$|(F_{k'}^\alpha \Delta F^\alpha) \cap \Omega^y| \to 0.$$

Now, the sets F^α has finite perimeter in Ω^y as a consequence of Proposition 6.1. By letting $\zeta = \chi_F|_{\Omega^y}$ we then have that $(y, \zeta) \in \mathbb{Q}_{(y_0, \Gamma_0)}$.

One is left to check that $\mathcal{F}_0(y, \zeta) \le \liminf \mathcal{F}_0(y_k, \zeta_k)$, which follows from the lower semicontinuity of $\mathcal{F}_0$. Indeed, the lower semicontinuity of bulk part of $\mathcal{F}_0$ follows by the argument of Lemma 6.2. As concerns the interface term, the lower semicontinuity with respect to local convergence in measure is proven in [1, Example 2.5]. ☐

In Proposition 6.4 a lim inf inequality for the interface functional has been established. Combined with the lower semicontinuity of the bulk energy (Lemma 6.2), we conclude that the whole energy functional satisfies a $\Gamma - \liminf$ inequality w.r.t. the convergence notion of Lemma 6.2. Under the full Dirichlet conditions on the boundary of the domain, we shall prove Theorem 2.3 by using a Modica-Mortola [30] recovery sequence deeply generalized by Baldo in [3]. The Γ-convergence allows to prove the convergence of the phase field solutions to the sharp interface solution.

Proof of Theorem 2.3. We first claim that, if $(y, \zeta) \in \mathbb{Q}_{y_0}$, $\Omega^{y_0} \subset \mathbb{R}^n$ being a Lipschitz domain, and if we let $F = (F_1, \dots, F_m)$ with $F_\alpha = \{\xi \in \Omega^{y_0} : \zeta(\xi) = p_\alpha\}$, there exists a sequence $(\zeta_k)_k \subset W^{1,2}(\Omega^y; \mathbb{R}^h)$ such that $|\zeta_k| \le R$ for suitable $R > \max_{\alpha \in \{1, \dots m\}} |p_\alpha|$ and such that

$$\lim_{k\to\infty} \|\zeta_k - \zeta\|_{L^1(\Omega^y)} = 0 \quad \text{and} \quad \frac{1}{2}\sum_{\alpha,\beta=1}^{m} d_{\alpha,\beta}\mathcal{H}^{n-1}(F_{\alpha,\beta}) + \mathcal{F}^{\,\mathrm{bulk}}(y, \zeta) = \lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}(y, \zeta_k).$$

Indeed, since the y-component is a constant sequence, the claim completely rests on the construction of the recovery sequence $(\zeta_k)_k$ provided by Baldo [3] (such a sequence is also satisfying $\int_{\Omega^y} \zeta_k(\xi)\,d\xi = \int_{\Omega^y} \zeta(\xi)\,d\xi$ for any $k \in \mathbb{N}$, thus justifying our observations in Remark 2.5). In order to use this result, we need to assume the Lipschitz regularity of the deformed domain through the imposition of Dirichlet boundary conditions on the whole boundary of Ω. Moreover, by inspecting the construction of the recovery sequence from [3, Section 3], we see that we can obtain a sequence $(\zeta_k)_k$ that is uniformly bounded, i.e., such that $|\zeta_k| \le R_0$ for large enough R_0 (only depending on the multiwell potential Φ). Then, since $\zeta_k \circ y \to \zeta \circ y$ in $L^1(\Omega;\mathbb{R}^h)$ follows by Lemma 5.3, the convergence of the bulk part is obtained by dominated convergence by means of assumptions (2.6) and (2.7). The claim is proved.

The rest of the proof follows the one in [18]. Here we give a summary of it. Let $k \in \mathbb{N}$, let (y_k, ζ_k) be a minimizer (provided by Theorem 2.1) for $\mathcal{F}_{\varepsilon_k}$ over $\widetilde{\mathbb{Q}}^R_{y_0}$, $R > R_0$, and let $(y^*, \zeta^*) \in \mathbb{Q}_{y_0}$ be a state of finite energy for $\mathcal{F}_0$ whose recovery sequence is $(y^*, \zeta^*_k) \subset \widetilde{\mathbb{Q}}^R_{y_0}$. Using $\mathcal{F}_{\varepsilon_k}(y_k, \zeta_k) \le \mathcal{F}_{\varepsilon_k}(y^*, \zeta^*_k)$ and the fact that $\mathcal{F}_{\varepsilon_k}(y^*, \zeta^*_k) \to \mathcal{F}_0(y^*, \zeta^*)$ as $k \to \infty$, we conclude that $\mathcal{F}_{\varepsilon_k}(y_k, \zeta_k) \le C$. The coercivity (2.4) along with Friedrichs inequality ensures that $y_k \to y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$ for some not relabeled subsequence. Moreover, $y \in \mathbb{Y}$ and $y = y_0$ on $\partial\Omega$ (hence, $\Omega^{y_k} = \Omega^y = \Omega^{y_0}$). The uniform bound on $\mathcal{F}^{\rm int}_{\varepsilon_k}(y_k, \zeta_k) = \mathcal{F}^{\rm int}_{\varepsilon_k}(y, \zeta_k)$ also yields strong $L^1(\Omega^y;\mathbb{R}^h)$ compactness for the sequence ζ_k. This implies the existence of $\zeta \in L^\infty(\Omega^y;\mathbb{R}^h)$, such that $|\zeta| \le R$ and $\|\zeta_k - \zeta\|_{L^1(\Omega^y)} \to 0$ for some not relabeled subsequence. By Proposition 6.4, ζ is takes values in P and

$$\mathcal{F}^{\rm int}_0(y, \zeta) \le \liminf_{k\to\infty} \mathcal{F}^{\rm int}_{\varepsilon_k}(y_k, \zeta_k).$$

Now, we show that (y, ζ) is a minimizer $\mathcal{F}_0$ on $\mathbb{Q}_{y_0}$. In fact, for any $(\tilde{y}, \tilde{\zeta}) \in \mathbb{Q}_{y_0}$, let $(\tilde{y}, \tilde{\zeta}_k)$ be its recovery sequence: $\mathcal{F}_{\varepsilon_k}(\tilde{y}, \tilde{\zeta}_k) \to \mathcal{F}_0(\tilde{y}, \tilde{\zeta})$ as $k \to \infty$. By the lower semicontinuity of the bulk term $\mathcal{F}^{\rm bulk}$,

$$\mathcal{F}_0(y, \zeta) \le \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(y_k, \zeta_k) \le \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(\tilde{y}, \tilde{\zeta}_k) = \mathcal{F}_0(\tilde{y}, \tilde{\zeta}),$$

which proves the assertion. □

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