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Vertical Differentation beyond the Uniform Distribution

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ABSTRACT. The assessment of the way distributive shocks, such as increased polarization or higher inequality, affect vertically differentiated markets has been severely hampered by the standard reference to uniform distributions. In this paper we offer the first proof of existence of a subgame perfect Nash equilibrium in a vertically differentiated duopoly with uncovered market, for a large set of symmetric and asymmetric distributions of consumers, including, among others, all logconcave distributions. The proof relies on the 'income share elasticity' representation of the consumers' density function. Some illustrative examples are also provided to assess the impact of distributive shocks on market equilibrium.

KEYWORDS. Vertical differentiation, duopoly, non-uniform distribution, subgame perfect equilibrium, income share elasticity.

JEL CODES: L13, L11, D43, C72

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1 Introduction

There are two main reasons for products to be differentiated. The first is that consumers love variety, so that product differentiation is the market response to this property of individual preferences. The second is that consumers are different from each other, and product differentiation is the market response to this diversity. In these two explanations – often referred to as the non-address and the address approach to product differentiation – agents' heterogeneity plays quite a different role. Indeed, in the modern literature the key reference for the former is the Dixit and Stiglitz (1977) model of monopolistic competition, which is based on a representative agent assumption – though recent extensions of their basic framework allowed to capture the effects of consumers' heterogeneity on several aspects of market competitiveness (see, among others, Benassi *et al,* 2005; Osharin *et al*, 2014; Bertoletti and Etro, 2017). By contrast, within the approach based on consumers' diversity, the representation of agents' heterogeneity, along with the properties of preferences, is the key building block of any model. Actually, it is the shape of the distribution of consumers which defines, in a framework of discrete choice, the market environment faced by firms – the distribution of consumers in the space of the product characteristics within horizontal differentiation, the distribution of the willingness to pay for quality or the distribution of income under vertical differentiation.[1] As a consequence, any sound modeling of the market implications of consumers' diversity should be robust to a wide range of assumptions about the func-

[1] The key insight along these lines was presented in the seminal papers by Gabszewicz and Thisse (1979, 1980).

tional form of the consumers' distribution – a necessary condition to ensure realism and predictive power.

This objective has already been attained in the case of horizontal differentiation. Indeed, the well known paper by Anderson *et al* (1997) generalizes the linear city duopoly model by identifying a set of conditions that a general spatial distribution of consumers' must satisfy to ensure that a subgame perfect equilibrium in prices and locations exist and be unique. For vertical differentiation, however, the state of the art is far less satisfactory – which is in a way surprising, given the economic and social relevance of the involved heterogeneity. As shown in the same paper by Anderson *et al*,[2] there is indeed an isomorphism between the linear city and the vertical differentiation models, which allows for an extension of their generalization to the latter framework; but this occurs under appropriate conditions, and only under the assumption of full market coverage, i.e. when consumers split between firms producing goods of different quality levels, but are all 'rich enough' to buy at least the lowest quality good at the equilibrium prices.[3] However, the isomorphism does not apply to the uncovered market case, where some consumers may be 'too poor' to buy at the equilibrium prices, so that market segmentation includes market exclusion.

For vertically differentiated markets exhibiting this essential feature, the attempts at going beyond the uniform distribution have offered only partial or preliminary results. By using the Mussa and Rosen (1978) specification of the indirect utility function, the existence of a duopolistic equilibrium in

[2] See also Cremer and Thisse (1991).

[3] In an oligopolistic setting the existence of an equilibrium in prices for given qualities has been proved by Bonnisseau and Lahmandi-Ayed (2007) when consumers' distribution is concave.

prices has been proved by Furth (2011) under logconcave densities. The full solution has been developed only for two specific densities: Yurko (2011) develops an algorithm to solve numerically a oligopoly model of the Shaked-Sutton type (Shaked and Sutton, 1982) under a lognormal distribution of consumers, while Benassi *et al* (2006) offer an analytical solution of the price-quality game under a trapezoid distribution. Clearly, for models of vertical differentiation with uncovered market there is still an open problem in terms of robustness, manageability and empirical relevance.

This paper aims at offering an effective contribution along these three lines. On the one hand, it generalizes the solution of the duopoly model with vertical differentiation and uncovered market for a wide set of distributions of the consumers' willingness to pay for quality. It proves the existence of the subgame perfect price-quality equilibrium for a range of symmetric and asymmetric distributions, which includes – but is not limited to – all logconcave distributions. On the other hand, our proof makes use of simple but powerful analytical tools which allow us to deal with the complexity of the analytics, and to develop a manageable solution procedure which can be applied to the whole set of the admissible distributions. Finally, the possibility to obtain explicit solutions in terms of the parameters of the distribution makes it easier to analyze how distributive shocks affect the market prices, the quality spread, as well as the profitability and the market shares of the two firms.

We are able to prove the existence of equilibrium at the aforementioned degree of generality by uncovering a key feature of the model, namely that the optimality conditions at the two stages of the game can be seen as a block-recursive system of equations. Block-recursiveness ensures that the equilibrium market shares of the two firms are determined, under appropriate constraints, by the optimal behavior of the low quality firm at the two stages, while the optimal behavior of the high quality firm determines, given those shares, the equilibrium levels of qualities. The block-recursiveness property shows up once the density function describing the consumers' distribution is represented through the so-called income share elasticity (Esteban, 1986), an analytical tool which has already proved to be extremely useful in the analysis of the relation between consumers' heterogeneity and market structure, whenever market demand is intrinsically shaped by the distribution itself (e.g., Benassi *et al*, 2002).

The usefulness of Esteban's formulation goes beyond the existence proof. By clearly bringing out the above block-recursiveness property, it has the additional advantage of simplifying considerably the actual computation of the equilibrium solutions. Indeed, we provide some applications of our solution procedure, by analyzing the effects of some distributive shocks in two illustrative cases with a symmetric and an asymmetric consumers' distribution. The regularities which emerge from our examples suggest that for both symmetric and asymmetric distributions, a reduction in consumers' heterogeneity is associated with an increase in the quality spread, in the overall market coverage, and in the market share and profitability of the high quality firm. Interestingly enough, a side effect of higher concentration is that the threshold level of the willingness to pay for quality required to enter the market increases, while that required to afford the high quality may either increase or decrease.

In the sequel of the paper we proceed as follows. The basic framework is set in Section 2. In Section 3 we prove the existence and uniqueness of the equilibrium in prices. Section 4 is devoted to proving the existence of the subgame perfect equilibrium in prices and qualities. The effects of distributive shocks on the equilibrium solution are discussed through some examples in Section 5, while in Section 6 we gather some conclusions.

2 The basic framework

We consider a duopolistic market for a vertically differentiated product in which two firms play a simultaneous two-stage game with respect to prices and qualities. At the first stage firms set the quality of their product, $s \in (0, s_{\max}]$. We denote with H the firm choosing the higher quality s_H, and with L the firm choosing the lower quality s_L. Once the qualities have been set, firms produce at a cost independent of s and normalized to zero. At the second stage firms compete with respect to prices, respectively p_H and p_L. The game is solved by backward induction.

2.1 Preferences and demand

In this market demand stems from a continuum of heterogeneous consumers, whose size is normalized to 1. We assume that the generic consumer's utility function is of the Mussa and Rosen (1978) type:

$$
\begin{aligned}
U &= \theta s - p, && \text{if she purchases one unit of a good of quality } s \\
U &= 0, && \text{if she does not purchase.}
\end{aligned}
$$

Consumers differ across θ, the preference parameter which denotes the willingness to pay for quality. According to a standard view (Tirole 1988, pp. 96-97), θ can also be seen as the marginal rate of substitution between income and quality, with richer consumers exhibiting higher values of θ. In this sense, the heterogeneity of the consumers' preference for quality can be interpreted as reflecting income heterogeneity. In the sequel, we assume that θ is distributed according to a continuously differentiable density function $f(\theta)$ defined over the support $[0, \theta_{\max}]$, and we denote with $F(\theta) = \int_0^\theta f(\tau)\, d\tau$ the corresponding cumulative distribution function.

Given the above utility function, a consumer enters the market and buys one unit of the good only if characterized by a θ greater than the hedonic price (p/s) set by at least one of the two firms. In particular, the demand faced by the two firms can be written as follows.

a) If $0 < p_L/s_L < p_H/s_H$, the market is uncovered and both firms are active in the market. We can therefore define $\theta_L \equiv p_L/s_L$ as the marginal valuation of quality of the consumer which is indifferent between buying from L and not buying, and $\theta_H \equiv (p_H - p_L) / (s_H - s_L)$ as the marginal valuation of quality of the consumer which is indifferent between buying from L and buying from H. Therefore, the demand functions faced by firm L and H are respectively:

$$D_L = \int_{\theta_L}^{\theta_H} f(\theta)\, d\theta = F(\theta_H) - F(\theta_L), \qquad D_H = \int_{\theta_H}^{\theta_{\max}} f(\theta)\, d\theta = 1 - F(\theta_H).$$

b) If $0 = p_L/s_L < p_H/s_H$ the market is covered, and the demand functions are given by

$$D_L = \int_0^{\frac{p_H}{s_H - s_L}} f(\theta)\, d\theta = F\left(\frac{p_H}{s_H - s_L}\right), \quad D_H = \int_{\frac{p_H}{s_H - s_L}}^{\theta_{\max}} f(\theta)\, d\theta = 1 - F\left(\frac{p_H}{s_H - s_L}\right).$$

c) If $p_L/s_L \geq p_H/s_H > 0$, the market is uncovered and only firm H is active, so that:

$$D_L = 0, \qquad D_H = \int_{p_H/s_H}^{\theta_{\max}} f(\theta)\, d\theta = 1 - F\left(\frac{p_H}{s_H}\right).$$

d) If $p_L/s_L \geq p_H/s_H = 0$, then the market is covered with $D_H = 1$ and $D_L = 0$.

It is straightforward to verify that no equilibrium can fall into cases b), c), and d). In cases b) and d) firm L and firm H respectively earn zero profits and obviously perceive an incentive to set a positive price consistent with a positive market share; in case c) firm L earns zero profits and perceives an incentive to decrease its price up to a level which ensures a positive market share and positive profits. Therefore in the sequel we shall investigate the existence of equilibrium under case a).[4]

2.2 The distribution of θ

In the above setting, the distribution of θ is the key element shaping market demand and therefore the properties of the firms' market interactions. In order to preserve a high degree of generality, we do not set any definite functional form for this distribution, but we simply assume that $f(\theta)$ satisfies the following general properties:

[4] Notice that the possibility of ruling out full market coverage, corner solutions or only one firm being active in the market depends on our assumption on the support of the distribution of θ, namely that the minimum willingness to pay is 0. We are restating in our general framework a well-known property highlighted in the uniform case by, e.g., Wauthy (1996) and Motta (1993).

Property 1. $f(\theta) > 0$ for all $\theta \in (0, \theta_{\max})$.

Property 2. $\eta(\theta) \equiv \frac{\theta f(\theta)}{1-F(\theta)}$ is increasing in θ for all $\theta \in (0, \theta_{\max})$, with $\lim_{\theta \to \theta_{\max}} \eta(\theta) > 1$.

Property 3. $\frac{1}{1-F(\theta)}$ is convex in θ for all $\theta \in (0, \theta_{\max})$.

Notice that $1 - F(\theta)$ can be seen as the share of active (buying) consumers (i.e. the degree of market coverage or overall market demand) if the hedonic price $p/s = \theta$ is the lowest hedonic price available in the market. Properties 2 and 3 can therefore be interpreted as reasonable restrictions on the shape of this peculiar demand curve. Since $\eta(\theta)$ is the absolute value of the elasticity of $1 - F(\theta)$, Property 2 can be seen as ruling out that the elasticity of market demand might be constant or decreasing in the hedonic price. This condition is indeed satisfied for most of the commonly used distributions (e.g. normal, lognormal, Beta, etc.), with the notable exception of the Pareto distribution, which generates a function $1 - F(\theta)$ with constant elasticity. Property 3 imposes that the $1 - F(\theta)$ function be not too convex, which is generally (and also here) required for the profit functions to be concave. Notice that Properties 2 and 3 are satisfied by all f distributions (including the logconcave) which generate logconcave $1 - F$ functions; the latter, however, are only a subset of the admissible distributions (see An, 1998).

The following results will be useful in the sequel.

Remark 1. For a given $f(\theta)$, consider the following function, known as income share elasticity (Esteban, 1986):

$$\pi(\theta) = 1 + \frac{\theta f'(\theta)}{f(\theta)}.$$

By means of the $\pi(\theta)$ function, it is possible to write Property 2 as:

$$\eta(\theta) + \pi(\theta) > 0 \text{ for all } \theta \in (0, \theta_{\max}).$$

Moreover, it can be checked that Property 3 boils down to:

$$2\eta(\theta) + \pi(\theta) > 1 \text{ for all } \theta \in (0, \theta_{\max}).$$

Notice that logconcavity of the $1-F(\theta)$ function would imply $\eta(\theta)+\pi(\theta) > 1$, which is more restrictive than both Property 2 and Property 3.

Should θ be strictly interpreted as income, the function π would measure the relative marginal change in the share of income accruing to class θ, brought about by a marginal increase in θ; in general, Esteban (1986) shows that a one-to-one relationship exists between the π function and the underlying density f.[5] Representing Properties 2 and 3 in terms of η and π is relevant in our framework, since this elasticity formulation remarkably contributes to the analytical tractability of the two-stage game.

3 The Price Stage

In this Section we analyse the equilibrium at the price stage of the game, when the demand faced by the two firms is shaped by a distribution of θ satisfying the properties discussed above.

[5] It should be noticed that the π-formulation of the density often allows simpler representations of the relevant features of the distribution. For example, $\pi(\theta) = 1$ identifies the modal value of θ, while "the Pareto, Gamma and Normal density functions correspond to constant, linear and quadratic elasticities, respectively" (Esteban, 1986, p.442).

At the price stage of the game firms compete with respect to prices, for given qualities s_H and s_L. We have already shown that if an equibrium exists, it occurs at an uncovered market configuration with two active firms, i.e. with $p_H/s_H > p_L/s_L$. Therefore the profit functions are:

$$\begin{aligned} \Pi_H (p_H, p_L; s_H, s_L) &= p_H (1 - F(\theta_H)); \\ \Pi_L (p_H, p_L; s_H, s_L) &= p_L (F(\theta_H) - F(\theta_L)). \end{aligned}$$

The First Order Conditions (FOCs) for firms H and L are respectively:

$$\frac{\partial \Pi_H}{\partial p_H} = 1 - F(\theta_H) - \frac{p_H}{\Delta} f(\theta_H) = 0; \tag{1}$$

$$\frac{\partial \Pi_L}{\partial p_L} = F(\theta_H) - F(\theta_L) - p_L \left(\frac{f(\theta_H)}{\Delta} + \frac{f(\theta_L)}{s_L} \right) = 0, \tag{2}$$

where $\Delta \equiv s_H - s_L$. Consider first the FOC of firm H**.** By defining $\varphi_H \equiv (\partial \theta_H / \partial p_H)(p_H/\theta_H) = p_H/(p_H - p_L) > 1$, equation (1) can be rewritten as:

$$1 - F(\theta_H) - \varphi_H \theta_H f(\theta_H) = 0$$

and, using the definition of $\eta(\theta)$, as

$$\eta(\theta_H) \varphi_H = 1. \tag{3}$$

Equation (3) states the standard condition that with zero costs, the FOC of firm H requires that (in absolute value) the elasticity of its demand with respect to its price – given by the product of the elasticity of its demand with respect to θ_H and the elasticity of θ_H with respect to p_H – be equal to 1. The key implication of equation (3) is that whenever it holds, firm H sets a price such that θ_H takes a value at which $\eta(\theta_H) < 1$.

Consider now the FOC of firm L. By defining $\varphi_L \equiv (\partial\theta_H/\partial p_L)(p_L/\theta_H) = -p_L/(p_H - p_L) = 1 - \varphi_H < 0$, and using again the definition of $\eta(\theta)$, equation (2) can be formulated as:

$$\eta(\theta_L)\frac{1 - F(\theta_L)}{F(\theta_H) - F(\theta_L)} - \eta(\theta_H)\varphi_L\frac{1 - F(\theta_H)}{F(\theta_H) - F(\theta_L)} = 1. \tag{4}$$

Also for firm L the FOC implies that the absolute value of its demand elasticity with respect to p_L be equal to 1.[6] In the sequel it will be useful to refer to the following reformulation of equation (4):

$$(1 - \varphi_L\eta(\theta_H))(1 - F(\theta_H)) - (1 - F(\theta_L))(1 - \eta(\theta_L)) = 0 \tag{5}$$

Given the FOCs of the two firms in equations (3) and (5), we turn now to equilibrium at the price stage.

3.1 Equilibrium at the price stage: existence and uniqueness

Commonly, the equilibrium at the price stage is explicitly defined in terms of a price pair. However, we recall that for given qualities, any pair (θ_L, θ_H) univocally delivers a price pair (p_L, p_H). In our setting – where the distribution of θ and its properties are the crucial element – it turns out to be particularly useful to exploit this relation, and reformulate equilibrium precisely in terms of the indifferent consumers θ_L and θ_H. Indeed, for given qualities (s_L, s_H), an equilibrium at the price stage can be defined as a pair (θ_L^*, θ_H^*) at which (a) equations (3) and (5) are satisfied, (b) the Second

[6] It is easy to check that equation (4) actually states this condition, once the demand for L is written as $D_L = (1 - F(\theta_L)) - (1 - F(\theta_H))$. Notice that $\eta(\theta_L)$ is the p_L-elasticity of overall market demand – the elasticity of demand with respect to θ evaluated at θ_L multiplied by the (unit) elasticity of θ_L with respect to p_L – while $\eta(\theta_H)\varphi_L$ is the p_L-elasticity of the demand accruing to H – the elasticity of $1 - F(\theta)$ evaluated at θ_H, multiplied by the elasticity of θ_H with respect to p_L.

Order Conditions (SOCs) for profit maximization are verified, and (c) firm H perceives no incentive to undercut its rival in order to become the only firm active in the market. Clearly, the equibrium prices p_L^* and p_H^* can then be recovered from the definitions of θ_L and θ_H.

In order to prove that for all distributions exhibiting Properties 1-3 a unique equilibrium exists, we start by introducing some preliminary results. First of all we notice that, for given qualities, the system of equations (3) and (5) can actually be reformulated in terms of θ_L and θ_H only. Consider first equation (3). By using the definitions of φ_H, θ_H and θ_L, we can express φ_H in terms of θ_L and θ_H as:

$$\varphi_H = 1 + \frac{s_L}{\Delta}\frac{\theta_L}{\theta_H}.$$

By substituting the above expression into (3), and solving for θ_L, we obtain that for given qualities:

$$\theta_L = \frac{\Delta}{s_L}\left(\frac{1}{\eta(\theta_H)} - 1\right)\theta_H \tag{3'}$$

Consider now equation (5). By using $\varphi_L = 1 - \varphi_H$ and by substituting for φ_H from (3), equation (5) can be rewritten as:

$$(2 - \eta(\theta_H))(1 - F(\theta_H)) - (1 - F(\theta_L))(1 - \eta(\theta_L)) = 0. \tag{5'}$$

Notice that while equation (3') is a simple restatement of the FOC of firm H, equation (5') restates the FOC of firm L, making use of the condition stated in equation (3) that the elasticity of the demand of H with respect to p_H must be equal to 1. In any case, however, the system of equations (3) and (5) can be replaced with the system of equations (3') and (5').

For the ease of notation, we denote the relation in (3') as $\theta_L = h(\theta_H)$. In the sequel we shall make use of the following properties of the $h(\theta_H)$ function.

Lemma 1. The function $\theta_L = h(\theta_H)$ is decreasing in θ_H. Moreover, along $h(\bullet)$ the requisite that both firms are active, i.e. $\theta_L < \theta_H$, is satisfied for $\theta_H > \underline{\theta}_H$ where $\underline{\theta}_H$ solves $\eta(\underline{\theta}_H) = \Delta/s_H$, while the requisite that the market is uncovered, i.e. $\theta_L > 0$, is satisfied for $\theta_H < \overline{\theta}_H$, where $\overline{\theta}_H$ solves $\eta(\overline{\theta}_H) = 1$, with $0 < \underline{\theta}_H < \overline{\theta}_H < \theta_{\max}$.

Proof. By differentiating (3') we get:

$$h'(\theta_H) = \frac{\Delta}{s_L}\frac{1}{\eta(\theta_H)}(1 - \pi(\theta_H) - 2\eta(\theta_H))$$

which implies that

$$\text{sign}\{h'(\theta_H)\} = \text{sign}\{1 - \pi(\theta_H) - 2\eta(\theta_H)\}.$$

Therefore, along (3') $\theta_L = h(\theta_H)$ is decreasing if Property 3 holds.

The relation described by (3') is consistent with two firms being active and market being uncovered if the following two requisites are satisfied: (i) $\theta_L = h(\theta_H) < \theta_H$, and (ii) $\theta_L = h(\theta_H) > 0$. Consider first requisite (i):

$$\frac{\Delta}{s_L}\left(\frac{1}{\eta(\theta_H)} - 1\right)\theta_H < \theta_H,$$

i.e.,

$$\eta(\theta_H) > \frac{\Delta}{s_H}.$$

Since the $\eta(\theta)$ function is increasing according to Property 2, along the domain of (3') this requisite is satisfied for all $\theta_H > \underline{\theta}_H$, where $\underline{\theta}_H$ solves the boundary condition $\eta(\theta_H) = \Delta/s_H$. Requisite (ii) amounts to:

$$\frac{\Delta}{s_L}\left(\frac{1}{\eta(\theta_H)} - 1\right)\theta_H > 0$$

i.e.

$$\eta(\theta_H) < 1.$$

Again, Property 2 of the $\eta(\theta)$ function ensures that requisite (ii) is satisfied for all $\theta_H < \overline{\theta}_H$, where $\overline{\theta}_H$ solves the boundary condition $\eta(\theta_H) = 1$. Since $\eta(0) = 0$, $\Delta/s_H < 1$, and $\lim_{\theta\to\theta_{\max}} \eta(\theta) > 1$, the $\eta(\theta)$ function being increasing ensures that the following inequalities hold: $0 < \underline{\theta}_H < \overline{\theta}_H < \theta_{\max}$.■

Therefore, with given qualities, we shall consider equation (3') over the domain $\left(\underline{\theta}_H, \overline{\theta}_H\right)$.

We can now prove the following Proposition.

Proposition 1. For all consumers' distributions satisfying Properties 1-3, there exists a unique equilibrium at the price stage for any given pair (s_L, s_H).

Proof. The proof is developed in three steps. At the first step we prove that for all distributions of θ exhibiting Properties 1-3, there exists a candidate equilibrium pair (θ_L^*, θ_H^*) which satisfies the FOCs and SOCs of the price stage; at the second step we prove that this candidate equilibrium is unique; at the third step we verify that the candidate equilibrium with two active firms is deviation proof.

Step I. At the candidate equilibrium both (3') and (5') must be satisfied. By replacing θ_L in (5') with the $h(\theta_H)$ function given by (3'), a candidate equilibrium exists if there exists a value $\theta_H^* \in \left(\underline{\theta}_H, \overline{\theta}_H\right)$ at which:

$$\left(2-\eta\left(\theta_H^*\right)\right)\left(1-F\left(\theta_H^*\right)\right)-\left(1-F\left(h\left(\theta_H^*\right)\right)\right)\left(1-\eta\left(h\left(\theta_H^*\right)\right)\right)=0$$

and the SOCs are satisfied at θ_H^* and $\theta_L^* = h\left(\theta_H^*\right)$. Define now the continuous function:

$$\lambda\left(\theta_H\right) \equiv \left(2-\eta\left(\theta_H\right)\right)\left(1-F\left(\theta_H\right)\right)-\left(1-F\left(h\left(\theta_H\right)\right)\right)\left(1-\eta\left(h\left(\theta_H\right)\right)\right) \quad (6)$$

where $\theta_H \in \left(\underline{\theta}_H, \overline{\theta}_H\right)$. Since

$$\lim_{\theta_H \to \underline{\theta}_H} \lambda\left(\theta_H\right) = 1 - F\left(\underline{\theta}_H\right) > 0,$$

and

$$\lim_{\theta_H \to \overline{\theta}_H} \lambda\left(\theta_H\right) = -F\left(\overline{\theta}_H\right) < 0,$$

continuity of $\lambda\left(\theta_H\right)$ implies there exists a value $\theta_H^* \in \left(\underline{\theta}_H, \overline{\theta}_H\right)$ such that $\lambda\left(\theta_H^*\right) = 0$. Given θ_H^* and $\theta_L^* = h\left(\theta_H^*\right)$, and given s_L and s_H, the corresponding prices p_L^* and p_H^* can be obtained. The latter are candidate equilibrium solutions of the price stage, if the SOCs of firms H and L are satisfied at θ_H^* and θ_L^*. This is indeed the case, as verified in Appendix A.

Step II. We now prove that the candidate equilibrium is unique. This is indeed the case if the condition $\lambda\left(\theta_H^*\right) = 0$ implies that the first order derivative $\lambda'\left(\cdot\right)$ is negative at θ_H^*. By evaluating the derivative of (6) at θ_H^* we obtain

$$\begin{aligned}
\lambda'\left(\theta_H^*\right) &= -\left(1-F\left(\theta_H^*\right)\right)\frac{\eta\left(\theta_H^*\right)}{\theta_H^*}\left(\pi\left(\theta_H^*\right)+2\right)+ \\
&\quad +h'\left(\theta_H^*\right)\left(1-F\left(\theta_L^*\right)\right)\frac{\eta\left(\theta_L^*\right)}{\theta_L^*}\left(\pi\left(\theta_L^*\right)+1\right).
\end{aligned}$$

If Properties 1-3 hold, Lemma 1 ensures that $h'(\theta_H^*) < 0$. Moreover, according to Property 2, $\pi(\theta) > -\eta(\theta)$. Since at equilibrium $\eta(\theta_L^*) < \eta(\theta_H^*) < 1$, it follows that $\lambda'(\theta_H^*) < 0$.

Step III. The prices at the candidate equilibrium are the optimal response of each firm to the rival's price over a restricted domain: in particular, we have restricted firm H choice to the set of prices such that $\theta_H > \theta_L$, so that both firms are active. For this candidate equilibrium to be a Nash equilibrium over the entire strategy space, we have to verify that firm H has no incentive to deviate from it, by setting such a low price – namely, a price $p_H \leq p_L^*(s_H/s_L)$ – at which firm L is driven out of the market, and H remains the only active firm. Indeed, given p_L^*, firm H earns the duopoly profits

$$\Pi_H^D = p_H\left(1 - F\left(\frac{p_H - p_L^*}{s_H - s_L}\right)\right)$$

for $p_H > p_L^*(s_H/s_L)$, while it earns the monopoly profits

$$\Pi_H^M = p_H\left(1 - F\left(\frac{p_H}{s_H}\right)\right)$$

for $p_H < p_L^*(s_H/s_L)$, with $\Pi_H^D = \Pi_H^M$ when $p_H = p_L^*(s_H/s_L)$.

Notice that the unique maximum of the Π_H^M function occurs at a price p_H^M such that $\eta\left(p_H^M/s_H\right) = 1$, while we have already proved that the unique maximum of the Π_H^D function occurs at a price p_H^* such that $\eta\left(\left(p_H^* - p_L^*\right)/\left(s_H - s_L\right)\right) < 1$. Property 2 thus ensures that $p_H^M/s_H > \left(p_H^* - p_L^*\right)/\left(s_H - s_L\right) > p_L^*/s_L$. Since both Π_H^M and Π_H^D have a unique maximum at a price greater than the threshold price $p_L^*(s_H/s_L)$ at which they intersect, they actually cross along their increasing branch; therefore, given p_L^*, the unique maximum of the profit function of firm H over the whole strategy space occurs at p_H^*. This concludes the proof.■

We conclude our discussion of the price stage by highlighting in the following Corollaries some useful properties of the price stage equilibrium

Corollary 1. At the price equilibrium $\Pi_H > \Pi_L$.

Proof. Using the FOCs (3) and (5') into the definitions of Π_H and Π_L we obtain

$$\frac{\Pi_L}{\Pi_H} = (1 - \eta(\theta_H)) \left(\frac{2 - \eta(\theta_H)}{1 - \eta(\theta_L)} - 1 \right) < 1$$

since $(1 - \eta(\theta_H)) / (1 - \eta(\theta_L)) < 1$, given Property 2.■

Corollary 2 collects the results of some comparative statics exercises on the reaction of the optimal prices to changes in the quality levels.

Corollary 2. At the price stage equilibrium the price of firm H is decreasing in s_L (i.e. $dp_H^*/ds_L < 0$), while the price of firm L is increasing in s_H (i.e. $dp_L^*/ds_H = dp_L^*/d\Delta > 0$). Moreover, $dp_L^*/ds_L - \theta_L^* < 0$ and $dp_H^*/ds_H - \theta_H^* > 0$.

Proof. See Appendix B.■

4 The subgame perfect equilibrium

As discussed in Section 3, our price equilibrium is defined in terms of a pair of indifferent consumers, (θ_H^*, θ_L^*), which, given qualities, obviously delivers the firms' optimal prices. When dealing with the two-stage price-quality game, a subgame perfect equilibrium can be defined as a pair of indifferent consumers and a pair of quality levels, such that both the Nash equilibrium conditions with respect to prices for given qualities, and the Nash equilibrium conditions with respect to quality – which take into account the effect of quality on equilibrium prices – are verified. Again, prices can be recovered residually from the definitions of θ_H and θ_L.

In order to prove the existence of a subgame perfect equilibrium, we proceed as follows. By assuming that no leapfrogging occurs – i.e. that the best reply to s_L implies $s_H > s_L$, and the best reply to s_H implies $s_L < s_H$ – we derive in subsection 4.1 the firms' FOCs at the quality stage. In subsection 4.2 we first prove that a candidate equilibrium exists under the no leapfrogging assumption, and then that this candidate equilibrium is indeed the subgame perfect equilibrium over the whole strategy space.

4.1 The quality stage

If no leapfrogging occurs, at the first stage of the game the derivatives of the firms' profits with respect to quality can be written as:

$$\frac{\partial \Pi_H}{\partial s_H} = (1 - F(\theta_H))\, p_{HH} - p_H f(\theta_H) \left(\frac{\partial \theta_H}{\partial p_H} p_{HH} + \frac{\partial \theta_H}{\partial p_L} p_{LH} + \frac{\partial \theta_H}{\partial s_H} \right); \quad (7)$$

$$\begin{aligned} \frac{\partial \Pi_L}{\partial s_L} &= p_{LL} \left[F(\theta_H) - F(\theta_L) \right] + p_L f(\theta_H) \left(\frac{\partial \theta_H}{\partial p_H} p_{HL} + \frac{\partial \theta_H}{\partial p_L} p_{LL} + \frac{\partial \theta_H}{\partial s_L} \right) + \\ & \quad - p_L f(\theta_L) \left(\frac{\partial \theta_L}{\partial p_L} p_{LL} + \frac{\partial \theta_L}{\partial s_L} \right); \end{aligned} \quad (8)$$

where $p_{HH} \equiv \partial p_H^* / \partial s_H$, $p_{HL} \equiv \partial p_H^* / \partial s_L$, $p_{LH} \equiv \partial p_L^* / \partial s_H$, and $p_{LL} \equiv \partial p_L^* / \partial s_L$. All these derivatives of prices with respect to qualities can be obtained by total differentiation of the solution of the price stage, as in Corollary 2.

Consider first firm H. Collecting p_{HH} and using (1), equation (7) can be rewritten as

$$\frac{\partial \Pi_H}{\partial s_H} = \frac{p_H f(\theta_H)}{\Delta}(p_{LH} + \theta_H) = \frac{\theta_H f(\theta_H)}{\eta(\theta_H)}(p_{LH} + \theta_H).$$

The above expression is positive, since $p_{LH} > 0$ according to Corollary 2. The profits of firm H being monotonically increasing in s_H implies that at the quality stage firm H chooses:

$$s_H^{**} = s_{\max}. \tag{9}$$

Equation (9) confirms in our general setting the results obtained in the literature under explicit formulations of the distribution of θ, according to which the quality set by firm H is independent of that chosen by firm L and, in the absence of costs for quality, coincides with the highest quality $s_{\max}$.

Consider now firm L. Collecting p_{LL} and using (2), equation (8) can be rewritten as:

$$\frac{\partial \Pi_L}{\partial s_L} = \frac{p_L}{\Delta} f(\theta_H)(p_{HL} + \theta_H) + f(\theta_L)\theta_L^2,$$

so that, using the definition of η and equation (1), we obtain the following FOC for profit maximization of firm L:

$$\theta_H \left(\frac{1}{\eta(\theta_H)} - 1\right) f(\theta_H)(p_{HL} + \theta_H) + f(\theta_L)\theta_L^2 = 0.$$

Given equation (5') the above can be rewritten as

$$(1 - \eta(\theta_H))(p_{HL}(\theta_H, \theta_L) + \theta_H) + \frac{2 - \eta(\theta_H)}{1 - \eta(\theta_L)}\eta(\theta_L)\theta_L = 0. \tag{10}$$

This is in terms θ_H and θ_L only, given the expression for $p_{HL} = p_{HL}(\theta_H, \theta_L)$ obtained in equation (B2) of Appendix B.

4.2 Existence of a subgame perfect equilibrium

Once we have stated the necessary conditions for profit maximization at the quality stage, we turn now to discussing the existence of a subgame perfect equilibrium. Under the no leapfrogging assumption, a candidate subgame perfect equilibrium is a solution of the set of FOCs at the price and quality stages, at which the SOCs are satisfied. Since we have stated all these conditions in terms of indifferent consumers and quality levels only, our candidate equilibrium is a pair of indifferent consumers (θ_H^{**}, θ_L^{**}) and a pair of quality levels (s_H^{**}, s_L^{**}) such that equations (3'), (5'), (9) and (10) hold, and the SOCs for profit maximization are satisfied. Again, the prices can then be derived from the definitions of θ_H and θ_L.

An important property of the system of FOCs given by equations (3'), (5'), (9) and (10) is its recursiveness. Equation (9) defines s_H^{**}' independently of all other variables; equations (5') and (10) are a subsystem in terms of θ_H and θ_L only, while equation (3') determines s_L for given s_H, θ_H and θ_L. This recursiveness reflects an important economic property of the model: provided that the above system actually delivers an equilibrium, the market shares of the two firms are determined by the price and quality FOCs of firm L, under the constraint imposed by equation (3) that at the equilibrium value of θ_H the elasticity of the market share of firm H with respect to price must be equal to $1/\eta(\theta_H)$.

Before studying the solution of the above recursive system, we point out that equation (5'), repeated here for convenience

$$(2 - \eta(\theta_H))(1 - F(\theta_H)) = (1 - F(\theta_L))(1 - \eta(\theta_L)) \qquad (5')$$

defines an implicit relation, denoted as $\theta_L = g(\theta_H)$, the properties of which are detailed in Lemma 2.

Lemma 2. The relation $\theta_L = g(\theta_H)$ implicit in (5') has the following properties: (a) along $g(\bullet)$ the requisite that the market is uncovered, i.e. $\theta_L > 0$, is satisfied for $\theta_H > \theta_H^{low}$, where θ_H^{low} solves $\left(2 - \eta\left(\theta_H^{low}\right)\right)\left(1 - F\left(\theta_H^{low}\right)\right) = 1$; moreover, $\theta_H^{low} < \overline{\theta}_H$, where $\overline{\theta}_H$ solves $\eta\left(\overline{\theta}_H\right) = 1$ as in Lemma 1. (b) For $\theta_H \in \left(\theta_H^{low}, \overline{\theta}_H\right)$ the implicit relation $\theta_L = g(\theta_H)$ is increasing and the requisite that both firms are active, i.e. $\theta_L < \theta_H$, is verified.

Proof. Consider first point (a). If $\theta_L = 0$, so that $\eta(\theta_L) = 0$ and $F(\theta_L) = 0$, equation (5') collapses to $(2 - \eta(\theta_H))(1 - F(\theta_H)) = 1$. For θ_L to be positive the LHS of the above expression must be lower than one. Since it is decreasing in θ_H, the requisite that the market is uncovered is satisfied for $\theta_H > \theta_H^{low}$, where $\theta_H^{low} > 0$ solves $\left(2 - \eta\left(\theta_H^{low}\right)\right)\left(1 - F\left(\theta_H^{low}\right)\right) = 1$. The check that $\theta_H^{low} < \overline{\theta}_H$ is straightforward. Indeed, $\left(2 - \eta\left(\theta_H^{low}\right)\right)\left(1 - F\left(\theta_H^{low}\right)\right) = 1$ implies $\eta\left(\theta_H^{low}\right) = \left(1 - 2F\left(\theta_H^{low}\right)\right) / \left(1 - F\left(\theta_H^{low}\right)\right)$. This expression is positive by definition of η, and lower than 1. Property 2 then ensures that $\theta_H^{low} < \overline{\theta}_H$.

Consider now point (b). By implicit differentiation of (5'), the elasticity of θ_L with respect to θ_H along $g(\bullet)$ is

$$g'(\theta_H)\frac{\theta_H}{\theta_L} = \frac{(1 - F(\theta_H))}{(1 - F(\theta_L))}\frac{\eta(\theta_H)}{\eta(\theta_L)}\frac{(2 + \pi(\theta_H))}{(1 + \pi(\theta_L))}$$

Notice now that, since η is increasing, $\eta(\theta_H) < 1$ for $\theta_H \in \left(\theta_H^{low}, \overline{\theta}_H\right)$; this in turn implies, given equation (5') that over the same domain also $\eta(\theta_L) < 1$: as a result, since by Remark 1 $\pi(\theta) > -\eta(\theta)$, the above expression is positive and hence g is increasing. Finally, it is easy to check that the requisite $\theta_L < \theta_H$ is always satisfied for $\theta_H \in \left(\theta_H^{low}, \overline{\theta}_H\right)$: inspection of (5') shows that $\theta_H = \theta_L$ cannot occur along the $g(\bullet)$ function. Since $g(\bullet)$ is increasing in θ_H, with $\theta_H^{low} > 0$ and $g\left(\theta_H^{low}\right) = 0$, this implies that $\theta_L = g(\theta_H) < \theta_H$.■

Given Lemma 2, we now prove the following Proposition:

Proposition 2. For all consumers' distributions satisfying Properties 1-3, there exists a subgame perfect equilibrium of the price-quality game.

Proof. To prove the above Proposition we proceed by steps. We start by proving the existence of a candidate equilibrium. In particular, at the first step we prove that the subsystem (5') and (10) in terms of θ_H and θ_L has a solution $(\theta_H^{**}, \theta_L^{**})$; at the second step we show that, given θ_H^{**} and θ_L^{**} and $s_H^{**} = s_{\max}$, equation (3') delivers an optimal quality level $s_L^{**} \in (0, s_{\max})$; at the third step we check that the SOCs for profit maximization are satisfied at that solution. Finally, at the fourth step, we prove that the candidate equilibrium obtained under the no leapfrogging assumption is the subgame perfect equilibrium on the whole strategy space.

Step I. In order to prove that the subsystem (5') and (10) has a solution, we replace θ_L in (10) with the $g(\theta_H)$ implicit function from (5'). Then a solution exists, if there exists a value $\theta_H^{**} \in \left(\theta_H^{low}, \overline{\theta}_H\right)$ at which:

$$\left(1-\eta\left(\theta_H^{**}\right)\right)\left(p_{HL}\left(\theta_H^{**}, g\left(\theta_H^{**}\right)\right)+\theta_H^{**}\right)+\frac{2-\eta\left(\theta_H^{**}\right)}{1-\eta\left(g\left(\theta_H^{**}\right)\right)}\eta\left(g\left(\theta_H^{**}\right)\right)g\left(\theta_H^{**}\right)=0$$

Define now the following continuous function:

$\Psi(\theta_H) = \psi(\theta_H, g(\theta_H)) =$

$$= (1-\eta(\theta_H))(p_{HL}(\theta_H, g(\theta_H)) + \theta_H) + \tfrac{2-\eta(\theta_H)}{1-\eta(g(\theta_H))}\eta(g(\theta_H))\, g(\theta_H). \qquad (11)$$

where $\theta_H \in \left(\theta_H^{low}, \overline{\theta}_H\right)$.

Using the expression for p_{HL} given in equation (B2) in Appendix B, and recalling equation (3), it can be checked that:

$$\lim_{\theta_H \to \theta_H^{low}} \Psi(\theta_H) = \theta_H^{low}\left(1-\eta\left(\theta_H^{low}\right)\right)\left(1-\frac{1}{\eta\left(\theta_H^{low}\right)}\right) < 0$$

and

$$\lim_{\theta_H \to \overline{\theta}_H} \Psi(\theta_H) = \frac{\eta\left(g\left(\overline{\theta}_H\right)\right) g\left(\overline{\theta}_H\right)}{1-\eta\left(g\left(\overline{\theta}_H\right)\right)} > 0.$$

Therefore, continuity of $\Psi(\theta_H)$ implies that there exists a θ_H^{**} such that $\Psi(\theta_H^{**}) = 0$, with $\Psi'(\theta_H^{**}) > 0$. Given θ_H^{**}, the corresponding value of θ_L is $\theta_L^{**} = g(\theta_H^{**})$.

Step II. Substituting $s_H^{**} = s_{\max}$ and the pair $(\theta_H^{**}, \theta_L^{**})$ into (3') we obtain

$$\theta_L^{**} = \frac{s_{\max} - s_L}{s_L}\left(\frac{1}{\eta(\theta_H^{**})} - 1\right)\theta_H^{**},$$

i.e.

$$\frac{s_{\max}}{s_L} = \frac{\theta_L^{**}}{\theta_H^{**}}\frac{\eta(\theta_H^{**})}{1-\eta(\theta_H^{**})} + 1, \qquad (12)$$

which, solved for s_L, implies that $0 < s_L^{**} < s_{\max}$.

Step III. Given $(\theta_H^{**}, \theta_L^{**})$ from (5') and (10), and s_L^{**} from (12), we have now to verify whether the SOCs are satisfied at this solution. We have already proved in Appendix A that the SOCs for profit maximization with respect to prices are satisfied for any quality pair. Therefore they are satisfied at $(s_L^{**}, s_{\max})$. Given that at the quality stage firm H chooses a corner solution, we need simply to check that the SOC for profit maximization with respect to quality is verified for firm L. This is proved in Appendix C.

Step IV. Through the previous steps we have proved the existence of a candidate equilibrium, where the best replies have been constrained to be respectively a best reply from above to s_L and a best reply from below to s_H. For this candidate equilibrium to be a Nash equilibrium, we have to check that s_L^{**} is the best reply to $s_{\max}$, and $s_{\max}$ is the best reply to s_L^{**} on the whole strategy space. The proof of the former is trivial, since should both firms produce $s_{\max}$, product homogeneity would imply zero profits. As far as the best reply to s_L^{**} is concerned, notice that at the price equilibrium the profits of the low quality firm are increasing in the high quality level:

$$\frac{d\Pi_L}{ds_H} = \frac{\partial \Pi_L}{\partial p_L}\frac{\partial p_L}{\partial s_H} + \frac{\partial \Pi_L}{\partial p_H}\frac{\partial p_H}{\partial s_H} + \frac{\partial \Pi_L}{\partial s_H} = \frac{p_L}{\Delta} f_H \left(\frac{\partial p_H}{\partial s_H} - \theta_H \right) > 0$$

given Corollary 2.

Therefore, should firm H respond from below to s_L^{**} its maximum profits would be lower than the profits earned by firm L at the candidate equilibrium, which, in turn are lower than the profits earned by H at the candidate equilibrium (Corollary 1). Firm H's reply from above is therefore deviation proof on the whole strategy space. ■

Proposition 2 establishes that the duopoly model of vertical differentiation with uncovered market has a solution under a wide range of symmetric and asymmetric distributions, which includes – but is not limited to – all logconcave distributions. Besides its theoretical relevance, we believe that this result may have significant implications in terms of our understanding of actual market behavior. We have shown that, given a specific density function $f(\theta)$, the solution procedure is in principle quite simple, in that it makes use of the associated $\eta(\theta)$ and $\pi(\theta)$ functions only. This provides a useful instrument for the analysis of the effects of distributive phenomena on the configurations of markets with vertically differentiated products. We offer some examples in the sequel.

5 Applications: Beta and Dagum distributions

In this section we explicitly solve our duopoly model assuming that the consumers' θs are distributed according to different parametrizations of the symmetric Beta, and the Dagum distributions. We consider the former since the uniform distribution standardly used in vertical differentiation models is indeed a special case of the symmetric Beta (Johnson *et al*, 1995, p. 276). The Dagum distribution is well known to be a fairly good representation of actual (asymmetric) income distributions.[7] Our exercise will focus on the effects of equal-mean, second-order stochastic-dominance shifts of the distribution, as this allows to consider how the firms' equilibrium choices react to a well-defined change in the dispersion of the consumers' willingness to pay.[8]

The density of the Beta symmetric distribution over the unit interval $[0,1]$ is given by

$$f(\theta,\gamma) = \beta(\gamma)^{-1}\theta^{\gamma-1}(1-\theta)^{\gamma-1},$$

[7] The distribution F used in this paper obviously refers to the consumers' willingness to pay: a natural question is then that of the relationship between F and the actual *income* distribution – a relationship which is of course conditioned by the consumers' preferences. We are grateful to an anonymous referee for drawing our attention to this point, a short discussion of which is presented at the end of this section.

[8] Second-order stochastic dominance is well known to have noteworthy normative implications in terms of inequality rankings. In particular, equal-mean, second-order stochastic dominance amounts to Lorenz dominance (Atkinson, 1970).

where $\beta(\gamma)$ is the symmetric Beta function,[9] and $\gamma \geq 1$ is an index of second-order stochastic dominance for given mean, $\mu = 1/2$ – accordingly, higher γ distributions will Lorenz-dominate lower γ distributions. The shape of the symmetric Beta density for $\gamma = 2$, $\gamma = 3$ and $\gamma = 4$ is shown in Figure 1; notice that the dotted line represents the uniform distribution case ($\gamma = 1$). By solving our block-recursive system (equations (3'), (5'), (9) and (10)), we can calculate the equilibrium values of different relevant variables associated to the above values of γ. We report in Table 1 our results, calculated under the assumption $s_H^* = s_{\max} = 1$.

The same exercise can be performed for the Dagum distribution, which in our case one can conveniently write as

$$g(\theta, \delta) = 2\theta \left(1 + \theta^\delta K(\delta)\right)^{-\frac{2+\delta}{\delta}} K(\delta)^{2/\delta},$$

where the average willingness to pay has been normalized to 1, and $\delta > 1$ is a concentration parameter such that a higher δ distribution stochastically dominates a lower δ distribution in the second order sense; $K(\cdot)$ is a constant parameter depending on δ itself.[10] The shape of the Dagum density for $\delta = 2$, $\delta = 3$ and $\delta = 4$ is shown in Figure 2. The system of the firms' FOCs can be solved numerically, and for the above values of δ, with $s_H^* = s_{\max} = 1$, we find the results reported in Table 2.

Fig.1 Beta density for different values of γ

[9] The Beta function with parameters (p, q) is given by $B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\,du$, while symmetry requires $p = q = \gamma$; in the text $\beta(\gamma) = B(\gamma, \gamma)$. On the Beta distribution see Johnson *et al* (1995, ch.25). It is easily seen that $\gamma = 1$ delivers the uniform distribution

[10] In particular, $K(\delta) = [\Gamma(3/\delta)\,\Gamma((\delta-1)/\delta)/\Gamma(2/\delta)]^\delta$, where $\Gamma(\cdot)$ is the Gamma functon such that $\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} dz$. On the Dagum distribution and its properties see Kleiber (2008).

Fig.2 Dagum density for different values of δ

Table 1: Values of relevant variables under different concentration parameters: the Beta distribution

	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$
θ_H^{**}	0.36725	0.35902	0.35900
θ_L^{**}	0.12398	0.13087	0.13878
$1 - F(\theta_H^{**})$	0.69445	0.75066	0.78506
$F(\theta_H^{**}) - F(\theta_L^{**})$	0.26325	0.23110	0.2058
p_H^{**}	0.24235	0.25307	0.26615
p_L^{**}	6.3655×10^{-2}	6.0776×10^{-2}	5.8513×10^{-2}
$s_H^{**} - s_L^{**}$	0.48657	0.5356	0.57838
Π_H^{**}	0.16830	0.18997	0.20894
Π_L^{**}	1.6757×10^{-2}	1.4045×10^{-2}	1.2042×10^{-2}
Π_H^{**}/Π_L^{**}	10.044	13.526	17.351

Table 2: Values of relevant variables under different concentration parameters: the Dagum distribution

	$\delta = 2$	$\delta = 3$	$\delta = 4$
θ_H^{**}	0.48336	0.59110	0.64862
θ_L^{**}	0.17222	0.20747	0.22412
$1 - F(\theta_H^{**})$	0.63433	0.69171	0.71092
$F(\theta_H^{**}) - F(\theta_L^{**})$	0.29748	0.2655	0.25305
p_H^{**}	0.32542	0.39859	0.43754
p_L^{**}	8.7424×10^{-2}	0.10411	0.11144
$s_H^{**} - s_L^{**}$	0.49237	0.4982	0.50275
Π_H^{**}	0.20642	0.27571	0.31106
Π_L^{**}	2.6007×10^{-2}	2.7641×10^{-2}	2.8200×10^{-2}
Π_H^{**}/Π_L^{**}	7.9371	9.9747	11.03

The numbers considered in Tables 1 and 2, though clearly limited to very specific cases, lend themselves to some observations. In our exercise asymmetry does not seem to play a relevant role. The pattern of most variables is indeed the same under both distributions, i.e. irrespective of symmetry. In both cases an increase in the concentration of the willingness to pay leads to widening the degree of market coverage and narrowing the market for low quality goods – and hence to a larger market for high quality commodities. In both cases the quality spectrum gets larger as the willingness to pay becomes more concentrated. Indeed, at unchanged qualities, lower dispersion implies a positive shift in demand for both firms, making more convenient an aggressive price behavior of the high quality firm. The low quality firm protects its market share by enlarging the quality differential, and adjusting its price accordingly. While the equilibrium price of the high quality increases in all our examples, the price of the low quality increases (in the

Dagum case) or decreases (in the symmetric Beta case), depending on the balance between the initial demand stimulus and the effect of the reduction of quality.

In both our examples the market share of the high quality firm increases, while that of the low quality one decreases. However, due to the different ways in which the dominance parameter modifies the shapes of the two distributions, this pattern of the market shares is accompanied by a decrease of the threshold level of the willingness to pay supporting access to the high quality market (θ_H^{**}) in the case of the Beta distribution, while in the case of the Dagum distribution this threshold level increases. Also, and in our view most interestingly, under both distributions the minimum level of the willingness to pay required to enter the market (θ_L^{**}) increases: the shrinking of the left tail of the distribution decreases the incentive to trade-off the intensive margin on buyers for the extensive margin on excluded consumers.

As a final remark, one might ask to what extent one can interpret changes in the distribution of the willingness to pay as changes in the income distribution. We have already mentioned the view (Tirole 1988, pp. 96-97) according to which θ can be accounted for as the marginal rate of substitution between quality and income. If the marginal utility of quality is constant, the relation between the willingness to pay for quality and income can be written as $\theta = 1/u'(y)$, where $u(y)$ denotes the utility of income.[11] Suppose e.g. that $u(y)$ is a CRRA function, so that $\theta = y^{\rho}$, where $\rho > 0$ is the constant elasticity of the marginal utility of income. If $\rho = 1$ (i.e., $u(y)$ is logarithmic), then $\theta = y$ and the distribution F of the willingness to pay

[11] This is the formulation suggested by Tirole (1988, p.97, fn. 1), on the basis of a separable representation of the consumer's preferences.

coincides with the income distribution. If $\rho \neq 1$ the relationship between the distribution of income and the distribution of the willingness to pay is obviously affected by the concavity of the utility function. However, in the examples discussed above, it can be checked that in this case the reduction in the dispersion of the θ's is unambiguously associated to a reduction in the variance of the underlying distribution of incomes.[12]

6 Conclusions

The relationship between consumers' heterogeneity and market structure is an intriguing issue in modern economic theory. On the one hand, it calls for a renewed attention to the role of market demand in shaping the firms' competitive environment; on the other hand, it creates a link between several dimensions of firms' behavior and the economic and social phenomena affecting the consumers' heterogeneity – such as inequality, poverty, or income polarization. However, there is a key obstacle for a full development of this stream of economic analysis, which concerns its analytical tractability at high levels of generality.

This paper contributes in this direction by offering a general proof for the existence of a subgame perfect equilibrium in pure strategies in a duopolistic model with vertical differentiation and uncovered market. We extend the existing literature by showing that under very mild conditions on the shape of the consumers' distribution – which are indeed less restrictive than those

[12] This is true for all cases considered in our examples, subject to a finite variance constraint for the Dagum distribution requiring $\delta\rho > 2$. It may be also worth noting that our mean preserving shifts of the distribution of the willingness to pay are associated to lower or higher mean income, depending on the income concavity or convexity of θ: ρ smaller (larger) than one would imply a smaller (larger) mean income.

implied by logconcavity – it is possible to go beyond the proof of existence of a unique equilibrium in prices. By making use of the 'Esteban elasticity' representation of the consumers' density function we are able to prove the existence of a two-stage equilibrium in prices and qualities, and to offer a manageable algorithm to actually compute the above solution. Our application of this algorithm to different configurations of the Beta and Dagum distributions envisages the multiplicity of exercises that can in principle be performed to evaluate the effects of distributional changes in our basic setup.

The proof of the existence of equilibrium in a vertically differentiated duopoly for a wide set of density functions is a fundamental step, but it leaves open other interesting questions. Further investigation is required to extend our existence result to an oligopolistic setting and to formulate non density-specific propositions about the relationship between consumers' heterogeneity and the number of qualities offered in the market.

Appendix A. The Second Order Conditions at the price stage

In this and in the following Appendices we shall make use of the following simplifying notation: $f_i \equiv f(\theta_i)$, $F_i \equiv F(\theta_i)$, $\eta_i \equiv \eta(\theta_i)$, $\pi_i \equiv \pi(\theta_i)$, $i = H, L$.

The SOC of firm H requires:

$$\frac{\partial^2 \Pi_H}{\partial p_H^2} = -\frac{1}{\Delta}\left(2f_H + \frac{p_H}{\Delta} f'_H\right) < 0.$$

By using (1) and recalling the definitions of $\eta(\theta)$ and $\pi(\theta)$ it can be rewritten as

$$-\frac{1}{\Delta}\frac{(1-F_H)}{\theta_H}(2\eta_H + \pi_H - 1) < 0,$$

which is indeed the case for all $\theta_H > 0$, if Property 3 holds.

The SOC of firm L is satisfied if

$$\frac{\partial^2 \Pi_L}{\partial p_L^2} = -2\left(\frac{f_H}{\Delta} + \frac{f_L}{s_L}\right) + p_L\left(\frac{f'_H}{\Delta^2} - \frac{f'_L}{s_L^2}\right) < 0,$$

i.e.

$$f_H + \frac{\Delta}{s_L} f_L + \frac{1}{2}\frac{p_L}{\Delta}\left(f'_H - \left(\frac{\Delta}{s_L}\right)^2 f'_L\right) > 0. \tag{A1}$$

Equation (3) allows to reformulate Δ/s_L and p_L/Δ in terms of θ_H and θ_L only:

$$\frac{\Delta}{s_L} = \frac{\theta_L}{\theta_H}\frac{\eta_H}{1-\eta_H}, \tag{A2}$$

$$\frac{p_L}{\Delta} = \frac{\theta_H}{\eta_H}(1-\eta_H), \tag{A3}$$

so that using (5') and the definitions of $\eta(\theta)$ and $\pi(\theta)$, the inequality (A1) can be rewritten as

$$\eta_H + \frac{1}{2}(1-\eta_H)(\pi_H - 1) + \frac{1}{2}\frac{2-\eta_H}{1-\eta_L}\eta_L\frac{\eta_H}{1-\eta_H}(1+\pi_L) > 0.$$

At equilibrium, the first two terms are positive since equation (3) ensures $\eta_H < 1$ and Property 3 ensures that $(\pi_H - 1) > -2\eta_H$. The last term is positive since Property 2 and equation (3) imply (a) $\eta_L < \eta_H < 1$, and (b) $\pi_L > -\eta_L > -1$

Appendix B. The comparative statics of the price stage

By totally differentiating (1) and (2), we obtain the following linear system:

$$\begin{pmatrix} \frac{\partial^2 \Pi_H}{\partial p_H^2} & \frac{\partial^2 \Pi_H}{\partial p_H \partial p_L} \\ \frac{\partial^2 \Pi_L}{\partial p_L \partial p_H} & \frac{\partial^2 \Pi_L}{\partial p_L^2} \end{pmatrix} \begin{pmatrix} dp_H \\ dp_L \end{pmatrix} = -\begin{pmatrix} \frac{\partial^2 \Pi_H}{\partial p_H \partial \Delta} \\ \frac{\partial^2 \Pi_L}{\partial p_L \partial \Delta} \end{pmatrix} d\Delta_s - \begin{pmatrix} \frac{\partial^2 \Pi_H}{\partial p_H \partial s_L} \\ \frac{\partial^2 \Pi_L}{\partial p_L \partial s_L} \end{pmatrix} ds_L$$

where the above derivatives can be written as:

- $\frac{\partial^2 \Pi_H}{\partial p_H^2} = -2f_H\frac{1}{\Delta} - \frac{p_H}{\Delta^2}f'_H$; $\frac{\partial^2 \Pi_H}{\partial p_H \partial p_L} = f_H\frac{1}{\Delta} + \frac{p_H}{\Delta^2}f'_H$;
- $\frac{\partial^2 \Pi_L}{\partial p_L \partial p_H} = f_H\frac{1}{\Delta} - \frac{p_L}{\Delta^2}f'_H$;
- $\frac{\partial^2 \Pi_L}{\partial p_L^2} = -\frac{2}{\Delta}f_H - \frac{2}{s_L}f_L + \frac{p_L}{\Delta^2}f'_H - f'_L\frac{\theta_L}{s_L}$;
- $\frac{\partial^2 \Pi_H}{\partial p_H \partial \Delta} = f_H\left(\frac{\theta_H}{\Delta} + \frac{p_H}{\Delta^2}\right) + \frac{p_H}{\Delta}f'_H\frac{\theta_H}{\Delta}$;
- $\frac{\partial^2 \Pi_H}{\partial p_H \partial s_L} = -f_H\left(\frac{\theta_H}{\Delta} + \frac{p_H}{\Delta^2}\right) - \frac{p_H}{\Delta}f'_H\frac{\theta_H}{\Delta}$;

- $\dfrac{\partial^2 \Pi_L}{\partial p_L \partial \Delta} = f_H \left(\dfrac{p_L}{\Delta^2} - \dfrac{\theta_H}{\Delta} \right) + \dfrac{p_L}{\Delta} f'_H \dfrac{\theta_H}{\Delta}$;
- $\dfrac{\partial^2 \Pi_L}{\partial p_L \partial s_L} = f_H \left(\dfrac{\theta_H}{\Delta} - \dfrac{p_L}{\Delta^2} \right) - \dfrac{p_L}{\Delta} f'_H \dfrac{\theta_H}{\Delta} + 2\dfrac{\theta_L}{s_L} f_L + \dfrac{\theta_L^2}{s_L} f'_L$,

and are evaluated at equilibrium.

Denoting with A the coefficients matrix, its determinant can be written as:

$$\begin{aligned} |A| &= \frac{1}{\Delta^2} f_H^2 \left(2 + \pi_H\right) + \\ &+ \frac{1}{\Delta^2} \left\{ \frac{\Delta}{s_L} \left(\left(1 + \pi_L\right) f_L \right) \left(\left(1 + \pi_H\right) f_H + \frac{p_L}{\Delta} f'_H \right) \right\}. \end{aligned}$$

Property 2 ensures that $(1 + \pi_H) > 0$, provided that $\eta_H < 1$, which is indeed the case at the price stage equilibrium. Since $\theta_L < \theta_H$, Property 2 also ensures that at equilibrium $\eta_L < 1$ so that $(1 + \pi_L) > 0$. Therefore $|A| > 0$ if:

$$\left(1 + \pi_H\right) f_H + \frac{p_L}{\Delta} f'_H > 0.$$

Recalling equation (A3), at equilibrium we can reformulate this inequality as

$$\frac{f_H}{\eta_H} \left(2\eta_H + \pi_H - 1\right) > 0,$$

which is indeed the case under Property 3.

We can now perform the following comparative statics exercises.

The effect on p_L^* of a change in s_H.

First notice that for given s_L, $dp_L^*/ds_H = dp_L^*/d\Delta$. By applying Cramer's rule:

$$\frac{dp_L^*}{d\Delta} = \frac{1}{|A|} \frac{p_L}{\Delta^3} f_H^2 \left(2 + \pi_H\right),$$

where the RHS is evaluated at equilibrium, and is positive under Property 2.

The effect on p_H^* of a change in s_L.

By Cramer's rule and using the definition of $\pi(\theta)$, we have:

$$\frac{dp_H^*}{ds_L} = -\frac{\frac{p_H}{\Delta} f_H^2 (2+\pi_H) + \frac{\Delta}{s_L}(1+\pi_L) f_L \left(\frac{p_H}{\Delta}(\theta_H - \theta_L) f_H' + (\theta_H - \theta_L) f_H + \frac{p_H}{\Delta} f_H\right)}{f_H^2 (2+\pi_H) + \frac{\Delta}{s_L}(1+\pi_L) f_L \left((1+\pi_H) f_H + \frac{p_L}{\Delta} f_H'\right)}, \tag{B1}$$

where the RHS is evaluated at equilibrium. By using (A2) for Δ/s_L, (A3) for p_L/Δ, $p_H/\Delta = p_L/\Delta + \theta_H$, the definitions of η for f, the definition of π for f', and then equation (5') for $(1 - F_L)$, we obtain that at the price equilibrium equation (B1) collapses to:

$$\frac{dp_H^*}{ds_L} = -\frac{\theta_H}{\eta_H} \frac{(2+\pi_H)(1-\eta_H) + (1+\pi_L)\eta_L \frac{2-\eta_H}{1-\eta_L}\left(\pi_H + \eta_H - (\pi_H + \eta_H - 1)\frac{\theta_L}{\theta_H}\right)}{(2+\pi_H)(1-\eta_H) + (1+\pi_L)\eta_L \frac{2-\eta_H}{1-\eta_L}\left(\frac{2\eta_H + \pi_H - 1}{\eta_H}\right)}, \tag{B2}$$

which under Properties 2 and 3 is unambiguously negative.

The sign of $\left(\frac{dp_L^*}{ds_L} - \theta_L^*\right)$.

Consider now dp_L^*/ds_L. By applying Cramer's rule and using the definition of $\pi(\theta)$, we obtain

$$\frac{dp_L^*}{ds_L} = \theta_L \frac{\frac{\Delta}{s_L}(1+\pi_L) f_L \left[(1+\pi_H) f_H + \frac{p_L}{\Delta_s} f_H'\right] - \frac{s_L}{\Delta} f_H^2 (2+\pi_H)}{\frac{\Delta}{s_L}\left[(1+\pi_L) f_L\right]\left[(1+\pi_H) f_H + \frac{p_L}{\Delta_s} f_H'\right] + f_H^2 (2+\pi_H)}. \tag{B3}$$

By using (A2) for Δ/s_L, (A3) for p_L/Δ, the definitions of η for f, the definition of π for f', and then equation (5') for $(1 - F_L)$, we obtain that at the price equilibrium equation (B3) collapses to:

$$\frac{dp_L^*}{ds_L} = \theta_L \frac{(1+\pi_L)\,\eta_L \frac{2-\eta_H}{1-\eta_L}\,(2\eta_H + \pi_H - 1) - \frac{\theta_H (1-\eta_H)^2}{\theta_L}\,(2+\pi_H)}{(1+\pi_L)\,\eta_L \frac{2-\eta_H}{1-\eta_L}\,(2\eta_H + \pi_H - 1) + \eta_H\,(2+\pi_H)\,(1-\eta_H)}. \tag{B4}$$

Since under Property 3 the ratio in (B4) is surely less than one, we get:

$$\frac{dp_L^*}{ds_L} - \theta_L^* = \theta_L \left(\frac{(1+\pi_L)\eta_L \frac{2-\eta_H}{1-\eta_L}(2\eta_H+\pi_H-1) - \frac{\theta_H(1-\eta_H)^2}{\theta_L}(2+\pi_H)}{(1+\pi_L)\eta_L \frac{2-\eta_H}{1-\eta_L}(2\eta_H+\pi_H-1) + \eta_H(2+\pi_H)(1-\eta_H)} - 1 \right) < 0, \tag{B5}$$

where the RHS is evaluated at equilibrium.

The sign of $\left(\frac{dp_H^*}{ds_H} - \theta_H^*\right)$.

Consider now $dp_H^*/ds_H = dp_H^*/d\Delta$. By applying Cramer's rule and using the definition of $\pi(\theta)$, we obtain:

$$\frac{dp_H^*}{d\Delta} = \frac{\left(f_H \left(\frac{p_L}{\Delta}\pi_H - \theta_H \right) f_H \left(1 + \frac{p_H}{\Delta} \frac{(\pi_H - 1)}{\theta_H} \right) - f_H \left(\theta_H + \frac{p_H}{\Delta}\pi_H \right) \left(f_H \left(\frac{p_L}{\Delta} \frac{(\pi_H-1)}{\theta_H} - 2 \right) - \frac{f_L \Delta}{s_L}(\pi_L + 1) \right) \right)}{f_H^2 (2+\pi_H) + \frac{\Delta}{s_L} [(1+\pi_L) f_L] \left[(1+\pi_H) f_H + \frac{p_L}{\Delta} f_H' \right]}$$
(B6)

By using (A2) for Δ/s_L, (A3) for p_L/Δ, the definitions of η for f, the definition of π for f', equation (5') for $(1 - F_L)$, and noting that $p_H/\Delta = p_L/\Delta + \theta_H = \theta_H/\eta_H$, we obtain that at the price equilibrium equation (B6) collapses to:

$$\frac{dp_H^*}{d\Delta} = \theta_H \frac{\left(\left(\frac{1-\eta_H}{\eta_H} \right) \pi_H - 1 \right) (\eta_H + \pi_H - 1)(1-\eta_H) - (\eta_H + \pi_H) \left(\left((1-\eta_H) \frac{(\pi_H - 1)}{\eta_H} - 2 \right) (1-\eta_H) - \frac{\eta_L (2-\eta_H)}{(1-\eta_L)} (\pi_L + 1) \right)}{\eta_H (1-\eta_H)(2+\pi_H) + (1+\pi_L) \frac{\eta_L (2-\eta_H)}{(1-\eta_L)} (\pi_H + 2\eta_H - 1)}$$
(B7)

Since under Property 3 the ratio in (B7) is surely greater than one, we get:

$$\frac{dp_H^*}{d\Delta} - \theta_H > 0. \tag{B8}$$

Appendix C. The Second Order Condition at the quality stage

The proof that the Second Order Condition at the quality stage for firm L is satisfied, relies on three preliminary steps.

STEP I. It is useful to calculate $\Psi'(\theta_H)$ at $\Psi(\theta_H)=0$. By deriving equation (11) and recalling that $\theta_L = g(\theta_H)$ according to (5'), we get that at $\Psi(\theta_H)=0$:

$$\Psi'(\theta_H) = \frac{\eta'_H\eta_L\theta_L}{1-\eta_L}\frac{2-\eta_H}{1-\eta_H} + (1-\eta_H)\left(\frac{\partial p_{HL}}{\delta\theta_H} + \frac{\partial p_{HL}}{\delta\theta_L}g'(\theta_H)+1\right)+$$

$$-\frac{\eta'_H}{1-\eta_L}\eta_L\theta_L + \frac{2-\eta_H}{1-\eta_L}\left(\frac{\eta_L\theta_L\eta'_L}{1-\eta_L} + \eta'_L\theta_L + \eta_L\right)g'(\theta_H).$$

Collecting terms the above expression can be simplified into:

$$\Psi' = \frac{\eta'_H\eta_L\theta_L}{(1-\eta_L)(1-\eta_H)} + (1-\eta_H)\left(\frac{\partial p_{HL}}{\delta\theta_H} + \frac{\partial p_{HL}}{\delta\theta_L}g'(\theta_H)+1\right)+$$

$$+\frac{2-\eta_H}{1-\eta_L}\left(\frac{\theta_L\eta'_L}{1-\eta_L} + \eta_L\right)g'(\theta_H).$$

Using now the definitions of η and π, we can write:

$$\eta'_L = \frac{\eta_L}{\theta_L}(\pi_L+\eta_L); \quad \eta'_H = \frac{\eta_H}{\theta_H}(\pi_H+\eta_H),$$

which used into $\Psi'(\theta_H)$ leads to:

$$\Psi'(\theta_H) = \frac{\theta_L}{\theta_H}\frac{\eta_H(\pi_H+\eta_H)\eta_L}{(1-\eta_L)(1-\eta_H)} + (1-\eta_H)\left(\frac{\partial p_{HL}}{\partial\theta_H} + \frac{\partial p_{HL}}{\partial\theta_L}g'(\theta_H)+1\right)+$$

$$+\frac{2-\eta_H}{1-\eta_L}\eta_L\frac{\pi_L+1}{1-\eta_L}g'(\theta_H). \tag{C1}$$

Step II. Notice that at the price equilibrium the effects of changes in s_L on the value of θ_H and θ_L can be written as follows:

$$\theta_{LL} \equiv \frac{\partial \theta_L^*}{\partial s_L} = \frac{1}{s_L}\left(p_{LL} - \theta_L\right) < 0, \tag{C2}$$

$$\theta_{HL} \equiv \frac{\partial \theta_H^*}{\partial s_L} = \frac{1}{\Delta}\left(p_{HL} + \theta_H - p_{LL}\right). \tag{C3}$$

The sign of θ_{LL} can be easily established from (B5). As far as θ_{HL} is concerned, we recall from Lemma 2 that for (5') to be satisfied:

$$\theta_{LL} = g'\theta_{HL} \tag{C4}$$

so that, given $g' > 0$, $\theta_{HL} < 0$.

Step III. For future reference, it is useful to solve the FOC of firm L at the quality stage in terms of p_{LL}. From equation (8), this FOC can be written as:

$$p_{LL}\left(F_H - F_L\right) + p_L f_H \theta_{HL} = p_L f_L \theta_{LL}.$$

By using (C4) for θ_{LL} and (C3) for θ_{HL}, the above can be reformulated as:

$$p_{LL} = \frac{\left(\frac{p_L}{\Delta} f_L g' - \frac{p_L}{\Delta} f_H\right)}{\left(F_H - F_L + \frac{p_L}{\Delta} f_L g' - \frac{p_L}{\Delta} f_H\right)}\left(p_{HL} + \theta_H\right).$$

Consider now the ratio at the RHS of the above expression. Recalling equations (5') and (A3), the expression for g' and the definitions of η, we obtain:

$$\frac{\left(\frac{p_L}{\Delta} f_L g' - \frac{p_L}{\Delta} f_H\right)}{\left(F_H - F_L + \frac{p_L}{\Delta} f_L g' - \frac{p_L}{\Delta} f_H\right)} = \frac{(1-\eta_H)(1+\pi_H-\pi_L)(1-\eta_L)}{(1-\eta_H+\eta_L)(1+\pi_L)+(1-\eta_H)(1-\eta_L)(1+\pi_H-\pi_L)},$$

so that, using equation (10) for $(p_{HL} + \theta_H)$ we get:

$$p_{LL} = -\theta_L \frac{\eta_L(2-\eta_H)(1+\pi_H-\pi_L)}{(1-\eta_H+\eta_L)(1+\pi_L)+(1-\eta_H)(1-\eta_L)(1+\pi_H-\pi_L)}. \tag{C5}$$

Now we proceed to evaluate the SOC for firm L at the quality stage. The first order derivative of the profits of firm L with respect to quality (equation (8) in text) can be written as:

$$\frac{\partial \Pi_L}{\partial s_L} = \frac{p_L}{\Delta} f_H \left(p_{HL} + \theta_H\right) + f_L \theta_L^2 + \left(F_H - F_L - \frac{p_L}{\Delta} f_H - \theta_L f_L\right) p_{LL}.$$

Therefore, given equation (2), the second order derivative is:

$$\begin{aligned}\frac{\partial^2 \Pi_L}{\partial s_L^2} &= \frac{\partial}{\partial s_L}\left(\frac{p_L}{\Delta} f_H \left(p_{HL} + \theta_H\right)\right) - p_{LL}\frac{\partial}{\partial s_L}\left(\frac{p_L}{\Delta} f_H\right) + \\ &\quad + \frac{\partial}{\partial s_L}\left(f_L \theta_L^2\right) + p_{LL}\frac{\partial}{\partial s_L}\left(F_H - F_L - \theta_L f_L\right). \qquad \text{(C6)}\end{aligned}$$

Using (C2) in the first two terms, and the definition of π_L in the last two terms of (C6), we obtain:

$$\frac{\partial^2 \Pi_L}{\partial s_L^2} = \left(\left(2p_{LL} + \frac{p_L}{\Delta}\right) f_H + \frac{p_L}{\Delta} f_H' \left(p_{HL} + \theta_H - p_{LL}\right)\right)\theta_{HL} +$$

$$+ f_L \left(\theta_L - p_{LL}\right)\left(1 + \pi_L\right)\theta_{LL} + \tfrac{p_L}{\Delta} f_H \left(\frac{\partial p_{HL}}{\partial \theta_H} + 1\right)\theta_{HL} + \frac{p_L}{\Delta} f_H \frac{\partial p_{HL}}{\partial \theta_L}\theta_{LL}.$$

which, using (A3), (C4), and the definition of η, results into:

$$\frac{\partial^2 \Pi_L}{\partial s_L^2} = \left(\left(2p_{LL} + \tfrac{p_L}{\Delta}\right) f_H + \tfrac{p_L}{\Delta} f_H' \left(p_{HL} + \theta_H - p_{LL}\right)\right)\theta_{HL} +$$

$$f_L \left(\theta_L - p_{LL}\right)\left(1 + \pi_L\right) g' \theta_{HL} + \left(1 - F_H\right)\left(1 - \eta_H\right)\left(\frac{\partial p_{HL}}{\partial \theta_H} + 1\right)\theta_{HL} +$$

$$+ \frac{\partial p_{HL}}{\partial \theta_L} g' \theta_{HL}. \qquad \text{(C7)}$$

Consider now (C1). By multiplying both sides by θ_{HL} and rearranging terms, it can be rewritten as:

$$(1-\eta_H)\left(\left(\frac{\partial p_{HL}}{\partial \theta_H}+1\right)\theta_{HL}+\frac{\partial p_{HL}}{\partial \theta_L}g'\theta_{HL}\right)=$$

$$=\theta_{HL}\Psi'(\theta_H)-\left(\frac{\theta_L}{\theta_H}\frac{\eta_H(\pi_H+\eta_H)\eta_L}{(1-\eta_L)(1-\eta_H)}\theta_{HL}+\frac{2-\eta_H}{1-\eta_L}\eta_L\frac{\pi_L+1}{1-\eta_L}g'\theta_{HL}\right). \quad \text{(C1')}$$

Using (C1') into (C7) we obtain:

$$\frac{\partial^2\Pi_L}{\partial s_L^2}=(1-F_H)\theta_{HL}\Psi'(\theta_H)+f_L(\theta_L-p_{LL})(1+\pi_L)g'\theta_{HL}+$$

$$+\left(\left(2p_{LL}+\tfrac{p_L}{\Delta}\right)f_H+\tfrac{p_L}{\Delta}f_H'(p_{HL}+\theta_H-p_{LL})\right)\theta_{HL}+$$

$$-(1-F_H)\left(\frac{\theta_L\eta_H(\pi_H+\eta_H)\eta_L}{\theta_H(1-\eta_L)(1-\eta_H)}\theta_{HL}+\frac{2-\eta_H}{1-\eta_L}\eta_L\frac{\pi_L+1}{1-\eta_L}g'\theta_{HL}\right). \quad \text{(C8)}$$

The first term in (C8) is negative, since $\theta_{HL}<0$ (see Step II of this Appendix) and $\Psi'>0$ at equilibrium. In the sequel we prove that the sum of the remaining terms is equal to zero at equilibrium. By using repeatedly equations (5'), (10), (A3) as well as the expression for g' and the definitions of η and π, tedious algebra shows that this amounts to proving that:

$$2p_{LL}\frac{\eta_H}{\theta_H}+(1-\eta_H)+(1-\eta_H)\frac{1}{\theta_H}(\pi_H-1)(p_{HL}+\theta_H)+$$

$$-(1-\eta_H)\frac{1}{\theta_H}(\pi_H-1)p_{LL}+\frac{\eta_H}{\theta_H}(2+\pi_H)\theta_L-p_{LL}\frac{\eta_H}{\theta_H}(2+\pi_H)+$$

$$-\left(\frac{\theta_L}{\theta_H}\frac{\eta_H(\pi_H+\eta_H)\eta_L}{(1-\eta_L)(1-\eta_H)}+\eta_H\frac{(2+\pi_H)}{1-\eta_L}\frac{\theta_L}{\theta_H}\right)=0.$$

By substituting $p_{HL}+\theta_H$ from equation (10), collecting terms, and substituting for p_{LL} from equation (C5), this expression can be transformed into:

$$(1-\eta_H)+$$

$$-\frac{\theta_L\eta_L(2-\eta_H)}{\theta_H(1-\eta_L)(1-\eta_H)}\left(\frac{(1-\eta_H+\eta_L)(1+\pi_L)(2\eta_H+\pi_H-1)+\eta_H(1-\eta_L)(1-\eta_H)(1+\pi_H-\pi_L)}{(1-\eta_H+\eta_L)(1+\pi_L)+(1-\eta_L)(1-\eta_H)(1+\pi_H-\pi_L)}\right)=0.$$

The term $\theta_L\eta_L\left(2-\eta_H\right)/\theta_H\left(1-\eta_L\right)\left(1-\eta_H\right)$ can again be substituted from equation (10), so that we get:

$$(1-\eta_H)+$$

$$+\left(\frac{p_{HL}}{\theta_H}+1\right)\left(\frac{(1-\eta_H+\eta_L)(1+\pi_L)(2\eta_H+\pi_H-1)+\eta_H(1-\eta_L)(1-\eta_H)(1+\pi_H-\pi_L)}{(1-\eta_H+\eta_L)(1+\pi_L)+(1-\eta_L)(1-\eta_H)(1+\pi_H-\pi_L)}\right)=0. \quad \text{(C9)}$$

Now, we deal with the term $(p_{HL}/\theta_H)+1$. At equilibrium, given (B2) and (10):

$$\left(\frac{p_{HL}}{\theta_H}+1\right)=$$

$$-\left(1-\eta_H\right)\frac{(1+\pi_L)\eta_L\frac{2-\eta_H}{1-\eta_L}+(1-\eta_H)(2+\pi_H)}{(1+\pi_L)\eta_L\frac{2-\eta_H}{1-\eta_L}(2\eta_H+\pi_H-1)+(1-\eta_H)(2+\pi_H)\eta_H+(1+\pi_L)(\pi_H+\eta_H-1)(1-\eta_H)}$$

which, substituted into (C9) yields:

$$1-\frac{(1-\eta_H+\eta_L)(1+\pi_L)(2\eta_H+\pi_H-1)+\eta_H(1-\eta_L)(1-\eta_H)(1+\pi_H-\pi_L)}{(1+\pi_L)\eta_L(2-\eta_H)(2\eta_H+\pi_H-1)+(1-\eta_H)[2+\pi_H]\eta_H(1-\eta_L)+(1+\pi_L)(\pi_H+\eta_H-1)(1-\eta_H)(1-\eta_L)}=0$$

which is actually true. Therefore equation (C8) collapses to

$$\frac{\partial^2\Pi_L}{\partial s_L^2}=\left(1-F_H\right)\theta_{HL}\Psi'\left(\theta_H\right)<0.$$

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