Experimental characterisation of a CuAg alloy for thermo-mechanical applications. Part 2: Design strain-life curves estimated via statistical analysis

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Abstract. Strain-life fatigue data on copper alloys, especially type CuAg, are seldom available in the literature. This work fills this gap by estimating the strain-life curves of a CuAg alloy used for thermo-mechanical applications, from isothermal low-cycle fatigue tests at three temperatures (room temperature, 250 °C, 300 °C). Regression analysis is used to estimate the median fatigue curves at 50% survival probability. The comparison of median curves with the Universal Slopes Equation model, calibrated on monotonic tensile properties, shows a fairly good agreement. Design strain-life curves with a lower failure probability and given confidence are estimated by several approximate statistical methods ("Equivalent Prediction Interval", univariate tolerance interval, Owen's tolerance interval for regression). When higher survival probabilities are considered, the results show a marked decrease in the allowable design strain at a prescribed fatigue life. The suggested procedure thus improves the durability analysis of components loaded thermo-mechanically.

Keywords

strain-life fatigue curve; regression analysis; design fatigue curve; tolerance interval; prediction interval.

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Nomenclature

A, B	intercept and slope in regression model
A_d, B_d	intercept and slope of design fatigue curve
b, c	exponents of fatigue curve
b_d, c_d	exponents of design fatigue curve
D	ductility
E	elastic modulus
K	correction factor of design curve
N_f	number of cycles to failure
$2N_f$	number of reversals to failure
$\%RA$	area reduction
s^2	sample variance
t	Student variable
$x=\log(\varepsilon_a)$	transformed strain amplitude
$y=\log(2N_f)$	transformed fatigue life
z	standard normal variable
α	failure probability
γ	confidence
δ	random variable in regression model
ε_a, $\varepsilon_{el,a}$, $\varepsilon_{pl,a}$	amplitude of strain (total, elastic, plastic)
$\Delta\varepsilon$, $\Delta\varepsilon_{el}$, $\Delta\varepsilon_{pl}$	strain range (total, elastic, plastic)
ε'_f	fatigue ductility coefficient
σ^2	variance of δ
σ_{uts}	ultimate tensile strength
σ'_f	fatigue strength coefficient

$\left(\dfrac{\sigma'_{\mathrm{f}}}{E}\right)$, $\varepsilon'_{\mathrm{f}}$	intercepts of regression Manson-Coffin curve in log-log diagram
$\left(\dfrac{\sigma'_{\mathrm{f}}}{E}\right)_{\mathrm{d}}$, $(\varepsilon'_{\mathrm{f}})_{\mathrm{d}}$	intercepts of design Manson-Coffin curve in log-log diagram
^	estimated quantity

1. Introduction

Copper alloys have a favourable combination of high thermal conductivity and relatively high mechanical properties, which makes them suitable for structures simultaneously exposed to high thermal flux and mechanical loads.[1] A typical example is a mould for the continuous casting of steel, which is a long hollow component (with a square or circular cross section), usually fabricated from copper alloys of types CuAg, CuCrZr or CuNiBe.[1]

During each casting sequence, the mould is exposed to a cyclic thermal flux from the molten steel and may then be subjected to thermal fatigue damage, represented by a network of cracks on the surface directly in contact with steel. Elasto-plastic finite element simulations can be used to compute the mould thermo-mechanical response under cyclic thermal loadings.[2]

Part 1 of this article showed how material parameters of non-linear kinematic and isotropic plasticity models could be identified from experimental cyclic data at three different temperatures.[3]

To estimate the mould service life, computed stress and strain cycles need to be compared with the material fatigue curve. Unfortunately, as with cyclic plasticity data, experimental strain-life curves for Cu alloys (and especially for CuAg type) are also rare in the literature.

When experimental data are scarce (as often occurs at early design phases, before fatigue tests are planned), an initial estimate of strain-life curves is possible from monotonic tensile properties, which are easier, faster and cheaper to obtain. One of the earliest attempts was the Universal Slopes Equation (USE), proposed in 1965 and calibrated on strain-life data for ferrous (low-alloy and high-strength steel, stainless steel) and non-ferrous (silver, magnesium, beryllium, titanium, aluminium and high-temperature metals) alloys, but not including copper alloys.[4]

This work evaluates the strain-life fatigue curves of a CuAg alloy tested at three temperatures, as described in Part 1. Firstly, a regression analysis is applied to experimental data to identify the median (average) fatigue curve at 50% survival probability (50% failure probability). The USE model is then compared with the estimated median curves, in order to check its validity for copper alloys as well.

Several approximate statistical methods ("Equivalent Prediction Interval", univariate tolerance interval, Owen's tolerance interval for regression) are then used to determine the design fatigue curves at a higher survival probability and prescribed confidence, which account for the uncertainty in regression parameters. The determination of the design curve by statistical analysis is particularly important if only a small number (<10) of experimental samples is available, as with the CuAg data used in this study. By assuming a constant scatter over all strain amplitudes, such approximate methods give a straight design strain-life trend expressed by simple equations, which are much more practical even for non-specialists with respect to exact but more complex statistical methods. Although this work focuses on a specific copper alloy, the suggested procedure can be also extended to other applications with thermo-mechanical loads.[5,6,7,8]

2. Experimental testing

A CuAg alloy, classified in ASTM B 124,[9] was tested under isothermal low-cycle fatigue (LCF) tests at three temperatures: room temperature, 250 ºC, 300 ºC. The latter two temperatures were chosen as they represent the typical range in the inner surface of continuous casting CuAg moulds operating in service. At each temperature, smooth cylindrical specimens were loaded under strain-controlled fully-reversed cycles with different values of strain amplitude, ε_a. Tests were interrupted before the complete failure of the specimen, and the number of reversals to failure $2N_f$ was defined by an 80% drop in the maximum stress recorded in the test. Due to an anomalous behaviour of some specimens in the fatigue tests, a relatively small number of fatigue data were available for subsequent analysis: 8 points at room temperature, 7 points at both 250 °C and 300 °C. Details on the experiments and equipment are given elsewhere.[1,10]

3. Strain-life fatigue curves

The results of LCF tests provide the necessary information for estimating parameters of the strain-life equation, which relates the amplitude of total strain ε_a to the number of reversals to failure $2N_f$. Based on Morrow's notation, the strain-life equation (Manson–Coffin curve) is:[11]

$$\varepsilon_{\mathrm{a}} = \varepsilon_{\mathrm{el,a}} + \varepsilon_{\mathrm{pl,a}} = \frac{\sigma'_{\mathrm{f}}}{E}\left(2N_{\mathrm{f}}\right)^{b} + \varepsilon'_{\mathrm{f}}\left(2N_{\mathrm{f}}\right)^{c} \tag{1}$$

where $\varepsilon_{el,a}$ and $\varepsilon_{pl,a}$ are the amplitudes of elastic and plastic strain, respectively. Other symbols in (1) are σ'_f = fatigue strength coefficient, b = fatigue strength exponent, ε'_f = fatigue ductility coefficient, c = fatigue ductility exponent, and E = elasticity modulus.

Equation (1) is represented by two straight lines in a log-log diagram. The intercepts of the two lines at $2N_f$=1.0 are (σ'_f/E) for the elastic component, and ε'_f for the plastic component.

The slopes of the elastic and plastic line are b and c, respectively. Four coefficients (two intercepts and two slopes) are then needed, which are estimated by curve fitting (regression) applied separately to elastic and plastic experimental data.

3.1. Linear regression and median strain-life equation

Curve fitting through regression analysis is based on the following linear model:

$$y = A + Bx + \delta \tag{2}$$

in which x=log(ε_a) is the statistically independent variable, y=log($2N_f$) the statistically dependent variable, and δ is a random variable that accounts for the scatter of experimental data. Variable δ is assumed to be normally distributed, with zero mean and constant variance σ^2 (i.e. not a function of x), which implies that strain-line data on a log-log plot would have a constant "scatter band" (the regression model is called "homoscedastic"). According to the model (2), at each value of strain amplitude x, the fatigue life y is normally distributed with mean A+Bx and variance σ^2.

The model (2) equally applies to both the elastic and plastic strain-life equations in (1), provided that the appropriate strain amplitude (elastic $\varepsilon_{el,a}$ or plastic $\varepsilon_{pl,a}$) is used to define x. According to a log-transformation from (1) to (2), the intercept A and the slope B take one of the two following expressions (subscript "el" is for elastic strain-life equation, "pl" for the plastic one):

$$\begin{aligned} A_{el} &= -\frac{1}{b}\log\left(\frac{\sigma'_f}{E}\right) \quad ; \quad B_{el} = \frac{1}{b} \\ A_{pl} &= -\frac{1}{c}\log\left(\varepsilon'_f\right) \quad ; \quad B_{pl} = \frac{1}{c} \end{aligned} \tag{3}$$

while σ^2_{el} and σ^2_{pl} denote the variance in the elastic and plastic strain-life model.

The parameters A, B and σ^2 of the model (2) are not known in advance, but can be estimated from n experimental data ($\varepsilon_{a,i}$, $2N_{f,i}$) i=1,…n. For each value of $2N_{f,i}$, the total strain amplitude $\varepsilon_{a,i}$ needs to be further separated into elastic $\varepsilon_{el,a,i}$ and plastic $\varepsilon_{pl,a,i}$ parts. A log-transformation provides the pairs x_i=log($\varepsilon_{el,a,i}$), y_i=log($2N_{f,i}$) when fitting the elastic strain-life line, and the pairs x_i=log($\varepsilon_{pl,a,i}$), y_i=log($2N_{f,i}$) when fitting the plastic one. The least-squares estimators of parameters in (3), marked with the symbol ^, are:[12,13]

$$\hat{A} = \bar{y} - \hat{B}\bar{x} \qquad \hat{B} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad s^2 = \frac{1}{n-2}\sum_{i=1}^{n}\left[y_i - (\hat{A} + \hat{B}x_i)\right]^2 \tag{4}$$

where $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$ and $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ are the sample means of x_i and y_i. The sample variance s^2 is the unbiased estimator of σ^2. As they are obtained from a finite set of data, the estimators (4) generally deviate from the "true" parameters in (2). The deviation depends on the particular sample considered and becomes smaller as the sample size n increases (the "true" values would be obtained in the limit $n \rightarrow \infty$). The statistical uncertainty of estimators is quantified by a confidence level of estimations.[13]

Expressions (4), separately applied to both elastic and plastic strain-life data, provide the estimated regression parameters (intercept $\hat{A}$ and slope $\hat{B}$) which define the median ("average") line at 50% of survival probability:

$$\hat{y} = \log(2\hat{N}_{\mathrm{f}}) = \hat{A} + \hat{B}\,x \tag{5}$$

If the median strain-life line is written as in Eq. (1), the intercepts and slopes of the elastic and plastic components can be recovered by inverting Eq. (3):

$$\begin{aligned} \frac{\hat{\sigma}'_{\mathrm{f}}}{E} &= 10^{\left(-\hat{A}_{\mathrm{el}}/\hat{B}_{\mathrm{el}}\right)} \quad ; \quad \hat{b} = \frac{1}{\hat{B}_{\mathrm{el}}} \\ \hat{\varepsilon}'_{\mathrm{f}} &= 10^{\left(-\hat{A}_{\mathrm{pl}}/\hat{B}_{\mathrm{pl}}\right)} \quad ; \quad \hat{c} = \frac{1}{\hat{B}_{\mathrm{pl}}} \end{aligned} \tag{6}$$

The values of the intercept and slope for the median strain-life lines at RT, 250 °C and 300 °C are summarised in Table 1. The regression analysis also returned, for each temperature, the values of the sample standard deviation for the elastic ($s_{\mathrm{el,RT}}$=0.1728, $s_{\mathrm{el,250}}$=0.1952, $s_{\mathrm{el,300}}$=0.1172) and plastic ($s_{\mathrm{pl,RT}}$=0.2241, $s_{\mathrm{pl,250}}$=0.1276, $s_{\mathrm{el,300}}$=0.1218) strain-life curves.

For example, Figure 1(a) shows the median strain-life equation fitted to experimental data at room temperature (RT). The figure plots the lines for total, elastic and plastic strain amplitudes. The regression lines for elastic and plastic strain-life equations show a good agreement with experimental data. A similar fit is also observed for fatigue data at 250 °C and 300 °C (not shown).

Figure 1(b) compares the median strain-life equations at different temperatures. The two curves at 250 and 300 °C are close together, as it may be expected from tests obtained at similar temperatures. Such curves are not simply shifted, but intersect at around $2N_f > 10^4$. Surprisingly, an increase from RT to 250-300 °C does not cause the fatigue life to decrease over the whole range of strain amplitudes. In fact, the RT curve also intersects at 10^4 reversals to failure. For example, only at strain amplitudes below $\varepsilon_a = 3\cdot 10^{-3}$ (in the region $2N_f > 10^4$) is the fatigue life at 250 °C and 300 °C lower than the life at RT. Instead, at strain amplitudes higher than $\varepsilon_a = 3\cdot 10^{-3}$, the lowest fatigue life is observed at RT.

3.2. Approximate model: the Universal Slopes Equation

Low-cycle fatigue experiments are generally lengthy and costly, thus several investigators have attempted to devise approximate methods for estimating the strain-life curves from monotonic tensile properties, which are easier to obtain. One of the early attempts is the Universal Slopes Equation (USE):[11,14,18]

$$\Delta\varepsilon = \Delta\varepsilon_{\mathrm{el}} + \Delta\varepsilon_{\mathrm{pl}} = 3.5\frac{\sigma_{\mathrm{uts}}}{E} N_{\mathrm{f}}^{-0.12} + D^{0.6} N_{\mathrm{f}}^{-0.6} \qquad (7)$$

which correlates the strain-life curve to the modulus of elasticity E, ultimate tensile strength σ_{uts}, and ductility $D = \ln[100/(100 - \%RA)]$. Ductility is an indirect measure of the area reduction $\%RA$ in the specimen cross section. The name of the model comes from the constant slopes (–0.12 and –0.6) assumed for the elastic and plastic lines, regardless of material type. The tensile properties in (7) need to be evaluated at the temperature at which the strain-life equation is estimated.[14] Although the model (7) was defined for strain range $\Delta\varepsilon$ and cycles to failure N_f, the conversion to strain amplitude and reversals to failure is straightforward.

Equation (7) was originally correlated to fatigue data for 29 ferrous and non-ferrous materials (except copper alloys), covering a wide range of strength and ductility. Later, a wider set of material properties became available for best-fitting a modified equation, which resulted in slightly different coefficients and slopes. After Manson's pioneering approach, over the years other investigators have proposed other fitting models.[15,16,17]

Owing to its simplicity and widespread use, only the original USE model is considered here and an attempt is made to compare it with the CuAg fatigue data analysed in this study. Table 3 lists the tensile properties used to estimate the USE in (7).

Figure 2(a)-(c) compare the median strain-life curves (in elastic, plastic and total strain amplitude) with the approximation by USE. The comparison shows that, for RT data, the USE is not conservative in the region $2N_f<2\cdot10^4$ and slightly less conservative elsewhere. On the other hand, it provides a much better estimate of the overall strain-life curve for data at 250 °C and 300 °C.

Figure 2(d) compares the total strain-life curves as estimated by the USE model. The USE curves at 250 °C and 300° almost overlap, whereas the curve at room temperature is shifted to the right (i.e. according to the USE model, at room temperature the material always exhibits a higher fatigue strength, contrary to what is shown in Figure 1(b)).

Although quite promising, the observed agreement (especially for high temperature data) does not provide a definitive proof that the USE also represents a good approximation for copper alloys. Although the results seem encouraging, more experimental data, if possible extended to other copper alloys, are needed to further support the preliminary conclusions obtained here. It should also be emphasised that Universal Slopes should be viewed as a useful tool for screening materials, but cannot replace actual laboratory fatigue tests, in which stress and temperature histories expected in service may also be included.[18]

4. Design strain-life curves: methods

The median strain-life curve (5), fitted by regression, gives an average fatigue life $\hat{y}=\log(2\hat{N}_\mathrm{f})$ associated with a 50% failure probability, which means that a future observation will fall below $\hat{y}$ half of the time. This clearly makes the median curve unsafe in design. A lower bound curve shifted to the left of median is needed so that fatigue failures would occur with a much lower probability.

In addition, due to the finite size n of the experimental data used in regression analysis, the parameters $\hat{A}$, $\hat{B}$ and therefore the median line are also characterised by a statistical scatter. For instance, if a different sample were used in regression, slightly different estimates would be obtained (the uncertainty is negligible only in the limit $n\rightarrow\infty$). The scatter increases as long as the number of data used in regression is relatively small (<10), as with the CuAg data available in this study. A confidence level on the estimated parameters thus needs to be quantified.

A design (or characteristic) strain-life curve[b], targeting a failure probability α and confidence γ, is defined as[12]:

$$\hat{y}_{\mathrm{d}} = \hat{y} - K_{\alpha,\gamma,\mathrm{n},\mathbf{x},\mathrm{x}} \cdot s \tag{8}$$

where $\hat{y}$ is the median life from Eq. (5), s is the sample standard deviation from Eq. (4), and $K_{\alpha,\gamma,\mathrm{n},\mathbf{x},\mathrm{x}}$ is a correction factor that establishes the distance on the left-hand side from the median. In expression (8), $\hat{y}_{\mathrm{d}} = \log(2\hat{N}_{\mathrm{f,d}})$ relates to the number of reversals to failure $2\hat{N}_{\mathrm{f,d}}$.

In the most general cases, factor $K_{\alpha,\gamma,\mathrm{n},\mathbf{x},\mathrm{x}}$ depends on probability α and confidence level γ, on the particular strain value x at which life y_{d} is computed, as well as on the number n, and the specific amplitude values $\mathbf{x}=(x_1, x_2, \ldots, x_n)$ used in testing.

Being a function of x, factor $K_{\alpha,\gamma,\mathrm{n},\mathbf{x},\mathrm{x}}$ is not constant over the strain amplitude values, which thus gives a "hyperbolic" strain-life curve at a variable distance from the median. Since "hyperbolic" curves may be difficult to obtain, especially by non-specialists, a straight line approximation (assuming K=const.) is often preferred. This provides ready-to-use equations for calculating the strain-life parameters, with only a small loss in confidence compared to the exact statistical solution.[19]

By combining Eqs. (5) and (8), the design fatigue line is written as $\hat{y}_{\mathrm{d}} = \hat{A}_{\mathrm{d}} + \hat{B}_{\mathrm{d}}\, x$, where $\hat{B}_{\mathrm{d}} = \hat{B}$ is the slope (identical to that of the regression line) and $\hat{A}_{\mathrm{d}} = \hat{A} - K_{\alpha,\gamma,\mathrm{n},\mathbf{x}} \cdot s$ the new intercept (which is a constant, if K=const.). The intercept $\hat{A}_{\mathrm{d}}$ depends on K, which in turn is a function of the particular statistical method adopted. There are two intercepts and slopes for the elastic line ($\hat{A}_{\mathrm{el,d}}$, $\hat{B}_{\mathrm{el,d}}$) and plastic line ($\hat{A}_{\mathrm{pl,d}}$, $\hat{B}_{\mathrm{pl,d}}$). Unlike $\hat{A}_{\mathrm{d}}$, the slope remains unchanged for any K (i.e. all design lines share the same slope of the regression line). By analogy with Eq. (6), the following transformations:

$$\begin{aligned} \left(\frac{\hat{\sigma}'_{\mathrm{f}}}{E}\right)_{\mathrm{d}} &= 10^{\left(-\frac{\hat{A}_{\mathrm{el}} - K\cdot s_{\mathrm{el}}}{\hat{B}_{\mathrm{el}}}\right)} \quad ; \quad \hat{b}_{\mathrm{d}} = \hat{b} \\ \left(\hat{\varepsilon}'_{\mathrm{f}}\right)_{\mathrm{d}} &= 10^{\left(-\frac{\hat{A}_{\mathrm{pl}} - K\cdot s_{\mathrm{pl}}}{\hat{B}_{\mathrm{pl}}}\right)} \quad ; \quad \hat{c}_{\mathrm{d}} = \hat{c} \end{aligned} \tag{9}$$

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[b] the term "design curve" is often used to designate a characteristic curve factored by a safety factor.

are used to convert the intercepts and slopes of the elastic and plastic lines into the coefficients $(\sigma'_f/E)_d$, $(\varepsilon'_f)_d$, b_d, c_d of the design fatigue curve, written in Morrow's notation as:

$$\varepsilon_a = \varepsilon_{el,a} + \varepsilon_{pl,a} = \left(\frac{\hat{\sigma}'_f}{E}\right)_d (2N_f)^{\hat{b}_d} + (\hat{\varepsilon}'_f)_d (2N_f)^{\hat{c}_d} \tag{10}$$

In previous expressions, subscript "d" is used to denote the parameters of the design strain-life curve.

Several approaches (deterministic, tolerance interval, prediction interval) are reviewed in the next sections and then applied to the experimental data. Of these, the confidence band approach (see ASTM code[20]) is not considered, as it only provides a statement on the unknown "true" median curve (regardless of the prescribed probability α) and is thus not suitable for defining a design fatigue curve.

The next section briefly reviews each statistical method and the main statistical principles behind each one. The references cited provide more theoretical details and also apply each method to other sets of experimental data.

4.1. Deterministic approach (lower 2-sigma or 3-sigma)

This method neglects the statistical uncertainty of regression estimators (4) and postulates that $\hat{A}$, $\hat{B}$ and s^2 coincide with the "true" values A, B and σ^2. In the "lower 2-sigma" or "lower 3-sigma" approach, $\sigma=s$ and K=2 or 3, so that the design strain-life line is shifted to the left of the median by 2 or 3 standard deviations.[19] Another version of the method selects K based on a prescribed failure probability α, so that the design line is:[19]

$$\hat{y}_{det} = \hat{y} - z_{1-\alpha} s \tag{11}$$

where $z_{1-\alpha}=\Phi^{-1}(1-\alpha)$ is the quantile of the standard normal cumulative distribution function Φ(-). The values of $z_{1-\alpha}$ are tabulated; for example $z_{0.95}$=1.6449 by choosing α=5%.

The deterministic method would be correct only if the "true" parameters A, B, σ were known exactly, which only occurs in the limit case of infinitely large samples ($n\to\infty$). For small samples, the method is unsafe because it neglects the confidence γ of the estimated parameters,[19] which in fact does not appear in (11).

4.2. Tolerance interval

Using the parameters estimated by regression, this method defines an interval that bounds a selected portion of the probability distribution of the fatigue life $2N_f$.

Several methods can be applied. For example, the one-dimensional tolerance limit assumes that samples are tested at a single level x of strain amplitude. The K factor in (8) is computed from a univariate probability distribution. This method is thus approximate when applied to the regression case, where multiple values of x and y are used. In this case, the tolerance interval for regression (Owen's tolerance factor) is the most appropriate approach,[19] although it is not straightforward. For this reason, an approximation based on closed-form expressions has been proposed in Ref. [21].

4.2.1. One-dimensional (1D) tolerance limit

This method computes the K factor from the one-sided tolerance limit of a univariate normal distribution for y, so that the design fatigue line is:[12,19]

$$\hat{y}_{1D} = \hat{y} - K_{\alpha,\gamma,n}\, s \tag{12}$$

where $K_{\alpha,\gamma,n}$ only depends on failure probability α, confidence level γ and sample size n. The values of $K_{\alpha,\gamma,n}$ are tabulated;[22,23] for example, by choosing α=1%, γ=95% and n=8, it results $K_{0.01,0.95,8}$=4.354.

This method guarantees that, γ% of time, the actual fatigue life is lower than $\hat{y}_{1D}$ with a probability equal to or lower than α. The design fatigue line (12) is straight, since $K_{\alpha,\gamma,n}$ is constant and thus independent of x. This also implies that the method is approximate, as it assumes that samples are tested at a fixed value x of strain amplitude. In the regression case, (where both x and y are variables), the method described in the next section is preferable.

4.2.2. Tolerance limit in regression case and Owen's approximation

This method considers a factor $K_{\alpha,\gamma,n,\mathbf{x},x}$ that depends on both the actual strain amplitude x and the specific values $\mathbf{x}=(x_1, x_2, \ldots, x_n)$ used in fatigue tests, according to the definition (8). Given that it is a function of x, this method gives a "hyperbolic" design strain-life curve. Details can be found, for example, in Ref. [19].

A straight-line approximation (with a constant K given by a $K_{\alpha,\gamma,n,\mathbf{x},x}$ averaged over the x range of interest) has been proposed in Ref. [21] to overcome the practical difficulty of using the tolerance limit for the regression case. This approximation, however, results in a lower confidence at the extremes value of x used in experiments. In Owen's approximation, the design strain-life line is:

$$\hat{y}_{\text{Owen}} = \hat{y} - K_{\text{Owen}}\, s \tag{13}$$

where K_{Owen} is computed from the following expressions (for details, see Ref.[19]):

$$\begin{aligned} &K_{\text{Owen}} = K_D R_{\text{Owen}} \quad ; \quad K_D = c_1 K_{1\text{-}\alpha} + K_\gamma \sqrt{c_3 K_{1\text{-}\alpha}^2 + c_2 a} \\ &R_{\text{Owen}} = b_1 + \frac{b_2}{f^{b_3}} + b_4 \exp(-f), \quad \text{for} \quad f \geq 3 \\ &K_{1\text{-}\alpha} = \Phi^{-1}(1-\alpha) \quad ; \quad K_\gamma = \Phi^{-1}(\gamma) \\ &f = n-2 \quad ; \quad a = \frac{1.82}{n} \end{aligned} \tag{14}$$

Appendix A reports the values of coefficients b_i and c_i, as given in Ref. [19,23,24]. Based on these values, inserted in expressions (14), factor K_{Owen} has been computed for different levels of probability α and confidence γ, see Ref. [23,24]. Note that these references list different values of K_{Owen} for a confidence γ=90% and probability α=1%, 5%, 10%. The correct values are those of Ref. [23].

4.3. Prediction interval

This method defines a band that encloses, with a given confidence, a new value of fatigue life observed in a future experiment. The band then accounts for both the uncertainty of regression estimators and the scatter associated with the future observed fatigue life (this explains why the prediction band is always wider than the confidence band).

4.3.1. Prediction interval for regression analysis

In this approach, the design strain-life curve is: [13,25]

$$\hat{y}_{\text{pred}} = \hat{y} - K_{\text{pred}} \cdot s \quad ; \quad K_{\text{pred}} = t_{1-\alpha;n\text{-}2} \sqrt{1 + \frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}} \tag{15}$$

in which $t_{1\text{-}\alpha;n\text{-}2}$ is the quantile at probability (1-α) from the Student t-distribution with n-2 degrees of freedom. The values of $t_{1\text{-}\alpha;n\text{-}2}$ are tabulated;[13] for example, with n=10 and by choosing α=5%, it results $t_{0.95;8}$=1.8595.

Expression (15) depends on x and gives an "hyperbolic" design curve $\hat{y}_{\text{pred}}$, which is closest to the median at $x=\bar{x}$. This is interpreted as a chance α that the new fatigue life at level x would fall below $\hat{y}_{\text{pred}}$. When $n \rightarrow \infty$, it results $t_{1-\alpha;n-2} \rightarrow z_{1-\alpha}$ and the design curve (15) becomes straight, approaching the straight line (11) by the deterministic method.

Equation (15) will not be applied. In fact, providing a hyperbolic design fatigue curve that depends on x, is not easy to be handled for design purposes as it does not permit constant coefficients in Eq. (10) to be obtained. Nevertheless, Eq. (15) is reported as it is helpful in explaining the approximation behind the EPI method described in the next section.

4.3.2. Equivalent Prediction Interval (EPI)

This is an approximate method that attempts to further simplify the prediction interval in expression (15). It assumes that the uncertainty in fatigue life y follows a normal distribution with mean $\hat{y}$ and standard deviation σ_e (in fact, Eq. (15) shows that $\hat{y}$ follows a Student *t*-distribution). For this standard deviation σ_e, the deterministic method would compute a straight design line as $\hat{y}_{\text{Det,eq}} = \hat{y} - z_{1-\alpha}\sigma_{\text{e}}$. By comparing Eq. (11) and (15), the standard deviation σ_e can be determined as:

$$\sigma_{\text{e}} = s\left(\frac{t_{1-\alpha;\,\text{n-2}}}{z_{1-\alpha}}\right)\sqrt{1+\frac{1}{n}+\frac{(x-\bar{x})^2}{\sum_{\text{i}=1}^{n}\left(x_{\text{i}}-\bar{x}\right)^2}} \tag{16}$$

Given that it is a function of x, σ_e does not seem very practical. Ref. [25], shows how the design curve (15) becomes relatively flat when $n \geq 15$. The idea was thus to approximate the prediction interval curve (15) with a straight line, obtained by taking for y an equivalent and constant standard deviation σ_0 over the whole range of x values.[25] The band defined by this approximating straight line is called "equivalent prediction interval" (EPI).

The equivalent standard deviation was expressed as $\sigma_0 = s \cdot g_{\alpha,\text{n}}$, where s is the sample standard deviation from regression analysis (4) and $g_{\alpha,\text{n}}$ is an adjustment factor that accounts for the error in the estimates $\hat{A}$, $\hat{B}$ and s, which is determined as:[25]

$$
\begin{aligned}
g_{\alpha,\mathrm{n}} &= \exp\left[\,\Lambda(\alpha)\{\ln n\}^{-\Psi(\alpha)}\,\right] \\
\Lambda(\alpha) &= 1.56\left[\tanh^{-1}(1-\alpha)\right]^{1.12} \\
\Psi(\alpha) &= 3.32-1.7\alpha
\end{aligned}
\tag{17}
$$

Expressions (17) were fitted by numerical simulations and are only applicable for values $6 \le n \le 50$ and $1\% \le \alpha \le 15\%$.[25]

5. Results

The statistical methods described in the previous sections were applied to estimate the coefficients $(\sigma'_f/E)_d$, $(\varepsilon'_f)_d$, b_d, c_d of the design strain-life curve in Eq. (10), with reference to a failure probability of α=5% and confidence γ=90% (when applicable). Results for each temperature are listed in Table 2, which also reports the *K* factor values. Note that *K* depends on the sample size *n*, except for the deterministic method.

Figure 3 shows the average regression line (for plastic strain only) compared to several design lines, resulting from the coefficients in Table 2. The figure refers to data at room temperature and 250 °C. A similar comparison is shown in Figure 4 for the elastic lines.

In Figure 3, the tolerance interval for regression (Owen's method) and the univariate (1D) tolerance interval provide the most conservative design lines, i.e. those on the very left. The EPI method provides the strain-life line immediately to the right, while as expected, the deterministic method provides the least conservative design line, i.e. it is closest to the regression line. Note that the EPI curve is independent of the assigned confidence value, whereas the tolerance interval curves would shift to the right if the confidence level decreased. The same relative positioning of design lines can also be seen in Figure 4 for the elastic strain-life lines, although the lines are now more closely spaced.

A comparison of the design curves for total strain amplitude is presented in Figure 5. These curves permit the design strain amplitude at the prescribed fatigue life to be directly determined.

In fact, in design it is often of interest to know the allowable value of total strain amplitude $(\varepsilon_a)_d$ at a prescribed fatigue life $(2N_f)_d$. Table 4 compares, for each statistical method, the values of design amplitude $(\varepsilon_a)_d$ corresponding to $(2N_f)_d=10^5$ reversals to failure, for a failure probability of α=5% and 1% (with confidence γ=90%). For convenience, the values of $(\varepsilon_a)_d$ are scaled by 10^6 and thus given in $\mu\varepsilon$ (microstrain).

As expected, the design strain amplitude $(\varepsilon_a)_d$ decreases as the failure probability takes values lower than 50%. The downward shift of $(\varepsilon_a)_d$ also depends on the statistical method used to define the design line.

The amplitude $(\varepsilon_a)_d$ read on each design curves results to be several percentage points lower than the mean value provided by the regression curve. At room temperature, for example, Owen's method (the most conservative) gives (for probability 5%) a strain amplitude of 1217 με, which is about 28% lower than 1697με on the median curve – this difference being as high as 35% for a failure probability 1%. A less marked decrease is observed for the other methods, which are less conservative than Owen's approach. Similar trends are also observed at higher temperatures.

6. Conclusions

This work estimated the strain-life curves of a CuAg alloy based on isothermal low-cycle fatigue tests at three temperatures (room temperature, 250 °C, 300 °C). The literature is lacking in low-cycle fatigue experimental data, despite this alloy being widely used in several thermo-mechanical applications, particularly in steel-making plants. A regression analysis was first used to estimate the median strain-life curves (50% survival probability). An attempt was also made to compare the median curve with the Universal Slopes Equation (USE) model, calibrated on monotonic tensile properties. A satisfactory agreement was observed for curves at 250 °C and 300 °C, while a larger deviation characterised room temperature data.

Although quite promising, these results cannot be generalised. Additional experimental fatigue data on several types of copper alloys are needed to confirm the validity of the USE model for copper alloys as well. In this case, the USE model would be useful for an initial estimate of strain-life curves especially in the early design stage, when experimental low-cycle fatigue data are not usually available.

In the second part of the work, several statistical approximate models (e.g. "Equivalent Prediction Interval", univariate tolerance interval, Owen's tolerance interval for regression) were used for estimating design strain-life curves at a prescribed value of survival probability and confidence. The determination of design curves was particularly important in this study, given the relatively small number of experimental fatigue data, which made the statistical scatter not at all negligible.

An overall comparison confirmed that the strain-life curves given by the tolerance interval approach were the most conservative (i.e. the most shifted to the left from the median fatigue curve). By contrast, the strain-life curve estimated by the deterministic approach was the least conservative. Having obtained the design strain-life curves, it was straightforward to determine the design strain amplitude at $(2N_f)_d=10^5$ reversals to failure, for assigned values of probability (α=5% and 1%) and confidence γ=90%. For the room temperature data, the most conservative method (Owen's approach) gives a design strain amplitude at 5% probability, which is 28% lower than the median strain amplitude value, the reduction being even higher (35%) for a lower failure probability of α=1%. Comparable reductions were also obtained for high temperature strain-life data.

These results thus suggest that the use of design fatigue curves is recommended even for low-cycle fatigue data, especially when small samples with only high statistical scatter are available. The proposed procedure, which enables the durability analysis of thermo-mechanically loaded components to rely on design fatigue curves, can be easily extended to other materials (especially Cu alloys used in steel-making plants), if experimental data are available.

7. Appendix A – Parameters of Owen's approximate tolerance limit approach

Table A1. Empirical coefficients b_i (i=1,2,3,4) for K_{Owen} (from Ref. [19,21]).

Confidence level γ	b_1	b_2	b_3	b_4
0.95	0.9968	0.1596	0.60	−2.636
0.90	1.0030	−6.0160	3.00	1.099
0.85	1.0010	−0.7212	1.50	−1.486
0.80	1.0010	−0.6370	1.25	−1.554

Table A2. Empirical coefficients c_i (i=1,2,3) for K_{Owen} (from Ref. [19,21]).

	c_1	c_2	c_3
$f<2$	1	1	$\frac{1}{2f}$
$f\geq 2$	$1+\frac{3}{4(f-1.042)}$	$\frac{f}{f-2}$	$c_2-c_1^2$

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TABLES

Table 1. Strain-life parameters estimated by linear regression analysis (failure probability α=50%).

Temperature	Parameters of strain-life equation			
	$\hat{\sigma}'_f / E$	$\hat{b}$	$\hat{\varepsilon}'_f$	$\hat{c}$
RT	0.00311	-0.1065	0.09485	-0.4167
250 °C	0.00264	-0.1133	0.36665	-0.5551
300 °C	0.00244	-0.1125	0.57468	-0.6035

Table 2. Parameters of design strain-life curves estimated by various statistical methods (α=failure probability, γ = confidence).

Temperature	Statistical method	Parameters of strain-life equation				
		K	$(\hat{\sigma}'_{\mathrm{f}}/E)_{\mathrm{d}}$	$\hat{b}_{\mathrm{d}}$	$(\hat{\varepsilon}'_{\mathrm{f}})_{\mathrm{d}}$	$\hat{c}_{\mathrm{d}}$
RT	Deterministic approach (α=5%)	1.6449	0.00290	-0.1065	0.06660	-0.4167
	EPI (α=5%, *n*=8)	2.1932	0.00284	-0.1065	0.05919	-0.4167
	1D tolerance interval (α=5%, γ=90%, *n*=8)	2.7540	0.00277	-0.1065	0.05247	-0.4167
	Owen's method (α=5%, γ=90%, *n*=8)	2.9860	0.00274	-0.1065	0.04991	-0.4167
250 °C	Deterministic approach (α=5%)	1.6449	0.00243	-0.1133	0.28040	-0.5551
	EPI (α=5%, *n*=7)	2.3497	0.00234	-0.1133	0.24996	-0.5551
	1D tolerance interval (α=5%, γ=90%, *n*=7)	2.8940	0.00228	-0.1133	0.22873	-0.5551
	Owen's method (α=5%, γ=90%, *n*=7)	3.1888	0.00225	-0.1133	0.21800	-0.5551
300 °C	Deterministic approach (α=5%)	1.6449	0.00232	-0.1125	0.43503	-0.6035
	EPI (α=5%, *n*=7)	2.3497	0.00227	-0.1125	0.38611	-0.6035
	1D tolerance interval (α=5%, γ=90%, *n*=7)	2.8940	0.00223	-0.1125	0.35213	-0.6035
	Owen's method (α=5%, γ=90%, *n*=7)	3.1888	0.00221	-0.1125	0.33499	-0.6035

Table 3. Tensile parameters used to estimate the Universal Slopes Equation model.

Temperature	Tensile properties			
	E [MPa]	σ_{uts} [MPa]	*%RA*	*D*
RT	118692	234	78.5	1.54
250 °C	92196	176	64.7	1.04
300 °C	80662	158	64.6	1.03

Table 4. Comparison of design strain amplitudes (at $2N_f$=2×10^5 reversals to failure) estimated by different methods at different failure probability α (when applicable, confidence γ=90%).

Temperature	**Statistical method**	**Design strain amplitude $(\varepsilon_a)_d \times 10^6$**	
		α=5%	α=1%
RT	Linear regression (α=50%)	1697	1697
	Deterministic approach	1402	1303
	EPI	1321	1155
	1D tolerance interval	1246	1126
	Owen's method	1217	1098
250 °C	Linear regression (α=50%)	1333	1333
	Deterministic approach	1130	1058
	EPI	1056	915
	1D tolerance interval	1003	908
	Owen's method	976	880
300 °C	Linear regression (α=50%)	1220	1220
	Deterministic approach	1054	995
	EPI	993	879
	1D tolerance interval	950	874
	Owen's method	928	852

FIGURE CAPTIONS

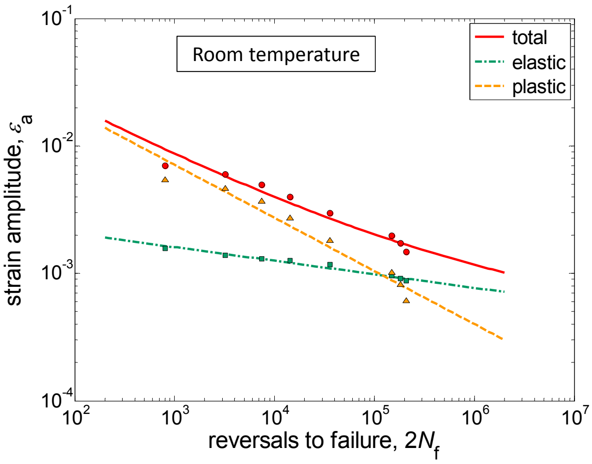
Figure 1. (a) Median strain-life curves (elastic, plastic, and total strain) fitted to experimental fatigue data at RT; (b) comparison of experimental data and median strain-life curves at RT, 250 °C and 300 °C.

Figure 2. Comparison between regression strain-life curves and Universal Slopes Equation curves: (a) room temperature; (b) 250 °C; (c) 300 °C.

Figure 3. Comparison between regression line (plastic strain) and design strain-life lines estimated by different methods: (a) room temperature; (b) 250 °C. Probability α=5%, confidence γ=90%.

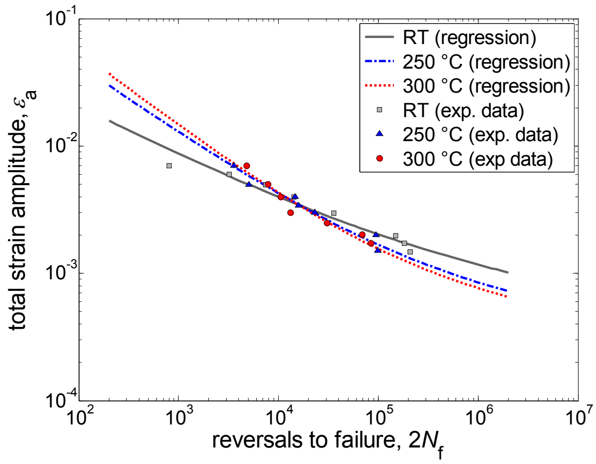
Figure 4. Comparison between regression line (elastic strain) and design strain-life lines estimated by different methods: (a) room temperature; (b) 250 °C. Probability α=5%, confidence γ=90%.

Figure 5. Comparison between regression line (total strain) and design strain-life lines estimated by different methods: (a) room temperature; (b) 250 °C. Probability α=5%, confidence γ=90%.



Room temperature
total
elastic
plastic
strain amplitude, ε_a
reversals to failure, $2N_f$

Fig. 1(a)



RT (regression)
250 °C (regression)
300 °C (regression)
RT (exp. data)
250 °C (exp. data)
300 °C (exp data)
total strain amplitude, ε_a
reversals to failure, $2N_f$

Fig. 1(b)

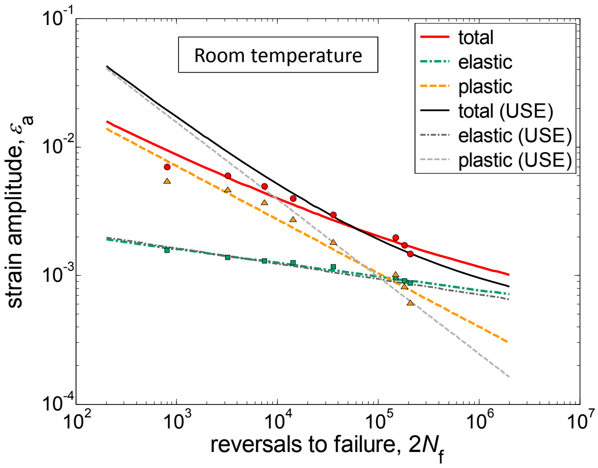


Fig. 2(a)

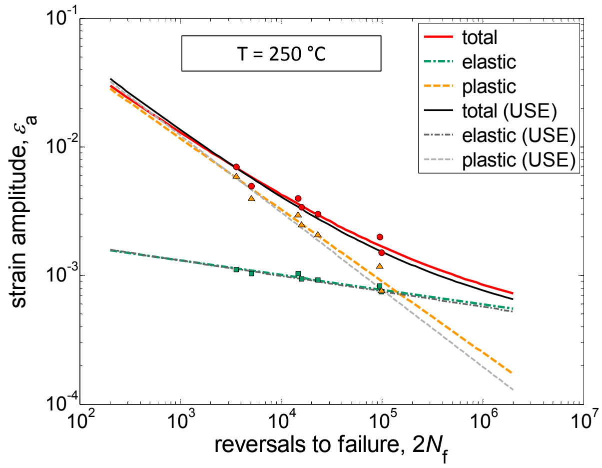


Fig. 2(b)

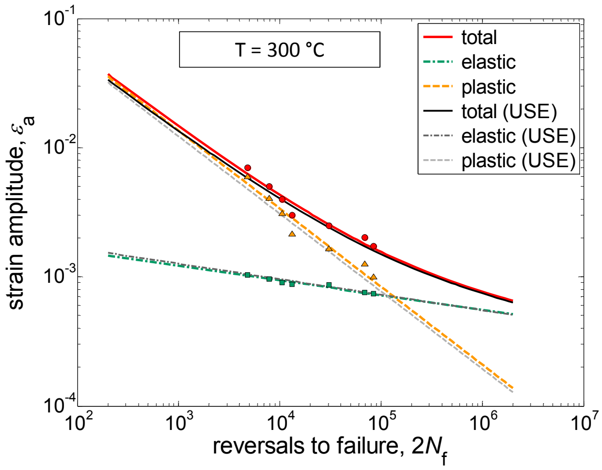


Fig. 2(c)

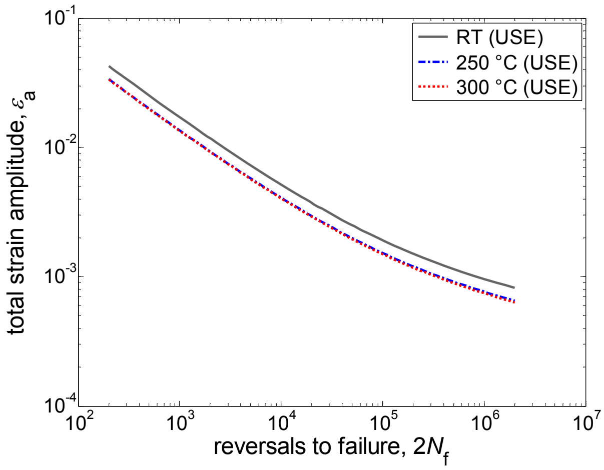


Fig. 2(d)

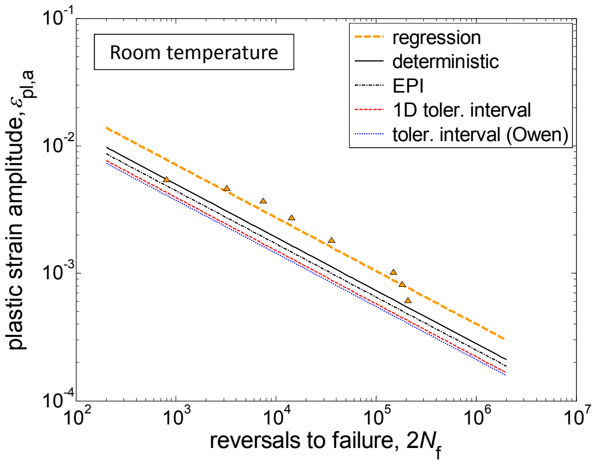

Room temperature
regression
deterministic
EPI
1D toler. interval
toler. interval (Owen)
plastic strain amplitude, $\varepsilon_{pl,a}$
reversals to failure, $2N_f$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}

Fig. 3(a)

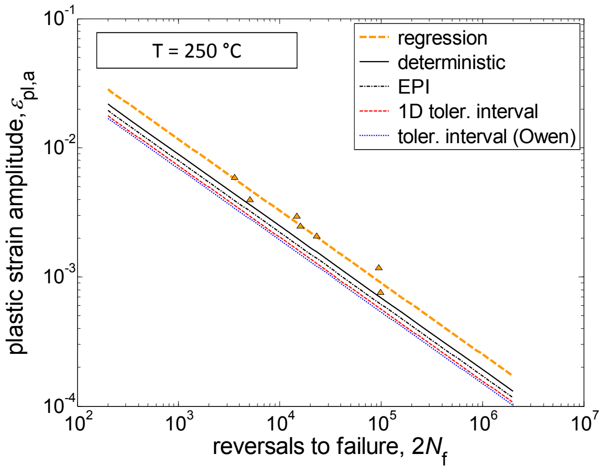
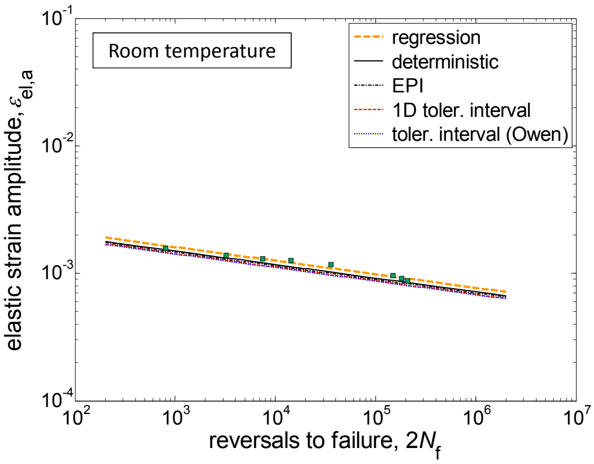
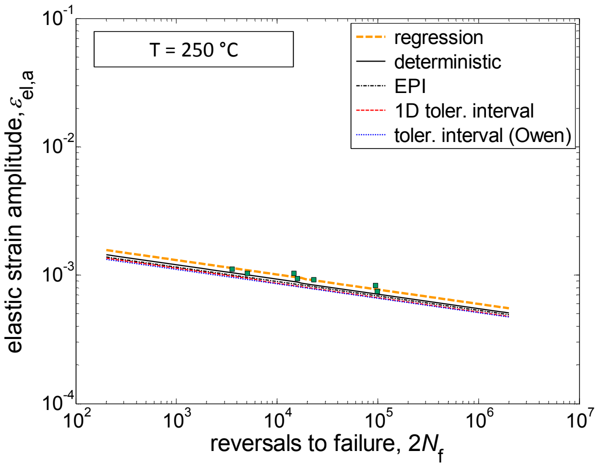

T = 250 °C
regression
deterministic
EPI
1D toler. interval
toler. interval (Owen)
plastic strain amplitude, $\varepsilon_{pl,a}$
reversals to failure, $2N_f$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}

Fig. 3(b)



Room temperature
regression
deterministic
EPI
1D toler. interval
toler. interval (Owen)
elastic strain amplitude, $\varepsilon_{el,a}$
reversals to failure, $2N_f$

Fig. 4(a)



T = 250 °C
regression
deterministic
EPI
1D toler. interval
toler. interval (Owen)
elastic strain amplitude, $\varepsilon_{el,a}$
reversals to failure, $2N_f$

Fig. 4(b)

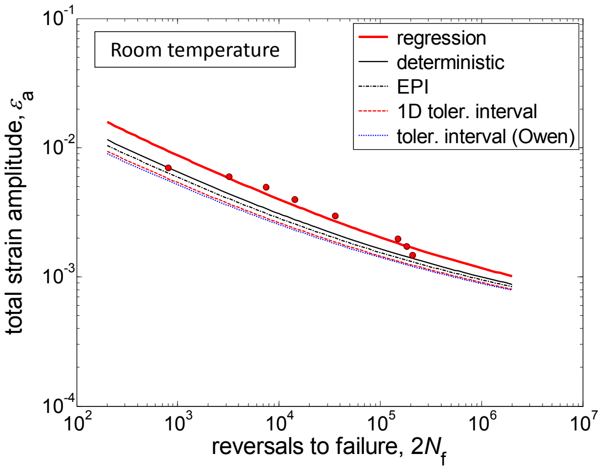

Room temperature
regression
deterministic
EPI
1D toler. interval
toler. interval (Owen)
total strain amplitude, ε_a
reversals to failure, $2N_f$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}

Fig. 5(a)

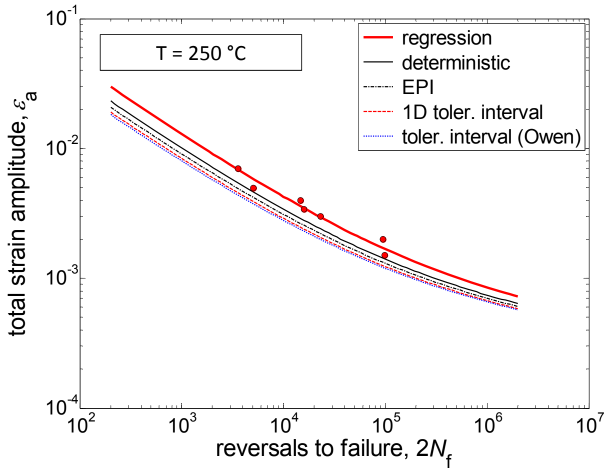

T = 250 °C
regression
deterministic
EPI
1D toler. interval
toler. interval (Owen)
total strain amplitude, ε_a
reversals to failure, $2N_f$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
10^{7}

Fig. 5(b)