*Technical note*

# Fatigue damage assessment in wide-band uniaxial random loadings by PSD decomposition: outcomes from recent research

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**Abstract**

Spectral decomposition of a PSD into narrow frequency bands has been suggested as a promising way for estimating the fatigue damage of uniaxial wide-band random loadings. The basic idea has been formulated in some recent publications, which also proposed different combination rules to sum up the damage of each narrow frequency band. The purpose of this technical note is to clarify the analogies, relationships and differences among the approaches developed in such publications.

**Keywords**

random loading; Power Spectral Density (PSD); frequency-domain approach; frequency band

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## 1 Introduction

A uniaxial wide-band random loading $x(t)$ can be characterised in the frequency domain by a Power Spectral Density (PSD) function $S(\omega)$. Analytical expressions can be used to estimate the fatigue damage of $x(t)$ directly from the spectral parameters of $S(\omega)$. Over the last decades, several spectral methods have been formulated for stationary Gaussian uniaxial random loadings [1-3] and then extended to multi-axial ones [4,5].

Recently, some papers proposed to split up the PSD of $x(t)$ into narrow frequency bands (*band-splitting*) and to sum up the damage of each band by means of a suitable combination rule, which provided a damage expression for estimating the fatigue damage of $x(t)$. The damage expression followed from the particular combination rule adopted.

For example, Refs. [6,7] used the non-linear combination rule of the "Projection-by-Projection" (PbP) multi-axial criterion [8,9] to derive a damage expression, which coincides with the "empirical" damage formula of the "single moment" (SM) spectral method [10,11]. Other than being a mathematical proof of the SM method, this approach then revealed the analogies existing between uniaxial and multi-axial spectral methods.

More recently, in others papers [12,13] another combination rule, which shifted the central frequencies of each infinitesimal narrow-band PSD, was proposed to define a fatigue damage criterion named *Bands Method*.

The purpose of this technical note is to briefly summarize the method of PSD decomposition (Section 2) and to review two different combination rules (Section 3 and 4) for estimating the fatigue damage of a uniaxial wide-band random loading, as proposed in several recent publications.

## 2 Spectral decomposition of a PSD

Assume that $S(\omega)$ is the one-sided PSD of a uniaxial wide-band random process $x(t)$. Then imagine to divide $S(\omega)$ into frequency bands with narrow (infinitesimal) width $d\omega$, see Figure 1. Each frequency band defines a narrow-band PSD $S_i(\omega)$ centred around the frequency $\omega_i$. Each $S_i(\omega)$ equals $S(\omega)$ within the frequency band centred around $\omega_i$ and it is zero outside.

The total number of PSDs $S_i(\omega)$ depends on the width of frequency bands and on the whole frequency range spanned by $S(\omega)$. If the frequency range of $S(\omega)$ extends from zero to infinite (with no cut-off frequency), then the spectral decomposition transforms $S(\omega)$ into an infinite set of PSDs

$S_i(\omega)$, $i$=1,...,∞. More commonly, a cut-off frequency $\omega_c$ bounds the frequency range of $S(\omega)$, thus the spectral decomposition gives a finite set $S_i(\omega)$, $i$=1,...,$n$.

After performing the spectral decomposition, the original PSD can be reconstructed as $S(\omega)=S_1(\omega)+S_2(\omega)+...+S_n(\omega)$, $0\leq\omega<\omega_c$ , which can also be written as:

$$S(\omega)=\sum_{\mathrm{i}=1}^{n} S_{\mathrm{i}}(\omega)=\mathrm{Trace}(\mathbf{S}_{\mathrm{n}}(\omega)) \tag{1}$$

where Trace(-) is the trace operator and $\boldsymbol{S}_n(\omega)$ is the following $n\times n$ diagonal matrix:

$$\mathbf{S}_{\mathrm{n}}(\omega)=\begin{bmatrix} S_1(\omega) & 0 & 0 & 0 \\ 0 & S_2(\omega) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & S_{\mathrm{n}}(\omega) \end{bmatrix} \tag{2}$$

which collects the narrow-band spectra $S_i(\omega)$, $i$=1,...,$n$ along the main diagonal (all out-of-diagonal elements are all zero).

A narrow-band random process $x_i(t)$ can be associated to each spectrum $S_i(\omega)$. The fatigue damage intensity (damage/s) of each $x_i(t)$ can be computed by the *Rayleigh* formula [14,15]:

$$d_{\mathrm{i}}=\frac{\nu_{0,\mathrm{i}}}{C}\left(\sqrt{2\,\lambda_{0,\mathrm{i}}}\right)^{\mathrm{k}}\Gamma\left(1+\frac{k}{2}\right) \tag{3}$$

where $Var(x(t))=\lambda_{0,i}$ is the variance of $x_i(t)$ and $\nu_{0,\mathrm{i}}=\sqrt{\lambda_2/\lambda_0}/2\pi$ is the frequency of up-ward crossings of the mean. For a narrow-band PSD $S_i(\omega)$, the zero moment is $\lambda_{0,\mathrm{i}}=S_{\mathrm{i}}(\omega)\cdot \mathrm{d}\omega$ and the mean up-crossing rate is $\nu_{0,i}\cong\omega_i/2\pi$. In Eq. (3), symbols $k$, $C$ are the parameters of the S/N line $s^k N=C$.

In [6,7] it was suggested to estimate the damage of $x(t)$ by combining the damage contributed by each narrow-band process $x_i(t)$. Different combination rules can be used, as discussed in the next two Sections.

## 3 Spectral combination by the "Projection-by-Projection" criterion to obtain the "single moment" spectral method

The random processes $x_i(t)$, $i$=1,2,...,∞ are jointly independent, as their PSDs $S_i(\omega)$ are two by two not overlapped. This is further confirmed by the zero out-of-diagonal elements of matrix $\mathbf{S}_n(\omega)$ in Eq. (2), which represent the cross-PSDs functions $S_{ij}(\omega)$ of processes $x_i(t)$ and $x_j(t)$.

The total damage caused by the independent random processes $x_i(t)$, $i$=1,2,...,∞ can be calculated by applying the non-linear summation rule of the "Projection-by-Projection" multi-axial criterion [8,9]:

$$d_{\mathrm{PbP}} = \left( \sum_{i=1}^{5} d_{\mathrm{i}}^{2/\mathrm{k}_{\mathrm{ref}}} \right)^{\frac{\mathrm{k}_{\mathrm{ref}}}{2}} \tag{4}$$

where $d_i$ is the damage of $x_i(t)$ in Eq. (3), which is calculated with the S/N parameters $k_{ref}$, $C_{ref}$ of a "reference S/N line" in the so-called Modified Wöhler diagram (see Ref. [8,9]).

As shown in [6,7], the expression (4) can be rearranged to give the damage expression of the "single moment" spectral method [10,11]:

$$d_{\mathrm{SM}} = \frac{2^{\mathrm{k}_{\mathrm{ref}}/2}}{2\pi C}\, \Gamma\left(1 + \frac{k_{\mathrm{ref}}}{2}\right) \left(\lambda_{2/\mathrm{k}_{\mathrm{ref}}}\right)^{\mathrm{k}_{\mathrm{ref}}/2} \tag{5}$$

This result shows that the fatigue damage of process $x(t)$ estimated by the "single moment" method implies a non-linear summation of damage contributions of narrow frequency bands given by a spectral decomposition of the PSD of $x(t)$.

## 4 Damage evaluation in frequency domain (*Bands Method*)

In 2015, Braccesi et al. [12] formulated a damage evaluation criterion, called *Bands Method,* starting from the hypothesis to have an uniaxial random load condition, that is to have a single PSD function $S(\omega)$, and by adopting the classical *Rayleigh* formula [14,15] and the linear damage rule of Palmgren-Miner.

By considering the spectral decomposition of Figure 1 and equation (1), the authors have suggested defining a common value of frequency $\nu_{0_r}$ (arbitrarily definable) for all the bands, that is a common value of the frequency of up-ward crossings of the mean. By imposing an equivalence between the damage of real $i$-th condition and the equivalent one, it is possible defining to define $\lambda_{0_{r_i}}$, that is the zero order spectral moment for each $i$-th term:

$$\lambda_{0_{r_i}} = \left(\frac{\nu_{0_i}}{\nu_{0_r}}\right)^{2/k} \lambda_{0_i} \tag{6}$$

In the previous equation, $\lambda_{0_i}$ and $\nu_{0_i}$ are, respectively, the zero order spectral moment and the central frequency of the $i$-th band.

By substituting Eq. (6) into Eq. (3), it is possible to obtain the damage of the $i$-th band as follows:

$$d_i = \frac{2^{k/2}}{C}\,\Gamma\left(1+\frac{k}{2}\right)\nu_{0_r}\lambda_{0_{r_i}}{}^{k/2} \tag{7}$$

where symbols $k$, $C$ are the parameters of the S/N line.

The damage value $d_i$, related to the single band, is now a function only of the reference value $\lambda_{0_{r_i}}$ of the zero order spectral moment of the band itself. The zero order moment (*variance*) $\lambda_{0_r}$ of a combination process expressed by Eq. (1) of uncorrelated processes (characterized each one by a zero order moment $\lambda_{0_{r_i}}$) is the following:

$$\lambda_{0_r} = \sum_i^n \lambda_{0_{r_i}} \tag{8}$$

where $n$ represents the finite number of adopted bands (it could even be infinite if not cut-off frequency is present). It is then possible to obtain the simplified form of *Bands Method* (9) by substituting equation (8) into equation (7):

$$d = \frac{2^{k/2}}{C}\,\Gamma\left(1+\frac{k}{2}\right)\nu_{0_r}\,\lambda_{0_r}{}^{k/2} \tag{9}$$

where $\lambda_{0_r}$ defined in (10) represents the sum of the reference spectral moments $\lambda_{0_{r_i}}$ in Eq. (6), as illustrated by (8):

$$\lambda_{0_r} = \sum_i^n \left(\frac{\nu_{0_i}}{\nu_{0_r}}\right)^{2/k} \lambda_{0_i} \tag{10}$$

and $\nu_{0_r}$ is the reference frequency, arbitrarily defined.

It is simple to note that equation (9) can also be represented in a more compact form similar to Eq. (4):

$$d = \left[\sum_i^n d_i{}^{2/k}\right]^{k/2} \tag{11}$$

The fundamental characteristic of this approach is its simplicity (which stays into the adoption of only zero order moments) as well as its rapidity [13], greater than criteria considered as a reference [3,15].

## 5 Conclusions

The fatigue damage assessment performed by finite element models, usually with a huge number (millions) of *dofs* and often subjected to complex multi-axial loading, have induced the authors to elaborate simple and fast damage evaluation rules, which are based on equally simple calculation and material behaviour models.

The main idea discussed in this paper is the *band-splitting* of a PSD followed by a damage combination rule, which gives a simple tool for estimating the fatigue damage of a wide-band random process $x(t)$. Starting from the same idea of PSD *band-splitting*, different combinations rules proposed in different papers have been reviewed.

For example, the non-linear damage rule of the PbP multi-axial criterion gave exactly the damage formula of the "single moment" (SM) method. Another approach, instead, defined the damage combination by a shift of central frequencies of each infinitesimal narrow-band PSD. The adoption of only zero order spectral moments and of linear combination techniques, both of different power spectra and of parts of these, has showed to be an *economically* advantageous approach in terms of calculation speed and to obtain results characterized by small damage evaluation errors.

These approaches represent very useful evaluation tools in the first stage of the design process of whatever mechanical system subjected to both uniaxial and multi-axial stress state.

Regardless of the damage combination rule adopted, however, the proposed band-splitting and damage combination pointed out an interesting analogy between spectral methods for multi-axial and uniaxial random loadings, which opens up a new perspective for estimating the fatigue damage of wide-band uniaxial random loadings.

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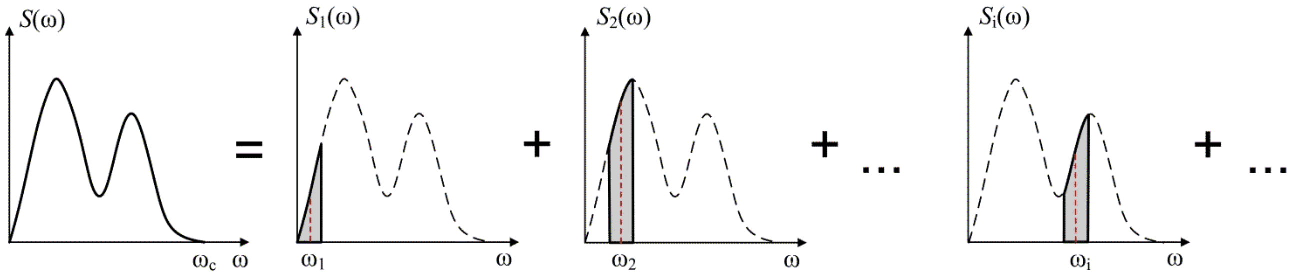


**Figure 1**. Spectral decomposition of the wide-band PSD $S(\omega)$ into a finite set of narrow-band PSDs $S_i(\omega)$, $i$=1,...,$n$.