

Implicit Gradient and Integral Average Effective Stresses: Relationships and Numerical Approximations

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ABSTRACT: This paper investigates two possible definitions of a non-local effective stress: the weighted average value and the implicit gradient solution. These definitions are usually applied for strength assessment, particularly in the high cycle fatigue regime, in the presence of notches, defects, cracks or welded joint toes or roots. The present research analyses their general relationship in plane notches. Then the paper presents a numerical method for their assessment by means of a general FE tool. Different solutions are calculated and compared, by evaluating the results obtained from different FEM commercial software.

KEYWORDS: IMPLICIT GRADIENT, LOCAL TENSION FIELD, NOTCH, WELDING, FATIGUE, NON-LOCAL, CRACK, EFFECTIVE STRESS.

NOMENCLATURE:

$\bar{\epsilon}_{eq}$	non-local equivalent strain
ϵ_{eq}	equivalent strain
ζ	scalar field
g	weight function
β	correlation function for sharp notches
β_ρ	correlation function for rounded notches
ψ	Gaussian weight function
s	distance between point X and Y of body
c	constant depending on material
L	constant depending on c
σ_{eq}	equivalent local tension

σ_{eff}	effective stress
$\sigma_{\text{eff,IG}}$	effective stress calculated by implicit gradient
$\sigma_{\text{eff,int}}$	effective stress calculated by an integral
$\sigma_{\text{eff,sum}}$	effective stress calculated by a summatory
δ	Dirac function
2α	opening angle
N	weight coefficient
ρ	filled radius
R	matrix built on r/c ratio
A	matrix built by angle 2α
P	coefficient matrix
X	investigated point
Y	general location point
r	distance from notch tip
s	Euclidian distance between X and Y

1. INTRODUCTION

In real applications, it is usual for an engineer to deal with strength assessment at stress raisers, i.e. notches, defects, welding toes or roots and cracks. In the literature, several authors addressed this problem by means of effective stress definitions at the notch tip [1–7]. Among the most recent outcomes in the literature, a number of contributions define the effective values by means of a non-local investigation of the stress/strain field close to the notch tip; these approaches include the implicit gradient [6–10], strain energy density [11–12] and the theory of critical distances [3].

In particular, the authors presented an approach capable of estimating the fatigue life of notched structures and welded joints based on an effective stress value computed numerically by solving the Helmholtz differential equation linked to the implicit gradient method [8–10].

This method is particularly suitable for a numerical estimation of the component fatigue life and its relation with its geometrical feature and stress raiser. The idea is rather simple and enables an original application of the average stress damage that was first formulated in the 1930s by Neuber [2] and referred to geometry where fatigue crack propagation direction is on the bisector of the notch. This idea succeeds in attaining the fatigue limit of a plane component with notches by simple manual calculation [4,14,15]. On the other hand, it is difficult and awkward when the notch is under complex loading, in this case the maximum stress or the

crack initiation and propagation direction can be outside the bisector; additionally the problem can be even more challenging when the component is three dimensional or geometrically complex.

Note that every effective stress definition, from Neuber [2] to more recent developments, has a heuristic character, since it does not describe the actual failure mechanism, but simply establishes a phenomenological relation between the strength of a component and a stress field quantity.

The implicit gradient method reinforces the idea that the damage should be related to an average value of the stress components occurring on the body; in such average calculations, stress values close to the critical point are more important and active than those close to the far away field (this suggests the possible use of a weight function). The computed effective stress is representative of the overall damage in the process zone. The influence zone dimension is simply regulated by the material properties and it is indicated by the length c .

In a uniaxial fatigue situation, the authors proposed to only average the first principal stress; for multi-axial fatigue strength assessment, it is necessary to use a multi-axial criterion by using, for instance, stress invariants or critical plane approaches [9].

By using the implicit gradient approach, it is possible to directly obtain the maximum value of the effective stress range and the location of the point where the fatigue crack initiates without imposing *a priori* the location of failure points. However, if the position of the critical point is known, it could sometimes be convenient to evaluate the “non-local” effective stress without taking advantage of specific Partial Differential Equation software for the integration of the Helmholtz differential equation.

This research has two objectives. The first is the quantitative clarification of the relationship between the implicit gradient effective stress and the weighted integral average stress values at the notch tip. The research will focus mainly on the mode I loading.

The second objective is to investigate a numerical method for the evaluation of the non-local field in a fast and simple way, using the local stress field that is available in all commercial FEM software without a request of specific routine for the integration of the Helmholtz partial differential equations.

For this purpose, starting from the analytical and theoretical fundamentals, two commercial software tools were used: Comsol Multiphysics and Ansys.

Comsol Multiphysics has a pre-loaded module for an ordinary differential equation solution (Helmholtz equation) that can be used to calculate effective stress values for non-local strength assessment all-over the investigated domain. Alternatively, different conventional FE software (for instance, Ansys FE software) can compute and export the nodal coordinates and nodal results turned out from a conventional local

structural analysis, so that a specific numerical procedure has to be developed for the evaluation of the effective stress in the selected critical points of the two-dimensional or three-dimensional components.

2 Theoretical framework

In a structural component containing a crack or a sharp V-notch such as in a welded joint, the maximum stress tends towards infinity because the tip radius is null. As indicated by Williams [13], the stress components σ_{ij} is proportional to a power of the distance r from the notch tip of type: $\sigma_{ij}(r) \propto r^{\lambda-1}$, where λ is the Williams' eigenvalue. A technique to compare a finite value of the stress with a reference strength (fatigue limit, yield stress, etc.) is by moving away from the tip or by some way averaging the stress field and using these averages or, in any case, non-local values, for strength assessment [2–15]. In particular, the implicit gradient approach proposed in [16–18] is very general, and, from a mathematical point of view, the effective stress is correctly defined for every geometry without imposing *a priori* the position of critical points.

According to ref. [7], a gradient formulation can be derived directly from non-local theory. Traditionally, the first application and definition have been applied to the strain values [5, 6, 7] but the value of the non-local (i.e. the effective value) equivalent stress $\bar{\sigma}_{eq}$ can be given in a similar way. In a point X of a generic mechanical components of volume V , $\bar{\zeta}$ is a weighted average of the local scalar ζ on V through the expression:

$$\bar{\zeta}(X) = \frac{1}{V_r} \int_V g(s) \zeta(Y) dV \quad \text{with} \quad V_r = \int_V g(s) dV \quad (1)$$

In which $g(s)$ is a weight function and s denotes the Euclidean distance $\|X - Y\|$ between the points X and Y.

The non-local equivalent stress given in Eq. (1) is expanded into a Taylor series. Assuming that the generalised derivatives of a certain scalar field ζ exist, the stress scalar at point $\mathbf{y}=\mathbf{x}+\mathbf{s}$ can be developed in a Taylor series in a neighbourhood of point \mathbf{x} of radius $\|\mathbf{s}\|$, with $\mathbf{s}=(s_1, s_2, s_3)$ (e.g. Ciarlet [26]):

$$\zeta(\mathbf{y}) = \sum_{|\mathbf{m}|=0}^{\infty} \frac{1}{\mathbf{m}!} \zeta^{(\mathbf{m})}(\mathbf{x}) s^{(\mathbf{m})} \quad (2)$$

In Equation (5), according to the multi-index notation for the R_n case, $\mathbf{m}=(m_1, m_2, \dots, m_n)$ is a sequence of n natural numbers of which the factorial is $\mathbf{m}!=m_1! m_2! \dots m_n!$ and such that $|\mathbf{m}|=m_1+m_2+\dots+m_n$; the symbol $\zeta^{(\mathbf{m})}$ represents the sets formed by all partial derivatives of a given order \mathbf{m} , namely

$$\sigma^{(\mathbf{m})} = \frac{\partial^{|\mathbf{m}|} \sigma}{\partial \mathbf{x}^{|\mathbf{m}|}} = \frac{\partial^{m_1}}{\partial x_1^{m_1}} \frac{\partial^{m_2}}{\partial x_2^{m_2}} \dots \frac{\partial^{m_n}}{\partial x_n^{m_n}} \zeta, \text{ and, finally, } s^{(\mathbf{m})} = s^{m_1} s^{m_2} \dots s^{m_n} \text{ (Ciarlet [26]).}$$

Subsequently, the product of $\zeta(\mathbf{y})$, expressed by Taylor's expansion (4), times the weight function is integrated over a symmetric domain

V

$$\bar{\zeta}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})} \int_V \alpha(\|\mathbf{s}\|) \sum_{|\mathbf{m}|=0}^{\infty} \frac{1}{\mathbf{m}!} \zeta^{(\mathbf{m})}(\mathbf{x}) s^{(\mathbf{m})} ds \quad (3)$$

where $ds=ds_1 ds_2 ds_3$.

By solving Eq. (3) with a symmetric weight function α , the odd terms vanish, and neglecting higher than a third-order derivative in the Taylor's expansion, Eq. (6) becomes the definition (1) of the non-local equivalent stress and can be replaced by

$$\bar{\zeta} = \zeta + c \nabla^2 \zeta \quad (4)$$

The coefficient c plays the role of a diffusive length and ∇^2 indicates the Laplacian operator, such that

$$\nabla^2 \zeta = \text{div grad } \sigma = \frac{\partial^2 \zeta}{\partial x_1^2} + \dots + \frac{\partial^2 \zeta}{\partial x_n^2} \quad (5)$$

According to Peerlings et al. [7], the implicit gradient approximation of eq. (4) can be derived by applying the Laplacian operator to both sides of eq. (4):

$$\nabla^2 \bar{\zeta}(\mathbf{x}) \cong \nabla^2 \zeta(\mathbf{x}) + c^2 \nabla^4 \zeta(\mathbf{x}) \quad (6)$$

Consistently with the approximation performed to obtain eq. (4), the fourth-order term of the non-local variable $\nabla^4 \bar{\zeta}$ can be neglected in eq. (6). Subsequently, the Laplacian of the local term can be replaced with the Laplacian of the non-local variable within the explicit gradient expansion eq. (4). Finally, we obtain the differential approximation

$$\bar{\zeta}(\mathbf{x}) \cong \zeta(\mathbf{x}) + c^2 \nabla^2 \bar{\zeta}(\mathbf{x}) \quad (7)$$

Note that, by construction, the implicit gradient formulation Eq. (7) and the explicit gradient formulation (4) imply the same degree of approximation, as they have both been obtained by neglecting fourth order derivatives in the Taylor expansion (5) (Peerlings et al. [7]).

In order to solve the Helmholtz partial differential equation (7), the boundary conditions concerning the scalar field ζ have to be specified. From a mathematical point of view, the Neumann boundary condition was taken into account:

$$\nabla \bar{\zeta} \cdot \mathbf{n} = 0 \quad (8)$$

where \mathbf{n} denotes the normal to the surface of V . These types of boundary conditions have been adopted in many other papers such as Lasry and Belytschko [19] and Mühlhaus and Aifantis [20].

In the following, the previously summarised theoretical framework will be applied to a specific weight function, without any lack of applicability of previous equations.

It is possible to define a non-local effective stress in a generic point X as an integral average of an equivalent local tension σ_{eq} , weighted by a Gaussian function $\psi(X, Y)$ depending on the distance s between points X and Y of the body:

$$\sigma_{\text{eff,int}}(\mathbf{X}) = \frac{1}{V_r(\mathbf{X})} \int_V \psi(\mathbf{X}, \mathbf{Y}) \sigma_{\text{eq}}(\mathbf{Y}) dV = \frac{\int_V \psi(\mathbf{X}, \mathbf{Y}) \sigma_{\text{eq}}(\mathbf{Y}) dV}{\int_V \psi(\mathbf{X}, \mathbf{Y}) dV} \quad \text{in } V \quad \text{with} \quad V_r(\mathbf{X}) = \int_V \psi(\mathbf{X}, \mathbf{Y}) dV \quad (9)$$

This effective value will be indicated as $\sigma_{\text{eff,int}}$; in Eq. (9) it is possible to use two different weight functions depending on the space dimension of the investigated structural problem (two dimensional plane investigations or three dimensional problems):

$$\psi = \frac{\frac{-s^2}{e^{2L^2}}}{2\pi L^2} \quad L=c\sqrt{2} \quad \text{2D problem} \quad (10)$$

$$\psi = \frac{\frac{-s^2}{e^{2L^2}}}{(L\sqrt{2\pi})^3} \quad L=c\sqrt{2} \quad \text{3D problem} \quad (11)$$

with s equal to the Euclidian distance between \mathbf{X} and \mathbf{Y} .

By approximating equation (9), similarly to eqs (1) and (2), it is possible to define an effective stress by means of the Helmholtz equation [8,18,21] using Neuman boundary conditions ($\nabla \sigma_{\text{eff}} \cdot \mathbf{n} = 0$). This effective value will be indicated as $\sigma_{\text{eff,IG}}$:

$$\sigma_{\text{eff,IG}} - c^2 \nabla^2 \sigma_{\text{eff,IG}} = \sigma_{\text{eq}} \quad \text{in } V \quad (12)$$

where σ_{eq} will be the first principal stress and c a material coefficient [8] (for instance, it is 0.2 mm for weldable construction steel). Now, for the sake of simplicity, c^2 in Eq. (12) agrees with c of the previous Eq. (7). The Gaussian weight function ψ is not the only applicable function, for instance it could be replaced, for example, by Green's weight function [21, 22]:

$$G_f(\mathbf{X}, \mathbf{Y}) = \frac{\frac{-s}{e^{\sqrt{c}}}}{4\pi c \rho} \quad (13)$$

Green's function is defined as the solution of the partial differential equation (7) with the source term being replaced by a Dirac function $\delta(\mathbf{X}-\mathbf{Y})$:

$$G(X,Y) - c\nabla^2 G(X,Y) = \delta(X - Y) \quad (14)$$

To obtain G in a 3D problem, G_f has to be added to the homogeneous partial differential equation because G_f does not satisfy the boundary condition [22].

By integrating G , a value equal to the implicit gradient can be found, by the theoretical framework for every weight function, unlike G , an integral that is different from the implicit gradient is obtained. Fig. 1 shows a comparison between the Gauss weight function Ψ and the Green function (13) that is not a continuous function in the origin.

The use of the Green function (13) in eq. (9) would be the most correct choice by establishing a clear and plane equality between the Weighted Integral and Implicit Gradient effective stress. Unfortunately, due to its singularity, its use is even more complex, from a numerical point of view, than the PDE solution. Hence, in the following, only Gaussin-like functions (10) and (11) will be used, and the relation between the Weighted Average and Implicit Gradient requires detailed investigation.

3 COMPARISON OF STRESS AVERAGE VALUES WITH IMPLICIT GRADIENT EFFECTIVE STRESS

The relationship between an integral value and the solution of a partial differential equation can be generally very complex. In the following, we will take advantage of some properties of the stress field at notches.

It has been analytically demonstrated [23] that, with good approximation, the stress field surrounding the tip of a notch is only dependent on the notch tip radius, the opening angle and the overall intensity. The overall stress intensity, under the mode I loading, is uniquely quantified by one stress field parameter, usually called the Notch Stress Intensity Factor (NSIF) [23] and, for instance, these parameters are even related to the fatigue strength of different welded joints [23–25].

When the whole geometry of the notch is sufficiently larger than the constant “ c ” and “ L ” or the integration field “ V ”, in any notch with a given notch opening angle and tip radius, the effective stress values are exclusively related to the NSIF value (see, for example, the analytical solution in reference [8] in the case of sharp V-notches).

Hence, the ratio between two possible definitions of the effective value, at a loaded notch, does not depend on the NSIF, but is a constant value, which depends simply on the main geometrical parameters, i.e. the

notch opening angle and the tip radius. This constant ratio can be computed in a specific geometry, but, under the above-mentioned assumption, it is the same for any other geometry and can be applied generally. According to such properties, a first plane geometry is considered. The initially investigated geometry is a simple 2D geometry (see Fig. 2), under a linear elastic plane stress condition and remote tensile gross stress σ_{nom} . The geometry has a notch-opening angle ranging from 0° to 180° and a notch tip radius varying from 0 to 1 mm, the initially investigated parameter being the null radius.

By means of a PDE solver that includes the Helmholtz equation solver, the $(\sigma_{eff,IG})$ can be directly solved in all points of the model. The linear elastic stress solution provides the local stress field. The level curves of $\sigma_{eff,IG}$ is given in Fig. 3. By changing the angle of the notch and plotting the results, it is possible to compare the differences at the tip (Fig. 4). Except for smaller angles, $\sigma_{eff,IG}$ decreases as the opening angle of the notch is increased.

Alternatively, by considering the integral value, even equation (9) can be evaluated by numerical integration; Fig. 4 also shows this type of result ($\sigma_{eff,int}$). In this specific case, the integral (9) was performed by means of the integration tools present in the Comsol software. The obtained results are given in table 1.

The ratio between the implicit gradient results and its integral values ranges from 1 to 1.25, depending on the opening angle. It is possible to fit a correlation function β that describes the ratio between $\sigma_{eff,IG}$ and $\sigma_{eff,int}$ at the notch tip as a function of 2α (Fig. 5). Eq. (15) fits the points of Fig. 5 and its maximum error at computed points is less than 0.2%.

$$\beta(2\alpha) = \frac{\sigma_{eff,IG}}{\sigma_{eff,int}} = 1.235 + 8.22e-05 \cdot (2\alpha) - 3.31e-06 \cdot (2\alpha)^2 - 2.44e-08 \cdot (2\alpha)^3 \quad [\alpha \text{ in deg.}] \quad (15)$$

Since the effective values are computed at the notch tip, only local stress values close to the tip are actually meaningful for effective stress computation. To achieve a sufficient approximation, it is not necessary to compute the integral across the whole domain, but it is faster to only make the integration in a subdomain (a circular subdomain with a radius equal to $6.5 c$ has been verified as suitable to void the weight function).

Differences between these two estimations can be assumed to be the intrinsic scatter between the two methods so that if the “integral” value is available, the “IG effective” value can be computed by this correction.

Note that in Eq. (15) when $2\alpha=0$, $\sigma_{eff,IG} / \sigma_{eff,int} = 1.235$, this value is very close to that estimated in ref. [21] for the non-local equivalent strain.

4 ROUNDED NOTCHES

By modifying the ratio of the tip radius ρ over c , it is possible to compute effective stress in rounded geometries as well. The reference geometry is reported in Fig. 6. The results related to the opening-angle equal to 90° are given in Fig. 7.

The scatter between the implicit gradient effective stress and the integral value decreases as the notch tip radius is increased. In any case, the error between the integral value and its numerical approximation is very low.

Hence, we can argue that the integral value of Eq. (9) can be evaluated by means of generic FE software also in the case of rounded notches. Eq. (15) can be modified in order to take into account the general relationship between the implicit gradient $\sigma_{\text{eff,IG}}$ and integral value $\sigma_{\text{eff,int}}$ by combining different values of the opening angle and notch tip radius. The ratio between $\sigma_{\text{eff,IG}}$ and $\sigma_{\text{eff,int}}$ is given in Fig. 8.

It is possible to create a correlation function β_ρ that describes the graph on Fig. 8 depending on the angle and filled radius ρ giving the ratio $\sigma_{\text{eff,IG}}/\sigma_{\text{eff,int}}$:

$$\beta_\rho = \frac{\sigma_{\text{eff,IG}}}{\sigma_{\text{eff,int}}} = (\mathbf{R} \cdot \mathbf{P}) \cdot \mathbf{A} \quad (16)$$

$$\mathbf{R} = [(\rho/c)^4 \quad (\rho/c)^3 \quad (\rho/c)^2 \quad (\rho/c) \quad 1]$$

$$\mathbf{P} = \begin{bmatrix} -5.35\text{E-}12 & 2.06\text{E-}10 & -3.13\text{E-}7 & 1.48\text{E-}4 & -0.013 \\ 4.70\text{E-}11 & 2.28\text{E-}10 & 2.74\text{E-}6 & -1.45\text{E-}3 & 0.127 \\ -1.07\text{E-}10 & -1.27\text{E-}10 & -6.67\text{E-}6 & 4.29\text{E-}3 & -0.385 \\ -1.6\text{E-}10 & 9.76\text{E-}8 & -2.20\text{E-}6 & -3.45\text{E-}3 & 0.307 \\ 1.89\text{E-}10 & -1.28\text{E-}7 & 1.55\text{E-}5 & -1.22\text{E-}3 & 1.26 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} (2\alpha)^4 \\ (2\alpha)^3 \\ (2\alpha)^2 \\ (2\alpha) \\ 1 \end{bmatrix} \quad (2\alpha \text{ in degree})$$

With Eq. (16) the maximum fitting error is 0.7% on $2\alpha=0$ and $\rho/c=0$

5. A NEW PROCEDURE FOR NUMERICAL EVALUATION OF INTEGRAL VALUES

In order to obtain the value of the effective stress $\sigma_{\text{eff,IG}}$ in selected critical points, a numerical procedure substituting the solution of the Helmholtz equation (12) with a local integration (9) will be presented in this section. This integration can be performed with a specific software or by means of a procedure developed herein and based on the use of the FE code used for the standard stress analysis.

As analysed in the previous section, the knowledge of the integral $\sigma_{\text{eff,int}}$ at the notch tip allows us to obtain the value of the implicit gradient by means of β parameter defined in Eqs (15 and 16). However, most FE software does not have the ability to make an integral over a domain, so that a new method is presented in order to solve Eq. (9).

By introducing the assumed weight coefficient N , we transform the integral into a proper weighted sum, with a quadrature rule approximation such as:

$$\sigma_{\text{eff,int}}(\mathbf{X}) = \frac{\int_V \psi(\mathbf{X}, \mathbf{Y}) \sigma_{\text{eq}}(\mathbf{Y}) \, dV}{\int_V \psi(\mathbf{X}, \mathbf{Y}) \, dV} \cong \sigma_{\text{eff,sum}} = \frac{\sum N_i \psi(\mathbf{X}, \mathbf{Y}_i) \sigma_{\text{eq}}(\mathbf{Y}_i)}{\sum N_i \psi(\mathbf{X}, \mathbf{Y}_i)} \quad (17)$$

For this calculation, it is necessary to define the weight coefficient N which is built on the distance between the nodes that depend only on the mesh size.

In a uniform mesh, i.e. with elements of the same size and with a constant distance between the nodes, N should be constant. In any other case, N will be computed depending on the dimension of the element and the actual distances among the nodes: the larger the size of the elements, the higher the weight of the related nodal values.

This approach suggests using, as an approximation of N , the nodal reaction (at each node constrained), when the whole body is loaded by a uniform distributed load, such as the gravity load. That is to say, in the same model used for stress analysis, each node is fully constrained, a uniform volume load is applied (for instance gravity load) and the reaction forces at the nodes are computed. Consequently, the larger the element, the larger the reaction and the suitable coefficient N is the nodal reaction at each constraint.

As a numerical example, a simple 50 x 100 rectangle is considered (see Fig. 9). If a unitary uniform load is applied, the sum of the load constrained, i.e. the sum of coefficient N , is equal to the area of the rectangle.

In the definition of the proposed elements, the area of element C is 2.5 times larger than the area of elements A and B . The weight of each node N , depends on the size of the contiguous area. For instance, in this

example:

$$N_1 = 25; N_3 = 50; N_2 = 87,5; N_{21} = 100; N_8 = 175; N_{44} = 125; N_{45} = 250.$$

From an operative point of view, we can make a mesh as in figure 10 with a regular subdivision in the neighbourhood of the notch tip zone. This zone is defined as an auxiliary mesh and represents the zone where the integral will be calculated. Now, we consider a new model defined only with the auxiliary mesh (all the others nodes will be erased). All nodes will be fixed and a unitary uniform load per unit volume is applied.

For any kind of FE software, it is simply necessary to export the coordinates of the nodes inside the auxiliary mesh and the reaction load at the same nodes: the reaction represents the N unknown coefficient. Then, any mathematical tool can compute $\sigma_{\text{eff,sum}}$ from equation (16). Obviously, the equivalent stress σ_{eq} is evaluated in the actual FE model inside the auxiliary mesh.

The proposed $\sigma_{\text{eff,sum}}$, in the considered example of Fig. 2, numerically analysed by Ansys structural analysis is actually equal to the Comsol integration for each opening angle taken into consideration (numerical result are given in Fig. 11 and Tab. 1). The accuracy in the calculation of the integral by means of Eq. (16) is very clear.

6 MESH SIZE EFFECT

A further problem regarding the numerical assessment of non-local effective stress is the sensitivity of the results to the dimension of the elements. Taking the 120° geometry and through mesh analysis, it is possible to look at the relationship between the maximum element size d and the errors that occur. Tab. 3 shows the dimension of the element at the notch tip. In this investigation “Solid 183” elements in Ansys and “quadrangular” elements in Comsol, were used.

As can be seen with a mesh of an element size equal to $c/5$ in the subdomain (of size $\sim 6.3c$), it is sufficient to have the same result of the integral solution.

7. CHECKING TEST

In order to check the procedure previously defined and the synthesis relationship, four different geometries in a two-dimensional or three-dimensional space have been created.

7.1 Two-dimensional problem

Fig. 13 shows a 2D specimen with an opening angle of 134° and a notch tip radius of 0.75 mm under a couple of forces. The result at the notch tip confirms the fitting equation, even under a different type of load:

$$\sigma_{\text{eff,int}} = 8.92 \text{ MPa} \quad [\text{from Eq. (17)}]$$

$$\beta_\rho = 1.08 \quad [(R_{\rho/c=3.75 \cdot P}) \cdot A_{2\alpha=134^\circ} = 1.08 \quad \text{from Eq. (16)}]$$

$$\sigma_{\text{eff}} = \beta_\rho \sigma_{\text{eff,int}} = 9.63 \text{ MPa}$$

$$\text{Alternatively: } \sigma_{\text{eff,IG}} = 9.64 \text{ MPa} \quad [\text{from Eq. (12) with FE results}]$$

The ratio between $\sigma_{\text{eff,int}} / \sigma_{\text{eff,IG}}$ given by Eq. (16) is in good agreement with FE results.

7.2 Three-dimensional problem

Three 3D geometries were also analysed, the point of interest X was always in the middle of the width at the tip of the crack/welding.

First of all, the author tried to replace the 2D analysis with 3D analysis, choosing the worst case: $2\alpha=0$ and $\rho/c=0$.

Using the right weight function for 3D geometry (Eq. (11)) the results are very similar to the 2D case. As shown in Table 4, the error in the 3D case is very low and comparable to the 2D geometry, even the difference seems to be in the same range.

More complicated geometries have been analysed:

- case A – welded joint with longitudinal attachment
- case B – non-load-carrying cruciform welded joint

(see Figs 14–15).

Table 5 summarises the effective stresses for 3D cases A and B. The error of the proposed quadrature equation (16) is less than 0.5 %.

8. GENERAL PROCEDURE

According to the previous theoretical framework, it is possible to sketch a procedure for effective non-local stress assessment by means of a common FE code able to solve the classical stress analysis under linear elastic hypothesis. First of all, it is suitable to make a subdomain ($\geq 7c$) around the interested zone: this makes it easier to create a good quality mesh (element size close to $c/5$) and to only export the required results. After that, it is only necessary to make an appropriate model and to export the nodal data: the first principal stress of the classical stress analysis (or any other equivalent stress value or stress invariant, σ_{eq}) and the nodal coordinates (x_i, y_i) for the evaluation of the weight functions ψ . Then, it is necessary to create a new model, called an *auxiliary model*, made with the nodes of the subdomain. All nodes of the auxiliary model will be fixed and a uniform load will be applied. Finally, the reaction solutions of the auxiliary model (N) are exported.

Eqs (10 and 11) define ψ for a two-dimensional or three-dimensional model and the data are combined into Eqs (15–17).

9. CONCLUSIONS:

The implicit gradient method has already been demonstrated to be effective particularly in fatigue strength assessments of sharp V-notches such as welded joints. With the procedure proposed in this paper, through the definition of two new correlation functions, it is possible to determinate the effective stress in selected critical points. In this way, the designer avoids the use of a PDE tool for the complete solution of the differential equation that is on the basis of the implicit gradient approach. The paper provided information regarding how to create the mesh and the subdomain, which data to export and how to use them in the proper method in either two-dimensional or three-dimensional geometries under mode I loadings.

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2α [°]	$\sigma_{\text{eff,IG}}$ [MPa]	$\sigma_{\text{eff,int}}$ [MPa]	$\sigma_{\text{eff,sum}}$ [MPa]	$(\sigma_{\text{eff,int}} - \sigma_{\text{eff,IG}}) / \sigma_{\text{eff,IG}}$ [%]	$(\sigma_{\text{eff,sum}} - \sigma_{\text{eff,int}}) / \sigma_{\text{eff,int}}$ [%]
0	7.85	6.36	6.36	-19.0	0.0
30	8.54	6.91	6.91	-19.0	0.0
60	8.92	7.31	7.32	-18.0	0.1
90	8.61	7.19	7.20	-16.5	0.1
120	7.03	6.08	6.08	-13.5	0.0
135	5.69	5.05	5.06	-11.1	0.2
150	4.09	3.76	3.76	-8.2	0.0
180	1.00	1.00	1.00	0.0	0.0

Table 1: Value of $\sigma_{\text{eff,IG}}$, $\sigma_{\text{eff,int}}$ and $\sigma_{\text{eff,sum}}$ at any tested angle of Fig. 2 ($\sigma_{\text{nom}} = 1$ MPa)

r/c	$\sigma_{\text{eff,IG}}$ [MPa]	$\sigma_{\text{eff,int}}$ [MPa]	$(\sigma_{\text{eff,int}} - \sigma_{\text{eff,IG}}) / \sigma_{\text{eff,IG}}$ [%]
0	8.61	7.19	-16.53
0.125	8.67	7.21	-16.78
0.25	8.72	7.25	-16.90
0.4	8.76	7.28	-16.86
0.5	8.78	7.31	-16.75
1	8.78	7.42	-15.58
2	8.57	7.53	-12.17
3	8.26	7.51	-9.16
5	7.64	7.21	-5.59

Table 2: Value of $\sigma_{\text{eff,IG}}$, $\sigma_{\text{eff,int}}$ and $\sigma_{\text{eff,sum}}$ at any tested ratio ($2\alpha=90^\circ$, $\sigma_{\text{nom}} = 1$ MPa)

Mesh	Comsol	Ansys	Error	
d/c	$\sigma_{\text{eff,IG}}$ [MPa]	$\sigma_{\text{eff,int}}$ [MPa]	$\sigma_{\text{eff,sum}}$ [MPa]	$(\sigma_{\text{eff,sum}} - \sigma_{\text{eff,int}}) / \sigma_{\text{eff,int}}$ [%]
1	7.03	6.09	6.04	-0.7%
0.5	7.03	6.09	6.06	-0.4%
0.2	7.03	6.08	6.08	0.0%

Table 3: Error in the mesh analysis of the specimen reported in Fig. 2 ($2\alpha=120^\circ$; d: maximum element size, $\sigma_{\text{nom}} = 1$ MPa)

	$\sigma_{\text{eff,IG}}$ [MPa]	$\sigma_{\text{eff,int}}$ [MPa]	$\sigma_{\text{eff,sum}}$ [MPa]	$\sigma_{\text{eff,IG}} / \sigma_{\text{eff,int}}$	$(\sigma_{\text{eff,sum}} - \sigma_{\text{eff,int}}) / \sigma_{\text{eff,int}}$ [%]
2D	7.85	6.36	6.36	1.23	0.0
3D	8.42	6.84	6.83	1.23	-0.15

Table 4: Comparison between 2D (Fig. 2) and 3D (Fig. 12) in the crack case for $\sigma_{\text{nom}} = 1$ MPa.

	$\sigma_{\text{eff,IG}}$ [MPa]	$\sigma_{\text{eff,int}}$ [MPa]	$\sigma_{\text{eff,sum}}$ [MPa]	$\sigma_{\text{eff,IG}} / \sigma_{\text{eff,int}}$	$(\sigma_{\text{eff,sum}} - \sigma_{\text{eff,int}}) / \sigma_{\text{eff,int}}$ [%]
A	2.57	2.29	2.30	1.12	0.4
B	1.98	1.76	1.76	1.12	0.1

Table 5: Effective stresses for 3D cases A and B ($\sigma_{\text{nom}} = 1$ MPa)