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Decay estimates for elastic cylinders with mixed boundary conditions

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Abstract The classical Saint–Venant's problem in cylinders is considered under various boundary conditions and some estimates are proven. In particular, if the cylinder is force free on the mantle and is subject to assigned displacements on the bases, satisfying exact compatibility conditions, then a unique elementary Saint–Venant's solution can be selected in such a way that it differs from the rigorous solution to the mixed problem only away from the bases.

Keywords Linear elastostatics · Cylinders · Saint–Venant's problems · Mixed boundary value problems

Mathematics Subject Classification (2000) MSC 35J57 · 74B05 · 74G50

1 Introduction

In the middle of nineteenth century, Adhémar Jean Claude Barré, count of Saint–Venant, proposed an ingenious and elegant theory, *monumentum aere*

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perennius [1], to approximate the solution to the traction problem of elastostatics[1]

$$\begin{aligned} \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\mathrm{div}\,\boldsymbol{u} &= \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{s}(\boldsymbol{u}) = 2\mu(\hat{\nabla}\boldsymbol{u})\boldsymbol{n} + \lambda(\mathrm{div}\,\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \mathcal{M}, \\ \boldsymbol{s}(\boldsymbol{u}) &= \boldsymbol{\sigma} \quad \text{on } \mathcal{C}_{-h}\cup\mathcal{C}_h, \end{aligned} \tag{1}$$

in the cylinder

$$\Omega = \{x = (x', x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \mathcal{C},\ |x_3| < h\}, \tag{2}$$

where λ, μ are the Lamé moduli assumed to satisfy $\mu > 0$, $3\lambda + 2\mu > 0$ and $\boldsymbol{\sigma}$ is an assigned field on $\partial\Omega$, vanishing on $\mathcal{M} = \partial\Omega \setminus (\mathcal{C}_{-h}\cup\mathcal{C}_h)$ and such that

$$\int_{\partial\Omega}\boldsymbol{\sigma} = \boldsymbol{0}, \quad \int_{\partial\Omega}\boldsymbol{x}\times\boldsymbol{\sigma} = \boldsymbol{0}. \tag{3}$$

Saint-Venant supposed that, due to the physical nature of the problem, a solution to $(1)_{1,2}$ should satisfy in Ω

$$\boldsymbol{s}(\boldsymbol{u})\boldsymbol{e}_i = \boldsymbol{0}, \quad i = 1, 2. \tag{4}$$

He was able to find a simple analytic expression of all the solutions $\mathfrak{S}$ to $(1)_{1,2}$, (4) (elementary Saint–Venant's solutions, see also [16]), depending on six arbitrary scalars, he used to select the unique field in $\boldsymbol{u}' \in \mathfrak{S}$ having the same net force and moment on $\mathcal{C}_h$ as the assigned field $\boldsymbol{\sigma}$,

$$\begin{aligned} \int_{\mathcal{C}_h}\boldsymbol{s}(\boldsymbol{u}') &= \int_{\mathcal{C}_h}\boldsymbol{\sigma}, \\ \int_{\mathcal{C}_h}\boldsymbol{x}\times\boldsymbol{s}(\boldsymbol{u}') &= \int_{\mathcal{C}_h}\boldsymbol{x}\times\boldsymbol{\sigma}. \end{aligned} \tag{5}$$

Of course, $\boldsymbol{u}'$ corresponds to a particular distribution of tractions on the bases. To overcome this difficulty, on the basis on his great physical intuition, Saint-Venant supposed that if $\boldsymbol{u}$ and $\boldsymbol{u}'$ are solutions to $(1)_{1,2}$, then

$$\left.\begin{aligned} \int_{\mathcal{C}_h}\boldsymbol{s}(\boldsymbol{u}-\boldsymbol{u}') &= \boldsymbol{0} \\ \int_{\mathcal{C}_h}\boldsymbol{x}\times\boldsymbol{s}(\boldsymbol{u}-\boldsymbol{u}') &= \boldsymbol{0} \end{aligned}\right\} \Rightarrow \hat{\nabla}\boldsymbol{u} \simeq \hat{\nabla}\boldsymbol{u}' \text{ away from the bases.} \tag{6}$$

[1] For the main notation we follow the classical monograph [8]. In particular, italic light–face letters, small upper–case letters and capital upper–case letters indicate scalars, vectors in $\mathbb{R}^3$ and second–order tensors (linear maps from $\mathbb{R}^3$ into itself), respectively; Lin is the linear space of second order tensors and Sym, Skw are the sets of its symmetric and skew elements, respectively. If $\boldsymbol{u}$ is a regular vector field, then $\nabla\boldsymbol{u}$ is the second order tensor with components $(\nabla\boldsymbol{u})_{ij} = \partial_j u_i$ in a orthonormal base $\{\boldsymbol{e}_i\}_{i=1,2,3}$; $\mathrm{div}\,\boldsymbol{u} = \mathrm{tr}\nabla\boldsymbol{u}$, $\Delta\boldsymbol{u} = \mathrm{div}\,\nabla\boldsymbol{u}$, $\nabla\boldsymbol{u}^\top$ is the transpose of $\nabla\boldsymbol{u}$ and $\hat{\nabla}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)$, $\tilde{\nabla}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^\top)$. The cylinder (1) has length $2h$ and axis $\boldsymbol{e}_3$, passing through the centroids of the sections $\mathcal{C}_{x_3} \equiv \mathcal{C}$ of the cylinder at $(0, x_3)$.

Hence, if (6) holds true (since then called *Saint–Venant's principle*), then the solution to the traction problem and the corresponding elementary Saint–Venant's solution agree away from the bases. Saint–Venant's principle was proven in 1965 by R. Toupin [19] for the semi–infinite cylinder and general elastic bodies. Some years later, G. Fichera [5] observed that Toupin's argument can be easily modify to cover the case of a finite cylinder. The Toupin and Fichera theorems can be stated as follows [8], [12][2].

Theorem 1 (Toupin) *Let* $\tilde{\Omega} = \{x \in \mathbb{R}^3 : x \in \mathcal{C}, x_3 > 0\}$ *and let* $\boldsymbol{u}$ *be a solution to*

$$\begin{aligned} \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\mathrm{div}\,\boldsymbol{u} = \boldsymbol{0} \quad & \text{in } \tilde{\Omega}, \\ \boldsymbol{s}(\boldsymbol{u}) = 2\mu(\hat{\nabla}\boldsymbol{u})\boldsymbol{n} + \lambda(\mathrm{div}\,\boldsymbol{u})\boldsymbol{n} = \boldsymbol{0} \quad & \text{on } \tilde{\mathcal{M}}, \\ \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{\sigma} \quad & \text{on } \mathcal{C}_0 \end{aligned} \tag{7}$$

with $\hat{\nabla}\boldsymbol{u} \in L^2(\tilde{\Omega})$, $\tilde{\mathcal{M}} = \partial\tilde{\Omega} \setminus \mathcal{C}_0$, *and*

$$\int_{\mathcal{C}_0} \boldsymbol{\sigma} = \int_{\mathcal{C}_0} \boldsymbol{x} \times \boldsymbol{\sigma} = \boldsymbol{0}. \tag{8}$$

Then, $\boldsymbol{u}$ *satisfies the decay estimate*

$$\int_{\tilde{\Omega}^R} |\hat{\nabla}\boldsymbol{u}|^2 \le e^{-\frac{(R-\delta)}{\gamma(\delta)}} \int_{\tilde{\Omega}} |\hat{\nabla}\boldsymbol{u}|^2, \tag{9}$$

where $\tilde{\Omega}^R = \{x \in \tilde{\Omega} : x_3 > R\}$,

$$\gamma(\delta) = \sqrt{\frac{\mu_M^2}{\mu_m \omega(\delta)}}, \tag{10}$$

μ_M, μ_m *maximum and minimum elastic moduli, and* $\omega(\delta)$ *the lowest (non–zero) characteristic value for the free vibrations of the cylinder* $T_\delta = \{x : x' \in \mathcal{C}, |x_3| < \delta\}$, *with* δ *assigned small positive constant.*

Theorem 2 (Fichera)[3] *If* $\boldsymbol{u}$ *is a solution to* (1), *with* $\hat{\nabla}\boldsymbol{u} \in L^2(\Omega)$ *and*

$$\int_{\mathcal{C}_h} \boldsymbol{\sigma} = \int_{\mathcal{C}_h} \boldsymbol{x} \times \boldsymbol{\sigma} = \boldsymbol{0}, \tag{11}$$

[2] $W^{1,2}(\Omega)$ is the ordinary Sobolev space [4], $W^{1/2,2}(\partial\Omega)$ is its trace space and $W^{-1/2,2}(\partial\Omega)$ is the dual space of $W^{1/2,2}(\partial\Omega)$, $W^{-1/2,2}(\partial\Omega) = [W^{1/2,2}(\partial\Omega)]^*$. If Σ is a subsurface of $\partial\Omega$, $W^{1/2,2}(\Sigma) = \{\varphi_{|\Sigma}, \varphi \in W^{1/2,2}(\partial\Omega)\}$ and $W^{-1/2,2}(\Sigma) = \{\varphi \in W^{1/2,2}(\partial\Omega) : \mathrm{supp}\,\varphi \subset \Sigma\}^*$. To keep notation to the minimum, if $\psi \in W^{-1/2,2}(\partial\Omega)$ and $\varphi \in W^{1/2,2}(\partial\Omega)$, we use the integral $\int_{\partial\Omega} \psi\varphi$ to denote the value of the functional ψ at φ. It will be clear from the context when the integral keeps its usual meaning.

[3] More recently, Theorem 2 has been extended to cover the case of concentrated forces $\boldsymbol{s}(\boldsymbol{u}) \in W^{-1,q}(\partial\Omega)$ for some $q < 2$ [17]. In such a case, (12) writes

$$\int_{\Omega_R} |\hat{\nabla}\boldsymbol{u}|^2 \le c\, e^{c_0(R-h)} \|\boldsymbol{s}(\boldsymbol{u})\|^2_{W^{-1,q}(\partial\Omega)}.$$

then $\boldsymbol{u}$ *satisfy*

$$\int_{\Omega_R} |\hat{\nabla}\boldsymbol{u}|^2 \leq c_0\, e^{-\frac{(R-\delta)}{\gamma(\delta)}} \int_{\Omega} |\hat{\nabla}\boldsymbol{u}|^2, \tag{12}$$

where $\Omega_R = \{x \in \Omega : |x_3| < R\}$, γ *is defined by* (10) *and* c_0 *is positive constant depending only on the elastic moduli and* T_δ.

The determination of the best constant appearing in (9), (12) is a relevant issue of the Saint–Venant's principle. For this question and other ones related to Saint–Venant's theory we quote [6], [7], [9], [11], [12], [13], [15], [17].

In this paper, we shall deal with Saint–Venant's problem under general boundary conditions, looking for assumptions on data assuring that the corresponding solution satisfies estimates of the type (12). In particular, denoting by $\mathfrak{R}$ the linear space of infinitesimal rigid displacements, for the mixed problem

$$\begin{aligned} \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\mathrm{div}\,\boldsymbol{u} &= \boldsymbol{0} \quad \text{in } \Omega,\\ \boldsymbol{s}(\boldsymbol{u}) &= \boldsymbol{0} \quad \text{on } \mathcal{M},\\ \boldsymbol{u} &= \boldsymbol{\sigma} \quad \text{on } \mathcal{C}_{-h}\cup\mathcal{C}_h, \end{aligned} \tag{13}$$

which is of some interest in the applications, we prove

Theorem 3 *Let* $\boldsymbol{\sigma}$ *be the trace of a field in* $W^{1,2}(\Omega)$. *The solution* $\boldsymbol{u} \in W^{1,2}(\Omega)$ *to* (13) *satisfies the decay estimate*

$$\int_{\Omega_R} |\hat{\nabla}\boldsymbol{u}|^2 \leq c\, e^{c_0(R-h)} \|\boldsymbol{\sigma}\|^2_{W^{1/2,2}(\partial\Omega)}, \tag{14}$$

for $R \ll h$, *where* c *and* c_0 *are positive constants independent of* R, *if and only if*

$$\int_{\mathcal{C}_{-h}\cup\mathcal{C}_h} \boldsymbol{\sigma}\cdot\boldsymbol{s}(\boldsymbol{v}) = 0, \tag{15}$$

for every solution $\boldsymbol{v}$ *to the equations*

$$\begin{aligned} \mu\Delta\boldsymbol{v} + (\lambda+\mu)\nabla\mathrm{div}\,\boldsymbol{u} &= \boldsymbol{0} \quad \text{in } \Omega,\\ \boldsymbol{s}(\boldsymbol{v}) &= \boldsymbol{0} \quad \text{on } \mathcal{M},\\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \mathcal{C}_{-h},\\ \boldsymbol{u} &= \boldsymbol{\varrho} \quad \text{on } \mathcal{C}_h, \end{aligned} \tag{16}$$

with $\boldsymbol{\varrho}$ *arbitrary field of* $\mathfrak{R}$.

2 The boundary value problems of elastostatics for cylinders

We assume that the cylinder (2) enjoys the properties assuring application of the divergence theorem and coincides with the stress free reference configuration of a linearly elastic body $\mathcal{B}$ with a symmetric elasticity tensor C, *i.e.*, a map from $\Omega \times$ Lin into Lin, linear on Lin, such that $\mathsf{C}[\boldsymbol{W}] = \boldsymbol{0}$, for every $\boldsymbol{W} \in$ Skw,

$$|\boldsymbol{E}|^2 \leq \boldsymbol{E}\cdot\mathsf{C}[\boldsymbol{E}] = E_{ij}\mathsf{C}_{ijhk}E_{hk} \leq \mu_0|\boldsymbol{E}|^2, \quad \forall\,\boldsymbol{E} \in \mathrm{Sym},$$

for some positive scalar μ_0, and

$$\boldsymbol{E}\cdot\mathsf{C}[\boldsymbol{F}]=\boldsymbol{F}\cdot\mathsf{C}[\boldsymbol{E}],\quad \forall\,\boldsymbol{E},\boldsymbol{F}\in\mathrm{Lin}. \tag{17}$$

If $\mathcal{B}$ is homogeneous, then C is independent of x; moreover, if $\mathcal{B}$ is isotropic, then

$$\mathsf{C}[\boldsymbol{E}]=2\mu\,\mathrm{sym}\boldsymbol{E}+(\mathrm{tr}\boldsymbol{E})\mathbf{1}, \tag{18}$$

where λ and μ are the Lamé moduli, satisfying $\mu>0$, $3\lambda+2\mu>0$.

If $\boldsymbol{u}(x)$ is a displacement yielding the body from Ω to a new equilibrium configuration, then $\boldsymbol{u}$ is a solution to the equations

$$\mathrm{div}\,\mathsf{C}[\nabla\boldsymbol{u}]=\mathbf{0}\quad[\partial_j(\mathsf{C}_{ijhk}\partial_k u_h)=0]\quad \text{in } \Omega. \tag{19}$$

If the body is homogeneous and isotropic, then the equations (19) takes the form $(1)_1$.

A weak solution to (19) is a field $\boldsymbol{u}\in W^{1,q}(\Omega)$, $q\in(1,+\infty)$, wich satisfies

$$\int_\Omega \nabla\boldsymbol{\varphi}\cdot\mathsf{C}[\nabla\boldsymbol{u}]=0,\quad \forall\boldsymbol{\varphi}\in C_0^\infty(\Omega). \tag{20}$$

For $q=2$ the solution is said to be *variational*. To a variational solution $\boldsymbol{u}$ a traction $\boldsymbol{s}(\boldsymbol{u})=\mathsf{C}[\nabla\boldsymbol{u}]\boldsymbol{n}\in W^{-1/2,2}(\partial\Omega)$ on the boundary is associated [4], where $\boldsymbol{n}$ is the unit outward (with respect to Ω) normal to $\partial\Omega$. If $\mathcal{B}$ is isotropic, then $\boldsymbol{s}(\boldsymbol{u})=2\mu(\hat{\nabla}\boldsymbol{u})\boldsymbol{n}+\lambda(\mathrm{div}\,\boldsymbol{u})\boldsymbol{n}$.

Let $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ be two variational solutions to (19). The following relations are classical

$$\begin{aligned}\int_\Omega \nabla\boldsymbol{u}\cdot\mathsf{C}[\nabla\boldsymbol{u}]&=\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{s}(\boldsymbol{u})\quad \textit{work and energy theorem},\\ \int_{\partial\Omega}\tilde{\boldsymbol{u}}\cdot\boldsymbol{s}(\boldsymbol{u})&=\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{s}(\tilde{\boldsymbol{u}})\quad \textit{Betti's reciprocity relation.}\end{aligned} \tag{21}$$

To uniquely determine a solution to (19) we have to require some boundary conditions. We shall be concerned with the following general boundary–value problem

$$\begin{aligned}\mathrm{div}\,\mathsf{C}[\nabla\boldsymbol{u}]&=\mathbf{0} &&\text{in } \Omega,\\ \alpha_1\boldsymbol{u}_n+\alpha_2\boldsymbol{s}(\boldsymbol{u})_n&=\mathbf{0} &&\text{on } \mathcal{M},\\ \alpha_3\boldsymbol{u}_t+\alpha_4\boldsymbol{s}(\boldsymbol{u})_t&=\mathbf{0} &&\text{on } \mathcal{M},\\ \beta_1\boldsymbol{u}_n+\beta_2\boldsymbol{s}(\boldsymbol{u})_n&=\boldsymbol{\sigma}_n &&\text{on } \mathcal{C}_{-h}\cup\mathcal{C}_h,\\ \beta_3\boldsymbol{u}_t+\beta_4\boldsymbol{s}(\boldsymbol{u})_t&=\boldsymbol{\sigma}_t &&\text{on } \mathcal{C}_{-h}\cup\mathcal{C}_h,\end{aligned} \tag{22}$$

where α_i,β_i are assigned scalars such that the products $\alpha_1\alpha_2$, $\alpha_3\alpha_4$, $\beta_1\beta_2$, $\beta_3\beta_4$ are nonnegative, $\boldsymbol{\sigma}$ is an assigned field on the bases and

$$\boldsymbol{\sigma}_n=(\boldsymbol{\sigma}\cdot\boldsymbol{n})\boldsymbol{n},\quad \boldsymbol{\sigma}_t=\boldsymbol{\sigma}-\boldsymbol{\sigma}_n.$$

In particular, by suitably choosing the scalars α_i,β_i, from (22) we have the classical problems

(I) $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 = 0,\ \alpha_2 = \alpha_4 = \beta_2 = \beta_4 = 1$ *traction problem*
(II) $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 > 0,\ \alpha_2 = \alpha_4 = \beta_2 = \beta_4 > 0$ *Robin problem*
(III) $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = 0,\ \alpha_1 = \alpha_4 = \beta_1 = \beta_4 = 1$ *first contact problem*
(IV) $\alpha_1 = \alpha_4 = \beta_1 = \beta_4 = 0,\ \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = 1$ *second contact problem*
(V) $\alpha_1 = \alpha_3 = \beta_2 = \beta_4 = 0,\ \alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 1$ *mixed problem*

Let us recall that in (I) we must require conditions (3).

To deal with (22) we need Korn's inequality (see [4], [10], [13], [14]) that we state in the form we shall use.

Lemma 1 *Let $\mathcal{T}$ be the cylinder of section $\mathcal{C}$ and length δ, and let p be a semi–norm on $W^{1,2}(\mathcal{T})$ which vanishes on $\mathfrak{R}$. There is a positive constant c_κ, depending only on $\mathcal{C}$, δ and p, such that*

$$\int_{\mathcal{T}} |\nabla \boldsymbol{u}|^2 \leq c_\kappa \left\{ \int_{\mathcal{T}} |\hat{\nabla} \boldsymbol{u}|^2 + p(\boldsymbol{u}) \right\}. \tag{23}$$

Choosing $p(\boldsymbol{u}) = \int_{\mathcal{M}} |\boldsymbol{u}_t|^2$, (23) writes

$$\int_{\mathcal{T}} |\nabla \boldsymbol{u}|^2 \leq c_\kappa \left\{ \int_{\mathcal{T}} |\hat{\nabla} \boldsymbol{u}|^2 + \int_{\mathcal{M}} |\boldsymbol{u}_t|^2 \right\}. \tag{24}$$

If $\mathcal{C}$ is not a disk, then

$$\int_{\mathcal{T}} |\nabla \boldsymbol{u}|^2 \leq c_\kappa \left\{ \int_{\mathcal{T}} |\hat{\nabla} \boldsymbol{u}|^2 + \int_{\mathcal{M}} |\boldsymbol{u}_n|^2 \right\}. \tag{25}$$

If $\mathcal{C}$ is a disk, then

$$\int_{\mathcal{T}} \left| \nabla \boldsymbol{u} - \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} (\tilde{\nabla} \boldsymbol{u}) \boldsymbol{e}_3 \right|^2 \leq c_\kappa \left\{ \int_{\mathcal{T}} |\hat{\nabla} \boldsymbol{u}|^2 + \int_{\mathcal{M}} |\boldsymbol{u}_n|^2 \right\}. \tag{26}$$

Under natural conditions (for instance, $\boldsymbol{\sigma} \in W^{-1/2,2}(\partial\Omega)$ in (I) and $\boldsymbol{\sigma} \in W^{1/2,2}(\mathcal{C}_{-h} \cup \mathcal{C}_h)$, $\boldsymbol{\sigma} \in W^{-1/2,2}(\mathcal{M})$ in (V)) by coupling standard techniques of functional analysis together with Korn's inequality (23), one shows that a unique variational solution to (22) exists (up to an additive field of $\mathfrak{R}$ in (I) and and additive field $\alpha \boldsymbol{e}_3 \times \boldsymbol{x}$, $\alpha \in \mathbb{R}$ in (III), if $\mathcal{C}$ is a disk). Moreover, one shows that if C, $\mathcal{C}$ and $\boldsymbol{\sigma}$ are more regular, then so the weak solutions do. In particular, if C, $\mathcal{C}$ and $\boldsymbol{\sigma}$ are of class C^∞, then $\boldsymbol{u}$ is of class C^∞ in $\overline{\Omega}$ except on ∂C_{-h} and ∂C_h.

3 Decay estimates for system (22)

Estimates of Saint–Venant's type could be of interest also in the boundary–value problems expressed by (22) different from (I). Let us show that for the weak solutions to the equations

$$\begin{aligned} \operatorname{div} \mathsf{C}[\nabla \boldsymbol{u}] &= \boldsymbol{0} \quad \text{in } \Omega, \\ \alpha_1 \boldsymbol{u}_n + \alpha_2 \boldsymbol{s}(\boldsymbol{u})_n &= \boldsymbol{0} \quad \text{on } \mathcal{M}, \\ \alpha_3 \boldsymbol{u}_t + \alpha_4 \boldsymbol{s}(\boldsymbol{u})_t &= \boldsymbol{0} \quad \text{on } \mathcal{M}, \end{aligned} \tag{27}$$

with $\mathcal{C}$ not a disk, a Saint–Venant's estimate holds without any compatibility condition on the data on the bases, under suitable assumptions on the scalars α_i.

Theorem 4 *Let $\mathcal{C}$ be not a disk, let at least one of the scalars α_1, α_3 be nonzero and let $\boldsymbol{u}$ be a weak solution to (27). There is a constant c_0 depending only on μ_0, α_i and the Korn constant of a small cylinder of length δ such that for $R < h - \delta$*

$$\int_{\Omega_R} |\hat{\nabla}\boldsymbol{u}|^2 \le c\, e^{c_0(R-h)} \int_{\Omega} |\hat{\nabla}\boldsymbol{u}|^2. \tag{28}$$

Proof - We give the details of the proof for the problem corresponding to the choice $\alpha_1\alpha_2 > 0$, $\alpha_3\alpha_4 \ge 0$ and one of α_3, α_4 different from zero. The proof in the other cases is analogous. Let $g(x_3)$ be the function vanishing for $|x_3| \ge L - R$, equal to 1 for $|x_3| \le L - R - \delta$ and equal to $\delta^{-1}(L - R - |x_3|)$ for $L-R-\delta < |x_3| < L-R$. Setting $\mathcal{T}_\delta^+ = \{x \in \Omega : L-R-\delta < x_3 < L-R\}$ and $\mathcal{T}_\delta^- = \{x \in \Omega : -L + R + \delta < x_3 < -L + R\}$, a standard computation yields

$$G(R) = \int_{\Omega} g|\hat{\nabla}\boldsymbol{u}|^2 + \alpha \int_{\mathcal{M}} |\boldsymbol{u}_n|^2 \le \frac{1}{\delta}\left\{\int_{\mathcal{T}_\delta^+} \boldsymbol{u}\cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3 + \int_{\mathcal{T}_\delta^-} \boldsymbol{u}\cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3\right\}.$$

where $\alpha = \alpha_1/\alpha_2$. Hence, since by Cauchy's, Poincaré's and Korn's inequalities and the basic calculus

$$\begin{aligned}\int_{T^{\pm}_{R,\delta}} \boldsymbol{u}\cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3 &\le \frac{1}{2}\int_{T^{\pm}_{R,\delta}} (|\boldsymbol{u}|^2 + \mu_0^2|\hat{\nabla}\boldsymbol{u}|^2) \\ &\le \frac{1}{c_0}\left\{\int_{T^{\pm}_{R,\delta}} |\hat{\nabla}\boldsymbol{u}|^2 + \alpha\int_{\mathcal{M}\cap\partial T^{\pm}_{R,\delta}} |\boldsymbol{u}_n|^2\right\} = \frac{\delta}{c_0}G'(R)\end{aligned}$$

we have

$$c_0 G(R) \le -G'(R). \tag{29}$$

and (33) follows by a simple integration. □

If $\mathcal{C}$ is a disk we consider the equations

$$\begin{aligned} \operatorname{div}\mathbf{C}[\nabla \boldsymbol{u}] &= \boldsymbol{0} && \text{in } \Omega,\\ \boldsymbol{u}_n &= \boldsymbol{0} && \text{on } \mathcal{M},\\ \boldsymbol{s}(\boldsymbol{u})_t &= \boldsymbol{0} && \text{on } \mathcal{M},\\ \boldsymbol{u}_n &= \boldsymbol{\sigma}_n && \text{on } \mathcal{C}_{-h}\cup\mathcal{C}_h,\\ \boldsymbol{s}(\boldsymbol{u})_t &= \boldsymbol{\sigma}_t && \text{on } \mathcal{C}_{-h}\cup\mathcal{C}_h. \end{aligned} \tag{30}$$

Since for $\boldsymbol{\varrho} = \boldsymbol{x}\times \boldsymbol{e}_3$

$$\begin{aligned} &\int_{\mathcal{C}_{-h}} \boldsymbol{\varrho}\cdot \boldsymbol{s}(\boldsymbol{u}) = \int_{\mathcal{C}_{x_3}} \boldsymbol{\varrho}\cdot \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{e}_3\cdot\int_{\mathcal{C}_{-h}} \boldsymbol{x}\times\boldsymbol{\sigma}_t, \qquad x_3\in[-h,0]\\ &\int_{\mathcal{C}_{h}} \boldsymbol{\varrho}\cdot \boldsymbol{s}(\boldsymbol{u}) = \int_{\mathcal{C}_{x_3}} \boldsymbol{\varrho}\cdot \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{e}_3\cdot\int_{\mathcal{C}_{h}} \boldsymbol{x}\times\boldsymbol{\sigma}_t, \qquad x_3\in[0,h]\end{aligned}$$

and

$$\int_{T_\delta^-} \boldsymbol{\varrho}^- \cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3 = \int_{\mathcal{C}_{-L+R+\delta}} \boldsymbol{\varrho}^- \cdot \boldsymbol{s}(\boldsymbol{u}),$$
$$\int_{T_\delta^+} \boldsymbol{\varrho}^+ \cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3 = \int_{\mathcal{C}_{L-R-\delta}} \boldsymbol{\varrho}^+ \cdot \boldsymbol{s}(\boldsymbol{u}),$$

we have

$$\int_{\mathcal{C}_{-h}} \boldsymbol{x} \times \boldsymbol{\sigma}_t = \int_{\mathcal{C}_h} \boldsymbol{x} \times \boldsymbol{\sigma}_t = \mathbf{0} \ \Rightarrow \ \int_{T_\delta^\pm} \boldsymbol{\varrho}^\pm \cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3 = \mathbf{0}, \tag{31}$$

with $\boldsymbol{\varrho}^\pm = \alpha^\pm \boldsymbol{x} \times \boldsymbol{e}_3$, for all scalars α^+ and α^-.

Theorem 5 *Let $\mathcal{C}$ be disk and let $\boldsymbol{\sigma}_n \in W^{1/2,2}(\partial\Omega)$, $\boldsymbol{\sigma}_t \in W^{-1/2,2}(\partial\Omega)$. If*

$$\int_{\mathcal{C}_{-h}} \boldsymbol{x} \times \boldsymbol{\sigma}_t = \int_{\mathcal{C}_h} \boldsymbol{x} \times \boldsymbol{\sigma}_t = \mathbf{0}, \tag{32}$$

then the solution to (30), unique modulo an additive infinitesimal rotation around the axis $\boldsymbol{e}_3$, satisfies

$$\int_{\Omega_R} |\hat{\nabla} \boldsymbol{u}|^2 \le c\, e^{c_0(R-h)} \big\{ \|\boldsymbol{\sigma}_n\|_{W^{1/2,2}(\partial\Omega)} + \|\boldsymbol{\sigma}_t\|_{W^{-1/2,2}(\partial\Omega)} \big\}. \tag{33}$$

where c_0 is a positive constant independent of R.

Proof - By (26), (31), Schwarz's inequality and by a suitably choosing of $\boldsymbol{\varrho}^\pm$, we have

$$\left| \int_{T_{R,\delta}^\pm} \boldsymbol{u} \cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3 \right| = \left| \int_{T_{R,\delta}^\pm} (\boldsymbol{u} - \boldsymbol{\varrho}^\mp) \cdot \mathbf{C}[\nabla \boldsymbol{u}]\boldsymbol{e}_3 \right| \le \mu_0 \sqrt{c_\kappa} \int_{T_\delta^\pm} |\hat{\nabla} \boldsymbol{u}|^2.$$

Therefore, we easily arrive at (29) (with $\alpha = 0$) and so to the desired result. □

If $\mathcal{C}$ is a polygon and $\mathbf{C}$ is constant and isotropic, in (30) we can require $\mathbf{C}$ to be only strongly elliptic, *i.e.*, $\mu(\lambda + 2\mu) > 0$ [18].

4 Saint–Venant's problem with mixed boundary conditions

It is clear that among problems (22) the closest to the historical Saint–Venant's one and to the applications in engineering are those in which the mantle of the cylinder is force free

$$\begin{aligned} \operatorname{div} \mathbf{C}[\nabla \boldsymbol{u}] &= \mathbf{0} && \text{in } \Omega, \\ \boldsymbol{s}(\boldsymbol{u}) &= \mathbf{0} && \text{on } \mathcal{M}, \\ \beta_1 \boldsymbol{u}_n + \beta_2 \boldsymbol{s}(\boldsymbol{u})_n &= \boldsymbol{\sigma}_n && \text{on } \mathcal{C}_{-h} \cup \mathcal{C}_h, \\ \beta_3 \boldsymbol{u}_t + \beta_4 \boldsymbol{s}(\boldsymbol{u})_t &= \boldsymbol{\sigma}_t && \text{on } \mathcal{C}_{-h} \cup \mathcal{C}_h. \end{aligned} \tag{34}$$

where, for simplicity, we assume[4] $\beta_1 = \beta_3 = 1$. Of course, conditions on $\boldsymbol{\sigma}$ assuring

$$\int_{\mathcal{C}_h} \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0}, \qquad \int_{\mathcal{C}_h} \boldsymbol{x} \times \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0}, \tag{35}$$

imply (12).

Let $\boldsymbol{v}_\varrho$ be the solution to

$$\begin{aligned} \operatorname{div} \mathbf{C}[\nabla \boldsymbol{v}] &= \boldsymbol{0} && \text{in } \Omega, \\ \boldsymbol{s}(\boldsymbol{v}) &= \boldsymbol{0} && \text{on } \mathcal{M}, \\ \beta_1 \boldsymbol{v}_n + \beta_2 \boldsymbol{s}(\boldsymbol{v})_n &= \boldsymbol{0} && \text{on } \mathcal{C}_{-h}, \\ \beta_3 \boldsymbol{v}_t + \beta_4 \boldsymbol{s}(\boldsymbol{v})_t &= \boldsymbol{0} && \text{on } \mathcal{C}_{-h} \\ \beta_1 \boldsymbol{v}_n + \beta_2 \boldsymbol{s}(\boldsymbol{v})_n &= \boldsymbol{\varrho}_n && \text{on } \mathcal{C}_h, \\ \beta_3 \boldsymbol{v}_t + \beta_4 \boldsymbol{s}(\boldsymbol{v})_t &= \boldsymbol{\varrho}_t && \text{on } \mathcal{C}_h, \end{aligned} \tag{36}$$

with $\boldsymbol{\varrho} \in \mathfrak{R}$. From Betti's theorem

$$\begin{aligned} &\int_{\mathcal{C}_{-h} \cup \mathcal{C}_h} \boldsymbol{u} \cdot \boldsymbol{s}(\boldsymbol{v}) = \int_{\mathcal{C}_{-h} \cup \mathcal{C}_h} \boldsymbol{v} \cdot \boldsymbol{s}(\boldsymbol{u}) = \int_{\mathcal{C}_{-h} \cup \mathcal{C}_h} [\boldsymbol{u}_n \cdot \boldsymbol{s}(\boldsymbol{v})_n + \boldsymbol{u}_t \cdot \boldsymbol{s}(\boldsymbol{v})_t] \\ &= \int_{\mathcal{C}_{-h} \cup \mathcal{C}_h} [(\boldsymbol{\sigma}_n - \beta_2 \boldsymbol{s}(\boldsymbol{u})_n) \cdot \boldsymbol{s}(\boldsymbol{v})_n + (\boldsymbol{\sigma}_t - \beta_4 \boldsymbol{s}(\boldsymbol{u})_t) \cdot \boldsymbol{s}(\boldsymbol{v})_t] \\ &= - \int_{\mathcal{C}_{-h}} [\beta_2 \boldsymbol{s}(\boldsymbol{v})_n \cdot \boldsymbol{s}(\boldsymbol{u})_n + \beta_4 \boldsymbol{s}(\boldsymbol{v})_t \cdot \boldsymbol{s}(\boldsymbol{u})_t] \\ &+ \int_{\mathcal{C}_h} [(\boldsymbol{\varrho}_n - \beta_2 \boldsymbol{s}(\boldsymbol{v})_n) \cdot \boldsymbol{s}(\boldsymbol{u})_n + (\boldsymbol{\varrho}_t - \beta_4 \boldsymbol{s}(\boldsymbol{v})_t) \cdot \boldsymbol{s}(\boldsymbol{u})_t] . \end{aligned} \tag{37}$$

Hence, it follows

$$\int_{\mathcal{C}_{-h} \cup \mathcal{C}_h} \boldsymbol{\sigma} \cdot \boldsymbol{s}(\boldsymbol{v}) = \int_{\mathcal{C}_h} \boldsymbol{\varrho} \cdot \boldsymbol{s}(\boldsymbol{u}). \tag{38}$$

Therefore, we have

Theorem 6 *Let $\boldsymbol{u}$ be the variational solution to (34). It holds*

$$\int_{\mathcal{C}_{-h} \cup \mathcal{C}_h} \boldsymbol{\sigma} \cdot \boldsymbol{s}(\boldsymbol{v}_\varrho) = 0, \ \forall \boldsymbol{\varrho} \in \mathfrak{R} \implies \boldsymbol{u} \text{ satisfies (12)}, \tag{39}$$

where $\boldsymbol{v}_\varrho$ is the solution to (36).

Clearly, (39) is only qualitative, unless we are able to specify analytically the solutions to (36). To get this information, let us consider the mixed problem

$$\begin{aligned} \operatorname{div} \mathbf{C}[\nabla \boldsymbol{u}] &= \boldsymbol{0} && \text{in } \Omega, \\ \boldsymbol{s}(\boldsymbol{u}) &= \boldsymbol{0} && \text{on } \mathcal{M}, \\ \boldsymbol{u} &= \boldsymbol{\sigma} && \text{on } \mathcal{C}_{-h} \cup \mathcal{C}_h, \end{aligned} \tag{40}$$

which, as is clear enough, is of interest in applications.

[4] Clearly, for $\beta_1 = \beta_3 = 0$ we recover the classical Saint–Venant problem.

If

$$\partial_j \mathsf{C}_{ij3k} = 0, \tag{41}$$

then the fields

$$\boldsymbol{v}_i = \kappa(x_3 + h)\boldsymbol{e}_i, \quad i = 1, 2, 3, \tag{42}$$

with κ suitable constant, are the solutions to (36) corresponding to translations and (38) yields

$$\left[(41) \text{ and } \int_{\mathcal{C}_{-h}} \boldsymbol{\sigma} = \int_{\mathcal{C}_h} \boldsymbol{\sigma}\right] \implies \int_{\mathcal{C}_h} \boldsymbol{s}(\boldsymbol{u}) = \mathbf{0}.$$

If C is constant and isotropic, then

$$\begin{aligned} \boldsymbol{v}_4 &= (x_3 + h)\Big(0, \frac{\kappa}{2}(x_3 + h), -x_2\Big), \\ \boldsymbol{v}_5 &= (x_3 + h)\Big(-\frac{\kappa}{2}(x_3 + h), 0, x_1\Big), \\ \boldsymbol{v}_6 &= (x_3 + h)(x_2, -x_1, 0) \end{aligned} \tag{43}$$

and (37) implies

Theorem 7 *Let* C *be constant and isotropic and let* $\boldsymbol{\sigma} \in W^{1/2,2}(\partial\Omega)$*. If*

$$\int_{\mathcal{C}_{-h}\cup\mathcal{C}_h} \boldsymbol{\sigma} \cdot \boldsymbol{s}(\boldsymbol{v}_i) = 0, \tag{44}$$

where the fields $\boldsymbol{v}_i$*,* $i = 1, \ldots, 6$ *are expressed by (42), (43), then the solution to (40) satisfies the estimate*

$$\int_{\Omega_R} |\hat{\nabla}\boldsymbol{u}|^2 \leq c\, e^{c_0(R-h)} \|\boldsymbol{\sigma}\|_{W^{1/2,2}(\partial\Omega)}, \tag{45}$$

for some positive constant c_0 *independent of* R*.*

• Let C and $\boldsymbol{\sigma}$ satisfy the hypotheses of Theorem 7 and let $\boldsymbol{u}'$ be the Saint–Venant elementary solution such that

$$\int_{\mathcal{C}_{-h}\cup\mathcal{C}_h} (\boldsymbol{\sigma} - \boldsymbol{u}') \cdot \boldsymbol{s}(\boldsymbol{v}_i) = 0.$$

By virtue of (45) if $1 \gg |\mathcal{C}|/h$, then $\hat{\nabla}\boldsymbol{u} \simeq \hat{\nabla}\boldsymbol{u}'$ away from the bases.

5 A counter–example

From a physical point of view it is clear that the space of *elastic fields with finite potential energy* $W^{1,2}(\Omega)$ is the natural setting for the traction problem of elastostatics and so for the Saint–Venant one under the hypothesis $\mathbf{C} \in L^\infty(\Omega)$. In this section we aim at pointing out that $W^{1,2}(\Omega)$ is natural also from a mathematical point of view. Indeed, we shall point out that uniqueness is lost for $q < 2$. To be precise, we show that, for every q in a left neighborhood of 2, there is an elasticity tensor $\mathbf{C}_0$ such that the problem

$$\begin{aligned} \operatorname{div} \mathbf{C}_0[\nabla \boldsymbol{u}] &= \mathbf{0} \quad \text{in } \Omega, \\ \alpha_1 \boldsymbol{u}_n + \alpha_2 \boldsymbol{s}(\boldsymbol{u})_n &= \mathbf{0} \quad \text{on } \mathcal{M}, \\ \alpha_3 \boldsymbol{u}_t + \alpha_4 \boldsymbol{s}(\boldsymbol{u})_t &= \mathbf{0} \quad \text{on } \mathcal{M}, \\ \beta_1 \boldsymbol{u}_n + \beta_2 \boldsymbol{s}(\boldsymbol{u})_n &= \mathbf{0} \quad \text{on } \mathcal{C}_{-h} \cup \mathcal{C}_h, \\ \beta_3 \boldsymbol{u}_t + \beta_4 \boldsymbol{s}(\boldsymbol{u})_t &= \mathbf{0} \quad \text{on } \mathcal{C}_{-h} \cup \mathcal{C}_h, \end{aligned} \tag{46}$$

has a non zero solution in $W^{1,q}(\Omega)$.

Let

$$\mathbf{C}_0[\boldsymbol{E}] = (\boldsymbol{L} \otimes \boldsymbol{L})\boldsymbol{E} + \xi^2 \operatorname{sym} \boldsymbol{E}. \tag{47}$$

where

$$\boldsymbol{E} = \mathbf{1} + 3\boldsymbol{e}_r \otimes \boldsymbol{e}_r.$$

Observe that $\mathbf{C}_0$ is bounded on $\mathbb{R}^3$ and is of class C^∞ in $\mathbb{R}^3 \setminus \{o\}$. Since

$$\boldsymbol{E} \cdot \mathbf{C}_0(\boldsymbol{E}) = (\boldsymbol{L} \cdot \boldsymbol{E})^2 + \xi^2 |\boldsymbol{E}|^2 \geq \xi^2 |\boldsymbol{E}|^2, \quad \forall\, \boldsymbol{E} \in \mathrm{Sym},$$

$\mathbf{C}_0$ satisfies (17) with $\mu_0 = (16 + \xi^2)/\xi^2$. By a simple computation [3] we see that the system

$$\operatorname{div} \mathbf{C}_0[\nabla \boldsymbol{u}] = \mathbf{0} \tag{48}$$

admits the solution

$$\tilde{\boldsymbol{u}} = r^\alpha \boldsymbol{e}_r, \quad \alpha = -\frac{3}{2}\left(\frac{|\xi|}{\sqrt{16+\xi^2}} + 1\right). \tag{49}$$

Clearly, $\tilde{\boldsymbol{u}} \in W^{1,q}(\Omega)$ with $\lim_{\xi \to 0} q(\xi) = 2^-$. It is simple to see that $\tilde{\boldsymbol{u}}$ is a weak solution to $(46)_1$ [3]. Let $\boldsymbol{u} \in W^{1,2}(\Omega)$ be the solution to (22) corresponding to data $\boldsymbol{\sigma}_n = \beta_1 \tilde{\boldsymbol{u}}_n + \beta_2 \boldsymbol{s}(\tilde{\boldsymbol{u}})_n$, $\boldsymbol{\sigma}_t = \beta_3 \tilde{\boldsymbol{u}}_n + \beta_4 \boldsymbol{s}(\tilde{\boldsymbol{u}})_n$. Hence it follows that the field $\tilde{\boldsymbol{u}} - \boldsymbol{u} \in W^{1,q}(\Omega)$ is a nonzero solution to (46).

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