A NECESSARY CONDITION FOR H^∞ WELL-POSEDNESS OF p-EVOLUTION EQUATIONS

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ABSTRACT. We consider p-evolution equations, for $p \geq 2$, with complex valued coefficients. We prove that a necessary condition for H^∞ well-posedness of the associated Cauchy problem is that the imaginary part of the coefficient of the subprincipal part (in the sense of Petrowski) satisfies a decay estimate as $|x| \to +\infty$.

1. Introduction and main result

Given an integer $p \geq 2$, we consider in $[0,T] \times \mathbb{R}$ the linear partial differential operator P of the form

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t,x)D_x^j\,, \tag{1.1}$$

with $D = \frac{1}{i}\partial$, $a_p \in C([0,T];\mathbb{R})$ and $a_j \in C([0,T];\mathcal{B}^\infty)$ for $0 \leq j \leq p-1$, (here $\mathcal{B}^\infty = \mathcal{B}^\infty(\mathbb{R}_x)$ is the space of complex valued functions which are bounded on $\mathbb{R}_x$ together with all their x-derivatives). We are dealing with a non-kowalewskian evolution operator; anisotropic evolution operators of the form (1.1) are usually called $p-$evolution operators. The condition that a_p is real valued means that the principal symbol (in the sense of Petrowski) of P has the real characteristic $\tau = -a_p(t)\xi^p$; by the Lax-Mizohata theorem (cf. [24]), this is a necessary condition to have a unique solution, in Sobolev spaces, of the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = g(x) & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

in a neighborhood of $t = 0$. We notice that for $p = 2$ the operator is of Schrödinger type, for $p = 3$ we have the same principal part as the Korteweg-De Vries equation. Many results of well-posedness in Sobolev spaces of (1.2) are available under the assumption that all the coefficients a_j of (1.1) are real (see, for instance, [1], [2], [3], [9], [14], [15]). On the contrary, when the coefficients $a_j(t,x)$ for $1 \leq j \leq p-1$ are not real, the theory is well developed only in the case $p = 2$: we know from the pioneering papers [20], [21] that a decay condition as $|x| \to +\infty$ on $\operatorname{Im} a_1$ is necessary and sufficient for well-posedness of the Cauchy problem (1.2) in H^∞. Sufficient conditions for well-posedness in H^∞ and/or Gevrey classes for 2 or 3$-$evolution equations have been given by many authors (see, for instance, [19], [25], [11], [22], [10], [16], [17], [13]). The general case $p \geq 2$ has been recently considered in [6], proving H^∞ well-posedness of the Cauchy problem (1.2) under suitable decay conditions, as $|x| \to +\infty$, on $\operatorname{Im} D_x^\beta a_j$, for $j \leq p-1$ and $[\beta/2] \leq j-1$. These results have been extended to the case of weighted Sobolev spaces in [8], to the case of first order systems of pseudo-differential operators

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in [4], to the case of higher order equations in [5], and to semi-linear 3-evolution equations in [7].

As far as we know, there are no results available about necessary conditions for H^∞ well-posedness for p-evolution equations, $p \geq 3$.

In this paper we give a necessary condition for well-posedness of the Cauchy problem (1.2) in H^∞, generalizing to the case $p \geq 2$ the necessary condition given by Ichinose in [20] for $p = 2$. More precisely, in [20] Ichinose considered, for $x \in \mathbb{R}^n$, the operator

$$P = D_t - a_2\triangle_x + \sum_{j=1}^{n} a_1^{(j)}(x)D_{x_j} + c(x), \tag{1.3}$$

with $a_2 \in (0,1]$ and $a_1^{(j)}, c \in \mathcal{B}^\infty(\mathbb{R}^n)$. He proved that a necessary condition for H^∞ well-posedness of the associated Cauchy problem is the existence of non-negative constants M, N such that

$$\sup_{x\in\mathbb{R}^n,\, \omega\in S^{n-1}} \left| \sum_{j=1}^{n} \int_0^{\varrho} \operatorname{Im} a_1^{(j)}(x + 2a_2\theta\omega)\omega_j d\theta \right| \leq M\log(1+\varrho) + N \qquad \forall\varrho > 0, \tag{1.4}$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. The same condition is also sufficient (cf. [21]) only in the case of space dimension $n = 1$.

In this paper we assume that there exists a constant $m > 0$ such that

$$|a_p(t)| \geq m \qquad \forall t \in [0,T] \tag{1.5}$$

and prove the following:

Theorem 1.1. *Let P be the operator in* (1.1) *with $a_p \in C([0,T];\mathbb{R})$ satisfying* (1.5) *and $a_j \in C([0,T];\mathcal{B}^\infty)$ for $0 \leq j \leq p-1$. A necessary condition for the Cauchy problem* (1.2) *to be well-posed in H^∞ is the existence of constants $M, N > 0$ such that:*

$$\sup_{x\in\mathbb{R}} \min_{0\leq\tau\leq t\leq T} \int_{-\varrho}^{\varrho} \operatorname{Im} a_{p-1}(t, x + pa_p(\tau)\theta)d\theta \leq M\log(1+\varrho) + N, \qquad \forall\varrho > 0. \tag{1.6}$$

Remark 1.2. If the coefficient $a_p(t)$ vanishes at some point of the interval $[0,T]$, the well-posedness in H^∞ of the Cauchy problem (1.2) may fail to be true also if the necessary condition (1.6) is satisfied (see [18], [6]). In [6], for instance, a Levi-type condition of the form

$$|\operatorname{Im} a_{p-1}(t,x)| \leq Ca_p(t)/\langle x\rangle \tag{1.7}$$

is needed to weaken (1.5) into $a_p(t) \geq 0$ for proving H^∞ well-posedness of (1.2). Notice that condition (1.7) is consistent with (1.6). In the present paper we focus on the fact that the dependence on space of the coefficient of the subprincipal part is allowed only accompanied by a decay condition at infinity.

2. Idea of the proof and auxiliary tools.

We prove Theorem 1.1 by contradiction, taking $f \equiv 0$ without any loss of generality.

We assume the Cauchy problem (1.2) to be well-posed, so that for every $g \in H^\infty$, there exists a unique $u \in C([0,T];H^\infty)$ solution of (1.2) and there exists $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $C > 0$ such that

$$\|u(t,\cdot)\|_0 \leq C\|g\|_q \qquad \forall t \in [0,T], \tag{2.1}$$

where $\|\cdot\|_s$ stands for the norm in the Sobolev space H^s (we shall write $\|\cdot\| := \|\cdot\|_0$ for simplicity).

Then we assume, by contradiction, that (1.6) does not hold. This implies that, for every $M > 0$ and $k \in \mathbb{N}$ there exist a sequence of points $x_k \in \mathbb{R}$ and a sequence $\varrho_k \to +\infty$ such that

$$\int_{-\varrho_k}^{\varrho_k} \operatorname{Im} a_{p-1}(t, x_k + pa_p(\tau)\theta)d\theta \geq M \log(1 + \varrho_k) + k \qquad \forall 0 \leq \tau \leq t \leq T. \tag{2.2}$$

We can then construct a sequence of initial data g_k localized at high frequency $n_k := \varrho_k^a$, for suitable $a > 0$, so obtaining a sequence u_k of solutions of the corresponding Cauchy problem. Further localizing these solutions in the phase space along the trajectory of the hamiltonian $a_p(t)\xi^p$, we produce a sequence of functions $v_k^{\alpha,\beta}$ (for $\alpha, \beta \in \mathbb{N}_0$) satisfying some energy estimates, because of (2.1).

Taking, finally, a suitable linear combination $\sigma_k(t)$ of the L^2-norms $\|v_k^{\alpha,\beta}(t,\cdot)\|$, we obtain, in Section 3, that (2.2) implies an estimate from below of $\sigma_k(t)$; this estimate will contradict an estimate from above for $\sigma_k(t)$ which is stated and proved in Section 4.

In this section we discuss condition (1.6), construct the sequence $\{v_k^{\alpha,\beta}\}$ and collect some estimates that will be crucial in the proofs of the contradictory estimates from below and from above of $\sigma_k(t)$.

The next section is completely devoted to the proof of the estimate from below (3.40).

In Section 4 we give the estimate from above (4.1), and finally prove Theorem 1.1.

Let us start by remarking that if condition (1.6) does not hold, then at least one of the following two conditions does not hold:

$$\sup_{x\in\mathbb{R}} \min_{0\leq\tau\leq t\leq T} \int_0^{\varrho} \operatorname{Im} a_{p-1}(t, x + pa_p(\tau)\theta)d\theta \leq M \log(1 + \varrho) + N, \qquad \forall \varrho > 0, \tag{2.3}$$

or

$$\sup_{x\in\mathbb{R}} \min_{0\leq\tau\leq t\leq T} \int_{-\varrho}^{0} \operatorname{Im} a_{p-1}(t, x + pa_p(\tau)\theta)d\theta \leq M \log(1 + \varrho) + N, \qquad \forall \varrho > 0. \tag{2.4}$$

Since

$$\int_{-\varrho}^{0} \operatorname{Im} a_{p-1}(t, x + pa_p(\tau)\theta)d\theta = \int_0^{\varrho} \operatorname{Im} a_{p-1}(t, x - pa_p(\tau)\theta)d\theta,$$

we can assume, without any loss of generality, that (2.3) does not hold and obtain then a contradiction (if (2.4) does not hold we argue in the same way taking $-a_p$ instead of a_p).

The following lemma will be the key to obtain the desired estimate from below (3.40):

Lemma 2.1. *If (2.3) does not hold, then for every $M > 0$ and $k \in \mathbb{N}$ there esist $x_k \in \mathbb{R}$ and $\varrho_k > 0$ such that:*

(i) $\varrho_k \to +\infty$;

(ii) $\displaystyle\int_0^{\varrho_k} \operatorname{Im} a_{p-1}(t, x_k + pa_p(\tau)\theta)d\theta \geq M \log(1 + \varrho_k) + k \qquad \forall 0 \leq \tau \leq t \leq T$;

(iii) $\displaystyle\int_0^{\varrho} \operatorname{Im} a_{p-1}(t, x_k + pa_p(\tau)\theta)d\theta \geq 0 \qquad \forall \varrho \in [0, \varrho_k],\ 0 \leq \tau \leq t \leq T.$

Proof. If (2.3) fails to be true, then for every $M > 0$, $k \in \mathbb{N}$ there exist $y_k \in \mathbb{R}$ and $\delta_k > 0$ such that for all $0 \leq \tau \leq t \leq T$

$$\int_0^{\delta_k} \operatorname{Im} a_{p-1}(t, y_k + p\theta a_p(\tau))d\theta \geq M \log(1 + \delta_k) + k. \tag{2.5}$$

Let us set, for $s \in [0, \delta_k]$,

$$F_k(s) := \int_0^{s} \operatorname{Im} a_{p-1}(t, y_k + p\theta a_p(\tau))d\theta$$

and let s_k be point of minimum of F_k on $[0,\delta_k]$. Define then

$$\begin{aligned} x_k &:= y_k + p s_k a_p(\tau) \\ \varrho_k &:= \delta_k - s_k. \end{aligned}$$

Remark that all $y_k, \delta_k, s_k, x_k, \varrho_k$ depend also on M.

For all $s \in [0,\varrho_k] \subseteq [0,\delta_k]$:

$$\begin{aligned} \int_0^s \operatorname{Im} a_{p-1}(t, x_k + p a_p(\tau)\theta) d\theta &= \int_{s_k}^{s+s_k} \operatorname{Im} a_{p-1}(t, y_k + p a_p(\tau)\theta') d\theta' \\ &= F_k(s+s_k) - F_k(s_k) \geq 0 \end{aligned} \tag{2.6}$$

by definition of s_k. This proves (iii).

Moreover, $F_k(s_k) \leq F_k(0) = 0$ and hence, from (2.6) and (2.5):

$$\begin{aligned} \int_0^{\varrho_k} \operatorname{Im} a_{p-1}(t, x_k + p a_p(\tau)\theta) d\theta &= \int_0^{\delta_k - s_k} \operatorname{Im} a_{p-1}(t, x_k + p a_p(\tau)\theta) d\theta \\ &= F_k(\delta_k) - F_k(s_k) \geq F_k(\delta_k) \\ &\geq M \log(1+\delta_k) + k \geq M \log(1+\varrho_k) + k, \end{aligned}$$

proving (ii).

Finally, the last inequality implies, for $k \to +\infty$,

$$\int_0^{\varrho_k} \operatorname{Im} a_{p-1}(t, x_k + p a_p(\tau)\theta) d\theta \geq k \to +\infty$$

and hence $\varrho_k \to +\infty$, because $a_{p-1} \in \mathcal{B}^\infty$. □

2.1. **Solutions with high frequency initial data.** Let us fix, here and throughout all the paper, a cut-off function $h \in C^\infty(\mathbb{R})$, such that

$$h(y) = \begin{cases} 1 & |y| \leq 1/4 \\ 0 & |y| \geq 1/2, \end{cases} \tag{2.7}$$

and a rapidly decreasing function ψ such that $\psi(0) = 2$ and

$$\operatorname{supp} \hat{\psi} \subseteq \{\xi \in \mathbb{R} : \ h(\xi) = 1\}.$$

Define then

$$g_k(x) = e^{i(x-x_k)n} \psi(x - x_k), \tag{2.8}$$

where

$$n := \varrho_k^a \tag{2.9}$$

for some $a > 0$ to be chosen later on (see (3.31)), and x_k, ϱ_k as in Lemma 2.1. Note that

$$\hat{g}_k(\xi) = e^{-i x_k \xi} \hat{\psi}(\xi - n), \tag{2.10}$$

so g_k is localized in the phase space around the point (x_k, n).

Denote by $u_k \in C([0,T]; H^\infty)$ the solution of the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x) u_k(t,x) = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ u_k(0,x) = g_k(x) & x \in \mathbb{R}. \end{cases} \tag{2.11}$$

Then, by (2.1) and (2.10) we have, for all $t \in [0, T]$:

$$\begin{aligned} \|u_k(t,\cdot)\| \le C\|g_k\|_q &= C(2\pi)^{-1/2}\left(\int_{\mathbb{R}} \langle\xi\rangle^{2q}|\hat{g}_k(\xi)|^2 d\xi\right)^{1/2} \\ &\le C_q(2\pi)^{-1/2}\langle n\rangle^q \left(\int_{\mathbb{R}} \langle\theta\rangle^{2q}|\hat{\psi}(\theta)|^2 d\theta\right)^{1/2} \\ &\le C'_q n^q, \end{aligned} \tag{2.12}$$

for some $C_q, C'_q > 0$.

2.2. **A localizing operator.** In this subsection we define, by giving its symbol $w_{n,k}(t,x,\xi)$, a pseudo-differential operator $W_{n,k}(t,x,D_x)$ which localizes the solutions of (1.2) in the phase space along the trajectory of the hamiltonian $a_p(t)\xi^p$.

Let $w_{n,k}(t,x,\xi)$ be the solution of the Hamilton's equation of motion

$$\begin{cases} \partial_t w_{n,k} = \{w_{n,k}, -a_p(t)\xi^p\} \\ w_{n,k}(0,x,\xi) = w_{0,n,k}(x,\xi) := \varrho_k^{1/2} h\left(\varrho_k(x-x_k)\right) h\left(\varrho_k^\mu(\xi/n-1)\right), \end{cases} \tag{2.13}$$

with $\mu > 0$ to be chosen later (see (3.31)), where $\{\cdot,\cdot\}$ denotes the Poisson brackets defined by

$$\{p(x,\xi), q(x,\xi)\} = \partial_x p(x,\xi)\partial_\xi q(x,\xi) - \partial_\xi p(x,\xi)\partial_x q(x,\xi).$$

Computing the Poisson brackets, equation (2.13) reduces to

$$\begin{cases} \left(\partial_t + p a_p(t)\xi^{p-1}\partial_x\right) w_{n,k} = 0 \\ w_{n,k}(0,x,\xi) = w_{0,n,k}(x,\xi), \end{cases} \tag{2.14}$$

which admits the solution

$$w_{n,k}(t,x,\xi) = w_{0,n,k}(x - pA_p(t)\xi^{p-1}, \xi), \quad A_p(t) = \int_0^t a_p(\tau)d\tau.$$

We thus obtain

$$w_{n,k}(t,x,\xi) := \varrho_k^{1/2} h\left(\varrho_k(x - x_k - pA_p(t)\xi^{p-1})\right) h\left(\varrho_k^\mu(\xi/n-1)\right). \tag{2.15}$$

The following lemma shows that the symbol $w_{n,k}(t,x,\xi)$ is supported in a neighborhood of the solution $(x_k + pA_p(t)n^{p-1}, n)$ of the Hamilton's canonical equation with initial data (x_k, n); moreover it introduces the sequence of symbols $w_{n,k}^{\alpha,\beta}$ which naturally appear in the computation of $\partial_\xi^\alpha D_x^\beta w_{n,k}$.

Lemma 2.2. *Let us define, for $\alpha, \beta \in \mathbb{N}_0$, $\mu \ge 2$ and n as in (2.9), the symbols*

$$w_{n,k}^{\alpha,\beta}(t,x,\xi) := \varrho_k^{1/2}(\partial_x^\alpha h)(x)(\partial_\xi^\beta h)(\xi)\Big|_{\substack{x=\varrho_k(x-x_k-pA_p(t)\xi^{p-1}) \\ \xi=\varrho_k^\mu(\xi/n-1)}}. \tag{2.16}$$

Then, for every $t \in \left[0, \frac{\varrho_k}{n^{p-1}}\right]$ we have that

$$\operatorname{supp} w_{n,k}^{\alpha,\beta}(t) \subseteq \left\{(x,\xi) : \ |x - (x_k + pA_p(t)n^{p-1})| \le \frac{c_p}{\varrho_k}, \ |\xi/n - 1| \le \frac{1}{2\varrho_k^\mu}\right\},$$

for $c_p = \max\{1, p2^{p-1}\sup_{[0,T]}|a_p|\}$, if k is large enough.

Proof. The estimate $|\xi/n-1|\le 1/(2\varrho_k^\mu)$ trivially follows by definition (2.16) and by (2.7). Moreover, (2.7) implies, for $t\in[0,\varrho_k/n^{p-1}]$, $\mu\ge 2$ and $(x,\xi)\in\operatorname{supp} w_{n,k}^{\alpha,\beta}$:

$$\begin{aligned}|x-(x_k+pA_p(t)n^{p-1})| &\le |x-(x_k+pA_p(t)\xi^{p-1})|+p|A_p(t)|n^{p-1}\left|\left(\frac{\xi}{n}\right)^{p-1}-1\right|\\ &\le \frac{1}{2\varrho_k}+p\sup_{[0,T]}|a_p|tn^{p-1}\cdot\left|\frac{\xi}{n}-1\right|\cdot\left|\left(\frac{\xi}{n}\right)^{p-2}+\left(\frac{\xi}{n}\right)^{p-3}+\ldots+1\right|\\ &\le \frac{1}{2\varrho_k}+p\sup_{[0,T]}|a_p|\frac{\varrho_k}{2\varrho_k^\mu}\left[2^{p-2}+2^{p-3}+\ldots+1\right]\\ &\le \frac{1}{2\varrho_k}+p\sup_{[0,T]}|a_p|\frac{1}{2\varrho_k^{\mu-1}}2^{p-1}\le\frac{c_p}{\varrho_k},\end{aligned}$$

for $c_p=\max\{1,p2^{p-1}\sup_{[0,T]}|a_p|\}$, since $\xi/n\le|\xi/n-1|+1\le 1/(2\varrho_k^\mu)+1\le 2$ for k large enough. □

As a consequence of Lemma 2.2, we localize, in the phase space, the solution of (2.11), defining

$$v_k^{\alpha,\beta}(t,x):=W_{n,k}^{\alpha,\beta}(t,x,D_x)u_k(t,x), \tag{2.17}$$

where $W_{n,k}^{\alpha,\beta}(t,x,D_x)$ is the pseudo-differential operator with symbol $w_{n,k}^{\alpha,\beta}(t,x,\xi)$. We shall denote, throughout all the paper, $W_{n,k}:=W_{n,k}^{0,0}(t,x,D_x)$ and $v_k:=v_k^{0,0}$ for simplicity.

2.3. **Useful estimates.**

In the next sections we need estimates of the L^2-norms of the functions $v_k^{\alpha,\beta}$ and of both operators $W_{n,k}^{\alpha,\beta}$ and $[a_j,W_{n,k}]$ acting on u_k . In this subsection we state and prove all these estimates. Proofs are quite technical, and the main tools for obtaining them, collected in Appendix A, are the Calderon-Vaillantcourt's Theorem A.3 and a skillful use of the expansion formula of the symbol of the product of two pseudo-differential operators (Theorems A.1 and A.2). To avoid losing his train of thought, the reader can skip these estimates at a first reading, passing directly to Section 3 and coming back to the estimates at the moment of their application.

To estimate the L^2-norms of v_k and of $v_k^{\alpha,\beta}$ we first need estimates of the semi-norms $|\cdot|^0_{\ell,\ell}$ of the symbols $w_{n,k}^{\alpha,\beta}\in S^0_{0,0}$, defined in formula (A.2) of the Appendix.

Lemma 2.3. *Let $n=\varrho_k^a$ with $a\ge\mu\ge 2$, and $t\in\left[0,\frac{\varrho_k}{n^{p-1}}\right]$. Then, for every $\alpha,\beta\in\mathbb{N}_0$ we have, for k large enough:*

(i) *for every $\gamma,\sigma\in\mathbb{N}_0$ there exists a constant $C_{\alpha,\beta,\gamma,\sigma}>0$ such that, for all $(t,x,\xi)\in[0,\frac{\varrho_k}{n^{p-1}}]\times\mathbb{R}^2$:*

$$|\partial_\xi^\gamma\partial_x^\sigma w_{n,k}^{\alpha,\beta}(t,x,\xi)|\le C_{\alpha,\beta,\gamma,\sigma}\varrho_k^{\frac12+\sigma}\left(\frac{\varrho_k^\mu}{n}\right)^\gamma;$$

(ii) *for every $\ell\in\mathbb{N}$ there exists $C_{\alpha,\beta,\ell}>0$ such that*

$$\left|w_{n,k}^{\alpha,\beta}\right|^0_{\ell,\ell}\le C_{\alpha,\beta,\ell}\varrho_k^{\frac12+\ell};$$

(iii) *for every $h\in\mathbb{N}_0$ and $\nu,\ell\in\mathbb{N}$ there exists $C_{\alpha,\beta,\nu,\ell}>0$ such that*

$$|\xi^h\partial_\xi^\nu w_{n,k}^{\alpha,\beta}|^0_{\ell,\ell}\le C_{\alpha,\beta,\nu,\ell}\,n^h\varrho_k^{\frac12+\ell}\left(\frac{\varrho_k^\mu}{n}\right)^\nu. \tag{2.18}$$

Proof. Let us write

$$\begin{aligned}\partial_\xi^\gamma\partial_x^\sigma w_{n,k}^{\alpha,\beta}(t,x,\xi) &= \varrho_k^\sigma\partial_\xi^\gamma w_{n,k}^{\alpha+\sigma,\beta}(t,x,\xi)\\ &= \varrho_k^{\sigma+\frac12}\sum_{\gamma_1+\gamma_2=\gamma} C_\gamma\partial_\xi^{\gamma_1}h^{(\alpha+\sigma)}(\varrho_k(x-x_k-pA_p(t)\xi^{p-1}))\cdot\partial_\xi^{\gamma_2}h^{(\beta)}(\varrho_k^\mu(\xi/n-1)).\end{aligned}\tag{2.19}$$

Since $|\xi|\le 2n$ on $\operatorname{supp} w_{n,k}^{\alpha,\beta}$, by Lemma 2.2 we have that

$$\begin{aligned}\left|\partial_\xi^{\gamma_1}h^{(\alpha+\sigma)}\left(\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\right)\right| &\le C_{\alpha,\sigma,\gamma_1}(A_p(t)|\xi|^{p-2}\varrho_k)^{\gamma_1}\\ &\le C_{\alpha,\sigma,\gamma_1}\left(\sup_{[0,T]}|a_p|\cdot t\cdot|\xi|^{p-2}\varrho_k\right)^{\gamma_1}\le C'_{\alpha,\sigma,\gamma_1}\left(\frac{\varrho_k^2}{n}\right)^{\gamma_1}\end{aligned}$$

for $t\in[0,\varrho_k/n^{p-1}]$. Moreover,

$$\left|\partial_\xi^{\gamma_2}h^{(\beta)}\left(\varrho_k^\mu(\xi/n-1)\right)\right|\le C_{\gamma_2,\beta}\left(\frac{\varrho_k^\mu}{n}\right)^{\gamma_2}.$$

Substituting in (2.19) we thus obtain (i), since $\mu\ge 2$.

From (i) and $a\ge\mu$ we get

$$\left|w_{n,k}^{\alpha,\beta}\right|_{\ell,\ell}^0=\sup_{\substack{\gamma,\sigma\le\ell\\ x,\xi\in\mathbb{R}}}\left|\partial_\xi^\gamma\partial_x^\sigma w_{n,k}^{\alpha,\beta}(t,x,\xi)\right|\le\sup_{\substack{\gamma,\sigma\le\ell\\ x,\xi\in\mathbb{R}}}C_{\alpha,\beta,\gamma,\sigma}\,\varrho_k^{\frac12+\sigma}\left(\frac{\varrho_k^\mu}{n}\right)^\gamma\le C_{\alpha,\beta,\ell}\,\varrho_k^{\frac12+\ell}$$

i.e. also (ii) is satisfied.

Finally, (iii) follows from (i), since $|\xi|\le 2n$ on $\operatorname{supp} w_{n,k}^{\alpha,\beta}$ and $a\ge\mu$. □

We are now ready to estimate $\|v_k\|$. By Calderon-Vaillalntcourt's Theorem A.3, (ii) of Lemma 2.3 and (2.12), we have that for all $t\in[0,\varrho_k/n^{p-1}]$

$$\|v_k(t,\cdot)\|=\|W_{n,k}^{0,0}(t,\cdot,D_x)u_k(t,\cdot)\|\le C\,|w_{n,k}(t,x,\xi)|_{2,2}^0\,\|u_k(t,\cdot)\|\le C'\varrho_k^{\frac12+2}n^q\tag{2.20}$$

for some $C,C'>0$; similarly, for every $\alpha,\beta\in\mathbb{N}_0$, it follows that

$$\|v_k^{\alpha,\beta}(t,\cdot)\|\le C_{\alpha,\beta}\varrho_k^{\frac12+2}n^q=C_{\alpha,\beta}\varrho_k^{\frac12+2+aq}\qquad\forall t\in[0,\varrho_k/n^{p-1}]\tag{2.21}$$

for some $C_{\alpha,\beta}>0$. To estimate also the derivatives of $v_k^{\alpha,\beta}$ we need the following:

Lemma 2.4. *Let $n=\varrho_k^a$ with $a\ge\mu$. For every $\nu,r\in\mathbb{N}$ and $\alpha,\beta\in\mathbb{N}_0$ there exists $C_{\alpha,\beta,r,\nu}>0$ such that for all $t\in[0,\varrho_k/n^{p-1}]$ with k large enough:*

$$\|D_x^r v_k^{\alpha,\beta}(t,\cdot)\|\le c_1 n^r\|v_k^{\alpha,\beta}\|+C_{\alpha,\beta,r,\nu}\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{r+q},$$

for a fixed constant $c_1>0$.

Proof. We define the function

$$\chi_{1,k}(\xi)=h\left(\frac{\varrho_k^\mu}{3}\left(\frac{\xi}{n}-1\right)\right).\tag{2.22}$$

By definition (2.7), we have that

$$\operatorname{supp}\chi_{1,k}\subseteq\left\{\xi\ :\ \left|\frac{\xi}{n}-1\right|\le\frac{3}{2\varrho_k^\mu}\right\}\subseteq\{\xi\ :\ |\xi|\le 3n\},\tag{2.23}$$

and

$$\operatorname{supp}(1-\chi_{1,k})\subseteq\left\{\xi\ :\ \left|\frac{\xi}{n}-1\right|\ge\frac{3}{4\varrho_k^\mu}\right\}.\tag{2.24}$$

This implies, by Lemma 2.2, that

$$\operatorname{supp}(1-\chi_{1,k})\cap \operatorname{supp} w_{n,k}^{\alpha,\beta}=\emptyset. \tag{2.25}$$

Localizing now at frequency n

$$\begin{aligned} D_x^r v_k^{\alpha,\beta} &= \chi_{1,k}(D_x)D_x^r v_k^{\alpha,\beta}+(1-\chi_{1,k}(D_x))D_x^r v_k^{\alpha,\beta}\\ &= \chi_{1,k}(D_x)D_x^r v_k^{\alpha,\beta}+\sum_{j=0}^{r}\binom{r}{j}(1-\chi_{1,k}(D_x))(D_x^j W_{n,k}^{\alpha,\beta})D_x^{r-j}u_k, \end{aligned}$$

and applying Calderón-Vaillancourt's Theorem A.3, we come to:

$$\begin{aligned} \|D_x^r v_k^{\alpha,\beta}(t,\cdot)\| \le\ & |\chi_{1,k}(\xi)\xi^r|_{2,2}^0\cdot\|v_k^{\alpha,\beta}\|\\ &+\sum_{j=0}^{r}\binom{r}{j}\varrho_k^j\left|\sigma\left((1-\chi_{1,k}(D_x))W_{n,k}^{\alpha+j,\beta}D_x^{r-j}\right)\right|_{2,2}^0\cdot\|u_k\|. \end{aligned}\tag{2.26}$$

Note that $|\chi_{1,k}(\xi)\xi^r|_{2,2}^0\le c_1n^r$ for some $c_1>0$, because of (2.23); to estimate the second term of (2.26), by Theorem A.1 and (2.25) we write, for every integer $\nu\ge 1$:

$$\begin{aligned} \sigma\left((1-\chi_{1,k}(D_x))W_{n,k}^{\alpha+j,\beta}D_x^{r-j}\right) &= \sum_{0\le\gamma\le\nu-1}\frac{1}{\gamma!}\partial_\xi^\gamma(1-\chi_{1,k}(\xi))D_x^\gamma(w_{n,k}^{\alpha+j,\beta}\xi^{r-j})\\ &+\int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!}\iint e^{-iy\eta}\partial_\xi^\nu(1-\chi_{1,k}(\xi+\theta\eta)D_x^\nu(w_{n,k}^{\alpha+j,\beta}(t,x+y;\xi)\xi^{r-j})dy đ\eta d\theta\\ &= \int_0^1\frac{(1-\theta)^{\nu-1}}{(\nu-1)!}\mathcal{O}_\nu(t,\theta,x,\xi)d\theta, \end{aligned}\tag{2.27}$$

where

$$\mathcal{O}_\nu(t,\theta,x,\xi):=\iint e^{-iy\eta}\partial_\xi^\nu(1-\chi_{1,k}(\xi+\theta\eta)D_x^\nu w_{n,k}^{\alpha+j,\beta}(t,x+y;\xi)\xi^{r-j}dy đ\eta.$$

Writing $\xi^{r-j}=\sum_{h=0}^{r-j}\binom{r-j}{h}(\xi+\theta\eta)^h(-\theta\eta)^{r-j-h}$ and $e^{-iy\eta}(-\eta)^{r-j-h}=D_y^{r-j-h}e^{-iy\eta}$, we have, integrating by parts:

$$\begin{aligned} \mathcal{O}_\nu =\ & -\sum_{h=0}^{r-j}\binom{r-j}{h}\theta^{r-j-h}\iint\partial_\xi^\nu\chi_{1,k}(\xi+\theta\eta)\cdot(\xi+\theta\eta)^hD_x^\nu w_{n,k}^{\alpha+j,\beta}(t,x+y;\xi)\\ &\cdot D_y^{r-j-h}e^{-iy\eta}dy đ\eta\\ =\ & \sum_{h=0}^{r-j}(-1)^{r-j-h+1}\binom{r-j}{h}\theta^{r-j-h}\iint e^{-iy\eta}\partial_\xi^\nu\chi_{1,k}(\xi+\theta\eta)\cdot(\xi+\theta\eta)^h\\ &\cdot D_y^{\nu+r-j-h}w_{n,k}^{\alpha+j,\beta}(t,x+y;\xi)dy đ\eta\\ =\ & \sum_{h=0}^{r-j}(-1)^{r-j-h+1}\binom{r-j}{h}\theta^{r-j-h}\varrho_h^{\nu+r-j-h}\iint e^{-iy\eta}\partial_\xi^\nu\chi_{1,k}(\xi+\theta\eta)\cdot(\xi+\theta\eta)^h\\ &\cdot w_{n,k}^{\alpha+\nu+r-h,\beta}(t,x+y;\xi)dy đ\eta. \end{aligned}$$

By Theorem A.2, (2.22), (2.23) and Lemma 2.3, for $\theta \in [0,1]$ we have that

$$
\begin{aligned}
|\mathcal{O}_\nu(t,\theta)|^0_{2,2} \le & \sum_{h=0}^{r-j} c_h \varrho_k^{\nu+r-j-h} |\partial_\xi^\nu \chi_{1,k}(\xi)\xi^h|^0_{4,4} \cdot |w_{n,k}^{\alpha+\nu+r-h,\beta}(t,x;\xi)|^0_{4,4} \\
\le & \sum_{h=0}^{r-j} C_{\alpha,\nu,r,h,\beta} \varrho_k^{\nu+r-j-h} \left(\frac{\varrho_k^\mu}{n}\right)^\nu n^h \varrho_k^{\frac{1}{2}+4} \\
\le & \sum_{h=0}^{r-j} C'_{\alpha,\beta,\nu,r,h} \left(\frac{\varrho_k^\mu}{n}\right)^\nu \left(\frac{n}{\varrho_k}\right)^{r-j} \varrho_k^{\nu+r-j+\frac{1}{2}+4} \\
= & \; C_{\alpha,\beta,\nu,r,j}\, \varrho_k^{\frac{1}{2}+4} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{r-j}
\end{aligned}
$$

for some $c_h, C_{\alpha,\nu,r,h,\beta}, C'_{\alpha,\beta,\nu,r,h}, C_{\alpha,\beta,\nu,r,j} > 0$, since $(n/\varrho_k)^h \le (n/\varrho_k)^{r-j}$ for $0 \le h \le r-j$.

Substituting in (2.27) and integrating with respect to θ we thus have that

$$
\left|\sigma\left((1-\chi_{1,k}(D_x))W_{n,k}^{\alpha+j,\beta} D_x^{r-j}\right)\right|^0_{2,2} \le |\mathcal{O}_\nu|^0_{2,2} \le C_{\alpha,\beta,\nu,r,j}\varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{r-j}. \tag{2.28}
$$

Substituting in (2.26), and taking into account (2.12), we have that

$$
\begin{aligned}
\|D_x^r v_k^{\alpha,\beta}(t,\cdot)\| \le & \; c_1 n^r \|v_k^{\alpha,\beta}\| + \sum_{j=0}^{r} C'_{\alpha,\beta,\nu,r,j}\varrho_k^{4+\frac{1}{2}+j} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{r-j+q} \\
\le & \; c_1 n^r \|v_k^{\alpha,\beta}\| + C_{\alpha,\beta,\nu,r}\varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{r+q}
\end{aligned}
$$

for some $C'_{\alpha,\beta,\nu,r,j}, C_{\alpha,\beta,\nu,r} > 0$, since $\left(\frac{\varrho_k}{n}\right)^j \le 1$ for every j. □

The following two lemmas give estimates of some pseudo-differential operators acting on the functions u_k.

Lemma 2.5. *Let $n = \varrho_k^a$ with $a \ge \mu$. Then for every $\sigma, \gamma, \lambda \in \mathbb{N}_0$ the operators $W_{n,k}^{\sigma,\gamma}(t,x,D_x)$ satisfy*

$$
W_{n,k}^{\sigma,\gamma} D_x^\lambda = \sum_{j=0}^{\lambda} c_j \varrho_k^j D_x^{\lambda-j} W_{n,k}^{\sigma+j,\gamma}, \tag{2.29}
$$

for some $c_0, \dots, c_\lambda > 0$. Moreover, there are constants $C_\lambda > 0$ and, for all $\nu \in \mathbb{N}_0$, $C_{\sigma,\gamma,\lambda,\nu} > 0$ such that for all $t \in [0, \varrho_k/n^{p-1}]$ with k large enough:

$$
\|W_{n,k}^{\sigma,\gamma}(t,\cdot,D_x)D_x^\lambda u_k(t,\cdot)\| \le C_\lambda \sum_{j=0}^{\lambda} \varrho_k^j n^{\lambda-j} \|v_k^{\sigma+j,\gamma}\| + C_{\sigma,\gamma,\lambda,\nu}\varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{\lambda+q}. \tag{2.30}
$$

Proof. Let us first prove (2.29) by induction on $\lambda \in \mathbb{N}$.

For $\lambda = 1$ we clearly have $W_{n,k}^{\sigma,\gamma} D_x = D_x W_{n,k}^{\sigma,\gamma} - \varrho_k W_{n,k}^{\sigma+1,\gamma}$.

Let us assume (2.29) to be true for every $\lambda' < \lambda$ and let us prove it for λ. By Theorem A.1:

$$\begin{aligned} W^{\sigma,\gamma}_{n,k} D^\lambda_x &= D^\lambda_x W^{\sigma,\gamma}_{n,k} + [W^{\sigma,\gamma}_{n,k}, D^\lambda_x] \\ &= D^\lambda_x W^{\sigma,\gamma}_{n,k} - \mathrm{op}\left(\sum_{\alpha=1}^{\lambda} \frac{1}{\alpha!}\partial^\alpha_\xi \xi^\lambda \cdot D^\alpha_x w^{\sigma,\gamma}_{n,k}\right) \\ &= D^\lambda_x W^{\sigma,\gamma}_{n,k} - \sum_{\alpha=1}^{\lambda}\binom{\lambda}{\alpha}\varrho_k^\alpha \left(W^{\sigma+\alpha,\gamma}_{n,k} D^{\lambda-\alpha}_x\right). \end{aligned}$$

By the inductive assumption, we thus have that

$$\begin{aligned} W^{\sigma,\gamma}_{n,k} D^\lambda_x &= D^\lambda_x W^{\sigma,\gamma}_{n,k} - \sum_{\alpha=1}^{\lambda}\binom{\lambda}{\alpha}\varrho_k^\alpha \left(\sum_{\ell=0}^{\lambda-\alpha} C_\ell \varrho_k^\ell D^{\lambda-\alpha-\ell}_x W^{\sigma+\alpha+\ell,\gamma}_{n,k}\right) \\ &= D^\lambda_x W^{\sigma,\gamma}_{n,k} - \sum_{\alpha=1}^{\lambda}\sum_{\ell=0}^{\lambda-\alpha} C_{\alpha,\lambda,\ell}\varrho_k^{\alpha+\ell} D^{\lambda-\alpha-\ell}_x W^{\sigma+\alpha+\ell,\gamma}_{n,k} \\ &= \sum_{\alpha'=0}^{\lambda} C_{\alpha',\lambda}\varrho_k^{\alpha'} D^{\lambda-\alpha'}_x W^{\sigma+\alpha',\gamma}_{n,k}. \end{aligned}$$

Therefore (2.29) is proved and, applying Lemma 2.4 for $j \le \lambda - 1$, we have that for every $\nu \in \mathbb{N}$:

$$\begin{aligned} \|W^{\sigma,\gamma}_{n,k}(t,\cdot,D_x) D^\lambda_x u_k(t,\cdot)\| &\le \sum_{j=0}^{\lambda} c_j \varrho_k^j \|D^{\lambda-j}_x v_k^{\sigma+j,\gamma}\| \\ &\le \sum_{j=0}^{\lambda-1} c_j \varrho_k^j \left(c_1 n^{\lambda-j}\|v_k^{\sigma+j,\gamma}\| + C_{\sigma,j,\gamma,\lambda,\nu}\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{\lambda-j+q}\right) \\ &\quad + c_\lambda \varrho_k^\lambda \|v_k^{\sigma+\lambda,\gamma}\| \\ &\le C'_\lambda \sum_{j=0}^{\lambda} \varrho_k^j n^{\lambda-j}\|v_k^{\sigma+j,\gamma}\| + C_{\sigma,\gamma,\lambda,\nu}\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{\lambda+q}, \end{aligned}$$

for some $C'_\lambda, C_{\sigma,\gamma,\lambda,\nu} > 0$. This proves (2.30). □

Lemma 2.6. *Let $a_j = a_j(t,x)$, for $0 \le j \le p-1$, be the coefficients of the operator* (1.1), *and let $n = \varrho_k^a$ with $a \ge \mu + 1 \ge 2$. Then, for every $\nu \in \mathbb{N}$ there exists $C_\nu > 0$ such that*

$$\|[a_j, W_{n,k}] D^j_x u_k(t,\cdot)\| \le C_\nu\, n^j \sum_{1\le\alpha_1+\alpha_2\le(\nu-1)(p-1)+j} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\| + C_\nu \varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+j}$$

for all $t \in [0, \varrho_k/n^{p-1}]$ with k large enough.

Proof. By Theorem A.1, for all $\nu \in \mathbb{N}$

$$\begin{aligned} &\sigma\left([a_j(t,x), W_{n,k}(t,x,D_x)] D^j_x\right) = \sigma([a_j, W_{n,k}]) \cdot \xi^j \\ &= -\left(\sum_{1\le\alpha\le\nu-1} \frac{1}{\alpha!}\partial^\alpha_\xi w_{n,k} \cdot D^\alpha_x a_j\right)\xi^j - \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!}\tilde{\mathcal{O}}_\nu(t,\theta,x,\xi)\,d\theta, \end{aligned} \tag{2.31}$$

where

$$\tilde{\mathcal{O}}_\nu(t,\theta,x,\xi) := \iint e^{-iy\eta}\partial^\nu_\xi w_{n,k}(t,x;\xi+\theta\eta) D^\nu_x a_j(t,x+y)\cdot \xi^j\, dy\, đ\eta.$$

Arguing as in the proof of Lemma 2.4 we can estimate, by Theorem A.2 and Lemma 2.3:

$$
\begin{aligned}
|\tilde{\mathcal{O}}_\nu|^0_{2,2} &= \left|\sum_{h=0}^{j}\binom{j}{h}\theta^{j-h}\iint \partial_\xi^\nu w_{n,k}(t,x;\xi+\theta\eta)\cdot(\xi+\theta\eta)^h D_x^\nu a_j(t,x+y)D_y^{j-h}e^{-iy\eta}dy đ\eta\right|^0_{2,2}\\
&\le \sum_{h=0}^{j}\binom{j}{h}\left|\iint e^{-iy\eta}\partial_\xi^\nu w_{n,k}(t,x;\xi+\theta\eta)\cdot(\xi+\theta\eta)^h D_x^{\nu+j-h}a_j(t,x+y)dy đ\eta\right|^0_{2,2}\\
&\le \sum_{h=0}^{j} C_j\left|\xi^h\partial_\xi^\nu w_{n,k}(t,x;\xi)\right|^0_{4,4}\cdot\left|D_x^{\nu+j-h}a_j(t,x)\right|^0_{4,4}\\
&\le C_\nu n^j\varrho_k^{4+\frac12}\left(\frac{\varrho_k^\mu}{n}\right)^\nu \qquad (2.32)
\end{aligned}
$$

for some $C_j, C_\nu > 0$, since $a_j \in C([0,T];\mathcal{B}^\infty)$ for $0 \le j \le p-1$.

In order to estimate now the first term of (2.31), we previously compute, by the Faà di Bruno formula:

$$
\begin{aligned}
\partial_\xi^\alpha w_{n,k} = &\sum_{\alpha_1+\alpha_2=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!}\varrho_k^{1/2}\cdot\partial_\xi^{\alpha_1}h\Big(\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\Big)\cdot\partial_\xi^{\alpha_2}h\Big(\varrho_k^\mu\Big(\frac{\xi}{n}-1\Big)\Big)\\
&=\varrho_k^{1/2}h\Big(\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\Big)\cdot\partial_\xi^{\alpha}h\Big(\varrho_k^\mu\Big(\frac{\xi}{n}-1\Big)\Big)\\
&+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge1}}\frac{\alpha!}{\alpha_1!\alpha_2!}\sum_{\substack{r_1+\ldots+r_s=\alpha_1\\ r_h\ge1}}C_{s,r}\varrho_k^{1/2}h^{(s)}\Big(\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\Big)\\
&\cdot\partial_\xi^{r_1}\Big[\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\Big]\cdots\partial_\xi^{r_s}\Big[\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\Big]\\
&\cdot\Big(\frac{\varrho_k^\mu}{n}\Big)^{\alpha_2}h^{(\alpha_2)}\Big(\varrho_k\Big(\frac{\xi}{n}-1\Big)\Big)\\
=&\Big(\frac{\varrho_k^\mu}{n}\Big)^{\alpha}\varrho_k^{1/2}h\Big(\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\Big)h^{(\alpha)}\Big(\varrho_k^\mu\Big(\frac{\xi}{n}-1\Big)\Big)\\
&+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge1}}\frac{\alpha!}{\alpha_1!\alpha_2!}\sum_{\substack{r_1+\ldots+r_s=\alpha_1\\ 1\le r_h\le p-1}}C'_{s,r}\Big(\varrho_kA_p(t)\Big)^{\alpha_1}\cdot\xi^{s(p-1)-\alpha_1}\\
&\cdot\Big(\frac{\varrho_k^\mu}{n}\Big)^{\alpha_2}\varrho_k^{1/2}h^{(s)}\Big(\varrho_k(x-x_k-pA_p(t)\xi^{p-1})\Big)h^{(\alpha_2)}\Big(\varrho_k\Big(\frac{\xi}{n}-1\Big)\Big)
\end{aligned}
$$

for some $C_{s,r}, C'_{s,r} > 0$. Coming back to the first term of (2.31) and taking into account the definition (2.15) of $w_{n,k}$:

$$
\begin{aligned}
&\Bigg(\sum_{1\le\alpha\le\nu-1}\frac{1}{\alpha!}\partial_\xi^\alpha w_{n,k}\cdot D_x^\alpha a_j\Bigg)\xi^j \le \sum_{1\le\alpha\le\nu-1}\frac{D_x^\alpha a_j}{\alpha!}\Big(\frac{\varrho_k^\mu}{n}\Big)^\alpha w_{n,k}^{0,\alpha}\cdot\xi^j\\
&+\sum_{1\le\alpha\le\nu-1}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge1}}\frac{D_x^\alpha a_j}{\alpha_1!\alpha_2!}\sum_{\substack{r_1+\ldots+r_s=\alpha_1\\ 1\le r_h\le p-1}}C'_{s,r}\varrho_k^{\alpha_1}A_p(t)^{\alpha_1}\Big(\frac{\varrho_k^\mu}{n}\Big)^{\alpha_2}w_{n,k}^{s,\alpha_2}\cdot\xi^{s(p-1)-\alpha_1+j}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\Big\| \mathrm{op}\Big[\Big(\sum_{1\le\alpha\le\nu-1}\frac{1}{\alpha!}\partial_\xi^\alpha w_{n,k}\cdot D_x^\alpha a_j\Big)\xi^j\Big]u_k\Big\| \\
\le\ & \sum_{1\le\alpha\le\nu-1} C_{\alpha,j}\left(\frac{\varrho_k^\mu}{n}\right)^\alpha \left\|W_{n,k}^{0,\alpha}D_x^j u_k\right\| \\
&+\sum_{1\le\alpha\le\nu-1}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge1}} C_{\alpha,j}\sup_{[0,\varrho_k/n^{p-1}]}|A_p(t)|^{\alpha_1}\varrho_k^{\alpha_1}\left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_2}\sum_{s=1}^{\alpha_1}\left\|W_{n,k}^{s,\alpha_2}D_x^{s(p-1)-\alpha_1+j}u_k\right\|.
\end{aligned}
$$

Applying (2.30), since $s\le\alpha_1$, $\mu\ge2$ and $0\le j\le p-1$, we thus obtain, for $t\in[0,\varrho_k/n^{p-1}]$:

$$
\begin{aligned}
&\Big\| \mathrm{op}\Big[\Big(\sum_{1\le\alpha\le\nu-1}\frac{1}{\alpha!}\partial_\xi^\alpha w_{n,k}\cdot D_x^\alpha a_j\Big)\xi^j\Big]u_k\Big\| \\
\le\ & \sum_{1\le\alpha\le\nu-1} C'_{\alpha,j}\left(\frac{\varrho_k^\mu}{n}\right)^\alpha\left[\sum_{h=0}^{j}\varrho_k^h n^{j-h}\|v_k^{h,\alpha}\| + C_{\alpha,j,\nu}\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{j+q}\right] \\
&+\sum_{1\le\alpha\le\nu-1}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge1}} C'_{\alpha,j}\left(\frac{\varrho_k}{n^{p-1}}\right)^{\alpha_1}\varrho_k^{\alpha_1}\left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_2} \\
&\cdot\sum_{s=1}^{\alpha_1}\Big[C_{s,\alpha,j}\sum_{h=0}^{s(p-1)-\alpha_1+j}\varrho_k^h n^{s(p-1)-\alpha_1+j-h}\|v_k^{s+h,\alpha_2}\| \\
&+C_{s,\alpha,j,\nu}\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{s(p-1)-\alpha_1+j+q}\Big] \\
\le\ & C_\nu n^j\sum_{1\le\alpha\le\nu-1}\left(\frac{\varrho_k^\mu}{n}\right)^\alpha\sum_{h=0}^{j}\left(\frac{\varrho_k}{n}\right)^h\|v_k^{h,\alpha}\| + C_\nu\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{j+q} \\
&+C_\nu\sum_{1\le\alpha\le\nu-1}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1\ge1}}\left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2}\frac{1}{n^{\alpha_1(p-2)}} \\
&\cdot\left[n^{\alpha_1(p-2)+j}\sum_{s=1}^{\alpha_1}\sum_{h=0}^{s(p-1)-\alpha_1+j}\left(\frac{\varrho_k}{n}\right)^h\|v_k^{s+h,\alpha_2}\| + \varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{\alpha_1(p-2)+j+q}\right] \\
\le\ & C_\nu n^j\sum_{1\le\alpha_2\le\nu-1}\sum_{h=0}^{j}\left(\frac{\varrho_k^\mu}{n}\right)^{h+\alpha_2}\|v_k^{h,\alpha_2}\| + C'_\nu\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+j} \\
&+C_\nu n^j\sum_{\substack{1\le\alpha_1+\alpha_2\le\nu-1\\ \alpha_1\ge1}}\ \sum_{s=1}^{\alpha_1}\ \sum_{h=0}^{s(p-1)-\alpha_1+j}\left(\frac{\varrho_k^\mu}{n}\right)^{s+h+\alpha_2}\|v_k^{s+h,\alpha_2}\|
\end{aligned}
$$

$$
(2.33)\qquad \le C_\nu n^j\sum_{1\le\alpha_1+\alpha_2\le(\nu-1)(p-1)+j}\left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2}\|v_k^{\alpha_1,\alpha_2}\| + C'_\nu\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+j}
$$

for some $C_{\alpha,j}, C_{\alpha,j,\nu}, C'_{\alpha,j}, C'_{\alpha,j,\nu}, C_{s,\alpha,j}, C_{s,\alpha,j,\nu}, C_\nu, C'_\nu > 0$.

By the Calderón-Vaillancourt's Theorem A.3, by (2.31), (2.32) and (2.33) we get:

$$\|[a_j, W_{n,k}]D_x^j u_k\| \leq C|\tilde{\mathcal{O}}_\nu|^0_{2,2} \cdot \|u_k\| + \Big\| \operatorname{op}\Big[\Big(\sum_{1\leq\alpha\leq\nu-1} \frac{1}{\alpha!}\partial_\xi^\alpha w_{n,k} \cdot D_x^\alpha a_j\Big)\xi^j\Big] u_k \Big\|$$

$$\leq\ C'_\nu \varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+j} + C_\nu n^j \sum_{1\leq\alpha_1+\alpha_2\leq(\nu-1)(p-1)+j} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\|$$

for some $C, C_\nu, C'_\nu > 0$. □

3. Estimates from below

In this section we want to produce estimates from below of the L^2-norms of the functions v_k and $v_k^{\alpha,\beta}$, and then of a linear combination $\sigma_k(t)$ of the L^2-norms of $v_k^{\alpha,\beta}$, $\alpha+\beta\geq 0$.

We start with the estimate of $\|v_k(0,\cdot)\|$. For n as in (2.9) and k large enough, from (2.10) we have that

$$\operatorname{supp}\hat{g}_k = \operatorname{supp}\hat{\psi}(\xi-n) \subseteq \{\xi\in\mathbb{R}:\ h(\varrho_k^\mu(\xi/n-1)) = 1\}. \tag{3.1}$$

Therefore

$$\begin{aligned} v_k(0,x) = W_{n,k}u_k(0,x) &= \int e^{ix\xi} w_{n,k}(0,x,\xi)\widehat{g_k}(\xi)đ\xi \\ &= \int e^{ix\xi}\varrho_k^{1/2} h\left(\varrho_k(x-x_k)\right) \underbrace{h\left(\varrho_k^\mu(\xi/n-1)\right)}_{1} e^{-ix_k\xi}\hat{\psi}(\xi-n)đ\xi \\ &= \varrho_k^{1/2} h\left(\varrho_k(x-x_k)\right) e^{i(x-x_k)n}\psi(x-x_k) \end{aligned}$$

and

$$\begin{aligned} \|v_k(0,\cdot)\|^2 &= \int \varrho_k |h\left(\varrho_k(x-x_k)\right)|^2 |\psi(x-x_k)|^2 dx = \int |h(y)|^2|\psi(y/\varrho_k)|^2 dy \\ &\geq \int |h(y)|^2 dy = \|h\|^2 > 0 \end{aligned} \tag{3.2}$$

if k is large enough, since $\psi(0)=2$ and $\varrho_k\to+\infty$.

Now, to produce an estimate from below of $\|v_k(t,\cdot)\|$, our idea is to follow the energy method, producing a "reverse energy estimate". To this aim, denoting by $\langle\cdot,\cdot\rangle$ the scalar product on L^2, we consider

$$\begin{aligned} \frac{d}{dt}\|v_k(t,\cdot)\|^2 =&\ 2\operatorname{Re}\langle\partial_t v_k, v_k\rangle \\ =&\ 2\operatorname{Re} i\langle Pv_k, v_k\rangle - 2\operatorname{Re} i a_p(t)\langle D_x^p v_k, v_k\rangle - 2\operatorname{Re} i\sum_{j=0}^{p-1}\langle (a_j(t,x)D_x^j v_k, v_k\rangle. \end{aligned} \tag{3.3}$$

We compute separately estimates from below of each term in formula (3.3). By definition of v_k we have that

$$\begin{aligned} Pv_k = PW_{n,k}u_k = W_{n,k}Pu_k + [P, W_{n,k}]u_k = \\ = 0 + [D_t + a_p(t)D_x^p, W_{n,k}]u_k + \sum_{j=0}^{p-1}[a_j(t,x)D_x^j, W_{n,k}]u_k, \end{aligned}$$

since $Pu_k = 0$.

Developing the symbol of the commutator $[D_t + a_p(t)D_x^p, W_{n,k}]$ and using the fact that $w_{n,k}$ is the solution of Hamilton's equation (2.14) we obtain, by Theorem A.1:

$$\begin{aligned}\sigma\left([D_t + a_p(t)D_x^p, W_{n,k}]\right)(t,x,\xi) &= D_t w_{n,k} + a_p(t)\sigma\left([D_x^p, W_{n,k}]\right)\\ &= D_t w_{n,k} + a_p(t)\sum_{\alpha=1}^{p}\frac{1}{\alpha!}\partial_\xi^\alpha \xi^p \cdot D_x^\alpha w_{n,k}\\ &= (D_t + pa_p(t)\xi^{p-1}D_x)w_{n,k} + a_p(t)\sum_{\alpha=2}^{p}\binom{p}{\alpha}\xi^{p-\alpha}D_x^\alpha w_{n,k}\\ &= a_p(t)\sum_{\alpha=2}^{p}\binom{p}{\alpha}\xi^{p-\alpha}D_x^\alpha w_{n,k}.\end{aligned}$$

Defining then

$$f_k := \operatorname{op}\left(a_p(t)\sum_{\alpha=2}^{p}\binom{p}{\alpha}\xi^{p-\alpha}D_x^\alpha w_{n,k}\right)u_k + \sum_{j=0}^{p-1}[a_j(t,x)D_x^j, W_{n,k}]u_k, \tag{3.4}$$

we have that

$$Pv_k = f_k$$

and hence from (3.3) we get

$$\begin{aligned}\frac{d}{dt}\|v_k(t,\cdot)\|^2 &= 2\operatorname{Re} i\langle f_k, v_k\rangle - 2\operatorname{Re} ia_p(t)\langle D_x^p v_k, v_k\rangle - \sum_{j=0}^{p-1}2\operatorname{Re} i\langle a_jD_x^j v_k, v_k\rangle\\ &= 2\operatorname{Re} i\langle f_k, v_k\rangle - \sum_{j=0}^{p-1}\langle (ia_jD_x^j + (ia_jD_x^j)^*)v_k, v_k\rangle \end{aligned}\tag{3.5}$$

since $\operatorname{Re} i\langle D_x^p v_k, v_k\rangle = 0$. Now,

$$\sigma(ia_j(t,x)D_x^j)^* = \sum_{\alpha\geq 0}\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha(\overline{ia_j(t,x)\xi^j}) = \sum_{\alpha=0}^{j}\binom{j}{\alpha}D_x^\alpha(-i\operatorname{Re} a_j - \operatorname{Im} a_j(t,x))\xi^{j-\alpha},$$

and hence

$$\begin{aligned}\sum_{j=0}^{p-1}\sigma[(ia_jD_x^j) + (ia_jD_x^j)^*] &= \sum_{j=0}^{p-1}\left[-2\operatorname{Im} a_j\xi^j + \sum_{\alpha=1}^{j}\binom{j}{\alpha}D_x^\alpha\left(-i\operatorname{Re} a_j - \operatorname{Im} a_j\right)\xi^{j-\alpha}\right]\\ &= -2\sum_{j=0}^{p-1}\operatorname{Im} a_j\xi^j + \sum_{h=0}^{p-2}\sum_{j=h+1}^{p-1}\binom{j}{h}D_x^{j-h}\left(-i\operatorname{Re} a_j - \operatorname{Im} a_j\right)\xi^h\\ &= -2\operatorname{Im} a_{p-1}\xi^{p-1}\\ &\quad + \sum_{h=0}^{p-2}\left[-2\operatorname{Im} a_h + \sum_{j=h+1}^{p-1}\binom{j}{h}D_x^{j-h}\left(-i\operatorname{Re} a_j - \operatorname{Im} a_j\right)\right]\xi^h.\end{aligned}$$

Substituting in (3.5), we have that there exist postive constants A_1, c' such that

$$\begin{aligned}\frac{d}{dt}\|v_k(t,\cdot)\|^2 \geq\ & -2\|f_k\|\cdot\|v_k\| + 2\langle \operatorname{Im} a_{p-1}D_x^{p-1}v_k, v_k\rangle - A_1\|v_k\|^2 \\ & + \sum_{h=1}^{p-2}\left[2\langle \operatorname{Im} a_h D_x^h v_k, v_k\rangle + \sum_{j=h+1}^{p-1}\binom{j}{h}\langle (D_x^{j-h}(i\operatorname{Re} a_j + \operatorname{Im} a_j))D_x^h v_k, v_k\rangle\right] \\ \geq\ & 2\langle \operatorname{Im} a_{p-1}D_x^{p-1}v_k, v_k\rangle - 2\|f_k\|\cdot\|v_k\| - A_1\|v_k\|^2 - c'\frac{n^{p-1}}{\varrho_k}\|v_k\|^2, \qquad (3.6)\end{aligned}$$

since

$$|\langle \operatorname{Im} a_h D_x^h v_k, v_k\rangle| \leq cn^h\|v_k\|^2 \leq cn^{p-2}\|v_k\|^2 \leq c\frac{n^{p-1}}{\varrho_k}\|v_k\|^2$$

because of the support of $w_{n,k}$, and analogously

$$|\langle (D_x^{j-h}(i\operatorname{Re} a_j + \operatorname{Im} a_j))D_x^h v_k, v_k\rangle| \leq c\frac{n^{p-1}}{\varrho_k}\|v_k\|^2.$$

Now we want to give estimates of the terms in (3.6). This is done in the following Propositions 3.1 and 3.2.

Proposition 3.1. *Let $n = \varrho_k^a$ with $a \geq \mu \geq 2$. Then, for all $\nu \in \mathbb{N}$ there exists $C_\nu > 0$ such that, for every $t \in \left[0, \frac{\varrho_k}{n^{p-1}}\right]$ with k large enough:*

$$\begin{aligned}\langle \operatorname{Im} a_{p-1}(t,x)D_x^{p-1}v_k, v_k\rangle \geq & \left(\operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1} - C\frac{n^{p-1}}{\varrho_k}\right)\|v_k\|^2 \\ & -C_\nu {\varrho_k}^{4+\frac{1}{2}}\left(\frac{{\varrho_k}^{\mu+1}}{n}\right)^\nu n^{q+p-1}\|v_k\|, \qquad (3.7)\end{aligned}$$

for some fixed $C > 0$.

Proof. We split

$$\begin{aligned}\operatorname{Im} a_{p-1}(t,x)D_x^{p-1} =\ & \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1} \\ & + \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})(D_x^{p-1} - n^{p-1}) \qquad (3.8) \\ & + \left(\operatorname{Im} a_{p-1}(t,x) - \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})\right) D_x^{p-1}\end{aligned}$$

and set

$$\begin{aligned}I_1 &:= \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1}, \\ I_2 &:= \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})(D_x^{p-1} - n^{p-1}) \\ I_3 &:= (\operatorname{Im} a_{p-1}(t,x) - \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1}))\, D_x^{p-1}.\end{aligned}$$

We have

$$\langle I_1 v_k, v_k\rangle = \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1}\|v_k\|^2. \qquad (3.9)$$

To estimate $\langle I_2 v_k, v_k\rangle$, we localize at frequency n by means of the function $\chi_{1,k}$ defined in (2.22) and write

$$\begin{aligned}I_2 v_k &= \chi_{1,k}(D_x)I_2 v_k + (1 - \chi_{1,k}(D_x))I_2 v_k \\ &= \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})[\chi_{1,k}(D_x)(D_x^{p-1} - n^{p-1})v_k \\ &\quad + (1 - \chi_{1,k}(D_x))\,(D_x^{p-1} - n^{p-1})v_k],\end{aligned}$$

so, denoting by

$$J_1 := \|\chi_{1,k}(D_x)(D_x^{p-1} - n^{p-1})v_k\|, \tag{3.10}$$

$$J_2 := \| \left(1 - \chi_{1,k}(D_x)\right) (D_x^{p-1} - n^{p-1})v_k\|, \tag{3.11}$$

we have

$$\|I_2 v_k\| \leq |\operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})|(J_1 + J_2). \tag{3.12}$$

By Calderon-Vaillantcourt's Theorem A.3,

$$J_1 \leq C|\chi_{1,k}(\xi)(\xi^{p-1} - n^{p-1})|^0_{2,2}\|v_k\| \leq C'\frac{n^{p-1}}{\varrho_k^\mu}\|v_k\| \tag{3.13}$$

for some $C, C' > 0$, since by (2.23):

$$\begin{aligned} |\chi_{1,k}(\xi)(\xi^{p-1} - n^{p-1})| =& \ |\chi_{1,k}(\xi)(\xi - n)(\xi^{p-2} + n\xi^{p-3} + n^2\xi^{p-4} + \ldots + n^{p-2})| \\ \leq& \ c\frac{n}{\varrho_k^\mu}(p-1)n^{p-2} = c'\frac{n^{p-1}}{\varrho_k^\mu}, \end{aligned}$$

for some $c, c' > 0$, and for all $\gamma = \gamma_1 + \gamma_2$ with $|\gamma| \leq 2$ there are constants C_{γ_1}, $C_\gamma > 0$ such that:

$$|\partial_\xi^{\gamma_1}\chi_{1,k}(\xi)\partial_\xi^{\gamma_2}(\xi^{p-1} - n^{p-1})| \leq \begin{cases} C_{\gamma_1}\dfrac{n^{p-1}}{\varrho_k^\mu} & \gamma_2 = 0 \\ C_\gamma n^{p-1-\gamma_2} \leq C_\gamma\dfrac{n^{p-1}}{\varrho_k^\mu} & \gamma_2 \geq 1. \end{cases}$$

As it concerns (3.11), by definition of v_k we write

$$(D_x^{p-1} - n^{p-1})v_k = \left(W_{n,k}(D_x^{p-1} - n^{p-1}) + [D_x^{p-1} - n^{p-1}, W_{n,k}]\right) u_k. \tag{3.14}$$

Since $\sigma([D_x^{p-1} - n^{p-1}, W_{n,k}]) = \sum_{\alpha=1}^{p-1}\binom{p-1}{\alpha}\xi^{p-1-\alpha}\varrho_k^\alpha w_{n,k}^{\alpha,0}$, we have that

$$[D_x^{p-1} - n^{p-1}, W_{n,k}] = \sum_{\alpha=1}^{p-1}\binom{p-1}{\alpha}\varrho_k^\alpha W_{n,k}^{\alpha,0}D_x^{p-1-\alpha} \tag{3.15}$$

and therefore, by (3.11), (3.14), (3.15), the Calderon-Vaillantcourt's Theorem A.3 and (2.28), for every $\nu \in \mathbb{N}$ there are constants $C, C'_\nu, C''_\nu > 0$ such that:

$$\begin{aligned} J_2 \leq& \| \left(1 - \chi_{1,k}(D_x)\right) W_{n,k}(D_x^{p-1} - n^{p-1})u_k\| \\ &+ \sum_{\alpha=1}^{p-1}\binom{p-1}{\alpha}\varrho_k^\alpha\|(1 - \chi_{1,k}(D_x))W_{n,k}^{\alpha,0}D_x^{p-1-\alpha}u_k\| \\ \leq& C\Bigg(\left|\sigma\left(\left(1 - \chi_{1,k}(D_x)\right) W_{n,k}(D_x^{p-1} - n^{p-1})\right)\right|^0_{2,2} \\ &+ \sum_{\alpha=1}^{p-1}\varrho_k^\alpha\left|\sigma\left(\left(1 - \chi_{1,k}(D_x)\right) W_{n,k}^{\alpha,0}D_x^{p-1-\alpha}\right)\right|^0_{2,2}\Bigg)\|u_k\| \\ \leq& C'_\nu\varrho_k^{4+\frac{1}{2}}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu (n^{p-1} + \varrho_k n^{p-2} + \ldots + \varrho_k^{p-1})n^q \\ \leq& C''_\nu\varrho_k^{4+\frac{1}{2}}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+p-1}. \end{aligned} \tag{3.16}$$

Substituting now (3.13) and (3.16) in (3.12) we come to

$$\|I_2 v_k\| \le C\frac{n^{p-1}}{\varrho_k^{\mu}}\|v_k\| + C_\nu \varrho_k^{4+\frac{1}{2}}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\nu} n^{q+p-1} \tag{3.17}$$

for some $C, C_\nu > 0$, and hence

$$\langle I_2 v_k, v_k\rangle \ge -C\frac{n^{p-1}}{\varrho_k^{\mu}}\|v_k\|^2 - C_\nu \varrho_k^{4+\frac{1}{2}}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\nu} n^{q+p-1}\|v_k\|. \tag{3.18}$$

Finally, to estimate $\langle I_3 v_k, v_k\rangle$, we localize in a neighborhood of $x_k + pA_p(t)\xi^{p-1}$ by defining, for h as in (2.7), the function

$$\chi_{2,k}(x) := h\left(\varrho_k \frac{x - x_k - pA_p(t)\xi^{p-1}}{4pc_p}\right), \tag{3.19}$$

where c_p is the constant defined in Lemma 2.2. We have that

$$\operatorname{supp}\chi_{2,k} \subseteq \left\{x \ :\ |x - x_k - pA_p(t)\xi^{p-1}| \le \frac{2pc_p}{\varrho_k}\right\} \tag{3.20}$$

and

$$\operatorname{supp}(1-\chi_{2,k}) \subseteq \left\{x \ :\ |\, x - x_k - pA_p(t)\xi^{p-1}| \ge \frac{pc_p}{\varrho_k}\right\}. \tag{3.21}$$

We now claim that

$$\operatorname{supp}(1-\chi_{2,k}) \cap \operatorname{supp} W_{n,k}^{\alpha,\beta} = \emptyset \qquad \forall t \in \left[0, \frac{\varrho_k}{n^{p-1}}\right]. \tag{3.22}$$

This holds true because on the support of $w_{n,k}^{\alpha,\beta}$, given by Lemma 2.2, we have that, for all $t \in \left[0, \frac{\varrho_k}{n^{p-1}}\right]$,

$$\begin{aligned}
|x - x_k - pA_p(t)\xi^{p-1}| &\le |x - x_k - pA_p(t)n^{p-1}| + p|A_p(t)||\xi^{p-1} - n^{p-1}| \\
&\le \frac{c_p}{\varrho_k} + p\sup_{[0,T]}|a_p|\cdot t\cdot|\xi - n|\cdot|\xi^{p-2} + n\xi^{p-3} + \ldots + n^{p-2}| \\
&\le \frac{c_p}{\varrho_k} + c_p\frac{\varrho_k}{n^{p-1}}\frac{n}{2\varrho_k^{\mu}}(p-1)n^{p-2} \le p\frac{c_p}{\varrho_k},
\end{aligned}$$

by the definition of c_p. Therefore (3.22) is proved and

$$I_3 v_k = (1-\chi_{2,k}(x))I_3 v_k + \chi_{2,k}(x)I_3 v_k = \chi_{2,k}(x)I_3 v_k.$$

Then, by Lemma 2.4:

$$\begin{aligned}
\|I_3 v_k\| =\ & \|\chi_{2,k}(x)I_3 v_k\| = |\operatorname{Im} a_{p-1}(t,x) - \operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})|\cdot\|\chi_{2,k}(x)D_x^{p-1}v_k\| \\
\le\ & \left(\sup_{[0,T]\times\mathbb{R}}|\operatorname{Im}\partial_x a_{p-1}(t,x)|\right)\cdot|x - x_k - pA_p(t)n^{p-1}|\cdot\|\chi_{2,k}(x)D_x^{p-1}v_k\| \\
\le\ & \frac{c}{\varrho_k}\|D_x^{p-1}v_k\| \le \frac{c}{\varrho_k}\left(c_1 n^{p-1}\|v_k\| + C_\nu\varrho_k^{4+\frac{1}{2}}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\nu} n^{q+p-1}\right) \\
\le\ & C'\frac{n^{p-1}}{\varrho_k}\|v_k\| + C'_\nu {\varrho_k}^{3+\frac{1}{2}}\left(\frac{{\varrho_k}^{\mu+1}}{n}\right)^{\nu} n^{q+p-1},
\end{aligned}$$

for some $c, C', C'_\nu > 0$, and so

$$\langle I_3 v_k, v_k\rangle \ge -C'\frac{n^{p-1}}{\varrho_k}\|v_k\|^2 - C'_\nu {\varrho_k}^{3+\frac{1}{2}}\left(\frac{{\varrho_k}^{\mu+1}}{n}\right)^{\nu} n^{q+p-1}\|v_k\|. \tag{3.23}$$

Summing up (3.9), (3.18) and (3.23) we finally get the desired estimate (3.7). □

Proposition 3.2. *Let $n = \varrho_k^a$ with $a > \mu+1$. Then for all $\nu \in \mathbb{N}$ there exists $C_\nu > 0$ such that the function f_k defined in (3.4) satisfies*

$$\|f_k(t,\cdot)\| \leq C\varrho_k^2 n^{p-2}\sum_{j=1}^{p}\|v_k^{j,0}\| + C_\nu n^{p-1}\sum_{1\leq\alpha_1+\alpha_2\leq\nu(p-1)}\left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2}\|v_k^{\alpha_1,\alpha_2}\|$$
$$+C_\nu\, n^{q+p-1}\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu$$

for some fixed $C>0$ and for every $t\in[0,\varrho_k/n^{p-1}]$ with k large enough.

Proof. Let us recall that

$$f_k = \operatorname{op}\left(a_p(t)\sum_{\alpha=2}^{p}\binom{p}{\alpha}\xi^{p-\alpha}D_x^\alpha w_{n,k}\right)u_k + \sum_{j=0}^{p-1}[a_j(t,x)D_x^j, W_{n,k}]u_k, \tag{3.24}$$

and estimate the above terms separately. For $\alpha = p$

$$\begin{aligned}\operatorname{op}(a_p(t)D_x^p w_{n,k})u_k &= \int e^{ix\xi}a_p(t)D_x^p w_{n,k}(t,x;\xi)\hat u_k(t,\xi)đ\xi\\ &= a_p(t)\varrho_k^p\int e^{ix\xi}w_{n,k}^{p,0}(t,x;\xi)\hat u_k(t,\xi)đ\xi\\ &= a_p(t)\varrho_k^p W_{n,k}^{p,0}(t,x;D_x)u_k(t,x) = a_p(t)\varrho_k^p v_k^{p,0}(t,x)\end{aligned}$$

and hence

$$\|\operatorname{op}(a_p(t)D_x^p w_{n,k})u_k\| \leq C\varrho_k^p\|v_k^{p,0}\| \tag{3.25}$$

for some $C>0$.

For $2\leq\alpha\leq p-1$, by (2.30) we have:

$$\begin{aligned}\|\operatorname{op}(a_p(t)\xi^{p-\alpha}D_x^\alpha w_{n,k})u_k(t,\cdot)\| &\leq C\varrho_k^\alpha\|W_{n,k}^{\alpha,0}D_x^{p-\alpha}u_k\|\\ &\leq C'\varrho_k^\alpha\left(n^{p-\alpha}\sum_{j=0}^{p-\alpha}\|v_k^{\alpha+j,0}\| + C_\nu\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+p-\alpha}\right)\\ &\leq C''\varrho_k^2 n^{p-2}\sum_{s=2}^{p}\|v_k^{s,0}\| + C'_\nu\varrho_k^{4+\frac12}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+p-1}\end{aligned} \tag{3.26}$$

for some $C, C', C'', C_\nu, C'_\nu > 0$, since $(\varrho_k/n)^\alpha \leq (\varrho_k/n)^2$ and $\varrho_k^2/n^2 \leq 1/n = \varrho_k^{-a}$ for $2\leq\alpha\leq p-1$ and $a\geq 2$.

In order to estimate the second addend of (3.24) we compute, for $0\leq j\leq p-1$:

$$\begin{aligned}[a_jD_x^j, W_{n,k}]u_k &= a_j\sum_{h=0}^{j}\binom{j}{h}(D_x^{j-h}W_{n,k})D_x^h u_k - W_{n,k}a_jD_x^j u_k\\ &= a_j\sum_{h=0}^{j-1}\binom{j}{h}\varrho_k^{j-h}W_{n,k}^{j-h,0}D_x^h u_k + [a_j, W_{n,k}]D_x^j u_k.\end{aligned}$$

Then, by Lemmas 2.5 and 2.6, for $0 \leq j \leq p-1$, we have that:

$$\begin{aligned}
\|[a_j D_x^j, W_{n,k}]u_k\| \leq\ & C \sum_{h=0}^{j-1} \varrho_k^{j-h} \|W_{n,k}^{j-h,0} D_x^h u_k\| + \|[a_j, W_{n,k}] D_x^j u_k\| \\
\leq\ & \sum_{h=0}^{j-1} C_h \varrho_k^{j-h} n^h \sum_{s=0}^{h} \|v_k^{j-h+s,0}\| + C_\nu \varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+j} \\
& + C_\nu n^j \sum_{1 \leq \alpha_1+\alpha_2 \leq (\nu-1)(p-1)+j} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\| \\
\leq\ & C \varrho_k n^{j-1} \sum_{s=1}^{j} \|v_k^{s,0}\| + C_\nu \varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+j} \qquad (3.27) \\
& + C_\nu n^j \sum_{1 \leq \alpha_1+\alpha_2 \leq (\nu-1)(p-1)+j} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\|
\end{aligned}$$

for some $C, C_\nu > 0$.

By (3.24), (3.25), (3.26) and (3.27):

$$\begin{aligned}
\|f_k(t,\cdot)\| \leq\ & C \varrho_k^p \|v_k^{p,0}\| + C'' \varrho_k^2 n^{p-2} \sum_{s=2}^{p} \|v_k^{s,0}\| + C'_\nu \varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+p-1} \\
& + \sum_{j=0}^{p-1} \Bigg[C \varrho_k n^{j-1} \sum_{s=1}^{j} \|v_k^{s,0}\| + C'_\nu n^j \sum_{1 \leq \alpha_1+\alpha_2 \leq (\nu-1)(p-1)+j} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\| \\
& + C'_\nu \varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+j} \Bigg] \\
\leq\ & \tilde{C} \varrho_k^2 n^{p-2} \sum_{s=1}^{p} \|v_k^{s,0}\| + \tilde{C}_\nu n^{p-1} \sum_{1 \leq \alpha_1+\alpha_2 \leq \nu(p-1)} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\| \\
& + \tilde{C}_\nu \varrho_k^{4+\frac{1}{2}} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^\nu n^{q+p-1}
\end{aligned}$$

for some $\tilde{C}, \tilde{C}_\nu > 0$. □

Summing up, from (3.6), by Propositions 3.1 and 3.2, for every $\nu \in \mathbb{N}$ we come to the estimate:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v_k(t,\cdot)\|^2 \geq\ & \left(\operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1}) n^{p-1} - A\left(1 + \frac{n^{p-1}}{\varrho_k}\right)\right) \|v_k\|^2 \\
& - C_\nu {\varrho_k}^{4+\frac{1}{2}} \left(\frac{{\varrho_k}^{\mu+1}}{n}\right)^\nu n^{q+p-1} \|v_k\| - C \varrho_k^2 n^{p-2} \sum_{j=1}^{p} \|v_k^{j,0}\| \cdot \|v_k\| \qquad (3.28) \\
& - C_\nu n^{p-1} \sum_{1 \leq \alpha_1+\alpha_2 \leq \nu(p-1)} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\| \cdot \|v_k\|
\end{aligned}$$

for some $A, C, C_\nu > 0$. Now, for $a > \mu + 1$, it is possible to take $\nu \in \mathbb{N}$ sufficiently large so that

$$\sup_k {\varrho_k}^{4+\frac{1}{2}} \left(\frac{{\varrho_k}^{\mu+1}}{n}\right)^\nu n^{q+p-1} \leq M_\nu \qquad (3.29)$$

for some $M_\nu > 0$. After substituting (3.29) in (3.28), we finally choose a and μ such that

$$\frac{d}{dt}\|v_k(t,\cdot)\| \geq \left(\operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1} - A\left(1 + \frac{n^{p-1}}{\varrho_k}\right)\right)\|v_k\| - M'_\nu \\ -C'_\nu n^{p-1} \sum_{1\leq\alpha_1+\alpha_2\leq\nu(p-1)} \left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\|, \tag{3.30}$$

for some $M'_\nu, C'_\nu > 0$; this can be done for

$$\begin{cases} \mu > p+1 \\ \mu + 1 < a \leq \dfrac{p\mu - 2}{p-1} = \mu + 1 + \dfrac{\mu - p - 1}{p-1}, \end{cases} \tag{3.31}$$

since $\varrho_k^2 n^{p-2} \leq n^{p-1}\left(\frac{\varrho_k^\mu}{n}\right)^j$ for all $1 \leq j \leq p$ if $2 \leq p\mu - a(p-1)$, and this implies, together with $a > \mu + 1$, that we must take $\mu > p + 1$.

Using now $\varrho_k\left(\frac{\varrho_k^\mu}{n}\right)^{\alpha_1+\alpha_2} \leq \left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha_1+\alpha_2}$, we come to

$$\frac{d}{dt}\|v_k(t,\cdot)\| \geq \left(\operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1} - A\left(1 + \frac{n^{p-1}}{\varrho_k}\right)\right)\|v_k\| - M' \\ -C'\frac{n^{p-1}}{\varrho_k} \sum_{1\leq\alpha_1+\alpha_2\leq\nu(p-1)} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha_1+\alpha_2} \|v_k^{\alpha_1,\alpha_2}\|$$

for some constants $M', C' > 0$, since ν has been fixed in (3.29).

Arguing in the same way for the functions $v_k^{\alpha,\beta}$ instead of v_k, we finally get:

Proposition 3.3. *Let n be as in* (2.9)*, a, μ as in* (3.31)*, $\nu \in \mathbb{N}$ sufficiently large so that* (3.29) *is satisfied. Then, for every $\alpha, \beta \in \mathbb{N}_0$ there exists $C_{\alpha,\beta} > 0$ such that for all $t \in [0, \varrho_k/n^{p-1}]$ with k large enough:*

$$\frac{d}{dt}\|v_k^{\alpha,\beta}(t,\cdot)\| \geq \left(\operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1} - A\left(1 + \frac{n^{p-1}}{\varrho_k}\right)\right)\|v_k^{\alpha,\beta}\| - C_{\alpha,\beta} \\ -C_{\alpha,\beta}\frac{n^{p-1}}{\varrho_k} \sum_{1\leq\tilde\alpha+\tilde\beta\leq\nu(p-1)} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\tilde\alpha+\tilde\beta} \|v_k^{\alpha+\tilde\alpha,\beta+\tilde\beta}\|. \tag{3.32}$$

From Proposition 3.3 it follows that:

$$\frac{d}{dt}\left(\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha+\beta}\|v_k^{\alpha,\beta}\|\right) \geq \left(\operatorname{Im} a_{p-1}(t, x_k + pA_p(t)n^{p-1})n^{p-1} - A\left(1 + \frac{n^{p-1}}{\varrho_k}\right)\right) \\ \cdot\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha+\beta}\|v_k^{\alpha,\beta}\| - C_{\alpha,\beta} \\ -C_{\alpha,\beta}\frac{n^{p-1}}{\varrho_k} \sum_{1\leq\tilde\alpha+\tilde\beta\leq\nu(p-1)} \left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha+\tilde\alpha+\beta+\tilde\beta} \|v_k^{\alpha+\tilde\alpha,\beta+\tilde\beta}\|. \tag{3.33}$$

We now choose $s \in \mathbb{N}$ sufficiently large so that, for all $\bar\alpha + \bar\beta \geq s+1$, using (2.21) and $a > \mu+1$, we have

$$\frac{n^{p-1}}{\varrho_k}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\bar\alpha+\bar\beta}\|v_k^{\bar\alpha,\bar\beta}\| \leq c_s\frac{n^{p-1}}{\varrho_k}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{s+1}\varrho_k^{\frac{1}{2}+2}n^q \leq c'_s \tag{3.34}$$

for some $c_s, c'_s > 0$. In order to satisfy (3.34) it's enough to take s such that

$$a(q+p-1)+\frac{1}{2}+1+(s+1)(\mu+1-a)\leq 0,$$

i.e.

$$s\geq\frac{a(q+p-2)+\mu+\frac{5}{2}}{a-\mu-1}. \tag{3.35}$$

With this choice of s we define:

$$\sigma_k(t):=\sum_{0\leq\alpha+\beta\leq s}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha+\beta}\|v_k^{\alpha,\beta}\|. \tag{3.36}$$

From (3.33) we have that:

$$\begin{aligned}
\frac{d}{dt}\sigma_k(t)=&\sum_{0\leq\alpha+\beta\leq s}\frac{d}{dt}\left[\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha+\beta}\|v_k^{\alpha,\beta}\|\right]\\
\geq&\sum_{0\leq\alpha+\beta\leq s}\left(\operatorname{Im}a_{p-1}(t,x_k+pA_p(t)n^{p-1})n^{p-1}-A\left(1+\frac{n^{p-1}}{\varrho_k}\right)\right)\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha+\beta}\|v_k^{\alpha,\beta}\|\\
&-C_s\sum_{1\leq\bar\alpha+\bar\beta\leq s}\frac{n^{p-1}}{\varrho_k}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\bar\alpha+\bar\beta}\|v_k^{\bar\alpha,\bar\beta}\|-C_s\\
\geq&\left(\operatorname{Im}a_{p-1}(t,x_k+pA_p(t)n^{p-1})n^{p-1}-A_s\left(1+\frac{n^{p-1}}{\varrho_k}\right)\right)\sigma_k(t)-C_s
\end{aligned}$$

for some $C_s, A_s > 0$, because of (3.34).

We have thus obtained for the function σ_k the following differential inequality:

$$\begin{gathered}
\sigma'_k(t)-B_k(t)\sigma_k(t)+C_s\geq 0\qquad t\in\left[0,\frac{\varrho_k}{n^{p-1}}\right],\ k\gg 1,\\
B_k(t):=\operatorname{Im}a_{p-1}(t,x_k+pA_p(t)n^{p-1})n^{p-1}-A_s\left(1+\frac{n^{p-1}}{\varrho_k}\right),
\end{gathered}$$

which clearly implies that

$$\sigma_k(t)\geq e^{\int_0^t B_k(\theta)d\theta}\left[\sigma_k(0)-C_s\int_0^t e^{-\int_0^\tau B_k(\theta)d\theta}d\tau\right]\qquad t\in\left[0,\frac{\varrho_k}{n^{p-1}}\right],\ k\gg 1.$$

For $t=\varrho_k/n^{p-1}$ we have

$$\sigma_k\left(\frac{\varrho_k}{n^{p-1}}\right)\geq e^{\int_0^{\frac{\varrho_k}{n^{p-1}}}B_k(\theta)d\theta}\left[\sigma_k(0)-C_s\int_0^{\frac{\varrho_k}{n^{p-1}}}e^{-\int_0^\tau B_k(\theta)d\theta}d\tau\right]. \tag{3.37}$$

Let us focus on the term $\int_0^{\frac{\varrho_k}{n^{p-1}}}B_k(\theta)d\theta$; the choice of x_k, ϱ_k of Lemma 2.1 gives for it, by the change of variables $\theta'=n^{p-1}\theta$ and for k large enough, the following estimate from below:

$$
\begin{aligned}
\int_0^{\frac{\varrho_k}{n^{p-1}}} B_k(\theta)d\theta &= \int_0^{\frac{\varrho_k}{n^{p-1}}} \operatorname{Im} a_{p-1}(\theta, x_k + pA_p(\theta)n^{p-1})n^{p-1}d\theta - A_s \int_0^{\frac{\varrho_k}{n^{p-1}}} \left(1 + \frac{n^{p-1}}{\varrho_k}\right) d\theta \\
&\geq \int_0^{\varrho_k} \operatorname{Im} a_{p-1}\left(\frac{\theta'}{n^{p-1}}, x_k + pA_p\left(\frac{\theta'}{n^{p-1}}\right)n^{p-1}\right) d\theta' - 2A_s \\
&= \int_0^{\varrho_k} \operatorname{Im} a_{p-1}\left(\frac{\theta'}{n^{p-1}}, x_k + pa_p(\tau_k)\theta'\right) d\theta' - 2A_s \\
&\geq M\log(1+\varrho_k) + k - 2A_s, \qquad (3.38)
\end{aligned}
$$

for some $\tau_k \in [0, \theta'/n^{p-1}]$, since $A_p(\theta'/n^{p-1})n^{p-1} = \theta' a_p(\tau_k)$ by the mean value theorem for integration.

Similarly it follows that for every $\tau \in [0, \frac{\varrho_k}{n^{p-1}}]$:

$$
\int_0^\tau B_k(\theta)d\theta \geq \int_0^{n^{p-1}\tau} \operatorname{Im} a_{p-1}\left(\frac{\theta'}{n^{p-1}}, x_k + pa_p(\tau_k')\theta'\right) d\theta' - 2A_s \geq -2A_s \qquad (3.39)
$$

for some $\tau_k' \in [0, \theta'/n^{p-1}]$, because of Lemma 2.1, since $n^{p-1}\tau \leq n^{p-1}\frac{\varrho_k}{n^{p-1}} \leq \varrho_k$.

Finally, from (3.36) and (3.2) we have $\|\sigma_k(0)\| \geq \|v_k(0)\| \geq \|h\| > 0$; therefore, substituiting the estimates (3.38) and (3.39) into (3.37), we have proved the following desired estimate from below for the function $\sigma_k(t)$:

Proposition 3.4. *For every $M > 0$ and $k \in \mathbb{N}$ let x_k, ϱ_k be as in Lemma 2.1. Taking $\mu \geq 2$ in* (2.13) *and n as in* (2.9) *with a, μ satisfying* (3.31)*, it is possible to construct the functions $v_k^{\alpha,\beta}$ in* (2.17) *and then to choose s great enough (see* (3.35)*) such that the function $\sigma_k(t)$ defined in* (3.36) *satisfies the following estimate from below:*

$$
\sigma_k\left(\frac{\varrho_k}{n^{p-1}}\right) \geq c(1+\varrho_k)^M, \qquad k \gg 1, \qquad (3.40)
$$

for some $c > 0$.

4. Estimate from above and proof of the main Theorem.

The estimate from above is now quite simple to be obtained and it is shown in the following:

Proposition 4.1. *For every $M > 0$ and $k \in \mathbb{N}$ let x_k, ϱ_k be as in Lemma 2.1. Taking $\mu \geq 2$ in* (2.13) *and n as in* (2.9) *with a, μ satisfying* (3.31)*, it is possible to construct the functions $v_k^{\alpha,\beta}$ in* (2.17) *and then to choose s great enough (see* (3.35)*) such that the function $\sigma_k(t)$ defined in* (3.36) *satisfies the following estimate from above for all $t \in [0, \frac{\varrho_k}{n^{p-1}}]$:*

$$
\sigma_k(t) \leq C\varrho_k^{\frac{1}{2}+2+aq}, \qquad k \gg 1, \qquad (4.1)
$$

for some $C > 0$.

Proof. The estimate (2.21) obtained in Section 2 and definition (3.36) immediately give:

$$
\sigma_k(t) \leq \sum_{0 \leq \alpha+\beta \leq s} C_{\alpha,\beta}\left(\frac{\varrho_k^{\mu+1}}{n}\right)^{\alpha+\beta} \varrho_k^{\frac{1}{2}+2+aq} \leq C\varrho_k^{\frac{1}{2}+2+aq}
$$

for some $C > 0$, since s has been fixed in (3.35). □

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us assume, by contradiction, that the Cauchy problem (1.2) is well-posed in H^∞ but (1.6) does not hold true. Then at least one of the two conditions (2.3) or (2.4) does not hold true. As we remarked in Section 2, we can assume, without loss of generality, that (2.3) does not hold and apply Lemma 2.1. By Propositions 3.4 and 4.1 we come to the estimate:

$$c(1+\varrho_k)^M \leq \sigma_k\left(\frac{\varrho_k}{n^{p-1}}\right) \leq C\varrho_k^{\frac{1}{2}+2+aq},$$

for positive constants c, C not depending on k, giving rise to a contradiction for k large enough, if we choose

$$M > \frac{1}{2}+2+aq.$$

Therefore condition (1.6) must be satisfied and the proof is complete. □

Appendix A

The localized pseudo-differential operators $W^{\alpha,\beta}_{n,k}(t,x,D_x)$ of the present paper have symbols $w^{\alpha,\beta}_{n,k}(t,x,\xi)$ depending on the parameter t and belonging to the class $S^0_{0,0}$ of all functions $p(x,\xi) \in C^\infty(\mathbb{R}^2)$ such that for every $\alpha, \beta \geq 0$

$$|D^\beta_x \partial^\alpha_\xi p(x,\xi)| \leq C_{\alpha,\beta}; \tag{A.1}$$

$S^0_{0,0}$ is a Fréchet space with semi-norms

$$|p|^0_{\ell,\ell'} := \max_{\alpha\leq\ell,\beta\leq\ell'} \sup_{x,\xi\in\mathbb{R}} |\partial^\alpha_\xi D^\beta_x p(x,\xi)|. \tag{A.2}$$

The class $S^0_{0,0}$ corresponds to the classical class $S^m_{\varrho,\delta}$ (defined by $|D^\beta_x\partial^\alpha_\xi p(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\varrho\alpha+\delta\beta}$ instead of (A.1); see [23]) with $m=\varrho=\delta=0$. In the $S^m_{0,0}$ classes the usual asymptotic expansion formula

$$p(x,\xi) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!}\partial^\alpha_\xi p_1(x,\xi) D^\beta_x p_2(x,\xi)$$

fails to be true, and we need to use the expansion formula with a remainder, as in [23, Thm. 3.1, Chap. 2] (see also [20, Thm. A]):

Theorem A.1. *Let $P_j(x,D_x)$ be pseudo-differential operators with symbols $p_j(x,\xi) \in S^{m_j}_{0,0}$, $j=1,2$. Then the operator $P(x,D_x) = P_1(x,D_x)\circ P_2(x,D_x)$ has symbol given by the oscillatory integral*

$$p(x,\xi) = \iint e^{-iy\eta} p_1(x,\xi+\eta)p_2(x+y,\xi)dy\,đ\eta \in S^{m_1+m_2}_{0,0},$$

where $đ\eta = (2\pi)^{-1}d\eta$.

Moreover, the following expansion formula holds for every $\nu \in \mathbb{N}$:

$$p(x,\xi) = \sum_{\alpha\leq\nu-1} \frac{1}{\alpha!}\partial^\alpha_\xi p_1(x,\xi)D^\beta_x p_2(x,\xi) + \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} r_{\theta,\nu}(x,\xi)d\theta,$$

where

$$r_{\theta,\nu}(x,\xi) := \iint e^{-iy\eta}\partial^\nu_\xi p_1(x,\xi+\theta\eta)D^\nu_x p_2(x+y,\xi)dy\,đ\eta \in S^{m_1+m_2}_{0,0}.$$

We recall from [23, Lemm 2.2, Chap. 7], (see also [20, Thm. B]):

Theorem A.2. *Let $p_j(x,\xi)\in S^0_{0,0}$ for $j=1,2$ and define*

$$p_\theta(x,\xi) := \iint e^{-iy\eta} p_1(x,\xi+\theta\eta)p_2(x+y,\xi)dy\,đ\eta.$$

Then for every $\ell\in\mathbb{N}_0$ there exists a constant $C_\ell>0$ such that

$$|p_\theta|^0_{\ell,\ell} \le C_\ell |p_1|^0_{\ell+2,\ell+2}|p_2|^0_{\ell+2,\ell+2}$$

for all $\theta\in[0,1]$.

We conclude the appendix with the statement of the Calderón-Vaillancourt's Theorem about continuity of pseudo-differential operators with symbols in the class $S^0_{0,0}$ acting on L^2 (see [12] or [21, Thm. C]):

Theorem A.3. *Let $p(x,D_x)$ be a pseudo-differential operator with symbol $p(x,\xi)\in S^0_{0,0}$. Then:*

$$\|p(x,D_x)u\| \le C|p|^0_{2,2}\;\|u\|$$

for all $u\in L^2$, with a positive constant C independent of p and u.

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