Coherence and Chattering of a One-Way Valve

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Abstract

In this paper we analyze the effects of a one-way valve on the isothermal gas flow through a pipe. The valve keeps the flow at a constant value $q_* > 0$, if possible; otherwise it is closed. First, for fixed q_*, we define a Riemann solver and characterize the coherence of its initial data; coherence is a necessary condition for the construction of solutions to a general initial-value problem based on a wave-front tracking scheme. We also give an example of an invariant and coherent domain where the valve can be either open and closed. Second, for suitable compact sets of initial data we precise the range of values q_* that guarantee the coherence. At last, in the case of a real valve with finite reaction time, we show the chattering (rapid switch on and off) of the valve in correspondence of incoherent initial data.

Keywords: systems of conservation laws, gas flow, valve, Riemann problem, coupling conditions, chattering.

2010 AMS subject classification: 35L65, 35L67, 76B75.

1 Introduction

We continue in this paper the study of an isothermal gas flow through a straight pipe in presence of valves, which was begun in [12]. As in that paper, we assume that the gas fills the whole pipe and its velocity is constant on every cross-section; moreover, we neglect the friction effects exerted by the walls of the pipe. The flow is then governed by the system

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0, \\ \partial_t(\rho\, v) + \partial_x\big(\rho\, v^2 + p(\rho)\big) = 0, \end{cases} \tag{1.1}$$

where $t > 0$ is the time and $x \in \mathbb{R}$ is the space position along the pipe. We denoted by ρ the mass density of the gas and by v its velocity; we also define the momentum by $q \doteq \rho\, v$. At last, the pressure is defined by the isothermal law

$$p(\rho) \doteq a^2 \rho, \tag{1.2}$$

where the constant $a > 0$ is the sound speed. We point out that our results can be easily extended to more general pressure laws. We refer to [5, 6, 9, 19] for the use of these equations in analogous modelings.

The main feature of this paper is the presence of a control valve, which we assume located along the pipe at $x = 0$. On the one hand, the modeling of valves regulating gas flows is huge in the applied literature; as far as this paper is concerned, we refer to [20]. Other useful sources of information are [18, 23, 24], namely for optimization problems. On the other hand, a comprehensive mathematical treatment of this subject through system (1.1) seems missing and, to the best of our knowledge, our previous paper [12] was the first one toward this direction. More precisely, in [12] we modeled a two-way valve that was open or closed according to a threshold of the pressure gradient; we proposed a suitable Riemann solver and studied its coherence, consistence, continuity with respect to the initial data and invariant domains. We refer to that paper for more introductory information. For completeness, we point out that in the engineering literature, valves are often modeled by systems of ordinary differential equations, see for instance [1] and references therein.

Differently than in [12], in this paper we focus on one-way valves, by far the most common type. In these cases the gas can flow through the valve only in one direction, say the positive direction of the x axis. We focus on the simple case of a valve that aims at keeping a fixed outgoing flow $q_* > 0$; when this is not possible, for instance because the incoming flow is too weak, then the valve closes. Then, we introduce a *coupling Riemann solver*; our main concern is the study of its *coherence*. This property means, roughly speaking, the following. Any solution u provided by the solver is self-similar, i.e., it only depends on t and x through the ratio $\xi = x/t$ and can then be thought as a function $u = u(\xi)$ of ξ only. Coherence prescribes that any two traces $u(\xi_o^\pm)$ at $\xi_o \in \mathbb{R}$ give rise, when considered as initial data of another Riemann problem, to a solution that is defined as $u(\xi_o^-)$ for $\xi < \xi_o$ and $u(\xi_o^+)$ for $\xi \geq \xi_o$. In a few words, the solution of the Riemann problem *for the traces of* u reproduces the same *local* behavior of u. This property, which is obvious for a classic Lax solution, may fail for more complicated solvers and, in particular, for coupling Riemann solvers associated to valves. From a mathematical point of view, coherence is an unavoidable condition for the construction of global solutions for general initial data with bounded variation (**BV**); in turn, the well-posedness of the initial-value problem for **BV** data is the starting point for the control, by the flow parameter q_*.

Here follows an outline of the paper together with the most important results. Some background material about system (1.1) is provided in Section 2; we refer to [22] for more information. Section 3 introduces the basic notion of coupling Riemann solver in the case of a general valve. Section 4 deals with the aforementioned model of a one-way valve with fixed q_* and contains the most important results of this paper. The framework, as in [12], is that of systems of conservation laws with point constraints, which has been developed up to now only for vehicular and pedestrian flows, see [10, 11, 25] and references therein. There, we state Theorem 4.5, which characterizes the coherence of the coupling Riemann solver, and Proposition 4.8, which shows an invariant and coherent domain consisting of states for which the valve can be both open and closed. This paper is just the first step toward a mathematical analysis of either the management and the optimal control of a gas flow through a valve. Section 5 is a first step toward this direction: for a fixed suitable set of initial data, we study the range of values q_* which make those data coherent. Next, Section 6 is both of applied and numerical interest. There we consider the case of a real-world valve with finite reaction time $\tau > 0$, and give an explicit example of chattering (rapid switch opening and closing of the valve, a phenomenon one would like to avoid) for incoherent initial data. We refer to [26] for a real-world example of chattering occurring at low flows. This establishes the relationship between incoherence and chattering. Notice that the numerical time-stepping schemes used to solve (1.1) with the valve considered in Section 4 usually do introduce such a delay. We show that the limit $\tau \to 0$ of the solutions may not converge (because of chattering) to the expected solution. Several technical and lengthy proofs are deferred to Section 7 while the shorter ones are provided just below the corresponding statements. This section also contains some minor additional results and comments. Section 8 resumes our conclusions and proposes some open problems.

We point out that, in this paper, all figures representing Lax curves are obtained by using the explicit mathematical expression of these curves.

2 Preliminary results and notation

In this section we provide some background results and give several definitions. System (1.1), together with the pressure law (1.2), can be written in the conservative (ρ, q)-variables, with $q \doteq \rho\, v$, as

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + a^2 \rho \right) = 0. \end{cases} \tag{2.1}$$

In the following we usually refer to (2.1) and denote $u \doteq (\rho, q)$; then u takes values in $\Omega \doteq \{(\rho, q) \in \mathbb{R}^2 : \rho > 0\}$. For any pair of constant states $u_\ell, u_r \in \Omega$ the Riemann problem for (2.1) is the initial-value problem with initial condition

$$u(0, x) = \begin{cases} u_\ell & \text{if } x < 0, \\ u_r & \text{if } x \geq 0. \end{cases} \tag{2.2}$$

Definition 2.1. *A function* $u \in \mathbf{C^0}([0,\infty);\mathbf{L^\infty}(\mathbb{R};\Omega))$ *is a* weak solution *of Riemann problem* (2.1), (2.2) *in* $[0,\infty)\times\mathbb{R}$ *if for any test function* $\varphi \in \mathbf{C_c^\infty}([0,\infty)\times\mathbb{R};\mathbb{R})$ *we have*

$$\int_0^\infty \int_{\mathbb{R}} \Big[\rho\,\partial_t\varphi + q\,\partial_x\varphi\Big]\mathrm{d}x\,\mathrm{d}t + \rho_\ell \int_{-\infty}^0 \varphi(0,x)\,\mathrm{d}x + \rho_r \int_0^\infty \varphi(0,x)\,\mathrm{d}x = 0,$$

$$\int_0^\infty \int_{\mathbb{R}} \left[q\,\partial_t\varphi + \left(\frac{q^2}{\rho^2}+a^2\right)\rho\,\partial_x\varphi\right]\mathrm{d}x\,\mathrm{d}t + q_\ell \int_{-\infty}^0 \psi(0,x)\,\mathrm{d}x + q_r \int_0^\infty \psi(0,x)\,\mathrm{d}x = 0.$$

We denote by $\mathbf{BV}(\mathbb{R};\Omega)$ the space of Ω-valued functions with bounded variation. We can assume that any function in $\mathbf{BV}(\mathbb{R};\Omega)$ is right continuous by possibly changing its values at countably many points.

Definition 2.2. *Let* $\mathsf{D} \subseteq \Omega\times\Omega$ *and a map* $\mathcal{RS} : \mathsf{D} \to \mathbf{BV}(\mathbb{R};\Omega)$*.*

- *We say that* $\mathcal{RS}$ *is a* Riemann solver *for* (2.1) *if for any* $(u_\ell, u_r) \in \mathsf{D}$ *the function* $(t,x) \mapsto \mathcal{RS}[u_\ell,u_r](x/t)$ *is a self-similar weak solution to* (2.1), (2.2) *in* $[0,\infty)\times\mathbb{R}$*.*
- *A Riemann solver* $\mathcal{RS}$ *is* coherent *at* $(u_\ell,u_r) \in \mathsf{D}$ *if* $u \doteq \mathcal{RS}[u_\ell,u_r]$ *satisfies for any* $\xi_o \in \mathbb{R}$

$$\Big(u(\xi_o^-),u(\xi_o^+)\Big) \in \mathsf{D} \qquad \text{and} \qquad \mathcal{RS}\Big[u(\xi_o^-),u(\xi_o^+)\Big](\xi) = \begin{cases} u(\xi_o^-) & \text{if } \xi<\xi_o,\\ u(\xi_o^+) & \text{if } \xi \geq \xi_o. \end{cases} \tag{ch}$$

The coherence domain $\mathsf{CH} \subseteq \mathsf{D}$ *of* $\mathcal{RS}$ *is the set of all pairs* $(u_\ell,u_r) \in \mathsf{D}$ *where* $\mathcal{RS}$ *is coherent.*

According to the previous definition, a Riemann solver $\mathcal{RS}$ is coherent at an initial datum $(u_\ell,u_r) \in \mathsf{D}$ if the ordered pair $\big(\mathcal{RS}[u_\ell,u_r](\xi_o^-),\mathcal{RS}[u_\ell,u_r](\xi_o^+)\big)$ of the traces of the corresponding solution belongs to D and, in a sense, is a fixed point of $\mathcal{RS}$. In particular, if a Riemann solver $\mathcal{RS}$ is coherent at $(u_o,u_o) \in \mathsf{D}$, then $\mathcal{RS}[u_o,u_o] \equiv u_o$, see [12, Proposition 2.3].

On the one hand, the coherence of a Riemann solver is a desirable property for a numerical scheme. Indeed, a numerical time-stepping scheme based on a Riemann solver that fails to be consistent may not produce the expected solution of a Riemann problem, see for instance [14]. On the other hand, the lack of coherence has a physical counterpart, which is typical when dealing with real valves: it can induce commuting. In Section 6 we give an example to better explain both issues.

Coherence may fail in presence of a valve, see [12] and Section 4 below. However, every Lax Riemann solver is coherent [12, Proposition 2.5]. In particular, the Lax Riemann solver $\mathcal{RS}_{\mathrm{p}} : \Omega\times\Omega \to \mathbf{BV}(\mathbb{R};\Omega)$ of (2.1), see [12, 22], is coherent in $\Omega\times\Omega$; by $(\text{ch})_2$ we deduce

$$\mathcal{RS}_{\mathrm{p}}[u_o,u_o] \equiv u_o. \tag{2.3}$$

We denote

$$u_{\mathrm{p}} \doteq \mathcal{RS}_{\mathrm{p}}[u_\ell,u_r], \qquad\qquad u_{\mathrm{p}}^\pm \doteq u_{\mathrm{p}}(0^\pm).$$

The eigenvalues of (2.1) are $\lambda_1(u) \doteq \frac{q}{\rho} - a = v-a$, and $\lambda_2(u) \doteq \frac{q}{\rho}+a = v+a$. System (2.1) is easily proved to be strictly hyperbolic in Ω and both characteristic fields are genuinely nonlinear [22]. Any smooth discontinuity curve $x=\gamma(t)$ of a weak solution u of (2.1) satisfies the Rankine-Hugoniot conditions

$$\Big(\rho^+ - \rho^-\Big)\dot\gamma = q^+ - q^-, \tag{2.4}$$

$$\Big(q^+ - q^-\Big)\dot\gamma = \left(\frac{(q^+)^2}{\rho^+} + a^2\,\rho^+\right) - \left(\frac{(q^-)^2}{\rho^-} + a^2\,\rho^-\right), \tag{2.5}$$

where $u^\pm(t) \doteq u(t,\gamma(t)^\pm)$ are the traces of u, see [8, 13].

For any fixed $u_o \in \Omega$ we define $\mathcal{S}_i^{u_o}, \mathcal{R}_i^{u_o} : (0,\infty) \to \mathbb{R}$, $i \in \{1,2\}$, as

$$\mathcal{S}_i^{u_o}(\rho) \doteq \rho\left[\frac{q_o}{\rho_o} + (-1)^i\,a\left(\sqrt{\frac{\rho}{\rho_o}} - \sqrt{\frac{\rho_o}{\rho}}\right)\right], \qquad \mathcal{R}_i^{u_o}(\rho) \doteq \rho\left[\frac{q_o}{\rho_o} + (-1)^i\,a\,\log\left(\frac{\rho}{\rho_o}\right)\right]. \tag{2.6}$$

Then we define $\mathcal{FL}_i^{u_o}, \mathcal{BL}_i^{u_o} : (0,\infty) \to \mathbb{R}$, $i \in \{1,2\}$, by

$$\mathcal{FL}_1^{u_o}(\rho) \doteq \begin{cases} \mathcal{R}_1^{u_o}(\rho) & \text{if } \rho \in (0,\rho_o], \\ \mathcal{S}_1^{u_o}(\rho) & \text{if } \rho \in (\rho_o,\infty), \end{cases} \qquad \mathcal{FL}_2^{u_o}(\rho) \doteq \begin{cases} \mathcal{S}_2^{u_o}(\rho) & \text{if } \rho \in (0,\rho_o), \\ \mathcal{R}_2^{u_o}(\rho) & \text{if } \rho \in [\rho_o,\infty), \end{cases}$$

$$\mathcal{BL}_1^{u_o}(\rho) \doteq \begin{cases} \mathcal{S}_1^{u_o}(\rho) & \text{if } \rho \in (0,\rho_o), \\ \mathcal{R}_1^{u_o}(\rho) & \text{if } \rho \in [\rho_o,\infty), \end{cases} \qquad \mathcal{BL}_2^{u_o}(\rho) \doteq \begin{cases} \mathcal{R}_2^{u_o}(\rho) & \text{if } \rho \in (0,\rho_o], \\ \mathcal{S}_2^{u_o}(\rho) & \text{if } \rho \in (\rho_o,\infty). \end{cases}$$

The graphs of the functions $\mathcal{FL}_i^{u_o}$ and $\mathcal{BL}_i^{u_o}$ are the forward $\mathsf{FL}_i^{u_o}$ and backward $\mathsf{BL}_i^{u_o}$ Lax curves of the i-th family through u_o, see Figure 1. Analogously, the shock $\mathsf{S}_i^{u_o}$ and rarefaction $\mathsf{R}_i^{u_o}$ curves through u_o are the graphs of the functions $\mathcal{S}_i^{u_o}$ and $\mathcal{R}_i^{u_o}$. The shock speeds are

$$s_1^{u_o}(\rho) \doteq v_o - a\sqrt{\rho/\rho_o}, \qquad s_2^{u_o}(\rho) \doteq v_o + a\sqrt{\rho/\rho_o}.$$

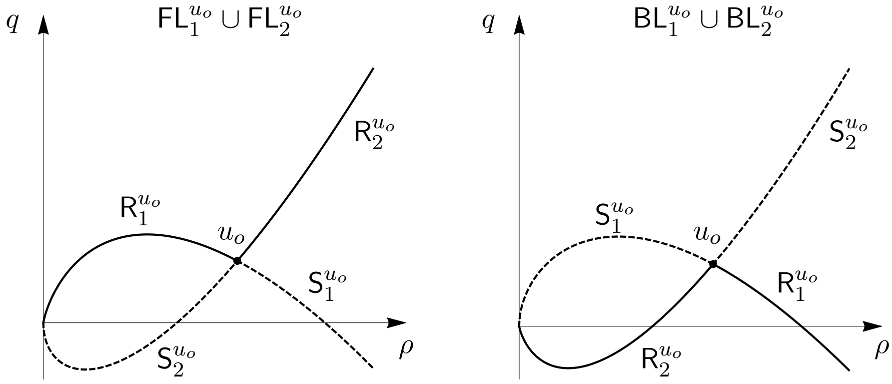

Figure 1: Forward and backward Lax curves.

We now provide the basic properties of the sets $\mathsf{S}_i^{u_o}$, $\mathsf{R}_i^{u_o}$, see [12, Proposition 2.4].

Proposition 2.3. *Let $u_o, u^o \in \Omega$ be two distinct states and $i \in \{1,2\}$. Then we have:*
(L1) $\mathsf{R}_i^{u_o} \cap \mathsf{R}_i^{u^o} \neq \emptyset$ *if and only if* $\mathsf{R}_i^{u_o} = \mathsf{R}_i^{u^o}$*;*
(L2) $\mathsf{S}_i^{u_o} \cap \mathsf{S}_i^{u^o}$ *has at most two elements, hence if* $u^o \in \mathsf{S}_i^{u_o}$ *then* $\mathsf{S}_i^{u^o} \cap \mathsf{S}_i^{u_o} = \{u^o, u_o\}$*;*
(L3) $\frac{\mathrm{d}\mathcal{S}_i^{u_o}}{\mathrm{d}\rho}(0^+) = (-1)^{i+1}\infty$ *and* $\frac{\mathrm{d}\mathcal{R}_i^{u_o}}{\mathrm{d}\rho}(0^+) = (-1)^{i+1}\infty$*;*
(L4) $\mathsf{R}_1^{u_o}$, $\mathsf{S}_1^{u_o}$, $\mathsf{FL}_1^{u_o}$ *and* $\mathsf{BL}_1^{u_o}$ *are strictly concave, while* $\mathsf{R}_2^{u_o}$, $\mathsf{S}_2^{u_o}$, $\mathsf{FL}_2^{u_o}$ *and* $\mathsf{BL}_2^{u_o}$ *are strictly convex;*
(L5) *we have, see Figure 2,*

$$\begin{cases} \mathcal{S}_2^{u_o}(\rho) = \mathcal{FL}_2^{u_o}(\rho) < \mathcal{R}_2^{u_o}(\rho) = \mathcal{BL}_2^{u_o}(\rho) < \mathcal{R}_1^{u_o}(\rho) = \mathcal{FL}_1^{u_o}(\rho) < \mathcal{S}_1^{u_o}(\rho) = \mathcal{BL}_1^{u_o}(\rho) & \text{if } \rho < \rho_o, \\ \mathcal{S}_1^{u_o}(\rho) = \mathcal{FL}_1^{u_o}(\rho) < \mathcal{R}_1^{u_o}(\rho) = \mathcal{BL}_1^{u_o}(\rho) < \mathcal{R}_2^{u_o}(\rho) = \mathcal{FL}_2^{u_o}(\rho) < \mathcal{S}_2^{u_o}(\rho) = \mathcal{BL}_2^{u_o}(\rho) & \text{if } \rho > \rho_o; \end{cases}$$

(L6) $\mathcal{FL}_i^{u_o}$ *and* $\mathcal{BL}_i^{u_o}$ *are* $\mathbf{C^2}$ *functions, moreover*

$$\frac{\mathrm{d}\mathcal{FL}_i^{u_o}}{\mathrm{d}\rho}(\rho_o) = \lambda_i(u_o) = \frac{\mathrm{d}\mathcal{BL}_i^{u_o}}{\mathrm{d}\rho}(\rho_o), \qquad \frac{\mathrm{d}^2\mathcal{FL}_i^{u_o}}{\mathrm{d}\rho^2}(\rho_o) = (-1)^i\,\frac{a}{\rho_o} = \frac{\mathrm{d}^2\mathcal{BL}_i^{u_o}}{\mathrm{d}\rho^2}(\rho_o).$$

Recall that $v = q/\rho$. A state (ρ,q) is called *subsonic* if $|v| < a$ and *supersonic* if $|v| > a$. The lines $q = \pm a\,\rho$ are called *sonic lines*. The following lemma shows that along a Lax curve of the first (second) family the velocity v is a decreasing (respectively, increasing) function of the density ρ.

Lemma 2.4. *If either $u_1, u_2 \in \mathsf{FL}_i^{u_o}$ or $u_1, u_2 \in \mathsf{BL}_i^{u_o}$ are distinct, then $(-1)^i\,(\rho_1-\rho_2)\,(v_1-v_2) > 0$.*

Proof. By (2.6) we have

$$(-1)^i\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\mathcal{R}_i^{u_o}(\rho)}{\rho}\right) = \frac{a}{\rho} > 0, \qquad (-1)^i\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\mathcal{S}_i^{u_o}(\rho)}{\rho}\right) = \frac{a}{2\rho}\left(\sqrt{\frac{\rho}{\rho_o}} + \sqrt{\frac{\rho_o}{\rho}}\right) > 0.$$

Hence, the functions $\mathcal{FL}_1^{u_o}(\rho)/\rho$, $\mathcal{BL}_1^{u_o}(\rho)/\rho$ are decreasing while $\mathcal{FL}_2^{u_o}(\rho)/\rho$, $\mathcal{BL}_2^{u_o}(\rho)/\rho$ are increasing. To conclude we notice that $v = \mathcal{FL}_i^{u_o}(\rho)/\rho$ if $u \in \mathsf{FL}_i^{u_o}$ while $v = \mathcal{BL}_i^{u_o}(\rho)/\rho$ if $u \in \mathsf{BL}_i^{u_o}$. □

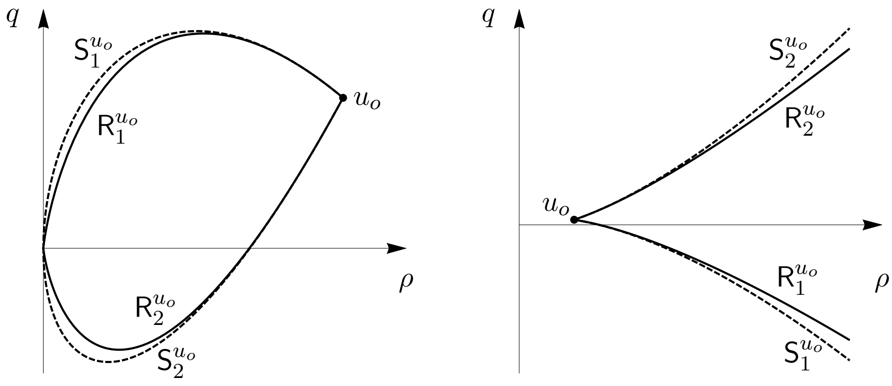

Figure 2: Mutual locations of the curves $\mathsf{R}_i^{u_o}$ (continuous line) and $\mathsf{S}_i^{u_o}$ (dashed line), $i = 1, 2$, for $\rho \leq \rho_o$ (left) and $\rho \geq \rho_o$ (right).

The following definitions are heavily used in the following; we refer to Figure 3.

Definition 2.5. *For $u_\ell, u_r \in \Omega$ we define:*

- $\bar{u}(u_\ell)$ *is the element of* $\mathsf{FL}_1^{u_\ell}$ *with the maximum q-coordinate;*
- $\underline{u}(u_r)$ *is the element of* $\mathsf{BL}_2^{u_r}$ *with the minimum q-coordinate;*
- $\tilde{u}(u_\ell, u_r)$ *is the (unique) element of* $\mathsf{FL}_1^{u_\ell} \cap \mathsf{BL}_2^{u_r}$*;*
- $\hat{u}(q_o, u_\ell)$*, for any $q_o \leq \bar{q}(u_\ell)$, is the intersection of* $\mathsf{FL}_1^{u_\ell}$ *and $q = q_o$ with the largest ρ-coordinate;*
- $\check{u}(q_o, u_r)$*, for any $q_o \geq \underline{q}(u_r)$, is the intersection of* $\mathsf{BL}_2^{u_r}$ *and $q = q_o$ with the largest ρ-coordinate.*

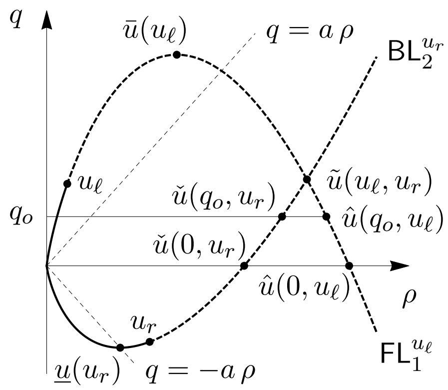

Figure 3: Notation. The dashed straight lines are the sonic lines.

We now show some simple properties of the above defined quantities; we denote $\big(\bar{\rho}(u_\ell), \bar{q}(u_\ell)\big) \doteq \bar{u}(u_\ell)$ and so on. First, notice that for any $u_\ell, u_r \in \Omega$ we have

$$\bar{q}(u_\ell) > 0 \qquad \text{and} \qquad \underline{q}(u_r) < 0. \tag{2.7}$$

Lemma 2.6. *We denote $v_\ell \doteq q_\ell/\rho_\ell$ and $v_r \doteq q_r/\rho_r$. Then we have:*

$$\rho_\ell > \bar{\rho}(u_\ell) \iff v_\ell < a \iff v_\ell < \bar{v}(u_\ell) \implies \bar{v}(u_\ell) = a, \tag{2.8}$$

$$\rho_\ell < \bar{\rho}(u_\ell) \iff v_\ell > a \iff v_\ell > \bar{v}(u_\ell) \iff \bar{v}(u_\ell) > a. \tag{2.9}$$

Analogously we have

$$\rho_r > \underline{\rho}(u_r) \iff v_r > -a \iff v_r > \underline{v}(u_r) \implies \underline{v}(u_r) = -a, \tag{2.10}$$

$$\rho_r < \underline{\rho}(u_r) \iff v_r < -a \iff v_r < \underline{v}(u_r) \iff \underline{v}(u_r) > -a. \tag{2.11}$$

Expressions (2.8) *and* (2.10) *hold true by replacing strict inequalities with equalities. At last,*

$$v_\ell \leq a \implies \bar{\rho}(u_\ell) = \frac{\rho_\ell}{e} \exp\left(\frac{v_\ell}{a}\right), \qquad \bar{v}(u_\ell) = a, \qquad \bar{q}(u_\ell) = a\,\bar{\rho}(u_\ell), \tag{2.12}$$

$$v_r \geq -a \qquad \Longrightarrow \qquad \underline{\rho}(u_r) = \frac{\rho_r}{e}\exp\left(-\frac{v_r}{a}\right), \qquad \underline{v}(u_r) = -a, \qquad \underline{q}(u_r) = -a\,\underline{\rho}(u_r). \tag{2.13}$$

The proof is deferred to Section 7.1. The proof of the following lemma is analogous to that of Lemma 2.6 (see also [12, (16)]) and is omitted.

Lemma 2.7. *Let $u_\ell, u_r \in \Omega$. Then we have:*

$$\hat{\rho}(0,u_\ell) = \begin{cases} \dfrac{\rho_\ell}{4a^2}\left[\sqrt{v_\ell^2+4a^2}+v_\ell\right]^2 & \text{if } v_\ell > 0,\\ \rho_\ell\,\exp\big(v_\ell/a\big) & \text{if } v_\ell \leq 0, \end{cases} \qquad \check{\rho}(0,u_r) = \begin{cases} \rho_r\,\exp\big(-v_r/a\big) & \text{if } v_r > 0,\\ \dfrac{\rho_r}{4a^2}\left[\sqrt{v_r^2+4a^2}-v_r\right]^2 & \text{if } v_r \leq 0. \end{cases} \tag{2.14}$$

We observe that $\hat{v}(0,u) = 0$ and therefore by (2.12) and (2.14) we deduce easily the following result.

Lemma 2.8. *We have*

$$\bar{q}\big(\hat{u}(0,u)\big) = \frac{a}{e}\,\hat{\rho}(0,u) = \begin{cases} \dfrac{a\,\rho}{e}\,\exp\left(\dfrac{v}{a}\right) & \text{if } v \leq 0,\\ \dfrac{\rho}{4\,a\,e}\left[\sqrt{v^2+4\,a^2}+v\right]^2 & \text{if } v > 0. \end{cases} \tag{2.15}$$

Moreover, the function $u \mapsto \bar{q}\big(\hat{u}(0,u)\big)$ is of class $\mathbf{C^1}$ in Ω.

In the following we use the notation

$$\mathring{q}(u) \doteq \frac{\rho}{4\,a\,e}\left[\sqrt{v^2+4\,a^2}+v\right]^2. \tag{2.16}$$

In the next lemma we investigate when $\hat{u}\big(q_o,\hat{u}(q_o,u_\ell)\big) = \hat{u}(q_o,u_\ell)$ (or the similar issue when $\check{u}$ and $\underline{q}$ replace $\hat{u}$ and $\bar{q}$, respectively); this result is used in the following sections to study the coherence. Of course, by Definition 2.5 we need $q_o \leq \bar{q}(u_\ell)$ and $q_o \leq \bar{q}\big(\hat{u}(q_o,u_\ell)\big)$ in order that both states are well defined.

Lemma 2.9. *Consider $u_\ell \in \Omega$ and suppose $q_o \leq \min\Big\{\bar{q}(u_\ell), \bar{q}\big(\hat{u}(q_o,u_\ell)\big)\Big\}$. Then $\hat{u}\big(q_o,\hat{u}(q_o,u_\ell)\big) = \hat{u}(q_o,u_\ell)$ if either $q_o \leq 0$ or one of the following equivalent conditions hold:*

$$\hat{\rho}(q_o,u_\ell) \geq \bar{\rho}\big(\hat{u}(q_o,u_\ell)\big), \quad \hat{v}(q_o,u_\ell) \leq a, \quad \hat{v}(q_o,u_\ell) \leq \bar{v}\big(\hat{u}(q_o,u_\ell)\big), \quad \bar{v}\big(\hat{u}(q_o,u_\ell)\big) = a. \tag{2.17}$$

Analogously, consider $u_r \in \Omega$ and $q_o \geq \min\Big\{\underline{q}(u_r), \underline{q}\big(\check{u}(q_o,u_r)\big)\Big\}$. Then $\check{u}\big(q_o,\check{u}(q_o,u_r)\big) = \check{u}(q_o,u_r)$ if either $q_o \geq 0$ or one of the equivalent conditions hold:

$$\check{\rho}(q_o,u_r) \geq \underline{\rho}\big(\check{u}(q_o,u_r)\big), \quad \check{v}(q_o,u_r) \geq -a, \quad \check{v}(q_o,u_r) \geq \underline{v}\big(\check{u}(q_o,u_r)\big), \quad \underline{v}\big(\check{u}(q_o,u_r)\big) = -a. \tag{2.18}$$

Proof. We only prove the first statement. Since $\hat{q}(q_o,u_\ell) = q_o$, then $\hat{u}(q_o,u_\ell) = \hat{u}\big(q_o,\hat{u}(q_o,u_\ell)\big)$ is equivalent to require that $\hat{u}(q_o,u_\ell)$ belongs to the decreasing branch of $\mathsf{FL}_1^{\hat{u}(q_o,u_\ell)}$. In turn, the latter statement is equivalent to $(2.17)_1$ and Lemma 2.6 with $\hat{u}(q_o,u_\ell)$ replacing u_ℓ shows that the conditions in (2.17) are equivalent. At last, $\hat{q}(q_o,u_\ell) = q_o \leq 0$ implies $\hat{v}(q_o,u_\ell) \leq 0 < a$; hence $(2.17)_2$ is trivially satisfied. □

We refer to Figure 4 for two cases where either $\hat{u}\big(q_o,\hat{u}(q_o,u_\ell)\big) \neq \hat{u}(q_o,u_\ell)$ or $\check{u}\big(q_o,\check{u}(q_o,u_r)\big) \neq \check{u}(q_o,u_r)$. Notice that $\hat{u}(q_o,u_\ell)$ lies on the increasing part of the curve $\mathsf{FL}_1^{\hat{u}(q_o,u_\ell)}$, see the proof of Lemma 2.9.

3 The mathematical modeling of the flow through a valve

In this section we recall the modeling of a gas flow through a valve located at $x = 0$; see [12] for more details. The valve can be either *inactive* or *active*. In the former case, system (2.1) fully describes the flow in the whole of $\mathbb{R}$: no additional condition is imposed and the flow takes place exactly as the valve is missing. In particular, the valve is understood as "fully open". In the latter case, the valve acts as an exterior force on the flow and then the conservation of momentum may be lost; such an action is modeled by considering two coupled boundary-value problems in $(0,\infty)\times(-\infty,0)$ and $(0,\infty)\times(0,\infty)$. On the contrary, conservation of the mass still occurs. As a consequence, along $x = 0$ only the first Rankine-Hugoniot condition (2.4) is imposed.

The following definitions provide a general framework to this modeling.

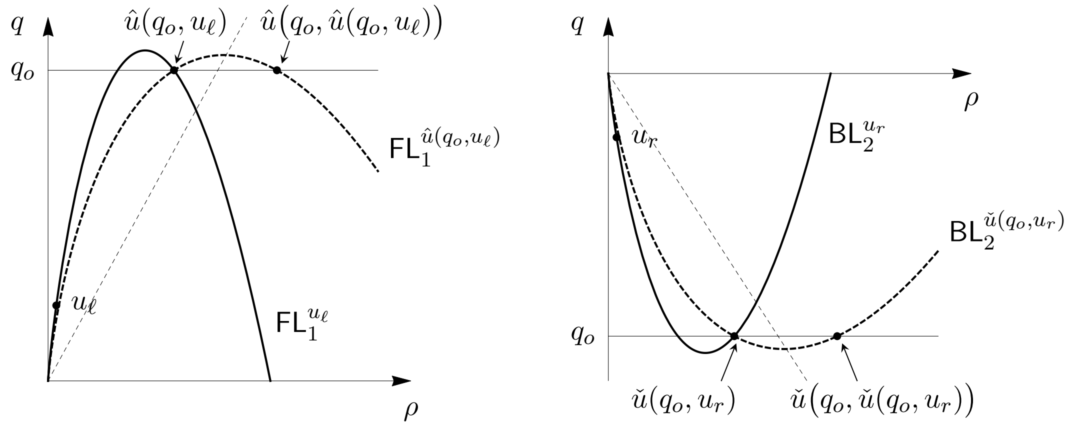

Figure 4: A case where $\hat{u}(q_o, u_\ell) \neq \hat{u}\big(q_o, \hat{u}(q_o, u_\ell)\big)$ (left) and $\check{u}(q_o, u_r) \neq \check{u}\big(q_o, \hat{u}(q_o, u_r)\big)$ (right). The dashed straight lines are the sonic lines.

Definition 3.1. *A function $u \in \mathbf{C^0}\big([0,\infty); \mathbf{BV}(\mathbb{R};\Omega)\big)$ is a* coupling solution *of Riemann problem* (2.1), (2.2) *if the restrictions of u to $[0,\infty) \times (-\infty, 0]$ and $[0,\infty) \times [0,\infty)$ are weak solutions (in the sense of [2, 7]) to the initial-boundary value problems for* (2.1) *with initial data*

$$u(0,x) = u_\ell \qquad \text{if } x < 0, \qquad\qquad u(0,x) = u_r \qquad \text{if } x > 0, \tag{3.1}$$

respectively, and coupling boundary conditions at the interface $x = 0$

$$q(t, 0^-) = q(t, 0^+) \qquad \text{for a.e. } t > 0. \tag{3.2}$$

This definition extends the notion of weak solution provided in Definition 2.1. In particular, it takes into account the possible presence of *stationary* discontinuities of u at $x = 0$, which satisfy the first Rankine-Hugoniot condition (2.4) with $\dot{\gamma} \equiv 0$, that is (3.2), but not necessarily the second Rankine-Hugoniot condition (2.5). Roughly speaking, these discontinuities can also be understood as under-compressive shock waves [21], because they do not necessarily satisfy the Lax conditions. At last, notice that the boundary $x = 0$ can be characteristic.

Every weak solution of (2.1), (2.2) is a coupling solution, but the converse does not necessarily hold true. This is in the same spirit of the solutions considered in [15, 16, 17].

For (2.1) we always use the Lax Riemann solver except at $x = 0$; at $x = 0$ we model the flow through the valve by a coupling Riemann solver. The extension of Definition 2.2 to this framework is the following.

Definition 3.2. *Let $\mathsf{D} \subseteq \Omega \times \Omega$ and a map $\mathcal{RS} : \mathsf{D} \to \mathbf{C^0}\big([0,\infty); \mathbf{BV}(\mathbb{R};\Omega)\big)$.*

- *We say that $\mathcal{RS}$ is a* coupling Riemann solver *for* (2.1) *if for any $(u_\ell, u_r) \in \mathsf{D}$ the function $(t,x) \mapsto \mathcal{RS}[u_\ell, u_r](t,x)$ is a coupling solution to Riemann problem* (2.1), (2.2) *in $[0,\infty) \times \mathbb{R}$.*
- *A coupling Riemann solver $\mathcal{RS}$ is* coherent *at $(u_\ell, u_r) \in \mathsf{D}$ if $u^\pm(t) \doteq \mathcal{RS}[u_\ell, u_r](t, 0^\pm)$ satisfies for a.e. $t > 0$*

$$\Big(u^-(t), u^+(t)\Big) \in \mathsf{D} \qquad \text{and} \qquad \mathcal{RS}\Big[u^-(t), u^+(t)\Big](t,x) = \begin{cases} u^-(t) & \text{if } x < 0, \\ u^+(t) & \text{if } x \geq 0. \end{cases} \tag{ch$_\text{v}$}$$

The coherence domain *$\mathsf{CH} \subseteq \mathsf{D}$ of $\mathcal{RS}$ is the set of all pairs $(u_\ell, u_r) \in \mathsf{D}$ where $\mathcal{RS}$ is coherent. The set $\mathsf{CH^C} \doteq \mathsf{D} \setminus \mathsf{CH}$ is the* incoherence domain*.*

Any (coherent) Riemann solver is a (coherent) coupling Riemann solver; the converse implications are not necessarily true. However, the solutions corresponding to a coupling Riemann solver are neither necessarily self-similar nor they are weak solutions at $x = 0$. For instance, the solution in Section 4 is self-similar while the one in Section 6 is not; none of them is a weak solution at $x = 0$.

The domain D of a coupling Riemann solver $\mathcal{RS}$ can be strictly included in $\Omega \times \Omega$. From a physical point of view, the domain D represents the Riemann data belonging to the *operating range* of the valve.

We now aim at defining a coupling Riemann solver $\mathcal{RS}_\text{v} : \mathsf{D}_\text{v} \to \mathbf{C^0}\big([0,\infty); \mathbf{BV}(\mathbb{R};\Omega)\big)$ in presence of a valve. We denote by $\mathsf{A} \subseteq \mathsf{D}_\text{v}$ the set of Riemann data for which the valve is active; it is assigned according to the valve under consideration. The set $\mathsf{D}_{\mathrm{v}} \setminus \mathsf{A}$ is the set of Riemann data for which the valve is inactive; see Section 4 for an example. For each $(u_\ell, u_r) \in \mathsf{A}$ we assign the flow $Q_{\mathsf{A}} = Q_{\mathsf{A}}(u_\ell, u_r) \in [\underline{Q}(u_r), \overline{Q}(u_\ell)]$ through the valve, where $\overline{Q}, \underline{Q} : \Omega \to \mathbb{R}$ are given by

$$
\begin{aligned}
\overline{Q}(u) &\doteq \begin{cases} \bar{q}(u) & \text{if } v \le a, \\ q & \text{if } v > a, \end{cases} \overset{(2.12)}{=} \begin{cases} \frac{a\,\rho}{e}\exp\left(\frac{v}{a}\right) & \text{if } v \le a, \\ q & \text{if } v > a, \end{cases} \\
\underline{Q}(u) &\doteq \begin{cases} \underline{q}(u) & \text{if } v \ge -a, \\ q & \text{if } v < -a, \end{cases} \overset{(2.13)}{=} \begin{cases} -\frac{a\,\rho}{e}\exp\left(-\frac{v}{a}\right) & \text{if } v \ge -a, \\ q & \text{if } v < -a. \end{cases}
\end{aligned}
\tag{3.3}
$$

We give a motivation of the choice for $\overline{Q}$ and $\underline{Q}$ in the following comment (c.4).

Lemma 3.3. *The functions $\overline{Q}$ and $\underline{Q}$ are of classe $\mathbf{C^1}$ in Ω. Moreover, for any $u \in \Omega$ we have*

$$\underline{q}(u) \le \underline{Q}(u) < 0 < \overline{Q}(u) \le \bar{q}(u). \tag{3.4}$$

The simple proof is omitted. At last, we define the coupling Riemann solver $\mathcal{RS}_{\mathrm{v}}$ as follows.

Definition 3.4. *Consider $\mathsf{A} \subseteq \mathsf{D}_{\mathrm{v}}$ and a function $Q_{\mathsf{A}} : \mathsf{A} \to \mathbb{R}$ such that $Q_{\mathsf{A}}(u_\ell, u_r) \in [\underline{Q}(u_r), \overline{Q}(u_\ell)]$ for every $(u_\ell, u_r) \in \mathsf{A}$. The corresponding coupling Riemann solver $\mathcal{RS}_{\mathrm{v}} : \mathsf{D}_{\mathrm{v}} \to \mathbf{BV}(\mathbb{R};\Omega)$ is defined as*

$$\mathcal{RS}_{\mathrm{v}}[u_\ell, u_r](\xi) \doteq \begin{cases} \mathcal{RS}_{\mathrm{p}}\Big[u_\ell, \hat{u}\big(Q_{\mathsf{A}}(u_\ell, u_r), u_\ell\big)\Big](\xi) & \textit{if } \xi < 0, \\ \mathcal{RS}_{\mathrm{p}}\Big[\check{u}\big(Q_{\mathsf{A}}(u_\ell, u_r), u_r\big), u_r\Big](\xi) & \textit{if } \xi \ge 0, \end{cases} \qquad \textit{if } (u_\ell, u_r) \in \mathsf{A}, \tag{3.5}$$

$$\mathcal{RS}_{\mathrm{v}}[u_\ell, u_r] \doteq \mathcal{RS}_{\mathrm{p}}[u_\ell, u_r], \qquad \textit{if } (u_\ell, u_r) \in \mathsf{D}_{\mathrm{v}} \setminus \mathsf{A}. \tag{3.6}$$

The behavior of a valve in correspondence to a Riemann datum $(u_\ell, u_r) \in \Omega \times \Omega$ is modeled by the flowchart in Figure 5. We denote

$$u_{\mathrm{v}} \doteq \mathcal{RS}_{\mathrm{v}}[u_\ell, u_r] \qquad \text{and} \qquad u_{\mathrm{v}}^{\pm} \doteq u_{\mathrm{v}}(0^{\pm}).$$

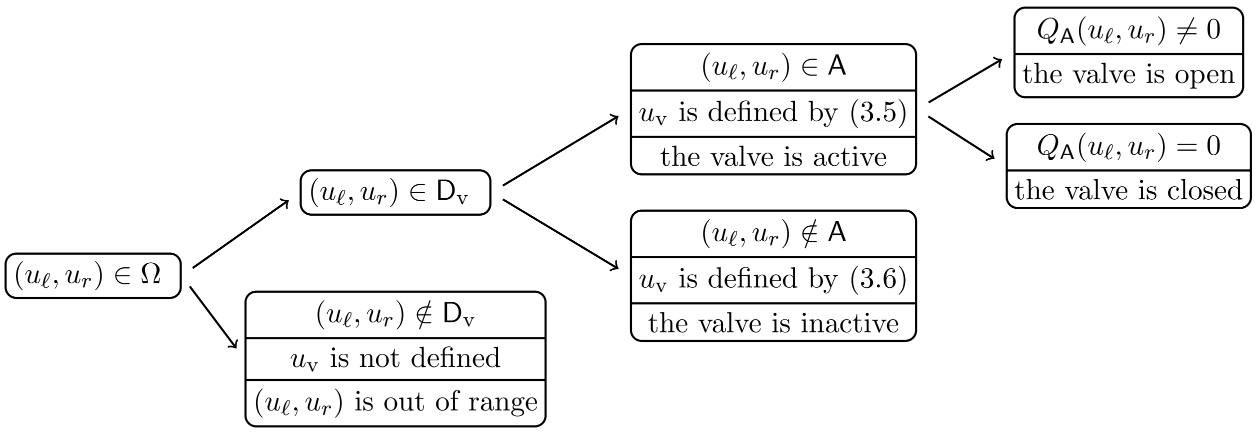

Figure 5: The flowchart corresponding to the valve considered in this paper.

Definition 3.4 deserves several comments.

(c.1) It may happen that there exists $(u_\ell, u_r) \in \mathsf{A}$ such that $u_{\mathrm{p}} \equiv u_{\mathrm{v}}$; this leads to define the sets

$$
\begin{aligned}
&\mathsf{A_N} \doteq \Big\{(u_\ell, u_r) \in \mathsf{A} : u_{\mathrm{v}} \equiv u_{\mathrm{p}}\Big\} = \Big\{(u_\ell, u_r) \in \mathsf{A} : \hat{u}\big(Q_{\mathsf{A}}(u_\ell, u_r), u_\ell\big) = \tilde{u}(u_\ell, u_r) = \check{u}\big(Q_{\mathsf{A}}(u_\ell, u_r), u_r\big)\Big\}, \\
&\mathsf{A_I} \doteq \mathsf{A} \setminus \mathsf{A_N}.
\end{aligned}
$$

They are the Riemann data for which the valve is active and either does not influence or influences the flow, respectively. Obviously $\{(u_\ell, u_r) \in \mathsf{D}_{\mathrm{v}} : u_{\mathrm{v}} \equiv u_{\mathrm{p}}\} = (\mathsf{D}_{\mathrm{v}} \setminus \mathsf{A}) \cup \mathsf{A_N}$.

(c.2) Consider $(u_\ell, u_r) \in \mathsf{A}$ and notice that $0 \in [\underline{Q}(u_\ell), \overline{Q}(u_r)]$ for every $(u_\ell, u_r) \in \Omega$, by Lemma 3.3. We say that at (u_ℓ, u_r) the valve is *closed* if $Q_{\mathsf{A}}(u_\ell, u_r) = 0$ and *open* if $Q_{\mathsf{A}}(u_\ell, u_r) \neq 0$. Clearly $Q_{\mathsf{A}}(u_\ell, u_r) = 0$ implies that the flow through the valve is null, i.e., $q_{\mathrm{v}}^{\pm} = 0$. However, notice that $q_{\mathrm{v}}^{\pm} = 0$ implies neither that the valve is closed nor, more generally, that it is active: for instance, we have $q_{\mathrm{v}}^{\pm} = 0$ if the valve is inactive and $\tilde{q}(u_\ell, u_r) = 0$.

(c.3) By Definition 2.5 and (3.4) we deduce

$$\overline{Q}(u_\ell) \geq q_\ell, \qquad \underline{Q}(u_r) \leq q_r \qquad \text{and} \qquad \underline{q}(u_r) \leq \underline{Q}(u_r) < 0 < \overline{Q}(u_\ell) \leq \bar{q}(u_\ell). \tag{3.7}$$

So, if $Q_{\mathsf{A}} = Q_{\mathsf{A}}(u_\ell, u_r) \in [\underline{Q}(u_r), \overline{Q}(u_\ell)]$, then $\hat{u}(Q_{\mathsf{A}}, u_\ell)$ and $\check{u}(Q_{\mathsf{A}}, u_r)$ are well defined. We use the notation

$$\hat{u}_{\mathsf{A}} = \hat{u}_{\mathsf{A}}(u_\ell, u_r) \doteq \hat{u}\big(Q_{\mathsf{A}}(u_\ell, u_r), u_\ell\big), \qquad \check{u}_{\mathsf{A}} = \check{u}_{\mathsf{A}}(u_\ell, u_r) \doteq \check{u}\big(Q_{\mathsf{A}}(u_\ell, u_r), u_r\big). \tag{3.8}$$

By Definition 2.5 it follows $\hat{\rho}_{\mathsf{A}} \geq \bar{\rho}(u_\ell)$, $\check{\rho}_{\mathsf{A}} \geq \underline{\rho}(u_r)$ and $\hat{q}_{\mathsf{A}} = Q_{\mathsf{A}} = \check{q}_{\mathsf{A}}$.

(c.4) By Definition 2.5 in order that $\hat{u}_{\mathsf{A}} = \hat{u}_{\mathsf{A}}(u_\ell, u_r)$ and $\check{u}_{\mathsf{A}} = \check{u}_{\mathsf{A}}(u_\ell, u_r)$ are well defined only the condition $Q_{\mathsf{A}}(u_\ell, u_r) \in [\underline{q}(u_r), \bar{q}(u_\ell)]$ is needed. It is easy to check that the more restrictive condition $Q_{\mathsf{A}}(u_\ell, u_r) \in [\underline{Q}(u_r), \overline{Q}(u_\ell)]$ required in Definition 3.4 is needed in order that $\xi \mapsto \mathcal{RS}_{\mathrm{p}}[u_\ell, \hat{u}_{\mathsf{A}}](\xi) \in \mathsf{FL}_1^{u_\ell}$ and $\xi \mapsto \mathcal{RS}_{\mathrm{p}}[\check{u}_{\mathsf{A}}, u_r](\xi) \in \mathsf{FL}_2^{\check{u}_{\mathsf{A}}}$ represent single waves with negative (≤ 0) and positive (≥ 0) speed, respectively.

Proposition 3.5. *The function* $(t, x) \mapsto u_{\mathrm{v}}(t, x) = \mathcal{RS}_{\mathrm{v}}[u_\ell, u_r](x/t)$ *is a self-similar coupling solution to Riemann problem* (2.1), (2.2) *for any* $(u_\ell, u_r) \in \mathsf{D}_{\mathrm{v}}$.

Proof. By definition, the restrictions of u_{v} to $[0, \infty) \times (-\infty, 0]$ and $[0, \infty) \times (0, \infty]$ coincide with $(t, x) \mapsto \mathcal{RS}_{\mathrm{p}}[u_\ell, \hat{u}_{\mathsf{A}}](x/t)$ and $(t, x) \mapsto \mathcal{RS}_{\mathrm{p}}[\check{u}_{\mathsf{A}}, u_r](x/t)$, respectively; they clearly are weak solutions of the initial-value problems (2.1), (3.1) in their respective domains of definition. It only remains to check (3.2).

If $(u_\ell, u_r) \in \mathsf{D}_{\mathrm{v}} \setminus \mathsf{A}$, then $u_{\mathrm{v}} = u_{\mathrm{p}}$, which is a weak solution and therefore also a coupling solution. If $(u_\ell, u_r) \in \mathsf{A}$ we have

$$\mathcal{RS}_{\mathrm{p}}[u_\ell, \hat{u}_{\mathsf{A}}](0^-) = \begin{cases} u_\ell (\neq \hat{u}_{\mathsf{A}}) & \text{if } Q_{\mathsf{A}} = q_\ell \text{ and } v_\ell > a, \\ \hat{u}_{\mathsf{A}} & \text{otherwise,} \end{cases} \qquad \mathcal{RS}_{\mathrm{p}}[u_\ell, \hat{u}_{\mathsf{A}}](0^+) = \hat{u}_{\mathsf{A}}, \tag{3.9}$$

$$\mathcal{RS}_{\mathrm{p}}[\check{u}_{\mathsf{A}}, u_r](0^-) = \check{u}_{\mathsf{A}}, \qquad \mathcal{RS}_{\mathrm{p}}[\check{u}_{\mathsf{A}}, u_r](0^+) = \begin{cases} u_r (\neq \check{u}_{\mathsf{A}}) & \text{if } Q_{\mathsf{A}} = q_r \text{ and } v_r < -a, \\ \check{u}_{\mathsf{A}} & \text{otherwise.} \end{cases} \tag{3.10}$$

Then $u_{\mathrm{v}}^- = \mathcal{RS}_{\mathrm{p}}[u_\ell, \hat{u}_{\mathsf{A}}](0^-)$ and $u_{\mathrm{v}}^+ = \mathcal{RS}_{\mathrm{p}}[\check{u}_{\mathsf{A}}, u_r](0^+)$ satisfy (3.2) because in any case we have $q_{\mathrm{v}}^- = Q_{\mathsf{A}} = q_{\mathrm{v}}^+$. □

4 A one-way valve

If two pipes are connected by a *one-way* valve, the flow at $x = 0$ occurs in a single direction only, say positive; in this case we consider coupling Riemann solvers of the form (3.5) with $Q_{\mathsf{A}} \geq 0$. The following lemma shows some general properties.

Lemma 4.1. *For any* $q_o \geq 0$ *and* $u_r \in \Omega$ *we have*

(i) $\check{u}(q_o, u_r)$ *is well defined;*

(ii) $q_o > \underline{Q}(u_r)$*;*

(iii) $\mathcal{RS}_{\mathrm{p}}[\check{u}(q_o, u_r), u_r](0) = \check{u}(q_o, u_r)$*;*

(iv) $\check{u}\big(q_o, \check{u}(q_o, u_r)\big) = \check{u}(q_o, u_r)$.

Proof. *(i)* By (2.7) we have $\underline{q}(u_r) < 0 \le q_o$ and therefore $\check{u}(q_o, u_r)$ is well defined by Definition 2.5. *(ii)* By (3.7) we have $\underline{Q}(u_r) < 0$ and therefore $q_o \ge 0 > \underline{Q}(u_r)$. *(iii)* Since $q_o \ge 0$, then $q_o = q_r$ implies $v_r \ge 0 > -a$; hence, by (3.10) we deduce

$$\mathcal{RS}_{\rm p}[\check{u}(q_o,u_r),u_r](0^+) = \check{u}(q_o,u_r) = \mathcal{RS}_{\rm p}[\check{u}(q_o,u_r),u_r](0^-).$$

(iv) This follows by Lemma 2.9. □

We now introduce our specific one-way valve. We fix $q_* > 0$; the valve keeps the flow at $x = 0$ equal to q_* if possible, otherwise it closes. By Lemma 4.1, (ii), we have $q_* \in [\underline{Q}(u_r), \overline{Q}(u_\ell)]$ if and only if $q_* \le \overline{Q}(u_\ell)$; this motivates the way Q_{A} is defined in (4.1) below.

Definition 4.2. *The coupling Riemann solver $\mathcal{RS}_{\rm v}$ corresponding to the above valve is given as in Definition 3.4 with $\mathsf{D}_{\rm v} \doteq \Omega \times \Omega \doteq \mathsf{A}$ and*

$$Q_{\mathsf{A}}(u_\ell) \doteq \begin{cases} q_* & \text{if } \overline{Q}(u_\ell) \ge q_*, \\ 0 & \text{if } \overline{Q}(u_\ell) < q_*. \end{cases} \tag{4.1}$$

The function Q_{A} is discontinuous along some curve in Ω; we explicitly find such a curve in the following Lemma 4.3. The valve corresponding to (4.1) is one-way because $Q_{\mathsf{A}} \ge 0$; moreover $\mathsf{D}_{\mathsf{v}} = \Omega \times \Omega = \mathsf{A}$ and $\mathsf{D}_{\mathsf{v}} \setminus \mathsf{A} = \emptyset$, that is, *the valve is always active.* We have

$$\begin{aligned} \mathsf{A}_{\mathsf{I}}^{\mathsf{C}} = \mathsf{A}_{\mathsf{N}} = \quad & \Big\{(u_\ell,u_r) \in \Omega\times\Omega : q_* \le \overline{Q}(u_\ell),\ \hat{u}(q_*,u_\ell) = \tilde{u}(u_\ell,u_r) = \check{u}(q_*,u_r)\Big\} \\ & \bigcup \Big\{(u_\ell,u_r) \in \Omega\times\Omega : q_* > \overline{Q}(u_\ell),\ \hat{u}(0,u_\ell) = \tilde{u}(u_\ell,u_r) = \check{u}(0,u_r)\Big\}. \end{aligned}$$

Since $Q_{\mathsf{A}} = Q_{\mathsf{A}}(u_\ell)$, the flow at $x = 0$ only depends on the upstream state u_ℓ. We stress that this holds for the valve under consideration; indeed, the flow across a general one-way valve may depend on both upstream and downstream states.

We denote

$$u_*^a \doteq (\rho_*^a, q_*) \doteq (q_*/a, q_*), \qquad\qquad u_*^0 \doteq (\rho_*^0, 0) \doteq (e\, q_*/a, 0), \tag{4.2}$$

see Figure 6. Notice that u_*^a is the intersection of the line $\{u \in \Omega : q = q_*\}$ with the sonic line $\{u \in \Omega : v = a\}$. Moreover, u_*^0 is the unique intersection of the curve $\mathsf{BL}_1^{u_*^a}$ with the line $\{u \in \Omega : q = 0\}$. In the region $\rho \ge \rho_*^a$ we have $\mathsf{BL}_1^{u_*^a} = \mathsf{R}_1^{u_*^a}$, while $\mathsf{FL}_1^{u_*^0} = \mathsf{R}_1^{u_*^0}$ in $\rho \le \rho_*^0$; hence $\mathsf{BL}_1^{u_*^a}$ and $\mathsf{FL}_1^{u_*^0}$ coincide in $\rho \in [\rho_*^a, \rho_*^0]$ by Proposition 2.3, (L1). As a consequence, u_*^0 is the unique state on $\{u \in \Omega : q = 0\}$ such that $\bar{u}(u_*^0) = u_*^a$; by (2.12) we deduce

$$\bar{u}(u_*^0) = u_*^a = \bar{u}(u_*^a). \tag{4.3}$$

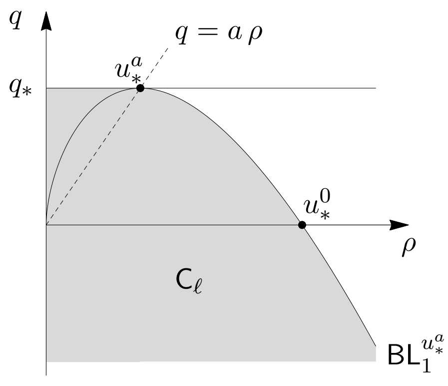

Figure 6: The shaded region represents the set C_ℓ of left states u_ℓ such that the valve corresponding to (4.1) is closed. The curve $\mathsf{BL}_1^{u_*^a}$ is a rarefaction if $\rho \ge \rho_*^a$.

The following lemma characterizes the states for which the valve is closed, that is, $Q_{\mathsf{A}} = 0$; its proof is a direct consequence of (3.3), (2.12) and is omitted.

Lemma 4.3. *The valve is closed if and only if one of the following equivalent conditions is satisfied:*

(i) $Q_{\mathsf{A}}(u_\ell) = 0$;

(ii) either $v_\ell > a$ *and* $q_\ell < q_*$, *or* $v_\ell \le a$ *and* $\bar{q}(u_\ell) = a\,\rho_\ell\, e^{v_\ell/a-1} < q_*$;

(iii) u_ℓ *belongs to the set*

$$\begin{aligned}\mathsf{C}_\ell &\doteq \Big\{u \in \Omega : v > a,\ q < q_*\Big\} \cup \Big\{u \in \Omega : v \le a,\ \rho\, e^{v/a} < \rho_*^0\Big\}\\ &= \Big((0, \rho_*^a] \times (-\infty, q_*)\Big) \cup \Big\{u \in \Omega : \rho > \rho_*^a,\ q < \mathcal{R}_1^{u_*^a}(\rho)\Big\}.\end{aligned}$$

We refer to Figure 6 for a graphical representation of the set C_ℓ. Since the function $Q_{\mathsf{A}} = Q_{\mathsf{A}}(u_\ell)$ has only two values, then it is discontinuous along the boundary of the set C_ℓ.

We now introduce the states $u_*^{\rm sup}$ and $u_*^{\rm sub}$, see Figure 7 on the left. Notice that $\mathsf{BL}_1^{u_*^0} = \mathsf{S}_1^{u_*^0}$ if $q \ge 0$.

Lemma 4.4. *The curve* $\mathsf{BL}_1^{u_*^0}$ *intersects the line* $q = q_*$ *in two points*

$$u_*^{\rm sup} \doteq (\rho_*^{\rm sup}, q_*) \qquad \text{and} \qquad u_*^{\rm sub} \doteq (\rho_*^{\rm sub}, q_*).$$

The state $u_*^{\rm sup}$ *is supersonic, the state* $u_*^{\rm sup}$ *is subsonic.*

Proof. By $(2.6)_1$, the equation $\mathcal{BL}_1^{u_*^0}(\rho) = q_*$ can be written as $\sqrt{w} - \frac{1}{\sqrt{w}} = \frac{w}{e}$, where $w \doteq \frac{e\,q_*}{a\,\rho}$. Since $w \ge 0$, we look for solutions satisfying $\sqrt{w} \ge \frac{1}{\sqrt{w}}$, i.e., $w \ge 1$. We multiply the equation by $\sqrt{w}\,e$ and then square to find $e^2\,(w-1)^2 = w^3$. The roots of this equation in $[1, \infty)$ are $w_1 \approx 2.20$ and $w_2 \approx 4.43$. Since $\rho = \frac{e\,q_*}{a\,w}$, $v = a\big(\sqrt{w} - 1/\sqrt{w}\big) = \frac{a\,w}{e}$, $w_1/e \approx 0.81$ and $w_2/e \approx 1.63$, the two intersection points satisfy

$$v_*^{\rm sup} \doteq \frac{a\,w_2}{e} > a > v_*^{\rm sub} \doteq \frac{a\,w_1}{e}, \qquad \rho_*^{\rm sup} \doteq \frac{e\,q_*}{a\,w_2} < \frac{q_*}{a} = \rho_*^a < \rho_*^{\rm sub} \doteq \frac{e\,q_*}{a\,w_1}.$$

This concludes the proof of the lemma. □

Lemma 4.4 implies that the inequality $\mathcal{BL}_1^{u_*^0}(\rho) > q_*$ holds if and only if $\rho \in (\rho_*^{\rm sup}, \rho_*^{\rm sub})$.

Now, we state the main result of this paper, where we explicitly characterize the incoherence domain $\mathsf{CH}^\complement$ of $\mathcal{RS}_{\rm v}$ and, as a consequence, the coherence domain CH. Since Q_{A} only depends on the upstream states, it is clear that

$$\mathsf{CH} = \mathsf{CH}_\ell \times \Omega, \qquad \mathsf{CH}^\complement = \mathsf{CH}_\ell^\complement \times \Omega,$$

where both $\mathsf{CH}_\ell \subseteq \Omega$ and $\mathsf{CH}_\ell^\complement \doteq \Omega \setminus \mathsf{CH}_\ell$ only contain left states u_ℓ.

Theorem 4.5 (Incoherence)**.** *The incoherence domain of* $\mathcal{RS}_{\rm v}$ *is* $\mathsf{CH}^\complement = \mathsf{CH}_\ell^\complement \times \Omega$, *where*

$$\mathsf{CH}_\ell^\complement = \Big\{u \in \Omega : v > v_*^{\rm sup},\ \mathcal{BL}_1^{u_*^0}(\rho) \le q < q_*\Big\} = \Big\{u \in \Omega : v > v_*^{\rm sup},\ q < q_* \le \bar{q}\big(\hat{u}(0, u)\big)\Big\}. \tag{4.4}$$

The proof is deferred to Section 7.2; we refer to Figure 7 on the left for a pictorial representation of $\mathsf{CH}_\ell^\complement$. Notice that the first writing of $\mathsf{CH}_\ell^\complement$ in (4.4) focuses on the range of values of q; conversely the second one shows the admissible values for q_*, which is useful in the next section.

An explicit expression of $\mathsf{CH}_\ell^\complement$ is obtained by noticing that since $q_*^0 = 0$ then $\mathsf{BL}_1^{u_*^0} = \mathsf{S}_1^{u_*^0}$ in the quadrant $q \ge 0$; hence by (2.6), (4.2) and (2.15) we deduce

$$\begin{aligned}\mathsf{CH}_\ell^\complement &= \left\{u \in \Omega : v > v_*^{\rm sup},\ a\,\rho\left(\sqrt{\frac{e\,q_*}{a\,\rho}} - \sqrt{\frac{a\,\rho}{e\,q_*}}\right) \le q < q_*\right\}\\ &= \big\{u \in \Omega : v > v_*^{\rm sup},\ q < q_* \le \mathring{q}(u)\big\},\end{aligned} \tag{4.5}$$

where $\mathring{q}$ is defined in (2.16).

By (4.5) we deduce $\mathsf{CH}_\ell = \mathsf{CH}_{\ell,1} \cup \mathsf{CH}_{\ell,2} \cup \mathsf{CH}_{\ell,3}$ where, see Figure 7 on the right,

$$\mathsf{CH}_{\ell,1} = \big\{u \in \Omega\,:\, v \le v_*^{\rm sup}\big\}, \tag{4.6}$$

$$\mathsf{CH}_{\ell,2} = \Big\{u \in \Omega\,:\, v > v_*^{\rm sup},\ q < \mathcal{S}_1^{u_*^0}(\rho)\Big\} = \big\{u \in \Omega\,:\, v > v_*^{\rm sup},\ \mathring{q}(u) < q_*\big\}, \tag{4.7}$$

$$\mathsf{CH}_{\ell,3} = \big\{u \in \Omega\,:\, v > v_*^{\rm sup},\ q \ge q_*\big\}. \tag{4.8}$$

We now show that if $\mathcal{RS}_{\rm v}$ is not coherent at (u_ℓ, u_r) then the valve is closed.

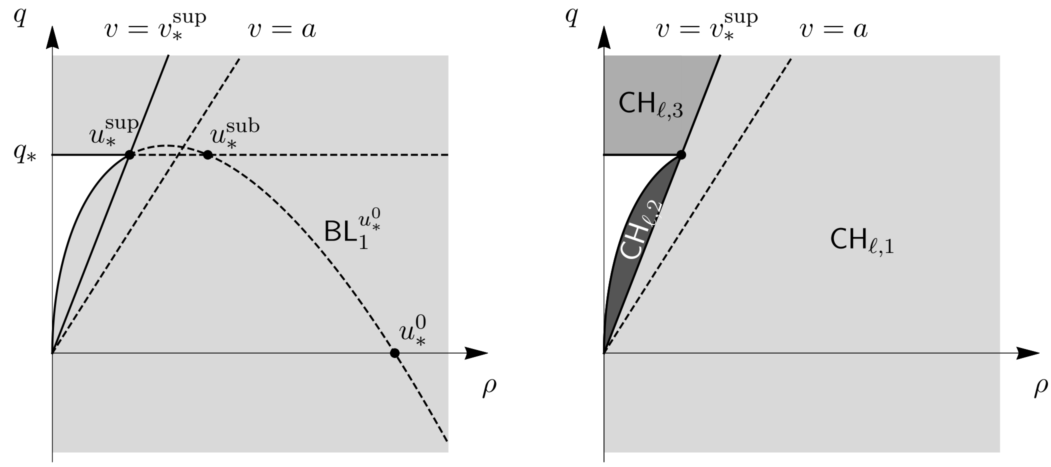

Figure 7: Left: the shaded region represents the coherence domain CH_ℓ, the white region the incoherence domain $\mathsf{CH}_\ell^\complement$. Right: the decomposition of CH_ℓ into the subsets (4.6)–(4.8).

Corollary 4.6. *We have* $\mathsf{CH}_\ell^\complement \subset \mathsf{C}_\ell$ *and* $\mathcal{RS}_{\rm v}[\mathsf{CH}_\ell^\complement, \Omega](0^-) \subseteq \Omega \setminus \mathsf{C}_\ell \subset \mathsf{CH}_\ell$.

Proof. By (4.4) and Lemma 4.3, *(iii)*, we have $\mathsf{CH}_\ell^\complement \subset \{u \in \Omega : v > a,\ q < q_*\} \subset \mathsf{C}_\ell$. The proof of the latter statement follows from the former one and the definition of coherence. Indeed, if $(u_\ell, u_r) \in \mathsf{CH}_\ell^\complement \times \Omega$, then by the former statement we have $u_\ell \in \mathsf{C}_\ell$, i.e., $Q_{\mathsf{A}}(u_\ell) = 0$, $q_{\rm v}^\pm = 0$ and the valve is closed. Now, if by contradiction $u_{\rm v}^- = \mathcal{RS}_{\rm v}[u_\ell, u_r](0^-) \in \mathsf{C}_\ell$, then $Q_{\mathsf{A}}(u_{\rm v}^-) = 0$ and hence $\hat{u}(0, u_{\rm v}^-) = u_{\rm v}^-$, $\check{u}(0, u_{\rm v}^+) = u_{\rm v}^+$, a contradiction because this would mean $u_\ell \in \mathsf{CH}_\ell$. □

In other words, Corollary 4.6 means that if $(u_\ell, u_r) \in \mathsf{CH}_\ell^\complement$, then in the solution $u_{\rm v} = \mathcal{RS}_{\rm v}[u_\ell, u_r]$ the valve is closed, while in the solution $\mathcal{RS}_{\rm v}[u_{\rm v}^-, u_{\rm v}^+]$ the valve is open and then $(u_{\rm v}^-, u_{\rm v}^+) \in \mathsf{CH}_\ell$; as a consequence by (ch$_{\rm v}$) we have $\mathcal{RS}_{\rm v}[u_{\rm v}^-, u_{\rm v}^+](0^\pm) = u_{\rm v}^\pm$.

We now address to invariant domains $\mathcal{I} \subseteq \Omega$ for the coupling Riemann solver $\mathcal{RS}_{\rm v}$; this means that $\mathcal{I} \times \mathcal{I} \subseteq \mathsf{D}_{\rm v}$ and $\mathcal{RS}_{\rm v}[\mathcal{I}, \mathcal{I}](\mathbb{R}) \subseteq \mathcal{I}$. The characterization of such domains is a very hard task; we just show a family of invariant domains that are *minimal* with respect to inclusion and constituted of coherent states; moreover, in correspondence of their states, the valve can be both open or closed.

In the construction below we need the following simple lemma.

Lemma 4.7. *We have the following results.*

(i) For every $u_1 = (\rho_1, q_*)$ *with* $\rho_1 \geq \rho_*^a$ *there exists a unique* $u_0 = (\rho_0, 0)$ *such that* $u_1 = \hat{u}(q_*, u_0)$.

(ii) Conversely, for every $u_0 = (\rho_0, 0)$ *there exists a unique* $u_1 = (\rho_1, q_*)$ *such that* $u_0 = \check{u}(0, u_1)$.

Proof. About *(i)*, since $q_* > 0$, if $u_0 = (\rho_0, 0)$ exists then $\rho_0 \geq \rho_1$. Hence u_1 should belong to $\mathsf{R}_1^{u_0}$ and, by (2.6), ρ_0 is determined by

$$q_* = \mathcal{R}_1^{u_0}(\rho_1) = -a\,\rho_1 \log\left(\frac{\rho_1}{\rho_0}\right) \qquad \Longleftrightarrow \qquad \rho_0 = \rho_1 \exp\left(\frac{q_*}{a\,\rho_1}\right) = \rho_1 \exp\left(\frac{\rho_*^a}{\rho_1}\right).$$

The above expression for ρ_0 and the hypothesis $\rho_1 \geq \rho_*^a$ imply $\frac{\mathrm{d}\mathcal{R}_1^{u_0}}{\mathrm{d}\rho}(\rho_1) = -a\left(1 - \frac{\rho_*^a}{\rho_1}\right) \leq 0$. The proof of *(ii)* is analogous; in this case the curve $\mathsf{BL}_2^{u_1}$ always intersects the line $q = 0$ because $q_* > 0 > \underline{q}(u_1)$. □

Remark that in both cases of Lemma 4.7 the states u_1 and u_0 lie on a rarefaction curve (of the first family in case *(i)*, of the second family in case *(ii)*). We exploit this remark in defining the curvilinear triangles in (4.11), (4.12).

We can begin the construction of our domain. We begin with $u_2 \doteq u_*^a$. By Lemma 4.7 we can implicitly define the states $u_3, u_4, \ldots$ by recursion as

$$u_2 = u_*^a, \qquad u_{2i} = \hat{u}(q_*, u_{2i+1}), \qquad u_{2i-1} = \check{u}(0, u_{2i}), \tag{4.9}$$

see Figure 8. We notice that $(4.9)_3$ with $i = 1$ makes sense and defines u_1; moreover, observe that $u_3 = u_*^0$.

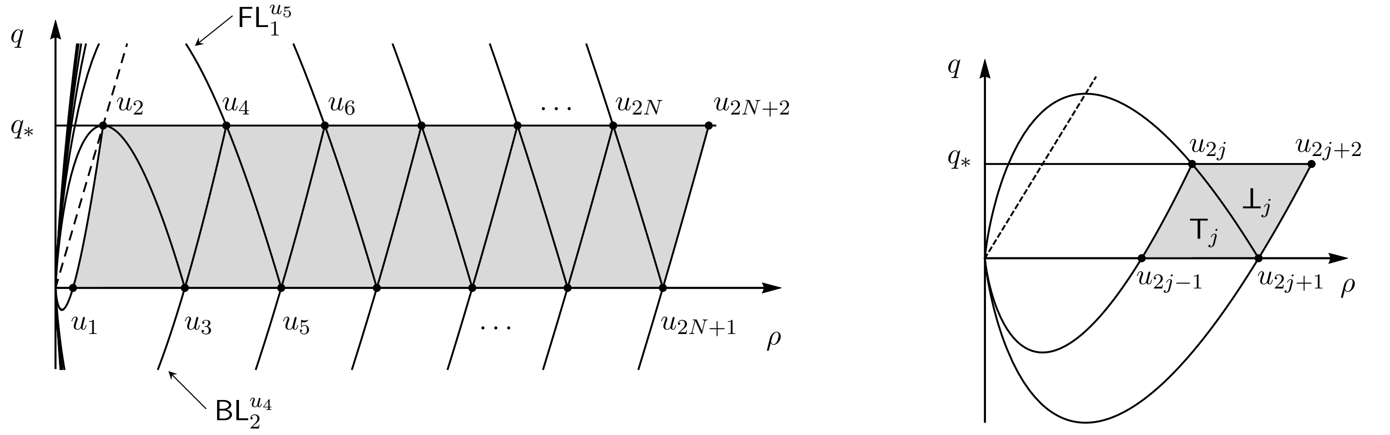

Figure 8: Left: the invariant domain I_N defined by (4.10). Right: the curvilinear triangles. The dashed line is the sonic line. The dashed line is the sonic line.

For any fixed $N > 1$ we define the set, see Figure 8 on the left,

$$\mathsf{I}_N \doteq \big\{u \in \Omega : q \in [0, q_*],\ \check{\rho}(0, u) \in [\rho_1, \rho_{2N+1}]\big\} = \bigcup_{j=1}^{N} \big(\perp_j \cup \mathsf{T}_j\big), \tag{4.10}$$

where T_j and $\perp_j$ are the curvilinear triangles, see Figure 8 on the right,

$$\mathsf{T}_j \doteq \Big\{u \in \Omega : \rho_{2j-1} \le \rho \le \rho_{2j+1},\ 0 \le q \le \min\Big\{\mathcal{R}_2^{u_{2j}}(\rho), \mathcal{R}_1^{u_{2j+1}}(\rho)\Big\}\Big\}, \tag{4.11}$$

$$\perp_j \doteq \Big\{u \in \Omega : \rho_{2j} \le \rho \le \rho_{2j+2},\ \max\Big\{\mathcal{R}_1^{u_{2j+1}}(\rho), \mathcal{R}_2^{u_{2j+2}}(\rho)\Big\} \le q \le q_*\Big\}. \tag{4.12}$$

Proposition 4.8. *For every $N > 1$ the set I_N is the minimal invariant domain containing $\{u_1, u_{2N+2}\}$. Furthermore, $\mathsf{I}_N \subset \mathsf{CH}_\ell$ and $\emptyset \neq \mathsf{I}_N \cap \mathsf{C}_\ell \subset \mathsf{T}_1$, $\mathsf{I}_N \cap \big(\Omega \setminus \mathsf{C}_\ell\big) \neq \emptyset$. At last, any state in I_N is subsonic, with the exception of the single sonic state $u_2 \doteq u_*^a$.*

The proof is deferred to Section 7.3. We notice that in real-world gas flows through pipes, the motion is almost never supersonic [20]. Also observe that I_N contains states with zero speed, a possibility that certainly occurs in real flows.

5 Flow ranges for uniform coherence

In Section 4 we *fixed* a flow q_* at the valve and characterized the coherent upstream states. We also investigated the upstream states that led to the closure of the valve and invariant domains. In this section we still focus on coherence but understand q_* as a parameter; for a fixed set Ω_ℓ of upstream states we want to determine the range of values of q_* that make Ω_ℓ coherent. We explicitly express the dependence on q_* by writing $\mathcal{RS}_{\rm v}^{q_*}$ in place of $\mathcal{RS}_{\rm v}$ and so on.

For any $u_\ell \in \Omega$, we define

$$\mathsf{Q}_*(u_\ell) \doteq \big\{q_* \in [0, \infty) : u_\ell \in \mathsf{CH}_\ell^{q_*}\big\}.$$

By definition, for any $u_\ell, u_r \in \Omega$ the set $\mathsf{Q}_*(u_\ell)$ is the set of flow values q_* that make the initial datum (u_ℓ, u_r) coherent. By (4.6), (4.7) and (4.8) we have

$$\mathsf{Q}_*(u_\ell) = \begin{cases} [0, \infty) & \text{if } u_\ell \in \mathsf{CH}_{\ell,1}, \\ \Big[0, q_\ell\Big] \bigcup \big(\mathring{q}(u_\ell), \infty\big) \neq [0, \infty) & \text{if } u_\ell \in \mathsf{CH}_{\ell,1}^{\mathsf{c}}, \end{cases} \tag{5.1}$$

where $\mathring{q}$ is defined in (2.16) and $\mathsf{CH}_{\ell,1}^{\mathsf{c}} \doteq \Omega \setminus \mathsf{CH}_{\ell,1}$. We emphasize that if $u_\ell \in \mathsf{CH}_{\ell,1}^{\mathsf{c}}$ then $q_\ell < \mathring{q}(u_\ell)$; as a consequence, there is a gap between the intervals $[0, q_\ell]$ and $(\mathring{q}(u_\ell), \infty)$. Therefore $\mathsf{Q}_*(u_\ell) \neq [0, \infty)$ if $u_\ell \in \mathsf{CH}_{\ell,1}^{\mathsf{c}}$.

Proposition 5.1. *We have*

$$\bigcap_{q_* \geq 0} \mathsf{CH}_\ell^{q_*} = \mathsf{CH}_{\ell,1}, \tag{5.2}$$

$$\bigcap_{u_\ell \in \mathsf{CH}_{\ell,1}^{\complement}} \mathsf{Q}_*(u_\ell) = \bigcap_{u_\ell \in \Omega} \mathsf{Q}_*(u_\ell) = \{0\}. \tag{5.3}$$

Proof. First, we prove (5.2). Let $u_\ell \in \mathsf{CH}_{\ell,1}$ and fix $q_* \geq 0$; by (5.1) we have $q_* \in [0,\infty) = \mathsf{Q}_*(u_\ell)$, hence $u_\ell \in \mathsf{CH}_\ell^{q_*}$. Conversely, if $u_\ell \in \bigcap_{q_* \geq 0} \mathsf{CH}_\ell^{q_*}$, then $q_* \in \mathsf{Q}_*(u_\ell)$ for all $q_* \geq 0$; as a consequence $\mathsf{Q}_*(u_\ell) \supseteq [0,\infty)$, which implies by (5.1) that $u_\ell \in \mathsf{CH}_{\ell,1}$.

Second, we prove (5.3). Obviously $\bigcap_{u_\ell \in \mathsf{CH}_{\ell,1}^{\complement}} \mathsf{Q}_*(u_\ell) \supseteq \bigcap_{u_\ell \in \Omega} \mathsf{Q}_*(u_\ell) \supseteq \{0\}$. We are left to show that for any $q_* > 0$ there exists $u_\ell \in \mathsf{CH}_{\ell,1}^{\complement}$ such that $q_* \notin \mathsf{Q}_*(u_\ell)$; by (5.1) this is equivalent to find u_ℓ such that $v_\ell > v_*^{\sup}$ and $q_\ell < q_* \leq \mathring{q}(u_\ell)$. To this aim it is sufficient to take $v_\ell = 2\,a$ and $\rho_\ell = \frac{e\,q_*}{(1+\sqrt{2})\,a}$, because then $v_\ell > v_*^{\sup}$ and $2\,a\,\rho_\ell < q_* = \mathring{q}(u_\ell)$. □

Formula (5.2) means that if the upstream states range in $\mathsf{CH}_{\ell,1}$, then we have coherence for any $q_* \geq 0$ and *conversely.* Notice that since $v_*^{\sup} > a$, if Ω_ℓ lies below the sonic line $q = a\,\rho$, then clearly $\Omega_\ell \subseteq \mathsf{CH}_{\ell,1}$. This remark applies to the invariant domain $\mathsf{I}_N^{\bar{q}_*}$ defined by (4.10) with $q_* = \bar{q}_*$; notice that $\mathsf{I}_N^{\bar{q}_*}$ is not an invariant domain of $\mathcal{RS}_{\rm v}^{q_*}$ if $q_* \neq \bar{q}_*$.

Formula (5.3) can be rephrased as follows: if the upstream states range *in the whole* of Ω (indeed, $\mathsf{CH}_{\ell,1}^{\complement}$ suffices), then we have coherence if and only if $q_* = 0$, i.e., the valve is always closed.

A more significative sufficient condition for the coherence of the states in a set Ω_ℓ is contained below. Clearly, if $\Omega_\ell \subseteq \mathsf{CH}_{\ell,1}$ then all states in Ω_ℓ are coherent for every $q_* \geq 0$. If Ω_ℓ is such that $\Omega_\ell \setminus \mathsf{CH}_{\ell,1} \neq \emptyset$ we introduce

$$Q_1(\Omega_\ell) \doteq \inf\big\{q : u \in \Omega_\ell \setminus \mathsf{CH}_{\ell,1}\big\}, \qquad Q_2(\Omega_\ell) \doteq \sup\big\{\mathring{q}(u) : u \in \Omega_\ell \setminus \mathsf{CH}_{\ell,1}\big\}.$$

We notice that it can happen either $Q_1(\Omega_\ell) = 0$ or $Q_2(\Omega_\ell) = \infty$. The following proposition gives a characterization of the intersection of the sets $\mathsf{Q}_*(u)$ as u varies in Ω_ℓ. Its importance lies in the fact that if q_* belongs to such an intersection then $\Omega_\ell \subseteq \mathsf{CH}_\ell^{q_*}$.

Proposition 5.2. *For any $\Omega_\ell \subset \Omega$ we have*

$$\bigcap_{u \in \Omega_\ell} \mathsf{Q}_*(u) = \begin{cases} [0,\infty) & \text{if } \Omega_\ell \subseteq \mathsf{CH}_{\ell,1}, \\ \big[0, Q_1(\Omega_\ell)\big] \cup \big[Q_2(\Omega_\ell),\infty\big) & \text{if } \Omega_\ell \setminus \mathsf{CH}_{\ell,1} \neq \emptyset. \end{cases}$$

If $\Omega_\ell \setminus \mathsf{CH}_{\ell,1} \neq \emptyset$ then $Q_1(u_\ell) < Q_2(u_\ell)$.

Proof. The case $\Omega_\ell \subseteq \mathsf{CH}_{\ell,1}$ follows from (5.1). If $\Omega_\ell \setminus \mathsf{CH}_{\ell,1} \neq \emptyset$, then it is sufficient to observe that $Q_2(\Omega_\ell) > Q_1(\Omega_\ell) \geq 0$ are well defined because both $u \mapsto q$ and $u \mapsto \mathring{q}(u)$ are continuous. □

6 A one-way valve with a reaction time

We consider in this section a one-way valve as in Section 4 but that, more realistically, needs a reaction time $\tau > 0$ to update its configuration; the resulting coupling Riemann solver $\mathcal{RS}_{\rm v}^{\tau}$, which turns out to be *non-selfsimilar*, is deduced from $\mathcal{RS}_{\rm v}$ by taking this delay into account. Our aim is to show by an explicit example how chattering arises rather than provide a detailed mathematical analysis of this valve with delay. Moreover, since the study of the wave interactions would be too heavy by using *exact* solutions, we deal with *approximate* solutions issued by a wave-front tracking algorithm. Our construction is not intended to be general, even if the overall picture of the solution is rather stable; the development of a general numerical method is beyond the purposes of the present paper and will be addressed in a future work. As a byproduct, the example shows that a numerical time-stepping scheme based on $\mathcal{RS}_{\rm v}$ to solve (2.1) with the ideal valve considered in Section 4 may not produce the expected solution.

Roughly speaking, here is our example. We consider an upstream flow with high (supersonic) velocity; however the gas is so rarefied that the flow is too low and then the valve cannot open, see [26] for real-world experiments where precisely this situation is taken into account. The downstream flow is zero. The interaction of the incoming gas with the closed valve produces a shock wave propagating backward; the state behind the shock is now stationary but the density has increased to a level that the valve opens. A complicated "periodic" interaction pattern arises until, after a certain time, the valve closes again and the whole process is then repeated.

More precisely, at time $t = 0$ the gas along the sections $(-\infty, 0)$ and $(0, \infty)$ of the pipe is described by the constant states u_ℓ and u_r, respectively. We assume

$$(u_\ell, u_r) \in \mathsf{CH}^{\mathsf{C}} \doteq \mathsf{CH}^{\mathsf{C}}_\ell \times \Omega, \qquad \text{with} \qquad v_\ell > a,\ q_\ell < q_*,\ v_r = 0, \tag{6.1}$$

see Figure 9, where CH^{C} is the incoherence domain given in Theorem 4.5. Since $\mathsf{CH}^{\mathsf{C}}_\ell \subset \mathsf{C}_\ell$ by Corollary 4.6, we have $Q_{\mathsf{A}}(u_\ell) = 0$.

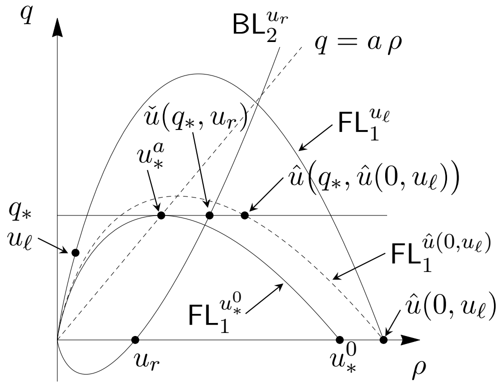

Figure 9: Notation used in Section 6.

The valve keeps closed at least for $t \in (0, \tau]$ because of $Q_{\mathsf{A}}(u_\ell) = 0$, and the gas motion is described by (3.5) with $Q_{\mathsf{A}}(u_\ell) = 0$. Moreover, we have $\check{u}(0, u_r) = u_r$ and then for $t \in (0, \tau]$ we deduce

$$u(t, x) \doteq \mathcal{RS}_{\rm v}[u_\ell, u_r](x/t) = \begin{cases} \mathcal{RS}_{\rm p}\big[u_\ell, \hat{u}(0, u_\ell)\big](x/t) & \text{if } x < 0, \\ u_r & \text{if } x \geq 0. \end{cases} \tag{6.2}$$

Clearly $\mathcal{RS}_{\rm p}\big[u_\ell, \hat{u}(0, u_\ell)\big]$ has a single 1-shock S_0^1, see Figure 10 on the left. The notation for the waves is explained in detail below.

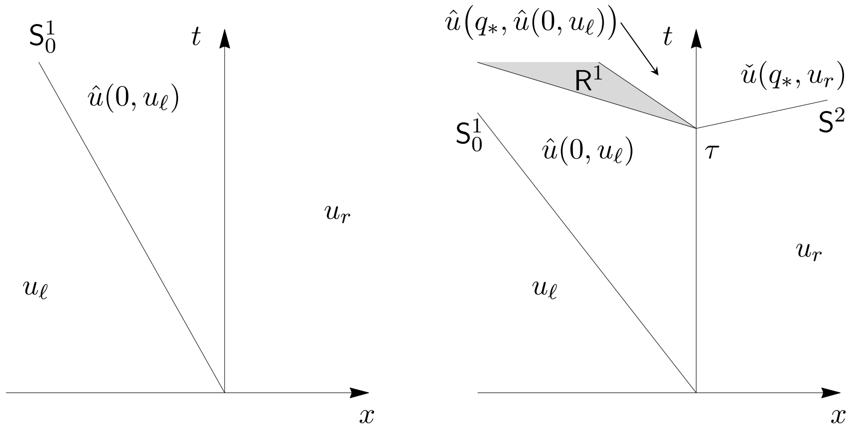

Figure 10: Left: $(t, x) \mapsto \mathcal{RS}_{\rm v}[u_\ell, u_r](x/t)$. Right: the solution u for $t \in [0, t_0^1)$.

By the assumption $(u_\ell, u_r) \in \mathsf{CH}^{\mathsf{C}}$ in (6.1) we deduce $u(0^+, 0^-) = \hat{u}(0, u_\ell) \notin \mathsf{C}_\ell$, see Figure 9 and Lemma 4.3, *(ii)*, namely $Q_{\mathsf{A}}\big(u(0^+, 0^-)\big) = q_*$. Therefore an ideal valve based on $\mathcal{RS}_{\rm v}$ given in Section 4 and with zero reaction time would be open; however, in the current case, the valve keeps closed until time

$t = \tau > 0$. It is important to emphasize that for coherent initial data there is no need to apply once more the Riemann solver after the initial time: the valve keeps open or closed at any time if it was so at time $t = 0^+$. In particular, for coherent Riemann initial data, chattering cannot arise.

At time $t = \tau$ we apply again $\mathcal{RS}_{\rm v}$ at $x = 0$ and find that, at least in a small neighborhood of $x = 0$ and for sufficiently small times $t > \tau$, we have

$$u(t,x) \doteq \mathcal{RS}_{\rm v}[\hat{u}(0,u_\ell), u_r]\left(\frac{x}{t-\tau}\right) = \begin{cases} \mathcal{RS}_{\rm p}\Big[\hat{u}(0,u_\ell), \hat{u}\big(q_*, \hat{u}(0,u_\ell)\big)\Big]\left(\frac{x}{t-\tau}\right) & \text{if } x < 0, \\ \mathcal{RS}_{\rm p}\big[\check{u}(q_*,u_r), u_r\big]\left(\frac{x}{t-\tau}\right) & \text{if } x \geq 0. \end{cases} \tag{6.3}$$

This time $\mathcal{RS}_{\rm p}\Big[\hat{u}(0,u_\ell), \hat{u}\big(q_*, \hat{u}(0,u_\ell)\big)\Big]$ is formed by a 1-rarefaction R^1 and $\mathcal{RS}_{\rm p}\big[\check{u}(q_*,u_r), u_r\big]$ by a 2-shock S^2, see Figures 9 and 10 on the right. The Riemann solver that we consider is then defined by (6.2) and (6.3), and makes sense at least in time intervals $(0,t)$ with $t - \tau > 0$ sufficiently small. By comparing (6.2) and (6.3) we see that it is non-selfsimilar.

Since the study of the interactions would be unfeasible by using exact expressions, we approximate the solutions by exploiting a wave-front tracking algorithm obtained by combining those proposed in [3] (unconstrained case) and [4] (constrained case). In particular, for a fixed $N \in \mathbb{N}$ we construct a piecewise constant approximate solution u_N obtained by replacing the large rarefactions with a fan of N waves with equal ρ-size and whose speeds of propagation equal the characteristic speeds of the states at the right. The rarefactions issued by the interactions of a shock wave with any discontinuity of these fans are considered "small" and kept as they are.

For waves we use the following notation. Here, it is not important to distinguish between families, as we did above. So, the apex k in the notation S_h^k or R_h^k accounts for the order of the waves, i.e., whether they are just generated ($k = 1$) or have undergone one ($k = 2$) or $k + 1$ interactions. The index h simply counts the waves from the left to the right, starting from 0. The notation $\mathsf{S}_h^{k,l}$ occurs when the shock wave S_h^k interacts with the l-th wave of a rarefaction fan. We denote below the states by an analogous notation.

We describe in details only the interactions in $(-\infty, 0]$; those in $[0, \infty)$ can be treated analogously but it is easy to see that all the waves in $(0, \infty)$ move with positive speed. We introduce some notation. First, we cut rarefaction R_1: we define $u_i^1 \in \mathsf{R}_1^{\hat{u}(0,u_\ell)}$ by

$$\rho_i^1 \doteq \frac{i}{N}\,\hat{\rho}\big(q_*, \hat{u}(0,u_\ell)\big) + \left(1 - \frac{i}{N}\right)\hat{\rho}(0,u_\ell), \qquad q_i^1 \doteq \mathcal{R}_1^{\hat{u}(0,u_\ell)}(\rho_i^1), \qquad i \in \{0,1,\ldots,N\}.$$

Then, see Figure 11, for $i \in \{0,1,\ldots,N\}$ and $j \in \{2,\ldots,N+1\}$ we define u_i^j as follows:

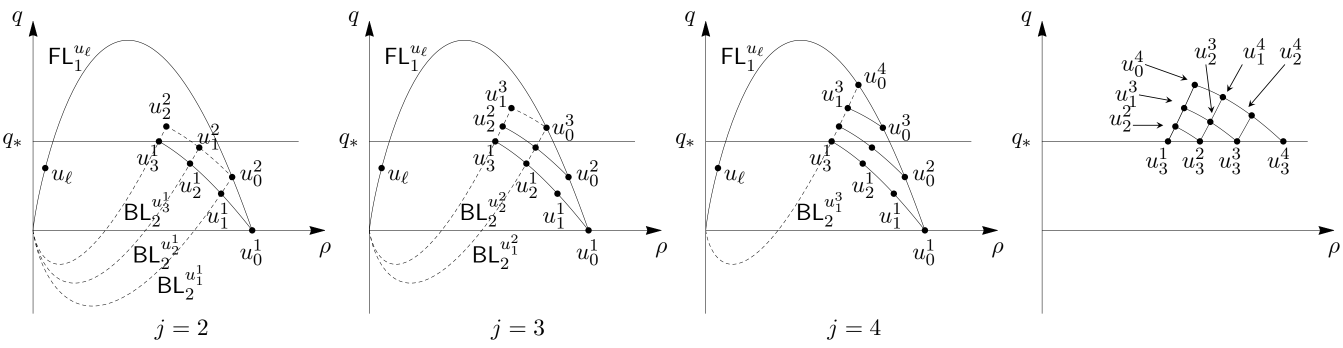

Figure 11: The states u_i^j in the case $N = 3$.

(s.1) for $j \in \{2,\ldots,N+1\}$ let $u_0^j \doteq \tilde{u}(u_\ell, u_1^{j-1})$ be the unique element of $\mathsf{FL}_1^{u_\ell} \cap \mathsf{BL}_2^{u_1^{j-1}}$;

(s.2) for $j \in \{2,\ldots,N\}$ and $i \in \{1,2,\ldots,N-1\}$ let $u_i^j \doteq \tilde{u}(u_{i-1}^j, u_{i+1}^{j-1})$ be the unique element of $\mathsf{FL}_1^{u_{i-1}^j} \cap \mathsf{BL}_2^{u_{i+1}^{j-1}}$;

(s.3) for $j \in \{2, \ldots, N+1\}$ let $u_N^j \doteq \hat{u}(q_*, u_{N-1}^j)$ be the element of $\mathsf{FL}_1^{u_{N-1}^j} \cap \{u \in \Omega : q = q_*\}$ with the largest ρ-coordinate.

More precisely, the u_i^j are defined recursively as follows. First we determine u_i^1 as above. By (s.1) we find u_0^2 from u_1^1. Recursively we define $u_i^2 \doteq \tilde{u}(u_{i-1}^2, u_{i+1}^1)$ for $i \in \{1, 2, \ldots, N-1\}$. At last we determine u_N^2 from u_{N-1}^2 by (s.3). So we defined u_i^2 for $i \in \{1, 2, \ldots, N\}$. We then iterate this procedure to determine u_i^3 for $i \in \{1, 2, \ldots, N\}$ and so on.

Notice that $u_0^1 = \hat{u}(0, u_\ell)$, $u_N^j = \hat{u}(q_*, u_{N-1}^j)$ and therefore

$$q_N^j = q_*, \quad j \in \{1, \ldots, N+1\}.$$

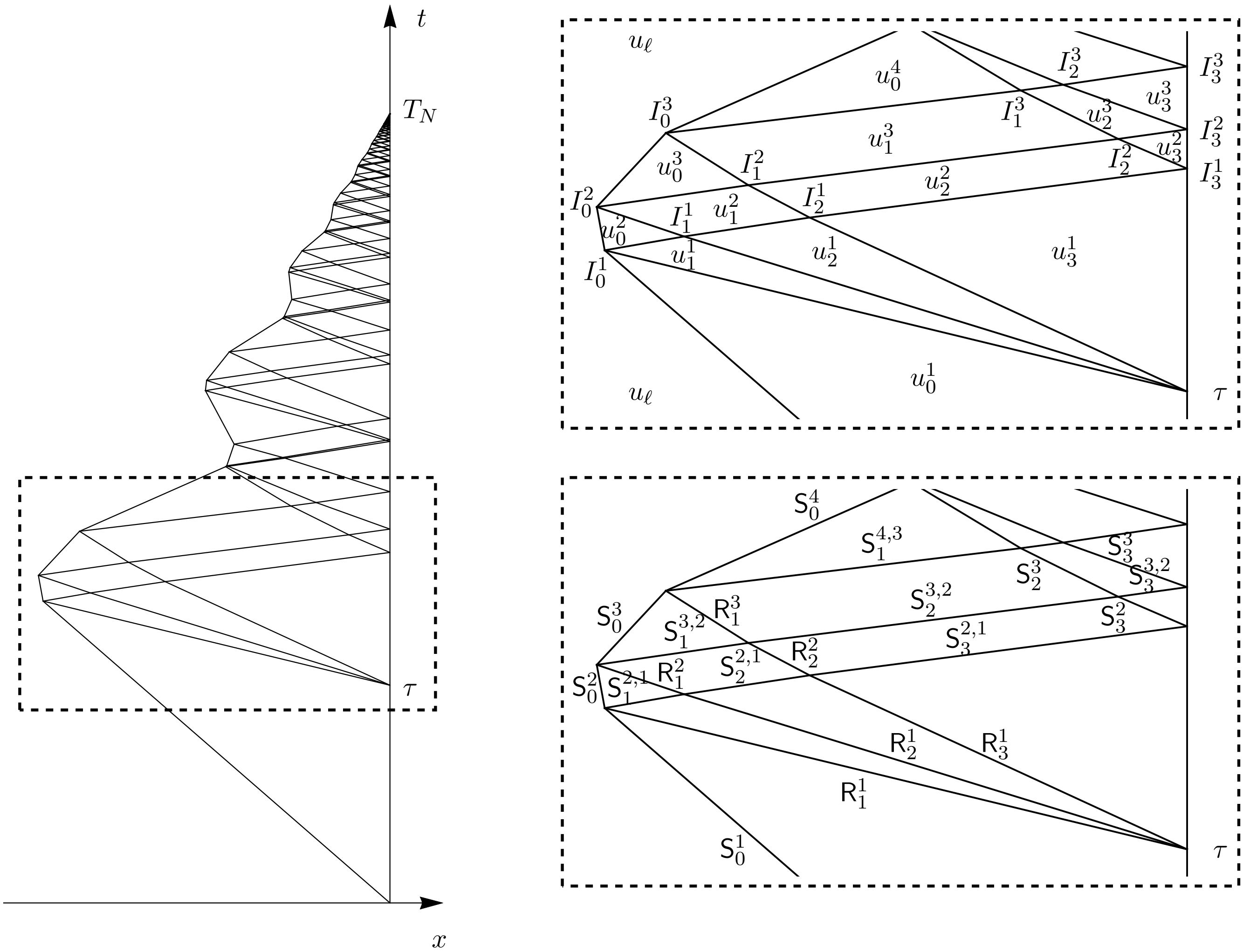

Figure 12: The approximate solution $(t, x) \mapsto u_N(t, x)$ for $t \in [0, T_N]$ and $x < 0$. Here $N = 3$.

For simplicity we focus on the case $N = 3$. Then in place of R^1 we consider the three waves $\mathsf{R}_i^1 \doteq (u_{i-1}^1, u_i^1)$ with speed of propagation $\lambda_1(u_i^1)$, $i \in \{1, 2, 3\}$. Next, we solve Riemann problems at each interaction between waves as follows, see Figure 12. We denote by I_h^k the point in the (x, t)-plane corresponding to the interaction occurring at time t_h^k.

(I.1) S_0^1 interacts with R_1^1 at time t_0^1: $\mathcal{RS}_\mathrm{p}[u_\ell, u_1^1]$ has a shock $\mathsf{S}_0^2 \doteq (u_\ell, u_0^2)$ and a shock $\mathsf{S}_1^{2,1} \doteq (u_0^2, u_1^1)$.

(I.2) $\mathsf{S}_1^{2,1}$ interacts with R_2^1 at time t_1^1: $\mathcal{RS}_\mathrm{p}[u_0^2, u_2^1]$ has a "small" rarefaction $\mathsf{R}_1^2 \doteq (u_0^2, u_1^2)$ and a shock $\mathsf{S}_2^{2,1} \doteq (u_1^2, u_2^1)$.

(I.3) $\mathsf{S}_2^{2,1}$ interacts with R_3^1 at time t_2^1: $\mathcal{RS}_{\rm p}[u_1^2, u_3^1]$ has a "small" rarefaction $\mathsf{R}_2^2 \doteq (u_1^2, u_2^2)$ and a shock $\mathsf{S}_3^{2,1} \doteq (u_2^2, u_3^1)$.

(I.4) $\mathsf{S}_3^{2,1}$ reaches $x = 0$ at time t_3^1: $\mathcal{RS}_{\rm v}[u_2^2, \check{u}(q_*, u_r)]$ coincides with $\mathcal{RS}_{\rm p}[u_2^2, u_3^2]$ in $(-\infty, 0)$ and with $\check{u}(q_*, u_r)$ in $(0, \infty)$. Therefore at time t_3^1 a backward shock $\mathsf{S}_3^2 \doteq (u_2^2, u_3^2)$ starts from $x = 0$.

(I.5) S_0^2 interacts with R_1^2 at time t_0^2: $\mathcal{RS}_{\rm p}[u_\ell, u_1^2]$ has a shock $\mathsf{S}_0^3 \doteq (u_\ell, u_0^3)$ and a shock $\mathsf{S}_1^{3,2} \doteq (u_0^3, u_1^2)$.

(I.6) $\mathsf{S}_1^{3,2}$ interacts with R_2^2 at time t_1^2: $\mathcal{RS}_{\rm p}[u_0^3, u_2^2]$ has a "small" rarefaction $\mathsf{R}_1^3 \doteq (u_0^3, u_1^3)$ and a shock $\mathsf{S}_2^{3,2} \doteq (u_1^3, u_2^2)$.

(I.7) $\mathsf{S}_2^{3,2}$ interacts with S_3^2 at time t_2^2: $\mathcal{RS}_{\rm p}[u_1^3, u_3^2]$ has a shock $\mathsf{S}_2^3 \doteq (u_1^3, u_2^3)$ and a shock $\mathsf{S}_3^{3,2} \doteq (u_2^3, u_3^2)$.

(I.8) $\mathsf{S}_3^{3,2}$ reaches $x = 0$ at time t_3^2: $\mathcal{RS}_{\rm v}[u_2^3, \check{u}(q_*, u_r)]$ coincides with $\mathcal{RS}_{\rm p}[u_2^3, u_3^3]$ in $(-\infty, 0)$ and with $\check{u}(q_*, u_r)$ in $(0, \infty)$. Therefore at time t_3^2 a backward shock $\mathsf{S}_3^3 \doteq (u_2^3, u_3^3)$ starts from $x = 0$.

(I.9) S_0^3 interacts with R_1^3 at time t_0^3: $\mathcal{RS}_{\rm p}[u_\ell, u_1^3]$ has a shock $\mathsf{S}_0^4 \doteq (u_\ell, u_0^4)$ and a shock $\mathsf{S}_1^{4,3} \doteq (u_0^4, u_1^3)$.

It should be now clear how to continue the construction of the approximate solution, see Figure 12. We observe that Figure 12 has not been obtained by a numerical scheme, but by using the exact expressions of the waves involved in the interactions for the values $a = 1$, $\tau = 1$, $q_* = 4$, $q_\ell = 14/5$, $\rho_\ell = 7/10$ and $\rho_r = 3$.

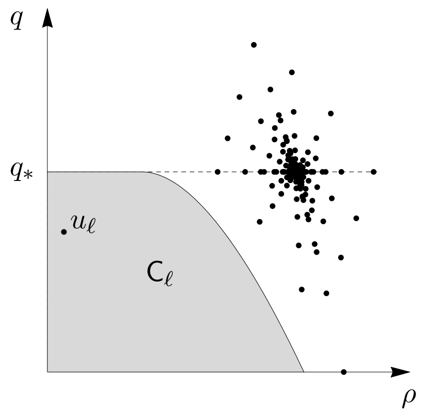

Figure 13: Values attained by the approximate solution u_N for $N = 3$ in $x < 0$.

We stress that $u_N^j \in \Omega \setminus \mathsf{C}_\ell$, see Lemma 4.3 and Figure 13; hence, the valve keeps open until time T_N when all the waves in $(-\infty, 0)$ cross $x = 0$. Immediately after time T_N we have $u_N(t, 0^-) = u_\ell$; therefore the valve closes at time $T_N + \tau$ and the above construction can be restarted, see Figure 14 on the left. During the time interval $(T_N, T_N + \tau)$ the maximal flow allowed through the valve is q_*; however $\overline{Q}(u_\ell) = q_\ell < q_*$, see Figure 14 on the right. Therefore, in a sufficiently small neighborhood of $x = 0$ and for sufficiently small times $t > T_N$, the gas is described by

$$\begin{cases} u_\ell & \text{if } x < 0, \\ \mathcal{RS}_{\rm p}\Big[\check{u}\big(q_\ell, \check{u}(q_*, u_r)\big), \check{u}(q_*, u_r)\Big]\left(\frac{x}{t - T_N}\right) & \text{if } x \geq 0. \end{cases}$$

We stress that for sufficiently small times $t > T_N$ the flow at the valve position is q_ℓ which differs from both 0 and q_*, see Figure 14 on the right.

By letting N go to infinity, the above example leads us to conjecture that there exists $T \doteq \lim_{N \to \infty} T_N > \tau > 0$ such that the valve is closed in the time interval $\big((n-1)(T + \tau), (n-1)T + n\tau\big]$, $n \in \mathbb{N}$; otherwise, it is open. This commuting between opening and closing represents the chattering of the valve.

Remark 6.1. *The above construction can be understood as a numerical time-stepping scheme, based on the wave-front tracking algorithm and with time step τ, to solve* (2.1) *with an ideal valve as in Section 4. If $\tau \to 0$, the above construction with $N = 3$ gives a numerical evidence that the approximate solution converges to $\mathcal{RS}_{\rm p}[u_\ell, u_r]$, which* differs *from $\mathcal{RS}_{\rm v}[u_\ell, u_r]$, see Figure 10. Hence, such a numerical scheme may not produce the expected solution of a Riemann problem.*

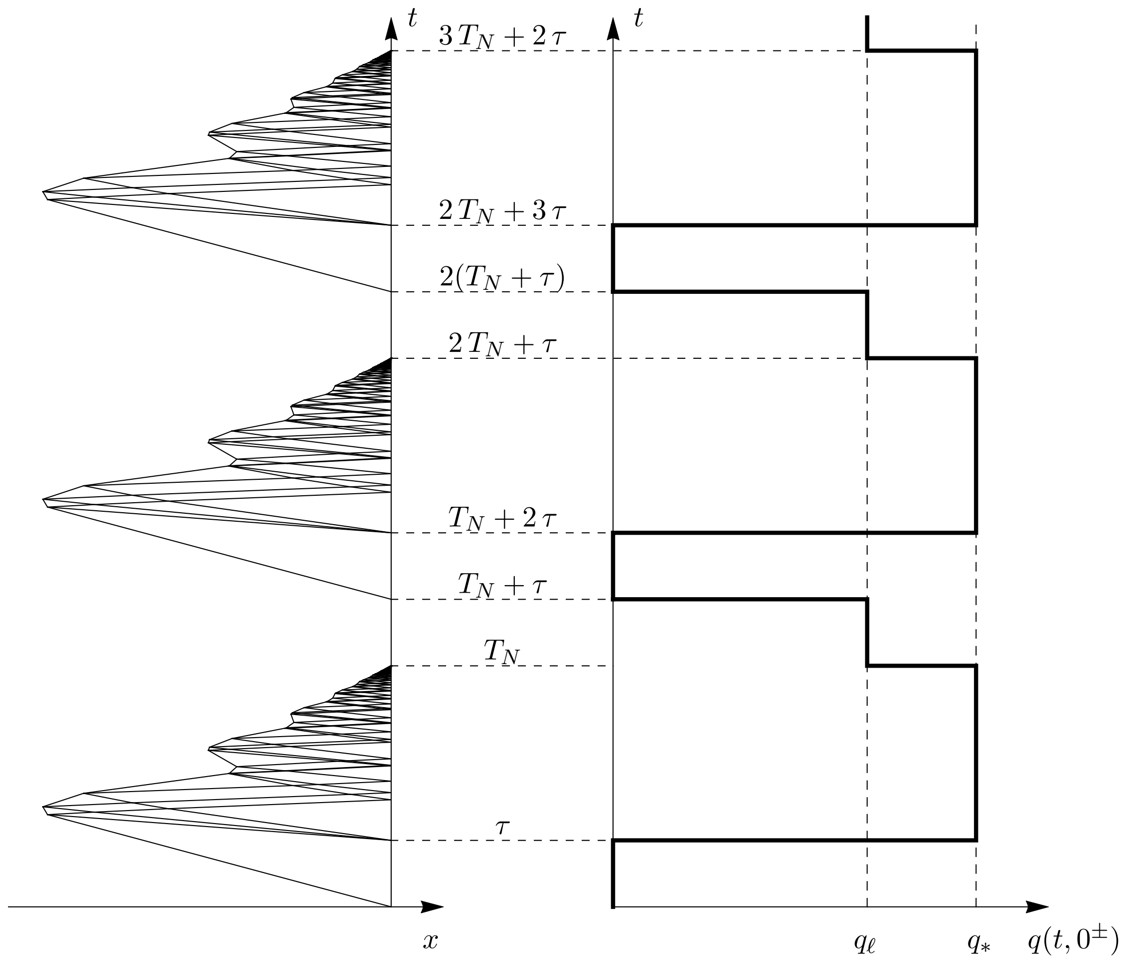

Figure 14: Left: the approximate solution $(t, x) \mapsto u_N(t, x)$ for $t \in [0, 3T_N + 2\tau]$ and $x < 0$, in the case $N = 3$. Right: the flow through the valve.

7 Technical proofs

7.1 Proof of Lemma 2.6

We focus on (2.8), (2.9) and (2.12). The function $\mathcal{FL}_1^{u_\ell}$ is strictly concave by Proposition 2.3, (L4), it attains its (unique) maximum value at $\bar\rho(u_\ell)$, and $\frac{\mathrm{d}\mathcal{FL}_1^{u_\ell}}{\mathrm{d}\rho}(\rho_\ell) = \lambda_1(u_\ell) = v_\ell - a$ by Proposition 2.3, (L6). Therefore we have

$$v_\ell \lesseqgtr a \iff \lambda_1(u_\ell) \lesseqgtr 0 \iff \rho_\ell \gtreqless \bar\rho(u_\ell) \iff v_\ell \lesseqgtr \bar v(u_\ell).$$

The last implication follows by Lemma 2.4. To complete the proof of (2.8), (2.9) it remains to show

$$(i)\ \rho_\ell > \bar\rho(u_\ell) \Longrightarrow \bar v(u_\ell) = a \qquad \text{and} \qquad (ii)\ \rho_\ell < \bar\rho(u_\ell) \iff \bar v(u_\ell) > a.$$

To this aim we use the obvious identities

$$(*)\ \frac{\mathcal{FL}_1^{u_\ell}\big(\bar\rho(u_\ell)\big)}{\bar\rho(u_\ell)} = \bar v(u_\ell) \qquad \text{and} \qquad (**)\ \frac{\mathrm{d}\mathcal{FL}_1^{u_\ell}}{\mathrm{d}\rho}\big(\bar\rho(u_\ell)\big) = 0.$$

(i) If $\rho_\ell > \bar\rho(u_\ell)$, then by (2.6) we have

$$0 \overset{(**)}{=} \frac{\mathrm{d}\mathcal{R}_1^{u_\ell}}{\mathrm{d}\rho}\big(\bar\rho(u_\ell)\big) = v_\ell - a - a\,\log\left(\frac{\bar\rho(u_\ell)}{\rho_\ell}\right) \implies \bar v(u_\ell) \overset{(*)}{=} \frac{\mathcal{R}_1^{u_\ell}\big(\bar\rho(u_\ell)\big)}{\bar\rho(u_\ell)} = v_\ell - a\,\log\left(\frac{\bar\rho(u_\ell)}{\rho_\ell}\right) = a.$$

This proves (2.8); moreover, it is clear that (2.8) still holds if we replace inequalities with equalities and then (2.12) follows from the above identities.

(*ii*) If $\rho_\ell < \bar\rho(u_\ell)$, then by (2.6) we have

$$0 \overset{(**)}{=} \frac{\mathrm{d}\mathcal{S}_1^{u_\ell}}{\mathrm{d}\rho}\big(\bar\rho(u_\ell)\big) = v_\ell - a\left(\sqrt{\frac{\bar\rho(u_\ell)}{\rho_\ell}} - \sqrt{\frac{\rho_\ell}{\bar\rho(u_\ell)}}\right) - \frac{a}{2}\left(\sqrt{\frac{\bar\rho(u_\ell)}{\rho_\ell}} + \sqrt{\frac{\rho_\ell}{\bar\rho(u_\ell)}}\right)$$

$$\implies \quad \bar v(u_\ell) \overset{(*)}{=} \frac{\mathcal{S}_1^{u_\ell}\big(\bar\rho(u_\ell)\big)}{\bar\rho(u_\ell)} = v_\ell - a\left(\sqrt{\frac{\bar\rho(u_\ell)}{\rho_\ell}} - \sqrt{\frac{\rho_\ell}{\bar\rho(u_\ell)}}\right) = \frac{a}{2}\left(\sqrt{\frac{\bar\rho(u_\ell)}{\rho_\ell}} + \sqrt{\frac{\rho_\ell}{\bar\rho(u_\ell)}}\right) > a.$$

The last estimate follows by the inequality $\sqrt{\xi} + (1/\sqrt{\xi}) > 2$, $\xi > 1$. To prove the converse it is sufficient to use (*i*) and what we remarked at the end of that issue: if $\rho_\ell < \bar\rho(u_\ell)$ fails, that is $\rho_\ell \geq \bar\rho(u_\ell)$, then we proved $\bar v(u_\ell) = a$, hence also $\bar v(u_\ell) > a$ fails.

7.2 Coherence

In this section we prove Theorem 4.5. Coherence is equivalent to condition $(\mathrm{ch_v})$. Condition $(\mathrm{ch_v})_1$ is obvious because $\mathsf{D}_\mathrm{v} = \Omega \times \Omega$. About condition $(\mathrm{ch_v})_2$, observe that $u_\mathrm{v} = \mathcal{RS}_\mathrm{v}[u_\ell, u_r]$ is always given by (3.5) and (4.1); we consider the six cases listed below, which encompass all possible situations. Notice that $u_\mathrm{v}^+ = \check u(Q_\mathsf{A}, u_r)$ by Lemma 4.1, (iii), and therefore, by *(iv)* of the same lemma, we deduce

$$\check u(Q_\mathsf{A}, u_\mathrm{v}^+) = \check u\big(Q_\mathsf{A}, \check u(Q_\mathsf{A}, u_r)\big) = \check u(Q_\mathsf{A}, u_r) = u_\mathrm{v}^+. \tag{7.1}$$

(1) Assume $v_\ell \leq a$ and $\bar q(u_\ell) \geq q_*$, see Figure 15 on the left. By (3.3) we have $\overline{Q}(u_\ell) = \bar q(u_\ell) \geq q_*$ and by (4.1) we deduce that $\mathcal{RS}_\mathrm{v}[u_\ell, u_r]$ is given by (3.5) with $Q_\mathsf{A} = q_*$. Then, we have $u_\mathrm{v}^- = \hat u(q_*, u_\ell)$ by (3.9) and $\overline{Q}(u_\mathrm{v}^-) \geq q_\mathrm{v}^- = \hat q(q_*, u_\ell) = q_*$ by $(3.7)_1$. By (4.1) we deduce that $\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-, u_\mathrm{v}^+]$ is also given by (3.8), (3.5) with $Q_\mathsf{A} = q_*$. Hence, by (7.1) and (2.3) we have

$$\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-, u_\mathrm{v}^+](\xi) = \begin{cases} \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-, \hat u(q_*, u_\mathrm{v}^-)](\xi) & \text{if } \xi < 0,\\ \mathcal{RS}_\mathrm{p}[\check u(q_*, u_\mathrm{v}^+), u_\mathrm{v}^+](\xi) & \text{if } \xi \geq 0,\end{cases}$$

$$= \begin{cases} \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-, \hat u(q_*, u_\mathrm{v}^-)](\xi) & \text{if } \xi < 0,\\ \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^+, u_\mathrm{v}^+](\xi) & \text{if } \xi \geq 0,\end{cases} = \begin{cases} \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-, \hat u(q_*, u_\mathrm{v}^-)](\xi) & \text{if } \xi < 0,\\ u_\mathrm{v}^+ & \text{if } \xi \geq 0.\end{cases}$$

Since $v_\ell \leq a$, by Lemma 2.6, the inequality $\bar\rho(u_\ell) \leq \hat\rho(q_*, u_\ell)$ and Lemma 2.4 we deduce $a = \bar v(u_\ell) \geq \hat v(q_*, u_\ell)$. Therefore, by $(2.17)_2$ we have $\hat u\big(q_*, \hat u(q_*, u_\ell)\big) = \hat u(q_*, u_\ell)$, that is, $\hat u(q_*, u_\mathrm{v}^-) = u_\mathrm{v}^-$. As a consequence, by (2.3) we obtain $\mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-, \hat u(q_*, u_\mathrm{v}^-)] \equiv u_\mathrm{v}^-$ and then $(\mathrm{ch_v})_2$ is proved.

(2) Assume $v_\ell \leq a$ and $\bar q(u_\ell) < q_*$, see Figure 15 on the right. By (3.3) we have $\overline{Q}(u_\ell) = \bar q(u_\ell) < q_*$ and by (4.1) we deduce that $\mathcal{RS}_\mathrm{v}[u_\ell, u_r]$ is given by (3.5) with $Q_\mathsf{A} = 0$. Then we have $u_\mathrm{v}^+ = \check u(0, u_r)$ by (7.1) and $u_\mathrm{v}^- = \hat u(0, u_\ell)$ by (3.9); hence $v_\mathrm{v}^- = \hat v(0, u_\ell) = 0 < a$. By (3.3) and Proposition 2.3, (L5), we have $\overline{Q}(u_\mathrm{v}^-) = \bar q(u_\mathrm{v}^-) = \bar q\big(\hat u(0, u_\ell)\big) \leq \bar q(u_\ell) < q_*$ and therefore by (4.1) we deduce that $\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-, u_\mathrm{v}^+]$ is still given by (3.5) with $Q_\mathsf{A} = 0$. Hence by (7.1) and the obvious fact that $\hat u(0, u_\mathrm{v}^-) = \hat u\big(0, \hat u(0, u_\ell)\big) = \hat u(0, u_\ell) = u_\mathrm{v}^-$, we have

$$\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-, u_\mathrm{v}^+](\xi) = \begin{cases} \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-, \hat u(0, u_\mathrm{v}^-)](\xi) & \text{if } \xi < 0,\\ \mathcal{RS}_\mathrm{p}[\check u(0, u_\mathrm{v}^+), u_\mathrm{v}^+](\xi) & \text{if } \xi \geq 0,\end{cases} = \begin{cases} u_\mathrm{v}^- & \text{if } \xi < 0,\\ u_\mathrm{v}^+ & \text{if } \xi \geq 0,\end{cases}$$

by (2.3). Then $(\mathrm{ch_v})_2$ is satisfied.

(3) Assume $v_\ell > a$ and $q_\ell = q_*$, see Figure 16 on the left. By (3.3) we have $\overline{Q}(u_\ell) = q_\ell = q_*$ and by (4.1) we deduce that $\mathcal{RS}_\mathrm{v}[u_\ell, u_r]$ is given by (3.5) with $Q_\mathsf{A} = q_*$. Then, by (3.9) we have $u_\mathrm{v}^- = u_\ell \neq \hat u(q_*, u_\ell)$, hence $v_\mathrm{v}^- = v_\ell > a$ and therefore $\overline{Q}(u_\mathrm{v}^-) = \overline{Q}(u_\ell) = q_\ell = q_*$. By (4.1) we have that $\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-, u_\mathrm{v}^+]$ is still given by (3.5) with $Q_\mathsf{A} = q_*$. Hence by (7.1) we deduce

$$\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-, u_\mathrm{v}^+](\xi) = \begin{cases} \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-, \hat u(q_*, u_\mathrm{v}^-)](\xi) & \text{if } \xi < 0,\\ \mathcal{RS}_\mathrm{p}[\check u(q_*, u_\mathrm{v}^+), u_\mathrm{v}^+](\xi) & \text{if } \xi \geq 0,\end{cases}$$

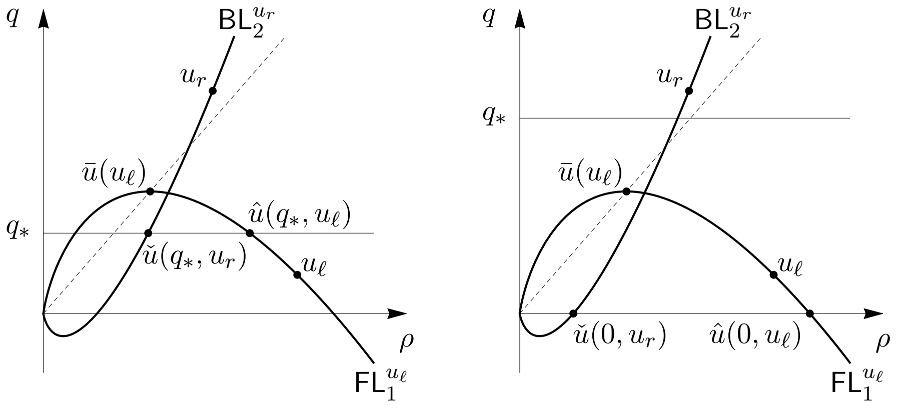

Figure 15: Cases (1) and (2) considered in Theorem 4.5.

$$= \begin{cases} \mathcal{RS}_{\rm p}[u_\ell, \hat u(q_*, u_\ell)](\xi) & \text{if } \xi < 0, \\ \mathcal{RS}_{\rm p}[u_{\rm v}^+, u_{\rm v}^+](\xi) & \text{if } \xi \geq 0, \end{cases} = \begin{cases} u_{\rm v}^- & \text{if } \xi < 0, \\ u_{\rm v}^+ & \text{if } \xi \geq 0, \end{cases}$$

because by (3.9) and $v_{\rm v}^- = v_\ell > a$ we deduce $\mathcal{RS}_{\rm p}[u_\ell, \hat u(q_*, u_\ell)] \equiv u_\ell = u_{\rm v}^-$ on $(-\infty, 0)$. In the case $\xi \geq 0$ we used (2.3). Therefore $(\mathrm{ch}_{\rm v})_2$ is satisfied.

(4) Assume $v_\ell > a$ and $q_\ell > q_*$, see Figure 16 on the right. By (3.3) we have $\overline{Q}(u_\ell) = q_\ell > q_*$ and by (4.1) we deduce that $\mathcal{RS}_{\rm v}[u_\ell, u_r]$ is given by (3.5) with $Q_{\mathsf A} = q_*$. By (7.1) we have $u_{\rm v}^+ = \check u(q_*, u_r)$ and by (3.9) we obtain $u_{\rm v}^- = \hat u(q_*, u_\ell)$. Hence, by $(3.7)_1$ we have $\overline{Q}(u_{\rm v}^-) \geq q_{\rm v}^- = \hat q(q_*, u_\ell) = q_*$; therefore by (4.1) we deduce that $\mathcal{RS}_{\rm v}[u_{\rm v}^-, u_{\rm v}^+]$ is still given by (3.5) with $Q_{\mathsf A} = q_*$. Hence, by (7.1) we have

$$\mathcal{RS}_{\rm v}[u_{\rm v}^-, u_{\rm v}^+](\xi) = \begin{cases} \mathcal{RS}_{\rm p}[u_{\rm v}^-, \hat u(q_*, u_{\rm v}^-)](\xi) & \text{if } \xi < 0 \\ \mathcal{RS}_{\rm p}[\check u(q_*, u_{\rm v}^+), u_{\rm v}^+](\xi) & \text{if } \xi \geq 0 \end{cases} = \begin{cases} \mathcal{RS}_{\rm p}[u_{\rm v}^-, \hat u(q_*, u_{\rm v}^-)](\xi) & \text{if } \xi < 0, \\ u_{\rm v}^+ & \text{if } \xi \geq 0, \end{cases}$$

where we used (2.3) in the case $\xi \geq 0$. If $\hat u(q_*, u_{\rm v}^-) = u_{\rm v}^-$, then $\mathcal{RS}_{\rm p}[u_{\rm v}^-, \hat u(q_*, u_{\rm v}^-)] \equiv u_{\rm v}^-$ and $(\mathrm{ch}_{\rm v})_2$ is proved by (2.3). Otherwise, if $\hat u(q_*, u_{\rm v}^-) \neq u_{\rm v}^-$, then the conclusion of the first part of Lemma 2.9 does not hold; as a consequence $(2.17)_2$ fails, i.e., we have $v_{\rm v}^- > a$. Together with the already observed fact $q_{\rm v}^- = q_*$, this implies by (3.9) that $\mathcal{RS}_{\rm p}[u_{\rm v}^-, \hat u(q_*, u_{\rm v}^-)] \equiv u_{\rm v}^-$ in $(-\infty, 0)$ and therefore also in this case $(\mathrm{ch}_{\rm v})_2$ is satisfied.

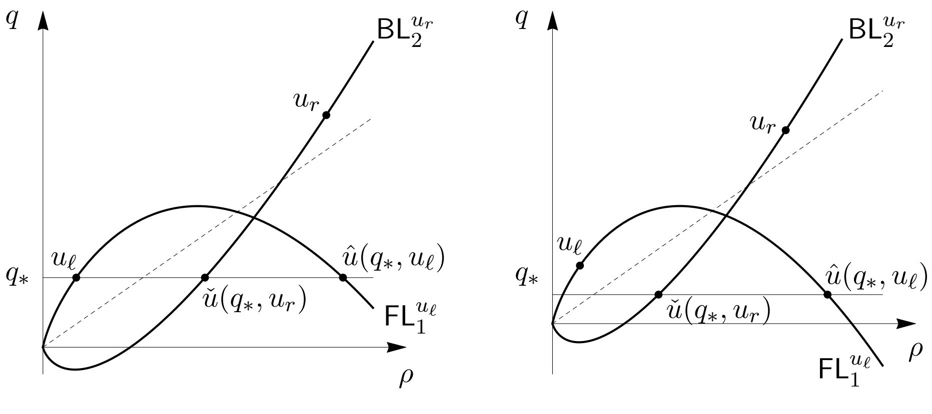

Figure 16: Cases (3) and (4) considered in Theorem 4.5.

(5) Assume $v_\ell > a$, $q_\ell < q_*$ and $\bar q\big(\hat u(0, u_\ell)\big) < q_*$, see Figure 17 on the left. By (3.3) we have $\overline{Q}(u_\ell) = q_\ell < q_*$ and by (4.1) we deduce that $\mathcal{RS}_{\rm v}[u_\ell, u_r]$ is given by (3.5) with $Q_{\mathsf A} = 0$. By (7.1) and (3.9) we have $u_{\rm v}^+ = \check u(0, u_r)$ and $u_{\rm v}^- = \hat u(0, u_\ell)$, respectively. Hence $v_{\rm v}^- = \hat v(0, u_\ell) = 0 < a$. By (3.3) we have $\overline{Q}(u_{\rm v}^-) = \bar q(u_{\rm v}^-) = \bar q\big(\hat u(0, u_\ell)\big) < q_*$ and therefore by (4.1) we have that $\mathcal{RS}_{\rm v}[u_{\rm v}^-, u_{\rm v}^+]$ is still given by (3.5) with $Q_{\mathsf A} = 0$. Hence, by the obvious fact that $\hat u(0, u_{\rm v}^-) = \hat u\big(0, \hat u(0, u_\ell)\big) = \hat u(0, u_\ell) = u_{\rm v}^-$ we have

$$\mathcal{RS}_{\rm v}[u_{\rm v}^-, u_{\rm v}^+](\xi) = \begin{cases} \mathcal{RS}_{\rm p}[u_{\rm v}^-, \hat u(0, u_{\rm v}^-)](\xi) & \text{if } \xi < 0, \\ \mathcal{RS}_{\rm p}[\check u(0, u_{\rm v}^+), u_{\rm v}^+](\xi) & \text{if } \xi \geq 0, \end{cases} = \begin{cases} u_{\rm v}^- & \text{if } \xi < 0, \\ u_{\rm v}^+ & \text{if } \xi \geq 0, \end{cases}$$

by (2.3). In conclusion $(\mathrm{ch_v})_2$ is satisfied also in this case.

(6) At last, assume $v_\ell > a$, $q_\ell < q_*$ and $\bar{q}(\hat{u}(0,u_\ell)) \ge q_*$, see Figure 17 on the right. As in case (5), by (3.3) we have $\overline{Q}(u_\ell) = q_\ell < q_*$ and by (4.1) we deduce that $\mathcal{RS}_\mathrm{v}[u_\ell,u_r]$ is given by (3.5) with $Q_\mathsf{A} = 0$. By (7.1) and (3.9) we have $u_\mathrm{v}^+ = \check{u}(0,u_r)$ and $u_\mathrm{v}^- = \hat{u}(0,u_\ell)$, respectively. Hence $v_\mathrm{v}^- = \hat{v}(0,u_\ell) = 0 < a$. However, differently from case (5), by (3.3) we have $\overline{Q}(u_\mathrm{v}^-) = \bar{q}(u_\mathrm{v}^-) = \bar{q}(\hat{u}(0,u_\ell)) \ge q_*$; therefore by (4.1) we have in this case that $\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-,u_\mathrm{v}^+]$ is given by (3.5) with $Q_\mathsf{A} = q_*$. Notice that $\mathcal{RS}_\mathrm{v}[u_\ell,u_r]$ had instead $Q_\mathsf{A} = 0$. Hence by (7.1) we deduce

$$\mathcal{RS}_\mathrm{v}[u_\mathrm{v}^-,u_\mathrm{v}^+](\xi) = \begin{cases} \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-,\hat{u}(q_*,u_\mathrm{v}^-)](\xi) & \text{if } \xi<0,\\ \mathcal{RS}_\mathrm{p}[\check{u}(q_*,u_\mathrm{v}^+),u_\mathrm{v}^+](\xi) & \text{if } \xi\ge 0,\end{cases} = \begin{cases} \mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-,\hat{u}(q_*,u_\mathrm{v}^-)](\xi) & \text{if } \xi<0,\\ u_\mathrm{v}^+ & \text{if } \xi\ge 0,\end{cases}$$

where we used (2.3) in the case $\xi \ge 0$. Since $q_\mathrm{v}^- = \hat{q}(0,u_\ell) = 0 \ne q_* = \hat{q}(q_*,u_\mathrm{v}^-)$, then $v_\mathrm{v}^- = \hat{v}(0,u_\ell) = 0$. Therefore by (3.9) we have that $\mathcal{RS}_\mathrm{p}[u_\mathrm{v}^-,\hat{u}(q_*,u_\mathrm{v}^-)](0^-) = \hat{u}(q_*,u_\mathrm{v}^-) \ne u_\mathrm{v}^-$. In conclusion $(\mathrm{ch_v})_2$ is *not* satisfied in this case.

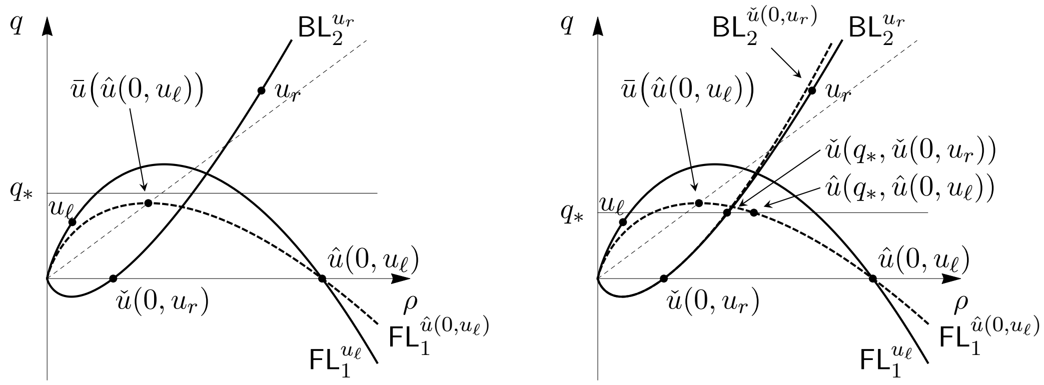

Figure 17: Cases (5) and (6) considered in Theorem 4.5.

The above analysis shows that $\mathcal{RS}_\mathrm{v}$ is not coherent at (u_ℓ,u_r) if and only if we are in case (6), i.e.,

$$v_\ell > a, \qquad\qquad q_\ell < q_* \le \bar{q}(\hat{u}(0,u_\ell)).$$

To conclude the proof of the theorem it only remains to show that in the region $v_\ell > a$, $q_\ell < q_*$, we have

$$q_* \le \bar{q}(\hat{u}(0,u_\ell)) \iff \mathcal{BL}_1^{u_*^0}(\rho_\ell) \le q_\ell.$$

By (2.15), (2.6) and $(4.2)_2$ this is equivalent to prove that for $\xi \doteq v_\ell/a$ and $\zeta \doteq (e\,q_*)/(a\,\rho_\ell)$, we have, for $\xi > 1$ and $\zeta > e\,\xi$,

$$4\,\zeta \le \left[\sqrt{\xi^2+4}+\xi\right]^2 \iff \sqrt{\zeta} - \frac{1}{\sqrt{\zeta}} \le \xi. \tag{7.2}$$

Since both functions $\zeta \mapsto \sqrt{\zeta} - \frac{1}{\sqrt{\zeta}}$ and $\xi \mapsto \sqrt{\xi^2+4}+\xi$ are increasing in $[1,\infty)$, then $(7.2)_1$ implies

$$\sqrt{\zeta} - \frac{1}{\sqrt{\zeta}} \le \frac{\sqrt{\xi^2+4}+\xi}{2} - \frac{2}{\sqrt{\xi^2+4}+\xi} = \xi,$$

while $(7.2)_2$ implies

$$\left[\sqrt{\xi^2+4}+\xi\right]^2 \ge \left[\sqrt{\left(\sqrt{\zeta}-\frac{1}{\sqrt{\zeta}}\right)^2+4}+\left(\sqrt{\zeta}-\frac{1}{\sqrt{\zeta}}\right)\right]^2 = 4\zeta.$$

This shows the double implication (7.2) and then Theorem 4.5 is completely proved.

7.3 Invariant domains

In this section we prove Proposition 4.8. We recall that the states $u_1, u_2, u_3, \ldots$ are implicitly defined as

$$u_2 = u_*^a, \qquad u_{2i} = \hat{u}(q_*, u_{2i+1}), \qquad u_{2i-1} = \check{u}(0, u_{2i}). \tag{7.3}$$

We recall that the definition of the states u_i, $i \in \mathbb{N} \doteq \{1, 2, 3, \ldots\}$ only uses rarefaction curves. Therefore by Proposition 2.3, (L1), for any u belonging to the stretch of curve $\mathsf{FL}_1^{u_{2i+1}}$ (or $\mathsf{BL}_2^{u_{2i}}$) joining u_{2i} with u_{2i+1} (resp., u_{2i-1} with u_{2i}), the piece of curve FL_1^{u} (resp., BL_2^{u}) joining u_{2i} (resp., u_{2i-1}) with u lies on the previous stretch.

Lemma 7.1. *For any $i \in \mathbb{N}$ we have:*

(i) $q_{2i} = q_*$ *and* $q_{2i-1} = 0$;

(ii) $\rho_i < \rho_{i+1}$;

(iii) $v_2 = a > v_{2i+2} > v_{2(i+1)+2} > 0 = v_{2i-1}$;

(iv) $\bar{q}(u_1) < q_* = \bar{q}(u_2) = \bar{q}(u_3) < \bar{q}(u_{2i+2}) = \bar{q}(u_{2i+3}) < \bar{q}(u_{2(i+1)+2}) = \bar{q}(u_{2(i+1)+3})$.

Proof. Clearly (i) follows by (7.3). Since $q_{2i-1} = 0 < q_* = q_{2i}$, by $(7.3)_2$ we have $\rho_{2i} < \rho_{2i+1}$ and by $(7.3)_3$ we deduce $\rho_{2i-1} < \rho_{2i}$; this proves (ii).

By $(4.2)_1$ and $(7.3)_1$ we have $v_2 = v_*^a = a$. By (i) we obtain $v_{2i+2} > 0 = v_{2i-1}$; moreover, since $q_{2i} = q_* = q_{2(i+1)+2}$ by (i) and $\rho_{2i+2} < \rho_{2(i+1)+2}$ by (ii), we have $v_{2i+2} > v_{2(i+1)+2}$. This proves (iii).

At last, by (iii) we have $v_i \leq a$, hence by (2.12), (i) and (ii) we have

$$\bar{q}(u_{2i-1}) = \frac{a}{e}\,\rho_{2i-1} < \frac{a}{e}\,\rho_{2i+1} = \bar{q}(u_{2i+1}).$$

By Proposition 2.3, (L1), we also obtain $\bar{q}(u_{2i}) = \bar{q}(u_{2i+1})$ and by (4.3) we deduce $\bar{q}(u_3) = q_2 = q_* = \bar{q}(u_2)$. This shows (iv) and concludes the proof. □

Now, we prove Proposition 4.8. By Lemma 7.1, (iii), any state in I_N is subsonic, with the exception of the sonic state $u_2 \doteq u_*^a$. Notice that $\mathsf{I}_N \cap \mathsf{C}_\ell = \mathsf{T}_1$; as a consequence, we have $\mathsf{I}_N \cap \mathsf{C}_\ell \neq \emptyset \neq \mathsf{I}_N \cap \big(\Omega \setminus \mathsf{C}_\ell\big)$.

We now prove that I_N is the minimal invariant domain containing $\{u_1, u_{2N+2}\}$. We first show that

$$\mathcal{I} \supseteq \mathsf{I}_N \qquad \text{for any invariant domain } \mathcal{I} \text{ containing } \{u_1, u_{2N+2}\}. \tag{7.4}$$

We prove (7.4) in four steps. We denote $\mathbb{R}_- \doteq (-\infty, 0)$ and $\mathbb{R}_+ \doteq (0, \infty)$.

S1: $\mathcal{RS}_{\rm v}[u_1, u_{2N+2}]$. Since $v_1 = 0 < a$, by (3.3) and (iv) in Lemma 7.1 we have $\overline{Q}(u_1) = \bar{q}(u_1) < q_*$. Then, by (4.1) we have that $\mathcal{RS}_{\rm v}[u_1, u_{2N+2}]$ is given by (3.5) with $Q_{\mathsf{A}}(u_1) = 0$. As a consequence we have

$$\begin{aligned}\mathcal{I} \supseteq \mathcal{RS}_{\rm v}[u_1, u_{2N+2}](\mathbb{R}) &= \mathcal{RS}_{\rm p}[u_1, \hat{u}(0, u_1)](\mathbb{R}_-) \cup \mathcal{RS}_{\rm p}[\check{u}(0, u_{2N+2}), u_{2N+2}](\mathbb{R}_+)\\ &= \Big\{u_1\Big\} \cup \Big\{u \in \mathsf{R}_2^{u_{2N+2}} : \rho \in [\rho_{2N+1}, \rho_{2N+2}]\Big\}.\end{aligned}$$

S2: $\mathcal{RS}_{\rm v}[u_o, u_{2N+2}]$ with $u_o \in \{u \in \mathsf{R}_2^{u_{2N+2}} : \rho \in [\rho_{2N+1}, \rho_{2N+2}]\}$. Any $u_o \in \mathsf{R}_2^{u_{2N+2}}$ with $\rho \in [\rho_{2N+1}, \rho_{2N+2}]$ is such that $v_o < a$; hence, by (3.3) and (iv) in Lemma 7.1 we have $\overline{Q}(u_o) = \bar{q}(u_o) \geq q_*$. Therefore by (4.1) we have that $\mathcal{RS}_{\rm v}[u_o, u_{2N+2}]$ is given by (3.5) with $Q_{\mathsf{A}}(u_o) = q_*$. As a consequence, we have

$$\begin{aligned}\mathcal{I} \supseteq \mathcal{RS}_{\rm v}[u_o, u_{2N+2}](\mathbb{R}) &= \mathcal{RS}_{\rm p}[u_o, \hat{u}(q_*, u_o)](\mathbb{R}_-) \cup \mathcal{RS}_{\rm p}[\check{u}(q_*, u_{2N+2}), u_{2N+2}](\mathbb{R}_+)\\ &= \Big\{u \in \mathsf{R}_1^{u_o} : \rho \in [\hat{\rho}(q_*, u_o), \rho_o]\Big\} \cup \Big\{u_{2N+2}\Big\}.\end{aligned}$$

By letting u_o vary in $\{u \in \mathsf{R}_2^{u_{2N+2}} : \rho \in [\rho_{2N+1}, \rho_{2N+2}]\}$ we deduce $\mathcal{I} \supseteq \perp_N$.

S3: $\mathcal{RS}_\mathrm{v}[u_1, u_o]$ with $u_o \in \bot_N$. As we already observed, we have $\overline{Q}(u_1) = \bar{q}(u_1) < q_*$. Therefore, by (4.1), for any $u_o \in \bot_N$ we have that $\mathcal{RS}_\mathrm{v}[u_1, u_o]$ is given by (3.5) with $Q_\mathsf{A}(u_1) = 0$. Hence we obtain

$$\begin{aligned}\mathcal{I} \supseteq \mathcal{RS}_\mathrm{v}[u_1, u_o](\mathbb{R}) &= \mathcal{RS}_\mathrm{p}[u_1, \hat{u}(0, u_1)](\mathbb{R}_-) \cup \mathcal{RS}_\mathrm{p}[\check{u}(0, u_o), u_o](\mathbb{R}_+) \\ &= \Big\{u_1\Big\} \cup \Big\{u \in \mathsf{R}_2^{u_o} : \rho \in [\check{\rho}(0, u_o), \rho_o]\Big\}.\end{aligned}$$

By letting u_o vary in $\bot_N$ we have $\mathcal{I} \supseteq \{u_1\} \cup \mathsf{T}_N \cup \bot_N$.

S4: iterate. It is now sufficient to repeat the above steps "starting" with u_{2N} instead of u_{2N+2} and so on to show $\mathcal{I} \supseteq \mathsf{T}_j \cup \bot_j$ for $j = 1, \ldots, N-1$.

This proves (7.4). To conclude the proof of Proposition 4.8 we must show that I_N is an invariant domain. This follows from Proposition 2.3, see (L1) and (L5), and the following considerations:

(1) if $u_\ell \in \mathsf{I}_N \cap \mathsf{C}_\ell \subset \mathsf{T}_1$, then $\mathcal{RS}_\mathrm{v}[u_\ell, \hat{u}(0, u_\ell)]$ is a 1-shock and $\hat{u}(0, u_\ell) \in \mathsf{T}_1 \subseteq \mathsf{I}_N$;

(2) if $u_\ell \in \mathsf{I}_N \setminus \mathsf{C}_\ell$, then $\mathcal{RS}_\mathrm{v}[u_\ell, \hat{u}(q_*, u_\ell)]$ is a 1-rarefaction and $\{(\rho, \mathcal{R}_1^{u_\ell}(\rho)) : \rho \in [\hat{\rho}(q_*, u_\ell), \rho_\ell]\} \subset \mathsf{I}_N$;

(3) if $u_r \in \mathsf{I}_N$, then $\mathcal{RS}_\mathrm{v}[\check{u}(0, u_r), u_r]$ is a 2-rarefaction and $\{(\rho, \mathcal{R}_2^{\check{u}(0,u_r)}(\rho)) : \rho \in [\check{\rho}(0, u_r), \rho_r]\} \subset \mathsf{I}_N$, while $\mathcal{RS}_\mathrm{v}[\check{u}(q_*, u_r), u_r]$ is a 2-shock and $\check{\rho}(q_*, u_r) \in \mathsf{I}_N$.

In conclusion, I_N is an invariant domain containing $\{u_1, u_{2N+2}\}$ and the proof is complete.

8 Conclusions

In this paper we proposed a simple model for the gas flow through a one-way valve; the motion takes place along a straight pipe and the flow is governed by system (1.1). We introduced the notions of *coupling Riemann solver*, which prescribes solutions in the whole pipe, and of *coherent initial data*. The latter amounts to require, roughly speaking, that further applications of the solver to adjacent states of the corresponding solution do not change the solution. From a mathematical point of view, coherence is a fundamental property when looking to solutions for initial data with bounded variation (**BV**).

We considered in details two types of valves.

(i) In the first (ideal) case, the valve acts with no time delay and is designed to only allow an outflow q_*; otherwise it is closed. The Riemann problem never leads to chattering, independently of the coherence or incoherence of the initial data: indeed, the Riemann solver shows a unique solution and interactions cannot arise. We characterized the coherent initial data and showed that incoherent data lead to the closure of the valve. For a fixed q_* we also studied the coherence of *sets* of initial data and, conversely, how to choose flow values q_* that make coherent a prescribed set of data.

(ii) In the second case, we considered an analogous valve but with a reaction time τ. The Riemann problem with incoherent initial data shows with a strong evidence that chattering can arise as a consequence of the interactions developed by the delayed working of the valve. Coherent Riemann initial data do not show this behavior.

Our results show that, at least for Riemann initial data, the occurrence of chattering (at low flows, as in [26]) depends on the reaction time of the valve.

Several questions may be risen. In case (i), it remains to understand the relationships between incoherence and chattering in a general initial-value problem with **BV** initial data. In case (ii), the open problems concern a rigorous proof of the aforementioned result and, more generally, the study of the general initial-value problem. Two limit problems also are worthy of consideration: indeed, it would be possible to recover the ideal valve from the delayed one in the limit $\tau \to \infty$. Both problems seem interesting from both the analytical and numerical point of views.

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