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CAROLINA ARAUJO, ALEX MASSARENTI, AND RICK RISCHTER

ABSTRACT. Let $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ be the Segre-Veronese variety given as the image of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ under the embedding induced by the complete linear system $\left| \mathcal{O}_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}}(d_1, \ldots, d_r) \right|$. We prove that asymptotically $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not h-defective for $h \leq (\min\{n_i\})^{\lfloor \log_2(d-1) \rfloor}$, where $d = d_1 + \cdots + d_r$.

CONTENTS

1. INTRODUCTION

Secant varieties are classical objects in algebraic geometry. The *h-secant variety* $\mathbb{S}ec_h(X)$ of a non-degenerate n-dimensional variety $X \subset \mathbb{P}^N$ is the Zariski closure of the union of all linear spaces spanned by collections of h points of X. The *expected dimension* of $\mathbb{S}ec_h(X)$ is

$$\text{expdim}(\mathbb{S}ec_h(X)) := \min\{nh + h - 1, N\}.$$

The actual dimension of $\mathbb{S}ec_h(X)$ may be smaller than the expected one. Following [Zak93], we say that X is *h-defective* if

$$\dim(\mathbb{S}ec_h(X)) < \text{expdim}(\mathbb{S}ec_h(X)).$$

Determining secant defectivity is an old problem in algebraic geometry, which goes back to the Italian school (see [Cas37, Chapter 10], [Sco08], [Sev01], [Ter11]).

In this paper we investigate secant defectivity for Segre-Veronese varieties. The problem is specially interesting in this case, in connection with problems of partially symmetric tensor decomposition (see [CGLM08], [Lan12]). Indeed, Segre-Veronese varieties parametrize rank one tensors. So their h-secant varieties parametrize tensors of a given rank depending on h. For this reason, they have been used to construct and study moduli spaces for additive decompositions of a general tensor into a given number of rank one tensors (see [Dol04], [DK93], [Mas16], [MM13], [RS00], [TZ11], [BGI11]).

The problem of secant defectivity for Veronese varieties was completely solved in [AH95]. In that paper, Alexander and Hirschowitz showed that, except for the degree 2 Veronese embedding, which

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is almost always defective, the degree d Veronese embedding of $\mathbb{P}^n$ is not h-defective except in the following cases:
$$(d, n, h) \in \{(4, 2, 5), (4, 3, 9), (3, 4, 7), (4, 4, 14)\}.$$

For Segre varieties, secant defectivity is classified in some special cases. Segre products of two factors $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \subset \mathbb{P}^{n_1 n_2 + n_1 + n_2}$ are almost always defective. For Segre products $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \subset \mathbb{P}^N$, the problem was completely settled in [CGG11]. In general, h-defectivity of Segre products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \subset \mathbb{P}^N$ is classified only for $h \leq 6$ ([AOP09]).

Next we turn to Segre-Veronese varieties. These are products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ embedded by the complete linear system $\left| \mathcal{O}_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}}(d_1, \ldots, d_r) \right|$, $d_i > 0$. The problem of secant defectivity for Segre-Veronese varieties has been solved in some very special cases, mostly for products of few factors (see [CGG05], [AB09], [Abo10], [BCC11], [AB12], [BBC12], [AB13]). Secant defectivity for Segre-Veronese products $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, with arbitrary number of factors and degrees, was classified in [LP13]. In general, h-defectivity is classified only for small values of h ([CGG05, Proposition 3.2]): except for the Segre product $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, Segre-Veronese varieties $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ are never h-defective for $h \leq \min\{n_i\} + 1$. In this paper we improve this bound by taking into account the embedding degrees $d_1, \ldots, d_r$. We show that, asymptotically for large $n_1 = \min\{n_i\}$ Segre-Veronese varieties are never h-defective for $h \leq n_1^{\lfloor \log_2(d-1) \rfloor}$, where $d = d_1 + \cdots + d_r$. More precisely, our main result in Theorem 4.8 can be rephrased as follows.

Theorem 1.1. *Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be two r-uples of positive integers, with $n_1 \leq \cdots \leq n_r$ and $d = d_1 + \cdots + d_r \geq 3$. Let $SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subset \mathbb{P}^N$ be the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ embedded by the complete linear system $\left| \mathcal{O}_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}}(d_1, \ldots, d_r) \right|$. Write*
$$d - 1 = 2^{\lambda_1} + \cdots + 2^{\lambda_s} + \epsilon,$$
with integers $\lambda_1 > \lambda_2 > \cdots > \lambda_s \geq 1$, $\epsilon \in \{0, 1\}$. Then $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not $(h+1)$-defective for
$$h \leq n_1((n_1 + 1)^{\lambda_1 - 1} + \cdots + (n_1 + 1)^{\lambda_s - 1}) + 1.$$

Our proof of Theorem 1.1 follows the strategy introduced in [MR16], which we now explain. Given a non-degenerate n-dimensional variety $X \subset \mathbb{P}^N$, and general points $x_1, \ldots, x_h \in X \subset \mathbb{P}^N$, consider the linear projection with center $\langle T_{x_1} X, \ldots, T_{x_h} X \rangle$,
$$\tau_{X,h} : X \subseteq \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_h},$$
where $N_h := N - 1 - \dim(\langle T_{x_1} X, \ldots, T_{x_h} X \rangle)$. By [CC02, Proposition 3.5], if $\tau_{X,h}$ is generically finite then X is not $(h+1)$-defective. In general, however, it is hard to control the dimension of the fibers of the tangential projections $\tau_{X,h}$ as h gets larger. In [MR16] a new strategy was developed, based on the more general *osculating projections* instead of just tangential projections. For a smooth point $x \in X \subset \mathbb{P}^N$, the *$k$-osculating space* $T_x^k X$ of X at x is roughly the smaller linear subspace where X can be locally approximated up to order k at x (see Definition 2.1). Given $x_1, \ldots, x_l \in X$ general points, we consider the linear projection with center $\langle T_{x_1}^{k_1} X, \ldots, T_{x_l}^{k_l} X \rangle$,
$$\Pi_{T_{x_1, \ldots, x_l}^{k_1, \ldots, k_l}} : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_{k_1, \ldots, k_l}},$$
and call it a $(k_1 + \cdots + k_l)$-*osculating projection*, where $N_{k_1, \ldots, k_l} := N - 1 - \dim(\langle T_{x_1}^{k_1} X, \ldots, T_{x_l}^{k_l} X \rangle)$. Under suitable conditions, one can degenerate the linear span of several tangent spaces $T_{x_i} X$ into a subspace contained in a single osculating space $T_x^k X$. So the tangential projections $\tau_{X,h}$ degenerates to a linear projection with center contained in the linear span of osculating spaces, $\langle T_{p_1}^k X, \ldots, T_{p_l}^k X \rangle$. If $\Pi_{T_{p_1, \ldots, p_l}^{k, \ldots, k}}$ is generically finite, then $\tau_{X,h}$ is also generically finite, and one concludes that X is not $(h+1)$-defective. The advantage of this approach is that one has to consider osculating spaces at much less points than h, allowing to control the dimension of the fibers of the projection. In [MR16], this strategy was successfully applied to study the problem of secant defectivity for Grassmannians. Here we apply it to Segre-Veronese varieties.

The paper is organized as follows. In Section 2 we describe explicitly osculating spaces of Segre-Veronese varieties. In Section 3 we study the relative dimension of general osculating projections. In Section 5 we study how many general tangent projections degenerate to osculating projections. Finally, in Section 4 we apply these result and the techniques developed in [MR16] to prove our main result on the dimension of secant varieties of Segre-Veronese varieties.

Notation and conventions. We always work over the field $\mathbb{C}$ of complex numbers. Varieties are always assumed to be irreducible. For a vector space V, we denote by $\mathbb{P}(V)$ its Grothendieck projectivization, i.e., the projective space of nonzero linear forms on V up to scaling.

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2. OSCULATING SPACES OF SEGRE-VERONESE VARIETIES

In this section we describe osculating spaces of Segre-Veronese varieties. We start by defining osculating spaces. They can also be defined intrinsically via jet bundles (see [MR16, Section 3]).

Definition 2.1. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n, and $p \in X$ a smooth point. Choose a local parametrization of X at p:

$$\phi: \quad \begin{aligned} \mathcal{U} \subseteq \mathbb{C}^n & \longrightarrow & \mathbb{C}^N \\ (t_1, \ldots, t_n) & \longmapsto & \phi(t_1, \ldots, t_n) \\ 0 & \longmapsto & p \end{aligned}$$

For a multi-index $I = (i_1, \ldots, i_n)$, set

$$(2.2) \qquad \phi_I = \frac{\partial^{|I|} \phi}{\partial t_1^{i_1} \ldots \partial t_n^{i_n}}.$$

For any $m \geq 0$, let $O_p^m X$ be the affine subspace of $\mathbb{C}^N$ centered at p and spanned by the vectors $\phi_I(0)$ with $|I| \leq m$.

The m-*osculating space* $T_p^m X$ of X at p is the projective closure of $O_p^m X$ in $\mathbb{P}^N$. Note that $T_p^0 X = \{p\}$, and $T_p^1 X$ is the usual tangent space of X at p. The m-*osculating dimension* of X at p is

$$\dim(T_p^m X) = \binom{n+m}{n} - 1 - \delta_{m,p},$$

where $\delta_{m,p}$ is the number of independent differential equations of order $\leq m$ satisfied by X at p.

Next we turn to Segre-Veronese varieties. In Proposition 2.5 below, we describe explicitly their osculating spaces at *coordinate points* by computing (2.2) for a suitable rational parametrization. Note that after a base change any point of a Segre-Veronese variety becomes a coordinate point. In order to do so, we recall the definition of Segre-Veronese varieties and fix some notation to be used throughout the paper.

Notation 2.3. In the rest of the paper, by a slight abuse of notation, we will denote by v both a vector in a vector space V and the corresponding point in $\mathbb{P}(V)$.

Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be two r-uples of positive integers, with $n_1 \leq \cdots \leq n_r$. Set $d = d_1 + \cdots + d_r$, $n = n_1 + \cdots + n_r$, and $N(\boldsymbol{n}, \boldsymbol{d}) = \prod_{i=1}^{r} \binom{n_i + d_i}{n_i} - 1$.

Let $V_1, \ldots, V_r$ be vector spaces of dimensions $n_1 + 1 \leq n_2 + 1 \leq \cdots \leq n_r + 1$, and consider the product

$$\mathbb{P}^{\boldsymbol{n}} = \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_r^*).$$

The line bundle

$$\mathcal{O}_{\mathbb{P}^{\boldsymbol{n}}}(d_1, \ldots, d_r) = \mathcal{O}_{\mathbb{P}(V_1^*)}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}(V_1^*)}(d_r)$$

induces an embedding

$$\begin{aligned} \sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}} : \quad \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_r^*) &\longrightarrow \mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_r} V_r^*) = \mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}, \\ (v_1, \ldots, v_r) &\longmapsto \quad v_1^{d_1} \otimes \cdots \otimes v_r^{d_r} \end{aligned}$$

where $v_i \in V_i$. We call the image

$$SV_{\boldsymbol{d}}^{\boldsymbol{n}} = \sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}}(\mathbb{P}^{\boldsymbol{n}}) \subset \mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}$$

a *Segre-Veronese variety*. It is a smooth variety of dimension n and degree $\frac{(n_1 + \cdots + n_r)!}{n_1! \ldots n_r!} d_1^{n_1} \ldots d_r^{n_r}$ in $\mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}$.

When $r = 1$, SV_d^n is a Veronese variety. In this case we write V_d^n for SV_d^n, and v_d^n for the Veronese embedding. When $d_1 = \cdots = d_r = 1$, $SV_{1,\ldots,1}^{\boldsymbol{n}}$ is a Segre variety. In this case we write $S^{\boldsymbol{n}}$ for $SV_{1,\ldots,1}^{\boldsymbol{n}}$, and $\sigma^{\boldsymbol{n}}$ for the Segre embedding. Note that

$$\sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}} = \sigma^{\boldsymbol{n}'} \circ \left(\nu_{d_1}^{n_1} \times \cdots \times \nu_{d_r}^{n_r} \right),$$

where $\boldsymbol{n}' = (N(n_1, d_1), \ldots, N(n_r, d_r))$, and $N(n_i, d_i) = \binom{n_i + d_i}{n_i} - 1$.

Given $v_1, \ldots, v_{d_j} \in V_j$, we denote by $v_1 \cdots \cdots v_{d_j} \in \mathrm{Sym}^{d_j} V_j$ the symmetrization of $v_1 \otimes \cdots \otimes v_{d_j}$.

Hoping that no confusion will arise, we write $(e_0, \ldots, e_{n_j})$ for a fixed basis of each V_j. Given a d_j-uple $I = (i_1, \ldots, i_{d_j})$, with $0 \leq i_1 \leq \cdots \leq i_{d_j} \leq n_j$, we denote by $e_I \in \mathrm{Sym}^{d_j} V_j$ the symmetric product $e_{i_1} \cdots \cdots e_{i_{d_j}}$.

For each $j \in \{1, \ldots, r\}$, consider a d_j-uple $I^j = (i_1^j, \ldots, i_{d_j}^j)$, with $0 \leq i_1^j \leq \cdots \leq i_{d_j}^j \leq n_j$, and set

$$I = (I^1, \ldots, I^r) = ((i_1^1, \ldots, i_{d_1}^1), (i_1^2, \ldots, i_{d_2}^2), \ldots, (i_1^r, \ldots, i_{d_r}^r)).$$

We denote by e_I the vector

$$e_I = e_{I^1} \otimes e_{I^2} \otimes \cdots \otimes e_{I^r} \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_r} V_r,$$

as well as the corresponding point in $\mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_r} V_r^*) = \mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}$. When $I^j = (i_j, \ldots, i_j)$ for every $j \in \{1, \ldots, r\}$, for some $0 \leq i_j \leq n_j$, we have

$$e_I = \sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}}(e_{i_1}, \ldots, e_{i_r}) \in SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subset \mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}.$$

In this case we say that e_I is a *coordinate point* of $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$.

Definition 2.4. Let n and d be positive integers, and set

$$\Lambda_{n,d} = \{I = (i_1, \ldots, i_d), 0 \leq i_1 \leq \cdots \leq i_d \leq n\}.$$

For $I, J \in \Lambda_{n,d}$, we define their distance $d(I, J)$ as the number of different coordinates. More precisely, write $I = (i_1, \ldots, i_d)$ and $J = (j_1, \ldots, j_d)$. There are $r \geq 0$, distinct indices $\lambda_1, \ldots, \lambda_r \subset \{1, \ldots, d\}$, and distinct indices $\tau_1, \ldots, \tau_r \subset \{1, \ldots, d\}$ such that $i_{\lambda_k} = j_{\tau_k}$ for every $1 \leq k \leq r$, and $\{i_\lambda | \lambda \neq \lambda_1, \ldots, \lambda_r\} \cap \{j_\tau | \tau \neq \tau_1, \ldots, \tau_r\} = \emptyset$. Then $d(I, J) = d - r$. Note that $\Lambda_{n,d}$ has diameter d and size $\binom{n+d}{n} = N(n, d) + 1$.

Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be two r-uples of positive integers, and set

$$\Lambda = \Lambda_{\boldsymbol{n}, \boldsymbol{d}} = \Lambda_{n_1, d_1} \times \cdots \times \Lambda_{n_r, d_r}.$$

For $I = (I^1, \ldots, I^r), J = (J^1, \ldots, J^r) \in \Lambda$, we define their distance as

$$d(I, J) = d(I^1, J^1) + \cdots + d(I^r, J^r).$$

Note that Λ has diameter d and size $\prod_{i=1}^{r} \binom{n_i + d_i}{n_i} = N(\boldsymbol{n}, \boldsymbol{d}) + 1$.

Such a distance, called the Hamming distance, was defined in [CGG02, Section 2] for Segre varieties. We can now state the main result of this section.

Proposition 2.5. *Let the notation and assumptions be as in Notation 2.3 and Definition 2.4. Set $I^1 = (i_1, \ldots, i_1), \ldots, I^r = (i_r, \ldots, i_r)$, with $0 \le i_j \le n_j$, and $I = (I^1, \ldots, I^r)$. Consider the point*

$$e_I = \sigma \nu_{\boldsymbol{d}}^{\boldsymbol{n}}(e_{i_1}, \ldots, e_{i_r}) \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}.$$

For any $s \ge 0$, we have

$$T_{e_I}^s(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) = \langle e_J \mid d(I, J) \le s \rangle.$$

In particular, $T_{e_I}^s(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) = \mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}$ for any $s \ge d$.

Proof. We may assume that $I^1 = (0, \ldots, 0), \ldots, I^r = (0, \ldots, 0)$. Write $(z_K)_{K \in \Lambda}$, for coordinates in $\mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}$, and consider the rational parametrization

$$\phi : \mathbb{A}^{\sum n_i} \to SV_{\boldsymbol{d}}^{\boldsymbol{n}} \cap \left(z_I \ne 0 \right) \subset \mathbb{A}^{N(\boldsymbol{n}, \boldsymbol{d})}$$

given by

$$A = (a_{j,i})_{j=1,\ldots,r, i=1,\ldots,n_j} \mapsto \left(\prod_{j=1}^{r} \prod_{k=1}^{d_j} a_{j, i_k^j} \right)_{K = (K^1, \ldots, K^r) \in \Lambda \setminus \{I\}},$$

where $K^j = (i_1^j, \ldots, i_{d_j}^j) \in \Lambda_{n_j, d_j}$ for each $j = 1, \ldots, r$.

For integers l and m, we write $\deg_{l,m} K$ for the degree of the polynomial

$$\phi(A)_K := \prod_{j=1}^{r} \prod_{k=1}^{d_j} a_{j, i_k^j}$$

with respect to $a_{l,m}$. Then $0 \le \deg_{l,m} K \le d_l$, and the degree of $\phi(A)_K$ with respect to all the variables $a_{j,i}$ is at most d. One computes:

$$\left(\frac{\partial^{\lambda_1 + \cdots + \lambda_t} \phi(A)}{\partial^{\lambda_1} a_{l_1, m_1} \ldots \partial^{\lambda_t} a_{l_t, m_t}} \right)_K = \begin{cases} 0 & \text{if } \deg_{l_j, m_j} K < \lambda_j \text{ for some } j. \\ \dfrac{\prod_{j=1}^{t} (\deg_{l_j, m_j} K)! \phi(A)_K}{\prod_{j=1}^{t} (\deg_{l_j, m_j} K - \lambda_j)! a_{l_j, m_j}^{\lambda_j}} & \text{otherwise.} \end{cases}$$

For $A = 0$ we get

$$\left(\frac{\partial^{\lambda_1 + \cdots + \lambda_t} \phi(0)}{\partial^{\lambda_1} a_{l_1, m_1} \ldots \partial^{\lambda_t} a_{l_t, m_t}} \right)_K = \begin{cases} 0 & \text{if } \deg_{l_j, m_j} K \ne \lambda_j \text{ for some } j. \\ \prod_{j=1}^{t} (\deg_{l_j, m_j} K)! & \text{otherwise.} \end{cases}$$

Therefore

$$\frac{\partial^{\lambda_1 + \cdots + \lambda_t} \phi(0)}{\partial^{\lambda_1} a_{l_1, m_1} \ldots \partial^{\lambda_t} a_{l_t, m_t}} = (\lambda_1!) \cdots (\lambda_t!) \, e_J,$$

where $J \in \Lambda$ is characterized by

$$\deg_{l,m} J = \begin{cases} \lambda_j & \text{if } (l, m) = (l_j, m_j) \text{ for some } j. \\ 0 & \text{otherwise.} \end{cases}$$

Note that $d(J, I) = \lambda_1 + \cdots + \lambda_t$. Conversely every $J \in \Lambda$ with $d(J, I) = \lambda_1 + \cdots + \lambda_t$ can be obtained in this way. Therefore, for every $0 \leq s \leq d$, we have

$$\left\langle \frac{\partial^s \phi(0)}{\partial^{\lambda_1} a_{l_1, m_1} \ldots \partial^{\lambda_t} a_{l_t, m_t}} \mid 1 \leq l_1, \ldots, l_t \leq r, 1 \leq m_j \leq n_j, j = 1, \ldots, t \right\rangle = \langle e_J | d(J, I) = s \rangle,$$

and hence $T_{e_I}^s(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) = \langle e_J \mid d(I, J) \leq s \rangle$. $\hfill \square$

Corollary 2.6. *For any point $p \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ we have*

$$\dim T_p^s SV_{\boldsymbol{d}}^{\boldsymbol{n}} = \sum_{l=1}^{s} \sum_{\substack{0 \leq l_1 \leq d_1, \ldots, 0 \leq l_r \leq d_r \\ l_1 + \cdots + l_r = l}} \binom{n_1 + l_1 - 1}{l_1} \cdots \binom{n_r + l_r - 1}{l_r}$$

for any $0 \leq s \leq d$, while $T_p^s(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) = \mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}$ for any $s \geq d$.
In particular, for the Veronese variety V_d^n we have

$$\dim T_p^s V_d^n = n + \binom{n+1}{2} + \cdots + \binom{n+s-1}{s}$$

for any $0 \leq s \leq d$.

3. Osculating projections

In this section we study linear projections of Segre-Veronese varieties from their osculating spaces. We follow the notation introduced in the previous section.

We start by analyzing projections of Veronese varieties from osculating spaces at coordinate points. We consider a Veronese variety $V_d^n \subset \mathbb{P}^{N(n,d)}$, $d \geq 2$, and a coordinate point $e_{\underline{i}} = e_{(i, \ldots, i)} \in V_d^n$ for some $i \in \{0, 1, \ldots, n\}$. We write $(z_I)_{I \in \Lambda_{n,d}}$ for the coordinates in $\mathbb{P}^{N(n,d)}$. The linear projection

$$\pi_i : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$
$$(x_j) \mapsto (x_j)_{j \neq i}$$

induces the linear projection

$$(3.1) \qquad\qquad \Pi_i : V_d^n \dashrightarrow V_d^{n-1}$$
$$(z_I)_{I \in \Lambda_{n,d}} \mapsto (z_I)_{I \in \Lambda_{n,d} \mid i \notin I}$$

making the following diagram commute

$$
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{\quad \nu_d^n \quad} & V_d^n \subseteq \mathbb{P}^{N(n,d)} \\
\downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \Pi_i} \\
\mathbb{P}^{n-1} & \xrightarrow{\quad \nu_d^{n-1} \quad} & V_d^{n-1} \subseteq \mathbb{P}^{N(n-1,d)}
\end{array}
$$

Lemma 3.2. *Consider the projection of the Veronese variety $V_d^n \subset \mathbb{P}^{N(n,d)}$, $d \geq 2$, from the osculating space $T_{e_{\underline{i}}}^s$ of order s at the point $e_{\underline{i}} = e_{(i, \ldots, i)} \in V_d^n$, $0 \leq s \leq d-1$:*

$$\Gamma_i^s : V_d^n \dashrightarrow \mathbb{P}^{N(n,d,s)}.$$

Then Γ_i^s is birational for any $s \leq d-2$, while $\Gamma_i^{d-1} = \Pi_i$.

Proof. The case $s = d - 1$ follows from Proposition 2.5 and the expression in (3.1) above, observing that, for any $J \in \Lambda_{n,d}$,

$$d(J, (i, \ldots, i)) = d \Leftrightarrow i \notin J.$$

Since Γ_i^{d-2} factors through Γ_i^j for every $0 \leq j \leq d-3$, it is enough to prove birationality of Γ_i^{d-2}. We may assume that $i \neq 0$, and consider the collection of indices

$$J_0 = (0, \ldots, 0, 0), J_1 = (0, \ldots, 0, 1), \ldots, J_n = (0, \ldots, 0, n) \in \Lambda_{(n,d)}.$$

Note that $d(J_j, (i, \ldots, i)) \geq d - 1$ for any $j \in \{1, \ldots, n\}$. So we can define the linear projection

$$\gamma : \mathbb{P}^{N(n,d,s)} \dashrightarrow \mathbb{P}^n$$
$$(z_J)_{J \mid d(I,J) > d-2} \mapsto (z_{J_0}, \ldots, z_{J_n}).$$

The composition

$$\gamma \circ \Gamma_i^{d-2} \circ \nu_d^n : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$
$$(x_0 : \cdots : x_n) \mapsto (x_0^{d-1} x_0 : \cdots x_0^{d-1} x_n) = (x_0 : \cdots : x_n)$$

is the identity, and thus Γ_i^{d-2} is birational. $\qquad\square$

Now we turn to Segre-Veronese varieties. Let $SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ be a Segre-Veronese variety, and consider a coordinate point

$$e_I = e_{i_1}^{d_1} \otimes e_{i_2}^{d_2} \otimes \cdots \otimes e_{i_r}^{d_r} \in SV_{\boldsymbol{d}}^{\boldsymbol{n}},$$

with $0 \leq i_j \leq n_j$, $I = ((i_1, \ldots, i_1), \ldots, (i_r, \ldots, i_r))$. We write $(z_I)_{I \in \Lambda}$ for the coordinates in $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$. Recall from Proposition 2.5 that the linear projection of $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ from the osculating space $T_{e_I}^s$ of order s at e_I is given by

$$(3.3) \qquad \Pi_{T_{e_I}^s} : SV_{\boldsymbol{d}}^{\boldsymbol{n}} \dashrightarrow \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d},s)}$$
$$(z_J) \mapsto (z_J)_{J \in \Lambda \mid d(I,J) > s}$$

for every $s \leq d - 1$. In order to study the fibers of $\Pi_{T_{e_I}^s}$, we define auxiliary rational maps

$$\Sigma_l : SV_{\boldsymbol{d}}^{\boldsymbol{n}} \dashrightarrow \mathbb{P}^{N_l}$$

for each $l \in \{1, \ldots, r\}$ as follows. The map Σ_1 is the composition of the product map

$$\Gamma_{i_1}^{d_1-2} \times \prod_{j=2}^{r} \Pi_{i_j} : V_{d_1}^{n_1} \times \cdots \times V_{d_r}^{n_r} \dashrightarrow \mathbb{P}^{N(n_1,d_1,d_1-2)} \times \prod_{j=2}^{r} \mathbb{P}^{N(n_j-1,d_j)}$$

with the Segre embedding

$$\mathbb{P}^{N(n_1,d_1,d_1-2)} \times \prod_{j=2}^{r} \mathbb{P}^{N(n_j-1,d_j)} \hookrightarrow \mathbb{P}^{N_1}.$$

The other maps Σ_l, $2 \leq l \leq r$, are defined analogously. In coordinates we have:

$$(3.4) \qquad \Sigma_l : SV_{\boldsymbol{d}}^{\boldsymbol{n}} \dashrightarrow \mathbb{P}^{N_l},$$
$$(z_J) \mapsto (z_J)_{J \in \Lambda_l}$$

where $\Lambda_l = \{ J = (J^1, \ldots, J^r) \in \Lambda \mid d(J^l, (i_l, \ldots, i_l)) \geq d_l - 1 \text{ and } i_j \notin J^j \text{ for } j \neq l \}$.

Proposition 3.5. *Consider the projection of the Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ from the osculating space $\Pi_{T_{e_I}^s}$ of order s at the point $e_I = e_{i_1}^{d_1} \otimes e_{i_2}^{d_2} \otimes \cdots \otimes e_{i_r}^{d_r} \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}$, $0 \leq s \leq d - 1$:*

$$\Pi_{T_{e_I}^s} : SV_{\boldsymbol{d}}^{\boldsymbol{n}} \dashrightarrow \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d},s)}.$$

Then $\Pi_{T_{e_I}^s}$ *is birational for any* $s \leq d-2$, *while* $\Pi_{T_{e_I}^{d-1}}$ *fits in the following commutative diagram:*

$$
\begin{array}{ccc}
\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} & \xrightarrow{\ \sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}}\ } & SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subseteq \mathbb{P}^{N(\boldsymbol{d},\boldsymbol{n})} \\
{\scriptstyle \pi_{i_1} \times \cdots \times \pi_{i_r}} \downarrow & & \downarrow {\scriptstyle \Pi_{T_{e_I}^{d-1}}} \\
\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_r-1} & \xrightarrow{\ \sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n-1}}\ } & SV_{\boldsymbol{d}}^{\boldsymbol{n-1}} \subseteq \mathbb{P}^{N(\boldsymbol{d},\boldsymbol{n-1})}
\end{array}
$$

where $\boldsymbol{n-1} = (n_1-1, \ldots, n_r-1)$. *Furthermore, the closure of the fiber of* $\Pi_{T_{e_I}^{d-1}}$ *is the Segre-Veronese variety* $SV_{\boldsymbol{d}}^{1,\ldots,1}$.

Proof. The case $s = d-1$ follows from the expressions in (3.1) and (3.3), and Lemma 3.2.

Since $\Pi_{T_{e_I}^{d-2}}$ factors through $\Pi_{T_{e_I}^j}$ for every $0 \leq j \leq d-3$, it is enough to prove birationality of $\Pi_{T_{e_I}^{d-2}}$.

First note that $\Pi_{T_{e_I}^{d-2}}$ factors the map Σ_l for any $l = 1, \ldots, r$. This follows from the expressions in (3.3) and (3.4), observing that

$$
J = (J^1, \ldots, J^r) \in \Lambda_l \Rightarrow d(J, I) \geq d_l - 1 + \sum_{j \neq l} d_j = d - 1 > d - 2.
$$

We write $\tau_l : \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d},d-2)} \dashrightarrow \mathbb{P}^{N_l}$ for the projection making the following diagram commute:

$$
\begin{array}{ccc}
SV_{\boldsymbol{d}}^{\boldsymbol{n}} & \xdashrightarrow{\ \Pi_{T_{e_I}^{d-2}}\ } & \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d},d-2)} \\
& {\scriptstyle \Sigma_l} \searrow & \downarrow {\scriptstyle \tau_l} \\
& & \mathbb{P}^{N_l}
\end{array}
$$

Take a general point

$$
x \in \overline{\Pi_{T_{e_I}^{d-2}}\left(SV_{\boldsymbol{d}}^{\boldsymbol{n}}\right)} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d},d-2)},
$$

and set $x_l = \tau_l(x)$, $l = 1, \ldots, r$. Denote by $F \subset \mathbb{P}^{\boldsymbol{n}}$ the closure of the fiber of $\Pi_{T_{e_I}^{d-2}}$ over x, and by F_l the closure of the fiber of Σ_l over x_l. Let $y \in F \subset F_l$ be a general point, and write $y = \sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}}(y_1, \ldots, y_r)$, with $y_j \in \mathbb{P}^{n_j}, j = 1, \ldots, r$. By Lemma 3.2, F_l is the image under $\sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}}$ of

$$
\langle y_1, e_{i_1} \rangle \times \cdots \times \langle y_{l-1}, e_{i_{l-1}} \rangle \times y_l \times \langle y_{l+1}, e_{i_{l+1}} \rangle \times \cdots \times \langle y_r, e_{i_r} \rangle \subset \mathbb{P}^{\boldsymbol{n}},
$$

$l = 1, \ldots, r$. It follows that $F = \{y\}$, and so $\Pi_{T_{e_I}^{d-2}}$ is birational. $\qquad\square$

Next we study the case of linear projections from the span of several osculating spaces at coordinate points, and investigate when they are birational.

We start with the case of a Veronese variety $V_d^n \subset \mathbb{P}^{N(n,d)}$, with coordinate points $e_{\underline{i}} = e_{(i,\ldots,i)} \in V_d^n$, $i \in \{0, 1, \ldots, n\}$. For $m \in \{1, \ldots, n\}$, let $\boldsymbol{s} = (s_0, \ldots, s_m)$ be an $(m+1)$-uple of positive integers, and set $s = s_0 + \cdots + s_m$. Let $e_{\underline{i_0}}, \ldots, e_{\underline{i_m}} \in V_d^n$ be distinct coordinate points, and denote by $T_{e_{\underline{i_0}}, \ldots, e_{\underline{i_m}}}^{s_0, \ldots, s_m} \subset \mathbb{P}^{N(n,d)}$ the linear span $\left\langle T_{e_{\underline{i_0}}}^{s_0}, \ldots, T_{e_{\underline{i_m}}}^{s_m} \right\rangle$. By Proposition 2.5, the projection of V_d^n from $T_{e_{\underline{i_0}}, \ldots, e_{\underline{i_m}}}^{s_0, \ldots, s_m} \subset \mathbb{P}^{N(n,d)}$ is given by:

$$
(3.6) \qquad \Gamma_{e_{\underline{i_0}}, \ldots, e_{\underline{i_m}}}^{s_0, \ldots, s_m} : V_d^n \dashrightarrow \mathbb{P}^{N(n,d,\boldsymbol{s})},
$$
$$
(z_I)_{I \in \Lambda_{n,d}} \mapsto (z_J)_{J \in \Lambda_{n,d}^{\boldsymbol{s}}}
$$

whenever $\Lambda_{n,d}^{\boldsymbol{s}} = \{J \in \Lambda_{n,d} \mid d(J, (j, \ldots, j)) > s_j \text{ for } j = 0, \ldots, m\}$ is not empty.

Lemma 3.7. *Let the notation be as above, and assume that $d \geq 2$ and $0 \leq s_j \leq d - 2$ for $j = 0, \ldots, m$.*

(a) *If $n \leq d$ and $s \leq n(d-1) - 2$, then $\Gamma^{s_0,\ldots,s_m}_{e_{i_0},\ldots,e_{i_m}}$ is birational onto its image.*

(b) *If $n \leq d$ and $s = n(d-1) - 1$, then $\Gamma^{s_0,\ldots,s_m}_{e_{i_0},\ldots,e_{i_m}}$ is a constant map.*

(c) *If $n > d$, then $\Gamma^{d-2,\ldots,d-2}_{e_{i_0},\ldots,e_{i_n}}$ is birational onto its image.*

Proof. Assume that $n \leq d$ and $s \leq n(d-1) - 2$. In order to prove that $\Gamma^{s_0,\ldots,s_m}_{e_{i_0},\ldots,e_{i_m}}$ is birational, we will exhibit $J_0, \ldots, J_n \in \Lambda^{\boldsymbol{s}}_{n,d}$, and linear projection

$$(3.8) \qquad \gamma : \mathbb{P}^{N(n,d)} \dashrightarrow \mathbb{P}^n$$
$$(z_I)_{I \in \Lambda^{\boldsymbol{s}}_{n,d}} \mapsto (z_{J_j})_{j=0,\ldots,n}$$

such that the composition $\gamma \circ \Gamma^{s_0,\ldots,s_m}_{e_{i_0},\ldots,e_{i_m}} \circ \nu^n_d : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the standard Cremona transformation of $\mathbb{P}^n$. The d-uples $J_j \in \Lambda^{\boldsymbol{s}}_{n,d}$ are constructed as follows. Since $n \leq d$ we can take n of the coordinates of J_j to be $0, 1, \ldots, \widehat{j}, \ldots, n$. The condition $s \leq n(d-1) - 2$ assures that we can complete the J_j's by choosing $d - n$ common coordinates in such a way that, for every $i, j \in \{0, \ldots, n\}$, we have $d(J_j, (i, \ldots, i)) > s_i$ (i.e., J_j has at most $(d - s_i - 1)$ coordinates equal to to i). This gives $J_j \in \Lambda^{\boldsymbol{s}}_{n,d}$ for every $j \in \{0, \ldots, n\}$. For the linear projection (3.8) given by these J_j's, we have that $\gamma \circ \Gamma^{s_0,\ldots,s_m}_{e_{i_0},\ldots,e_{i_m}} \circ \nu^n_d : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the standard Cremona transformation of $\mathbb{P}^n$.

Now assume that $n \leq d$ and $s = n(d-1) - 1$. If $J \in \Lambda^{\boldsymbol{s}}_{n,d}$, then J has at most $d - s_i - 1$ coordinates equal to i for any $i \in \{0, \ldots, n\}$. Since

$$\sum_{j=0}^{n} (d - s_j - 1) = (n+1)(d-1) - s = d,$$

there is only one possibility for J, i.e., $\Lambda^{\boldsymbol{s}}_{n,d}$ has only one element, and so $\Gamma^{s_0,\ldots,s_m}_{e_{i_0},\ldots,e_{i_m}}$ is a constant map.

Finally, assume that $n > d$. Set $K_0 = \{0, \ldots, n - d\}$. For any $j \in K_0$, set

$$(J_{K_0})_j := (j, n - d + 1, \ldots, n),$$

and note that $d((J_{K_0})_j, (i, \ldots, i)) > d - 2$ for every $i \in \{0, \ldots, n\}$. Thus $(J_{K_0})_j \in \Lambda^{\boldsymbol{d-2}}_{n,d}$ for every $j \in K_0$. So we can define the linear projection

$$\gamma_{K_0} : \mathbb{P}^{N(n,d,\boldsymbol{d-2})} \dashrightarrow \mathbb{P}^{n-d}.$$
$$(z_I)_{I \in \Lambda^{\boldsymbol{d-2}}_{n,d}} \mapsto (z_{(J_{K_0})_j})_{j \in K_0}$$

The composition $\gamma_{K_0} \circ \Gamma^{d-2,\ldots,d-2}_{e_{i_0},\ldots,e_{i_n}} \circ \nu^n_d : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-d}$ is the linear projection given by

$$\gamma_{K_0} \circ \Gamma^{d-2,\ldots,d-2}_{e_{i_0},\ldots,e_{i_n}} \circ \nu^n_d : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-d}.$$
$$(x_i)_{i \in \{0,\ldots,n\}} \mapsto (x_i)_{i \in K_0}$$

Analogously, for each subset $K \subset \{0, \ldots, n\}$ with $n - d + 1$ distinct elements, we define a linear projection $\gamma_K : \mathbb{P}^{N(n,d,\boldsymbol{d-2})} \dashrightarrow \mathbb{P}^{n-d}$ such that the composition $\gamma_K \circ \Gamma^{d-2,\ldots,d-2}_{e_{i_0},\ldots,e_{i_n}} \circ \nu^n_d : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-d}$ is the linear projection given by

$$\gamma_K \circ \Gamma^{d-2,\ldots,d-2}_{e_{i_0},\ldots,e_{i_n}} \circ \nu^n_d : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-d}.$$
$$(x_i)_{i \in \{0,\ldots,n\}} \mapsto (x_i)_{i \in K}$$

This shows that $\Gamma^{d-2,\ldots,d-2}_{e_{i_0},\ldots,e_{i_n}}$ is birational. $\qquad \square$

The following is an immediate consequence of Lemma 3.7.

Corollary 3.9. *Let the notation be as above, and assume that $d \geq 2$. Then*

(a) $\Gamma^{d-2,\ldots,d-2}_{e_{i_0},\ldots,e_{i_{n-1}}}$ *is birational.*

(b) *If $n \geq 2$ then* $\Gamma^{d-2,\ldots,d-2,\min\{n,d\}-2}_{e_{i_0},\ldots,e_{i_n}}$ *is birational, while* $\Gamma^{d-2,\ldots,d-2,\min\{n,d\}-1}_{e_{i_0},\ldots,e_{i_n}}$ *is not.*

(c) *If $d \geq 3$ then* $\Gamma^{d-3,\ldots,d-3,\min\{2n,d\}-2}_{e_{i_0},\ldots,e_{i_n}}$ *is birational, while* $\Gamma^{d-3,\ldots,d-3,\min\{2n,d\}-1}_{e_{i_0},\ldots,e_{i_n}}$ *is not.*

Now we turn to Segre-Veronese varieties. Let $SV^{\boldsymbol{n}}_{\boldsymbol{d}} \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ be a Segre-Veronese variety, and write $(z_I)_{I \in \Lambda_{n,d}}$ for coordinates in $\mathbb{P}^{N(n,d)}$. Consider the coordinate points $e_{I_0}, e_{I_1}, \ldots, e_{I_{n_1}} \in SV^{\boldsymbol{n}}_{\boldsymbol{d}}$, where

$$I_j = ((j,\ldots,j),\ldots,(j,\ldots,j)) \in \Lambda.$$

(Recall that $n_1 \leq \cdots \leq n_r$). Let $\boldsymbol{s} = (s_0,\ldots,s_m)$ be an $(m+1)$-uple of positive integers, and set $s = s_0 + \cdots + s_m$. Denote by $T^{s_0,\ldots,s_m}_{e_{I_0},\ldots,e_{I_m}} \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ the linear span of the osculating spaces $T^{s_0}_{e_{I_0}},\ldots,T^{s_m}_{e_{I_m}}$. By Proposition 2.5, the projection of $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ from $T^{s_0,\ldots,s_m}_{e_{I_0},\ldots,e_{I_m}}$ is given by:

$$(3.10) \qquad \Pi_{T^{s_0,\ldots,s_m}_{e_{I_0},\ldots,e_{I_m}}} : SV^{\boldsymbol{n}}_{\boldsymbol{d}} \dashrightarrow \mathbb{P}^{N(\boldsymbol{d},\boldsymbol{n},\boldsymbol{s})}$$

$$(z_J)_{J \in \Lambda} \mapsto (z_J)_{J \in \Lambda^{\boldsymbol{s}}}$$

whenever $\Lambda^{\boldsymbol{s}} = \{ J \in \Lambda \mid d(I_j, J) > s_j \ \forall j \}$ is not empty.

Proposition 3.11. *Let the notation be as above, and assume that $r, d \geq 2$. Then the projection* $\Pi_{T^{d-2,\ldots,d-2}_{e_{I_0},\ldots,e_{I_{n_1-1}}}} : SV^{\boldsymbol{n}}_{\boldsymbol{d}} \dashrightarrow \mathbb{P}^{N(\boldsymbol{d},\boldsymbol{n},\boldsymbol{d-2})}$ *is birational.*

Proof. For each $l \in \{1,\ldots,r\}$, set

$$\Lambda_l = \left\{ J = (J^1,\ldots,J^r) \in \Lambda \ \middle| \ \begin{cases} 0,\ldots,n_1-1 \notin J^j \text{ if } j \neq l \\ d(J^l,(i,\ldots,i)) \geq d_l - 1 \ \forall i \in \{0,\ldots,n_1-1\} \end{cases} \right\},$$

and consider the linear projection

$$(3.12) \qquad \Sigma_l : SV^{\boldsymbol{n}}_{\boldsymbol{d}} \dashrightarrow \mathbb{P}^{N_l}.$$

$$(z_J)_{J \in \Lambda} \mapsto (z_J)_{J \in \Lambda_l}$$

Note that $\Lambda_l \subset \Lambda^{\boldsymbol{d-2}}$, and so there is a linear projection $\tau_l : \mathbb{P}^{N(\boldsymbol{d},\boldsymbol{n},\boldsymbol{d-2})} \dashrightarrow \mathbb{P}^{N_l}$ such that $\Sigma_l = \tau_l \circ \Pi_{T^{d-2,\ldots,d-2}_{p_0,\ldots,p_{n_1-1}}}$.

The restriction of $\Sigma_l \circ \sigma\nu^{\boldsymbol{n}}_{\boldsymbol{d}}$ to

$$\{pt\} \times \ldots \{pt\} \times \mathbb{P}^{n_l} \times \{pt\} \times \ldots \{pt\}$$

is isomorphic to the osculating projection

$$\Gamma^{d_l-2,\ldots,d_l-2}_{e_{i_0},\ldots,e_{i_{n_1-1}}} : V^{n_l}_{d_l} \dashrightarrow \mathbb{P}^{N(n_l,d_l,\boldsymbol{d_l-2})}.$$

This is birational by Corollary 3.9. For $j \neq l$, the restriction of $\Sigma_l \circ \sigma\nu^{\boldsymbol{n}}_{\boldsymbol{d}}$ to

$$\{pt\} \times \ldots \{pt\} \times \mathbb{P}^{n_j} \times \{pt\} \times \ldots \{pt\}$$

is isomorphic to the projection with center $\langle e_0,\ldots,e_{n_1-1} \rangle$. Arguing as in the last part of the proof of Proposition 3.5, we conclude that $\Pi_{T^{d-2,\ldots,d-2}_{e_{I_0},\ldots,e_{I_{n_1-1}}}}$ is birational. $\square$

4. Non-secant defectivity of Segre-Veronese varieties

In this section we explain how osculating projections can be used to establish non-secant defectivity of Segre-Veronese varieties. We start by recalling the definition of secant varieties and secant defectivity.

Definition 4.1 (Secant varieties)**.** Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety of dimension n. Consider the rational map $\alpha : X \times \cdots \times X \dashrightarrow \mathbb{G}(h-1, N)$ mapping h general points to their linear span $\langle x_1, \ldots, x_h \rangle$. Let

$$\Gamma_h(X) \subset X \times \cdots \times X \times \mathbb{G}(h-1, N)$$

be the closure of the graph of α, with the natural projection $\pi_2 : \Gamma_h(X) \to \mathbb{G}(h-1, N)$. Set

$$\mathcal{S}_h(X) = \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h-1, N).$$

Both $\Gamma_h(X)$ and $\mathcal{S}_h(X)$ are irreducible of dimension hn. Now consider the incidence variety

$$\mathcal{I}_h = \{(x, \Lambda) \mid x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N),$$

and the associated diagram

$$
\begin{array}{ccc}
 & \mathcal{I}_h & \\
{\scriptstyle \pi_h} \swarrow & & \searrow {\scriptstyle \psi_h} \\
\mathbb{P}^N & & \mathbb{G}(h-1, N).
\end{array}
$$

Note that $\psi_h : \mathcal{I}_h \to \mathbb{G}(h-1, N)$ is a $\mathbb{P}^{h-1}$-bundle. The variety

$$(\psi_h)^{-1}(\mathcal{S}_h(X)) \subset \mathcal{I}_h \subset \mathbb{P}^N \times \mathbb{G}(h-1, N)$$

is an $(hn + h - 1)$-dimensional variety with a $\mathbb{P}^{h-1}$-bundle structure over $\mathcal{S}_h(X)$.

The *h-secant variety* of X is the variety

$$\mathbb{S}ec_h(X) = \pi_h\big((\psi_h)^{-1}(\mathcal{S}_h(X))\big) \subset \mathbb{P}^N.$$

We say that X is *h-defective* if

$$\dim \mathbb{S}ec_h(X) < \min\{nh + h - 1, N\}.$$

Determining secant defectivity is a classical problem in algebraic geometry. The following characterization of secant defectivity in terms of tangential projections is due to Chiantini and Ciliberto.

Definition 4.2. Let $x_1, \ldots, x_h \in X \subset \mathbb{P}^N$ be general points, with tangent spaces $T_{x_i} X$. We say that the linear projection

$$\tau_{X,h} : X \subseteq \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_h}$$

with center $\langle T_{x_1} X, \ldots, T_{x_h} X \rangle$ is a *general h-tangential projection.*

Proposition 4.3. [CC02, Proposition 3.5] *Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety of dimension n, and $x_1, \ldots, x_h \in X$ general points. Assume that*

$$N - \dim(\langle T_{x_1} X, \ldots, T_{x_h} X \rangle) - 1 \geq n.$$

Then the general h-tangential projection $\tau_{X,h} : X \dashrightarrow X_h$ is generically finite if and only if X is not $(h+1)$-defective.

In general, however, it is hard to control the dimension of the fibers of tangential projections $\tau_{X,h}$ when h is large. In [MR16] a new strategy was introduced, based on degenerating the linear span of several tangent spaces $T_{x_i} X$ into a subspace contained in a single osculating space $T_x^k X$. The more points one can use in this degeneration, the better the method works. To count the number of points that can be used, the following notion was introduced in [MR16, Definition 4.6 and Assumption 4.3].

Definition 4.4. Let $X \subset \mathbb{P}^N$ be a projective variety.

We say that X has *m-osculating regularity* if the following property holds. Given general points $p_1, \ldots, p_m \in X$ and integer $k \geq 0$, there exists a smooth curve C and morphisms $\gamma_j : C \to X$,

$j = 2, \ldots, m$, such that $\gamma_j(t_0) = p_1$, $\gamma_j(t_\infty) = p_j$, and the flat limit T_0 in $\mathbb{G}(\dim(T_t), N)$ of the family of linear spaces

$$T_t = \left\langle T_{p_1}^k, T_{\gamma_2(t)}^k, \ldots, T_{\gamma_m(t)}^k \right\rangle, \, t \in C \backslash \{t_0\}$$

is contained in $T_{p_1}^{2k+1}$.

We say that X has *strong 2-osculating regularity* if the following property holds. Given general points $p, q \in X$ and integers $k_1, k_2 \geq 0$, there exists a smooth curve $\gamma : C \to X$ such that $\gamma(t_0) = p$, $\gamma(t_\infty) = q$ and the flat limit T_0 in $\mathbb{G}(\dim(T_t), N)$ of the family of linear spaces

$$T_t = \left\langle T_p^{k_1}, T_{\gamma(t)}^{k_2} \right\rangle, \, t \in C \backslash \{t_0\}$$

is contained in $T_p^{k_1+k_2+1}$.

The method of [MR16] goes as follows. If $X \subset \mathbb{P}^N$ has m-osculating regularity, one degenerates a general m-tangential projection into a linear projection with center contained in $T_p^3 X$. Then one further degenerates a general osculating projection $T_{p_1, \ldots, p_m}^{(3, \ldots, 3)}$ into a linear projection with center contained in $T_q^7 X$. By proceeding recursively, one degenerates a general h-tangential projection into a linear projection with center contained in a suitable linear span of osculating spaces, and then checks whether this projection is generically finite (see the proof of [MR16, Theorem 5.3] for details). So one gets the following criterion:

Theorem 4.5. *Let $X \subset \mathbb{P}^N$ be a projective variety having m-osculating regularity. Let $k_1, \ldots, k_l \geq 1$ be integers such that the general osculating projection $\Pi_{T_{p_1, \ldots, p_l}^{k_1, \ldots, k_l}}$ is generically finite. Then X is not h-defective for*

$$h \leq \left(\sum_{j=1}^l m^{\lfloor \log_2(k_j+1) \rfloor - 1} \right) + 1.$$

If X in addition has strong 2-osculating regularity, then this can be done even more effectively. To state the criterion of [MR16], we introduce a function $h_m : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ counting how many tangent spaces can be degenerated into a higher order osculating space in this way.

Definition 4.6. Let $m \geq 2$ be an integer. Define the function

$$h_m : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$$

as follows: $h_m(0) = 0$. For any $k \geq 1$, write

$$k + 1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon,$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geq 1$ are integers, and $\varepsilon \in \{0, 1\}$. Then set

$$h_m(k) = m^{\lambda_1 - 1} + m^{\lambda_2 - 1} + \cdots + m^{\lambda_l - 1}.$$

Theorem 4.7. *[MR16, Theorem 5.3] Let $X \subset \mathbb{P}^N$ be a projective variety having m-osculating regularity and strong 2-osculating regularity. Let $k_1, \ldots, k_l \geq 1$ be integers such that the general osculating projection $\Pi_{T_{p_1, \ldots, p_l}^{k_1, \ldots, k_l}}$ is generically finite. Then X is not h-defective for $h \leq \sum_{j=1}^l h_m(k_j) + 1$.*

We now prove our main result on non-defectivity of Segre-Veronese varieties. We follow the notation introduced in the previous sections.

Theorem 4.8. *Let the notation be as above. The Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not h-defective for*

$$h \leq n_1 h_{n_1+1}(d - 2) + 1.$$

Proof. We will show in Propositions 5.1 and 5.10 that the Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ has strong 2-osculating regularity, and $(n_1 + 1)$-osculating regularity. The result then follows immediately from Proposition 3.11 and Theorem 4.7. $\square$

Remark 4.9. Write
$$d - 1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_s} + \epsilon$$
with integers $\lambda_1 > \lambda_2 > \cdots > \lambda_s \geq 1$, $\epsilon \in \{0, 1\}$, so that $\lambda_1 = \lfloor \log_2(d-1) \rfloor$. By Theorem 4.8 $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not h-defective for
$$h \leq n_1((n_1 + 1)^{\lambda_1 - 1} + \cdots + (n_1 + 1)^{\lambda_s - 1}) + 1.$$
So we have that asymptotically $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not h-defective for
$$h \leq n_1^{\lfloor \log_2(d-1) \rfloor}.$$

Recall [CGG05, Proposition 3.2]: except for the Segre product $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, the Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not h-defective for $h \leq n_1 + 1$, independently of $\boldsymbol{d}$. In the following table, for a few values of d, we compute the highest value of h for which Theorem 4.8 gives non h-defectivity of $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$.

$d = d_1 + \cdots + d_r$	h
3	$n_1 + 1$
5	$n_1(n_1 + 1) + 1$
7	$n_1((n_1 + 1) + 1) + 1$
9	$n_1(n_1 + 1)^2 + 1$
11	$n_1((n_1 + 1)^2 + 1) + 1$
13	$n_1((n_1 + 1)^2 + n_1 + 1) + 1$
15	$n_1((n_1 + 1)^2 + (n_1 + 1) + 1) + 1$
17	$n_1(n_1 + 1)^3 + 1$

Remark 4.10. Note that the bound of Theorem 4.8 is sharp in some cases. For instance, it is well known that $SV_{(2,2)}^{(1,1)}$, $SV_{(1,1,2)}^{(1,1,1)}$, $SV_{(1,1,1,1)}^{(1,1,1,1)}$ are 3-defective, and $SV_{(1,1,1)}^{(2,2,2)}$ is 4-defective. On the other hand $SV_{(2,2)}^{(1,1)}$, $SV_{(1,1,2)}^{(1,1,1)}$, $SV_{(1,1,1,1)}^{(1,1,1,1)}$ are not 2-defective, and $SV_{(1,1,1)}^{(2,2,2)}$ is not 3-defective.

Remark 4.11. By Proposition 5.10, the Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$, $\boldsymbol{n} = (n_1 \leq \cdots \leq n_r)$, has $(n_1 + 1)$-osculating regularity. We do not know in general what is the highest osculating regularity of $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$. Better osculating regularity results would yield better bounds for h in Theorem 4.8. Determining the highest osculating regularity of a variety can be a difficult problem. There are examples of singular ruled surfaces that do not have 2-osculating regularity (see [MR16, Example 4.4]). In the smooth case, we exhibit below a family of surfaces which do not have 3-osculating regularity, and a surface having 2-osculating regularity but not 3-osculating regularity. We do not have an example of a smooth surface that does not have 2-osculating regularity.

Example 4.12. Consider the rational normal scroll $X_{(1,a)} \subset \mathbb{P}^{a+2}$ with $a \geq 3$, which is locally parametrized by
$$\begin{array}{cccc} \phi: & \mathbb{A}^2 & \longrightarrow & \mathbb{A}^{a+2} \\ & (u, \alpha) & \mapsto & (\alpha u^a, \alpha u^{a-1}, \ldots, \alpha u, \alpha, u). \end{array}$$
Note that $\frac{\partial^s \phi}{\partial \alpha^j \partial u^{s-j}} = 0$ for any $s \geq j \geq 2$, and $\alpha \frac{\partial^s \phi}{\partial u^{s-1} \partial \alpha} = \frac{\partial^{s-1} \phi}{\partial u^{s-1}}$ for any $s \geq 3$. Therefore, at a general point $p \in X_{(1,a)}$ we have
$$\dim(T_p^m X_{(1,a)}) = m + 2$$
for $2 \leq m \leq a$. On the other hand, by [DeP96, Lemma 4.10] if $a \geq 5$ we have that
$$\dim(\mathbb{S}ec_3(X_{(1,a)})) = 7.$$

Hence by Terracini's lemma [Rus03, Theorem 1.3.1] the span of three general tangent spaces of $X_{(1,a)}$ has dimension seven, and since $\dim(T_p^3 X_{(1,a)}) = 5$ then $X_{(1,a)}$ does not have 3-osculating regularity for $a \geq 5$. Furthermore, we can check via a direct computation that three general tangent

spaces to $X_{(1,4)}$ span the whole of $\mathbb{P}^6$. Therefore, we conclude that $X_{(1,a)}$ does not have 3-osculating regularity for $a \geq 4$.

An interesting question is whether $X_{(1,a)}$ has 2-osculating regularity. Using Maple computations, we have checked that $X_{(1,a)}$ has 2-osculating regularity for small values of a.

For example, consider the case $a = 4$. Let $(u_0, \alpha_0), (u_1, \alpha_1) \in \mathbb{A}^2$ be two general points, and $p_t = \phi(u_0 + tu_1, \alpha_0 + t\alpha_1)$. Let $[X_0 : \cdots : X_6]$ be the homogeneous coordinates on $\mathbb{P}^6$. A direct Maple computation shows that the span $T_t = \left\langle T^1_{p_0} X_{(1,4)}, T^1_{p_t} X_{(1,4)} \right\rangle$ is cut out by a polynomial $F(X_i, t)$ which evaluated in $t = 0$ gives that

$$(4.13) \qquad T_0 = \lim_{t \mapsto 0} T_t = \{X_0 - 4u_0 X_1 + 6u_0^2 X_2 - 4u_0^3 X_3 + u_0^4 X_4 = 0\}.$$

Note that the equation above does not depend on (u_1, α_1). Furthermore, via another standard Maple computation we can show that $T^3_{p_0} X_{(1,4)} \subset \mathbb{P}^6$ is the hyperplane define by the equation in (4.13), and hence $T_0 = T^3_{p_0} X_{(1,4)}$. Finally, since $T^m_{p_0} X_{(1,4)} = \mathbb{P}^6$ for any $m \geq 4$ we conclude that $X_{(1,4)}$ has 2-osculating regularity.

5. Osculating regularity of Segre-Veronese varieties

In this section we show that the Segre-Veronese variety $SV^{\boldsymbol{n}}_{\boldsymbol{d}} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ has strong 2-osculating regularity, and $(n_1 + 1)$-osculating regularity. We follow the notation introduced in the previous sections.

Proposition 5.1. *The Segre-Veronese variety $SV^{\boldsymbol{n}}_{\boldsymbol{d}} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ has strong 2-osculating regularity.*

Proof. Let $p, q \in SV^{\boldsymbol{n}}_{\boldsymbol{d}} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ be general points. There is a projective automorphism of $SV^{\boldsymbol{n}}_{\boldsymbol{d}} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ mapping p and q to the coordinate points e_{I_0} and e_{I_1}. These points are connected by the degree d rational normal curve defined by

$$\gamma([t : s]) = (se_{I_0} + te_{I_1})^{d_1} \otimes \cdots \otimes (se_{I_0} + te_{I_1})^{d_r}.$$

We work in the affine chart $(s = 1)$, and set $t = (t : 1)$. Given integers $k_1, k_2 \geq 0$, consider the family of linear spaces

$$T_t = \left\langle T^{k_1}_{e_{I_0}}, T^{k_2}_{\gamma(t)} \right\rangle, \, t \in \mathbb{C} \backslash \{0\}.$$

We will show that the flat limit T_0 of $\{T_t\}_{t \in \mathbb{C} \backslash \{0\}}$ in $\mathbb{G}(\dim(T_t), N(\boldsymbol{n}, \boldsymbol{d}))$ is contained in $T^{k_1+k_2+1}_{e_{I_0}}$.

We start by writing the linear spaces T_t explicitly. For $j = 1, \ldots, r$, we define the vectors

$$e_0^t = e_0 + te_1, e_1^t = e_1, e_2^t = e_2, \ldots, e_{n_j}^t = e_{n_j} \in V_j.$$

Given $I^j = (i_1, \ldots, i_{d_j}) \in \Lambda_{n_j, d_j}$, we denote by $e^t_{I^j} \in \operatorname{Sym}^{d_j} V_j$ the symmetric product $e^t_{i_1} \cdot \ldots \cdot e^t_{i_{d_j}}$. Given $I = (I^1, \ldots, I^r) \in \Lambda = \Lambda_{\boldsymbol{n},\boldsymbol{d}}$, we denote by $e^t_I \in \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ the point corresponding to

$$e^t_{I^1} \otimes \cdots \otimes e^t_{I^r} \in \operatorname{Sym}^{d_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{d_r} V_r.$$

By Proposition 2.5 we have

$$T_t = \left\langle e_I \mid d(I, I_0) \leq k_1; \, e^t_I \mid d(I, I_0) \leq k_2 \right\rangle, \, t \neq 0.$$

We shall write T_t in terms of the basis $\{e_J | J \in \Lambda\}$. Before we do so, it is convenient to introduce some additional notation.

Notation 5.2. Let $I \in \Lambda_{n,d}$, and write:

$$(5.3) \qquad I = (\underbrace{0, \ldots, 0}_{a \text{ times}}, \underbrace{1, \ldots, 1}_{b \text{ times}}, i_{a+b+1}, \ldots, i_d),$$

with $a, b \geq 0$ and $1 < i_{a+b+1} \leq \cdots \leq i_d$. Given $l \in \mathbb{Z}$, define $\delta^l(I) \in \Lambda_{n,d}$ as

$$\delta^l(I) = (\underbrace{0, \ldots, 0}_{a-l \text{ times}}, \underbrace{1, \ldots, 1}_{b+l \text{ times}}, i_{a+b+1}, \ldots, i_d),$$

provided that $-b \leq l \leq a$.

Given $I = (I^1, \ldots, I^r) \in \Lambda$ and $\boldsymbol{l} = (l_1, \ldots, l_r) \in \mathbb{Z}^r$, define

$$\delta^{\boldsymbol{l}}(I) = (\delta^{l_1}(I^1), \ldots, \delta^{l_r}(I^r)) \in \Lambda,$$

provided that each $\delta^{l_j}(I_j)$ is defined. Let $l \in \mathbb{Z}$. If $l \geq 0$, set

$$\Delta(I, l) = \left\{ \delta^{\boldsymbol{l}}(I) \,|\, \boldsymbol{l} = (l_1, \ldots, l_r), l_1, \ldots, l_r \geq 0, \, l_1 + \cdots + l_r = l \right\} \subset \Lambda.$$

If $l < 0$, set

$$\Delta(I, l) = \{ J \,|\, I \in \Delta(J, -l) \} \subset \Lambda.$$

Define also:

$$s_I^+ = \max\{ \, l \,; \Delta(I, l) \neq \emptyset \} \in \{0, \ldots, d\} = d - d(I, I_0),$$
$$s_I^- = \max\{ \, l \,; \Delta(I, -l) \neq \emptyset \} \in \{0, \ldots, d\} = d - d(I, I_1),$$
$$\Delta(I)^+ = \bigcup_{0 \leq l} \Delta(I, l) = \bigcup_{0 \leq l \leq s_I^+} \Delta(I, l), \text{ and}$$
$$\Delta(I)^- = \bigcup_{0 \leq l} \Delta(I, -l) = \bigcup_{0 \leq l \leq s_I^-} \Delta(I, -l).$$

Note that if $J \in \Delta(I, l)$, then $d(J, I) = |l|$, $d(J, I_0) = d(I, I_0) + l$, and $d(J, I_1) = d(I, I_1) - l$. Note also that, if $J \in \Delta(I)^- \cap \Delta(K)^+$, then $d(I, K) = d(I, J) + d(J, K)$.

Now we write each vector e_I^t with $d(I, I_0) < k_2$ in terms of the basis $\{e_J | J \in \Lambda\}$.

First, we consider the Veronese case. Let $I = (i_1, \ldots, i_d) \in \Lambda_{n,d}$ be as in (5.3), so that $s_I^+ = a$. We have:

$$e_I^t = (e_0^t)^a (e_1^t)^b e_{i_{a+b+1}}^t \cdots e_{i_d}^t = (e_0 + te_1)^a e_1^b e_{i_{a+b+1}} \cdots e_{i_d} =$$
$$= e_0^a e_1^b e_{i_{a+b+1}} \cdots e_{i_d} + t\binom{a}{1} e_0^{a-1} e_1^{b+1} e_{i_{a+b+1}} \cdots e_{i_d} + \cdots + t^a e_1^{b+a} e_{i_{a+b+1}} \cdots e_{i_d} =$$
$$= \sum_{l=0}^{a} t^l \binom{a}{l} e_0^{a-l} e_1^{b+l} e_{i_{a+b+1}} \cdots e_{i_d} = \sum_{l=0}^{a} t^l \binom{a}{l} e_{\delta^l(I)}.$$

In the Segre-Veronese case, for any $I = (I^1, \ldots, I^r) \in \Lambda$, we have

$$(5.4) \qquad e_I^t = \sum_{J = (J^1, \ldots, J^r) \in \Delta(I)^+} t^{d(I, J)} c_{(I, J)} e_J,$$

where $c_{(I,J)} = \binom{s_{I^1}^+}{d(I^1, J^1)} \cdots \binom{s_{I^r}^+}{d(I^r, J^r)}$. So we can rewrite the linear subspace T_t as

$$(5.5) \qquad T_t = \Big\langle e_I \mid d(I, I_0) \leq k_1; \sum_{J \in \Delta(I)^+} t^{d(I, J)} c_{(I, J)} e_J \mid d(I, I_0) \leq k_2 \Big\rangle.$$

For future use, we define the set indexing coordinates z_I that do not vanish on some generator of T_t:

$$\Delta = \{ I \mid d(I, I_0) \leq k_1 \} \bigcup \left(\bigcup_{d(I, I_0) \leq k_2} \Delta(I)^+ \right) \subset \Lambda.$$

On the other hand, by Proposition 2.5, we have

$$T_{e_{I_0}}^{k_1 + k_2 + 1} = \langle e_I \mid d(I, I_0) \leq k_1 + k_2 + 1 \rangle = \{ z_I = 0 \mid d(I, I_0) > k_1 + k_2 + 1 \}.$$

In order to prove that $T_0 \subset T_{e_{I_0}}^{k_1+k_2+1}$, we will define a family of linear subspaces L_t whose flat limit at $t = 0$ is $T_{e_{I_0}}^{k_1+k_2+1}$, and such that $T_t \subset L_t$ for every $t \neq 0$. (Note that we may assume that $k_1 + k_2 \leq d - 2$, for otherwise $T_{e_{I_0}}^{k_1+k_2+1} = \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$.) For that, it is enough to exhibit, for each pair $(I, J) \in \Lambda^2$ with $d(I, I_0) > k_1 + k_2 + 1$, a polynomial $f(t)_{(I,J)} \in \mathbb{C}[t]$ so that the hyperplane $(H_I)_t \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ defined by

$$z_I + t \left(\sum_{J \in \Lambda, \ J \neq I} f(t)_{(I,J)} z_J \right) = 0$$

satisfies $T_t \subset (H_I)_t$ for every $t \neq 0$. If $I \notin \Delta$, then we can take $f(t)_{(I,J)} \equiv 0 \ \forall J \in \Lambda$. So from now on we assume that $I \in \Delta$. We claim that it is enough to find a hyperplane of type

$$(5.6) \qquad F_I = \sum_{J \in \Delta(I)^-} t^{d(I,J)} c_J z_J = 0,$$

with $c_J \in \mathbb{C}$ for $J \in \Delta(I)^-$, $c_I \neq 0$, and such that $T_t \subset (F_I = 0)$ for $t \neq 0$. Indeed, once we find such F_I's, we can take $(H_I)_t$ to be

$$z_I + \frac{t}{c_I} \left(\sum_{J \in \Delta(I)^-, \ J \neq I} t^{d(J,I)-1} c_J z_J \right) = 0.$$

In (5.6), there are $|\Delta(I)^-|$ indeterminates c_J. Let us analyze what conditions we get by requiring that $T_t \subseteq (F_I = 0)$ for $t \neq 0$. For any e_K^t with non-zero coordinate z_I, we have $I \in \Delta(K)^+$, and so $K \in \Delta(I)^-$. Given $K \in \Delta(I)^-$ we have

$$F_I(e_K^t) \overset{(5.4)}{=} F_I \left(\sum_{J \in \Delta(K)^+} t^{d(K,J)} c_{(K,J)} e_J \right) =$$

$$\overset{(5.6)}{=} \sum_{J \in \Delta(I)^- \cap \Delta(K)^+} t^{d(I,K)-d(K,J)} c_J \left(t^{d(K,J)} c_{(K,J)} \right) = t^{d(I,K)} \left[\sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_{(K,J)} c_J \right].$$

Thus:

$$F_I(e_K^t) = 0 \ \forall \, t \neq 0 \Leftrightarrow \sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_{(K,J)} c_J = 0.$$

This is a linear condition on the coefficients c_J, with $J \in \Delta(I)^-$. Therefore

$$(5.7) \qquad T_t \subset (F_I = 0) \text{ for } t \neq 0 \Leftrightarrow \begin{cases} F_I(e_L) = 0 & \forall L \in \Delta(I)^- \cap B[I_0, k_1] \\ F_I(e_K^t) = 0 \ \forall t \neq 0 & \forall K \in \Delta(I)^- \cap B[I_0, k_2] \end{cases}$$

$$\Leftrightarrow \begin{cases} c_L = 0 & \forall L \in \Delta(I)^- \cap B[I_0, k_1] \\ \displaystyle\sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_{(K,J)} c_J = 0 & \forall K \in \Delta(I)^- \cap B[I_0, k_2], \end{cases}$$

where $B[J, u] = \{K \in \Lambda \mid d(J, K) \leq u\}$. Set

$$c = \left| \Delta(I)^- \cap B[I_0, k_1] \right| + \left| \Delta(I)^- \cap B[I_0, k_2] \right|.$$

The problem is now reduced to finding a solution $(c_J)_{J \in \Delta(I)^-}$ of the linear system given by the c equations (5.7) with $c_I \neq 0$.

In the following we write for short $s = s_I^-$, $\bar{s} = s_I^+$ and $D = d(I, I_0) > k_1 + k_2 + 1$. We want to find $s + 1$ complex numbers $c_I = c_0, c_1, \ldots, c_s$ satisfying the following conditions

$$(5.8) \quad \begin{cases} c_j = 0 & \forall j = s, \ldots, D - k_1 \\ \displaystyle\sum_{l=0}^{d(I,K)} \left(c_{d(I,K)-l} \sum_{J \in \Delta(I)^- \cap \Delta(K,l)} c_{(K,J)} \right) = 0 & \forall K \in \Delta(I)^- \cap B[I_0, k_2]. \end{cases}$$

For $0 \le l \le d(I, K)$, we have

$$\sum_{J \in \Delta(I)^- \cap \Delta(K,l)} c_{(K,J)} = \sum_{J \in \Delta(I)^- \cap \Delta(K,l)} \binom{s_{K^1}^+}{d(K^1, J^1)} \cdots \binom{s_{K^r}^+}{d(K^r, J^r)} = \sum_{\substack{\boldsymbol{l}=(l_1,\ldots,l_r) \\ 0 \le l_1,\ldots,l_r \\ l_1 + \cdots + l_r = l}} \binom{s_{K^1}^+}{l_1} \cdots \binom{s_{K^r}^+}{l_r} =$$

$$= \binom{s_{K^1}^+ + \cdots + s_{K^r}^+}{l} = \binom{s_K^+}{l} = \binom{s_I^+ + d(I, K)}{l}.$$

Thus the system (5.8) can be written as

$$\begin{cases} c_j = 0 & \forall j = s, \ldots, D - k_1 \\ \displaystyle\sum_{k=0}^{j} \binom{\bar{s} + j}{j - k} c_k = 0 & \forall j = s, \ldots, D - k_2, \end{cases}$$

that is

$$(5.9) \quad \begin{cases} c_s = 0 \\ \vdots \\ c_{D-k_1} = 0 \end{cases} \quad \begin{cases} \binom{\bar{s}+s}{0} c_s + \binom{\bar{s}+s}{1} c_{s-1} + \cdots + \binom{\bar{s}+s}{s} c_0 = 0 \\ \vdots \\ \binom{\bar{s}+D-k_2}{0} c_{D-k_2} + \binom{\bar{s}+D-k_2}{1} c_{D-k_2-1} + \cdots + \binom{\bar{s}+D-k_2}{D-k_2} c_0 = 0. \end{cases}$$

We will show that the linear system (5.9) admits a solution with $c_0 \ne 0$. If $s < D - k_2$, then the system (5.9) reduces to $c_s = \cdots = c_{D-k_1} = 0$. In this case we can take $c_0 = 1, c_1 = \ldots, c_s = 0$.

From now on assume that $s \ge D - k_2$. Since $c_s = \cdots = c_{D-k_1} = 0$ in (5.9), we are reduced to checking that the following system admits a solution $(c_i)_{0 \le i \le D-k_1+1}$ with $c_0 \ne 0$:

$$\begin{cases} \binom{\bar{s}+s}{s-(D-k_1+1)} c_{D-k_1+1} + \binom{\bar{s}+s}{s-(D-k_1)} c_{D-k_1} + \cdots + \binom{\bar{s}+s}{s} c_0 = 0 \\ \vdots \\ \binom{\bar{s}+D-k_2}{k_1-1-k_2} c_{D-k_1+1} + \binom{\bar{s}+D-k_2}{k_1-k_2} c_{D-k_1} + \cdots + \binom{\bar{s}+D-k_2}{D-k_2} c_0 = 0. \end{cases}$$

Since $s \le D$ and $D > k_1 + k_2 + 1$, we have $s - D + k_2 + 1 < D - k_1 + 1$. Therefore, it is enough to check that the $(s - D + k_2 + 1) \times (D - k_1 + 1)$ matrix

$$M = \begin{pmatrix} \binom{\bar{s}+s}{s-(D-k_1+1)} & \binom{\bar{s}+s}{s-(D-k_1)} & \cdots & \binom{\bar{s}+s}{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{\bar{s}+D-k_2}{k_1-1-k_2} & \binom{\bar{s}+D-k_2}{k_1-k_2} & \cdots & \binom{\bar{s}+D-k_2}{D-k_2-1} \end{pmatrix}$$

has maximal rank. So it is enough to show that the $(s - D + k_2 + 1) \times (s - D + k_2 + 1)$ submatrix of M

$$M' = \begin{pmatrix} \binom{\overline{s}+s}{s-(s-D+k_2+1)} & \binom{\overline{s}+s}{s-(s-D+k_2)} & \cdots & \binom{\overline{s}+s}{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{\overline{s}+D-k_2}{D-k_2-(s-D+k_2+1)} & \binom{\overline{s}+D-k_2}{D-k_2-(s-D+k_2)} & \cdots & \binom{\overline{s}+D-k_2}{D-k_2-1} \end{pmatrix} =$$

$$= \begin{pmatrix} \binom{\overline{s}+s}{\overline{s}+s+1-D+k_2} & \binom{\overline{s}+s}{\overline{s}+s-D+k_2} & \cdots & \binom{\overline{s}+s}{\overline{s}+1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{\overline{s}+D-k_2}{\overline{s}+s+1-D+k_2} & \binom{\overline{s}+D-k_2}{\overline{s}+s-D+k_2} & \cdots & \binom{\overline{s}+D-k_2}{\overline{s}+1} \end{pmatrix}$$

has non-zero determinant. To conclude, observe that the determinant of M' is equal to the determinant of the matrix of binomial coefficients

$$M'' := \left(\binom{i}{j} \right)_{\substack{\overline{s}+D-k_2 \le i \le \overline{s}+s \\ \overline{s}+1 \le j \le \overline{s}+s+1-D+k_2}}.$$

Since $D - k_2 > k_1 + 1 \ge 1$, $\det(M') = \det(M'') \ne 0$ by [GV85, Corollary 2]. $\qquad\square$

Proposition 5.10. *The Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ has $(n_1 + 1)$-osculating regularity.*

Proof. We follow the same argument and computations as in the proof of Proposition 5.1. Given general points $p_0, \dots, p_{n_1} \in SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$, we may apply a projective automorphism of $SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subseteq \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ and assume that $p_j = e_{I_j}$ for every j. Each p_j, $j \ge 1$, is connected to p_0 by the degree d rational normal curve defined by

$$\gamma_j([t:s]) = (se_0 + te_j)^{d_1} \otimes \cdots \otimes (se_0 + te_j)^{d_r}.$$

We work in the affine chart $(s = 1)$, and set $t = (t : 1)$. Given $k \ge 0$, consider the family of linear spaces

$$T_t = \left\langle T_{p_0}^k, T_{\gamma_1(t)}^k, \dots, T_{\gamma_{n_1}(t)}^k \right\rangle, \ t \in \mathbb{C}\backslash\{0\}.$$

We will show that the flat limit T_0 of $\{T_t\}_{t \in \mathbb{C}\backslash\{0\}}$ in $\mathbb{G}(\dim(T_t), N(\boldsymbol{n},\boldsymbol{d}))$ is contained in $T_{p_0}^{2k+1}$.

We start by writing the linear spaces T_t explicitly in terms of the basis $\{e_J | J \in \Lambda\}$. As in the proof of Proposition 5.1, it is convenient to introduce some additional notation. Given $I \in \Lambda_{n,d}$, we define $\delta_j^l(I), l \ge 0$, as in Notation 5.2, with the only difference that this time we substitute 0's with j's instead of 1's. Similarly, for $I = (I^1, \dots, I^r) \in \Lambda$, $\boldsymbol{l} = (l_1, \dots, l_r) \in \mathbb{Z}^r$, and $l \in \mathbb{Z}$, we define the sets $\Delta(I, l)_j, \Delta(I)_j^+, \Delta(I)_j^- \subset \Lambda$, and the integers $s(I)_j^+, s(I)_j^- \in \{0, \dots, d\}$.

For $j = 1, \dots, r$, we define the vectors

$$e_0^{j,t} = e_0 + te_j, e_1^{j,t} = e_1, e_2^{j,t} = e_2, \dots, e_{n_j}^{j,t} = e_{n_j} \in V_j.$$

Given $I^l = (i_1, \dots, i_{d_j}) \in \Lambda_{n_j, d_j}$, we denote by $e_{I^j}^{j,t} \in \mathrm{Sym}^{d_j} V_j$ the symmetric product $e_{i_1}^{j,t} \cdot \ldots \cdot e_{i_{d_j}}^{j,t}$. Given $I = (I^1, \dots, I^r) \in \Lambda = \Lambda_{\boldsymbol{n},\boldsymbol{d}}$, we denote by $e_I^{j,t} \in \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ the point corresponding to

$$e_{I^1}^{j,t} \otimes \cdots \otimes e_{I^r}^{j,t} \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_r} V_r.$$

By Proposition 2.5 we have

$$T_t = \left\langle e_I \mid d(I, I_0) \le k; \ e_I^{j,t} \mid d(I, I_0) \le k, j = 1, \dots, n_1 \right\rangle, \ t \ne 0.$$

Now we write each vector $e_I^{j,t}$, with $I = (I^1, \dots, I^r) \in \Lambda$ such that $d(I, I_0) \le k$, in terms of the basis $\{e_J | J \in \Lambda\}$:

$$e_I^{j,t} = \sum_{J=(J^1, \dots, J^r) \in \Delta(I)_j^+} t^{d(I,J)} c_{(I,J)} e_J$$

where $c_{(I,J)} = \binom{s(I^1)_j^+}{d(I^1,J^1)} \cdots \binom{s(I^r)_j^+}{d(I^r,J^r)}$. So we can rewrite the linear subspace T_t as

$$T_t = \left\langle e_I \mid d(I,I_0) \le k; \sum_{J \in \Delta(I)_j^+} t^{d(I,J)} c_{(I,J)} e_J \mid d(I,I_0) \le k, \ j = 1, \ldots, n_1 \right\rangle,$$

and define the set

$$\Delta = \bigcup_{1 \le j \le n_1} \bigcup_{d(J,I_0) \le k} \Delta(J)_j^+ \subset \Lambda.$$

On the other hand, by Proposition 2.5, we have

$$T_{p_0}^{2k+1} = \langle e_I \mid d(I,I_0) \le 2k+1 \rangle = \{z_I = 0 \mid d(I,I_0) > 2k+1\}.$$

As in the proof of Proposition 5.1, in order to prove that $T_0 \subset T_{p_0}^{2k+1}$, it is enough to exhibit, for each $I \in \Delta$ with $d(I,I_0) > 2k+1$, a family of hyperplanes of the form

$$(5.11) \qquad \left(F_I = \sum_{J \in \Gamma(I)} t^{d(I,J)} c_J z_J = 0 \right)$$

such that $T_t \subset (F_I = 0)$ for $t \ne 0$, and $c_I \ne 0$. Here $\Gamma(I) \subset \Lambda$ is a suitable subset to be defined later. Let $I \in \Delta$ be such that $d(I,I_0) > 2k+1$. We claim that there is a unique j such that

$$(5.12) \qquad I \in \bigcup_{d(J,I_0) \le k} \Delta(J)_j^+.$$

Indeed, assume that $I \in \Delta(J,l)_i$ and $I \in \Delta(K,m)_j$, with $d(J,I_0), d(K,I_0) \le k$. If $i \ne j$, then we must have

$$d(J,I_0) \ge m \quad \text{and} \quad d(K,I_0) \ge l.$$

But then $d(I,I_0) = d(J,I_0) + l \le d(J,I_0) + d(K,I_0) \le 2k$, contradicting the assumption that $d(I,I_0) > 2k+1$. Let J and j be such that $d(J,I_0) \le k$ and $I \in \Delta(J)_j^+$. Note that $d(I,I_0) - s(I)_j^- = d(J,I_0) - s(J)_j^- \le k$, and hence $k+1 - d(I,I_0) + s(I)_j^- > 0$. We set $D = d(I,I_0)$ and define

$$(5.13) \qquad \Gamma(I) = \bigcup_{0 \le l \le k+1-D+s(I)_j^-} \Delta(I,-l)_j \subset \Lambda.$$

This is the set to be used in (5.11). First we claim that

$$(5.14) \qquad J \in \Gamma(I) \Rightarrow J \notin \bigcup_{\substack{1 \le i \le n_1 \\ i \ne j}} \bigcup_{d(I,I_0) \le k} \Delta(I)_i^+.$$

Indeed, assume that $J \in \Delta(I,-l)_j$ with $0 \le l \le k+1-D+s(I)_j^-$, and $J \in \Delta(K)_i^+$ for some K with $d(K,I_0) \le k$. If $i \ne j$, then

$$s(K)_j^- = s(J)_j^- = s(I)_j^- - l \ge D - (k+1) > k,$$

contradicting the assumption that $d(K,I_0) \le k$. Therefore, if F_I is as in (5.11) with $\Gamma(I)$ as in (5.13), then we have

$$\left\langle e_I \mid d(I,I_0) \le k; \sum_{J \in \Delta(I)_i^+} t^{d(I,J)} c_{(I,J)} e_J \mid d(I,I_0) \le k, \ i = 1, \ldots, n_1, i \ne j \right\rangle \subset (F_I = 0), t \ne 0,$$

and thus

$$T_t \subset (F_I = 0), t \ne 0 \iff \left\langle \sum_{J \in \Delta(I)_j^+} t^{d(I,J)} c_{(I,J)} e_J \mid d(I,I_0) \le k \right\rangle \subset (F_I = 0), t \ne 0.$$

The same computations as in the proof of Proposition 5.1 yield

$$(5.15) \qquad T_t \subset (F_I = 0), t \neq 0 \iff \sum_{J \in \Delta(K)_j^+ \cap \Gamma(I)} c_J c_{(K,J)} = 0 \quad \forall K \in \Delta(I)_j^- \cap B[I_0, k].$$

So the problem is reduced to finding a solution $(c_J)_{J \in \Gamma(I)}$ for the linear system (5.15) such that $c_I \neq 0$. We set $c_J = c_{d(I,J)}$ and reduce, as in the proof of Proposition 5.1, to the linear system

$$(5.16) \qquad \sum_{l=0}^{k+1-D+s(I)_j^-} \binom{d-i}{D-l-i} c_l = 0, \quad D - s(I)_j^- \leq i \leq k$$

in the variables $c_0, \ldots, c_{k+1-D+s(I)_j^-}$. The argument used in the end of Proposition 5.1 shows that the linear system (5.16) admits a solution with $c_0 \neq 0$. $\qquad \square$

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