MAXIMAL RANK OF SPACE CURVES IN THE RANGE A

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Abstract. We prove the following statement, which has been conjectured since 1985: *There exists a constant K such that for all natural numbers d, g with $g \leq Kd^{3/2}$ there exists an irreducible component of the Hilbert scheme of $\mathbb{P}^3$ whose general element is a smooth, connected curve of degree d and genus g of maximal rank.*

1. Introduction

The postulation of algebraic space curves has been the object of wide interest in the last thirty years (see for instance [1], [2], [24]). In particular, the following Conjecture was stated in 1985 in [2], p. 2 (see also [3], §6, Problem 4):

Conjecture 1. *There exists a constant K such that for all natural numbers d, g with $g \leq Kd^{3/2}$ there exists an irreducible component of the Hilbert scheme of $\mathbb{P}^3$ whose general element is a smooth, connected curve of degree d and genus g of maximal rank.*

Here we consider smooth and connected curves X with $h^1(\mathcal{I}_X(m)) = 0$, $h^0(\mathcal{I}_X(m-1)) = 0$, $\deg(X) = d$, $g(X) = g$ and $h^1(\mathcal{O}_X(m-2)) = 0$ (hence of maximal rank by Castelnuovo-Mumford regularity). Since $h^1(\mathcal{I}_X(m)) = 0$ and $h^1(\mathcal{O}_X(m)) = 0$, we have

$$1 + md - g \leq \binom{m+3}{3} \tag{1}$$

Let $d(m, g)_{\max}$ be the maximal integer d such that (1) is satisfied, i.e. set $d(m, g)_{\max} := \lfloor \binom{m+3}{3} + g - 1)/m \rfloor$. Since $h^0(\mathcal{I}_X(m-1)) = 0$ and $h^1(\mathcal{O}_X(m-1)) = 0$, we have

$$1 + (m-1)d - g \geq \binom{m+2}{3} \tag{2}$$

Let $d(m, g)_{\min}$ be the minimal integer d such that (2) is satisfied, i.e. set $d(m, g)_{\min} := \lceil \binom{m+2}{3} + g - 1)/(m-1) \rceil$.

For every integer $s > 0$ define the number $p_a(C_s) := s(s+1)(2s-5)/6+1$ (which is going to to be the genus of the curve C_s to be introduced later in Section 2). For

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all positive integers $m \geq 3$ set

$$\begin{aligned}\varphi(m) &= p_a(C_{\lfloor m/\sqrt{20}\rfloor-4}) + p_a(C_{\lfloor m/\sqrt{20}\rfloor-5}) \\ &= \frac{(\lfloor m/\sqrt{20}\rfloor-4)(\lfloor m/\sqrt{20}\rfloor-3)(2\lfloor m/\sqrt{20}\rfloor-13)}{6}+1 \\ &\quad + \frac{(\lfloor m/\sqrt{20}\rfloor-5)(\lfloor m/\sqrt{20}\rfloor-4)(2\lfloor m/\sqrt{20}\rfloor-15)}{6}+1.\end{aligned}$$

For any smooth curve $X \subset \mathbb{P}^3$ let N_X denote the normal bundle of X in $\mathbb{P}^3$. If $h^1(N_X) = 0$, then X is a smooth point of the Hilbert scheme of $\mathbb{P}^3$ and this Hilbert scheme has the expected dimension $h^0(N_X)$ at X.

Our main result is the following:

Theorem 1. *For every integer $m \geq 3$ and every (d, g) with $17052 \leq g \leq \varphi(m)$ and $d(m, g)_{\min} \leq d \leq d(m, g)_{\max}$ there exists a component of the Hilbert scheme of curves in $\mathbb{P}^3$ of genus g and degree d, whose general element X is smooth and satisfies $h^0(\mathcal{I}_X(m-1)) = 0$, $h^1(\mathcal{I}_X(m)) = 0$, $h^1(\mathcal{O}_X(m-2)) = 0$, and $h^1(N_X(-1)) = 0$.*

As an application of Theorem 1 we prove Conjecture 1. Indeed, if $g = 0$ we have just to quote [17]. Next, if $0 < g < 17052$ we may choose $K > 0$ such that $g \geq K(g+3)^{3/2}$. Hence from $K(g+3)^{3/2} \leq g \leq Kd^{3/2}$ we get $d \geq g+3$ and we are done by [1]. Finally, if $g \geq 17052$ we have the following:

Corollary 1. *Let $K = \frac{2}{3}\left(\frac{1}{10}\right)^{3/2}$ and $\epsilon = \frac{11}{20}+4\left(\frac{1}{20}\right)^{3/2}$. If $17052 \leq g \leq Kd^{3/2}-6\epsilon d$ then there exists an irreducible component of the Hilbert scheme of $\mathbb{P}^3$ whose general element X is a smooth, connected curve of degree d and genus g of maximal rank and with $h^1(N_X(-1)) = 0$.*

The constant K in Corollary 1 is certainly not optimal, but the exponent $d^{3/2}$ is sharp among the curves with $h^1(N_X) = 0$ (see [11], [25, Corollaire 5.18] and [18, II.3.6] for the condition $h^1(N_X(-2)) = 0$, [18, II.3.7] and [27] for the condition $h^1(N_X(-1)) = 0$, and [18, II.3.8] for the condition $h^1(N_X) = 0$).

If X is as in Theorem 1, then by Castelnuovo-Mumford regularity we have $h^1(\mathcal{I}_X(t)) = 0$ for all $t > m$ and the homogeneous ideal of X is generated by forms of degree m and degree $m+1$. A smooth curve $Y \subset \mathbb{P}^3$ with $h^0(\mathcal{I}_Y(m-1)) = 0$, $\frac{m^2+4m+6}{6} \leq \deg(Y) < \frac{m^2+4m+6}{3}$ and maximal genus among the curves with $h^0(\mathcal{I}_Y(m-1)) = 0$ satisfies $h^1(\mathcal{O}_Y(m-1)) = 0$ ([15, proof of Theorem 3.3 at p. 97]). In the statement of Theorem 1 we claim one shift more, namely, $h^1(\mathcal{O}_X(m-2)) = 0$, in order to apply Castelnuovo-Mumford regularity to X.

We describe here one of the main differences with respect to [17, 1, 2]. Fix integers d, g as in Theorem 1 or Corollary 1. Suppose that we have constructed two irreducible and generically smooth components W_1, W_2 of the Hilbert scheme of smooth space curves of degree d and genus g. Suppose also that we have proved the existence of $Y_1 \in W_1$ and $Y_2 \in W_2$ with $h^0(\mathcal{I}_{Y_2}(m-1)) = 0$, $h^1(\mathcal{I}_{Y_1}(m)) = 0$ and $h^1(N_{Y_i}) = h^1(\mathcal{O}_{Y_i}(m-3)) = 0$, $i = 1, 2$. If $W_1 = W_2$, then by the semicontinuity theorem for cohomology and Castelnuovo-Mumford regularity a general $X \in W_1$ satisfies $h^0(\mathcal{I}_X(m-1)) = 0$, $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq m$ and $h^1(N_X) = 0$. In particular a general element of W_1 has maximal rank. But we need to know that $W_1 = W_2$. If $d \geq g+3$ it was not known at that time that the Hilbert scheme of smooth space curves of degree d and genus g is irreducible ([6]), but it was obvious since at least Castelnuovo that its part parametrizing the non-special curves is irreducible (modulo the irreducibility of the moduli scheme $\mathcal{M}_g$ of genus g smooth curves). When $d < g+3$, the Hilbert scheme of smooth space curves of degree d and genus g is often reducible, even in ranges with d/g not small ([5, 19, 20, 21, 22, 23]). In [2] when $d \geq (g+2)/2$ we defined a certain irreducible component $Z(d,g)$ of the Hilbert scheme of smooth space curves of degree d and genus g and (under far stronger assumptions on d,g) we were able find Y_1 and Y_2 with $W_1 = W_2 = Z(d,g)$. Several pages of Section 5 are devoted to solve this problem.

We work over an algebraically closed field of characteristic zero.

2. Preliminaries

2.1. **The curves $C_{t,k}$.** For each locally Cohen-Macaulay curve $C \subset \mathbb{P}^3$ the index of speciality $e(C)$ of C is the maximal integer e such that $h^1(\mathcal{O}_C(e)) \neq 0$.

Fix an integer $s > 0$. Let $C_s \subset \mathbb{P}^3$ be any curve fitting in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-s-1) \to (s+1)\mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{I}_{C_s} \to 0 \tag{3}$$

Each C_s is arithmetically Cohen-Macaulay and in particular $h^0(\mathcal{O}_{C_s}) = 1$. By taking the Hilbert function in (3) we get $\deg(C_s) = s(s+1)/2$, $p_a(C_s) = s(s+1)(2s-5)/6 + 1$ and $e(C_s) = s-3$. Hence $h^i(\mathcal{I}_{C_s}(s-1)) = 0$, $i = 0,1,2$. By taking $d := \deg(C_s)$ we get $p_a(C_s) = 1 + d(s-1) - \binom{s+2}{3} = G_A(d,s)$. The set of all curves fitting in (3) is an irreducible variety and its general member is smooth and connected. Among them there are the stick-figures called $\mathbf{K}_s$ in [12], [13] and [4]. We have $h^1(N_{C_s}(-2)) = 0$ for all C_s ([10, Lemme 1], see also [9]). Unless otherwise stated we only use smooth C_s.

For any t,k let $C_{t,k} := C_t \sqcup C_s$ be the union of a smooth C_t and a smooth C_k with the only restriction that they are disjoint. By definition each $C_{t,k}$ is smooth. Let $d_{t,k} := \deg(C_{t,k}) = t(t+1)/2 + k(k+1)/2$ and $g_{t,k} := h^1(\mathcal{O}_{C_{t,k}}) = 2 + t(t+1)(2t-5)/6 + k(k+1)(2k-5)/6$ for $t \geq k > 0$. If $t \geq k > 0$ then we have

$$(t+k-1)d_{t,k} + 2 - g_{t,k} = \binom{t+k+2}{3} \tag{4}$$

Since each connected component A of $C_{t,k}$ satisfies $h^i(N_A(-2)) = 0$, $i = 0,1$, we have $h^i(N_{C_{t,k}}(-2)) = 0$, $i = 0,1$.

Lemma 1. *We have $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0,1,2$.*

Proof. Since $C_t \cap C_k = \emptyset$, we have $Tor^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}_{C_t}, \mathcal{I}_{C_k}) = 0$ and $\mathcal{I}_{C_t} \otimes \mathcal{I}_{C_k} = \mathcal{I}_{C_{t,k}}$. Therefore tensoring (3) with $s := t$ by $\mathcal{I}_{C_k}(t+k-1)$ we get

$$0 \to t\mathcal{I}_{C_k}(k-2) \to (t+1)\mathcal{I}_{C_k}(k-1) \to \mathcal{I}_{C_{t,k}}(t+k-1) \to 0 \tag{5}$$

We have $h^2(\mathcal{I}_{C_k}(k-2)) = h^1(\mathcal{O}_{C_k}(k-2))$ and the latter integer is zero, because $e(C_k) = k-3 < k-2$. We have $h^1(\mathcal{I}_{C_k}(k-1)) = 0$, because C_k is arithmetically Cohen-Macaulay. We have $h^0(\mathcal{I}_{C_k}(k-1)) = 0$, by the case $s = k$ of (3). Hence $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0,1,2$. □

Remark 1. In this paper we only need $k \in \{t-1, t\}$.

Remark 2. We have $e(C_{t,k}) = \max\{e(C_t), e(C_k) = \max\{t-3, k-3\} \leq t+c-4$. Recall that $d_{t,k} = \deg(C_{t,k})$. If $s := t+k$, then $d_{s-1,1} = (s^2-s+2)/2 \geq d_{t,k}$. If s is even then $d_{t,k} \geq s(s+2)/4 = d_{\frac{s}{2},\frac{s}{2}}$. If s is odd, then $d_{t,k} \geq (s+1)^2/4 = d_{\frac{s+1}{2},\frac{s-1}{2}}$.

Remark 3. Let X be a general smooth curve of genus g and degree $d \geq g+3$ such that $h^1(\mathcal{O}_X(1)) = 0$; if either $g \geq 26$ ([25, p. 67, inequality $D_P(g) \leq g+3$]) or $g \leq 25$ and $d \geq g+14$ ([25, p. 67]), then $h^1(N_X(-2)) = 0$.

2.2. **Smoothing.** We are going to apply standard smoothing techniques (see for instance [16] and [26]).

Lemma 2. *Fix $A \sqcup B$ with $A = C_t$ and $B = C_k$. Let X be a nodal curve with $X = A \cup B \cup Y$, Y a smooth curve of degree $d' \geq 2$ and genus g', $\sharp(A \cap Y) = 1$, $\sharp(B \cap Y) = 1$, $h^1(\mathcal{O}_Y(1)) = 0$ and $h^1(N_Y(-2)) = 0$. Then $h^1(N_X(-1)) = 0$ and X is smoothable.*

Proof. Set $C := A \cup B$. Write $\{p_1\} = A \cap Y$ and $\{p_2\} = B \cap Y$. We have an exact sequence

$$0 \to N_X(-1) \to N_X(-1)|_C \oplus N_X(-1)|_Y \to N_X(-1)|_{\{p_1,p_2\}} \to 0 \tag{6}$$

Since $N_X(-1)|_C$ is obtained from $N_C(-1)$ by making two positive elementary transformations and $h^1(N_C(-1)) = 0$, we have $h^1(N_X(-1)|_C) = 0$. Since $N_X(-2)|_Y$ is obtained from $N_Y(-2)$ by making two positive elementary transformations and $h^1(N_Y(-2)) = 0$, we have $h^1(N_X(-2)|_Y) = 0$. Let $H \subset \mathbb{P}^3$ be a general plane containing $\{p_1, p_2\}$. Since Y is not a line, $Y \cap H$ is a zero-dimensional scheme. Since $h^1(N_X(-2)|_Y) = 0$, the restriction map

$$H^0(Y, N_X(-1)|_Y) \to H^0(Y \cap H, N_X(-1)|_{H \cap Y})$$

is surjective. Since $\{p_1, p_2\} \subseteq Y \cap H$, the restriction map $H^0(Y \cap H, N_X(-1)|_{H \cap Y}) \to H^0(\{p_1, p_2\}, N_X(-1)|_{\{p_1,p_2\}})$ is surjective. Hence the restriction map

$$H^0(Y, N_X(-1)|_Y) \to H^0(\{p_1, p_2\}, N_X(-1)|_{\{p_1,p_2\}})$$

is surjective. From (6) we get $h^1(N_X(-1)) = 0$.

Since $h^1(N_X(-1)) = 0$, X is smoothable ([12, Corollary 1.2]). □

Call $U(t, k, d', g')$ the set of all curves $X = A \cup B \cup Y$ appearing in Lemma 2. For all integer $y \geq 0$ and $x \geq y+3$ the Hilbert scheme of smooth space curves of degree x and genus y is irreducible ([6, 7]). By Lemma 2 there is a unique irreducible component $W(t, k, d', g')$ of the Hilbert scheme of $\mathbb{P}^3$ containing the curve X of Lemma 2. A general $C \in W(t, k, d', g')$ is smooth and $h^1(N_C(-1)) = 0$. We have $\deg(C) = d' + \deg(C_t) + \deg(C_k) = d' + t(t+1)/2 + k(k+1)/2$ and genus $g(C) = g' + p_a(C_t) + p_a(C_k) = g' - 2 + t(t+1)(2t-5)/6 + k(k+1)(2k-5)/6$.

3. Assertion $M(s, t, k)$, $k \in \{t-1, t\}$

For any $t \geq 27$, set $c(2t+1, t, t) = t+3$, $d(2t+1, t, t) = 0$, $c(2t, t, t-1) = t+2$ and $d(2t, t, t-1) = t-1$. Set $g(t+k+1, t, k) := c(t+k+1, t, k) - 3$. Note that if $k \in \{t-1, t\}$ we have

$$t(t+1) + k(k+1) + d(t+k+1, t, k) = (t+k)(t+k+4 - c(t+k+1, t, k)) \tag{7}$$

Now fix an integer $s \geq t+k+3$ with $s - t - k - 1 \equiv 0 \pmod 2$ and define the integers $c(s, t, k)$, $g(s, t, k)$ and $d(s, t, k)$ by the relations $g(s, t, k) = c(s, t, k) - 3 - 3(s-t-k-1)/2$ and

$$s(d_{t,k} + c(s,t,k)) + 3 - g_{t,k} - g(s,t,k) + d(s,t,k) = \binom{s+3}{3}, \ 0 \leq d(s,t,k) \leq s-2 \tag{8}$$

Note that (8) holds even if $s = t + k + 1$. From (8) for the integers $s + 2$ and s and the equality $g(s + 2, t, k) - g(s, t, k) = c(s + 2, t, k) - c(s, t, k) - 3$ we get

$$\begin{aligned} &2d_{t,k} + 2c(s, t, k) + (s + 1)(c(s + 2, t, k) - c(s, t, k))+ \\ &d(s + 2, t, k) - d(s, t, k) + 3 = (s + 3)^2 \end{aligned} \tag{9}$$

Remark 4. We have $c(2t + 1, t, t) = t + 3$, $d(2t + 1, t, t) = 0$, $c(2t, t, t - 1) = t + 2$, $d(2t, t, t - 1) = t - 1$, $c(2t + 2, t, t - 1) = 2t + 6$, $d(2t + 2, t, t - 1) = 2t - 3$, $c(2t + 3, t, t) = 2t + 7$, $d(2t + 3, t, t) = 2t - 1$.

Remark 5. We explain here the main reason for the assumption $t \geq 27$ made in this section. Fix an integer $s \geq t + k + 1$ with $s \equiv t + k + 1 \pmod 2$. We work with a curve $X = C_{t,k} \sqcup A$ with A a general smooth curve of degree $c(s, t, k)$ and genus $g(s, t, k)$ and we need $h^1(N_X(-2)) = 0$, i.e. we need $h^1(N_A(-2)) = 0$. If $s = t + k + 1$, then A has genus 0. The normal bundle of a general smooth rational curve $A \subset \mathbb{P}^3$ of degree $c(t + k + 1, t, k) \geq 3$ is balanced, i.e. it is the direct sum of two line bundles of degree $2c(s, t, k) - 1$ ([8]), hence $h^1(N_A(-2)) = 0$. Now assume $s \geq t + k + 3$. By Lemma 3 below we have $g(s, t, k) \geq g(t + k + 1, t, k)$. We have $g(2t + 1, t, t) = t \geq 27$ and $g(2t, t, t - 1) = t - 1 \geq 26$. Since $g(s, t, k) \geq 26$, Remark 3 gives $h^1(N_A(-2)) = 0$.

Lemma 3. *For each* $s \geq t + k + 1$ *with* $s \equiv t + k - 1 \pmod 2$ *we have* $2(c(s + 2, t, k) - c(s, t, k)) \geq s + 4$.

Proof. Since $g_{t,k} + g(s, t, k) < g_{\lceil (s+1)/2 \rceil, \lfloor (s+1)/2 \rfloor}$, (8) for s, t, k and (1) for $t' = \lceil (s + 1)/2 \rceil$ and $k' = \lfloor (s + 1)/2 \rfloor$ imply $d_{t',k'} \geq c(s, t, k) + d_{t,k}$. Remark 4 gives $c(s + 2, t', k') = k' + 3$. Since $0 \leq d(s + 2, t, k) \leq s$ and $0 \leq d(s, t, k) \leq s - 2$, (9) and the difference between (8) for $s' := s + 2$ and (4) for t', k' imply $c(s + 2, t, k) - c(s, t, k) \geq -1 + c(s + 2, t', k') = \lfloor (s + 1)/2 \rfloor + 2$. □

Let $Q := \mathbb{P}^1 \times \mathbb{P}^1$. The elements of $|\mathcal{O}_Q(0, 1)|$ are the fibers of the projection $\pi_2 : Q \to \mathbb{P}^1$, so that each $D \in |\mathcal{O}_Q(1, 0)|$ contains exactly one point of each fiber of π_2.

Assertion $M(s, t, k)$, $k \in \{t - 1, t\}$, $s \geq t + k + 1$, $s \equiv t + k + 1 \pmod 2$: Set $e = 1$ if $0 \leq d(s, t, k) \leq c(s + 2, t, k) - c(s, t, k) - 3$ and $e = 2$ if $d(s, t, k) > c(s + 2, t, k) - c(s, t, k) - 3$. There is a 6-tuple $(X, Q, D_1, D_2, S_1, S_2)$ such that

(a) Q is a smooth quadric surface, $X = C_{t,k} \sqcup Y$, Y is a smooth curve of degree $c(s, t, k)$ and genus $g(s, t, k)$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X \cap Q$;

(b) D_1, D_2 are different elements of $|\mathcal{O}_Q(1, 0)|$, each of them containing one point of $Y \cap Q$, $S_i \subset D_i \setminus D_i \cap Y$, $1 \leq i \leq 2$, and $\sharp(S_1) + \sharp(S_2) = d(s, t, k)$; $\pi_2(S_2) \subseteq \pi_2(S_1)$; $S_2 = \emptyset$ and $\pi_2(S_1) \subseteq \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$ if $e = 1$, $\sharp(S_2) = d(s, t, k) - c(s + 2, t, k) + c(s, t, k) + 3$ and $\pi_2(S_2) \subseteq \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$ if $e = 2$;

(c) $h^i(\mathcal{I}_{X \cup S_1 \cup S_2}(s)) = 0$, $i = 0, 1$.

Remark 6. Fix lines $L, R \subset \mathbb{P}^3$ such that $L \cap R = \emptyset$ and $o \in \mathbb{P}^3 \setminus (L \cup R)$. Let $\ell : \mathbb{P}^3 \setminus \{o\} \to \mathbb{P}^2$ denote the linear projection from o. We have $\sharp(\ell(L) \cap \ell(R)) = 1$, i.e. there is a unique line $D(L, R, o) \subset \mathbb{P}^3$ such that $o \in D(L, R, o)$, $D(L, R, o) \cap L \neq \emptyset$ and $D(L, R, o) \cap R \neq \emptyset$. We have $\sharp(D(L, R, o) \cap L) = \sharp(D(L, R, o) \cap R) = 1$. The function $(L, R, o) \mapsto D(L, R, o)$ is regular.

Lemma 4. *For all $t \ge 27$ and $k \in \{t-1, t\}$ assertion $M(t+k+1, t, k)$ is true.*

Proof. Fix $C_{t,k}$ intersecting Q at $2d_{t,k}$ general points ([25]).

(a) Assume $k = t$. We have $c(2t+1, t, t) = t+3$ and $d(2t+1, t, t) = 0$ and so we take $e = 1$ with $S_1 = S_2 = \emptyset$. Take any $A \in |\mathcal{O}_Q(2, t+1)|$ with $A \cap C_{t,k} = \emptyset$. We have $\mathrm{Res}_Q(C_{t,t} \cup A) = C_{t,t}$ and thus $h^i(\mathcal{I}_{\mathrm{Res}_Q(C_{t,t}\cup A)}(2t-1)) = 0$, $i = 0, 1$. We have $h^i(Q, \mathcal{I}_{Q\cap(C\cap A)}(2t+1, 2t+1)) = h^i(Q, \mathcal{I}_{C_{t,t}\cap Q}(2t-1, t)) = 0$, $i = 0, 1$, by (7) and the generality of $C_{t,k} \cap Q$. Hence $h^i(\mathcal{I}_{C_{t,k}\cup A}(2t+1)) = 0$, $i = 0, 1$.

We deform A to a curve Y of degree $t+3$ and genus t with $Y \cap C_{t,k} = \emptyset$, Y intersecting transversally Q and with no line of Q containing ≥ 2 points of $Q \cap (C_{t,k} \cup Y)$. By the semicontinuity theorem for cohomology ([14, III.8.8]), for a general Y we have $h^i(\mathcal{I}_{C_{t,k}\cup Y}(2t+1)) = 0$, $i = 0, 1$. Set $X := C_{t,k} \cup Y$, $S_1 = S_2 = \emptyset$ and take as D_1 and D_2 any two different elements of $|\mathcal{O}_Q(1, 0)|$, each of them containing one point of $Y \cap Q$.

(b) Assume $k = t-1$. We have $c(2t, t, t-1) = t+2$, $d(2t, t, t-1) = t-1$ and $c(2t+2, t, t-1) - c(2t, t, t-1) = t+4$ (Remark 4). Hence $e = 1$. However, in the proof of $M(t+k+1, t, k)$ we will exchange the two rulings (as we will do below for the general proof that $M(s, t, k) \Longrightarrow M(s+2, t, k)$), so that $D_1, D_2 \in |\mathcal{O}_Q(0, 1)|$. Take lines $L_1, L_2 \in |\mathcal{O}_Q(1, 0)|$ such that $L_1 \ne L_2$ and $C_{t,t-1} \cap (L_1 \cup L_2) = \emptyset$, and t different lines $R_j \in |\mathcal{O}_Q(0, 1)|$, $1 \le j \le t$, none of them containing a point of $C_{t,t-1} \cap Q$. Fix $D_1, D_2 \in |\mathcal{O}_Q(0, 1)|$ containing no point of $C_{t,t-1} \cap Q$ and with $D_h \ne R_j$ for all h, j. Set $u_h := L_1 \cap D_h$, $h = 1, 2$. Fix $E_1 \subset D_1$ with $\sharp(E_1) = t-1$ and $E_1 \cap (L_1 \cup L_2) = \emptyset$. We have $h^1(Q, \mathcal{I}_{E_1}(2t-2, t)) = 0$. Since $C_{t,k} \cap Q$ is a general subset of Q with cardinality $2d_{t,k}$, we have $h^i(Q, \mathcal{I}_{Q\cap(C\cap A)\cup E_1}(2t, 2t)) = h^i(Q, \mathcal{I}_{(C_{t,t}\cap Q)\cup E_1}(2t-2, t)) = 0$, $i = 0, 1$, by (7). The residual sequence of Q gives $h^i(\mathcal{I}_{C_{t,k}\cup A\cup E_1}(2t)) = 0$, $i = 0, 1$.

Take an ordering $\{o_1, \dots, o_{t-1}\}$ of E_1 and let M_i the only element of $|\mathcal{O}_Q(1, 0)|$ with $o_i \in M_i$. Set $w_i := R_i \cap M_i$, $1 \le i \le t-1$. We fix a deformation $\{L_h(\lambda)\}_{\lambda\in\Lambda}$, $h = 1, 2$, of L_h with the following properties: Λ is a connected and affine smooth curve, $o \in \Lambda$, $L_h(o) = L_h$, $u_h \in L_h(\lambda)$ for all λ, $L_1(\lambda) \cap L_2(\lambda) = \emptyset$ for all λ and $L_h(\lambda)$ is transversal to Q for all $\lambda \ne o$. For each i with $1 \le i \le t-1$ there is a unique line $R_i(\lambda)$ containing w_i and intersecting both $L_1(\lambda)$ and $L_2(\lambda)$ (Remark 6). There is a deformation $\{R_t(\lambda)\}_{\lambda\in\Lambda}$ of R_t with $R_t(o) = R_t$, $R_t(\lambda)$ intersecting both $L_1(\lambda)$ and $L_2(\lambda)$. Taking instead of Λ a smaller neighborhood of o we may assume $R_i(\lambda) \cap R_j(\lambda) = \emptyset$ for all $i \ne j$ and all λ so that $A(\lambda) := L_1(\lambda) \cup L_2(\lambda) \cup R_1(\lambda) \cup \cdots \cup R_t(\lambda)$ is a connected nodal curve of degree $t+2$ and arithmetic genus $t-1$. By semicontinuity (restricting if necessary Λ to a neighborhood of o) we have $h^i(\mathcal{I}_{C_{t,k}\cup A(\lambda)\cup E_1}(2t)) = 0$, $i = 0, 1$, for all $\lambda \in \Lambda$. Fix $\lambda_0 \in \Lambda \setminus \{o\}$. Let $\{B_\delta\}_{\delta\in\Delta}$ be a smoothing of $A(\lambda_0)$ fixing u_1 and u_2, i.e. take a smooth and connected affine curve Δ and $a \in \Delta$ with $B_a = A(\lambda_0)$, B_δ a smooth curve of degree $t+2$ and genus $t-1$ and $\{u_1, u_2\} \subset B_\delta$ for all δ. Restricting if necessary Δ we may assume that B_δ is transversal to Q and disjoint from $C_{t,k} \cup E_1$ for all $\delta \in \Delta$ and (by semicontinuity) that $h^i(\mathcal{I}_{C_{t,k}\cup B_\delta\cup E_1}(2t)) = 0$, $i = 0, 1$. Since $A(\lambda_0)$ is transversal to Q, we may (up to a finite covering of Δ) find $t-1$ sections $s_1, \dots, s_{t-1}$ of the family $\{B_\delta \cap Q\}_{\delta\in\Delta}$ of $2t+4$ ordered points of Q with $s_i(a) = w_i$, $i = 1, \dots, t-1$. Let $M_j(\delta)$, $\delta \in \Delta$, be the only element of $|\mathcal{O}_Q(1, 0)|$ with $w_i \in M_i(\delta)$. Set $o_i(\delta) := L_1 \cap M_i(\delta)$ and $E_1(\delta) := \{o_1(\delta), \dots, o_{t-1}(\delta)\}$. By semicontinuity for a general $\delta \in \Delta \setminus \{a\}$ we have $h^i(\mathcal{I}_{C_{t,k}\cup B_\delta\cup E_1(\delta)}(2t)) = 0$. We fix such a δ and set $X := C_{t,k} \cup B_\delta$, $S_1 := E_1(\delta)$, $S_2 := \emptyset$. For $M(2t,t,t-1)$ we use the lines D_1, D_2 and $M_j(\delta)$, $1 \le j \le t-1$. □

Lemma 5. *For each integer $s \ge t+k+1$ such that $s \equiv t+k+1 \pmod 2$ we have $2c(s,t,k) \ge s+4$ and $2c(s,t,k) \ge s+6$ is $s \ge t+k+3$.*

Proof. The case $s = t+k+1$ is true by Remark 4. The general case follows by induction $s-2 \Longrightarrow s$ by Lemma 3. □

Lemma 6. *Assume $t \ge 27$ and $k \in \{t-1,t\}$. Fix an integer $s \ge t+k+1$ such that $s \equiv t+k+1 \pmod 2$. If $M(s,t,k)$ is true, then $M(s+2,t,k)$ is true.*

Proof. Let $e \in \{1,2\}$ be the integer arising in $M(s,t,k)$ and $f \in \{1,2\}$ the corresponding integer for $M(s+2,t,k)$. Take (X,Q,D_1,D_2,S_1,S_2) satisfying $M(s,t,k)$ with $X = C_{t,k} \sqcup Y$ and $D_1, D_2 \in |\mathcal{O}_Q(1,0)|$. The 6-tuple $(X',Q,D_1',D_2',S_1',S_2')$ will be a solution after exchanging the two rulings of Q, i.e. we will take $D_1', D_2' \in |\mathcal{O}_Q(0,1)|$ and we use π_1 instead of π_2. In each step with $d(s,t,k) \ne 0$ we obtain X' smoothing a curve W union of X, $\chi := \cup_{o \in S_1 \cup S_2} \chi(o)$, $e+1$ elements $|\mathcal{O}_Q(1,0)|$ and $c(s+2,t,k) - c(s,t,k) - e - 1$ elements of $|\mathcal{O}_Q(0,1)|$. See step (c) for the easier case $d(s,t,k) = 0$ (here to get W we add to X a line $D_0 \in |\mathcal{O}_Q(1,0)|$ and $c(s+2,t,k) - c(s,t,k) - 1$ elements of $|\mathcal{O}_Q(0,1)|$).

(a) Assume $e = 2$ and set $z := d(s,t,k) + 3 - c(s+2,t,k) + c(s,t,k)$. Since $d(s,t,k) \le s-2$, Lemma 3 gives $d(s,t,k) \le 2(c(s+2,t,k) - c(s,t,k) - 3)$, i.e. $z \le c(s+2,t,k) - c(s,t,k) - 3$. By assumption there is $E \subset Y \cap (Q \setminus (D_1 \cup D_2))$ such that $\sharp(E) = z$ and $\pi_2(E) = \pi_2(S_2) \subseteq \pi_2(S_1)$. Take a line $D_0 \in |\mathcal{O}_Q(1,0)|$ different from D_1, D_2, with $D_0 \cap E = \emptyset$, $D_0 \cap C_{t,k} \cap Q = \emptyset$ and $D_0 \cap Y \cap Q \ne \emptyset$; we use that $2c(s,t,k) \ge 3+z$ (Lemma 5). Take distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le c(s+2,t,k) - c(s,t,k) - 3$, such that $L_i \cap Y \ne \emptyset$ if and only if $i \le z$, $X \cap (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} L_i) = E$, $L_i \cap (C_{t,k} \cap Q) = \emptyset$ for all i. Set $J := (D_0 \cup D_1 \cup D_2) \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} L_i)$. We fix f general lines $R_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le f$, and $A_i \subset R_i$, $1 \le i \le f$, with the conditions $\sum_{i=1}^{f} \sharp(A_i) = b(s+2,t,k)$, $\pi_1(A_f) \subseteq \pi_1(A_1)$ and $\pi_1(A_f) \subseteq \pi_1(Y \cap (Q \setminus J))$. Set $\chi := \cup_{o \in S_1 \cup S_2} \chi(o)$, $A := A_1 \cup A_2$ and $W := X \cup J \cup \chi$. W is a flat degeneration of a disjoint union of $C_{t,k}$ and a smooth curve of degree $c(s+2,t,k)$ and genus $g(s+2,t,k)$, but to obtain a deformation compatible with the data A_1, A_2, see steps (a1) and (a2). We have $\mathrm{Res}_Q(W \cup A) = X \cup S_1 \cup S_2$ and so $h^i(\mathcal{I}_{\mathrm{Res}_Q(W \cup A)}(s)) = 0$, $i = 0,1$. We have $h^i(Q, \mathcal{I}_{(W \cap Q) \cup A}(s+2,s+2)) = h^i(Q, \mathcal{I}_{(X \cap (Q \setminus J)) \cup A}(s-1, s+5+c(s,t,k) - c(s+2,t,k)))$. We have $\sharp((X \cap (Q \setminus J)) \cup A) = h^0(Q, \mathcal{O}_Q(s-1, s+5+c(s,t,k) - c(s+2,t,k)))$. We have $h^1(Q, \mathcal{I}_A(s-1, s+5+c(s,t,k) - c(s+2,t,k))) = 0$, because $s+5+c(s,t,k) - c(s+2,t,k) > 0$, $f \le 2$ and $\sharp(A_1) \le s$; this is a key reason for our definition of $M(s+2,t,k)$. Therefore to prove that $h^i(Q, \mathcal{I}_{(X \cap (Q \setminus J)) \cup A}(s-1, s+5+c(s,t,k) - c(s+2,t,k))) = 0$, $i = 0,1$, it is sufficient to prove that we may take as $X \cap (Q \setminus J)$ a general subset of Q with its prescribed cardinality. By Remark 5 we have $h^1(N_X(-2)) = 0$. Since $h^1(N_X(-2)) = 0$, we may deform X keeping fixed E so that the other points are general in Q.

(a1) We have just proved that $h^i(\mathcal{I}_{W \cup A}(s+2)) = 0$, $i = 0,1$. If $d(s+2,t,k) = 0$, then $M(s+2,t,k)$ is proved for $e = 2$. Now assume $d(s+2,t,k) > 0$. To prove $M(s+2,t,k)$ when $e = 2$ we need to deform W to a smooth $X' = C_{t,k} \sqcup Y'$ intersecting transversally Q and (perhaps moving A) to obtain condition (b) of $M(s+2,t,k)$. Set $P_i := Y \cap D_i$, $i = 0,1,2$. Let $\{D_i(\lambda)\}_{\lambda \in \Lambda}$ be a deformation of D_i

with Λ a smooth and connected affine curve, $o \in \Lambda$, $D_i(o) = D_i$, $D_i(\lambda)$, $\lambda \in \Lambda \setminus \{o\}$, a line of $\mathbb{P}^3$ transversal to Q and containing P_i. Fix $i \in \{1, \dots, z\}$. By Remark 6 for each $\lambda \in \Lambda$ there is a unique line $L_i(\lambda) \subset \mathbb{P}^3$ such that $D_0 \cap L_i \in L_i(\lambda)$, $L_i(\lambda) \cap D_1(\lambda) \neq \emptyset$ and $L_i(\lambda) \cap D_2(\lambda) \neq \emptyset$; restricting if necessary Λ we may assume that all $L_i(\lambda)$, $\lambda \neq o$, are transversal to Q. Fix an integer i with $z < i \leq c(s+2,t,k) - c(s,t,k) - 3$ and fix a general $m_i \in L_i$. By Remark 6 there is a unique line $L_i(\lambda)$ such that $m_i \in L_i(\lambda)$, $L_i(\lambda) \cap D_1(\lambda) \neq \emptyset$ and $L_i(\lambda) \cap D_2(\lambda) \neq \emptyset$; restricting if necessary Λ we may assume that all $L_i(\lambda)$, $\lambda \neq o$, are transversal to Q. Restricting if necessary Λ to a smaller neighborhood of o in Λ we may assume that $L_i(\lambda) \cap L_j(\lambda) = \emptyset$ for all $i \neq j$, that $C_{t,k} \cap L_i(\lambda) = \emptyset$ for all i and all λ, that $L_i(\lambda) \cap D_0 \neq \emptyset$ if and only if $i \leq z$. Fix a general $\lambda \in \Lambda$ and set $J(\lambda) := D_0(\lambda) \cup D_1(\lambda) \cup D_2(\lambda) \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-3} L_i(\lambda))$. Let $\chi(\lambda)$ be the union of all $\chi(q)$ with either $q \in D_1(\lambda) \cap L_i(\lambda)$, $1 \leq i \leq c(s+2,t,k) - c(s,t,k) - 3$ or $q \in D_2(\lambda) \cap L_i(\lambda)$, $1 \leq i \leq z$. Set $W(\lambda) := X \cup J(\lambda) \cup \chi(\lambda)$. $W(\lambda)$ is the disjoint union of $C_{t,k}$ and of a degeneration of a flat family of smooth and connected curves of degree $c(s+2,t,k)$ and genus $g(s+2,t,k)$. As in the first part of step (a), restricting if necessary Λ, by semicontinuity we get $h^i(\mathcal{I}_{W(\lambda)\cup A}(s+2)) = 0$, $i = 0, 1$.

(a2) To prove $M(s+2,t,k)$ we need to prove that there is a set like A (call it A') satisfying both $h^i(\mathcal{I}_{W(\lambda)\cup A'}(s+2)) = 0$, $i = 0, 1$, and condition (b) of $M(s+2,t,k)$. First of all, instead of P_i, $0 \leq i \leq 2$, we take a family $\{P_i(\lambda)\}_{\lambda\in\Lambda}$ of points of Y with $P_i(o) = P_i$ and $P_i(\lambda) \in Y \setminus Y \cap Q$ for all $\lambda \in \Lambda \setminus \{o\}$. Assume for the moment $f = 2$. We modify the definition of $D_i(\lambda)$, because we impose that $P_i(\lambda) \in D_i(\lambda)$ (instead of $P_i \in D_i$), but we also impose that $D_1(\lambda) \cap R_1 \neq \emptyset$ and $D_2(\lambda) \cap R_2 \neq \emptyset$ (this is possible by Remark 6). Then we construct $L_i(\lambda)$ as above. With this new definition R_1 and R_2 are secant lines of $W(\lambda) \setminus (C_{t,k} \cup Y)$, $Y \subset W(\lambda)$, $\pi_1(A_2) \subseteq \pi_1(A_1)$ and $\pi_1(A_f) \subseteq \pi_1(Q \cap (Y \setminus J(\lambda) \cap Y))$; call $m_1, \dots, m_x$, $x = \sharp(A_f)$, the points of $Y \cap Q$ whose image is $\pi_1(A_f)$. We fix $\lambda \in \Lambda \setminus \{o\}$. Let $\{B_\delta\}_{\delta\in\Delta}$ be a smoothing of $W(\lambda)$ with Δ an affine and connected smooth curve, $a \in \Delta$, and $B_a = W(\lambda)$. Set $A(a) := A$. Since Y is transversal to Q, up to a finite covering of Δ we may find $x + 2$ sections $s_1, \dots, s_x, z_1, z_2$ of the total space of $\{B_\delta\}_{\delta\in\Delta}$ with $s_i(a) = m_i$, $z_1(a) = R_1 \cap D_1(\lambda)$, $z_2(a) = R_2 \cap D_2(\lambda)$, $s_i(\delta) \in B_\delta \cap Q$, $z_1(\delta) \in B_\delta \cap Q$ and $z_2(\delta) \in B_\delta \cap Q$ for all Δ. Let $R_h(\delta)$, $h = 1, 2$, be the only element of $|\mathcal{O}_Q(0,1)|$ containing $z_h(\delta)$. For each $\delta \in \Delta \setminus \{a\}$ and $i \in \{1, \dots, x\}$ let $M_i(\delta) \in |\mathcal{O}_Q(1,0)|$ be the only line of this ruling of Q containing $s_i(\delta)$. Set $A_1(\delta) := \cup_{i=1}^{x}(R_1(\delta) \cap M_i(\delta))$ and $A_2(\delta) := \cup_{i=1}^{d(s+2,t,k)-x}(R_2(\delta) \cap M_i(\delta))$. Set $X_\delta := C_{t,k} \cup B_\delta$. By construction $(X_\delta, Q, R_1, R_2, A_1(\delta), A_2(\delta))$ satisfies condition (b) of $M(s+2,t,k)$, exchanging the two rulings of Q. By semicontinuity we have $h^i(\mathcal{I}_{B_\delta\cup A(\delta)}(s+2)) = 0$, $i = 0, 1$, for a general $\delta \in \Delta$.

Now assume $f = 1$. In this case we only impose that $D_i(\lambda)$ meets R_1; we have $\pi_1(A_1) \subset \pi_1(Q \cap (Y \setminus J(\lambda) \cap Y))$ and $x = \sharp(A_1) = b(s+2,t,k)$.

(b) Assume $e = 1$ and $d(s,t,k) > 0$, i.e. assume $0 < d(s,t,k) \leq c(s+2,t,k) - c(s,t,k) - 3$. We set $S_2 := 0$ and ignore D_2. We fix $o \in S_1$. Take a line $D_0 \neq D_1$ meeting $Y \cap Q$ and $c(s+2,t,k) - c(s,t,k) - 2$ distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, with $L_i \cap (C_{t,k} \cap Q) = \emptyset$ for all i, $L_i \cap (Y \cap Q) \neq \emptyset$ if and only if $1 \leq i \leq d(s,t,k) - 1$ and $S_1 \setminus \{o\} = D_1 \cap (L_1 \cup \dots \cup L_{d(s,t,k)-1})$. Set $J := (D_0 \cup D_1) \cup (\bigcup_{i=1}^{c(s+2,t,k)+c(s,t,k)-2} L_i)$ and $\chi := \cup_{o\in S_1}\chi(o)$. Note that $\chi(X \cup J \cup \chi) - \chi(X) = c(s,t,k) - c(s+2,t,k) + 3$. To modify step (a2) we impose that $D_1(\lambda) \cap R_1 \neq \emptyset$ and $D_0(\lambda) \cap R_2 \neq \emptyset$.

(c) Assume $d(s,t,k) = 0$. Hence $S_1 = S_2 = \emptyset$. Take a line $D_0 \in |\mathcal{O}_Q(1,0)|$ different from D_1, D_2 and with $D_0 \cap Y \cap Q \neq \emptyset$. Take $c(s+2,t,k)-c(s,t,k)-1$ lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le c(s+2,t,k) - c(s,t,k) - 1$, such that $L_i \cap (C_{t,k} \cap Q) = \emptyset$ for all i and $L_i \cap (Y \cap Q) \neq \emptyset$ if and only if $1 \le i \le c(s+2,t,k) - c(s,t,k) - 3$. Set $J := D_0 \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(m,t,k)-1} L_i)$, $Y' := Y \cup J$ and $W := X \cup J$. Note that $\chi(W) - \chi(X) = c(s,t,k) - c(s+2,t,k) + 3$. The union Y' is a connected nodal curve, which is a flat degeneration of a family of smooth curves of degree $c(s+2,t,k)$ and genus $g(s+2,t,k)$ not intersecting $C_{t,k}$. As in step (a) we get $h^1(\mathcal{I}_W(s+2)) = 0$ and $h^0(\mathcal{I}_W(s+2)) = d(s+2,t,k)$. If $d(s+2,t,k) = 0$, then we are done, because $A = \emptyset$ and so condition (b) of $M(s+2,t,k)$ is trivially true. Now assume $d(s+2,t,k) > 0$.

First assume $f = 2$. As in step (a) we prove $M(s+2,t,k)$ interchanging the rulings of Q and set $x := c(s+4,t,k)-c(s+2,t,k)-3$. We fix general lines $R_1, R_2 \in |\mathcal{O}_Q(0,1)|$ and take $A_i \subset R_i$ such that $\pi_1(A_2) \subseteq \pi_1(A_1) \cap \pi_1(Q \cap (Y \setminus J \cap Y))$. Set $A := A_1 \cup A_2$. For a general X we have $h^i(\mathcal{I}_{W \cup A}(s+2)) = 0$, $i = 0,1$. Set $q := D_0 \cap Y$. By Remark 6 there is a family $\{D_0(\lambda)\}_{\lambda \in \Lambda}$ of lines of $\mathbb{P}^3$ and $o \in \Lambda$ with $D_0(o) = D_0$, $\sharp(D_0(\lambda) \cap Y) = 1$ for all λ, $D_0(\lambda) \cap Y \notin Q$ if $\lambda \neq 0$, $D_0(\lambda) \cap R_1 \neq \emptyset$ and $D_0(\lambda) \cap R_2 \neq \emptyset$. Up to a finite covering of Λ we may also find families $\{L_i(\lambda)\}_{\lambda \in \Lambda}$, $1 \le i \le c(s+2,t,k) - c(s,t,k) - 1$. Set $J(\lambda) = D_0(\lambda) := D_0 \cup (\bigcup_{i=1}^{c(s+2,t,k)-c(s,t,k)-1} L_i(\lambda))$. We do the smoothing of $Y \cup J(\lambda)$ as in step (a2).

Finally, if $f = 1$ we only need $D_0(\lambda) \cap R_1 \neq \emptyset$ for all λ. □

4. With a constant genus g

We fix an integer $t \ge 27$ and take $k \in \{t-1, t\}$. We fix an integer $g \ge g_{t,k} + g(t+k+5,t,k)$. Let y be the maximal integer $\ge t+k+5$ such that $y \equiv t+k-1 \pmod 2$ and $g_{t,k} + g(y,t,k) \le g$ (y exists, because $\lim_{u \to +\infty} g(t+k+1+2u,t,k) = +\infty$). By the definition of y we have $y \ge t+k+5$ and $y \equiv t+k-1 \pmod 2$. For all integers $x \ge y+2$ with $x \equiv y \pmod 2$ define the integers $a(x,t,k,y)$ and $b(x,t,k,y)$ by the relation

$$xd_{t,k} + 3 - g + xa(x,t,k,y) + b(x,t,k,y) = \binom{x+3}{3},\ 0 \le b(x,t,k,y) \le x-1 \tag{10}$$

If $x \ge y+4$, by taking the difference between equation (10) and the same equation for the integer $x' := x-2$ we get

$$\begin{aligned} &2d_{t,k} + 2a(x,t,k,y) + (x+2)(a(x+2,t,k,y) - a(x,t,k,y)) \\ &\qquad + b(x+2,t,k,y) - b(x,t,k,y) = (x+3)^2 \end{aligned} \tag{11}$$

Lemma 7. *For each $x \ge y+2$ with $x \equiv y \pmod 2$ we have $2(a(x+2,t,k,y) - a(x,t,k)) \ge x+5$.*

Proof. Assume by contradiction $2(a(x+2,t,k,y) - a(x,t,k)) \le x+4$. Recall that for all $u \ge v > 0$ we have

$$(u+v-1)d_{u,v} + 2 - g_{u,v} = \binom{u+v+2}{3} \tag{12}$$

First assume x odd, i.e. $k = t$. Since $g_{(x+1)/2,(x+1)/2} > g$, (12) and (10) give $d_{(x+1)/2,(x+1)/2} \ge d_{t,k} + a(x,t,k,y)$. Since $b(x+2,t,k,y) \le x+1$ and $b(x,t,k,y) \ge 0$ (11) gives

$$(x+1)(x+3)/2 + (x+2)(x+4)/2 + x + 1 \ge (x+3)^2,$$

which is false. Now assume x even, i.e. $k = t-1$. Since $g_{(x+2)/2,x/2} > g$, (12) and (10) gives $d_{(x+2)/2,x/2} \ge d_{t,k} + a(x,t,k,y)$. Since $b(x+2,t,k,y) \le x+1$ and $b(x,t,k,y) \ge 0$ (11) gives

$$(x+2)^2/2 + (x+2)(x+4)/2 + x + 1 \ge (x+3)^2,$$

which is false. □

Lemma 8. *We have* $2(a(y+2,t,k,y) - c(y,t,k)) \ge y+5$.

Proof. Define the integers w, z by the relations

$$(13) \qquad (y+2)(w+d_{t,k}) + 3 - g_{t,k} - g(y,t,k) + z = \binom{y+5}{3},\ 0 \le z \le y+1$$

Since $g \ge g_{t,k} + g(y,t,k)$, we have $w \le a(y+2,t,k)$. Hence it is sufficient to prove that $2(w - c(y,t,k)) \ge y+5$. Taking the difference between (13) and the case $s = y$ of (8) we get

$$2d_{t,k} + 2c(y,t,k) + (y+2)(w - c(y,t,k)) + z - d(y,t,k) = (y+3)^2$$

Then we continue as in the proof of Lemma 7 with $y+2$ instead of $x+2$. □

The next lemma follows at once by induction on x, the inequality $2c(y,t,k) \ge y+6$ and Lemmas 7 and 8.

Lemma 9. *We have* $2a(x,t,k,y) \ge x+6$ *for all integers* $x \ge y+2$ *with* $x \equiv y \pmod 2$.

Lemma 10. *For each* $x \ge y+2$ *with* $x \equiv y \pmod 2$ *we have* $a(x,t,k,y) \ge g - g_{t,k} + 3$.

Proof. First assume $x = y+2$. We have

$$(y+2)(d_{t,k} + c(y+2,t,k)) + 3 - g_{t,k} - g(y+2,t,k) + d(y+2,t,k) =$$
$$(y+2)(d_{t,k} + a(y+2,t,k,y)) + b(y+2,t,k,y) + 3 - g$$

hence

$$(y+2)(c(y+2,t,k) - a(y+2,t,k,y) + d(y+2,t,k) - b(y+2,t,k,y)$$
$$(14) \qquad = g(y+2,t,k) - g - g_{t,k}$$

By the definition of y the right hand side of (14) is negative. Since $c(y+2,t,k) \ge g(y+2,t,k)+3$, $b(y+2,t,k,y) \le y+1$, $d(y+2,t,k) \ge 0$, we have $c(y+2,t,k,y) \ge g - g_{t,k}$.

Now assume $x \ge y+4$. By Lemma 7 we have $a(x,t,k,y) \ge a(y+2,t,k,y)$. □

By Lemma 10 there is a non-special curve of degree $a(x,t,k,y)$ and genus $g - g_{t,k}$. We need this observation in the next statement.

Assertion $N(x,t,k,y)$, $x \ge y$, $x \equiv y \pmod 2$: Set $e = 1$ if $0 \le b(x,t,k,y) \le a(x+2,t,k,y) - a(s,t,k,y) - 1$ and $e = 2$ if $b(x,t,k,y) \ge a(x+2,t,k,y) - a(x,t,k,y)$. There is a 6-tuple $(X, Q, D_1, D_2, S_1, S_2)$ such that

(a) Q is a smooth quadric surface, $X = C_{t,k} \sqcup Y$, Y is a smooth non-special curve of degree $a(x,t,k,y)$ and genus $g - g_{t,k}$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X \cap Q$;

(b) D_1, D_2 are different elements of $|\mathcal{O}_Q(1,0)|$, each of them containing one point of $Y \cap Q$, $S_i \subset D_i \setminus D_i \cap Y$, $1 \le i \le 2$, and $\sharp(S_1) + \sharp(S_2) = b(x,t,k,y)$; $\pi_2(S_2) \subseteq \pi_2(S_1)$ and $\pi_2(S_e) \subset \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$; $S_2 = \emptyset$ if $e = 1$, $\sharp(S_2) = b(x+2,t,k,y) - a(x+2,t,k,y) + a(x,t,k,y) + 2$ if $e = 2$;

(c) $h^i(\mathcal{I}_{X \cup S_1 \cup S_2}(x)) = 0$, $i = 0, 1$.

Lemma 11. *If $N(x,t,k,y)$ is true, then $N(x+2,t,k,y)$ is true.*

Proof. We outline the modifications of the proof of Lemma 6 needed to get Lemma 11. Let $e \in \{1,2\}$ (resp. $f \in \{1,2\}$) be the integer arising in $N(x,t,k,y)$ (resp. $N(x+2,t,k,y)$). Take $(X, Q, D_1, D_2, S_1, S_2)$ satisfying $N(x,t,k,y)$. Set $w := a(x+2,t,k,y) - a(x,t,k,y)$.

(a) Assume $e = 2$. Set $z := b(x+2,t,k,y)+2-w$. Since $b(x+2,t,k,y) \le x+1$, Lemma 7 gives $z \le w-2$. Let $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le w-2$, be the lines such that $S_1 = D_1 \cap (L_1 \cup \cdots \cup L_{w-2})$ and $S_2 = D_2 \cap (L_1 \cup \cdots \cup L_z)$. Set $J := D_1 \cup D_2 \cup (\bigcup_{i=1}^{w-2} L_i)$ and $\chi := \cup_{o \in S_1 \cup S_2} \chi(o)$. Condition (b) gives $\sharp(L_i \cap Y) = 1$ for all i. Condition (a) gives $C_{t,k} \cap J = \emptyset$. Hence $W := X \cup J \cup \chi$ is a smoothable curve of degree $a(x+2,t,k,y)$ with $h^1(\mathcal{O}_W) = g$.

(b) Assume $e = 1$, i.e. assume $d(x+2,t,k,y) \le w-1$. Let $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le b(x,t,k,y)$, be the lines such that $S_1 = D_1 \cap (L_1 \cup \cdots \cup L_{b(x,t,k,y)})$. Take general lines $L_j \in |\mathcal{O}_Q(0,1)|$, $b(x,t,k,y) < j \le w-1$. Set $J := D_1 \cup (\bigcup_{i=1}^{w-1} L_i)$ and $\chi := \cup_{o \in S_1} \chi(o)$. Condition (a) gives $C_{t,k} \cap J = \emptyset$. Hence $W := X \cup J \cup \chi$ is a smoothable curve of degree $a(x+2,t,k,y)$ with $h^1(\mathcal{O}_W) = g$. □

Lemma 12. *$N(y+2,t,k,y)$ is true.*

Proof. Use the proof of Lemma 6 and Lemma 11 starting with $(X, Q, D_1, D_2, S_1, S_2)$ satisfying $M(y,t,k)$ and quoting Lemma 8 instead of Lemma 7. □

5. Proving Conjecture 1

In order to prove Theorem 1 and Corollary 1, first of all we notice that from the previous section we could deduce with a small effort the following two facts, but that (as explained at the end of the introduction) they would not prove Theorem 1 and Corollary 1.

For each integer d such that $g-3 \le d \le d(m,g)_{\max}$ there exists a smooth and connected curve $X_1 \subset \mathbb{P}^3$ such that $\deg(X_1) = d$, $g(X) = g$, $h^1(\mathcal{O}_{X_1}(m-2)) = 0$, $h^1(\mathcal{I}_{X_1}(m)) = 0$ and $h^1(N_{X_1}(-1)) = 0$.

For each integer $d \ge d(m,g)_{\min}$ there exists a smooth and connected curve $X_2 \subset \mathbb{P}^3$ such that $\deg(X_2) = d$, $g(X) = g$, $h^1(\mathcal{O}_{X_2}(m-2)) = 0$, $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

Now fix an integer d such that $d(m,g)_{\min} \le d \le d(m,g)_{\max}$. To prove Theorem 1 for the pair (d,g) it is sufficient to prove that we may find X_1, X_2 as above and with the additional condition that X_1 and X_2 are in the same irreducible component, Γ, of $\mathrm{Hilb}(\mathbb{P}^3)$. If we prove this statement, then by the semicontinuity theorem for cohomology ([14, III.8.8]) we get $h^1(\mathcal{I}_X(m)) = 0$ and $h^0(\mathcal{I}_X(m-1)) = 0$, hence we would conclude the proof for the pair (d,g). To get X_1 and X_2 in the same irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ we need to rewrite the proofs of the previous section with a few improvements. But first we need to distinguish between the case in which d is very near to $d(m,g)_{\min}$ and the case in which d is very near to $d(m,g)_{\max}$. In the first case (say $d(m,g)_{\min} \le d \le d'$) we will modify the proof of the existence of X_2 with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ to get (for the same curve X_2) also $h^1(\mathcal{I}_{X_2}(m)) = 0$. If d is very near to $d(m,g)_{\max}$ (say $d'' \le d \le d(m,g)_{\max}$) we will modify the proof of the existence of the curve X_1 to get a curve X_1 with $h^1(\mathcal{I}_{X_1}(m)) = 0$ and $h^0(\mathcal{I}_{X_1}(m-1)) = 0$. We use that $N(x,t,k,y)$ are true for $x = m-5, m-4, m-3, m-2$ (Lemma 13).

Set $\epsilon := 0$ if m is odd and $\epsilon := 1$ if m is even.

5.0.1. *Near* $d(m,g)_{\min}$. In this range the most difficult part is the proof of the existence of X_2. It is the construction of X_2 which says in which $W(t',k',d',b')$ we will try to find X_1. Recall that to get a curve X_2 with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ we started with a curve $C_{t,t-\epsilon}$ with $h^i(\mathcal{I}_{C_{t,t-\epsilon}}(2t-1-\epsilon)) = 0$, where t is the maximal integer $t > 0$ such that such that $g_{t,t-\epsilon} + g(2t+5-\epsilon, t, t-\epsilon) \le g$. Set $k := t-\epsilon$. Recall that an element W of $U(t,k,a_d,b)$ has degree d and $h^1(\mathcal{O}_W) = g$ if and only if $b = g - g_{t,k}$ and $a_d = d - d_{t,k}$. The component $W(t',k',d',b')$ is the component $W(t,k,a_d,b)$, where $b = g - g_{t,k}$ and $a_d = d - d_{t,k}$. The curve T satisfying $N(m-1,t,k,y)$ has $h^1(\mathcal{O}_T) = g$, 3 connected components, $h^0(\mathcal{I}_T(m-1)) = b(m-1,t,k,y)$ and $h^1(\mathcal{I}_T(m-1)) = 0$, hence $d > a(m-1,t,k,y) + d_{t,k}$. The minimum integer $d(m,g)_{\min}$ is $a(m-1,t,k,y) + d_{t,k} + 1$, unless $b(m-1,t,k,y) \in \{m-2, m-1\}$ (in the latter case we have $d(m,g)_{\min} = a(m-1,t,k,y) + d_{t,k} + 2$).

(a) We make the construction of Section 4 for the integer $m' := m-1 \equiv t+k-1 \pmod 2$ and the integer g (note that the numerology for g in Theorem 1 is such that we may do the construction of Section 4 for $m' := m-1$ and the integer g). We get an integer $y \le m'-4 = m-5$ with $y \equiv t+k-1 \equiv 0 \pmod 2$. Then for all integers $x \ge y+2$ with $x \equiv y \pmod 2$ we proved $N(x,t,k,y)$. Hence $N(m-5,t,k,y)$ and $N(m-3,t,k,y)$ are true (Lemma 13). Since $d \ge d(m,g)_{\min}$, we have $d > a(m-1,t,k,y) + d_{t,k}$, hence we want to add in a smooth quadric Q a certain union of $d - a(m-3,t,k,y) - d_{t,k}$ lines. We write $C_t \cup C'_k$ for a general (but fixed in this construction) $C_{t,k}$, because we need to distinguish the two connected components of $C_{t,k}$, even when $k = t$.

(a1) Assume $d = d(m,g)_{\min} = a(m-1,t,k,y) + d_{t,k} + 1$. Set $z := d - a(m-3,t,k,y) - d_{t,k} = 1 + a(m-1,t,k,y) - a(m-3,t,k,y)$. We need to modify $N(m-3,t,k,y)$ in the following way.

Assertion $N'(m-3,t,k,y)$, $m-3 \equiv y \pmod 2$: Set $e = 1$ if $b(m-3,t,k,y) \le z-3$ and $e = 2$ if $b(m-3,t,k,y) \ge z-2$. There is a 6-tuple (X,Q,D_1,D_2,S_1,S_2) such that

(a) Q is a smooth quadric surface, $X = C_t \sqcup C'_k \sqcup Y$, Y is a smooth curve of degree $a(m-3,t,k,y)$ and genus $g - g_{t,k}$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X \cap Q$;

(b) D_1, D_2 are different elements of $|\mathcal{O}_Q(1,0)|$, $D_1 \cap C_t \ne \emptyset$, $D_2 \cap C_k \ne \emptyset$, $S_i \subset D_i \setminus D_i \cap (C_t \cup C'_k)$, $1 \le i \le 2$, and $\sharp(S_1) + \sharp(S_2) = b(m-3,t,k,y)$; $\pi_2(S_2) \subseteq \pi_2(S_1)$, $\pi_2(S_e) \subset \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$; $S_2 = \emptyset$ if $e = 1$, $\sharp(S_2) = b(m-3,t,k,y) - z + 3$ if $e = 2$;

(c) $h^i(\mathcal{I}_{X \cup S_1 \cup S_2}(m-3)) = 0$, $i = 0,1$.

As in the proof of Lemma 6 and Lemma 11 we get (X,Q,D_1,D_2,S_1,S_2), $X = C_t \sqcup C'_k \sqcup Y$ satisfying $N'(m-3,t,k,y)$; in the proof of Lemma 6 we take R_1 containing a point of $C_t \cap Q$ instead of a point of $Y \cap Q$ and R_2 containing a point of $C'_k \cap Q$ instead of a point of $Y \cap Q$.

(a1.1) Assume $b(m-3,t,k,y)=0$. Take $D_0 \in |\mathcal{O}_Q(1,0)|$ containing one point of $Y \cap Q$, $L_1 \in |\mathcal{O}_Q(0,1)|$ containing a point of C_t, $L_2 \in |\mathcal{O}_Q(0,1)|$ containing a point of C'_k and general $L_i \in |\mathcal{O}_Q(0,1)|$, $3 \le i \le z-1$. Set $J := D_0 \cup (\bigcup_{i=1}^{z-1} L_i)$. Since $X \cap (Q \setminus J)$ is a general subset of Q with cardinality $2d_{t,k}+2a(m-3,t,k,y)-3$, we have $h^0(Q,\mathcal{I}_{Q\cap(X\cup J)}(m-1)) = h^0(Q,\mathcal{I}_{X\cap(Q\setminus J)}(m-2,m-z)) = 0$ (use (10) for $x = m-3$, that $z = 1+a(m-1,t,k,y)-a(m-3,t,k,y)$ and that $b(m-1,t,k,y) \le m-2$). Since $\mathrm{Res}_Q(X\cup Y) = X$ and $h^0(\mathcal{I}_X(m-3)) = 0$, we have $h^0(\mathcal{I}_{X\cup J}(m-1)) = 0$. The union $X \cup J$ is a nodal and connected smoothable curve of degree d and arithmetic genus g and $Y \cup J$ is a connected smoothable curve of degree $d-d_{t,k}$ and arithmetic genus $g-g_{t,k}-2 \ge 26$. We may smooth $Y \cup J$ in a family of curves, all of them containing the two points $(C_t \cup C'_k)\cap J$. Call E a general element of this smoothing. Since $\mathrm{Aut}(\mathbb{P}^3)$ is 2-transitive, we may see E as a general non-special space curve of its degree and its genus ≥ 26. By construction and Lemma 2 we have $C_t \cup C'_k \cup E \in U(t,k,a_d,b)$ and $h^1(N_{C_t\cup C'_k\cup E}(-1)) = 0$. By semicontinuity there is a smooth $X_2 \in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

(a1.2) Assume $0 < b(m-3,t,k,y) \le z-3$. Hence $S_2 = \emptyset$. We take D_1 and call $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le b(m-3,t,k,y)$, the elements of $|\mathcal{O}_Q(0,1)|$ such that $S_1 = D_1 \cap (L_1 \cup \cdots \cup L_{b(m-3,t,k,y)})$; note that each line L_i contains a point of $Y \cap Q$. Take any $L_{b(m-3,t,k,y)+1} \in |\mathcal{O}_Q(0,1)|$ with $C'_k \cap L_{b(m-3,t,k,y)+1} \ne \emptyset$, any $L_{b(m-3,t,k,y)+2} \in |\mathcal{O}_Q(0,1)|$ with $Y \cap L_{b(m-3,t,k,y)+2} \ne \emptyset$, $L_{b(m-3,t,k,y)+2} \ne L_i$ for $i \le b(m-3,t,k,y)$ and (if $b(m-3,t,k,y) < z-3$) take general $L_j \in |\mathcal{O}_Q(0,1)|$, $b(m-3,t,k,y)+3 \le j \le z-1$. Set $J := D_1 \cup (\bigcup_{i=1}^{z-1} L_i)$, $\chi := \cup_{o\in S_1}\chi(o)$ and $W := X \cup J \cup \chi$. We have $\mathrm{Res}_Q(W) = X \cup S_1$ and thus $h^0(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-3)) = 0$. Since $W \cap Q$ is the union of J and $2d_{t,k}+2a(m-3,t,k,y)-b(m-3,t,k,y)-3$ general points of Q and $b(m-1,t,k,y) \le m-1$, (11) gives $h^0(Q,\mathcal{I}_{W\cap Q}(m-1)) = h^0(Q,\mathcal{I}_{X\cap(Q\setminus J)}(m-2,m-z)) = 0$. Thus $h^0(\mathcal{I}_W(m-1)) = 0$. We first deform W to the union F of $C_t \cup C'_k \cup D_1 \cup Y \cup (\bigcup_{i=b(m-3,t,k,y)+1}^{z-1} L_i)$ and $b(m-3,t,k,y)$ disjoint lines $M_1,\dots,M_{b(m-3,t,k,y)}$, each of them containing one point of Y. The union F is a nodal and connected curve. Write $F = C_t \cup C'_k \cup G$. We have $\sharp(G\cap C_t) = \sharp(G\cap C'_k) = 1$. Let G' be a general smoothing of G fixing the 2 points of $(C_t \cup C'_k)\cap G$. $C_t \cup C'_k \cup G' \in U(t,k,a_d,b)$. By Lemma 2 and semicontinuity there is a smooth $X_2 \in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

(a1.3) Assume $b(m-3,t,k,y) \ge z-2$. Since $z = a(m-1,t,k,y)-a(m-3,t,k,y)+1$ and $b(m-3,t,k)) \le m-4$, Lemma 7 gives $2(z-3) \ge b(m-3,t,k,y)$. Let $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le z-3$, be the lines such that $S_1 = D_1 \cap (\bigcup_{i=1}^{z-3} L_i)$ and $S_2 := D_2 \cap (\bigcup_{i=1}^{w} L_i)$. Take $L_{z-2} \in |\mathcal{O}_Q(0,1)|$ containing one point of $Y \cap Q$ and different from the other lines L_i, $i \le z-3$. Set $J := D_1 \cup D_2 \cup (\bigcup_{i=1}^{z-2} L_i)$, $\chi := \cup_{o\in S_1}\chi(o)$ and $W := X \cup J \cup \chi$. We have $\mathrm{Res}_Q(W) = X \cup S_1 \cup S_2$ and thus $h^0(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-3)) = 0$. Since $W\cap Q$ is the union of J and $2d_{t,k}+2a(m,t,k,y)-w-3$ general points of Q and $b(m-1,t,k,y) \le m-1$ (11) gives $h^0(Q,\mathcal{I}_{W\cap Q}(m-1)) = h^0(Q,\mathcal{I}_{X\cap(Q\setminus J)}(m-2,m-z)) = 0$. Thus $h^0(\mathcal{I}_W(m-1)) = 0$. We first deform W to the union F of $C_t \cup C'_k \cup D_1 \cup D_2 \cup Y \cup (\bigcup_{i=w+1}^{z-2} L_i)$ and w disjoint lines $M_1,\dots,M_w$, each of them containing one point of Y. The union F is a nodal and connected curve. Write $F = C_t \cup C'_k \cup G$. We have $\sharp(G\cap C_t) = \sharp(G\cap C'_k) = 1$. Let G' be a general smoothing of G fixing the 2 points of $(C_t \cup C'_k)\cap G$. We have $C_t \cup C'_k \cup G' \in U(t,k,a_d,b)$. By Lemma 2 and semicontinuity there is a smooth $X_2 \in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

(a1.4) Assume $d(m,g)_{\min} = a(m-1,t,k,y) + d_{t,k} + 2$. We are in the set-up of step (a1.3) with the integer $z' := a(m-1,t,k,y) - a(m-3,t,k,y) + 2$ instead of the integer $z := a(m-1,t,k,y) - a(m-3,t,k,y) + 1$.

(a2) Assume $d > d(m,g)_{\min}$ and set $w := d - d(m,g)_{\min}$. By step (a1) there is a nodal curve $E = C_t \cup C'_k \cup F \in U(t,k,a_d - w, b)$ with $\sharp(C_t \cap F) = \sharp(C'_k \cap F) = 1$, $C_t \cap D'_k = \emptyset$, F and $h^0(\mathcal{I}_E(m-1)) = 0$. Take a general union G of F and w lines, each of them meeting F at exactly one point and quasi-transversally. By construction $E' := C_t \cup C'_k \cup G$ is nodal and $C_t \cap G = C_t \cap F$, $C'_k \cap G = C'_k \cap F$. Since $h^0(\mathcal{I}_E(m-1)) = 0$ and $E' \supset E$, we have $h^0(\mathcal{I}_{E'}(m-1)) = 0$. We may smooth G keeping fixed the points $C_t \cap F$ and $C'_k \cap F$, because $\mathrm{Aut}(\mathbb{P}^3)$ is 2-transitive. Hence there is a non-special smooth curve G'' of degree $d - d_{t,k}$ and genus $g - g_{t,k}$ with $C_t \cap G'' = C_t \cap F$, $C'_k \cap G'' = C'_k \cap F$ and which is a general member of a family with F' as its special member and with $C_t \cup C'_k \cup G''$ nodal. By semicontinuity we have $h^0(\mathcal{I}_{C_t \cup C'_k \cup G''}(m-1)) = 0$. We have $C_t \cup C'_k \cup G'' \in U(t,k,a_d,b)$.

(b) Set $\alpha := t(t-2)$ if $k = t$ and $\alpha := t^2 - 3t + 1$ if $k = t-1$. Fix a plane H, a smooth conic $D \subset H$ and general $C_{t,k}$. We have $D \cap C_{t,k} = \emptyset$ and $C_{t,k} \cap H$ is a general subset of H with cardinality $d_{t,k}$. Hence $h^0(H, \mathcal{I}_{H \cap (C_{t,k} \cup D)}(t+k)) = h^0(H, \mathcal{I}_{C_{t,k} \cap H}(t+k-1)) = \binom{t+k+1}{2} - d_{t,k} = \alpha$ and $h^1(H, \mathcal{I}_{H \cap (C_{t,k} \cup D)}(t+k)) = 0$. Then we continue the construction from the critical value $t+k$ to the critical value $t+k+2$, then to the critical value $t+k+4$, and so on up to the critical value $m-2$; in each step, say to arrive at the critical value x from a curve A' and a set S' with $h^1(\mathcal{I}_{A' \cup S'}(x-2)) = 0$ and $h^0(\mathcal{I}_{A' \cup S'}(x-2)) = \alpha$ and $0 \le \sharp(S') \le x-3$ (and so $\sharp(S') = \binom{x+1}{3} - (x-2)\deg(A') - 3 + g - \alpha$; we have bijectivity inside Q and get a curve A'' and a set S'' with $h^1(\mathcal{I}_{A'' \cup S''}(x)) = 0$ and $h^0(\mathcal{I}_{A'' \cup S''}(x)) \le \alpha$. In the last step we also need to connect the connected components of the curve and get an element $B \in U(t,k,a',b)$ for some a'; we need to check that at each step the numerical conditions are satisfied. Call $(X, Q, D_1, D_2, S_1, S_2)$ the curve we get for $\mathcal{O}_{\mathbb{P}^3}(m-2)$ and either $e = 1$ or $e = 2$. Set $S := S_1 \cup S_2$ and $\alpha' := \sharp(S)$. We have $0 \le \alpha' \le m-3$. Since S is a union of connected components of $X \cup S$, the restriction map $H^0(\mathcal{O}_{X \cup S}(m-2)) \to H^0(\mathcal{O}_X(m-2))$ is surjective and its kernel has dimension $\sharp(S)$. Since $h^1(\mathcal{I}_{X \cup S}(m-2)) = 0$, we have $h^1(\mathcal{I}_X(m-2)) = 0$ and $h^0(\mathcal{I}_X(m-2) = \alpha + \alpha' \le \alpha + m - 3$. We cover in this way the integers d such that $\binom{m+3}{3} + g - 1 - dm \ge \alpha + m - 3$. Hence we cover all d such that $d(m,g)_{\max} - d \ge 1 + \lfloor \alpha/m \rfloor$. If $t \le m/4$ we have $\alpha/m \le m/4$.

5.0.2. *Near $d(m,g)_{\max}$.* In this range the most difficult part is the existence of X_1 with $h^1(\mathcal{I}_{X_1}(m)) = 0$ and it is this part which dictates the component $W(t',k',a',b')$ in which we will find both X_1 and X_2. We stress that the integers t,k introduced in this subsection are not the same as in the previous one and hence also y may be different.

(a) In this step we prove the existence of X_1. We start with the maximal integer k such that $g_{k+1-\epsilon,k} + g(2k+6-\epsilon, k+1-\epsilon, k) \le g$ and set $t := k+1-\epsilon$. We use $N(x,t,k,y)$. In particular we have $N(m-4,t,k,y)$ and $N(m-2,t,k,y)$. Set $a_d := d - d_{t,k}$ and $b := g - g_{t,k}$. In this step we prove the existence of $A \in U(t,k,a_d,b)$ with $h^1(\mathcal{I}_A(m)) = 0$, hence by semicontinuity the existence of $X_1 \in W(t,t-1,a_d,b)$ with $h^1(\mathcal{I}_{X_1}(m)) = 0$. Set $z := d - a(m-2,t,k,y) - d_{t,k}$. We write $C_t \cup C'_k$ for a general (but fixed in this construction) $C_{t,k}$, because we need to distinguish the two connected components, even when $k = t$. Recall that we have (1).

(a1) Assume $d = d(m,g)_{\max}$. Let T be any curve satisfying $N(m,t,k,y)$. We have $\deg(T) = d_{t,k} + a(m,t,k,y)$, $h^1(\mathcal{O}_T) = g$, $h^1(\mathcal{O}_T(m)) = 0$, T has 3 connected components, $h^1(\mathcal{I}_T(m)) = 0$ and $h^0(\mathcal{I}_T(m)) = b(m,t,k,y)$. By (1) we have $d = a(m,t,k,y) + d_{t,k}$ if $b(m,t,k,y) \le m-3$ and $d = a(m,t,k,y) + d_{t,k} + 1$ if $m-2 \le b(m,t,k,y) \le m-1$. Hence $a(m,t,k,y) - a(m-2,t,k) \le z \le a(m,t,k,y) - a(m-2,t,k,y)+1$. Call η the difference between the right hand side and the left hand side of (1).

Assertion $N''(m-2,t,k,y)$, $m \equiv y \pmod 2$: Set $e = 1$ if $b(m-2,t,k,y) \le z-3$ and $e = 2$ if $b(x,t,k,y) \ge z-2$. There is a 6-tuple (X,Q,D_1,D_2,S_1,S_2) such that

- (a) Q is a smooth quadric surface, $X = C_t \sqcup C'_k \sqcup Y$, Y is a smooth curve of degree $a(m-2,t,k,y)$ and genus $g - g_{t,k}$ and Q intersects transversally X, with no line of Q containing ≥ 2 points of $X \cap Q$;
- (b) D_1, D_2 are different elements of $|\mathcal{O}_Q(1,0)|$, $D_1 \cap C_t \ne \emptyset$, $D_2 \cap C'_k \ne \emptyset$, $S_i \subset D_i \setminus D_i \cap (C_t \cup C'_k)$, $1 \le i \le 2$, and $\sharp(S_1) + \sharp(S_2) = b(x,t,k,y)$; $\pi_2(S_2) \subseteq \pi_2(S_1)$ and $\pi_2(S_e) \subset \pi_2(Y \cap (Q \setminus (D_1 \cup D_2)))$; $S_2 = \emptyset$ if $e = 1$, $\sharp(S_2) = b(m-2,t,k,y) - z + 2$ if $e = 2$;
- (c) $h^i(\mathcal{I}_{X \cup S_1 \cup S_2}(x)) = 0$, $i = 0,1$.

As in the proof of Lemma 6 and Lemma 11 we get (X,Q,D_1,D_2,S_1,S_2), $X = C_t \sqcup C'_k \sqcup Y$ satisfying $N''(m-2,t,k,y)$; in the proof of Lemma 6 we take R_1 containing a point of $C_t \cap Q$ instead of a point of $Y \cap Q$ and R_2 containing a point of $C'_k \cap Q$ instead of a point of $Y \cap Q$.

(a1.1) Assume $b(m-2,t,k,y) = 0$. Take $z-1$ distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le z-1$, such that $L_i \cap C_t = \emptyset$ for all i, $L_i \cap C'_k \ne \emptyset$ if and only if $i = 1$ and $L_i \cap Y \ne \emptyset$ if and only if $i = 2$. Set $J := D_1 \cup (\bigcup_{i=1}^{z-1} L_i)$. Since $X \cap (Q \setminus J)$ is a general subset of Q with cardinality $2d_{t,k} + 2a(m-3,t,k,y) - 3$, we have $h^1(Q, \mathcal{I}_{Q \cap (X \cup J)}(m)) = h^1(Q, \mathcal{I}_{X \cap (Q \setminus J)}(m-1, m+1-z)) = 0$ (use the generality of $X \cap (Q \setminus J)$ and the difference between (1) and the case $x := m-2$ of (10), which gives an upper bound for $\sharp(X \cap (Q \setminus J))$; we get an equality if and only if $\eta = 0$, i.e. $b(m,t,k,y) = m-2$ and $d = a(m,t,k,y) + d_{t,k} + 1$). Since $\mathrm{Res}_Q(X \cup J) = X$ and $h^1(\mathcal{I}_X(m-2)) = 0$, we have $h^1(\mathcal{I}_{X \cup J}(m)) = 0$. The union $X \cup J$ is a nodal and connected smoothable curve of degree d and arithmetic genus g and $Y \cup J$ is a smooth and connected curve of degree $d - d_{t,k}$ and arithmetic genus $g - g_{t,k} - 2 \ge 26$. We may smooth $Y \cup J$ in a family of curves, all of them containing the two points $(C_t \cup C'_k) \cap J$. Call E a general element of this smoothing. Since $\mathrm{Aut}(\mathbb{P}^3)$ is 2-transitive, we may see E as a general non-special space curve of its degree and its genus ≥ 26. By construction and Lemma 2 we have $C_t \cup C'_k \cup E \in U(t,k,a_d,b)$ and $h^1(N_{C_t \cup C'_k \cup E}(-1)) = 0$. By semicontinuity there is a smooth $X_1 \in W(t,k,a_d,b)$ with $h^1(\mathcal{I}_{X_1}(m)) = 0$ and $h^1(N_{X_1}(-1)) = 0$.

(a1.2) Assume $0 < b(m-2,t,k,y) \le z-3$. Hence $S_2 = \emptyset$. We take D_1 and call $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le b(m-2,t,k,y)$, the elements of $|\mathcal{O}_Q(0,1)|$ such that $S_1 = D_1 \cap (L_1 \cup \cdots \cup L_{b(m-2,t,k,y)})$; note that each line L_i contains a point of $Y \cap Q$. Take any $L_{b(m-2,t,k,y)+1} \in |\mathcal{O}_Q(0,1)|$ with $C'_k \cap L_{b(m-2,t,k,y)+1} \ne \emptyset$, any $L_{b(m-2,t,k,y)+2} \in |\mathcal{O}_Q(0,1)|$ with $Y \cap L_{b(m-2,t,k,y)+2} \ne \emptyset$, $L_{b(m-2,t,k,y)+2} \ne L_i$ for $i \le b(m-2,t,k,y)$ and (if $b(m-2,t,k,y) < z-3$) take general $L_j \in |\mathcal{O}_Q(0,1)|$, $b(m-2,t,k,y)+3 \le j \le z-1$. Set $J := D_1 \cup (\bigcup_{i=1}^{z-1} L_i)$, $\chi := \cup_{o \in S_1} \chi(o)$ and $W := X \cup J \cup \chi$. We have $\mathrm{Res}_Q(W) = X \cup S_1$ and thus $h^1(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-2)) = 0$. Since $\eta \ge 0$, (1) and the case $x = m-2$ of (11) give $2d_{t,k} + 2a(m,t,k,y) - b(m-

$2,t,k,y)-3 = m(m+3-z)-\eta \le h^0(Q,\mathcal{O}_Q(m-2,m+2-z))$. Since $W\cap Q$ is the union of J and $2d_{t,k}+2a(m,t,k,y)-b(m-2,t,k,y)-3$ general points of Q, we have $h^1(Q,\mathcal{I}_{W\cap Q}(m)) = h^1(Q,\mathcal{I}_{X\cap(Q\setminus J)}(m-1,m+1-z)) = 0$. Thus $h^1(\mathcal{I}_W(m)) = 0$. We first deform W to the union F of $C_t\cup C'_k\cup D_1\cup Y\cup(\bigcup_{i=b(m-3,t,k,y)+1}^{z-1} L_i)$ and $b(m-3,t,k,y)$ disjoint lines $M_1,\dots,M_{b(m-3,t,k,y)}$, each of them containing one point of Y. The union F is a nodal and connected curve. Write $F = C_t\cup C'_k\cup G$. We have $\sharp(G\cap C_t) = \sharp(G\cap C'_k) = 1$. Let G' be a general smoothing of G fixing the 2 points of $(C_t\cup C'_k)\cap G$. $C_t\cup C'_k\cup G' \in U(t,k,a_d,b)$. By Lemma 2 and semicontinuity there is a smooth $X_2\in W(t,k,a_d,b)$ with $h^1(\mathcal{I}_{X_2}(m)) = 0$ and $h^1(N_{X_2}(-1)) = 0$.

(a1.3) Assume $b(m-2,t,k,y)\ge z-2$. Since $z\ge a(m,t,k,y)-a(m-2,t,k)$ and $b(m-2,t,k,y)\le m-3$, the case $x = m-2$ of Lemma 7 gives $2(z-3)\ge b(m-2,t,k,y)$. Set $w := b(m-2,t,k)-z+3$. Let $L_i\in|\mathcal{O}_Q(0,1)|$, $1\le i\le z-3$, be the line such that $S_1 = D_1(\bigcup_{i=1}^{z-3} L_i)$ and $S_2 := D_2\cap(\bigcup_{i=1}^{w} L_i)$. Let $L_{z-2}\in|\mathcal{O}_Q(0,1)|$ be a line with $L_{z-2}\ne L_i$ for any $i\ne z-2$ and $L_{z-2}\cap Y\ne\emptyset$. Note that $L_j\cap Y\ne\emptyset$ if and only if either $j\le w$ or $j = z-2$. Set $J := D_1\cup D_2\cup(\bigcup_{i=1}^{z-2} L_i)$, $\chi := \cup_{o\in S_1\cup S_2}\chi(o)$ and $W := X\cup J\cup\chi$ and continue as in the last step.

(a2) Assume $d < d(m,g)_{\max}$ We have $\eta\ge m(d(m,g)_{\max}-d)\ge m$ and in particular $\eta\ge m\ge b(m-2,t,k,y)+2$. To prove the existence of X_1 in this component we only need that $z\ge 3$, i.e. that $d\ge a_{m-2,t,k,y}+d_{t,k}+3$, which is true because $1+(m-1)d-g\ge\binom{m+2}{3}$ and $(m-1)(a(m-2,t,k,y)+d_{t,k})+3-g = \binom{m+1}{2}-a(m-2,t,k)-d_{t,k}+b(m-2,t,k,y)\ge 3m$. Take (X,Q,D_1,D_2,S_1,S_2) satisfying $N(m-2,t,k,y)$ with $X = C_t\sqcup C'_k\sqcup Y$ and throw away D_1, D_2, S_1 and S_2. Fix $D\in|\mathcal{O}_Q(1,0)|$ containing one point of $Y\cap Q$ and $z-1$ distinct lines $L_i\in|\mathcal{O}_Q(0,1)|$ with $L_i\cap Y = \emptyset$ for all i, $L_i\cap C_t\ne\emptyset$ if and only if $i = 1$ and $L_i\cap C'_k\ne\emptyset$ if and only if $i = 2$. Set $J := D\cup(\bigcup_{i=1}^{z-1} L_i)$ and $W := X\cup J$. As in the previous steps it is sufficient to prove that $h^1(\mathcal{I}_W(m)) = 0$. We have $\mathrm{Res}_Q(W) = X$ and thus $h^1(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-2)) = 0$. Hence it is sufficient to prove that $h^1(Q,\mathcal{I}_{W\cap Q}(m)) = 0$. We have $h^1(Q,\mathcal{I}_{Q\cap W}(m)) = h^1(Q,\mathcal{I}_{X\cap(Q\setminus J)}(m-1,m+1-z))$. Since $X\cap Q$ is general in Q, it is sufficient to prove that $\sharp(X\cap(Q\setminus J))\le m(m+2-z)$. We have $\sharp(X\cap(Q\setminus J)) = 2d_{t,k}+2a(m-2,t,k,y)-3$. By the definition of η and (10) for $x = m-2$ we have $2d_{t,k}+2a(m-2,t,k,y)-3 = m(m+2-z)+b(m-2,t,k,y)+2-\eta\le m(m+2-z)$.

(b) In this part we get the existence of $A\in U(t,k,a_d,b)$ with $h^0(\mathcal{I}_A(m-1)) = 0$, $\deg(A) = d$ and $p_a(A) = g$, hence by semicontinuity the existence of $X_2\in W(t,k,a_d,b)$ with $h^0(\mathcal{I}_{X_2}(m-1)) = 0$. We have $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0,1$ and $m-1\equiv t+k \pmod 2$. Fix a plane H. Let c be the maximal integer such that $\binom{t+k+2-c}{2}\le d_{t,k}$. Let $E\subset H$ be a general linear projection of a general smooth and rational degree c curve $E'\subset\mathbb{P}^3$. The curve E is nodal and it has $(c-1)(c-2)/2$ singular points. Set $\chi := \cup_{p\in\mathrm{Sing}(E)}\chi(p)$. The union $E\cup\chi$ is the flat limit of a family of degree c smooth rational curves in $\mathbb{P}^3$ ([14, Fig. 11 at p. 260]. Hence to prove that a general union of some $C_{t,k}$ and a smooth rational curve of degree c is contained in no surface of degree $t+k$ it is sufficient to prove that $h^0(\mathcal{I}_{C_{t,k}\cup E\cup\chi}(t+k) = 0$ for a general $C_{t,k}$. Thus it is sufficient to prove that $h^0(\mathcal{I}_{C_{t,k}\cup E}(t+k)) = 0$ for a general $C_{t,k}$. For a general $C_{t,k}$ we have $C_{t,k}\cap E = \emptyset$ and $C_{t,k}\cap H$ is a general subset of H with cardinality $d_{t,k}$. By definition c is the minimal positive integer such that $h^0(H,\mathcal{I}_{C_{t,k}\cap H}(t+k-c)) = 0$. Set $\beta = h^0(\mathcal{O}_{C_{t,k}\cup E\cup\chi}(t+k))-\binom{t+k+3}{3}$. Since $\binom{t+k+2-c}{2}-\binom{t+k-1}{2} = t+k+1-c$, we have $\beta \le (c-1)(c-2)/2 + t + k + 1 - c$. Then we continue from the critical value $t+k$ to the critical value $t+k+2$ and so on.

At the end we obtain some $B \in U(t,k,a_d,b)$ with $h^0(\mathcal{I}_B(m-1)) = 0$ if $1 + d(m-1) - g \ge \binom{m+2}{3} + \beta$. In particular it is sufficient to assume $d \ge d(m,g)_{\min} + \lceil \beta/(m-1) \rceil$. We have $c \sim \sqrt{2}t$, because $\deg(C_{t,k}) \sim t^2$ and $\binom{t+k+2}{2} \sim 2t^2$. Hence $\beta \sim (c-1)(c-2)/2 \sim t^2$. Since $t \le m/4$, it is sufficient to have roughly $d \ge d(m,g)_{\min} + m/4$.

Lemma 13. *Fix t and $k \in \{t-1, t\}$ such that $y \equiv t+k-1 \pmod 2$ and let $g_{t,k} + g(t+k+5,t,k) \le g \le -1 + g_{t+1,k+1} + g(t+k+7,t+1,k+1)$. Then we have $y \le \sqrt{20}t - 1$. In particular, if $t \ge \lfloor m/\sqrt{20} \rfloor - 5$ then $y \le m-6$.*

Proof. We have $g_{t+1,k+1} - g_{t,k} = 2t^2 - 2$ if $k = t$ and $g_{t+1,k+1} - g_{t,k} = 2t^2 - 2t - 1$ if $k = t-1$. By definition of y, we have $y \ge k+t+5$ and $g \ge g_{t,k} + g(y,t,k) = g_{t,k} + c(y,t,k) - c(t+k+1,t,k) - 3(y-t-k-1)/2 - 3 = g_{t,k} + \sum_{i=1}^{(y-t-k-1)/2} (t+k+1+2i+3)/2 - 3(y-t-k-1)/2 - 3 = g_{t,k} + 1/8(t+k+y+9)(y-t-k-1) - 3(y-t-k-1)/2 - 3$. On the other hand, we have $g \le -1 + g_{t+1,k+1} + g(t+k+7,t+1,k+1) \le -1 + g_{t+1,k+1} + 3(t+k+7)$. Hence we get $1/8(t+k+y+9)(y-t-k-1) \le g_{t+1,k+1} - g_{t,k} + 3(y-t-k-1)/2 + 3 - 1 + 3(t+k+7)$ and in particular $(y+1)^2 \le 20t^2$. □

Proof of Theorem 1: We fix the integer g and we perform the above construction in both the odd and the even case, by taking either $k = t$ or $k = t-1$. We have $h^1(\mathcal{O}(C_{t,k}(t-1) = 0$, hence we get $h^1(\mathcal{O}(C_X(t-1) = 0$ by a repeated application of Mayer-Vietoris and semicontinuity. For every $t \ge 27$ such that $g \ge g_{t+3,k+3} \ge g_{t,k} + g(t+k+5,t,k)$ we get an integer $y \equiv t+k-1$ such that the statement of Theorem 1 holds for every $m \ge y+6$ with $m \equiv y \pmod 2$. By Lemma 13, the condition $m \ge y+6$ is satisfied for every $t \ge \lfloor m/\sqrt{20} \rfloor - 5$, hence we obtain our statement for every g with $2g_{30} = 17052 \le g \le \varphi(m)$. □

Proof of Corollary 1: Let m be the minimal non-negative integer such that

$$md + 1 - g \le \binom{m+3}{3} \tag{15}$$

The minimality of m gives

$$(m-1)d + 1 - g > \binom{m+2}{3}, \tag{16}$$

in particular $d \ge \frac{(m+2)(m+1)m}{6(m-1)} \le \frac{m^2}{6}$. From (15) and (16) we get $d \le \binom{m+2}{2}$. Since $g \le Kd^{3/2} - 6\epsilon d$, we have

$$\begin{aligned} g &\le \frac{2}{3}\left(\frac{1}{10}\right)^{3/2} \binom{m+2}{2}^{3/2} - 6\epsilon d \\ &\le \frac{2}{3}\left(\frac{1}{10}\right)^{3/2} \left(\frac{(m+2)^2}{2}\right)^{3/2} - 6\epsilon d \\ &\le \frac{2}{3}\left(\frac{1}{20}\right)^{3/2} (m+2)^3 - \epsilon m^2 \le \varphi(m) \end{aligned}$$

(notice that the coefficients of m^3 are controlled by our choice of K and the coefficients of m^2 are controlled by our choice of ϵ). Since $g \leq \varphi(m)$, Theorem 1 covers all degrees d_0 in the interval $d(m,g)_{\min} \leq d_0 \leq d(m,g)_{\max}$. In order to check that d is in this interval, just notice that $d \geq d(m,g)_{\min}$ by (16) and $d \leq d(m,g)_{\max}$ by (15). □

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