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**Abstract**

The paper is concerned with the derivation of a general imperfect interface law in a linear multiphysics framework for a composite, constituted by two solids, separated by a thin adhesive layer. The analysis is performed by means of the asymptotic expansions technique. After defining a small parameter $\varepsilon$, which will tend to zero, associated with the thickness and the constitutive coefficients of the intermediate layer, we characterize three different limit models and their associated limit problems: the soft interface model, in which the constitutive coefficients depend linearly on $\varepsilon$; the hard interface model, in which the constitutive properties are independent of $\varepsilon$; the rigid interface model, in which they depend on $\frac{1}{\varepsilon}$. The asymptotic expansion method is reviewed by taking into account the effect of higher order terms and by defining a general multiphysics interface law which comprises the above aforementioned models.
**Keywords.** Asymptotic analysis; interfaces; multiphysics materials.

# 1 Introduction

During the last decades, the interest in bonded structures, obtained by assembling different parts made of possibly different materials to compose a unique construction, is strongly increased. The advantages offered by the modern bonding techniques are numerous; among the others, it can be mentioned that they allow the assembly to enhance its mechanical properties, which are designed to be superior to those of the constituent materials, and to acquire specific requirements of resistance and comfort. On the other hand, in bonded structures as well as in composites, the bond between the various parts may generally be imperfect and discontinuities in physical fields can arise, which significantly change the mechanical and physical properties of the material. Hence, a correct theoretical modelling of imperfect bonding plays an important role in engineering design.

A classical theoretical treatment for bonded region is to assume the existence of a thin interphase between two adjacent parts. With a diminishing thickness, the interphase is replaced by a surface, the so-called *imperfect interface*, on which some of the physical fields undergo appropriately designed jump conditions, simulating the effect of the interphase layer. Various interface models have been developed throughout the years by means of classical variational tools and more refined mathematical techniques, spanning from uncoupled phenomena, such as thermal conduction and elasticity, to multifield and multiphysics theories, such as continua with microstructure and piezoelectricity.

In the context of uncoupled phenomena, lowly-conducting (LC) (thermal or electrical) interphases give rise to imperfect interfaces exhibiting a discontinuity in the temperature field but preserving the continuity of the normal component of the heat flux, the so-called Kapitza's resistance model, see, e.g., [4, 5, 29]. Conversely, at a highly-conducting (HC) interface, a jump in the normal component of the heat flux is found, proportional to the surface Laplacian of the interface potential, see e.g. [38, 39]. A unifying approach of a general imperfect interface model, involving the concurrent jump of both the temperature field and the normal heat flux, recovering both the LC and HC models, has been proposed by [26, 28]. In the setting of linear elasticity, many works have been devoted in the formulation of interface models, first built on engineering-based and phenomenological observations and then, rigorously justified with diverse mathematical tools. Three types of imperfect interface have been proposed, namely, the spring-layer interface model (SL) (also known as soft or weak interface), the coherent interface (CI) (also known as rigid or strong or membrane-type interface) and the general imperfect interface. According to the SL model, the traction vector is continuous across the interface, while the displacement vector suffers a discontinuity linearly proportional to the traction vector. This model represents a classical theoretical description of an imperfect interface, firstly proposed by Goland and Reissner [21] and widely used in engineering applications, see, e.g., [3, 25, 27]. According to the CI model, the displacement vector field is continuous across the interface, while the traction vector field suffers a jump discontinuity and satisfies a two-dimensional Laplace-Young equation. This particular model has been firstly developed for continuum theories with surface effects (see, e.g., [24, 40]) and then employed in nano-sized materials and structures (see, e.g., [51–53]). Finally, in the general imperfect model, both the displacement and normal traction fields are discontinuous across the interface [6–8, 27]. Within the framework of linear multiphysics theories, several works have suggested a generalization of the classical LC/SL and HC/CI interface models, including the effects of other physical interactions, see, for instance, [13, 16, 42, 50].

Moreover, a general multiphysics imperfect interface model, which can be specialized to electric (or thermal) conduction, elasticity, piezoelectricity and piezomagnetoelectricity, has been developed and numerically tested by [22, 23, 49], adapting the interface models conceived in [6, 7].

The asymptotic analysis represents a powerful mathematical technique, usually employed in the derivation and justification of classical thin structures and layered composites models [12, 18, 47, 48, 54]. This method relies on the definition of a small parameter $\varepsilon$, being the thickness of the interphase. Under the assumption that the interphase elastic coefficients rescale like the power of its small thickness, i.e., $\varepsilon^p$, the analysis allows to characterize different limit behaviors, by letting $\varepsilon$ tend to zero: for $p = 1$, a soft interface model can be recovered, which correspond to a SL elastic model, see, for instance, [2, 19, 20, 30, 35]; for $p = -1$, the membrane-type elastic or rigid interface model has been mathematically justified by means of strong convergence arguments in [9–11, 36]. Within the framework of a higher-order theory, assuming the interphase elastic constants independent of the small thickness ($p = 0$), the asymptotic analysis yields to a general stiff imperfect interface condition, prescribing both the jumps of the displacement and traction vector fields and recovering as a particular case the perfect contact conditions at the zero-th order [1, 14, 31, 32, 41, 43]. Using asymptotic analysis and strong convergence methods, M. Serpilli generalized the soft and rigid interface conditions to some particular multiphysics and multifield composites, having piezoelectric and magneto-electro-thermo-elastic couplings [44, 45] or presenting an internal microstructure (micropolar elasticity, [46]).

## 1.1 Main objectives and contributions of this work

The aim of this work is to provide a general imperfect interface law taking into account linear coupled multiphysics phenomena. The interface conditions will be given in terms of the jumps of the multiphysics state (kinematical variable) and of its conjugated counterpart (a generalized traction vector), respectively.

Differently from the approach followed in [6] and extended to linear multiphysics materials in [22, 23], in which Taylor's expansions of the relevant fields are considered, in the present paper the analysis is based on the asymptotic expansions method. Three different rescaling of the material parameters for the multiphysics thin interphase are considered and three limit regimes are analyzed: the soft case, where the material parameters rescale like the thickness; the hard case, where the material parameters are independent of the thickness; the rigid case, where the material parameters rescale like the inverse of the thickness. For each case, we compute the governing equations in the adherents and the transmission conditions on the imperfect interface at order 0 and at higher orders (order 1) of the asymptotic expansion. The results obtained at higher orders allow to define non trivial transmission conditions, which better approximate the linear coupled multifield behavior of the thin interphase. Afterwards, using an approach already introduced in [41] and based on matching conditions, we combine the results of the asymptotic analysis to obtain an original compact general imperfect interface model, which comprises in itself the three aforementioned limit behaviors. The general imperfect interface can be then particularized to any linear coupled or uncoupled multiphysics material. Moreover, the variational formulation for the general multiphysics interface problem is obtained: this is a key step to the numerical implementation of the interface conditions and to study the well-posedness of the limit problem. It is worth mentioning the unifying character of the general imperfect interface conditions, which leads to a straight-forward generalization of some previous works by the authors, see, e.g., [33, 41, 44–46].

Lastly, we must highlight the main difference between our general imperfect interface model (cf. eqns. (20)-(21)) and the one proposed, for instance, in [22, 23]: the present interface model depends just on the constitutive parameters of the interphase layer, whose thickness goes to zero in the asymptotic process; while concerning the model developed by S.T. Gu and co-authors [22, 23], which employes Y. Benveniste's methodology [6], consisting in the extension of the surrounding bodies to the separation surface, the interface conditions involve information not only on the thickness of the eliminated interphase, its material properties, but also on the material properties of the neighbouring media. These additional terms, depending on the adherents, could lead in some particular situations to contradictory results (cf. Sect. 6 in [33]).

### 1.2 Outline of the manuscript

The structure of the present paper is as follows. In Section 2 we give the preliminary notation and statement the balance equations, with the rescaling and asymptotic expansions method assumptions. In Section 3, Section 4 and Section 5, we characterize the three regimes at order 0 and order 1, namely the soft, hard and rigid cases, respectively. In Section 6, we show that it is possible to group the three limit cases of soft, rigid and hard interfaces in an unique, implicit, non-local, multifield imperfect interface law (cf. eqns. (20)-(21)), which recovers the three cases as particular cases. In Section 7, a variational formulation for the general multiphysics interface problem is obtained. In Section 8, we study the specialization of the general multiphysics imperfect interface equations to the cases of linear elastic, piezoelectric and thermoelastic material. In Section 9, we present an example, referring to the stretching of a piezoelectric composite. Finally, Section 10 contains some conclusive remarks.

## 2 Position of the problem

In the sequel, Greek indices range in the set $\{1, 2\}$, Latin indices range in the set $\{1, 2, 3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted. Let us consider a three-dimensional Euclidian space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. We also introduce the following notations for, respectively, the scalar product: $\mathbf{a} \cdot \mathbf{b} := a_i b_i$, for all vectors $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$.

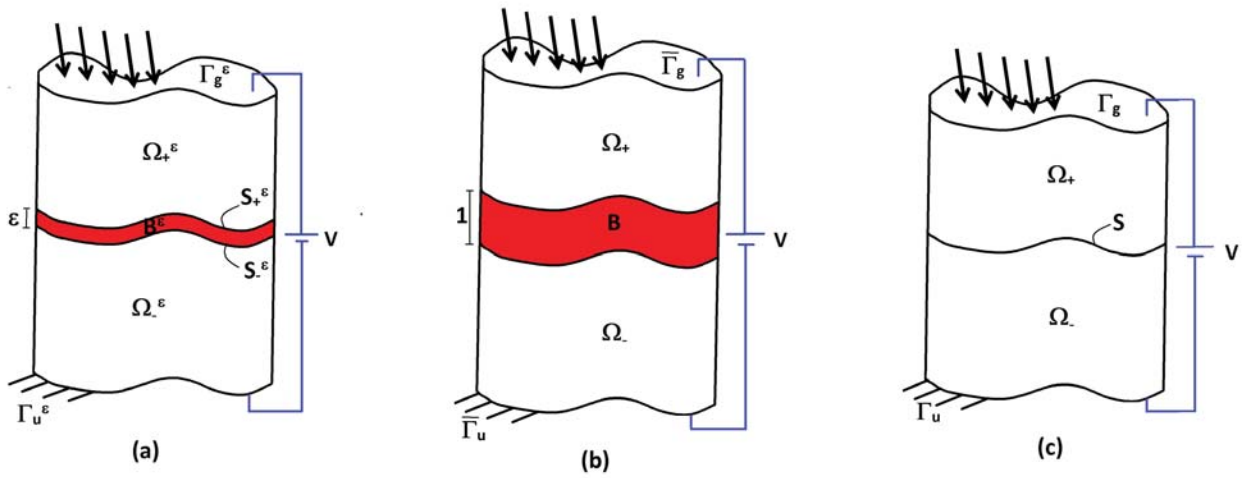


Figure 1: Initial (a), rescaled (b) and limit (c) configurations of the composite.

Let us define a small parameter $0 < \varepsilon < 1$. We consider the assembly constituted of two solids $\Omega_\pm^\varepsilon \subset \mathbb{R}^3$, called the adherents, bonded together by an intermediate thin layer $B^\varepsilon := S \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ of thickness $\varepsilon$, called the adhesive, with cross-section $S \subset \mathbb{R}^2$. In the following $B^\varepsilon$ and $S$ will be called interphase and interface, respectively. Let $S_\pm^\varepsilon$ be the plane interfaces between the interphase and the adherents and let $\Omega^\varepsilon := \Omega_+^\varepsilon \cup B^\varepsilon \cup \Omega_-^\varepsilon$ denote the composite system comprising the interphase and the adherents (cf. Figure 1.a).

In the sequel, we denote by $\|\cdot\|_{s,\Omega}$ the norm of the Sobolev space $H^s(\Omega;\mathbb{R}^d)$, for all $d \geq 1$, $|\cdot|_{0,\Omega}$ and $(\cdot,\cdot)_{0,\Omega}$ stand for the norm and the scalar product in $L^2(\Omega;\mathbb{R}^d)$, respectively. Obviously, the same holds in $\Omega_\pm^\varepsilon$, $B^\varepsilon$, $S$.

## 2.1 Constitutive law

We suppose that the composite is constituted of a material which presents a linear coupled multiphysics behaviour. We assume that the state at the equilibrium of the multiphysics material is characterized by a collection of order parameters, using the multifield theory jargon (see, e.g., [37]): $N$ vector state variables, namely $\mathbf{u}_1^\varepsilon, \ldots, \mathbf{u}_N^\varepsilon$, and $M$ scalar state variables, namely $\varphi_1^\varepsilon, \ldots, \varphi_M^\varepsilon$. Let us put together all the unknowns into a generalized vector field $\mathbf{s}^\varepsilon := (\mathbf{u}_1^\varepsilon, \ldots, \mathbf{u}_N^\varepsilon, \varphi_1^\varepsilon, \ldots, \varphi_M^\varepsilon)$, the so-called *multiphysics state*. With the multiphysics state $\mathbf{s}^\varepsilon$, we associate its conjugated physical quantity $\mathbf{t}^\varepsilon = \mathbf{t}^\varepsilon(\boldsymbol{\nabla}^\varepsilon \mathbf{s}^\varepsilon)$, where $(\boldsymbol{\nabla}^\varepsilon \mathbf{s}^\varepsilon)_i := \mathbf{s}_{,i}^\varepsilon = (\mathbf{u}_{1,i}^\varepsilon, \ldots, \mathbf{u}_{N,i}^\varepsilon, \varphi_{1,i}^\varepsilon, \ldots, \varphi_{M,i}^\varepsilon)$ denotes the gradient of $\mathbf{s}^\varepsilon$. The vector field $\mathbf{t}^\varepsilon := (\boldsymbol{\sigma}_1^\varepsilon, \ldots, \boldsymbol{\sigma}_N^\varepsilon, \mathbf{D}_1^\varepsilon, \ldots, \mathbf{D}_M^\varepsilon)$ represents a generalized stress field. We consider the following homogeneous and linear constitutive law:

$$\mathbf{t}^\varepsilon = \mathbb{K}^\varepsilon \boldsymbol{\nabla}^\varepsilon \mathbf{s}^\varepsilon,$$

where $\mathbb{K}^\varepsilon$ is a generalized linear constitutive matrix. Component-wise, one has

$$\begin{cases} \boldsymbol{\sigma}_J^\varepsilon = \mathbf{C}_{JK}^\varepsilon \nabla^\varepsilon \mathbf{u}_K^\varepsilon + \mathbf{P}_{JI}^\varepsilon \nabla^\varepsilon \varphi_I^\varepsilon, \\ \boldsymbol{D}_L^\varepsilon = \mathbf{R}_{LK}^\varepsilon \nabla^\varepsilon \mathbf{u}_K^\varepsilon + \mathbf{H}_{LI}^\varepsilon \nabla^\varepsilon \varphi_I^\varepsilon, \end{cases}$$

with $J, K = 1, \ldots, N$ and $I, L = 1, \ldots, M$. The constitutive tensor $\mathbb{K}^\varepsilon$ satisfies the classical symmetry and positivity properties.

## 2.2 Governing equilibrium equation

We assume that the adherents are subject to a generalized system of body forces $\mathbf{F}^\varepsilon : \Omega^\varepsilon_\pm \to \mathbb{R}^{3N\times M}$ and surface forces $\mathbf{G}^\varepsilon : \Gamma^\varepsilon_g \to \mathbb{R}^{3N\times M}$, where $\Gamma^\varepsilon_g \subset (\partial\Omega^\varepsilon_+ \setminus S^\varepsilon_+) \cup (\partial\Omega^\varepsilon_- \setminus S^\varepsilon_-)$. Body and surface forces are neglected in adhesive layer. On $\Gamma^\varepsilon_u := (\partial\Omega^\varepsilon_+ \setminus S^\varepsilon_+) \cup (\partial\Omega^\varepsilon_- \setminus S^\varepsilon_-) \setminus \Gamma^\varepsilon_g$ homogeneous boundary conditions are prescribed, so that $\mathbf{s}^\varepsilon = \mathbf{0}$ on $\Gamma^\varepsilon_u$. We assume that everywhere, near the interphase boundary $\Gamma_{lat} := \partial S \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, homogeneous Neumann boundary conditions are applied. The fields of the external forces are endowed with sufficient regularity to ensure the existence of equilibrium configuration. It is assumed that the adhesive and the adherents are perfectly bonded in order to ensure the continuity of the multiphysics state and generalized stress vector fields across $S^\varepsilon_\pm$. The differential formulation of the governing equations has the following structure:

$$\begin{cases} -\text{div } \mathbf{t}^\varepsilon = \mathbf{F}^\varepsilon & \text{in } \Omega^\varepsilon, \\ \mathbf{t}^\varepsilon \mathbf{n}^\varepsilon = \mathbf{G}^\varepsilon & \text{on } \Gamma^\varepsilon_g, \\ \mathbf{s}^\varepsilon = \mathbf{0} & \text{on } \Gamma^\varepsilon_u, \end{cases} \tag{1}$$

where $\mathbf{t}^\varepsilon \mathbf{n}^\varepsilon := (\boldsymbol{\sigma}^\varepsilon_1 \mathbf{n}^\varepsilon, \ldots, \boldsymbol{\sigma}^\varepsilon_N \mathbf{n}^\varepsilon, \mathbf{D}^\varepsilon_1 \cdot \mathbf{n}^\varepsilon, \ldots, \mathbf{D}^\varepsilon_M \cdot \mathbf{n}^\varepsilon)$ represents the generalized traction vector on the boundary $\Gamma^\varepsilon_g$ and $\mathbf{n}^\varepsilon$ the outer normal unit vector to $\Gamma^\varepsilon_g$. Let us introduce the functional space

$$V(\Omega^\varepsilon) := \{\mathbf{s}^\varepsilon \in H^1(\Omega^\varepsilon; \mathbb{R}^{3N\times M});\ \mathbf{s}^\varepsilon = \mathbf{0} \ \text{ on } \Gamma^\varepsilon_u\}.$$

The variational formulation of problem (1) defined on the variable domain $\Omega^\varepsilon$ reads as follows:

$$\begin{cases} \text{Find } \mathbf{s}^\varepsilon \in V(\Omega^\varepsilon) \ \text{ such that} \\ \bar{A}^\varepsilon_-(\mathbf{s}^\varepsilon, \mathbf{r}^\varepsilon) + \bar{A}^\varepsilon_+(\mathbf{s}^\varepsilon, \mathbf{r}^\varepsilon) + \hat{A}^\varepsilon(\mathbf{s}^\varepsilon, \mathbf{r}^\varepsilon) = L^\varepsilon(\mathbf{r}^\varepsilon), \end{cases} \tag{2}$$

for all $\mathbf{r}^\varepsilon := (\mathbf{v}^\varepsilon_1, \ldots, \mathbf{v}^\varepsilon_N, \psi^\varepsilon_1, \ldots, \psi^\varepsilon_M) \in V(\Omega^\varepsilon)$, where the bilinear forms $\bar{A}^\varepsilon_\pm(\cdot, \cdot)$, and $\hat{A}^\varepsilon(\cdot, \cdot)$ and the linear form $L^\varepsilon(\cdot)$ are defined by

$$\bar{A}^\varepsilon_\pm(\mathbf{s}^\varepsilon, \mathbf{r}^\varepsilon) := \int_{\Omega^\varepsilon_\pm} \bar{\mathbb{K}}^\varepsilon \boldsymbol{\nabla}^\varepsilon \mathbf{s}^\varepsilon \cdot \boldsymbol{\nabla}^\varepsilon \mathbf{r}^\varepsilon dx^\varepsilon, \quad \hat{A}^\varepsilon(\mathbf{s}^\varepsilon, \mathbf{r}^\varepsilon) := \int_{B^\varepsilon} \hat{\mathbb{K}}^\varepsilon \boldsymbol{\nabla}^\varepsilon \mathbf{s}^\varepsilon \cdot \boldsymbol{\nabla}^\varepsilon \mathbf{r}^\varepsilon dx^\varepsilon,$$

$$L^\varepsilon(\mathbf{r}^\varepsilon) := \int_{\Omega^\varepsilon_\pm} \mathbf{F}^\varepsilon \cdot \mathbf{r}^\varepsilon dx^\varepsilon + \int_{\Gamma^\varepsilon_g} \mathbf{G}^\varepsilon \cdot \mathbf{r}^\varepsilon d\Gamma^\varepsilon.$$

By virtue of the regularity of the loads, the positivity of the constitutive matrices and thanks to the Lax-Milgram's lemma, problem (2) admits one and only one solution.

## 2.3 Rescaling

In order to study the asymptotic behavior of the solution of problem (2) when $\varepsilon$ tends to zero, we rewrite the problem on a fixed domain $\Omega$

independent of $\varepsilon$. By using the approach of [12], we consider the bijection $\pi^\varepsilon : x \in \overline{\Omega} \mapsto x^\varepsilon \in \overline{\Omega}^\varepsilon$ given by

$$\pi^\varepsilon : \begin{cases} \bar{\pi}^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, x_3 \mp \frac{1}{2}(1-\varepsilon)), & \text{for all } x \in \overline{\Omega}_\pm, \\ \hat{\pi}^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3), & \text{for all } x \in \overline{B}, \end{cases}$$

where, after the change of variables, the adherents occupy $\Omega_\pm := \Omega^\varepsilon_\pm \pm \frac{1}{2}(1-\varepsilon)\mathbf{e}_3$ and the interphase $B = \{x \in \mathbb{R}^3 : \ (x_1, x_2) \in S, \ |x_3| < \frac{1}{2}\}$. The sets $S_\pm = \{x \in \mathbb{R}^3 : \ (x_1, x_2) \in S, \ x_3 = \pm\frac{1}{2}\}$ denote the interfaces between $B$ and $\Omega_\pm$ and $\Omega = \Omega_+ \cup \Omega_- \cup B$ is the rescaled configuration of the composite. Lastly, $\Gamma_g$ and $\Gamma_u$ indicate the images through $\pi^\varepsilon$ of $\Gamma^\varepsilon_g$ and $\Gamma^\varepsilon_u$ (cf. Figure 1.b). Consequently, $\frac{\partial}{\partial x^\varepsilon_\alpha} = \frac{\partial}{\partial x_\alpha}$ and $\frac{\partial}{\partial x^\varepsilon_3} = \frac{\partial}{\partial x_3}$ in $\Omega_\pm$, and $\frac{\partial}{\partial x^\varepsilon_\alpha} = \frac{\partial}{\partial x_\alpha}$ and $\frac{\partial}{\partial x^\varepsilon_3} = \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}$ in $B$.

We assume that the constitutive coefficients of $\Omega^\varepsilon_\pm$ are independent of $\varepsilon$, so that $\bar{\mathbb{K}}^\varepsilon = \bar{\mathbb{K}}$, while the constitutive coefficients of $B^\varepsilon$ present the following dependences on $\varepsilon$, so that $\hat{\mathbb{K}}^\varepsilon = \varepsilon^p \hat{\mathbb{K}}$, with $p \in \{-1, 0, 1\}$. Three different limit behaviors will be characterized according to the choice of the exponent $p$: in the case of $p = -1$, we derive a model for a *rigid* interface; when $p = 0$, we derive a model for a *hard* interface; by choosing $p = 1$, we deduce a model for a *soft* interface.

Finally, we suppose that the unknowns, test functions and data verify the following rescaling assumptions:

$$\begin{array}{llll} \mathbf{s}^\varepsilon(x^\varepsilon) = \mathbf{s}^\varepsilon(x) & x \in \Omega, & \mathbf{r}^\varepsilon(x^\varepsilon) = \mathbf{r}(x) & x \in \Omega, \\ \mathbf{F}^\varepsilon(x^\varepsilon) = \mathbf{F}(x) & x \in \Omega_\pm, & \mathbf{G}^\varepsilon(x^\varepsilon) = \mathbf{G}(x) & x \in \Gamma_g, \end{array}$$

so that $L^\varepsilon(\mathbf{r}^\varepsilon) = L(\mathbf{r})$. By virtue of the previous hypothesis, the rescaled problem can be written in the following form:

$$\begin{cases} \text{Find } \mathbf{s}^\varepsilon \in V(\Omega), \text{ such that} \\ \bar{A}_-(\mathbf{s}^\varepsilon, \mathbf{r}) + \bar{A}_+(\mathbf{s}^\varepsilon, \mathbf{r}) + \varepsilon^{p-1}\hat{a}(\mathbf{s}^\varepsilon, \mathbf{r}) + \varepsilon^p \hat{b}(\mathbf{s}^\varepsilon, \mathbf{r}) + \varepsilon^{p+1}\hat{c}(\mathbf{s}^\varepsilon, \mathbf{r}) = L(\mathbf{r}), \end{cases} \tag{3}$$

for all $\mathbf{r} \in V(\Omega) := \{\mathbf{s} \in H^1(\Omega; \mathbb{R}^{3N\times M}); \ \mathbf{s} = \mathbf{0} \ \text{ on } \Gamma_u\}$, where the bilinear forms $\bar{A}_\pm(\cdot,\cdot)$, $\hat{a}(\cdot,\cdot)$, $\hat{b}(\cdot,\cdot)$ and $\hat{c}(\cdot,\cdot)$ are defined by

$$\begin{aligned} \bar{A}_\pm(\mathbf{s}^\varepsilon, \mathbf{r}) &:= \int_{\Omega_\pm} \bar{\mathbb{K}} \boldsymbol{\nabla}\mathbf{s}^\varepsilon \cdot \boldsymbol{\nabla}\mathbf{r}\, dx, \\ \hat{a}(\mathbf{s}^\varepsilon, \mathbf{r}) &:= \int_B \hat{\mathbb{K}}_{33}\mathbf{s}^\varepsilon_{,3} \cdot \mathbf{r}_{,3}\, dx, \\ \hat{b}(\mathbf{s}^\varepsilon, \mathbf{r}) &:= \int_B \left\{ \hat{\mathbb{K}}_{3\alpha}\mathbf{s}^\varepsilon_{,3} \cdot \mathbf{r}_{,\alpha} + \hat{\mathbb{K}}_{\alpha 3}\mathbf{s}^\varepsilon_{,\alpha} \cdot \mathbf{r}_{,3} \right\} dx, \\ \hat{c}(\mathbf{s}^\varepsilon, \mathbf{r}) &:= \int_B \hat{\mathbb{K}}_{\alpha\beta}\mathbf{s}^\varepsilon_{,\beta} \cdot \mathbf{r}_{,\alpha}\, dx, \end{aligned}$$

and $\hat{\mathbb{K}}_{ij}$ denote the sub-matrices of $\hat{\mathbb{K}}$, defined by

$$\hat{\mathbb{K}} = \begin{bmatrix} \hat{\mathbb{K}}_{\alpha\beta} & \hat{\mathbb{K}}_{\alpha 3} \\ \hat{\mathbb{K}}_{3\alpha} & \hat{\mathbb{K}}_{33} \end{bmatrix}, \quad (\hat{\mathbb{K}}_{ij})^T = \hat{\mathbb{K}}_{ji}.$$

We can now perform an asymptotic analysis of the rescaled problem (3). Since the rescaled problem (3) has a polynomial structure with respect to the small parameter $\varepsilon$, we can look for the solution $\mathbf{s}^\varepsilon$ of the problem as a series of powers of $\varepsilon$:

$$\begin{aligned} \mathbf{s}^\varepsilon &= \mathbf{s}^0 + \varepsilon \mathbf{s}^1 + \varepsilon^2 \mathbf{s}^2 + \dots, \\ \bar{\mathbf{s}}^\varepsilon &= \bar{\mathbf{s}}^0 + \varepsilon \bar{\mathbf{s}}^1 + \varepsilon^2 \bar{\mathbf{s}}^2 + \dots, \\ \hat{\mathbf{s}}^\varepsilon &= \hat{\mathbf{s}}^0 + \varepsilon \hat{\mathbf{s}}^1 + \varepsilon^2 \hat{\mathbf{s}}^2 + \dots . \end{aligned} \tag{4}$$

where $\bar{\mathbf{s}}^\varepsilon = \mathbf{s}^\varepsilon \circ \bar{\pi}^\varepsilon$ and $\hat{\mathbf{s}}^\varepsilon = \mathbf{s}^\varepsilon \circ \hat{\pi}^\varepsilon$. By substituting (4) into the rescaled problem (3), and by identifying the terms with identical power of $\varepsilon$, we obtain, as customary, a set of variational problems to be solved in order to characterize the limit multiphysics state $\mathbf{s}^0$, the first order corrector term $\mathbf{s}^1$ and their associated limit problem, for $p \in \{-1, 0, 1\}$.

Lastly, matching conditions are introduced based on the continuity of the generalized traction $\mathbf{t}^\varepsilon \mathbf{e}_3$ and multiphyisic state $\mathbf{s}^\varepsilon$ at the interfaces $S^\varepsilon_\pm$ in the initial configuration and on the continuity of the traction and state $\bar{\mathbf{t}}^\varepsilon \mathbf{e}_3$, $\bar{\mathbf{s}}^\varepsilon$, $\hat{\mathbf{t}}^\varepsilon \mathbf{e}_3$, $\hat{\mathbf{s}}^\varepsilon$ at the interfaces $S_\pm$ in the rescaled configuration (see [33, 41]). Using the matching conditions, any transmission condition obtained in terms of the rescaled fields $\mathbf{t}^\varepsilon \mathbf{e}_3$ and $\mathbf{s}^\varepsilon$ can be reformulated on the limit configuration, which is the geometric limit of the initial configuration as the thickness of the adhesive interface $\varepsilon$ tends to zero. This is possible because the matching conditions provide a link between the fields evaluated at $x_3 = 0^\pm$ and the rescaled fields evaluated at $x_3 = (\pm 1/2)^\pm$. In particular, following the approach by [33, 41], since the adhesive and the adherents are perfectly bonded together, the continuity of the multiphysic state is verified, so that $\mathbf{s}^\varepsilon(\tilde{x}^\varepsilon, \pm\frac{\varepsilon}{2}) = \bar{\mathbf{s}}^\varepsilon(\tilde{x}, \pm\frac{1}{2}) = \hat{\mathbf{s}}^\varepsilon(\tilde{x}, \pm\frac{1}{2})$. By developing in Taylor's series along $x_3$ and taking into account the asymptotic expansions $(4)_1$, we have:

$$\begin{aligned} \mathbf{s}^\varepsilon\left(x^\varepsilon, \pm\tfrac{\varepsilon}{2}\right) &= \mathbf{s}^\varepsilon(x^\varepsilon, 0^\pm) \pm \tfrac{\varepsilon}{2} {=} s^\varepsilon_{,3}(x^\varepsilon, 0^\pm) + \dots = \\ &= \mathbf{s}^0(x^\varepsilon, 0^\pm) + \varepsilon \mathbf{s}^1(x^\varepsilon, 0^\pm) \pm \tfrac{\varepsilon}{2} \mathbf{s}^0_{,3}(x^\varepsilon, 0^\pm) + \dots . \end{aligned} \tag{5}$$

By substituting $(4)_{2,3}$ in (5), together with the continuity conditions, we can compute the following jumps and mean values:

$$\begin{aligned} &[[\mathbf{s}^0]] = [\bar{\mathbf{s}}^0], \quad [[\mathbf{s}^1]] = [\bar{\mathbf{s}}^1] - \langle\langle \mathbf{s}^0 \rangle\rangle_{,3}, \\ &[[\mathbf{s}^\varepsilon]] = [\bar{\mathbf{s}}^\varepsilon] - \varepsilon \langle\langle \mathbf{s}^\varepsilon_{,3} \rangle\rangle + o(\varepsilon), \\ &\langle\langle \mathbf{s}^0 \rangle\rangle = \langle \bar{\mathbf{s}}^0 \rangle, \quad \langle\langle \mathbf{s}^1 \rangle\rangle = \langle \bar{\mathbf{s}}^1 \rangle - \tfrac{1}{4}[[\mathbf{s}^0_{,3}]], \\ &\langle\langle \mathbf{s}^\varepsilon \rangle\rangle = \langle \bar{\mathbf{s}}^\varepsilon \rangle - \tfrac{\varepsilon}{4}[[\mathbf{s}^\varepsilon_{,3}]] + o(\varepsilon), \end{aligned} \tag{6}$$

where

$$\begin{aligned} &\langle f \rangle(\tilde{x}) := \tfrac{1}{2}(f(\tilde{x}, (1/2)^+) + f(\tilde{x}, -(1/2)^-), \quad \tilde{x} := (x_\alpha) \in S, \\ &[f]\,(\tilde{x}) := f(\tilde{x}, (1/2)^+) - f(\tilde{x}, -(1/2)^-), \\ &\langle\langle f \rangle\rangle(\tilde{x}) := \tfrac{1}{2}(f(\tilde{x}, 0^+) + f(\tilde{x}, 0^-)), \\ &[[f]](\tilde{x}) := f(\tilde{x}, 0^+) - f(\tilde{x}, 0^-), \end{aligned}$$

denote, respectively, the mean value and the jump functions at the interfaces. Following a similar analysis for the generalized traction vector, analogous results are obtained:

$$\begin{array}{l}[[\mathbf{t}^0\mathbf{e}_3]] = [\bar{\mathbf{t}}^0\mathbf{e}_3], \quad [[\mathbf{t}^1\mathbf{e}_3]] = [\bar{\mathbf{t}}^1\mathbf{e}_3] - \langle\langle \mathbf{t}^0_{,3}\mathbf{e}_3\rangle\rangle, \\ [[\mathbf{t}^\varepsilon\mathbf{e}_3]] = [\bar{\mathbf{t}}^\varepsilon\mathbf{e}_3] - \varepsilon\langle\langle \mathbf{t}^\varepsilon_{,3}\mathbf{e}_3\rangle\rangle + o(\varepsilon), \\ \langle\langle \mathbf{t}^0\mathbf{e}_3\rangle\rangle = \langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle, \quad \langle\langle \mathbf{t}^1\mathbf{e}_3\rangle\rangle = \langle\bar{\mathbf{t}}^1\mathbf{e}_3\rangle - \frac{1}{4}[[\mathbf{t}^0_{,3}\mathbf{e}_3]], \\ \langle\langle \mathbf{t}^\varepsilon\mathbf{e}_3\rangle\rangle = \langle\bar{\mathbf{t}}^\varepsilon\mathbf{e}_3\rangle - \frac{\varepsilon}{4}[[\mathbf{t}^\varepsilon_{,3}\mathbf{e}_3]] + o(\varepsilon).\end{array} \tag{7}$$

# 3 The soft multiphysics interface model

In this section we derive the limit model for a soft multiphysics interface, corresponding to an adhesive which is weaker with respect to the adherents. By choosing $p = 1$ and injecting (4) in (3), we obtain the following set of variational problems $\mathcal{P}_q$:

$$\begin{cases} \mathcal{P}_0: & \bar{A}_-(\mathbf{s}^0,\mathbf{r}) + \bar{A}_+(\mathbf{s}^0,\mathbf{r}) + \hat{a}(\mathbf{s}^0,\mathbf{r}) = L(\mathbf{r}), \\ \mathcal{P}_1: & \bar{A}_-(\mathbf{s}^1,\mathbf{r}) + \bar{A}_+(\mathbf{s}^1,\mathbf{r}) + \hat{a}(\mathbf{s}^1,\mathbf{r}) + \hat{b}(\mathbf{s}^0,\mathbf{r}) = 0, \\ \mathcal{P}_q: & \bar{A}_-(\mathbf{s}^q,\mathbf{r}) + \bar{A}_+(\mathbf{s}^q,\mathbf{r}) + \hat{a}(\mathbf{s}^q,\mathbf{r}) + \hat{b}(\mathbf{s}^{q-1},\mathbf{r}) + \hat{c}(\mathbf{s}^{q-2},\mathbf{r}) = 0, \ q \geq 2. \end{cases}$$

Let us consider the variational problem $\mathcal{P}_0$. We perform the following integration by parts:

$$-\int_{\Omega_\pm} \mathrm{div}\, \bar{\mathbf{t}}^0 \cdot \mathbf{r} dx - \int_B \hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^0_{,33} \cdot \mathbf{r} dx \mp \int_{S_\pm} \left(\bar{\mathbf{t}}^0\mathbf{e}_3 - \hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^0_{,3}\right)\Big|_{x_3=\pm\frac{1}{2}} \cdot \mathbf{r} d\Gamma = L(\mathbf{r}), \tag{8}$$

where $\bar{\mathbf{t}}^0 := \bar{\mathbb{K}}_\pm \boldsymbol{\nabla}\bar{\mathbf{s}}^0$. From equation (8), using standard arguments, the following equilibrium equations are obtained

$$\begin{cases} -\mathrm{div}\, \bar{\mathbf{t}}^0 = \mathbf{F} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^0\mathbf{n} = \mathbf{G} & \text{on } \Gamma_g, \\ \hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^0_{,33} = \mathbf{0} & \text{in } B, \\ \mp\left(\bar{\mathbf{t}}^0\mathbf{e}_3 - \hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^0_{,3}\right)\Big|_{x_3=\pm\frac{1}{2}} = \mathbf{0} & \text{on } S_\pm. \end{cases} \tag{9}$$

Equations $(9)_{1,2}$ represent the equilibrium equations on the adherents with the suitable boundary conditions, while from $(9)_3$, we show that $\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^0_{,3}$ is independent of the $x_3$-coordinate. Besides, by virtue of the continuity conditions of the multiphysisic state on $S_\pm$, we can characterize the explicit expression of $\hat{\mathbf{s}}^0$ as an affine function of $x_3$:

$$\hat{\mathbf{s}}^0(\tilde{x}, x_3) = \langle\bar{\mathbf{s}}^0\rangle(\tilde{x}) + x_3[\bar{\mathbf{s}}^0](\tilde{x}). \tag{10}$$

From $(9)_4$, we obtain that the generalized traction vector is continuous through the interphase $B$ and its mean value depends explicitily on $[\bar{\mathbf{s}}^0]$:

$$[\bar{\mathbf{t}}^0\mathbf{e}_3] = \mathbf{0}, \quad \langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle = \hat{\mathbb{K}}_{33}[\bar{\mathbf{s}}^0].$$

Let us consider the variational problem $\mathcal{P}_1$. Using the divergence and Gauss-Green's theorem, we have:

$$\begin{aligned} &-\int_{\Omega_\pm} \mathrm{div}\, \bar{\mathbf{t}}^1 \cdot \mathbf{r} dx - \int_B \left(\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^1_{,33} + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3})\hat{\mathbf{s}}^0_{,3\alpha}\right) \cdot \mathbf{r} dx \\ &\mp \int_{S_\pm} \left(\bar{\mathbf{t}}^1\mathbf{e}_3 - (\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^1_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^0_{,\alpha})\right)\Big|_{x_3=\pm\frac{1}{2}} \cdot \mathbf{r} d\Gamma + \int_{\Gamma_{lat}} \hat{\mathbb{K}}_{3\alpha}\hat{\mathbf{s}}^0_{,3}\nu_\alpha \cdot \mathbf{r} d\Gamma = 0, \end{aligned} \tag{11}$$

with $(\nu_\alpha)$ the outer unit normal vector to the boundary $\Gamma_{lat}$. Equation (11) yields to the following differential system

$$\begin{cases} -\text{div } \bar{\mathbf{t}}^1 = \mathbf{0} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^1 \mathbf{n} = \mathbf{0} & \text{on } \Gamma_g, \\ \hat{\mathbb{K}}_{33} \hat{\mathbf{s}}^1_{,33} + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3}) \hat{\mathbf{s}}^0_{,3\alpha} = \mathbf{0} & \text{in } B, \\ \mp \left( \bar{\mathbf{t}}^1 \mathbf{e}_3 - (\hat{\mathbb{K}}_{33} \hat{\mathbf{s}}^1_{,3} + \hat{\mathbb{K}}_{\alpha 3} \hat{\mathbf{s}}^0_{,\alpha}) \right) \Big|_{x_3 = \pm \frac{1}{2}} = \mathbf{0} & \text{on } S_\pm. \end{cases} \tag{12}$$

Equations $(12)_{1,2}$ represent the homogeneous equilibrium equations on the adherents at order 1. From $(12)_3$, we obtain that

$$\hat{\mathbb{K}}_{33} \hat{\mathbf{s}}^1_{,3} + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3}) \hat{\mathbf{s}}^0_{,\alpha} = \boldsymbol{\psi}(\tilde{x}),$$

where $\boldsymbol{\psi}(\tilde{x})$ is an arbitrary function independent of $x_3$. Hence, by using expression (10) and after an integration between $-\frac{1}{2}$ and $\frac{1}{2}$, we can determine that $\boldsymbol{\psi}(\tilde{x}) = \hat{\mathbb{K}}_{33}[\bar{\mathbf{s}}^1] + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3})\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha}$. Moreover, from $(12)_4$, we obtain the expression of the transmission conditions in terms of the jump and mean value of the traction vector at order 1:

$$[\bar{\mathbf{t}}^1 \mathbf{e}_3] = -\hat{\mathbb{K}}_{3\alpha} [\bar{\mathbf{s}}^0]_{,\alpha}, \quad \langle \bar{\mathbf{t}}^1 \mathbf{e}_3 \rangle = \hat{\mathbb{K}}_{33} [\bar{\mathbf{s}}^1] + \hat{\mathbb{K}}_{\alpha 3} \langle \bar{\mathbf{s}}^0 \rangle_{,\alpha}.$$

**Remark 1.** After the integration by parts (11), an integral on the lateral boundary $\Gamma_{lat}$ of the intermediate layer arises. This stress term corresponds to a non equilibrated load $\mathbf{f}$ and likely represents a classical boundary layer term, usually appearing at the higher orders of asymptotic analysis problem (see [34, 41]). The resultant of these boundary stresses can be considered as a force applied at boundary of interface $S$, and it can be evaluated as:

$$\int_{\Gamma_{lat}} \hat{\mathbb{K}}_{3\alpha} \hat{\mathbf{s}}^0_{,3} \nu_\alpha \cdot \mathbf{r} d\Gamma = \int_{\Gamma_{lat}} \mathbf{f} \cdot \mathbf{r} d\Gamma,$$

and, thus,

$$\mathbf{f} = \hat{\mathbb{K}}_{3\alpha} \hat{\mathbf{s}}^0_{,3} \nu_\alpha = \hat{\mathbb{K}}_{3\alpha} [\bar{\mathbf{s}}^0] \nu_\alpha = \hat{\mathbb{K}}_{3\alpha} (\hat{\mathbb{K}}_{33})^{-1} \langle \bar{\mathbf{t}}^0 \mathbf{e}_3 \rangle \nu_\alpha.$$

The presence of these forces is not directly taken into account by the interface laws. Thus, as in [34], additional terms have to be introduced in the expansions of the stress or the displacement field.

The transmission problems at order 0 and order 1 can be summarized as follows:

- Order 0

**Governing equations**

$$\begin{cases} -\text{div } \bar{\mathbf{t}}^0 = \mathbf{F} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^0 \mathbf{n} = \mathbf{G} & \text{on } \Gamma_g, \\ \bar{\mathbf{s}}^0 = \mathbf{0} & \text{on } \Gamma_u, \end{cases}$$

**Transmission conditions on $S_\pm$**

$$\begin{cases} [\bar{\mathbf{s}}^0] = (\hat{\mathbb{K}}_{33})^{-1} \langle \bar{\mathbf{t}}^0 \mathbf{e}_3 \rangle, \\ [\bar{\mathbf{t}}^0 \mathbf{e}_3] = \mathbf{0}. \end{cases} \tag{13}$$

- Order 1

**Governing equations**

$$\begin{cases} -\text{div } \bar{\mathbf{t}}^1 = \mathbf{0} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^1 \mathbf{n} = \mathbf{0} & \text{on } \Gamma_g, \\ \bar{\mathbf{s}}^1 = \mathbf{0} & \text{on } \Gamma_u, \end{cases}$$

**Transmission conditions on $S_\pm$**

$$\begin{cases} [\bar{\mathbf{s}}^1] = (\hat{\mathbb{K}}_{33})^{-1} \left( \langle \bar{\mathbf{t}}^1 \mathbf{e}_3 \rangle - \hat{\mathbb{K}}_{\alpha 3} \langle \bar{\mathbf{s}}^0 \rangle_{,\alpha} \right), \\ [\bar{\mathbf{t}}^1 \mathbf{e}_3] = -\hat{\mathbb{K}}_{3\alpha} [\bar{\mathbf{s}}^0]_{,\alpha}. \end{cases} \tag{14}$$

**Remark 2.** The transmission problems for a soft multiphysics interface at order 0 and order 1 represent a formal generalization of the soft interface models obtained by means of the asymptotic methods in linear elasticity [33, 41] and in other multifield frameworks, such as piezoelectricity [44], magneto-electro-thermo-elasticity [45] and micropolar elasticity [46]. The soft interface model presents the same structure at both order 0 and 1 as, for instance, in linear elastic asymptotic models. At order 0, the interface shows a discontinuity of the multiphysics state and behaves from a mechanical point of view as a series of linear springs, reacting to the gap between the top and bottom multiphysics states, while the generalized traction vector remains continuous. At order 1, we obtain a mixed interface model in which both the multiphysics state and the traction vector are discontinuous through the interface and they depend on the in-plane derivatives of the jump and mean values of $\bar{\mathbf{s}}^0$.

# 4 The hard multiphysics interface model

In this section we derive the limit model for a hard multiphysics interface, which corresponds to an intermediate layer having the same rigidities of the top and bottom bodies. Let $p = 0$, we substitute (4) in (3) and we obtain the following set of variational problems $\mathcal{P}_q$:

$$\begin{cases} \mathcal{P}_{-1}: & \hat{a}(\mathbf{s}^0, \mathbf{r}) = 0, \\ \mathcal{P}_0: & \bar{A}_-(\mathbf{s}^0, \mathbf{r}) + \bar{A}_+(\mathbf{s}^0, \mathbf{r}) + \hat{a}(\mathbf{s}^1, \mathbf{r}) + \hat{b}(\mathbf{s}^0, \mathbf{r}) = L(\mathbf{r}), \\ \mathcal{P}_1: & \bar{A}_-(\mathbf{s}^1, \mathbf{r}) + \bar{A}_+(\mathbf{s}^1, \mathbf{r}) + \hat{a}(\mathbf{s}^2, \mathbf{r}) + \hat{b}(\mathbf{s}^1, \mathbf{r}) + \hat{c}(\mathbf{s}^0, \mathbf{r}) = 0, \\ \mathcal{P}_q: & \bar{A}_-(\mathbf{s}^q, \mathbf{r}) + \bar{A}_+(\mathbf{s}^q, \mathbf{r}) + \hat{a}(\mathbf{s}^{q+1}, \mathbf{r}) + \hat{b}(\mathbf{s}^q, \mathbf{r}) + \hat{c}(\mathbf{s}^{q-1}, \mathbf{r}) = 0, \ q \geq 2. \end{cases}$$

Let us consider the variational problem $\mathcal{P}_{-1}$. After an integration by parts along $x_3$, we get

$$\begin{aligned} &\hat{\mathbb{K}}_{33} \hat{\mathbf{s}}^0_{,33} = \mathbf{0} && \text{in } B, \\ &\hat{\mathbb{K}}_{33} \hat{\mathbf{s}}^0_{,3}|_{x_3 = \pm\frac{1}{2}} = \mathbf{0} && \text{in } S_\pm. \end{aligned}$$

Thus, $\hat{\mathbf{s}}^0 = \hat{\mathbf{s}}^0(\tilde{x})$ is independent of the through-the-thickness coordinate and so, by the continuity on the upper and lower interfaces, one has $[\hat{\mathbf{s}}^0] = [\bar{\mathbf{s}}^0] = \mathbf{0}$ and $\hat{\mathbf{s}}^0 = \langle \hat{\mathbf{s}}^0 \rangle = \langle \bar{\mathbf{s}}^0 \rangle$.

Following the same steps of Section 3, let us apply the Gauss-Green's formulae to problem $\mathcal{P}_0$. We obtain again the equilibrium equations $(9)_{1,2}$ and the following additional conditions are recovered:

$$\begin{aligned} &\hat{\mathbb{K}}_{33} \hat{\mathbf{s}}^1_{,33} + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3}) \hat{\mathbf{s}}^0_{,3\alpha} = \mathbf{0} && \text{in } B, \\ &\mp \left( \bar{\mathbf{t}}^0 \mathbf{e}_3 - (\hat{\mathbb{K}}_{33} \hat{\mathbf{s}}^1_{,3} + \hat{\mathbb{K}}_{\alpha 3} \hat{\mathbf{s}}^0_{,\alpha}) \right) \Big|_{x_3 = \pm\frac{1}{2}} = \mathbf{0} && \text{on } S_\pm. \end{aligned} \tag{15}$$

Since $\hat{\mathbf{s}}^0$ does not depend on $x_3$, equation $(15)_1$ reduces to $\hat{\mathbf{s}}^1_{,33} = \mathbf{0}$, hence $\hat{\mathbf{s}}^1(\tilde{x}, x_3) = \langle \bar{\mathbf{s}}^1 \rangle(\tilde{x}) + x_3[\bar{\mathbf{s}}^1](\tilde{x})$, where the continuity conditions are taken into account. Moreover, from $(15)_2$, we obtain

$$[\bar{\mathbf{t}}^0\mathbf{e}_3] = \mathbf{0}, \quad \langle \bar{\mathbf{t}}^0\mathbf{e}_3 \rangle = \hat{\mathbb{K}}_{33}[\bar{\mathbf{s}}^1] + \hat{\mathbb{K}}_{\alpha 3}\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha}.$$

Let us consider the variational problem $\mathcal{P}_1$. By means of the Gauss-Green fomulae, we get the same equilibrium equations $(12)_{1,2}$ at order 1, with the following additional conditions:

$$\begin{aligned} &\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,33} + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3})\hat{\mathbf{s}}^1_{,3\alpha} + \hat{\mathbb{K}}_{\alpha\beta}\hat{\mathbf{s}}^0_{,\alpha\beta} = \mathbf{0} \quad \text{in } B, \\ &\mp\left(\bar{\mathbf{t}}^1\mathbf{e}_3 - (\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^1_{,\alpha})\right)\Big|_{x_3=\pm\frac{1}{2}} = \mathbf{0} \quad \text{on } S_\pm. \end{aligned} \tag{16}$$

By integrating $(14)_1$ along $x_3$ , we can obtain the following characterization, where $c$ is constant with respect to $x_3$:

$$\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^1_{,\alpha} = -x_3\hat{\mathbb{K}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha\beta} - \hat{\mathbb{K}}_{3\alpha}(\langle \bar{\mathbf{s}}^1 \rangle_{,\alpha} + x_3[\bar{\mathbf{s}}^1]_{,\alpha}) + c.$$

Its mean value takes the following form:

$$\langle \hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^1_{,\alpha} \rangle = -\hat{\mathbb{K}}_{3\alpha}\langle \bar{\mathbf{s}}^1 \rangle_{,\alpha} + c.$$

From Eq. $(14)_2$, we can say that

$$\langle \bar{\mathbf{t}}^1\mathbf{e}_3 \rangle = \langle \hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^1_{,\alpha} \rangle = -\hat{\mathbb{K}}_{3\alpha}\langle \bar{\mathbf{s}}^1 \rangle_{,\alpha} + c.$$

After another integration of Eq. $(14)_1$ along $x_3$, we can finally characterize $\hat{\mathbf{s}}^2$ and, by imposing the continuity conditions in $x_3 = \pm\frac{1}{2}$, we find an explicit expression of the constant $c$, as follows

$$c = \hat{\mathbb{K}}_{33}[\bar{\mathbf{s}}^2] + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3})\langle \bar{\mathbf{s}}^1 \rangle_{,\alpha}.$$

Hence, we can recover the following relations:

$$[\bar{\mathbf{t}}^1\mathbf{e}_3] = -\left(\hat{\mathbb{K}}_{3\alpha}[\bar{\mathbf{s}}^1]_{,\alpha} + \hat{\mathbb{K}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha\beta}\right), \quad \langle \bar{\mathbf{t}}^1\mathbf{e}_3 \rangle = \hat{\mathbb{K}}_{33}[\bar{\mathbf{s}}^2] + \hat{\mathbb{K}}_{\alpha 3}\langle \bar{\mathbf{s}}^1 \rangle_{,\alpha}.$$

The transmission condition related to $[\bar{\mathbf{t}}^1\mathbf{e}_3]$ can also be written as follows

$$[\bar{\mathbf{t}}^1\mathbf{e}_3] = -\hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}\langle \bar{\mathbf{t}}^0\mathbf{e}_3 \rangle_{,\alpha} - \hat{\mathbb{L}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha\beta},$$

with $\hat{\mathbb{L}}_{\alpha\beta} := \hat{\mathbb{K}}_{\alpha\beta} - \hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}\hat{\mathbb{K}}_{\beta 3}$.

**Remark 3.** The integration by parts of problem $\mathcal{P}_0$ leads to an integral on the lateral boundary $\Gamma_{lat}$, which can be interpreted as a distribution of stresses along the adhesive boundary. This stress term takes the following form, for the hard case:

$$\int_{\Gamma_{lat}} \left(\hat{\mathbb{K}}_{3\alpha}\hat{\mathbf{s}}^1_{,3} + \hat{\mathbb{K}}_{\alpha\beta}\hat{\mathbf{s}}^0_{,\beta}\right)\nu_\alpha \cdot \mathbf{r}d\Gamma = \int_{\Gamma_{lat}} \mathbf{f} \cdot \mathbf{r}d\Gamma,$$

and, thus,

$$\begin{aligned} \mathbf{f} &= \left(\hat{\mathbb{K}}_{3\alpha}\hat{\mathbf{s}}^1_{,3} + \hat{\mathbb{K}}_{\alpha\beta}\hat{\mathbf{s}}^0_{,\beta}\right)\nu_\alpha = \left(\hat{\mathbb{K}}_{3\alpha}[\bar{\mathbf{s}}^1] + \hat{\mathbb{K}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0 \rangle_{,\beta}\right)\nu_\alpha = \\ &= \left(\hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}\langle \bar{\mathbf{t}}^0\mathbf{e}_3 \rangle + \hat{\mathbb{L}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0 \rangle_{,\beta}\right)\nu_\alpha. \end{aligned} \tag{17}$$

The presence of these forces is not directly taken into account by the interface laws. Thus, as in [34], additional terms have to be introduced in the expansions of the stress or the displacement field.

The hard interface transmission problems at order 0 and order 1 can be summarized as follows:

- Order 0

**Governing equations**

$$\begin{cases} -\mathrm{div}\ \bar{\mathbf{t}}^0 = \mathbf{F} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^0\mathbf{n} = \mathbf{G} & \text{on } \Gamma_g, \\ \bar{\mathbf{s}}^0 = \mathbf{0} & \text{on } \Gamma_u, \end{cases}$$

**Transmission conditions on $S_\pm$**

$$\begin{cases} [\bar{\mathbf{s}}^0] = \mathbf{0}, \\ [\bar{\mathbf{t}}^0\mathbf{e}_3] = \mathbf{0}. \end{cases} \tag{18}$$

- Order 1

**Governing equations**

$$\begin{cases} -\mathrm{div}\ \bar{\mathbf{t}}^1 = \mathbf{0} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^1\mathbf{n} = \mathbf{0} & \text{on } \Gamma_g, \\ \bar{\mathbf{s}}^1 = \mathbf{0} & \text{on } \Gamma_u, \end{cases}$$

**Transmission conditions on $S_\pm$**

$$\begin{cases} [\bar{\mathbf{s}}^1] = (\hat{\mathbb{K}}_{33})^{-1}\left(\langle \bar{\mathbf{t}}^0\mathbf{e}_3\rangle - \hat{\mathbb{K}}_{\alpha 3}\langle \bar{\mathbf{s}}^0\rangle_{,\alpha}\right), \\ [\bar{\mathbf{t}}^1\mathbf{e}_3] = -\left(\hat{\mathbb{K}}_{3\alpha}[\bar{\mathbf{s}}^1]_{,\alpha} + \hat{\mathbb{K}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0\rangle_{,\alpha\beta}\right). \end{cases} \tag{19}$$

**Remark 4.** The hard multiphysics interface problems above present the same structures of the analogous linear elastic hard interface models [31–33, 41]. Concerning the order 0, the transmission conditions provide a continuity of the multiphysics state and of its conjugated counterpart, which is typical for adhesives having the same rigidity properties of the adherents. In this case, we do not perceive the presence of the thin layer and the upper and lower bodies a perfectly bonded together. While at order 1, we encounter a mixed interface model with a jump of the state and traction vector depending on the values of the multiphysics state and traction vector at order 0. These order 0 terms are known since they have been determined in the previous problem and they appear in the formulation as source terms.

# 5 The rigid multiphysics interface model

In this section we derive the limit model for a rigid multiphysics interface. Let $p = -1$, we substitute (4) in (3) and we obtain the following set of variational problems $\mathcal{P}_q$:

$$\begin{cases} \mathcal{P}_{-2}: & \hat{a}(\mathbf{s}^0,\mathbf{r}) = 0, \\ \mathcal{P}_{-1}: & \hat{a}(\mathbf{s}^1,\mathbf{r}) + \hat{b}(\mathbf{s}^0,\mathbf{r}) = 0, \\ \mathcal{P}_0: & \bar{A}_-(\mathbf{s}^0,\mathbf{r}) + \bar{A}_+(\mathbf{s}^0,\mathbf{r}) + \hat{a}(\mathbf{s}^2,\mathbf{r}) + \hat{b}(\mathbf{s}^1,\mathbf{r}) + \hat{c}(\mathbf{s}^0,\mathbf{r}) = L(\mathbf{r}), \\ \mathcal{P}_1: & \bar{A}_-(\mathbf{s}^1,\mathbf{r}) + \bar{A}_+(\mathbf{s}^1,\mathbf{r}) + \hat{a}(\mathbf{s}^3,\mathbf{r}) + \hat{b}(\mathbf{s}^2,\mathbf{r}) + \hat{c}(\mathbf{s}^1,\mathbf{r}) = 0, \\ \mathcal{P}_q: & \bar{A}_-(\mathbf{s}^q,\mathbf{r}) + \bar{A}_+(\mathbf{s}^q,\mathbf{r}) + \hat{a}(\mathbf{s}^{q+1},\mathbf{r}) + \hat{b}(\mathbf{s}^q,\mathbf{r}) + \hat{c}(\mathbf{s}^{q-1},\mathbf{r}) = 0,\ q \geq 2. \end{cases}$$

The asymptotic procedure for the rigid case is analogous to the one adopted in Section 4. All the computations and technical details are described in Appendix A. In what follows, we will present only the characterization of the limit transmission problems at order 0 and order 1 with the associated interface conditions:

- Order 0

**Governing equations**

$$\begin{cases} -\text{div } \bar{\mathbf{t}}^0 = \mathbf{F} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^0\mathbf{n} = \mathbf{G} & \text{on } \Gamma_g, \\ \bar{\mathbf{s}}^0 = \mathbf{0} & \text{on } \Gamma_u, \end{cases}$$

**Transmission conditions on $S_\pm$**

$$\begin{cases} [\bar{\mathbf{s}}^0] = \mathbf{0}, \\ [\bar{\mathbf{t}}^0\mathbf{e}_3] = -\hat{\mathbb{L}}_{\alpha\beta}\langle\bar{\mathbf{s}}^0\rangle_{,\alpha\beta}. \end{cases}$$

- Order 1

**Governing equations**

$$\begin{cases} -\text{div } \bar{\mathbf{t}}^1 = \mathbf{0} & \text{in } \Omega_\pm, \\ \bar{\mathbf{t}}^1\mathbf{n} = \mathbf{0} & \text{on } \Gamma_g, \\ \bar{\mathbf{s}}^1 = \mathbf{0} & \text{on } \Gamma_u, \end{cases}$$

**Transmission conditions on $S_\pm$**

$$\begin{cases} [\bar{\mathbf{s}}^1] = -(\hat{\mathbb{K}}_{33})^{-1}\hat{\mathbb{K}}_{\alpha 3}\langle\bar{\mathbf{s}}^0\rangle_{,\alpha} \\ [\bar{\mathbf{t}}^1\mathbf{e}_3] = -\hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle_{,\alpha} - \hat{\mathbb{L}}_{\alpha\beta}\langle\bar{\mathbf{s}}^1\rangle_{,\alpha\beta}. \end{cases}$$

**Remark 5.** The rigid multiphysics interface problems show the same features of the rigid interface asymptotic models obtained in different frameworks in [9–11, 44–46]. Concerning the order 0 model, we obtain a continuity of the multiphysics state at the interface level, while the traction vector is discontinuous and depends on the divergence of a generalized membrane stress vector ($\mathbf{N}^0_\alpha$), defined as follows

$$\mathbf{N}^0_\alpha := \hat{\mathbb{L}}_{\alpha\beta}\langle\bar{\mathbf{s}}^0\rangle_{,\beta}, \quad [\bar{\mathbf{t}}^0\mathbf{e}_3] = -\mathbf{N}^0_{\alpha,\alpha}.$$

The order 1 presents a discontinuity on both the multiphysics state and traction vector. Analogously to the order 0 model, we obtain a generalized equilibrium membrane problem defined on the plane of the interface.

# 6 Implicit form of the interface transmission conditions

In [41], it has been shown that it is possible to obtain a condensed form of transmission conditions summarizing both the orders 0 and 1 of the soft and hard cases in only one couple of equations in terms of the jump of the displacement field and tractions at the interface. Equivalently, we show that it is possible to define an implicit general multiphysics interface law starting from the rigid case, comprising the order 0 and the order 1 of the soft and hard multiphysics interface models.

To this end, we denote by $\tilde{\mathbf{s}}^\varepsilon := \bar{\mathbf{s}}^0 + \varepsilon\bar{\mathbf{s}}^1 + \varepsilon^2\bar{\mathbf{s}}^2$ and $\tilde{\mathbf{t}}^\varepsilon := \bar{\mathbf{t}}^0 + \varepsilon\bar{\mathbf{t}}^1$, two suitable approximations for $\bar{\mathbf{s}}^\varepsilon$ and $\bar{\mathbf{t}}^\varepsilon$.

Let us consider the rigid multiphysics interface conditions, as starting point. After rescaling back the constitutive coefficients $\hat{\mathbb{K}} = \varepsilon\hat{\mathbb{K}}^\varepsilon$ in $B^\varepsilon$, we can write $[\tilde{\mathbf{s}}^\varepsilon]$ and $[\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3]$, following the results of Section 5. Indeed, one has

$$
\begin{aligned}
[\tilde{\mathbf{s}}^\varepsilon] &:= [\bar{\mathbf{s}}^0] + \varepsilon[\bar{\mathbf{s}}^1] + \varepsilon^2[\bar{\mathbf{s}}^2] = \\
&= -\varepsilon(\hat{\mathbb{K}}^\varepsilon_{33})^{-1}\hat{\mathbb{K}}^\varepsilon_{\alpha 3}\langle\bar{\mathbf{s}}^0\rangle_{,\alpha} + \varepsilon^2\left(\tfrac{1}{\varepsilon}(\hat{\mathbb{K}}^\varepsilon_{33})^{-1}(\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle - \varepsilon\hat{\mathbb{K}}^\varepsilon_{\alpha 3}\langle\bar{\mathbf{s}}^1\rangle_{,\alpha})\right) = \\
&= -\varepsilon(\hat{\mathbb{K}}^\varepsilon_{33})^{-1}\left(\hat{\mathbb{K}}^\varepsilon_{\alpha 3}\langle\tilde{\mathbf{s}}^\varepsilon\rangle_{,\alpha} - \langle\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\rangle\right) + o(\varepsilon^2), \\
\left[\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\right] &:= [\bar{\mathbf{t}}^0\mathbf{e}_3] + \varepsilon[\bar{\mathbf{t}}^1\mathbf{e}_3] = \\
&= -\varepsilon\hat{\mathbb{L}}^\varepsilon_{\alpha\beta}\langle\bar{\mathbf{s}}^0\rangle_{,\alpha\beta} - \varepsilon\left(\hat{\mathbb{K}}^\varepsilon_{3\alpha}(\hat{\mathbb{K}}^\varepsilon_{33})^{-1}\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle_{,\alpha} + \varepsilon\hat{\mathbb{L}}^\varepsilon_{\alpha\beta}\langle\bar{\mathbf{s}}^1\rangle_{,\alpha\beta}\right) = \\
&= -\varepsilon\hat{\mathbb{K}}^\varepsilon_{3\alpha}(\hat{\mathbb{K}}^\varepsilon_{33})^{-1}\langle\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\rangle_{,\alpha} - \varepsilon\hat{\mathbb{L}}^\varepsilon_{\alpha\beta}\langle\tilde{\mathbf{s}}^\varepsilon\rangle_{,\alpha\beta} + o(\varepsilon^2).
\end{aligned}
\tag{20}
$$

An alternative expression of the above transmission conditions can be given in terms of $\langle\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\rangle$ and $\left[\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\right]$, which will be useful to write the variational formulation of the interface multiphysics problem:

$$
\begin{aligned}
\langle\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\rangle &= \tfrac{1}{\varepsilon}\hat{\mathbb{K}}^\varepsilon_{33}[\tilde{\mathbf{s}}^\varepsilon] + \hat{\mathbb{K}}^\varepsilon_{\alpha 3}\langle\tilde{\mathbf{s}}^\varepsilon\rangle_{,\alpha} + o(\varepsilon^2), \\
\left[\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\right] &= -\hat{\mathbb{K}}^\varepsilon_{3\alpha}[\tilde{\mathbf{s}}^\varepsilon]_{,\alpha} - \varepsilon\hat{\mathbb{K}}^\varepsilon_{\alpha\beta}\langle\tilde{\mathbf{s}}^\varepsilon\rangle_{,\alpha\beta} + o(\varepsilon^2).
\end{aligned}
\tag{21}
$$

It is now shown that this interface law is general enough to describe the interface laws at order 0 and order 1 prescribing the multiphysics state jump and traction jump in the cases of the soft and hard interfaces, after a suitable rescaling of the matrices $\hat{\mathbb{K}}^\varepsilon$ and up to neglecting higher order terms in $\varepsilon$.

Indeed, to simulate the case of a soft interface, let us choose matrices $\hat{\mathbb{K}}^\varepsilon = \varepsilon\hat{\mathbb{K}}$ in (20):

$$
\begin{aligned}
[\tilde{\mathbf{s}}^\varepsilon] &:= [\bar{\mathbf{s}}^0] + \varepsilon[\bar{\mathbf{s}}^1] + o(\varepsilon^2) = \\
&= -(\hat{\mathbb{K}}_{33})^{-1}\left(\varepsilon\hat{\mathbb{K}}_{\alpha 3}(\langle\bar{\mathbf{s}}^0\rangle_{,\alpha} + \varepsilon\langle\bar{\mathbf{s}}^1+\rangle_{,\alpha}) - \langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle - \varepsilon\langle\bar{\mathbf{t}}^1\mathbf{e}_3\rangle\right) + o(\varepsilon^2), \\
\left[\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\right] &:= [\bar{\mathbf{t}}^0\mathbf{e}_3] + \varepsilon[\bar{\mathbf{t}}^1\mathbf{e}_3] = \\
&= -\varepsilon\hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}(\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle_{,\alpha} + \varepsilon\langle\bar{\mathbf{t}}^1\mathbf{e}_3\rangle_{,\alpha}) - \varepsilon^2\hat{\mathbb{L}}_{\alpha\beta}(\langle\bar{\mathbf{s}}^0\rangle_{,\alpha\beta} + \varepsilon\langle\bar{\mathbf{s}}^1\rangle_{,\alpha\beta}) + o(\varepsilon^2),
\end{aligned}
$$

By identifying the terms with identical power of $\varepsilon$ on the right-hand and left-hand sides of the above relations, we can derive the soft interface conditions at order 0 and 1, as customary,

$$
\text{Order 0}: \begin{cases} [\bar{\mathbf{s}}^0] = (\hat{\mathbb{K}}_{33})^{-1}\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle \\ [\bar{\mathbf{t}}^0\mathbf{e}_3] = \mathbf{0} \end{cases}
\quad
\text{Order 1}: \begin{cases} [\bar{\mathbf{s}}^1] = (\hat{\mathbb{K}}_{33})^{-1}\left(\langle\bar{\mathbf{t}}^1\mathbf{e}_3\rangle - \hat{\mathbb{K}}_{\alpha 3}\langle\bar{\mathbf{s}}^0\rangle_{,\alpha}\right), \\ [\bar{\mathbf{t}}^1\mathbf{e}_3] = -\hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle_{,\alpha}. \end{cases}
$$

The same procedure can be applied in order to identify the transmission conditions of the hard interface, by choosing $\hat{\mathbb{K}}^\varepsilon = \hat{\mathbb{K}}$ in (20):

$$
\begin{aligned}
[\tilde{\mathbf{s}}^\varepsilon] &:= [\bar{\mathbf{s}}^0] + \varepsilon[\bar{\mathbf{s}}^1] + o(\varepsilon^2) = \\
&= -\varepsilon(\hat{\mathbb{K}}_{33})^{-1}\left(\hat{\mathbb{K}}_{\alpha 3}(\langle\bar{\mathbf{s}}^0\rangle_{,\alpha} + \varepsilon\langle\bar{\mathbf{s}}^1+\rangle_{,\alpha}) - \langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle - \varepsilon\langle\bar{\mathbf{t}}^1\mathbf{e}_3\rangle\right) + o(\varepsilon^2), \\
\left[\tilde{\mathbf{t}}^\varepsilon\mathbf{e}_3\right] &:= [\bar{\mathbf{t}}^0\mathbf{e}_3] + \varepsilon[\bar{\mathbf{t}}^1\mathbf{e}_3] = \\
&= -\varepsilon\hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}(\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle_{,\alpha} + \varepsilon\langle\bar{\mathbf{t}}^1\mathbf{e}_3\rangle_{,\alpha}) - \varepsilon\hat{\mathbb{L}}_{\alpha\beta}(\langle\bar{\mathbf{s}}^0\rangle_{,\alpha\beta} + \varepsilon\langle\bar{\mathbf{s}}^1\rangle_{,\alpha\beta}) + o(\varepsilon^2),
\end{aligned}
$$

and, hence, by considering the terms with same power of $\varepsilon$, we get:

$$
\text{Order 0}: \begin{cases} [\bar{\mathbf{s}}^0] = \mathbf{0} \\ [\bar{\mathbf{t}}^0\mathbf{e}_3] = \mathbf{0} \end{cases}
\quad
\text{Order 1}: \begin{cases} [\bar{\mathbf{s}}^1] = (\hat{\mathbb{K}}_{33})^{-1}\left(\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle - \hat{\mathbb{K}}_{\alpha 3}\langle\bar{\mathbf{s}}^0\rangle_{,\alpha}\right), \\ [\bar{\mathbf{t}}^1\mathbf{e}_3] = -\hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}\langle\bar{\mathbf{t}}^0\mathbf{e}_3\rangle_{,\alpha} - \hat{\mathbb{L}}_{\alpha\beta}\langle\bar{\mathbf{s}}^0\rangle_{,\alpha\beta}. \end{cases}
$$

The above interface conditions, derived from the rigid case, are formally equivalent to (13)-(14) and (18)-(19): this proves that the general implicit multiphysics transmission conditions can also describe the soft and hard cases at order 0 and order 1 with a good approximation.

# 7 The variational formulation of the general multiphysics interface problem

In order to write the variational formulation of the general multiphysics interface problem, we will employ the expression of the general transmission conditions presented in (21). These relations can be transformed into interface equations defined on $S$, by making use of the matching relations (6)-(7), up to higher order terms:

$$\begin{aligned}
\langle\langle \mathbf{t}^\varepsilon \mathbf{e}_3\rangle\rangle &= \tfrac{1}{\varepsilon}\hat{\mathbb{K}}^\varepsilon_{33}[[\mathbf{s}^\varepsilon]] + \hat{\mathbb{K}}^\varepsilon_{\alpha 3}\langle\langle \mathbf{s}^\varepsilon\rangle\rangle_{,\alpha} + \\
&\quad +\varepsilon\left(\tfrac{1}{\varepsilon}\hat{\mathbb{K}}^\varepsilon_{33}[[\mathbf{s}^\varepsilon_{,3}]] + \hat{\mathbb{K}}^\varepsilon_{\alpha 3}\langle\langle \mathbf{s}^\varepsilon_{,3}\rangle\rangle_{,\alpha}\right) - \tfrac{\varepsilon}{4}[[\mathbf{t}^\varepsilon_{,3}\mathbf{e}_3]],\\
[[\mathbf{t}^\varepsilon \mathbf{e}_3]] &= -\hat{\mathbb{K}}^\varepsilon_{3\alpha}[[\mathbf{s}^\varepsilon]]_{,\alpha} - \varepsilon\hat{\mathbb{K}}^\varepsilon_{\alpha\beta}\langle\langle \mathbf{s}^\varepsilon\rangle\rangle_{,\alpha\beta} + \\
&\quad +\varepsilon\left(-\hat{\mathbb{K}}^\varepsilon_{3\alpha}[[\mathbf{s}^\varepsilon_{,3}]]_{,\alpha} - \varepsilon\hat{\mathbb{K}}^\varepsilon_{\alpha\beta}\langle\langle \mathbf{s}^\varepsilon_{,3}\rangle\rangle_{,\alpha\beta}\right) - \varepsilon\langle\langle \mathbf{t}^\varepsilon_{,3}\mathbf{e}_3\rangle\rangle.
\end{aligned}$$

In what follows, for the sake of simplicity, we will neglect the higher order terms of the interface conditions and omit the indices $\varepsilon$ on the geometrical and mechanical quantities. Thus,

$$\begin{aligned}
\langle\langle \mathbf{t}\mathbf{e}_3\rangle\rangle &= \tfrac{1}{\varepsilon}\hat{\mathbb{K}}_{33}[[\mathbf{s}]] + \hat{\mathbb{K}}_{\alpha 3}\langle\langle \mathbf{s}\rangle\rangle_{,\alpha},\\
[[\mathbf{t}\mathbf{e}_3]] &= -\hat{\mathbb{K}}_{3\alpha}[[\mathbf{s}]]_{,\alpha} - \varepsilon\hat{\mathbb{K}}_{\alpha\beta}\langle\langle \mathbf{s}\rangle\rangle_{,\alpha\beta}.
\end{aligned} \tag{22}$$

Let us write the variational form of the equilibrium equations on each sub-domain $\Omega_+$ and $\Omega_-$. The sum of the two equations leads to

$$\int_{\Omega_\pm} \bar{\mathbb{K}}\boldsymbol{\nabla}\mathbf{s}\cdot\boldsymbol{\nabla}\mathbf{r}dx - \int_S \mathbf{t}(\tilde{x},0^+)\mathbf{n}(\tilde{x},0^+)\cdot\mathbf{r}d\Gamma - \int_S \mathbf{t}(\tilde{x},0^-)\mathbf{n}(\tilde{x},0^-)\cdot\mathbf{r}d\Gamma = $$
$$= \int_{\Omega_\pm}\mathbf{F}\cdot\mathbf{r}dx + \int_{\Gamma_g}\mathbf{G}\cdot\mathbf{r}d\Gamma,$$

which can be written

$$\int_{\Omega_\pm} \bar{\mathbb{K}}\boldsymbol{\nabla}\mathbf{s}\cdot\boldsymbol{\nabla}\mathbf{r}dx + \int_S [[\mathbf{t}\mathbf{e}_3\cdot\mathbf{r}]]d\tilde{x} = L(\mathbf{r}),$$

letting $\mathbf{e}_3 = \mathbf{n}(\tilde{x},0^-) = -\mathbf{n}(\tilde{x},0^+)$ and $d\Gamma = d\tilde{x}$. Then, using the property $[[ab]] = \langle\langle a\rangle\rangle[[b]] + [[a]]\langle\langle b\rangle\rangle$ and relations (22), we obtain

$$\begin{aligned}
&\int_{\Omega_\pm} \bar{\mathbb{K}}\boldsymbol{\nabla}\mathbf{s}\cdot\boldsymbol{\nabla}\mathbf{r}dx + \int_S\left(\frac{1}{\varepsilon}\hat{\mathbb{K}}_{33}[[\mathbf{s}]] + \hat{\mathbb{K}}_{\alpha 3}\langle\langle \mathbf{s}\rangle\rangle_{,\alpha}\right)\cdot[[\mathbf{r}]]d\tilde{x} - \\
&\quad - \int_S\left(\hat{\mathbb{K}}_{3\alpha}[[\mathbf{s}]]_{,\alpha} + \varepsilon\hat{\mathbb{K}}_{\alpha\beta}\langle\langle \mathbf{s}\rangle\rangle_{,\alpha\beta}\right)\cdot\langle\langle \mathbf{r}\rangle\rangle d\tilde{x} = L(\mathbf{r}).
\end{aligned}$$

After an integration by parts, the variational formulation states as follows

$$\left\{\begin{array}{l} \text{Find } \mathbf{s}\in W(\tilde{\Omega}), \text{ such that}\\ \bar{A}_-(\mathbf{s},\mathbf{r}) + \bar{A}_+(\mathbf{s},\mathbf{r}) + \mathcal{A}(\mathbf{s},\mathbf{r}) = \mathcal{L}(\mathbf{r}),\end{array}\right. \tag{23}$$

for all $\mathbf{r} \in W(\tilde{\Omega})$, where $W(\tilde{\Omega}) := \{\mathbf{s} \in H^1(\tilde{\Omega}; \mathbb{R}^{3N\times M}),\ \mathbf{s}|_S \in H^1(S; \mathbb{R}^{3N\times M}), \mathbf{s} = \mathbf{0} \ \text{ on } \Gamma_u\}$, with $\tilde{\Omega} := \Omega_+ \cup S \cup \Omega_-$ and

$$\mathcal{A}(\mathbf{s},\mathbf{r}) := \int_S \Big( \frac{1}{\varepsilon}\hat{\mathbb{K}}_{33}[[\mathbf{s}]] \cdot [[\mathbf{r}]] + \hat{\mathbb{K}}_{\alpha 3}\langle\langle \mathbf{s}\rangle\rangle_{,\alpha} \cdot [[\mathbf{r}]] + \hat{\mathbb{K}}_{3\alpha}[[\mathbf{s}]] \cdot \langle\langle \mathbf{r}\rangle\rangle_{,\alpha} + \\ + \varepsilon\hat{\mathbb{K}}_{\alpha\beta}\langle\langle \mathbf{s}\rangle\rangle_{,\alpha} \cdot \langle\langle \mathbf{r}\rangle\rangle_{,\beta} \Big)\, d\tilde{x},$$

$$\mathcal{L}(\mathbf{r}) := \int_{\partial\Omega_\pm} \mathbf{F}\cdot \mathbf{r} dx + \int_{\Gamma_g} \mathbf{G}\cdot\mathbf{r} d\Gamma + \int_{\partial S} \mathbf{f}\cdot \langle\langle \mathbf{r}\rangle\rangle d\gamma,$$

where $\mathbf{f} := \left(\hat{\mathbb{K}}_{3\alpha}[[\mathbf{s}]] + \varepsilon\hat{\mathbb{K}}_{\alpha\beta}\langle\langle \mathbf{s}\rangle\rangle_{,\beta}\right)\nu_\alpha$ denotes the load on the lateral boundary of the interface, which can be evaluated with an analogous procedure of Section 6, using (17) and (6)-(7). By virtue of the positivity of the constitutive tensors, a Poincaré-type inequality, the continuity of the linear form and by considering that $|\mathbf{s}(\tilde{x},0^+) \pm \mathbf{s}(\tilde{x},0^-)|^2_{0,S} = |\mathbf{s}(\tilde{x},0^+)|^2_{0,S} + |\mathbf{s}(\tilde{x},0^-)|^2_{0,S} \pm 2(\mathbf{s}(\tilde{x},0^+),\mathbf{s}(\tilde{x},0^-))_{0,S}$, there exist two positive constants $c_1$ and $c_2$ such that

$$c_1\left\{\|\mathbf{s}\|^2_{1,\Omega_+} + \|\mathbf{s}\|^2_{1,\Omega_-} + |\mathbf{s}(\tilde{x},0^+)|^2_{0,S} + |\mathbf{s}(\tilde{x},0^-)|^2_{0,S}\right\} = \\ = c_1\left\{\|\mathbf{s}\|^2_{1,\Omega_+} + \|\mathbf{s}\|^2_{1,\Omega_-} + |\mathbf{s}(\tilde{x},0^+) - \mathbf{s}(\tilde{x},0^-)|^2_{0,S} + |\mathbf{s}(\tilde{x},0^+) + \mathbf{s}(\tilde{x},0^-)|^2_{0,S}\right\} = \\ = c_1\left\{\|\mathbf{s}\|^2_{1,\Omega_+} + \|\mathbf{s}\|^2_{1,\Omega_-} + |[[\mathbf{s}]]|^2_{0,S} + |\langle\langle \mathbf{s}\rangle\rangle|^2_{0,S}\right\} \le \bar{A}_-(\mathbf{s},\mathbf{s}) + \bar{A}_+(\mathbf{s},\mathbf{s}) + \mathcal{A}(\mathbf{s},\mathbf{s}) = \\ = \mathcal{L}(\mathbf{s}) \le c_2\left\{\|\mathbf{s}\|^2_{1,\Omega_+} + \|\mathbf{s}\|^2_{1,\Omega_-} + |\mathbf{s}(\tilde{x},0^+)|^2_{0,S} + |\mathbf{s}(\tilde{x},0^-)|^2_{0,S}\right\}^{1/2}.$$

Thus, thanks to the Lax-Milgram lemma, we can infer that the interface variational problem (23) admits one and only one solution.

**Remark 6.** It is interesting to notice that the bilinear form $\mathcal{A}(\cdot,\cdot)$, related to the interface energy, takes into account simultaneously two different behaviours with some coupling terms: the first one, depending on the jump of the multiphysics state $\mathcal{A}_{spring}(\mathbf{s},\mathbf{r}) := \int_S \frac{1}{\varepsilon}\hat{\mathbb{K}}_{33}[[\mathbf{s}]]\cdot[[\mathbf{r}]]d\tilde{x}$, is classically associated with a soft interface, corresponding to spring-type conditions; the second one, depending on the mean value of the state $\mathcal{A}_{membrane}(\mathbf{s},\mathbf{r}) := \int_S \varepsilon\hat{\mathbb{K}}_{\alpha\beta}\langle\langle \mathbf{s}\rangle\rangle_{,\alpha}\cdot\langle\langle \mathbf{r}\rangle\rangle_{,\beta}d\tilde{x}$, corresponds to a membrane interface energy and it appears usually in hard and rigid interface problems [44–46].

# 8 Applications to multiphysics materials

## 8.1 The elastic case

A linear elastic material represents the simplest choice of multiphysics material in which the only order parameter is given by the displacement field $\mathbf{u}$, i.e., $\mathbf{s} = \mathbf{u}$. The corresponding constitutive law is classically defined by:

$$\boldsymbol{\sigma} = \mathbf{Ce}(\mathbf{u}),$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\mathbf{e}(\mathbf{u}) := \mathrm{Sym}\nabla\mathbf{u}$ represent, respectively, the Cauchy stress tensor and the linearized strain tensor, while $\mathbf{C} = (C_{ijk\ell})$ is the elasticity tensor. The transmission conditions (22) on $S$ can be written as follows:

$$\begin{cases} \langle\langle \boldsymbol{\sigma}\mathbf{e}_3\rangle\rangle = \frac{1}{\varepsilon}\hat{\mathbf{C}}^{33}[[\mathbf{u}]] + \hat{\mathbf{C}}^{\alpha 3}\langle\langle \mathbf{u}\rangle\rangle_{,\alpha}, \\ [[\boldsymbol{\sigma}\mathbf{e}_3]] = -\hat{\mathbf{C}}^{3\alpha}[[\mathbf{u}]]_{,\alpha} - \varepsilon\hat{\mathbf{C}}^{\alpha\beta}\langle\langle \mathbf{u}\rangle\rangle_{,\alpha\beta}, \end{cases}$$

where $\hat{\mathbf{C}}^{j\ell} = (\hat{C}^{j\ell}_{ki}) = (\hat{C}_{ijk\ell})$.

If we consider a linear elastic isotropic material, with Lamé constants $\hat{\lambda}$ and $\hat{\mu}$, the constitutive matrices $\hat{\mathbf{C}}^{j\ell}$ reduces to

$$\hat{\mathbf{C}}^{33} := \begin{bmatrix} \hat{\mu} & 0 & 0 \\ 0 & \hat{\mu} & 0 \\ 0 & 0 & 2\hat{\mu}+\hat{\lambda} \end{bmatrix}, \quad \hat{\mathbf{C}}^{13} := \begin{bmatrix} 0 & 0 & \hat{\mu} \\ 0 & 0 & 0 \\ \hat{\lambda} & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{C}}^{23} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\mu} \\ 0 & \hat{\lambda} & 0 \end{bmatrix},$$

$$\hat{\mathbf{C}}^{11} := \begin{bmatrix} 2\hat{\mu}+\hat{\lambda} & 0 & 0 \\ 0 & \hat{\mu} & 0 \\ 0 & 0 & \hat{\mu} \end{bmatrix}, \quad \hat{\mathbf{C}}^{22} := \begin{bmatrix} \hat{\mu} & 0 & 0 \\ 0 & 2\hat{\mu}+\hat{\lambda} & 0 \\ 0 & 0 & \hat{\mu} \end{bmatrix}, \quad \hat{\mathbf{C}}^{12} := \begin{bmatrix} 0 & \hat{\mu} & 0 \\ \hat{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and, hence, the transmission conditions become

$$\begin{cases} \langle\langle \boldsymbol{\tau}\rangle\rangle = \frac{1}{\varepsilon}\hat{\mu}[[\mathbf{u}_H]] + \hat{\mu}\nabla_s\langle\langle u_3\rangle\rangle, \quad \text{with} \quad \mathbf{u}_H = (u_\alpha), \\ \langle\langle \sigma_{33}\rangle\rangle = \frac{1}{\varepsilon}(2\hat{\mu}+\hat{\lambda})[[u_3]] + \hat{\lambda}\ \mathrm{div}_s\langle\langle \mathbf{u}_H\rangle\rangle, \\ [[\boldsymbol{\tau}]] = -\hat{\lambda}\nabla_s[[u_3]] - \varepsilon\left(\hat{\mu}\Delta_s\langle\langle \mathbf{u}_H\rangle\rangle + (\hat{\mu}+\hat{\lambda})\nabla_s\mathrm{div}_s\langle\langle \mathbf{u}_H\rangle\rangle\right), \\ [[\sigma_{33}]] = -\hat{\mu}\mathrm{div}_s[[\mathbf{u}_H]] - \varepsilon\hat{\mu}\Delta_s\langle\langle u_3\rangle\rangle, \end{cases}$$

where $\boldsymbol{\tau} := (\sigma_{\alpha 3})$ denotes the shear stress vector, and $\nabla_s$, $\mathrm{div}_s$ and $\Delta_s$ represent the two-dimensional gradient, divergence and Laplacian operators, respectively.

## 8.2 The piezoelectric case

A piezoelectric material represents one of the most peculiar multiphysics material, combining the linear elastic behavior with the electric counterpart. The piezoeletric state reduces to a pair $\mathbf{s} = (\mathbf{u}, \varphi)$, constituted by the displacement field $\mathbf{u} = (u_i)$ and the electric potential $\varphi$. The constitutive law takes the following form:

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{Ce}(\mathbf{u}) + \mathbf{PE}, \\ \mathbf{D} = -(\mathbf{P})^T\mathbf{e}(\mathbf{u}) + \mathbf{HE}, \end{cases} \tag{24}$$

where $\mathbf{D} = (D_i)$ represent the electric displacement field, $\mathbf{E} := -\nabla\varphi$ represent the electric field, while $\mathbf{C} = (C_{ijk\ell})$, $\mathbf{P} = (P_{ijk})$ and $\mathbf{H} = (H_{ij})$ denote, respectively, the elasticity tensor, the piezoelectric coupling tensor and the dielectric tensor. The governing equations of the interface problems become two elasticity and electrostatic equilibrium problems on $\Omega_+$ and $\Omega_-$. By considering the generalized body loads $\mathbf{F} := (\mathbf{f}, \varrho_e)$ and surface loads $\mathbf{G} := (\mathbf{g}, d)$, where $\mathbf{f}$ and $\mathbf{g}$ represent the mechanical volume and surface loads, while $\varrho_e$ and $d$ denote the volume and surface charge densities, one has the following field equations:

$$\begin{cases} -\mathrm{div}\ \bar{\boldsymbol{\sigma}} = \mathbf{f} & \text{in } \Omega_\pm, \\ \mathrm{div}\ \bar{\mathbf{D}} = \varrho_e & \text{in } \Omega_\pm, \\ \bar{\boldsymbol{\sigma}}\mathbf{n} = \mathbf{g} & \text{on } \Gamma_g, \\ \bar{\mathbf{D}}\cdot\mathbf{n} = d & \text{on } \Gamma_g, \\ \mathbf{u} = \mathbf{0}, \ \varphi = 0 & \text{on } \Gamma_u. \end{cases} \tag{25}$$

The generalized traction vector takes the form $\mathbf{t}\mathbf{e}_3 = (\boldsymbol{\sigma}\mathbf{e}_3, \mathbf{D}\cdot\mathbf{e}_3)$. Hence, the transmission conditions (22) on $S$ can be adapted as follows:

$$\begin{cases} \langle\langle\boldsymbol{\sigma}\mathbf{e}_3\rangle\rangle = \frac{1}{\varepsilon}\left(\hat{\mathbf{C}}^{33}[[\mathbf{u}]] + \hat{\mathbf{p}}^{33}[[\varphi]]\right) + \hat{\mathbf{C}}^{\alpha 3}\langle\langle\mathbf{u}\rangle\rangle_{,\alpha} + \hat{\mathbf{p}}^{\alpha 3}\langle\langle\varphi\rangle\rangle_{,\alpha} \\ \langle\langle\mathbf{D}\cdot\mathbf{e}_3\rangle\rangle = \frac{1}{\varepsilon}\left(-\hat{\mathbf{p}}^{33}\cdot[[\mathbf{u}]] + \hat{H}_{33}[[\varphi]]\right) - \hat{\mathbf{p}}^{3\alpha}\cdot\langle\langle\mathbf{u}\rangle\rangle_{,\alpha} + \hat{H}_{\alpha 3}\langle\langle\varphi\rangle\rangle_{,\alpha} \\ [[\boldsymbol{\sigma}\mathbf{e}_3]] = -\hat{\mathbf{C}}^{3\alpha}[[\mathbf{u}]]_{,\alpha} - \hat{\mathbf{p}}^{3\alpha}[[\varphi]]_{,\alpha} - \varepsilon\hat{\mathbf{C}}^{\alpha\beta}\langle\langle\mathbf{u}\rangle\rangle_{,\alpha\beta} - \varepsilon\hat{\mathbf{p}}^{\alpha\beta}\langle\langle\varphi\rangle\rangle_{,\alpha\beta} \\ [[\mathbf{D}\cdot\mathbf{e}_3]] = \hat{\mathbf{p}}^{3\alpha}\cdot[[\mathbf{u}]]_{,\alpha} - \hat{H}_{\alpha 3}[[\varphi]]_{,\alpha} + \varepsilon\hat{\mathbf{p}}^{\alpha\beta}\cdot\langle\langle\mathbf{u}\rangle\rangle_{,\alpha\beta} - \varepsilon\hat{H}_{\alpha\beta}\langle\langle\varphi\rangle\rangle_{,\alpha\beta}, \end{cases}$$

where $\hat{\mathbf{C}}^{j\ell} = (\hat{C}^{j\ell}_{ki}) = (\hat{C}_{ijk\ell})$ and $\hat{\mathbf{p}}^{ki} = (\hat{p}^{ki}_j) = (\hat{P}_{kij})$. As we already proved in Section 6, the above interface conditions contains the soft, hard and rigid interface conditions and are equivalent to those obtained in [44], for weak lowly-conducting and strong highly-conducting piezoelectric interfaces.

Considering, for instance, a transversally isotropic material with poling axis $\mathbf{e}_3$, such as PZT-4, the constitutive matrices take the following form:

$$\hat{\mathbf{C}}^{33} := \begin{bmatrix} c_{55} & 0 & 0 \\ 0 & c_{55} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}, \quad \hat{\mathbf{C}}^{13} := \begin{bmatrix} 0 & 0 & c_{55} \\ 0 & 0 & 0 \\ c_{13} & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{C}}^{23} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{55} \\ 0 & c_{13} & 0 \end{bmatrix},$$

$$\hat{\mathbf{p}}^{33} := \begin{bmatrix} 0 \\ 0 \\ e_{33} \end{bmatrix}, \quad \hat{\mathbf{p}}^{13} := \begin{bmatrix} e_{15} \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{p}}^{23} := \begin{bmatrix} 0 \\ e_{15} \\ 0 \end{bmatrix}, \hat{\mathbf{p}}^{31} := \begin{bmatrix} e_{31} \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{\mathbf{p}}^{32} := \begin{bmatrix} 0 \\ e_{31} \\ 0 \end{bmatrix}, \quad \hat{\mathbf{C}}^{11} := \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & \frac{c_{11}-c_{12}}{2} & 0 \\ 0 & 0 & c_{55} \end{bmatrix}, \quad \hat{\mathbf{C}}^{22} := \begin{bmatrix} \frac{c_{11}-c_{12}}{2} & 0 & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & c_{55} \end{bmatrix}$$

$$\hat{\mathbf{C}}^{12} := \begin{bmatrix} 0 & \frac{c_{11}-c_{12}}{2} & 0 \\ c_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{p}}^{11} := \begin{bmatrix} 0 \\ 0 \\ e_{15} \end{bmatrix}, \quad \hat{\mathbf{p}}^{12} := \mathbf{0}, \quad \hat{H}_{\alpha 3} = 0, \quad \hat{H}_{11} = \hat{H}_2$$

and, hence, the transmission conditions become

$$\begin{cases} \langle\langle\boldsymbol{\tau}\rangle\rangle = \frac{1}{\varepsilon}c_{55}[[\mathbf{u}_H]] + c_{55}\nabla_s\langle\langle u_3\rangle\rangle + e_{15}\nabla_s\langle\langle\varphi\rangle\rangle, \\ \langle\langle\sigma_{33}\rangle\rangle = \frac{1}{\varepsilon}\left(c_{33}[[u_3]] + e_{33}[[\varphi]]\right) + c_{13}\ \mathrm{div}_s\langle\langle\mathbf{u}_H\rangle\rangle, \\ \langle\langle D_3\rangle\rangle = \frac{1}{\varepsilon}\left(-e_{33}[[u_3]] + \hat{H}_{33}[[\varphi]]\right) - e_{31}\mathrm{div}_s\langle\langle\mathbf{u}_H\rangle\rangle, \\ [[\boldsymbol{\tau}]] = -\left(c_{13}\nabla_s[[u_3]] - e_{31}\nabla_s[[\varphi]]\right) - \frac{\varepsilon}{2}\left((c_{11}-c_{12})\Delta_s\langle\langle\mathbf{u}_H\rangle\rangle + (c_{11}+c_{12})\nabla_s\mathrm{div}_s\langle\langle \right. \\ [[\sigma_{33}]] = -c_{55}\mathrm{div}_s[[\mathbf{u}_H]] - \varepsilon\left(c_{55}\Delta_s\langle\langle u_3\rangle\rangle + e_{15}\Delta_s\langle\langle\varphi\rangle\rangle\right), \\ [[D_3]] = -e_{15}\mathrm{div}_s[[\mathbf{u}_H]] - \varepsilon\left(e_{15}\Delta_s\langle\langle u_3\rangle\rangle + \hat{H}_{11}\Delta_s\langle\langle\varphi\rangle\rangle\right). \end{cases} \tag{26}$$

## 8.3 The thermoelastic case

The interface problem can be easily adapted in the case of linear elasticity with thermal effect. The thermoelastic state is defined by the pair $\mathbf{s} = (\mathbf{u}, \theta)$, constituted by the displacement field $\mathbf{u} = (u_i)$ and the variation of temperature $\theta$. The corresponding constitutive law takes the following form:

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{C}\mathbf{e}(\mathbf{u}) - \mathbf{X}\theta, \\ \mathbf{q} = -\mathbf{K}\nabla\theta, \end{cases}$$

where $\mathbf{q} = (q_i)$ denotes the thermal flow, $\mathbf{X} = (X_{ij})$ and $\mathbf{K} = (K_{ij})$ represent, respectively, the thermal expansion tensor and the thermal conductivity tensor. By noting with $\mathbf{F} := (\mathbf{f}, w)$ and $\mathbf{G} := (\mathbf{g}, h)$, the thermo-mechanical volume and surface loads, we obtain the following governing equations:

$$\begin{cases} -\text{div}\ \bar{\boldsymbol{\sigma}} = \mathbf{f} & \text{in } \Omega_\pm, \\ \text{div}\ \bar{\mathbf{q}} = w & \text{in } \Omega_\pm, \\ \bar{\boldsymbol{\sigma}}\mathbf{n} = \mathbf{g} & \text{on } \Gamma_g, \\ \bar{\mathbf{q}} \cdot \mathbf{n} = h & \text{on } \Gamma_g, \\ \mathbf{u} = \mathbf{0},\ \theta = 0 & \text{on } \Gamma_u. \end{cases}$$

The generalized traction vector takes the form $\mathbf{te}_3 = (\boldsymbol{\sigma}\mathbf{e}_3, \mathbf{q} \cdot \mathbf{e}_3)$. Hence, the transmission conditions (22) on $S$ can be adapted as follows:

$$\begin{cases} \langle\langle \boldsymbol{\sigma}\mathbf{e}_3 \rangle\rangle = \frac{1}{\varepsilon}\hat{\mathbf{C}}^{33}[[\mathbf{u}]] + \hat{\mathbf{C}}^{\alpha 3}\langle\langle \mathbf{u} \rangle\rangle_{,\alpha} - \hat{\mathbf{X}}^3\langle\langle \theta \rangle\rangle \\ \langle\langle \mathbf{q} \cdot \mathbf{e}_3 \rangle\rangle = \frac{1}{\varepsilon}\hat{K}_{33}[[\theta]] \\ [[\boldsymbol{\sigma}\mathbf{e}_3]] = -\hat{\mathbf{C}}^{3\alpha}[[\mathbf{u}]]_{,\alpha} - \varepsilon\hat{\mathbf{C}}^{\alpha\beta}\langle\langle \mathbf{u} \rangle\rangle_{,\alpha\beta} + \varepsilon\hat{\mathbf{X}}^{\alpha}\langle\langle \theta \rangle\rangle_{,\alpha} \\ [[\mathbf{q} \cdot \mathbf{e}_3]] = -\varepsilon\hat{K}_{\alpha\beta}\langle\langle \theta \rangle\rangle_{,\alpha\beta}, \end{cases}$$

where $\hat{\mathbf{X}}^k = (\hat{X}_i^k) := (\hat{X}_{ik})$. The same interface problem can be also associated to other multiphysics and multifield theories such as poroelasticiy (in this case $\theta$ stands for the pore pressure) or elasticity with voids and microstretch elasticity (in this case $\theta$ represent the microstretch state).

By choosing an isotropic thermoelastic material, one has $\hat{X}_{ij} = \hat{\alpha}\delta_{ij}$ and $\hat{K}_{ij} = \hat{k}\delta_{ij}$, with $\delta_{ij}$ the Kronecker's tensor. Thus, the transmission conditions takes the following form:

$$\begin{cases} \langle\langle \boldsymbol{\tau} \rangle\rangle = \frac{1}{\varepsilon}\hat{\mu}[[\mathbf{u}_H]] + \hat{\mu}\nabla_s\langle\langle u_3 \rangle\rangle, \\ \langle\langle \sigma_{33} \rangle\rangle = \frac{1}{\varepsilon}(2\hat{\mu} + \hat{\lambda})[[u_3]] + \hat{\lambda}\ \text{div}_s\langle\langle \mathbf{u}_H \rangle\rangle - \hat{\alpha}\langle\langle \theta \rangle\rangle, \\ \langle\langle q_3 \rangle\rangle = \frac{1}{\varepsilon}\hat{k}[[\theta]], \\ [[\boldsymbol{\tau}]] = -\hat{\lambda}\nabla_s[[u_3]] - \varepsilon\left(\hat{\mu}\Delta_s\langle\langle \mathbf{u}_H \rangle\rangle + (\hat{\mu} + \hat{\lambda})\nabla_s\text{div}_s\langle\langle \mathbf{u}_H \rangle\rangle + \hat{\alpha}\nabla_s\langle\langle \theta \rangle\rangle\right), \\ [[\sigma_{33}]] = -\hat{\mu}\text{div}_s[[\mathbf{u}_H]] - \varepsilon\hat{\mu}\Delta_s\langle\langle u_3 \rangle\rangle, \\ [[q_3]] = -\varepsilon\hat{k}\Delta_s\langle\langle \theta \rangle\rangle. \end{cases}$$

# 9 A closed-form solution for the stretching of a piezoelectric composite

Consider a three-dimensional composite body made of two piezoelectric parallelepipeds, $\Omega_-$ and $\Omega_+$, called the adherents, having identical lateral dimensions but different heights, $h_-$ and $h_+$ respectively. The two adherents are joined by a thin piezoelectric adhesive of thickness $\varepsilon$, whose behavior is described by the transmission conditions (26). The total height of the composite is defined by $H := h_+ + h_- + \varepsilon$. The whole body is subject to a tensile load $q$ acting on the upper and lower boundary, denoted $\Gamma_+$ and $\Gamma_-$ respectively (cf. Figure 2). The union of the lateral boundaries of $\Omega_-$ and $\Omega_+$ is denoted $\Gamma_l$, and $\Gamma_g = \Gamma_\pm \cup \Gamma_l$. Moreover, $\Gamma_u = \emptyset$.

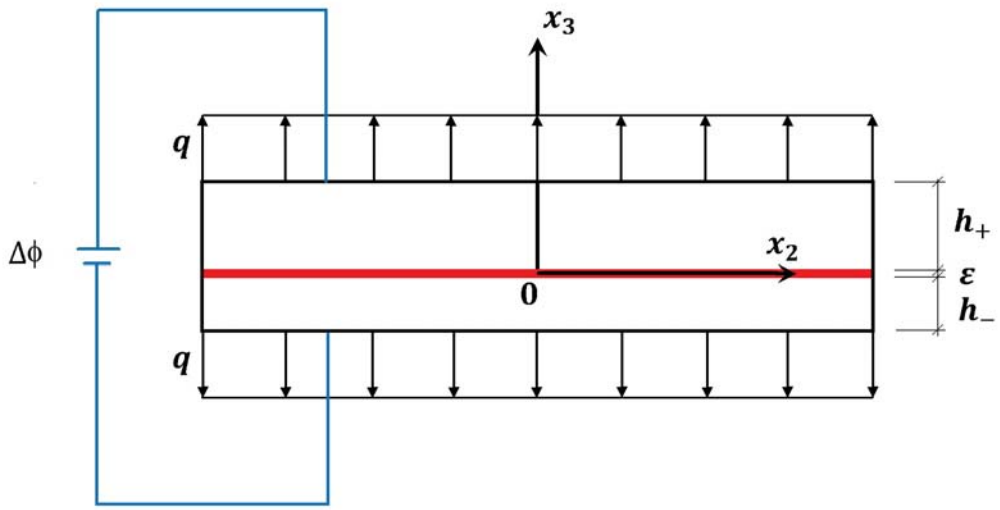


Figure 2: Geometry of the piezoelectric three-layered composite body.

The governing equations are given by the equilibrium equations (25) with $\mathbf{f} = \mathbf{0}$, $\varrho_e = 0$, $\mathbf{g}_\pm := \pm q\ \mathbf{e}_3$ on $\Gamma_\pm$ and $d = 0$, together with the constitutive laws (24), specialized for the orthotropic symmetry with poling axis $\mathbf{e}_3$, and the transmission conditions (26).

The following choice:

$$\left\{\begin{array}{ll} \mathbf{u} = q\ \sum_{i=1}^{3} \bar{C}_i x_i \mathbf{e}_i + u_3^{\pm}\mathbf{e}_3 & \text{in } \Omega_\pm, \\ \varphi = -q\bar{C}_4 x_3 + \varphi^{\pm} & \text{in } \Omega_\pm, \\ \bar{\boldsymbol{\sigma}} = q(\mathbf{e}_3 \otimes \mathbf{e}_3) & \text{in } \Omega_+ \cup \Omega_-, \\ \bar{\mathbf{D}} = \mathbf{0} & \text{in } \Omega_+ \cup \Omega_-, \end{array}\right. \tag{27}$$

with the costants $\bar{C}_i, i = 1, 2, 3, 4$ given in Appendix B, satisfies the equilibrium equations (25) and the constitutive laws (24).

Note that the choice (27) corresponds to homogeneous piezoelectric states $(\mathbf{u}, \varphi)$ of the adherents superimposed to jump discontinuities $[[\mathbf{u}]] = (u_3^+ - u_3^-)\mathbf{e}_3$ and $[[\varphi]] = (\varphi^+ - \varphi^-)$ concentrated at the adhesive interface $S$.

By imposing the transmission conditions (26), the first, fourth, fifth and sixth conditions are identically satisfied, while the second and third conditions give the following values for the jumps $[[u_3]]$ and $[[\phi]]$:

$$\left\{\begin{array}{l} [[u_3]] = q\varepsilon \frac{\hat{H}_{33} + (\bar{C}_1 + \bar{C}_2)(\hat{e}_{31}\hat{e}_{33} - \hat{c}_{13}\hat{H}_{33})}{\hat{e}_{33}^2 + \hat{c}_{33}\hat{H}_{33}}, \\ [[\varphi]] = q\varepsilon \frac{\hat{e}_{33} + (\bar{C}_1 + \bar{C}_2)(\hat{e}_{33}\hat{c}_{13} - \hat{e}_{13}\hat{c}_{33})}{\hat{e}_{33}^2 + \hat{c}_{33}\hat{H}_{33}}. \end{array}\right. \tag{28}$$

The closed-form solution given by (27) and (28) allows to compute the macroscopic elastic modulus of the composite $E$, defined as

$$\begin{aligned} E &:= \frac{q}{\frac{u_3(x_1, x_2, h_+ + \varepsilon/2) - u_3(x_1, x_2, -h_- - \varepsilon/2)}{H}} \\ &= \frac{1}{\bar{C}_3 + \frac{\varepsilon}{H}\frac{\hat{H}_{33} + (\bar{C}_1 + \bar{C}_2)(\hat{e}_{31}\hat{e}_{33} - \hat{c}_{13}\hat{H}_{33})}{(\hat{e}_{33}^2 + \hat{c}_{33}\hat{H}_{33})}}. \end{aligned}$$

The difference of electric potential $\Delta\phi$, induced between the upper and lower surfaces of the composite, can be calculated as

$$\begin{aligned}\Delta\phi &:= \varphi(x_1, x_2, h_+ + \varepsilon/2) - \varphi(x_1, x_2, -h_- - \varepsilon/2) \\ &= -q\bar{C}_4(h_+ + h_-) + q\varepsilon\frac{\hat{e}_{33} + (\bar{C}_1 + \bar{C}_2)(\hat{e}_{33}\hat{c}_{13} - \hat{e}_{13}\hat{c}_{33})}{\hat{e}_{33}^2 + \hat{c}_{33}\hat{H}_{33}}.\end{aligned}$$

# 10 Summary and conclusion

General imperfect contact conditions have been proposed, simulating the behavior of a thin interphase undergoing linear coupled multiphysics phenomena. These conditions link the generalized stress vector field and its jump to the multiphysic state vector field and its jump at the interface, which is the geometric limit of the interphase as its thickness parameter $\varepsilon$ goes to zero.

The approach used to obtain the transmission conditions is based on the asymptotic expansions method and on energy minimization. Zero and higher order interface models have been derived for soft, hard and rigid interphases, meaning that the multiphysics parameters of the interphase material rescale as $\varepsilon^p$ with the interphase thickness $\varepsilon$, and $p = 1, 0, -1$, respectively. For the three regimes, the asymptotic expansions of the multiphysic state and generalized stress vector fields are introduced and the effect of higher ordererms is taken into account.

Using matching conditions, the transmission conditions for the three regimes at the various orders have been condensed into a single, implicit, non-local formulation, which recovers as particolar cases the cases of spring-layer and perfect interfaces and are expected to provide a better approximation of the behavior of the thin interphase. Indeed, considering the order 1 corrector terms of the asymptotic expansions, summed up with the leading terms at order 0, we can extend the application of the general myltiphysics interface law to moderately thick adhesive layers.

A variational formulation of the general multiphysics interface problem has been presented. The weak formulation represents a key step towards simulating numerically imperfect interface effects inside composite materials, exhibiting linear coupled phenomena, and it serves as a basis for the study of the well-posedness of the mathematical problem. The applicability of the proposed transmission conditions has been illustrated via a series of examples, the case of linear elasticity, the thermoelastic and the piezoeletric cases, for which we propose a closed-form solution. Other cases could be recovered, being the structure of the transmission conditions completely general and applicable to any situation involving linear coupled multiphysics phenomena.

# Appendix A

In the sequel, we briefly present the characterization of the order 0 and order 1 interface conditions for the rigid case.

Considering problem $\mathcal{P}_{-2}$, we can infer that $\hat{\mathbf{s}}^0 = \hat{\mathbf{s}}^0(\tilde{x})$ is independent of the through-the-thickness coordinate and so, by the continuity on the upper and lower interfaces, one has $[\hat{\mathbf{s}}^0] = [\bar{\mathbf{s}}^0] = \mathbf{0}$ and $\hat{\mathbf{s}}^0 = \langle \hat{\mathbf{s}}^0 \rangle = \langle \bar{\mathbf{s}}^0 \rangle$.

Let us take into account problem $\mathcal{P}_{-1}$. Since $\hat{\mathbf{s}}^0$ does not depend on $x_3$, we obtain that $\hat{\mathbf{s}}^1(\tilde{x}, x_3) = \langle \bar{\mathbf{s}}^1 \rangle(\tilde{x}) + x_3[\bar{\mathbf{s}}^1](\tilde{x})$, where the continuity conditions are taken into account. Moreover, one has

$$[\bar{\mathbf{s}}^1] = -(\hat{\mathbb{K}}_{33})^{-1}\hat{\mathbb{K}}_{\alpha 3}\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha}. \tag{29}$$

Let us consider problem $\mathcal{P}_0$ and perform an integration by parts along $x_3$ and the in-plane coordinates $x_\alpha$. Thus, we obtain

$$\begin{aligned} &\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,33} + (\hat{\mathbb{K}}_{3\alpha} + \hat{\mathbb{K}}_{\alpha 3})\hat{\mathbf{s}}^1_{,3\alpha} + \hat{\mathbb{K}}_{\alpha\beta}\hat{\mathbf{s}}^0_{,\alpha\beta} = \mathbf{0} \quad \text{in } B, \\ &\mp\Big(\bar{\mathbf{t}}^0\mathbf{e}_3 - (\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^1_{,\alpha})\Big)\Big|_{x_3=\pm\frac{1}{2}} = \mathbf{0} \quad \text{on } S_\pm. \end{aligned} \tag{30}$$

Integrating $(10)_1$ in the interval $x_3 = \pm\frac{1}{2}$, one can evaluate the following jump:

$$[\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^1_{,\alpha}] = -\hat{\mathbb{L}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha\beta}.$$

From $(10)_2$ and the above relation, we can compute

$$[\bar{\mathbf{t}}^0\mathbf{e}_3] = [\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^2_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^1_{,\alpha}] = -\hat{\mathbb{L}}_{\alpha\beta}\langle \bar{\mathbf{s}}^0 \rangle_{,\alpha\beta}.$$

Hence, together with $[\bar{\mathbf{s}}^0] = \mathbf{0}$, the rigid interface conditions at order 0 are retrieved. Moreover, by characterizing the explicit expression of $\hat{\mathbf{s}}^2$, we can infer that

$$\hat{\mathbb{K}}_{33}[\bar{\mathbf{s}}^2] = \langle \bar{\mathbf{t}}^0\mathbf{e}_3 \rangle - \hat{\mathbb{K}}_{\alpha 3}\langle \hat{\mathbf{s}}^1 \rangle_{,\alpha},$$

which will be useful in the sequel. Finally, from problem $\mathcal{P}_1$, by operating an integration by parts, we get

$$\begin{aligned} [\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^3_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^2_{,\alpha}] &= -(\hat{\mathbb{K}}_{\alpha\beta}\langle \bar{\mathbf{s}}^1 \rangle_{,\alpha\beta} + \hat{\mathbb{K}}_{3\alpha}[\bar{\mathbf{s}}^2]_{,\alpha}) = \\ &= -(\hat{\mathbb{L}}_{\alpha\beta}\langle \bar{\mathbf{s}}^1 \rangle_{,\alpha\beta} + \hat{\mathbb{K}}_{3\alpha}(\hat{\mathbb{K}}_{33})^{-1}\langle \bar{\mathbf{t}}^0\mathbf{e}_3 \rangle_{,\alpha}). \end{aligned}$$

Since

$$[\bar{\mathbf{t}}^1\mathbf{e}_3] = [\hat{\mathbb{K}}_{33}\hat{\mathbf{s}}^3_{,3} + \hat{\mathbb{K}}_{\alpha 3}\hat{\mathbf{s}}^2_{,\alpha}],$$

combined with (29), we can recover the order 1 rigid interface conditions.

# Appendix B

The constitutive constants $\bar{C}_i, i = 1, 2, 3, 4$, of the example in Section 9 take the following form:

$$\begin{aligned} \bar{C}_1 &= \tfrac{1}{\Delta}\left(c_{13}c_{22}H_{33} + c_{13}e_{32}^2 + c_{22}e_{31}e_{33} - c_{12}e_{32}e_{33}\right), \\ \bar{C}_2 &= -\tfrac{1}{\Delta}\left(c_{12}c_{13}H_{33} + c_{13}e_{31}e_{32} + c_{12}e_{31}e_{33} - c_{11}e_{32}e_{33}\right), \\ \bar{C}_3 &= \tfrac{1}{\Delta}\left(c_{12}^2H_{33} - c_{22}e_{31}^2 + 2c_{12}e_{31}e_{32} - c_{11}(c_{22}H_{33} + e_{32}^2)\right), \\ \bar{C}_4 &= \tfrac{1}{\Delta}\left(-c_{13}c_{22}e_{31} + c_{12}c_{13}e_{32} - c_{12}^2e_{33} + c_{11}c_{22}e_{33}\right), \end{aligned}$$

with

$$\begin{aligned} \Delta = \; & c_{13}^2(c_{22}H_{33} + e_{32}^2) - c_{33}(c_{11}c_{22}H_{33} + c_{22}e_{31}^2 + c_{11}e_{32}^2) \\ & + c_{11}c_{23}e_{32}e_{33} - c_{11}c_{22}e_{33}^2 + c_{12}e_{31}(2c_{33}e_{32} - c_{23}e_{33}) \\ & - c_{13}(c_{12}c_{23}H_{33} + c_{23}e_{31}e_{32} - 2c_{22}e_{31}e_{33} + 2c_{12}e_{32}e_{33}) \\ & + c_{12}^2(c_{33}H_{33} + e_{33}^2). \end{aligned}$$

# References


[1] R. Abdelmoula, M. Coutris, J. Marigo, Comportment asymptotique d'une interphase élastique mince, C. R. Aca. Sci. Paris Ser. IIb 326 237-242 (1998).

[2] E. Acerbi, G. Buttazzo, D. Percivale, Thin inclusions in linear elasticity: a variational approach, J. Reine Angew. Math. 386 99-115 (1988).

[3] Y. Benveniste, Effective mechanical behaviour of composite materials with imperfect contact between the constituents. Mechanics of Materials 4 197-208 (1985).

[4] Y. Benveniste, The effective conductivity of composites with imperfect thermal contact at constituent interfaces, International Journal of Engineering Science 24 1537-1552 (1986).

[5] Y. Benveniste, Effective thermal-conductivity of composites with a thermal contact resistance between the constituents-nondilute case, Journal of Applied Physics 61 2840-2843, (1987).

[6] Y. Benveniste, A general interface model for a three-dimensional curved thin anisotropic interphase between two anisotropic media, J. Mech. Phys. Solids 54 708-734 (2006).

[7] Y. Benveniste, An $O(h^N)$ interface model of a three-dimensional curved interphase in conduction phenomena, Proc. R. Soc. A 462, 1593-1617 (2006)

[8] Y. Benveniste, T. Miloh, Imperfect soft and stiff interfaces in two-dimensional elasticity, Mech. Mat. 33(6) 309-323 (2001).

[9] A.-L. Bessoud, F. Krasucki, G. Michaille, Multi-materials with strong interface: variational modelings, Asympt. Anal. 1 1-19 (2009).

[10] A.-L. Bessoud, F. Krasucki, M. Serpilli, Asymptotic analysis of shell-like inclusions with high rigidity, J.Elasticity 103 153-172 (2011).

[11] A.-L. Bessoud, F. Krasucki, M. Serpilli, Plate-like and shell-like inclusions with high rigidity, C. R. Aca. Sci. Paris Ser. I 346 697-702 (2008).

[12] P.G. Ciarlet, Mathematical Elasticity, vol. II: Theory of Plates, North-Holland, Amsterdam, (1997).

[13] T. Chen, Exact size-dependent connections between effective moduli of fibrous piezoelectric nanocomposites with interface effects, Acta Mechanica 196 205-217 (2008).

[14] S. Dumont, F. Lebon, M.L. Raffa, R. Rizzoni, H. Welemane, Multiscale Modeling of Imperfect Interfaces and Applications, In: Ibrahimbegovic A. (eds) Computational Methods for Solids and Fluids. Computational Methods in Applied Sciences, vol 41. Springer (2016).

[15] S. Dumont, R. Rizzoni, F. Lebon and E. Sacco, Soft and hard interface models for bonded elements,Composites Part B: Engineering 153, 480-490 (2018).

[16] S. Firooz, A. Javili, Understanding the role of general interfaces in the overall behavior of composites and size effects, Comp. Materials Sci. 162 245-254 (2019).

[17] G. Geymonat, S. Hendili, F. Krasucki, M. Serpilli, M. Vidrascu, Asymptotic expansions and domain decomposition in Domain Decomposition Methods XXI, Lecture notes in Computational Science and Engineering, 98, (Springer, 2014).

[18] G. Geymonat, F. Krasucki, Analyse asymptotique du comportement en flexion de deux plaques collées, C. R. Aca. Sci. Paris Ser. IIb, 325(6) 307-314 (1997).

[19] G. Geymonat, F. Krasucki, S. Lenci, Mathematical Analysis of a bonded joint with a soft thin adhesive, Math. Mech. Solids, 16 201-225 (1999).

[20] Y. Gilibert, A. Rigolot, Asymptotic analysis of double adhesive bonded joints loaded in shear tension (in French), J. Méc. Appl., 3(3) 341-372 (1979).

[21] M. Goland, E. Reissner, The stresses in cemented joints, J. Appl. Mech. 11, A17-A27 (1944).

[22] S. T. Gu, Q. C.He, Interfacial discontinuity relations for coupled multifield phenomena and their application to the modeling of thin interphases as imperfect interfaces, J. Mech. Phys. Solids 59, 1413-1426 (2011).

[23] S. T. Gu, J.-T. Liu, Q. C.He, The strong and weak forms of a general imperfect interface model for linear coupled multifield phenomena, Int. J. Engn. Sci 85 31-46 (2014)

[24] M. E. Gurtin, A.I. Murdoch, A continuum theory of elastic material surfaces,Arch. Ra.t Mech. Anal. 57(4), 291-323 (1975).

[25] Z. Hashin, Extremum principles for elastic heterogeneous media with imperfect interfaces and their application to bounding of effective moduli, Journal of the Mechanics and Physics of Solids 40 767-781 (1985).

[26] Z. Hashin, Thin interphase/imperfect interface in conduction, J. App. Phys. 89 (4), 2261-2267 (2001).

[27] Z. Hashin, Thin interphase/imperfect interface in elasticity with application to coated fiber composites, J. Mech. Phys. Solids 50, 2509-2537 (2002).

[28] A. Javili, S. Kaessmair, P. Steinmann, General imperfect interfaces, Comput. Methods Appl. Mech. Engrg. , 275 76-97 (2014).

[29] P. L. Kapitza, Collected papers of P. L. Kapitza (ed. D. ter Haar), 2, Pergamon (1941, reprinted 1965).

[30] A. Klarbring, Derivation of the adhesively bonded joints by the asymptotic expansion method, Int. J. Eng. Sci. 29 493-512 (1991).

[31] F. Lebon, R. Rizzoni, Asymptotic analysis of a thin interface: the case involving similar rigidity, Int. J. Eng. Sci. 48 473-486 (2010).

[32] F. Lebon, R. Rizzoni, Asymptotic behavior of a hard thin linear interphase: An energy approach, Int. J. Solids Struct. 48 441-449 (2011).

[33] F. Lebon, R. Rizzoni, Higher order interfacial effects for elastic waves in one dimensional phononic crystals via the Lagrange-Hamilton's principle, European Journal of Mechanics - A/Solids 67 57-70 (2018).

[34] H. Le Dret, A. Raoult, A., The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. J. Math. Pures Appl. 74 (6), 549-578 (1995).

[35] C. Licht, G. Michaille, A modelling of elastic adhesive bonded joints, Adv. Math. Sci. Appl., 7 (2) 711-740 (1997).

[36] M. Ljulj, J. Tambaca, 3D structure – 2D plate interaction model, Math. Mech. Solids, (published online) doi.org/10.1177/1081286519846202 (2019).

[37] P. Mariano, Multifield theories in mechanics of solids, Adv. Appl. Mech. 38, 1-93 (2002).

[38] T. Miloh, Y. Benveniste, On the effective conductivity of composites with ellipsoidal inhomogeneities and highly conducting interfaces, Proc. R. Soc. Lond. A 455, 2687-2706 (1999).

[39] H. Pham Huy, E. Sanchez-Palencia, Phenomenes de transmission à travers des couches minces de conductivité elevée, J. Math. Analysis Applic. 47 284-309 (1974).

[40] Y.Z. Povstenko, Theoretical investigation of phenomena caused by heterogeneous surface tension in solids, J. Mech. Phys. Solids, 41 1499-1514 (1993).

[41] R. Rizzoni, S. Dumont, F. Lebon, E. Sacco, Higher order model for soft and hard elastic interfaces, Int. J. Solids Struct. 51, 4137-4148 (2014).

[42] S. Saeb, P. Steinmann, A. Javili, Designing tunable composites with general interfaces, Int. J. Solid. Struct. 171 181-188 (2019).

[43] P. Schmidt, Modelling of adhesively bonded joints by an asymptotic method, Int. J. Eng. Sci., 46 1291-1324 (2008).

[44] M. Serpilli, Mathematical modeling of weak and strong piezoelectric interfaces, J. Elasticity 121(2), 235-254 (2015).

[45] M. Serpilli, Asymptotic interface models in magneto-electro-thermo-elastic composites, Meccanica 52(6), 1407-1424 (2017).

[46] M. Serpilli, On modeling interfaces in linear micropolar composites, Math. Mech. Solids 23(4), 667-685 (2018).

[47] M. Serpilli, S. Lenci, An overview of different asymptotic models for anisotropic three-layer plates with soft adhesive, Int. J. Solids Struct. 81, 130-140 (2016).

[48] M. Serpilli, S. Lenci, Asymptotic modelling of the linear dynamics of laminated beams Int. J. Solids Struct. 49(9), 1147-1157 (2012).

[49] B.R. Wang, J.-T. Liu, S. T. Gu, Q. C.He, Numerical evaluation of the effective conductivities of composites with interfacial weak and strong discontinuities, Int. J. Therm. Sci. 93 1-20 (2015).

[50] B.R. Wang, J.-T. Liu, S. T. Gu, An XFEM/level set strategy for simulating the piezoelectric spring-type interfaces with apparent physical background, Finite Elements in Analysis and Design 133 62-75 (2017).

[51] J. Yvonnet, Q.C. He, C. Toulemonde, Numerical modelling of the effective conductivities of composites with arbitrarily shaped inclusions and highly conducting interface, Compos. Sci. Tech. 68 2818-2825 (2008).

[52] J. Yvonnet, H. Le Quang, Q.C. He, An XFEM/level set approach to modelling surface/interface effects and to computing the size-dependent effective properties of nanocomposites. Comput. Mech. 42 119-131 (2008).

[53] J. Yvonnet, A. Mitrushchenkov, G. Chambaud, Q.C. He, Finite element model of ionic nanowires with size-dependent mechanical properties determined by ab initio calculations. Comput. Methods Appl. Mech. Eng. 200 614-625 (2011).

[54] F. Zaittouni, F. Lebon, C. Licht, Étude théorique et numérique du comportement d'un assemblage de plaques, C. R. Acad. Sci. Paris, t. 325, Série II b , 307-314, (1997).