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Local and Global Existence results for the Characteristic Problem for Linear and Semi-linear Wave Equations

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Introduction

> "Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way to bringing us at last to a more perfect knowledge."
>
> LEONHARD EULER

The wave operator

$$\Box := -\partial_t^2 + \sum_{j=1}^{n} \partial_j^2$$

on $\mathbb{R}^{n+1}$ is one of the fundamental differential operators of mathematical physics. In fact, the equation $\Box u = f$ is satisfied in many situations, by the amplitude u of a wave subject to a driving force f, with the speed of propagation normalized to be unity.

In the study of classical PDE's (Laplace, Heat, Wave Equations)there are very specific kinds of boundary conditions usually associated with each of these equations. In the case of the wave equation, the most classical boundary value problem is the Cauchy problem which prescribes both the initial position and the initial velocity (at $t = 0$).

The ground for telling whether a boundary condition is appropriate for a given PDE is often physically suggested, but it has to be clarified by a fundamental mathematical insight.

In a very simplified model, the light in the Universe propagates in accordance to the Wave Equation and with finite speed. The fact that the speed of light is finite is important in astronomy. In fact, due to the vast distances involved, it can take a very long time for light to travel from its source to Earth. The photographs of the Universe taken today capture images of the galaxies as they appeared billions of years ago. The fact that farther-away objects appear younger (due to the finite speed of light) raises the necessity to infer the evolution of stars, of galaxies and of the universe itself.

If we look at the sky and we take a picture of the Universe, what we have is nothing more than data assigned on a past light cone. If we want to know the past history of the Universe, we need to solve a Characteristic Initial Value Problem for the Wave Equations with these initial data on a past null cone.

In the mathematical literature there are a lot of results concerning the Cauchy Problem for the Wave Equation with data on the hyperplane $t = 0$, but just few works on the Characteristic Problem. Let's now retrace the history of the Characteristic Initial Value Problem for the Wave Equations.

An explicit solution of the characteristic initial value problem for the homogeneous wave equation $\Box u = 0$ appeared in [9, p. 750], where Courant and Hilbert show that on the time axis the solution of this problem is obtainable by a method which depends on the Asgeirsson's Mean Value Theorem (see [2]) and the solution of an integral equation. Moreover Courant and Hilbert describe how this formula can be extended for a point not on the time axis by using the invariance properties of this solution with respect to the Lorentz Transformations. The same problem in two dimensions was solved by Protter [22] by using a modified Riemann's method. Employing this extended Riemann's method, Young [25] derives a formula which is valid for the n-dimensional case, $n \geq 2$.

Observe that in all these papers the formula is proved only for the homogeneous problem and the explicit solution is obtained only on the time axis.

The history of the problem goes on with a series of results obtained using Energy-like methods.

One of the most interesting results is [21]. In this paper Müller Zum Hagen solves the Characteristic initial value problem for hyperbolic systems of second order differential equations with data on the intersection of two conic hypersurfaces, avoiding the geometrical singularity at the tip of the cone.

Later Dossa (and others) studied at first the linear problem with variable coefficients, then the quasi-linear and semi-linear case including the tip of the cone.

The first result in this sense is [10], where Dossa obtains local and global solutions of the Cauchy problem with initial data on a characteristic cone for a class of system of quasi-linear partial differential equations of second order. These solutions are obtained on weighted and non-isotropic Sobolev spaces.

This result is generalized in [11], where Dossa and Bah consider a semi-linear hyperbolic Cauchy problem of the second order with data on a characteristic cone. Using an appropriate hypothesis on the structure of the nonlinear terms, they proved that the solution is not only defined on a small neighborhood of the vertex of the cone, but also on a neighborhood of the entire cone itself.

A global solution of a non-linear hyperbolic systems on a characteristic cone is then obtained in [13] by Dossa and Touadera. They prescribe on the cone initial data with small norm and verifying some decay property at infinity. Then they apply the Penrose Transformation to the problem, thus the unbounded domain correspondent to the interior of an infinite characteristic cone is turned into a finite domain.

In all the works of Dossa and his collaborators there are several layers of technical difficulties: the variable coefficients, the geometrical singularity at the tip of the cone, the characteristic hypersurface and the quasi-linear problem. The more interesting features of this problem got mixed with the technicalities and it becomes hard to understand the key ideas and the definition of tools to be used. For this reason we choose a simple differential operator (the pure Wave Operator) and we assigned data on the future light cone, including the tip.

This Thesis wants to highlight all the peculiarities of the characteristic problem by giving a detailed explanation of the steps that bring at first to the solution of the linear problem and then to the semi-linear one.

In the first part of this work, an explicit representation formula for the solution of the linear equation is given, extending the results known for the homogeneous equation and the trace on the time axis of the solution [22] and [25]. The result we got using the Protter's and Young's method is then compared with the explicit formula which is possible to derive in the four dimensional case using Distribution Theory and, in particular, the Fundamental Solutions.

Further, Energy Estimates are derived. In constructing such Estimates one encounters several difficulties due to the presence of a geometrical singularity at the tip of the cone. To manage the construction of the Energy Estimate, one introduces suitable Sobolev-like norms characterized by weights, which mitigates the difficulties in the origin. These Estimates are well posed only for functions which vanish of order high enough at the origin. This fact brings us to split the initial data in the sum of two terms. The first term consists of the Taylor polynomial of the initial datum, the second one consist of remainder regular function with the required vanishing order at the origin.

An interesting phenomenon observed here is a gap of differentiability between the solution and the initial data, which cannot be avoided as it is shown via suitable counterexamples.

The solution obtained using the Energy method is still incomplete, because of the splitting of the initial data. This fact brings us to solve the problem for purely polynomial data. For this purpose, it is used a generalization of the well-known harmonic polynomials.

The last part of the thesis is devoted to the semi-linear problem, in order to arrive at the result, we must incorporate to the earlier techniques of resolution (based on the fixed point method and energy inequalities) the tools developed in the previous chapters.

Chapter 1

Statement of the Characteristic Problem for the Wave Equation

This chapter will be devoted to the presentation of the Characteristic Cauchy Problem for the Wave Equation and to the description of its main features. In order to clarify the peculiarities of the problem, we will give a brief outline of the results related to this topic. Then in the last two sections we will develop the instruments which will be used to solve both the linear and the semi-linear problems.

1.1 Statement of the problem

Suppose $P(D)$ is a differential operator of degree m. To solve the Cauchy Problem for P with data on a hypersurface $S = \{x \in \mathbb{R}^n \text{ s.t. } \rho(x) = 0\}$ means to find a solution u of the equation

$$P(D)u = f \tag{1.1}$$

with given f, so that for another given function φ we have

$$u(x) - \varphi(x) = \mathcal{O}(\rho(x)^m) \quad \text{when} \quad \rho(x) \to 0\,. \tag{1.2}$$

If the function $u-\varphi$ is sufficiently differentiable, the condition (1.2) is equivalent to the vanishing on S of $u - \varphi$ and all its derivatives of order $< m$ in the direction transversal to S. This is the usual form in which we give the Cauchy problem for P.

Moreover the condition (1.2) means that the solution u can be written in the form $u = \varphi + \rho^m v$. We apply the operator P and when $\rho = 0$ a simple computation gives

$$m! P_m \left(\frac{\operatorname{grad} \rho}{i} \right) v = f - P(D)\varphi\,, \tag{1.3}$$

where P_m is the principal part of P.

If $P_m\left(\frac{\operatorname{grad}\rho}{i}\right) \neq 0$ and the data are in $\mathscr{C}^\infty$, one natural idea for attempting to find the function v consists of calculating v and its derivatives when $\rho = 0$. In this way we can compute the Taylor expansion of the solution u in a neighborhood of S in one and only one way so that (1.1) is fulfilled. This is exactly the method used in the proof of *Cauchy-Kowalewski Theorem*. For the Cauchy problem, the theorem of Cauchy-Kowalewski (see [7, p. 330]), proved for equations with analytic data, establishes the existence of solutions in power series for equations which are not characteristic with respect to the initial surface.

Theorem 1.1 (Cauchy-Kowalewski). *Suppose $P(x, D_x)$ is a differential operator of degree m with analytic coefficients in some neighborhood of a point $x_0 \in \mathbb{R}^n$. We take functions f, φ, ρ which are analytic in a neighborhood of x_0. Suppose that ρ is real-valued and satisfied $P_m(x_0, \operatorname{grad}\rho(x_0)) \neq 0$. We denote by S the hypersurface defined in a neighborhood of x_0 by the equation $\rho(x) = \rho(x_0)$. Then in some neighborhood of x_0 there exists a unique solution of the Cauchy Problem*

$$\left\{\begin{array}{l} P(x, D_x)u = f \\ D^\alpha(u - \varphi)|_S = 0 \end{array}\right. \qquad \textit{if } |\alpha| < m. \tag{1.4}$$

This result leads us immediately to introduce the following definition.

Definition 1.1 (Characteristic Hypersurface). Let $P(x, D_x)$ be a differential operator of degree m with $\mathscr{C}^\infty$ coefficients in $\mathbb{R}^n$. A hypersurface $S = \{x \in \Omega \text{ s.t. } \rho(x) = \rho(x_0)\}$ where $\rho \in \mathscr{C}^1(\Omega)$ and $\operatorname{grad}\rho(x_0) \neq 0$ is said to be *characteristic at* x_0 with respect to the operator P, if

$$P_m(x_0, \operatorname{grad}\rho(x_0)) = 0\,, \tag{1.5}$$

where P_m is the principal part of P.

Asking S to be non-characteristic for P is the natural hypothesis for the solvability of the problem (1.1) and the Cauchy problem prescribed on a time-like characteristic hypersurface is know to be ill posed, in fact not only existence, but also uniqueness can fail.

In the case where P has constant coefficients and S is a characteristic hyperplane the uniqueness of an analytic solution fails. The counterexample is immediately given by the following situation. Suppose P homogeneous of degree m and $P_m(0, \ldots, 0, 1) = 0$, so the hyperplane $x_n = 0$ is characteristic for P. Then the function $u(x) = x_n^m$ is a non-zero solution of the problem

$$\left\{\begin{array}{l} P(D)u = 0 \\ D_n^j u|_{x_n=0} = 0 \end{array}\right. \quad \text{for } j = 1, \ldots, m-1. \tag{1.6}$$

The same counterexample proves that the uniqueness of $\mathscr{C}^\infty$ solutions fails as well. Nonetheless, in certain particular cases it is possible to state a theorem of local existence and uniqueness with respect to a characteristic surface. This is as follows.

Theorem 1.2 (Goursat-Beudon)**.** *Suppose that all coefficients of the equation*

$$\sum_{|\alpha|\le m} a_\alpha D^\alpha u = f \tag{1.7}$$

are analytic in a neighborhood of the origin and that the coefficient of D_n^m *vanishes identically while the coefficient of* $D_j D_n^{m-1}$ *is* $\neq 0$ *when* $x_n = 0$ *for some* $j < n$*. For arbitrary functions* f *and* φ *which are analytic in a neighborhood of the origin there exists a unique solution* u *of* (1.7) *which is analytic in a neighborhood of the origin and satisfies the boundary conditions*

$$\begin{cases} D_n^k(u-\varphi)|_{x_n=0} = 0 & \text{if } k < m-1 \\ (u-\varphi)|_{x_j=0} = 0 \end{cases}.$$

It is possible to use the Cauchy-Kowalewski Theorem to prove that the Cauchy problem for an equation with analytic coefficients and data on a non-characteristic surface is also uniquely determinated if non-analytic solutions are allowed. This is the classical local uniqueness theorem of Hölmgren. For the proof see [7, p. 333].

Theorem 1.3 (Hölmgren)**.** *Suppose* $P(x, D_x)$ *is a differential operator of degree* m *with analytic coefficients in some neighborhood* V *of a point* $x_0 \in \mathbb{R}^n$*. Suppose that* S *is a hypersurface containing* x_0 *defined in a neighborhood of* x_0 *by the analytic equation* $\rho(x) = 0$*. Suppose that* S *is non-characteristic for* P *at* x_0*. We put* $V^+ = \{x \in V \text{ such that } \rho(x) \ge 0\}$*. Then there exists a neighborhood* W *of* x_0 *such that any function* $u \in \mathscr{C}^m(V^+)$ *which solves the Cauchy Problem*

$$\begin{cases} P(x, D_x)u = 0 & \text{in } V^+ \\ D^\alpha u|_S = 0 & \text{in } V \cap S, |\alpha| < m \end{cases} \tag{1.8}$$

is necessarily equal zero in $W^+ = \{x \in W \text{ such that } \rho(x) \ge 0\}$*.*

For more refined uniqueness theorems which can be proved with similar technique we refer the reader to [18].

In general there is no uniqueness for the Cauchy problem with data on a characteristic hyperplane, unless one considers solutions with suitable decay at infinity, such as discussed in Hörmander [17, p. 143]. There we can find also a characterization of these operators for which an existence theorem is valid.

By the discussion above we deduce that if P has constant coefficients and S is a hyperplane, in order to have local uniqueness it is necessary and sufficient that S is non-characteristic.

Assume that S is a characteristic hypersurface for an operator P (not a hyperplane) with constant coefficients. It is possible to choose S in order to have not only the existence but also the uniqueness of the solution of the Initial Value Problem hold. This strictly depends on the nature of the particular hypersurface considered.

1.1.1 The Characteristic Initial Value Problem

We ask now how the Initial Value Problem should be stated when the hypersurface S is characteristic. If we write $u = \varphi + v$, then the Cauchy Problem (1.1)-(1.2) is reduced to one with homogeneous Cauchy data. Changing notation, we may assume the boundary condition (1.2) has the homogeneous form

$$u(x) = \mathcal{O}(\rho(x)^m) \quad \text{when} \quad \rho(x) \to 0\,. \tag{1.9}$$

We set,

$$U_+ = \begin{cases} u & \text{if } \rho > 0 \\ 0 & \text{if } \rho < 0 \end{cases} \quad F_+ = \begin{cases} f & \text{if } \rho > 0 \\ 0 & \text{if } \rho < 0 \end{cases}. \tag{1.10}$$

Then, it follows from (1.1)-(1.9), that

$$P(D)U_+ = F_+$$

holds in the distribution sense; conversely, this implies (1.1) and (1.9) when $\rho > 0$, provided S is non-characteristic. Thus the Cauchy Problem with homogeneous Cauchy data is then equivalent to finding a solution of (1.1) vanishing on one side of S when f does.

This leads us to state the Initial Value problem when S is characteristic as follows.

Definition 1.2 (Characteristic Initial Value Problem)**.** Let $P(x, D_x)$ be a differential operator of degree m with $\mathscr{C}^\infty$ coefficients in $\mathbb{R}^n$ and $S = \{x \in \mathbb{R}^n \text{ s.t. } \rho(x) = 0\}$ be a characteristic hypersurface for P. The *Characteristic Initial Value Problem* for the operator P is the problem of finding a solution u of

$$\begin{cases} P(x, D_x)u = f \\ u|_S = \varphi \end{cases} \quad \text{in } \rho \geq 0 \tag{1.11}$$

with support in $\rho \geq 0$, when f is given with support in $\rho \geq 0$ as well.

Remark 1.1. As Cauchy Problem we intend an Initial Value Problem with initial datum assigned on a non-characteristic hypersurface. On the contrary, in the case of a characteristic surface, we will talk about a *Characteristic Initial Value Problem.*

It is easy to see that the vectors tangent to the cone

$$\partial\mathcal{C} = \{(t,x) \in [0,+\infty) \times \mathbb{R}^n \text{ s.t. } t = |x|\} \tag{1.12}$$

belong to the null cone in the frequencies space, which is the characteristic manifold of the Wave Operator

$$\Box := -\partial_t^2 + \sum_{j=1}^{n} \partial_j^2. \tag{1.13}$$

So the hypersurface $\partial\mathcal{C}$ is characteristic for the operator (1.13) in the sense of Definition 1.1.

Set

$$\mathcal{C} = \{(t,x) \in [0,+\infty) \times \mathbb{R}^n \text{ s.t. } |x| < t\}, \tag{1.14}$$

we consider the following Characteristic Initial Value Problem for the Wave Equation

$$\begin{cases} \Box u = f \\ u|_{\partial\mathcal{C}} = \varphi|_{\partial\mathcal{C}} \end{cases} \quad \text{in } \mathcal{C}$$

where φ and f are assigned functions in (t,x) variables.

Remark 1.2. To better handle previous problem, we assign as initial datum a function which is the restriction to $\partial\mathcal{C}$ of a regular function $\varphi : \mathbb{R}^{n+1} \longrightarrow \mathbb{C}$. The restriction to $\partial\mathcal{C}$ is defined as $\varphi|_{\partial\mathcal{C}} : \partial\mathcal{C} \longrightarrow \mathbb{C}$. If we project $\varphi|_{\partial\mathcal{C}}$ on $\mathbb{R}^n$ using the parametrization (1.12) of the hypersurface $\partial\mathcal{C}$, we get a function $\varphi^* : \mathbb{R}^n \longrightarrow \mathbb{C}$ defined as

$$\varphi^*(x) = \varphi(|x|, x),$$

which may not be smooth at the origin[1].

The goal of this work is to find regular local and global solutions of the Characteristic Initial Value Problem for the Linear Wave Equation

$$\begin{cases} \Box u = f \\ u|_{\partial\mathcal{C}} = \varphi^* \end{cases} \quad \text{in } \mathcal{C}, \tag{1.15}$$

where f, φ are regular assigned functions, and semi-linear Wave Equation

$$\begin{cases} \Box u = Q(\partial u, \partial u) \\ u|_{\partial\mathcal{C}} = \varphi^* \end{cases} \quad \text{in } \mathcal{C}$$

where $Q(\partial u, \partial u)$ denotes a generic quadratic form in ∂u.

[1]For example the smooth function $u(t,x) = t$ on the cone $\partial\mathcal{C}$ becomes the non-smooth function $u(|x|, x) = |x|$

1.2 Characteristic coordinates and derivatives

In this Section we describe the additional tools to better handle the features of this problem. In order to simplify the representation of the cone $\mathcal{C}$ and of the functions defined on it, we introduce the *Characteristic Coordinates.* These particular coordinated allow to clearly distinguish the derivatives tangential to the data hypersurface from the transversal ones. This distinction will be very important while deriving the Energy Estimates for the considered problem, where it will be needed to re-write the classical $\mathbb{R}^{n+1}$-derivatives $\partial = (\partial_t, \partial_1, \dots, \partial_n)$ in terms of tangential and transversal derivatives with respect to $\partial\mathcal{C}$.

1.2.1 Characteristic coordinates

We define (ξ, η, ω) as new variables according to the relations

$$\begin{cases} x = \frac{\xi-\eta}{2}\,\omega \\ t = \frac{\xi+\eta}{2} \end{cases} \iff \begin{cases} \xi = t + |x| \\ \eta = t - |x| \\ \omega = \frac{x}{|x|} \end{cases} . \tag{1.16}$$

Let us define functions v, g and ψ such that

$$u(t,x) = v(\xi, \eta, \omega)\ , \tag{1.17a}$$

$$f(t,x) = 4\,g(\xi, \eta, \omega)\ , \tag{1.17b}$$

$$\varphi(t,x) = \psi(\xi, \eta, \omega) \tag{1.17c}$$

using the well-know expression in polar coordinates of the Laplace-Beltrami operator

$$\Delta_{\mathbb{R}^n} = \partial_r^2 - \frac{n-1}{2(\xi-\eta)}\partial_r + \frac{1}{r^2}\Delta_{S^{n-1}},$$

the problem (1.15) becomes

$$\begin{cases} Lv := \partial^2_{\xi\eta} v - \dfrac{n-1}{2(\xi-\eta)}(\partial_\xi v - \partial_\eta v) - \dfrac{1}{(\xi-\eta)^2}\Delta_{S^{n-1}} v = g \\ v(\xi, 0, \omega) = \psi^*(\xi, \omega) \end{cases} . \tag{1.18}$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the n-dimensional sphere S^{n-1}.

The two variables ξ and η are called *Characteristic Coordinates* since ξ =const. and η =const. represent characteristic half-cones; in particular the hypersurface $\partial\mathcal{C}$ becomes the plane $\eta = 0$ in the (ξ, η, ω) space.

1.2.2 Derivatives with respect to the Characteristic Coordinates

As we have already said, the characteristic coordinates allow us to precisely individuate the *tangential and transversal derivatives* with respect to the initial data hypersurface $\partial\mathcal{C}$.

We need now to clearly define the derivatives with respect to the Characteristic Coordinates we have just introduced.

The relations (1.16) lead us immediately to define the operators ∂_ξ and ∂_η as follows.

Definition 1.3 (Derivative with respect to ξ)**.** We denote by ∂_ξ the differential operator defined as

$$\partial_\xi := \partial_t + \partial_r .$$

∂_ξ is called *derivative with respect to the characteristic variable* ξ.

Definition 1.4 (Derivative with respect to ξ)**.** We denote by ∂_η the differential operator defined as

$$\partial_\eta := \partial_t - \partial_r .$$

∂_ξ is called *derivative with respect to the characteristic variable* η.

To define the derivative with respect to ω, we recall the following definition.

Definition 1.5 (Space Rotations Generators)**.** The generators of the group of all *Space Rotations* in $\mathbb{R}^n$ are given by the set of differential operators $\{\Omega_{ij}\}_{i,j=1,\ldots,n}$ where

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i .$$

It can be verified that each operator Ω_{ij} commutes with the wave operator $\Box$ (see [20]). In addition, the operators Ω_{ij} are purely spherical (i.e. it is a vectorial field tangent to the spheres centered in the origin) and the spherical Laplace-Beltrami operator $\Delta_{S^{n-1}}$ can be written as

$$\Delta_{S^{n-1}} = \sum_{i<j} \Omega_{ij}^2 . \tag{1.19}$$

It is clear from the above that the solution u of (1.15) must satisfy

$$\left\{ \begin{array}{l} \Box \Omega_{ij}^k u = \Omega_{ij}^k f \\ \Omega_{ij}^k u|_{\partial C} = \Omega_{ij}^k \varphi^* \end{array} \right. , \quad \forall k \in \mathbb{N} ;$$

in particular from (1.19) we have that the operator $\Delta_{S^{n-1}}$ commutates with the restriction-projection operator *

$$(\Delta_{S^{n-1}} u)^* = \Delta_{S^{n-1}} \varphi^* .$$

Using the operators Ω_{ij}, we define the gradient with respect to the characteristic variable ω as follows.

Definition 1.6 (Gradient with respect to ω)**.** We denote by ∇_ω the vectorial differential operator whose component of order j is given by

$$\nabla_{\omega_j} := \omega_i \Omega_{ij},$$

where we used the Einstein Convention on the sum[2]. ∇_ω is called *"angular" gradient* or *gradient with respect to the characteristic variable ω.*

It is easy to see that the derivative ∂_ξ is parallel to the cone directrix, while ∇_ω follows the angular direction; so both ∂_ξ and ∇_ω are tangential to $\partial\mathcal{C}$. On the contrary, the derivative ∂_η is transversal to the data hypersurface.

Remark 1.3. Let ∇_x be the classical spatial gradient, then, given a smooth function f, the tangential component of $\nabla_x f$ to the sphere centered at the origin is given by

$$\not\nabla f := \nabla_x f - \partial_r f \omega \,,$$

where $\partial_r f\omega = (\nabla f \cdot \omega)\omega$ is the projection of $\nabla_x f$ on the radial direction ω.

In order to use operators with the same degree of homogeneity, we replace the adimensional operator ∇_ω with

$$\not\nabla := \frac{1}{\xi - \eta} \nabla_\omega \,.$$

We are now able to define as

$$\partial_+ := \left\{\partial_\xi, \not\nabla\right\} \tag{1.20}$$

the set of the *tangential derivatives* with respect to $\partial\mathcal{C}$ and as

$$\partial_- := \{\partial_\eta\} \tag{1.21}$$

the set of *transversal (i.e. non-tangential) derivatives* with respect to $\partial\mathcal{C}$.

Remark 1.4. Let $k \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ be a multiindex such that $|\alpha| = k$, we introduce the multiindex notation to indicate the following family of derivatives

$$\not\nabla^k v := \left\{ \frac{1}{(\xi - \eta)^k} \nabla^{\alpha_1}_{\omega_1} \cdots \nabla^{\alpha_n}_{\omega_n} v \mid \alpha \in \mathbb{Z}^n_+ \,, |\alpha| = k \right\}.$$

1.3 Problem features

1.3.1 Initial conditions

In the theory of partial differential equations, by the Initial Value Problem we mean the problem of finding the solutions of a given partial differential

[2] We omit the sum symbol on the repeated index.

equation with the appropriate number of initial conditions prescribed at an initial data surface.

In order to have a well-posed Initial Value Problem (1.1) it is necessary to assign a certain number of initial conditions on the hypersurface S.

It is reasonable to expect that the number of initial conditions is exactly equal to the order of transversal derivatives in the differential equation one gets when the equation (1.1) is evaluated on the initial data hypersurface S.

In case of the Cauchy Problem for the Wave Equation with data on the hyperplane $t = 0$ the transversal derivative is ∂_t; when we compute $\Box u = f$ on $t = 0$, we get

$$-\partial_t^2 u(0,x) + \Delta u(0,x) = f(0,x).$$

Thus the considered problem is well-posed once two initial conditions are assigned on the hyperplane $t = 0$.

In case of the Characteristic Initial Value Problem for the Wave Equation with data on the null cone $t = |x|$, the transversal derivative is ∂_η and on the cone $\partial\mathcal{C}$ we have

$$(\partial^2_{\xi\eta} v)^* - \frac{n-1}{2\xi}[(\partial_\xi v)^* - (\partial_\eta v)^*] = \frac{1}{\xi^2}(\Delta_{S^{n-1}} v)^* + g^*. \tag{1.22}$$

So only one initial condition should be needed on $\partial\mathcal{C}$.

1.3.2 Gap of regularity between the solution and the initial data

When solving the classical Cauchy Problem we note that the regularity of the initial data is inherited by the solution. In other words, to have a H^s-regular solution it suffices to assign initial data with H^s-regularity. This does not occur in the case of the Characteristic Initial Value Problem.

Let's rewrite the equation (1.22) considering that the projection-restriction operator * commutes with the ∂_+ derivatives

$$\partial_\xi(\partial_\eta v)^* + \frac{n-1}{2\xi}(\partial_\eta v)^* = \frac{n-1}{2\xi}\partial_\xi \psi^* + \frac{1}{\xi^2}\Delta_{S^{n-1}}\psi^* + g^*. \tag{1.23}$$

Focusing on the previous equation, we can clearly see that to estimate a ∂_η derivative of the solution on $\partial\mathcal{C}$ we need at most two ∂_+ derivatives[3] of the initial datum ψ; in fact at the right-hand term (1.23) we have the Laplace-Beltrami operator.

In general, to estimate k transversal derivatives, at most $2k-1$ tangential derivatives could be needed. This iterative estimate generates a *gap of regularity* between the solution and the initial data: to have a H^s-regular solution we need to assign φ and f with H^{2s-1} and H^{2s-3}-regularity respectively.

[3] Either one ∂_ξ derivative or two $\not\nabla$ derivatives, depending on the assigned data.

1.3.3 Main difficulties

From the mathematical point of view, the main difficulties we met in studying the Characteristic Initial Value Problem are due to the nature of the initial surface, in particular, we are referring to:

1. The presence of a geometrical singular point: the tip of the cone;

2. The fact that the classical derivatives $\partial = (\partial_t, \partial_1, \ldots, \partial_n)$ and $\nabla = (\partial_1, \ldots, \partial_n)$ do not commute with the restriction-projection operator $*$ on data surface in the sense that

$$\partial_k \varphi^* = \omega_k (\partial_t \varphi)^* + (\partial_k \varphi)^* \neq (\partial_k \varphi)^*.$$

Thus, while developing the method to solve the considered problem, we have to take into account that:

1. When we restrict a smooth function to $\partial\mathcal{C}$ we may get a singular function and additional singularities are generated by deriving terms of the form $|x|$;

2. The transversal (i.e. not tangential to $\partial\mathcal{C}$) derivatives are not immediately derived from the initial datum, but they are iteratively retrieved by the progressive differentiation of the *Transport Equation* (1.22) along $\partial\mathcal{C}$. This generates the gap of regularity previously described in Section 1.3.2.

1.4 Weighted Sobolev Spaces

It is useful to parametrize the cone $\mathcal{C}$ and its surface $\partial\mathcal{C}$ using balls and spheres in $\mathbb{R}^n$, in particular we have

$$\mathcal{C}_T = \{(t, x) \text{ s.t. } x \in B_t, t \in [0, T]\}$$
$$\partial\mathcal{C}_T = \{(t, x) \text{ s.t. } x \in S_t, t \in [0, T]\}$$

where S_t and B_t denote the $(n-1)$-dimensional sphere and the n-dimensional ball with radius t and centered in zero, respectively.

In the Classical Sobolev Imbedding Theorem (see Theorem B.1) the Sobolev constant depends on the domain Ω, which fulfills the *Cone Property* (see Definition B.1).

In case of domains[4] such as B_t and S_t it is easy to explicit this dependence in terms of the radium t, as shown in the results below.

[4]For the compact manifold S_t the cone property is proved using the parameterizations.

Theorem 1.4. *Let $q \in \mathbb{R}$ and s positive integer satisfy $s > \frac{n}{2}$ and $2 \leq q \leq \frac{2n}{n-2s}$, then, fixed $t > 0$,*

$$H^s(B_t) \hookrightarrow L^q(B_t)\,.$$

In fact, for all $u \in L^q(B_t)$ the following estimate holds

$$\|u\|_{L^q(B_t)} \leq Ct^{-\frac{n}{2}+\frac{n}{q}}(1+t)^s \|u\|_{H^s(B_t)}$$

with $C = C(q, n, s)$.

Proof. By virtue of Theorem B.1, for all $u \in L^q(B_1)$ we have

$$\|u\|_{L^q(B_1)} \leq C \|u\|_{H^s(B_1)}\ ,$$

with $C = C(q, n, s)$. For $t > 0$, consider the function v defined for all $x \in B_1$ as $u(x) = v(tx)$. Clearly $v : B_t \longrightarrow \mathbb{R}$, simple computations show that:

(A) $v \in L^q(B_t)$ and $\|u\|_{L^q(B_1)} = t^{-\frac{n}{q}} \|u\|_{L^q(B_t)}$;

(B) For all α multi-index, $|\alpha| < s$, $\partial^\alpha u(x) = t^{|\alpha|}(\partial^\alpha v)(tx)$ and

$$\|\partial^\alpha u\|_{L^2(B_1)} = t^{|\alpha|-\frac{n}{2}} \|\partial^\alpha v\|_{L^2(B_t)}\ .$$

By combining (A) and (B) we deduce that

$$t^{-\frac{n}{q}} \|v\|_{L^q(B_t)} \leq Ct^{-\frac{n}{2}} \sum_{|\alpha|<s} t^{|\alpha|} \|\partial^\alpha v\|_{L^2(B_t)}\ ,$$

which immediately leads to the thesis. □

Upon retracing the steps in the proof of the previous Theorem, one readly obtains the following result.

Theorem 1.5. *Let s positive integer satisfy $s > \frac{n}{2}$, then, fixed $t > 0$,*

$$H^s(B_t) \hookrightarrow L^\infty(B_t) \cap \mathscr{C}(B_t)\,.$$

In fact, for all $u \in L^\infty(B_t) \cap \mathscr{C}(B_t)$ the following estimate holds

$$\|u\|_{L^\infty(B_t)} \leq Ct^{-\frac{n}{2}}(1+t)^s \|u\|_{H^s(B_t)}$$

with $C = C(n, s)$.

The results stated in Theorems 1.4 and 1.5 can be easily extended to $L^q(S_t)$ and $H^s(S_t)$ spaces simply replacing n with $n-1$.

Corollary 1.1. *Let $q \in \mathbb{R}$ and s positive integer satisfy $s > \frac{n-1}{2}$ and $2 \leq q \leq \frac{2(n-1)}{n-1-2s}$, then, fixed $t > 0$,*

$$H^s(S_t) \hookrightarrow L^q(S_t)\,.$$

In fact, for all $u \in L^q(S_t)$ the following estimate holds

$$\|u\|_{L^q(S_t)} \leq C t^{-\frac{n-1}{2}+\frac{n-1}{q}} (1+t)^s \|u\|_{H^s(S_t)}$$

with $C = C(q, n, s)$.

Corollary 1.2. *Let s positive integer satisfy $s > \frac{n-1}{2}$, then, fixed $t > 0$,*

$$H^s(S_t) \hookrightarrow L^\infty(S_t) \cap \mathscr{C}(S_t)\,.$$

In fact, for all $u \in L^\infty(S_t) \cap \mathscr{C}(S_t)$ the following estimate holds

$$\|u\|_{L^\infty(S_t)} \leq C t^{-\frac{n-1}{2}} (1+t)^s \|u\|_{H^s(S_t)}$$

with $C = C(n, s)$.

As stated before, while deriving functions defined on the cone $\partial\mathcal{C}$, singularities occur. In order to control these singularities in Energy Estimates using a H^s_{loc}-like norm, we define the following generalized Sobolev norms.

$$\|u\|_{X^s_T} := \sup_{t\in[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|\leq s} \|\partial^\alpha u(t)\|_{L^2(B_t)}\,, \tag{1.24a}$$

$$\|u\|_{D^s_T} := \sum_{j=0}^{2s-1} \sup_{t\in[0,T]} t^{-\frac{n-1}{2}-s+j} \sum_{|\alpha|\leq j} \left\|\partial^\alpha_+ u(t)\right\|_{L^2(S_t)}\,, \tag{1.24b}$$

$$\|u\|_{Y^s_T} := \sum_{k=0}^{s-1} \sum_{j=0}^{2(s-k)-1} \sup_{t\in[0,T]} t^{-\frac{n-1}{2}-s+k+j} \sum_{|\alpha|\leq j} \left\|\partial^\alpha_+ \partial^k_- u(t)\right\|_{L^2(S_t)} + \|u\|_{X^s_T} =$$

$$= \sum_{k=0}^{s-1} \left\|\partial^k_- u\right\|_{D^{s-k}_T} + \|u\|_{X^s_T}\,; \tag{1.24c}$$

where ∂_+ contains only derivatives tangential to the data surface $\partial\mathcal{C}$, while ∂_- contains only transversal derivatives (see Section 1.2.2).

The definition of norms (1.24) leads us immediately to introduce the closures of the space $\mathscr{C}^\infty_c(\mathbb{R}^{n+1})$ with respect to them. In these spaces we will state the first existence and uniqueness result for the Linear Problem in Section 3.3.

Definition 1.7 (Space X^s_T)**.** We denote by X^s_T the the closure of the space $\mathscr{C}^\infty_c(\mathbb{R}^{n+1})$ with respect to the norm $\|\cdot\|_{X^s_T}$.

Definition 1.8 (Space D^s_T)**.** We denote by D^s_T the the closure of the space $\mathscr{C}^\infty_c(\mathbb{R}^{n+1})$ with respect to the norm $\|\cdot\|_{D^s_T}$.

Definition 1.9 (Space Y^s_T)**.** We denote by Y^s_T the the closure of the space $\mathscr{C}^\infty_c(\mathbb{R}^{n+1})$ with respect to the norm $\|\cdot\|_{Y^s_T}$.

The so called "weighted" norms (1.24) have been introduced for the purpose of handling functions with singular terms in the origin of the form $|x|$. In Chapter 3 we will see that if for a smooth function f the norms $\|\cdot\|_{D^s_T}$ or $\|\cdot\|_{Y^s_T}$ are finite then f satisfies vanishing conditions of certain order at the origin.

Some properties of the spaces X^s_T and Y^s_T are listed below.

Remark 1.5. Trivially, the spaces X^s_T, D^s_T and Y^s_T endowed with the correspondent norms are Banach spaces.

The following Lemma extends the Theorem B.3 to the space X^s_T.

Lemma 1.1. *X^s_T forms an algebra under pointwise multiplication provided $s > \frac{n}{2}$; in particular for all $u, v \in X^s_T$ we have*

$$\|uv\|_{X^s_T} \lesssim \|u\|_{X^s_T} \|v\|_{X^s_T} ,$$

where the implicit constant does not depend on T.

Proof. Consider

$$\sup_{t\in[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|\leq s} \|\partial^\alpha (uv)(t)\|_{L^2(B_t)} ,$$

by Leibniz's rule we get

$$\partial^\alpha(uv) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta u \partial^{\alpha-\beta} v,$$

so it is sufficient to show that for any $\beta \leq \alpha$, $|\alpha| \leq s$ and for all $t \in [0, T]$, results

$$t^{-\frac{n}{2}} \left\|\partial^\beta u(t)\partial^{\alpha-\beta} v(t)\right\|_{L^2(B_t)} \lesssim \|u\|_{X^s_T} \|v\|_{X^s_T} .$$

In view of Theorem 1.4, if $s - |\beta| \leq \frac{n}{2}$ and $2 \leq q \leq \frac{2n}{n-2(s-|\beta|)}$, then

$$\left\|\partial^\beta w(t)\right\|_{L^q(B_t)} \lesssim t^{-\frac{n}{2}+\frac{n}{q}} \left\|\partial^\beta w(t)\right\|_{H^{s-|\beta|}(B_t)} ; \tag{1.25}$$

moreover, by Theorem 1.5, if $s - |\beta| > \frac{n}{2}$, then

$$\left\|\partial^\beta w(t)\right\|_{L^\infty(B_t)} \lesssim t^{-\frac{n}{2}} \left\|\partial^\beta w(t)\right\|_{H^{s-|\beta|}(B_t)} . \tag{1.26}$$

Let k be the largest integer such that $s-k>\frac{n}{2}$, since $s>\frac{n}{2}$, we have that $k\geq 0$. If $|\beta|\leq k$, then $s-|\beta|>\frac{n}{2}$, so

$$\left\|\partial^\beta u(t)\partial^{\alpha-\beta}v(t)\right\|_{L^2(B_t)}\lesssim\left\|\partial^\beta u(t)\right\|_{L^\infty(B_t)}\left\|\partial^{\alpha-\beta}v(t)\right\|_{L^2(B_t)}\lesssim$$
$$\lesssim t^{-\frac{n}{2}}\left\|\partial^\beta u(t)\right\|_{H^{s-|\beta|}(B_t)}\left\|\partial^{\alpha-\beta}v(t)\right\|_{H^{s-|\alpha-\beta|}(B_t)};$$

hence

$$t^{-\frac{n}{2}}\left\|\partial^\beta u(t)\partial^{\alpha-\beta}v(t)\right\|_{L^2(B_t)}\lesssim\left(\sup_{t\in[0,T]}t^{-\frac{n}{2}}\sum_{|\gamma|=s}\|\partial^\gamma u(t)\|_{L^2(B_t)}\right)\cdot$$
$$\cdot\left(\sup_{t\in[0,T]}t^{-\frac{n}{2}}\sum_{|\gamma|=s}\|\partial^\gamma v(t)\|_{L^2(B_t)}\right)\lesssim\|u\|_{X_T^s}\|v\|_{X_T^s}\,. \quad (1.27)$$

The case $|\alpha-\beta|\leq k$ is analogous.

Now, if $|\beta|,|\alpha-\beta|>k$, then $s-|\beta|,s-|\alpha-\beta|\geq\frac{n}{2}$, moreover

$$\frac{n-2(s-|\beta|)}{n}+\frac{n-2(s-|\alpha-\beta|)}{n}=\frac{2n-2(2s-|\alpha|)}{n}\leq 2-\frac{2s}{n}<1\,.$$

Hence there exist positive numbers r,r', with $\frac{1}{r}+\frac{1}{r'}=1$, such that

$$2\leq 2r<\frac{n}{n-2(s-|\beta|)}\,,\quad 2\leq 2r'<\frac{n}{n-2(s-|\alpha-\beta|)}\,,$$

thus by *Generalized Hölder's Inequality* and (1.25) we have

$$\left\|\partial^\beta u(t)\partial^{\alpha-\beta}v(t)\right\|_{L^2(B_t)}\lesssim\left\|\partial^\beta u(t)\right\|_{L^{2r}(B_t)}\left\|\partial^{\alpha-\beta}v(t)\right\|_{L^{2r'}(B_t)}\lesssim$$
$$\lesssim t^{-\frac{n}{2}+\frac{n}{2r}}\left\|\partial^\beta u(t)\right\|_{H^{s-|\beta|}(B_t)}\cdot$$
$$\cdot\, t^{-\frac{n}{2}+\frac{n}{2r'}}\left\|\partial^{\alpha-\beta}v(t)\right\|_{H^{s-|\alpha-\beta|}(B_t)}\,.$$

This implies (1.27) and completes the proof. □

Chapter 2

An Explicit Formula for the Characteristic Problem for the Inhomogeneous Wave Equation

This chapter is divided into four sections. In section 2.1 the results [22] and [25] are extended to the inhomogeneous case. We generalize the Protter's and Young's method to derive an explicit formula which gives the value of the solution of the problem (1.15) in a point $(t_0, 0)$ on the t-axis.

The wave equation is invariant with respect to the Lorentz Transformation. In section 2.2 we describe the properties of these transformations, which will be used in section 2.3 to obtain a formula for the explicit solution of the problem (1.15) on a generic point $(T^*, X_1^*, 0')$.

In the last section we compare the result we got using the Protter's and Young's method with the explicit formula which is possible to derive in the four dimensional case using Distribution Theory and, in particular, the Fundamental Solutions.

2.1 The solution on the t-axis

Our aim is to solve the initial value problem (1.15) where the given functions φ and f are assumed to be smooth. In particular, we wish to compute the solution at a point $(t_0, 0)$ on the t-axis.

We transform (1.15) to characteristic coordinates by introducing (ξ, η, ω) as new variables according to the relations (1.16).

Let v and g be as in (1.17), so the equation $\Box u = f$ becomes (1.18).

Following this transformation, we introduce the *"reduced" formal adjoint operator* associated to L,

$$Mw := \partial^2_{\xi\eta} w + \frac{n-1}{2(\xi - \eta)} (\partial_\xi w - \partial_\eta w) \; , \tag{2.1}$$

that differs from the classical formal adjoint operator L^* in that it depends only on the variables ξ and η.

We can easily verify the identity

$$(\partial_\eta w - \partial_\xi w)Lv + (\partial_\eta v - \partial_\xi v)Mw = \\ = \partial_\xi(\partial_\eta v \partial_\eta w) - \partial_\eta(\partial_\xi v \partial_\xi w) - \frac{\partial_\eta w - \partial_\xi w}{(\xi - \eta)^2}\Delta_{S^{n-1}} v \,. \quad (2.2)$$

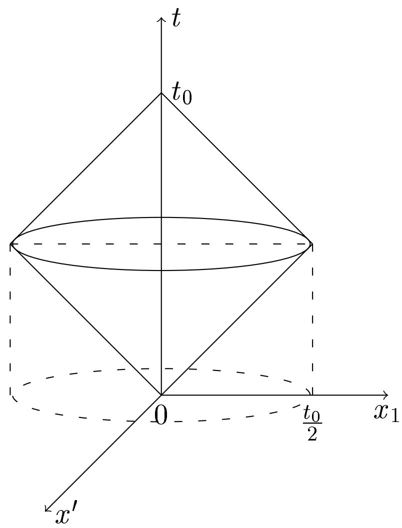

Figure 2.1: Domain D_0

Since we are interested in obtaining the solution at the point $(t_0, 0)$, in (t, x) space we consider the domain

$$D_0 : \begin{cases} |x| \le t \\ |x| \le t_0 - t \\ 0 \le t \le t_0 \end{cases} , \quad (2.3)$$

which corresponds in (ξ, η, ω) coordinates to the region $D_{\xi,\eta} \times S^{n-1}$, where

$$D_{\xi,\eta} : \begin{cases} \eta \le \xi \\ \eta \ge 0 \\ \xi \le t_0 \end{cases} . \quad (2.4)$$

Let us take v for the desired solution of $Lv = g$ and w for a regular solution of $Mw = 0$. We integrate the identity (2.2) on $D_{\xi,\eta} \times S^{n-1}$ to

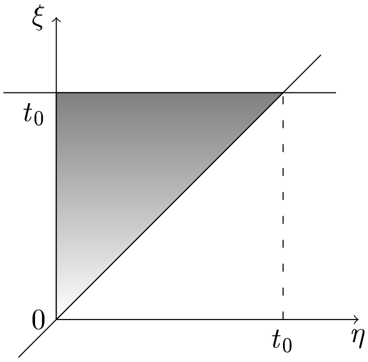

Figure 2.2: Domain $D_{\xi,\eta}$

obtain

$$\int_{S^{n-1}} \iint_{D_{\xi,\eta}} (\partial_\eta w - \partial_\xi w) g \, d\xi \, d\eta \, d\sigma_\omega =$$
$$= \int_{S^{n-1}} \iint_{D_{\xi,\eta}} \left[\partial_\xi(\partial_\eta v \partial_\eta w) - \partial_\eta(\partial_\xi v \partial_\xi w) - \frac{\partial_\eta w - \partial_\xi w}{(\xi-\eta)^2} \Delta_{S^{n-1}} v \right] d\xi \, d\eta \, d\sigma_\omega \, . \tag{2.5}$$

Using the fact that

$$\iint_{D_{\xi,\eta}} d\xi \, d\eta = \int_0^{t_0} \int_\eta^{t_0} d\xi \, d\eta = \int_0^{t_0} \int_0^{\xi} d\eta \, d\xi$$

and, by the Divergence Theorem,

$$\int_{S^{n-1}} \Delta_{S^{n-1}} v \, d\sigma_\omega = 0 \, ,$$

in the equation (2.5) we obtain

$$\int_{S^{n-1}} \iint_{D_{\xi,\eta}} (\partial_\eta w - \partial_\xi w) g \, d\xi \, d\eta \, d\sigma_\omega =$$
$$= \int_{S^{n-1}} \left[\int_0^{t_0} [\partial_\eta v \partial_\eta w]_\eta^{t_0} \, d\eta - \int_0^{t_0} [\partial_\xi v \partial_\xi w]_0^{\xi} \, d\xi \right] d\sigma_\omega \, . \tag{2.6}$$

Let us now take w for the special function

$$w(\xi,\eta) = (t_0 - \xi)^{\frac{n-1}{2}} (t_0 - \eta)^{\frac{n-1}{2}} \, , \tag{2.7}$$

which satisfies the equation $Mw = 0$, moreover

$$\partial_\xi w(\xi,\eta) = -\frac{n-1}{2} (t_0 - \xi)^{\frac{n-3}{2}} (t_0 - \eta)^{\frac{n-1}{2}} \, ,$$
$$\partial_\eta w(\xi,\eta) = -\frac{n-1}{2} (t_0 - \xi)^{\frac{n-1}{2}} (t_0 - \eta)^{\frac{n-3}{2}} \, .$$

Thus, substituting in (2.6) and introducing the Dirac measure δ, we get

$$\int_{S^{n-1}}\iint_{D_{\xi,\eta}}(\xi-\eta)[(t_0-\eta)(t_0-\xi)]^{\frac{n-3}{2}}g\,d\xi\,d\eta\,d\sigma_\omega=$$
$$=\int_{S^{n-1}}\iint_{D_{t,r}}[\partial_\eta v(\xi,\eta,\omega)+\partial_\xi v(\xi,\eta,\omega)](t_0-\eta)^{n-2}\delta(\xi-\eta)\,d\xi\,d\eta d\sigma_\omega+$$
$$-\int_{S^{n-1}}\iint_{D_{t,r}}\partial_\xi v(\xi,\eta,\omega)(t_0-\xi)^{\frac{n-3}{2}}(t_0-\eta)^{\frac{n-1}{2}}\;\delta(\eta)\,d\xi\,d\eta\,d\sigma_\omega\,. \quad (2.8)$$

We set

$$I_1=\int_{S^{n-1}}\iint_{D_{t,r}}[\partial_\eta v(\xi,\eta,\omega)+\partial_\xi v(\xi,\eta,\omega)](t_0-\eta)^{n-2}\delta(\xi-\eta)\,d\xi\,d\eta\,d\sigma_\omega\,,$$
$$I_2=\int_{S^{n-1}}\iint_{D_{t,r}}\partial_\xi v(\xi,\eta,\omega)(t_0-\xi)^{\frac{n-3}{2}}(t_0-\eta)^{\frac{n-1}{2}}\delta(\eta)\,d\xi\,d\eta\,d\sigma_\omega\,,$$
$$I_3=\int_{S^{n-1}}\iint_{D_{\xi,\eta}}(\xi-\eta)[(t_0-\eta)(t_0-\xi)]^{\frac{n-3}{2}}g(\xi,\eta,\omega)\,d\xi\,d\eta\,d\sigma_\omega\,,$$

so (2.8) can be written as $I_1=I_2+I_3$. We return now to (t,x)-coordinates, first setting $\xi=t+r$ and $\eta=t-r$, so $d\xi\,d\eta=2dt\,dr$, then $x=r\omega$, so

$$dr d\sigma_\omega=\frac{dx}{|x|^{n-1}}\,.$$

Hence, we obtain

$$I_1=4\int_{S^{n-1}}\iint_{D_{t,r}}\partial_t u(t,r\omega)(t_0-t)^{n-2}\delta(2r)\,dt\,dr d\sigma_\omega=$$
$$=2\int_{S^{n-1}}d\sigma_\omega\int_0^{t_0}\partial_t u(t,0)(t_0-t)^{n-2}dt=$$
$$=2\omega_n\int_0^{t_0}\partial_t u(t,0)(t_0-t)^{n-2}dt\,,$$

$$I_2=2\int_{S^{n-1}}\iint_{D_{t,r}}\partial_\xi u(t,r\omega)(t_0-t-r)^{\frac{n-3}{2}}(t_0-t+r)^{\frac{n-1}{2}}\;\delta(t-r)dt\,dr d\sigma_\omega=$$
$$=2t_0\int_{S^{n-1}}\iint_{D_{t,r}}\partial_\xi u(t,r\omega)\left[(t_0-t)^2-r^2\right]^{\frac{n-3}{2}}\;\delta(t-r)dt\,dr d\sigma_\omega=$$
$$=2t_0\iint_{D_0}(1,\omega)\cdot\partial u(t,x)\left[(t_0-t)^2-|x|^2\right]^{\frac{n-3}{2}}\;\delta(t-|x|)\frac{dt\,dx}{|x|^{n-1}}\,,$$

$$I_3=\int_{S^{n-1}}\iint_{D_{t,r}}4r\,f(t,r\omega)\left[(t_0-t)^2-r^2\right]^{\frac{n-3}{2}}\;dt\,dr d\sigma_\omega=$$
$$=\iint_{D_0}f(t,x)\left[(t_0-t)^2-|x|^2\right]^{\frac{n-3}{2}}\frac{dt\,dx}{|x|^{n-2}}\,.$$

where $\omega_n = \int_{S^{n-1}} d\sigma_\omega = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ and

$$D_{t,r} : \begin{cases} 0 \le r \le \frac{t_0}{2} \\ r \le t \le -r + t_0 \end{cases} .$$

In I_1 we have used the fact that δ is a homogeneous function of degree -1, hence $\delta(2r) = \frac{\delta(r)}{2}$.

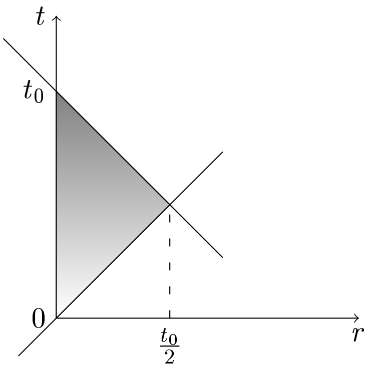

Figure 2.3: Domain $D_{t,r}$

Thus substituting these results in $I_1 = I_2 + I_3$, we immediately obtain

$$\int_0^{t_0} \partial_t u(t,0)(t_0 - t)^{n-2} dt =$$
$$= \frac{t_0}{\omega_n} \iint_D (t,x) \cdot \partial u(t,x) \left[(t_0-t)^2 - |x|^2\right]^{\frac{n-3}{2}} \frac{\delta(t-|x|)}{|x|} \frac{dt\,dx}{|x|^{n-1}} +$$
$$+ \frac{1}{2\omega_n} \iint_{D_0} f(t,x) \left[(t_0-t)^2 - |x|^2\right]^{\frac{n-3}{2}} \frac{dt\,dx}{|x|^{n-2}} .$$

Differentiating both sides of the above equation $n-2$ times with respect to the variable t_0, we at last get the following formula

$$u(t_0, 0) = \varphi(0,0) +$$
$$+ \frac{\partial_{t_0}^{n-2}}{(n-2)!\,\omega_n} t_0 \iint_{D_0} (t,x) \cdot \partial u(t,x) \left[(t_0-t)^2 - |x|^2\right]^{\frac{n-3}{2}} \frac{\delta(t-|x|)}{|x|} \frac{dt\,dx}{|x|^{n-1}}$$
$$+ \frac{\partial_{t_0}^{n-2}}{2(n-2)!\,\omega_n} \iint_{D_0} f(t,x) \left[(t_0-t)^2 - |x|^2\right]^{\frac{n-3}{2}} \frac{dt\,dx}{|x|^{n-2}} , \quad (2.9)$$

which gives the solution of the characteristic initial value problem (1.15) at points on the t-axis.

Remark 2.1. The delta functions we introduced in integrals (2.8) can be viewed as the *pull-backs* of standard delta distributions, or equivalently, as measures supported on a hypersurface (see [16, Theorem 6.1.5]). Let f be a regular function and S be the hypersurface defined by $\phi(x) = 0$, where ϕ is a smooth function with $\nabla\phi(x) \neq 0$ in $S \cap \operatorname{supp} f$ and denote by $d\sigma_x$ the induced area element on S, then we have

$$\int_{\mathbb{R}^{n+1}} f(x)\delta(\phi(x))dx = \int_S f(x)\frac{d\sigma_x}{|\nabla\phi(x)|}.$$

In the particular case of the null cone, the expression $\delta(t - |x|)$ is well defined, in fact we have $\phi(t, x) = t - |x|$, so $\nabla\phi(x) = (1, \omega)$, which gives $|\nabla\phi(x)| = \sqrt{2}$ and

$$\int_{\mathbb{R}^{n+1}} f(x)\delta(t - |x|)dx = \int_{\partial C} f(x)\frac{d\sigma_x}{\sqrt{2}}.$$

The formula (2.9) is a generalization of the results of Protter [22] and Young [25]. Note that in all these papers the formula is proved only for the homogeneous problem and the explicit solution is obtained only on the time axis.

For a point (T^*, X^*) not on the t-axis we obtain the solution by using a Lorentz transformation and the formula (2.9).

2.2 Lorentz transformations

We imagine the point (T^*, X^*) transported by a rotation of the coordinate system into the plane $X' = 0$, so without loss of generality, we can set $(T^*, X^*) = (T^*, X_1^*, 0')$, $T^* > X_1^*$.

The Lorentz transformation $\mathcal{L}$ such that

$$(T^*, X_1^*, 0') = \mathcal{L}(t_0, 0)$$

is given by

$$\mathcal{L} : \begin{cases} T = \dfrac{T^* t + X_1^* x_1}{\sqrt{T^{*2} - X_1^{*2}}} \\ X_1 = \dfrac{X_1^* t + T^* x_1}{\sqrt{T^{*2} - X_1^{*2}}} \\ X' = x' \end{cases}, \qquad \mathcal{L}^{-1} : \begin{cases} t = \dfrac{T^* T - X_1^* X_1}{\sqrt{T^{*2} - X_1^{*2}}} \\ x_1 = \dfrac{-X_1^* T + T^* X_1}{\sqrt{T^{*2} - X_1^{*2}}} \\ x' = X' \end{cases}, \tag{2.10}$$

where

$$t_0 = \sqrt{T^{*2} - X_1^{*2}}\,. \tag{2.11}$$

By the *Special Theory of Relativity* we know that Lorentz transformations are isometries for the Minkowski space. Thus, if $(T^*, X^*) = \mathcal{L}(t_0, x_0)$, we get

$$(t - t_0)^2 - |x - x_0|^2 = (T - T^*)^2 - |X - X^*|^2\,.$$

In other words, the Lorentz Transformation $\mathcal{L}$ changes the light cone with vertex (t_0, x_0) into the light cone with vertex (T^*, X^*), in particular $\mathcal{L}$ leaves the characteristic cone (1.12) unchanged, i.e. $\mathcal{L}^{-1}(\partial\mathcal{C}) = \partial\mathcal{C}$. Consequently, the transformation $\mathcal{L}$ changes the region D_0 into the region

$$\mathcal{D}(T^*, X_1^*) = \mathcal{L}(D_0) : \begin{cases} |X| \leq T \\ \sqrt{(X_1 - X_1^*)^2 + |X'|^2} \leq T^* - T \\ 0 \leq T \leq T^* \end{cases} .$$

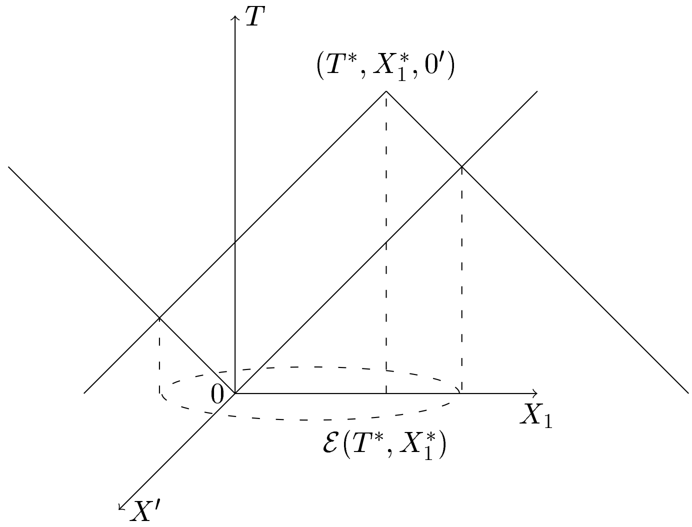

Figure 2.4: Domain D_*

It's important to note that the norm $|x|$ becomes

$$|x| = \sqrt{\frac{(T^*X_1 - X_1^*T)^2 + \left(T^{*2} - X_1^{*2}\right)|X'|^2}{T^{*2} - X_1^{*2}}} .$$

Let us take

$$U = u \circ \mathcal{L}^{-1} \quad \Rightarrow \quad u = U \circ \mathcal{L}, \tag{2.12a}$$

$$F = f \circ \mathcal{L}^{-1} \quad \Rightarrow \quad f = F \circ \mathcal{L}, \tag{2.12b}$$

$$\Phi = \varphi \circ \mathcal{L}^{-1} \quad \Rightarrow \quad \varphi = \Phi \circ \mathcal{L}; \tag{2.12c}$$

so by linearity we have

$$\partial u = L\partial U ,$$

where L is the matrix associated to the linear transformation (2.10), so we immediately get the following relation

$$(t, x) \cdot \partial u = L^{-1}(T, X) \cdot L\partial U = (T, X) \cdot \partial U.$$

In formula (2.9) also differential operators and measures appear. It is useful to study the behavior of these objects when a Lorentz Transformation is applied. The best approach is trying to re-write both the differential operators and the measures in term of invariant objects under Lorentz Transformations.

Lemma 2.1. *The measure* $\mu(t,x) = \dfrac{\delta(t-|x|)}{|x|}$ *is invariant under Lorentz Transformations.*

Proof. It suffices to show that μ is equivalent to the measure

$$\nu(t,x) = \delta(t^2 - |x|^2)\chi_{(t>0)}(t,x).$$

In particular, we have $d\mu = 2\,d\nu$. □

Let S be the first order differential operator defined as

$$S(t,x,\partial_t,\nabla) := t\partial_t + x\cdot\nabla. \tag{2.13}$$

We calculate

$$Su = \frac{(T^*T - X_1^*X_1)(T^*\partial_T U - X_1^*\partial_{X_1}U)}{T^{*2} - X_1^{*2}} + X'\cdot\nabla' U\ ,$$

where the relation between u and U is stated in (2.12a), hence

$$S(t_0,0,\partial_{t_0},0) = T^*\partial_{T^*} - X_1^*\partial_{X_1^*} = S(T^*,X_1^*,0',\partial_{T^*},(\partial_{X_1^*},0')). \tag{2.14}$$

Note that the operator S is a conformal Killing vector field for the Minkowski space. In particular, it generates the Lie group of all homothetic functions (see [20]).

Remark 2.2. Let us define $S_0 := S(t_0,0,\partial_{t_0},0) = t_0\partial_{t_0}$ and $S_* := T^*\partial_{T^*} - X_1^*\partial_{X_1^*}$. We can use the operator S_0 for expressing the derivative ∂_{t_0} as follows

$$\partial_{t_0} = \frac{1}{t_0}S_0 = \frac{1}{\sqrt{T^{*2} - X_1^{*2}}}S_*\ .$$

Lemma 2.2. *Let* P_n *be the differential operator of order* n *defined as follows*

$$P_n(T,X) := \sum_{k=0}^{n}\binom{n}{k}T^k X_1^{n-k}\partial_T^k\partial_{X_1}^{n-k};$$

then for all $n\in\mathbb{N}$ *we have* $SP_n = P_{n+1} + nP_n$, *hence*

$$\left(\frac{S}{\sqrt{T^2 - X_1^2}}\right)^n = \frac{P_n}{\left(T^2 - X_1^2\right)^{\frac{n}{2}}}\ , \tag{2.15}$$

where S *is given by* (2.13).

Proof. First of all, note that for all $\alpha \in \mathbb{R}$ we have

$$S\left(\left(T^2 - X_1^2\right)^{\frac{\alpha}{2}}\right) = \alpha\left(T^2 - X_1^2\right)^{\frac{\alpha}{2}};$$

then we pass to prove, using induction, that $SP_n = P_{n+1} + nP_n$, for all $n \in \mathbb{N}$.

A trivial computation gives $S = P_1$ and $S^2 = P_2 + P_1$, then we suppose that the previous equality holds for a certain $n > 1$ and we show that $(n) \Rightarrow (n+1)$. We consider

$$\begin{aligned}
SP_n = S\left(\sum_{k=0}^{n}\binom{n}{k}T^k X_1^{n-k}\partial_T^k\partial_{X_1}^{n-k}\right) = \\
= \sum_{k=0}^{n}\binom{n}{k}\Big[kT^k X_1^{n-k}\partial_T^k\partial_{X_1}^{n-k} + T^{k+1}X_1^{n-k}\partial_T^{k+1}\partial_{X_1}^{n-k} + \\
+(n-k)T^k X_1^{n-k}\partial_T^k\partial_{X_1}^{n-k} + T^k X_1^{n-k+1}\partial_T^k\partial_{X_1}^{n-k+1}\Big] = \\
\overset{(h=k+1)}{=} nP_n + \sum_{k=0}^{n}\binom{n}{k}T^k X_1^{n-k+1}\partial_T^k\partial_{X_1}^{n-k+1} + \\
+\sum_{h=1}^{n+1}\binom{n}{h-1}T^h X_1^{n-h+1}\partial_T^h\partial_{X_1}^{n-h+1} = \\
= nP_n + T^{n+1}\partial_T^{n+1} + X_1^{n+1}\partial_{X_1}^{n+1} + \\
+\sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]T^k X_1^{n-k+1}\partial_T^k\partial_{X_1}^{n-k+1} = nP_n + P_{n+1}.
\end{aligned}$$

Using induction again, we prove (2.15). The step $n = 1$ is trivial, we suppose that (2.15) holds for a certain $n > 1$ and we show that $(n) \Rightarrow (n+1)$,

$$\begin{aligned}
\left(\frac{S}{\sqrt{T^2 - X_1^2}}\right)^{n+1} = \frac{S}{\sqrt{T^2 - X_1^2}}\left(\frac{S}{\sqrt{T^2 - X_1^2}}\right)^n = \\
= \frac{S}{\sqrt{T^2 - X_1^2}}\left(\frac{P_n}{\left(T^2 - X_1^2\right)^{\frac{n}{2}}}\right) = \\
= \frac{\left(T^2 - X_1^2\right)^{-\frac{n}{2}}SP_n + S\left(\left(T^2 - X_1^2\right)^{-\frac{n}{2}}\right)P_n}{\sqrt{T^2 - X_1^2}} = \frac{P_{n+1}}{\left(T^2 - X_1^2\right)^{\frac{n+1}{2}}}.
\end{aligned}$$

□

Remark 2.3. Fix $k \in \mathbb{N}$, by using Lemma 2.2 it immediately follows that

$$\left(\frac{S}{\sqrt{T^2 - X_1^2}}\right)^n \left(T^2 - X_1^2\right)^{\frac{n+k}{2}} = \frac{(n+k)!}{k!}\left(T^2 - X_1^2\right)^{\frac{k}{2}}.$$

Moreover, let $G = G(T, X)$ be a smooth function, the following equality holds

$$\left(\frac{S}{\sqrt{T^2 - X_1^2}}\right)^n \left(\left(T^2 - X_1^2\right)^{\frac{n+k}{2}} G\right) = \left(T^2 - X_1^2\right)^{\frac{k}{2}} \left(P_n + n!\binom{n+k}{k}\right) G\,.$$

2.3 The general formula

Since the wave equation is invariant under the Lorentz transformation (2.10), the function U solves the problem (1.15) with the initial values F and Φ, given respectively by (2.12b) and (2.12c). Thus, applying (2.10) to the formula (2.9) and using the properties of Lorentz transformations stated in section 2.2, we can obtain the solution U at the point $(T^*, X_1^*, 0')$

$$\begin{aligned} U(T^*, X_1^*, 0') = {} & \Phi(0,0) + \\ & + \frac{1}{(n-2)!\,\omega_n} \left(\frac{S_*}{\sqrt{T^{*2} - X_1^{*2}}}\right)^{n-2} \left[W_1(T^*, X_1^*) + W_2(T^*, X_1^*)\right], \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} & \frac{1}{(n-2)!\,\omega_n} \left(\frac{S_*}{\sqrt{T^{*2} - X_1^{*2}}}\right)^{n-2} W_1(T^*, X_1^*) = \\ & \quad = \frac{1}{(n-2)!\,\omega_n} \left(\frac{S_*}{\sqrt{T^{*2} - X_1^{*2}}}\right)^{n-2} \iint_{\mathcal{D}(T^*, X_1^*)} (T, X) \cdot \partial U(T, X) \\ & \frac{\left[(T-T^*)^2 - (X_1 - X_1^*)^2 - |X'|^2\right]^{\frac{n-3}{2}} \left(T^{*2} - X_1^{*2}\right)^{\frac{n}{2}}}{\left[(T^* X_1 - X_1^* T)^2 + \left(T^{*2} - X_1^{*2}\right)|X'|^2\right]^{\frac{n-1}{2}}} \; \frac{\delta(T - |X|)}{|X|}\, dT\, dX \end{aligned}$$

solves the homogeneous problem and

$$\begin{aligned} & \frac{1}{(n-2)!\,\omega_n} \left(\frac{S_*}{\sqrt{T^{*2} - X_1^{*2}}}\right)^{n-2} W_2(T^*, X_1^*) = \\ & \quad = \frac{1}{2(n-2)!\,\omega_n} \left(\frac{S_*}{\sqrt{T^{*2} - X_1^{*2}}}\right)^{n-2} \iint_{\mathcal{D}(T^*, X_1^*)} F(T, X) \\ & \qquad \frac{\left[(T-T^*)^2 - (X_1 - X_1^*)^2 - |X'|^2\right]^{\frac{n-3}{2}} \left(T^{*2} - X_1^{*2}\right)^{\frac{n-2}{2}}}{\left[(T^* X_1 - X_1^* T)^2 + \left(T^{*2} - X_1^{*2}\right)|X'|^2\right]^{\frac{n-2}{2}}}\, dT\, dX \end{aligned}$$

solves the inhomogeneous problem with null initial datum.

We know that

$$(1,\omega)\cdot\partial U(|X|,X)=(1,\omega)\cdot\partial\Phi(|X|,X)=\partial_r\Phi^*(X)\,,$$
$$\delta(T-|X|)\,dT\,dX=\frac{d\sigma_{T,X}}{\sqrt{2}}\,,$$
$$d\sigma_{T,X}=\sqrt{2}dX\,,$$

thus

$$\begin{aligned}W_1(T^*,X_1^*)=&\iint_{\mathcal{D}(T^*,X_1^*)}(T,X)\cdot\partial U(T,X)\cdot\\ &\cdot\frac{\left[(T-T^*)^2-(X_1-X_1^*)^2-|X'|^2\right]^{\frac{n-3}{2}}\left(T^{*2}-X_1^{*2}\right)^{\frac{n}{2}}}{\left[(T^*X_1-X_1^*T)^2+\left(T^{*2}-X_1^{*2}\right)|X'|^2\right]^{\frac{n-1}{2}}}\frac{\delta(T-|X|)}{|X|}\,dT\,dX=\\ &=\int_{\substack{T=|X|\\ \sqrt{(X_1-X_1^*)^2+|X'|^2}\le T^*-T}}(1,\omega)\cdot\partial U(T,X)\cdot\\ &\cdot\frac{\left[(T-T^*)^2-(X_1-X_1^*)^2-|X'|^2\right]^{\frac{n-3}{2}}\left(T^{*2}-X_1^{*2}\right)^{\frac{n}{2}}}{\left[(T^*X_1-X_1^*T)^2+\left(T^{*2}-X_1^{*2}\right)|X'|^2\right]^{\frac{n-1}{2}}}\frac{d\sigma_{T,X}}{\sqrt{2}}=\\ =&\int_{\mathcal{E}(T^*,X_1^*)}\partial_r\Phi^*(X)\frac{\left[(T^{*2}-X_1^{*2})-2(T^*|X|-X_1^*X_1)\right]^{\frac{n-3}{2}}\left(T^{*2}-X_1^{*2}\right)^{\frac{n}{2}}}{(T^*|X|-X_1^*X_1)^{n-1}}\,dX\,,\end{aligned}$$

where

$$\mathcal{E}(T^*,X_1^*)=\left\{X\in\mathbb{R}^n\text{ s.t. }|X|+\sqrt{(X_1-X_1^*)^2+|X'|^2}\le T^*\right\}$$

is exactly the n-dimensional ellipsoid with foci in 0 and $(X_1^*,0')$ and with major axis T^*; in addition $\mathcal{E}(T^*,X_1^*)$ is the projection of the domain D_* onto $\mathbb{R}^n$ (see Figure 2.4).

Finally, applying Lemma 2.2 and Remark 2.3, the formula (2.16) becomes

$$\begin{aligned}U(T^*,X_1^*,0')=&\,\Phi(0,0)+\\ &+\frac{T^{*2}-X_1^{*2}}{\omega_n}\left(\frac{P^*_{n-2}}{(n-2)!}+\binom{n}{2}\right)\int_{\mathcal{E}(T^*,X_1^*)}\partial_r\Phi^*(X)\cdot\\ &\cdot\frac{\left[(T^{*2}-X_1^{*2})-2(T^*|X|-X_1^*X_1)\right]^{\frac{n-3}{2}}}{(T^*|X|-X_1^*X_1)^{n-1}}\,dX+\\ &+\frac{1}{2\omega_n}\left(\frac{P^*_{n-2}}{(n-2)!}+1\right)\iint_{\mathcal{D}(T^*,X_1^*)}F(T,X)\cdot\\ &\cdot\frac{\left[(T-T^*)^2-(X_1-X_1^*)^2-|X'|^2\right]^{\frac{n-3}{2}}}{\left[(T^*X_1-X_1^*T)^2+\left(T^{*2}-X_1^{*2}\right)|X'|^2\right]^{\frac{n-2}{2}}}\,dT\,dX\,.\end{aligned}$$

We can summarize the previous results as follows.

Theorem 2.1. *The solution of the Characteristic Initial Value problem* (1.15)*, where* $\varphi, f \in \mathscr{D}(\mathbb{R}^{n+1})$*, is given by*

$$u(t^*, x_1^*, 0) = \varphi(0,0) + \frac{t^{*2} - x_1^{*2}}{\omega_n}\left(\frac{P^*_{n-2}}{(n-2)!} + \binom{n}{2}\right)\int_{\mathcal{E}(t^*,x_1^*)} \partial_r \varphi^*(x)\cdot$$
$$\cdot\frac{\left[(t^{*2} - x_1^{*2}) - 2(t^*|x| - x_1^* x_1)\right]^{\frac{n-3}{2}}}{(t^*|x| - x_1^* x_1)^{n-1}}\, dx +$$
$$+\frac{1}{2\omega_n}\left(\frac{P^*_{n-2}}{(n-2)!} + 1\right)\iint_{\mathcal{D}(t^*,x_1^*)} f(t,x)\cdot$$
$$\cdot\frac{\left[(t-t^*)^2 - (x_1 - x_1^*)^2 - |x'|^2\right]^{\frac{n-3}{2}}}{\left[(t^* x_1 - x_1^* t)^2 + \left(t^{*2} - x_1^{*2}\right)|x'|^2\right]^{\frac{n-2}{2}}}\, dt\, dx\ ,$$

with $(t^*, x_1^*, 0) \in \mathcal{C}$*. The previous expression must be intended in the distribution sense.*

Remark 2.4. Up to now, the Theorem 2.1 must be intended in the distribution sense, in fact a differential operator acts on an integral function that can be a non-smooth object. In the following chapters we will prove that regular initial data generate a regular solution, thus, at last, the Theorem 2.1 holds in the classical sense.

2.4 Explicit weak solutions in the distribution sense

Among the basic tools for the study of the equation $\Box u = f$ there are the *Fundamental Solutions* for $\Box$, which are the distributions E that satisfy $\Box E = \delta$, where δ is the Dirac delta-function or point mass at the origin. Indeed, once one has a fundamental solution E, one can solve $\Box u = f$ for any compactly supported f, by taking u to be the convolution of f with E, for $\Box(f * E) = f * \Box E = f * \delta = f$.

In the physically relevant case $n = 3$ the Fundamental Solutions for the Wave Equation have a particularly simple expression; our goal is to re-obtain explicit formulas for the problem (1.15) using Distribution Theory and the Fundamental Solutions. We will prove that the two new formulas are exactly equivalent to the previous one (see Theorem 2.1). More precisely, the two new formulas are nothing more than the same formula derived using two different methods.

At first we need to rewrite the problem (1.15) in the sense of distributions.

Theorem 2.2. *Let* u *be the solution of* (1.15)*, then for all* $\psi \in \mathscr{D}(\mathbb{R}^{n+1})$*,* $\operatorname{supp}\psi \subset \mathcal{C}$ *we have*

$$\iint_{\mathcal{C}} \Box u \psi\, dx dt = \iint_{\mathcal{C}} u \Box \psi\, dx dt - \iint_{\mathcal{C}} \psi T_\varphi\, dx dt$$

where T_φ is the distribution

$$T_\varphi(t,x) = 2\left(\frac{\varphi(t,x)}{|x|} + (1,\omega)\cdot\partial\varphi(t,x)\right)\delta(t-|x|), \tag{2.17}$$

supported in $\mathcal{C}$.

Proof. For all $\psi \in \mathscr{D}(\mathbb{R}^{n+1})$, $\operatorname{supp}\psi \subset \mathcal{C}$, we have

$$\iint_{\mathcal{C}} \Box u\psi\, dxdt = \int_{\mathbb{R}^n}\int_{|x|}^{+\infty} \partial_t^2 u\psi\, dtdx - \int_0^{+\infty}\int_{B_t} \Delta u\psi\, dtdx \tag{2.18}$$

where

$$\int_{|x|}^{+\infty} \partial_t^2 u\psi\, dt = (u\partial_t\psi - \partial_t u\psi)|_{\partial\mathcal{C}} + \int_{|x|}^{+\infty} u\partial_t^2\psi\, dt$$

and

$$\int_{B_t} \Delta u\psi\, dx = \int_{S_t} (\Delta u\psi - u\Delta\psi)\cdot\omega d\sigma_x + \int_{B_t} u\Delta\psi\, dx.$$

Recalling that

$$(\partial_t u + \nabla u\cdot\omega)|_{\partial\mathcal{C}} = \nabla\varphi^*\cdot\omega = \partial_r\varphi^*$$

the (2.18) becomes,

$$\begin{aligned}\iint_{\mathcal{C}} \Box u\psi\, dxdt &= \iint_{\mathcal{C}} u\Box\psi\, dxdt + \\ &+ \int_{\partial\mathcal{C}} (\partial_t\psi + \nabla\psi\cdot\omega)u - (\partial_t u + \nabla u\cdot\omega)\psi\frac{d\sigma_{t,x}}{\sqrt{2}} = \\ &= \iint_{\mathcal{C}} u\Box\psi\, dxdt + \int_{\mathbb{R}^n} (\varphi^*\partial_r\psi^* - \psi^*\partial_r\varphi^*)dx.\end{aligned}$$

Consider

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi^*\partial_\omega\psi^* dx &= \int_{S^{n-1}}\int_0^{+\infty} \varphi^*\partial_r\psi^* r^2 drd\omega = \\ &= \int_{S^{n-1}} \left[\varphi^*\psi^* r^2\right]_0^{+\infty} d\omega - \int_{S^{n-1}}\int_0^{+\infty} \partial_r\varphi^*\psi^* r^2 drd\omega + \\ &\quad - 2\int_{S^{n-1}}\int_0^{+\infty} \varphi^*\psi^* rdrd\omega = \\ &= -\int_{\mathbb{R}^n} \psi^*\partial_r\varphi^* - 2\psi^*\frac{\varphi^*}{|x|}dx,\end{aligned}$$

thus

$$\iint_{\mathcal{C}} \Box u\psi\, dxdt = \iint_{\mathcal{C}} u\Box\psi\, dxdt - 2\int_{\mathbb{R}^n} \psi^*\left[\partial_r\varphi^* + \frac{\varphi^*}{|x|}\right]dx.$$

This leads immediately to (2.17). □

Since we are interested in solving the Wave Equation in the future light cone $\mathcal{C}$, we have to consider the *Retarded Fundamental Solution of the Wave Operator* in n dimensions, which is the distribution

$$E = \frac{\pi^{\frac{1-n}{2}}}{4} A * \chi_+^{\frac{1-n}{2}}$$

where $A(t,x) = t^2 - |x|^2$ and $\chi_+^\alpha(x) = \frac{x_+^\alpha}{\Gamma(1+\alpha)}$ is the homogeneous distribution (see [16, p.140]); thus, for $n = 3$ we have

$$E = \frac{1}{4\pi} A * \delta = \frac{1}{4\pi}\delta(t^2 - |x|^2). \tag{2.19}$$

Remark 2.5. By virtue of the proof of Lemma 2.1, the distribution (2.19) can be rewritten as follows

$$E = \frac{1}{4\pi} A * \delta = \frac{1}{8\pi}\frac{\delta(t-|x|)}{|x|}\chi_{(t>0)}(t).$$

2.4.1 Classical weak solution in the distribution sense

By virtue of Theorem 2.2, the weak solution in the distribution sense of the Characteristic Initial Value problem (1.15) is the distribution $u \in \mathscr{D}'(\mathbb{R}^{n+1})$ given by

$$u = E * (T_\varphi + F) \tag{2.20}$$

where F is the distribution

$$F = \begin{cases} f & \text{in } \mathcal{C} \\ 0 & \text{otherwise} \end{cases}.$$

Since we exactly know the distributions E, T_φ and F, we can derive an explicit formula for the solution u. For the sake of the simplicity we restrict ourselves to the physically important case $n = 3$. Consider $\psi \in \mathscr{D}(\mathbb{R}^{3+1})$, $\operatorname{supp}\psi \subset \mathcal{C}$, then, by virtue of Remark 2.5,

$$E * T_\varphi * \psi(t,x) = \frac{1}{8\pi}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3} T_\varphi * \psi(s,y)\frac{\delta(t-s-|x-y|)}{|x-y|}\chi_{(t-s>0)}dyds$$

where

$$T_\varphi * \psi(s,y) = 2\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\psi(\sigma,\eta)\frac{\varphi(s-\sigma,y-\eta)}{|y-\eta|}\delta(s-\sigma-|y-\eta|)d\eta d\sigma +$$
$$+ 2\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\psi(\sigma,\eta)(1,\omega)\cdot\partial\varphi(s-\sigma,y-\eta)\delta(s-\sigma-|y-\eta|)d\eta d\sigma.$$

Thus

$$E * T_\varphi * \psi(t,x) =$$
$$= \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty}\psi(\sigma,\eta)\Big[\frac{\varphi(s-\sigma,y-\eta)}{|y-\eta|}+(1,\omega)\cdot\partial\varphi(s-\sigma,y-\eta)\Big]$$
$$\delta(s-\sigma-|y-\eta|)\frac{\delta(t-s-|x-y|)}{|x-y|}\chi_{(t-s>0)}dydsd\eta d\sigma =$$
$$= \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty}\psi(\sigma,\eta)\Big[\frac{\varphi^*(y-\eta)}{|y-\eta|}+\partial_r\varphi^*(y-\eta)\Big]$$
$$\frac{\delta(t-\sigma-|y-\eta|-|x-y|)}{|x-y|}\chi_{(t-\sigma-|y-\eta|>0)}dyd\eta d\sigma \overset{(z=y-\eta)}{=}$$
$$= \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty}\psi(\sigma,\eta)\Big[\frac{\varphi^*(z)}{|z|}+\partial_r\varphi^*(z)\Big]$$
$$\frac{\delta(t-\sigma-|z|-|x-z-\eta|)}{|x-z-\eta|}\chi_{(t-\sigma-|z|>0)}dzd\eta d\sigma. \quad (2.21)$$

Set

$$K(t,x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\Big[\frac{\varphi^*(z)}{|z|}+\partial_r\varphi^*(z)\Big]\frac{\delta(t-|z|-|x-z|)}{|x-z|}\chi_{(t-|z|>0)}dz \quad (2.22)$$

then (2.21) becomes

$$E * T_\varphi * \psi(t,x) = \int_{\mathbb{R}^3}\int_{-\infty}^{+\infty}\psi(\sigma,\eta)K(t-\sigma,x-\eta)d\eta d\sigma = K * \psi(t,x).$$

Similarly, we have

$$E * F * \psi(t,x) = \frac{1}{8\pi}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}F*\psi(s,y)\frac{\delta(t-s-|x-y|)}{|x-y|}\chi_{(t-s>0)}dyds$$

where

$$F * \psi(s,y) = \int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\psi(\sigma,\eta)F(s-\sigma,y-\eta)d\eta d\sigma.$$

Thus

$$E * F * \psi(t,x) =$$
$$= \frac{1}{8\pi}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\psi(\sigma,\eta)F(s-\sigma,y-\eta)$$
$$\frac{\delta(t-s-|x-y|)}{|x-y|}\chi_{(t-s>0)}dydsd\eta d\sigma \overset{\substack{(z=y-\eta)\\(p=s-\sigma)}}{=}$$
$$= \frac{1}{8\pi}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\psi(\sigma,\eta)F(p,z)$$
$$\frac{\delta(t-p-\sigma-|x-z-\eta|)}{|x-z-\eta|}\chi_{(t-p-\sigma>0)}dzdpd\eta d\sigma. \quad (2.23)$$

Set

$$H(t,x) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{-\infty}^{+\infty} F(p,z)\frac{\delta(t-p-|x-z|)}{|x-z|}\chi_{(t-p>0)}dzdp \tag{2.24}$$

then (2.23) becomes

$$E * F * \psi(t,x) = \int_{\mathbb{R}^3}\int_{-\infty}^{+\infty} \psi(\sigma,\eta)H(t-\sigma,x-\eta)d\eta d\sigma = H * \psi(t,x).$$

Thus

$$\begin{aligned} u(t,x) &= E * (T_\varphi + F)(t,x) = K(t,x) + H(t,x) = \\ &= \frac{1}{4\pi}\int_{\partial\mathcal{E}(t,x)}\left[\frac{\varphi^*(z)}{|z||x-z|} + \frac{\partial_r\varphi^*(z)}{|x-z|}\right]d\sigma_z + \frac{1}{8\pi}\int_{\mathcal{E}(t,x)}\frac{F(t-|x-z|,z)}{|x-z|}dz, \end{aligned} \tag{2.25}$$

where

$$\mathcal{E}(t,x) = \left\{z \in \mathbb{R}^3 \text{ s.t. } |z| + |z-x| \le t\right\}$$

is exactly the 3-dimensional ellipsoid with foci in 0 and x and with major axis t.

We have just proved the following result.

Theorem 2.3. *Assume $\varphi, f \in \mathscr{D}(\mathbb{R}^4)$. The weak solution in the distribution sense of the Characteristic Initial Value problem* (1.15) *for $n = 3$ is the distribution $u \in \mathscr{D}'(\mathbb{R}^4)$ supported in $\mathcal{C}$ given by* (2.25).

2.4.2 The Kirchhoff Method for the weak solutions

Now we are going to describe how to get formula (2.25) using the Kirchhoff Method. This method is simply based on the fact that a distribution can always be represented as the limit of a sequence of regular functions.

Let u be the solution of the Characteristic Initial Value problem (1.15) for $n = 3$ and $\phi \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$\phi(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t > 1 \end{cases},$$

we denote by u_ε the function $u_\varepsilon(t,x) = \phi_\varepsilon(t,x)u(t,x)$, where

$$\phi_\varepsilon(t,x) = \phi\left(\frac{t-|x|}{\varepsilon}\right) = \begin{cases} 0, & \text{if } t < |x| \\ 1, & \text{if } t > \varepsilon + |x| \end{cases}.$$

To obtain u in $\mathcal{C}$, we observe that

$$\Box u_\varepsilon = \phi_\varepsilon f + \frac{2}{\varepsilon}\left(\partial_t u + \omega\cdot\nabla u + \frac{u}{|x|}\right)\phi'_\varepsilon. \tag{2.26}$$

When $t - |x| > \varepsilon$, we have $u(t,x) = u_\varepsilon(t,x) = E * \Box u_\varepsilon(t,x)$, or explicitly

$$u(t,x) = \frac{1}{8\pi}\int_0^{+\infty}\int_{\mathbb{R}^3}\Big[f(t-s,x-y)\phi\left(\frac{t-s-|x-y|}{\varepsilon}\right) + \\ + \frac{2}{\varepsilon}\phi'\left(\frac{t-s-|x-y|}{\varepsilon}\right) \\ \left(\partial_t u + \omega\cdot\nabla u + \frac{u}{|x-y|}\right)(t-s,x-y)\Big]\frac{\delta(s-|y|)}{|y|}dyds = \\ \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big[\frac{f(t-|y|,x-y)}{|y|}\phi\left(\frac{t-|y|-|x-y|}{\varepsilon}\right) + \\ + \frac{2}{\varepsilon}\phi'\left(\frac{t-|y|-|x-y|}{\varepsilon}\right) \\ \left(\frac{\partial_t u + \omega\cdot\nabla u}{|y|} + \frac{u}{|y||x-y|}\right)(t-|y|,x-y)\Big]dy.$$

Note that

$$\phi\left(\frac{t-|y|-|x-y|}{\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} \chi_{(t-|y|-|x-y|\geq 0)}(y)$$
$$\frac{1}{\varepsilon}\phi'\left(\frac{t-|y|-|x-y|}{\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} \delta(t-|y|-|x-y|)$$

so, letting $\varepsilon \to 0$ we obtain again the formula (2.25) using the *Kirchhoff's Method*

$$u(t,x) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big[\frac{f(t-|y|,x-y)}{|y|}\chi_{(t-|y|-|x-y|\geq 0)}(y) + \\ + 2\delta(t-|y|-|x-y|)\left(\frac{\partial_t u + \omega\cdot\nabla u}{|y|} + \frac{u}{|y||x-y|}\right)(t-|y|,x-y)\Big]dy = \\ = \frac{1}{4\pi}\int_{\partial\mathcal{E}(t,x)}\left[\frac{\partial_r\varphi^*(x-y)}{|y|} + \frac{\varphi^*(x-y)}{|y||x-y|}\right]d\sigma_y + \\ + \frac{1}{8\pi}\int_{\mathcal{E}(t,x)}\frac{f(t-|y|,x-y)}{|y|}dy.$$

2.4.3 Equivalence of the two formulas

The coexistence of the two formulas (2.9) and (2.25) raises the need of comparing them. They are apparently different one from the other: in the first one a differential operator is applied to an integral function, in the second one there is a convolution with singular kernels. We are going to check the equivalence between the two formulas by proving that they are different representations of the same expression.

Observe that, by virtue of the invariance under Lorentz Transformations, it is sufficient to prove the equivalence on the t-axis (i.e. for $x = 0$). Let's start rewriting the two considered formulas by applying the conditions $n = 3$ and $x = 0$. The (2.9) becomes the following

$$\begin{aligned} u(t,0) &= \varphi(0,0) + \frac{\partial_t}{\omega_3}\left(t\iint_{D_t}(\tau,y)\cdot\partial u(\tau,y)\frac{\delta(\tau-|y|)}{|y|}\frac{d\tau\,dy}{|y|^2}\right)+ \\ &\qquad + \frac{\partial_t}{2\omega_3}\iint_{D_t} f(\tau,y)\frac{dt\,dy}{|y|} = \\ &= \varphi(0,0) + \frac{\partial_t}{4\pi}\left(t\iint_{D_t}(\tau,y)\cdot\partial u(\tau,y)\frac{\delta(\tau-|y|)}{|y|^3}d\tau\,dy\right)+ \\ &\qquad + \frac{\partial_t}{8\pi}\iint_{D_t}\frac{f(\tau,y)}{|y|}d\tau\,dy, \end{aligned} \tag{2.27}$$

where

$$D_t : \begin{cases} |y| \le \tau \\ |y| \le t-\tau \\ 0 \le \tau \le t \end{cases} . \tag{2.28}$$

On the other hand, (2.25) gives

$$u(t,0) = \frac{1}{4\pi}\int_{S_{\frac{t}{2}}}\left[\frac{\partial_r\varphi^*(y)}{|y|}+\frac{\varphi^*(y)}{|y|^2}\right]d\sigma_y + \frac{1}{8\pi}\int_{B_{\frac{t}{2}}}\frac{f(t-|y|,y)}{|y|}dy. \tag{2.29}$$

We focus on (2.27), considering separately each term. For the inhomogeneous term, we have

$$\begin{aligned} \partial_t\iint_{D_t}\frac{f(\tau,y)}{|y|}d\tau\,dy &= \partial_t\int_0^{\frac{t}{2}}\int_{B_\tau}\left[\frac{f(\tau,y)}{|y|}+\frac{f(t-\tau,y)}{|y|}\right]dy\,d\tau = \\ &= \int_{B_{\frac{t}{2}}}\frac{f(\frac{t}{2},y)}{|y|}dy + \int_0^{\frac{t}{2}}\int_{B_\tau}\frac{\partial_t f(t-\tau,y)}{|y|}dy\,d\tau = \\ &= \int_{B_{\frac{t}{2}}}\frac{f(\frac{t}{2},y)}{|y|}dy - \int_{B_{\frac{t}{2}}}\int_{|y|}^{\frac{t}{2}}\frac{\partial_\tau f(t-\tau,y)}{|y|}d\tau\,dy = \\ &= \int_{B_{\frac{t}{2}}}\frac{f(\frac{t}{2},y)}{|y|}dy - \int_{B_{\frac{t}{2}}}\frac{f(\frac{t}{2},y)-f(t-|y|,y)}{|y|}dy = \\ &= \int_{B_{\frac{t}{2}}}\frac{f(t-|y|,y)}{|y|}dy. \end{aligned}$$

Proceeding as before, for the other term we get

$$\partial_t \left(t \iint_{D_t} (\tau, y) \cdot \partial u(\tau, y) \frac{\delta(\tau - |y|)}{|y|^3} d\tau\, dy \right) =$$

$$= \iint_{D_t} (\tau, y) \cdot \partial u(\tau, y) \frac{\delta(\tau - |y|)}{|y|^3} d\tau\, dy +$$

$$+ t\partial_t \int_0^{\frac{t}{2}} \int_{B_\tau} (\tau, y) \cdot \partial u(\tau, y) \frac{\delta(\tau - |y|)}{|y|^3} +$$

$$+ (t - \tau, y) \cdot \partial u(t - \tau, y) \frac{\delta(t - \tau - |y|)}{|y|^3} d\tau\, dy =$$

$$= \int_{B_{\frac{t}{2}}} \frac{\partial_r \varphi^*(y)}{|y|^2} dy + t \int_{B_{\frac{t}{2}}} (t - |y|, y) \cdot \partial u(t - |y|, y) \frac{\delta(t - 2|y|)}{|y|^3} dy =$$

$$= \int_{S_{\frac{t}{2}}} \frac{\varphi^*(y)}{|y|^2} d\sigma_y - 4\pi \varphi^*(0) + \int_{S_{\frac{t}{2}}} \frac{\partial_r \varphi^*(y)}{|y|} d\sigma_y.$$

Plugging these results back to (2.27), we get the equivalence of the two formulas.

Chapter 3

Energy Estimates for the Linear Problem

In the previous Chapter we proved the existence of a solution for the Linear Characteristic Initial Value Problem (1.15) by deriving an explicit formula for it. This proves, in case of regular initial data, the existence of the solution of the considered problem, at least in the distribution sense. Now we wonder if the problem admits also regular solutions, provided stronger hypotheses on the initial data. The additional hypotheses will be mainly a consequence of the geometrical singularity of the data surface.

Now we are going to build the Energy Estimates for the considered problem. Due to the particular features of the problem, this will be possible only for a particular set of functions with a specific vanishing order at the origin. So the Energy Estimates will allow us to prove the regularity and the uniqueness of solutions only in the considered space X_T^s.

3.1 Classical Energy Estimates

Due to the particular nature of the initial data surface $\partial\mathcal{C}$, it is possible to directly derive an Energy Estimate with usual H^s spaces only for first order derivatives. In other words, using classical methods we can directly get only the H^1 Energy Estimate. For the derivatives of order greater than two, we need to proceed step by step by building the estimates at first on the cone boundary and then in the interior.

3.1.1 An estimate for the gradient

Proposition 3.1. *Let u be the solution of the Characteristic Initial Value Problem* (1.15) *where the given functions φ and f are assumed to be smooth.*

Then the following Energy Estimate holds

$$\|\partial u(t)\|_{L^2(B_t)} \leq \|\nabla\varphi^*\|_{L^2(B_t)} + \int_0^t \|f(\tau)\|_{L^2(B_\tau)}\, d\tau\,, \tag{3.1}$$

where $|\partial u|^2 = |\partial_t u|^2 + |\nabla u|^2$ *and* B_t *is the* n*-dimensional ball with radius* t *and centered in* 0*.*

In case of the homogeneous wave equation $\Box u = 0$ *we have the "Energy Identity"*

$$\|\partial u(t)\|_{L^2(B_t)} = \|\nabla\varphi^*\|_{L^2(B_t)}\,. \tag{3.2}$$

Proof. Multiplying the wave equation by $\partial_t u$ we easily find the following identity

$$\partial_t|\partial u|^2 = 2\left[\partial_t u\Box u + \mathrm{div}_x(\nabla u\partial_t u)\right]\;.$$

Fixed $t_0 > 0$, let $\mathcal{C}_{t_0}$ be the cone

$$\mathcal{C}_{t_0} = \{(t,x) \in [0,t_0]\times\mathbb{R}^n \text{ s.t. } |x| < t\}\,,$$

then we integrate the previous identity on $\mathcal{C}_{t_0}$

$$\iint_{\mathcal{C}_{t_0}} \partial_t|\partial u|^2 dt\,dx = 2\iint_{\mathcal{C}_{t_0}} \partial_t u f\,dt\,dx + 2\iint_{\mathcal{C}_{t_0}} \mathrm{div}_x(\nabla u\partial_t u)dt\,dx\;. \tag{3.3}$$

and we perform a simple integration by parts argument

$$\begin{aligned}\iint_{\mathcal{C}_{t_0}} \partial_t|\partial u|^2 dt\,dx &= \int_{B_{t_0}}\int_{|x|}^{t_0} \partial_t|\partial u|^2 dt\,dx = \\ &= \int_{B_{t_0}} |\partial u(t_0,x)|^2 dx - \iint_{\mathcal{C}_{t_0}} |\partial u(t,x)|^2\delta(t-|x|)dt\,dx = \\ &= \int_{B_{t_0}} |\partial u(t_0,x)|^2 dx - \int_{\partial\mathcal{C}_{t_0}} |\partial u(t,x)|^2 \frac{d\sigma_{t,x}}{\sqrt{2}}\,,\end{aligned}$$

$$\begin{aligned}\iint_{\mathcal{C}_{t_0}} \mathrm{div}_x(\nabla u\partial_t u)dt\,dx &= \int_0^{t_0}\int_{B_t} \mathrm{div}_x(\nabla u\partial_t u)dx\,dt = \\ &= \int_0^{t_0}\int_{\partial B_t} (\nabla u\cdot\omega)\partial_t u\,d\sigma_x dt = \\ &= \int_{\partial\mathcal{C}_{t_0}} (\nabla u\cdot\omega)\partial_t u\,\frac{d\sigma_{t,x}}{\sqrt{2}}\,.\end{aligned}$$

Plugging these results back into the identity (3.3) we obtain

$$\begin{aligned}\int_{B_{t_0}} |\partial u(t_0,x)|^2 dx = 2\iint_{\mathcal{C}_{t_0}} &\partial_t u f\,dt\,dx + \\ &+ \int_{\partial\mathcal{C}_{t_0}} |\partial u(t,x)|^2 + 2(\nabla u\cdot\omega)\partial_t u\,\frac{d\sigma_{t,x}}{\sqrt{2}}\;.\end{aligned} \tag{3.4}$$

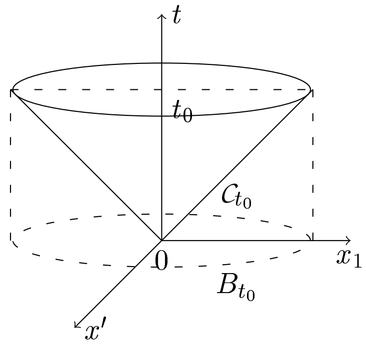

Figure 3.1: The cone C_{t_0}

Recalling that $\varphi^*(x) = u(|x|, x)$, it is easy to see that

$$\nabla\varphi^*(x) = \omega\partial_t u(|x|, x) + \nabla u(|x|, x) \ ,$$

hence

$$|\nabla\varphi^*(x)|^2 = |\partial u(|x|, x)|^2 + 2\partial_t u(|x|, x)(\omega \cdot \nabla u(|x|, x)) \ .$$

Therefore (3.4) can be rewritten as

$$\int_{B_{t_0}} |\partial u(t_0, x)|^2 dx = \int_{B_{t_0}} |\nabla\varphi^*(x)|^2 dx + 2 \iint_{C_{t_0}} \partial_t u f \, dt \, dx \ . \qquad (3.5)$$

We apply the *Schwartz Inequality* to (3.5), so we get

$$\|\partial u(t_0)\|^2_{L^2(B_{t_0})} \leq \|\nabla\varphi^*\|^2_{L^2(B_{t_0})} + 2 \int_0^{t_0} \|f(t)\|_{L^2(B_t)} \|\partial u(t_0)\|_{L^2(B_{t_0})} \, dt \, .$$

Finally, the Gronwall's Lemma A.1 applied to the previous inequality gives

$$\|\partial u(t_0)\|_{L^2(B_{t_0})} \leq \|\nabla\varphi^*\|_{L^2(B_{t_0})} + \int_0^{t_0} \|f(t)\|_{L^2(B_t)} \, dt \, ,$$

and this concludes the first part of the proof.

As regards the "Energy Identity" (3.2), it follows immediately from (3.5). □

It is possible to highlight the geometry of the problem by re-stating the Proposition 3.1 as follows.

Corollary 3.1. *Let u be the solution of the Characteristic Initial Value Problem* (1.15) *where the given functions φ and f are assumed to be smooth. Then the following Energy Estimate holds*

$$\|\partial u(t)\|_{L^2(B_t)} \leq \|\partial_+ u\|_{L^2(\partial C_t)} + \int_0^t \|f(\tau)\|_{L^2(B_\tau)} \, d\tau.$$

3.1.2 An estimate in H^1

The following computation allows us to extend the estimate (3.1) to H^1 space.

Proposition 3.2. *Let u be the solution of the inhomogeneous characteristic problem* (1.15) *where the given functions φ and f are assumed to be smooth. Then for all $t > 0$ the following Energy Estimate holds*

$$\|u(t)\|_{H^1(B_t)} \leq (1+t)\left(\|\varphi^*\|_{H^1(B_t)} + \int_0^t \|f(\tau)\|_{L^2(B_\tau)}\, d\tau\right), \qquad (3.6)$$

where B_t is the n-dimensional ball with radius t and centered in 0.

Proof. Since u satisfies the boundary condition $u|_{\partial C} = \varphi^*$, we have

$$[u(t,x)]^2 = 2\int_{|x|}^t \partial_t u(\tau,x)u(\tau,x)\, d\tau + [\varphi^*(x)]^2\ ; \qquad (3.7)$$

then we integrate the identity (3.7) on B_t

$$\begin{aligned}\|u(t)\|^2_{L^2(B_t)} &\leq 2\int_{B_t}\int_{|x|}^t |\partial_t u(\tau,x)||u(\tau,x)|\, dx d\tau + \int_{B_t} |\varphi^*(x)|^2 dx \leq \\ &\leq 2\int_0^t \int_{B_\tau} |\partial_t u(\tau,x)||u(\tau,x)|\, dx d\tau + \|\varphi^*\|^2_{L^2(B_t)} \leq \\ &\leq 2\int_0^t \|\partial u(\tau)\|_{L^2(B_\tau)} \|u(\tau)\|_{L^2(B_\tau)}\, d\tau + \|\varphi^*\|^2_{L^2(B_t)}\ .\end{aligned}$$

Finally, the Gronwall's Lemma A.1 applied to the previous inequality gives

$$\|u(t)\|_{L^2(B_t)} \leq \|\varphi^*\|_{L^2(B_t)} + \int_0^t \|\partial u(\tau)\|_{L^2(B_\tau)}\, d\tau\,,$$

then

$$\begin{aligned}\|u(t)\|_{L^2(B_t)} &\leq \|\varphi^*\|_{L^2(B_t)} + \int_0^t \left(\|\nabla\varphi^*\|_{L^2(B_\tau)} + \int_0^\tau \|f(s)\|_{L^2(B_s)}\, ds\right) d\tau \leq \\ &\leq \|\varphi^*\|_{L^2(B_t)} + t\,\|\nabla\varphi^*\|_{L^2(B_t)} + \int_0^t \int_s^t d\tau\, \|f(s)\|_{L^2(B_s)}\, ds \leq \\ &\leq \|\varphi^*\|_{L^2(B_t)} + t\left[\|\nabla\varphi^*\|_{L^2(B_t)} + \int_0^t \|f(s)\|_{L^2(B_s)}\, ds\right], \qquad (3.8)\end{aligned}$$

if we sum the inequalities (3.1) and (3.8) side by side, we get

$$\begin{aligned}\|u(t)\|_{H^1(B_t)} &\leq \|\varphi^*\|_{L^2(B_t)} + (1+t)\left[\|\nabla\varphi^*\|_{L^2(B_t)} + \int_0^t \|f(\tau)\|_{L^2(B_\tau)}\, d\tau\right] \leq \\ &\leq (1+t)\left[\|\varphi^*\|_{H^1(B_t)} + \int_0^t \|f(\tau)\|_{L^2(B_\tau)}\, d\tau\right].\end{aligned}$$

This concludes the proof. □

As before, it is possible to highlight the geometry of the problem by re-stating the Proposition 3.2 as follows.

Corollary 3.2. *Let u be the solution of the inhomogeneous characteristic problem* (1.15) *where the given functions φ and f are assumed to be smooth. Then for all $t > 0$ the following Energy Estimate holds*

$$\|u(t)\|_{H^1(B_t)} \leq (1+t)\left(\|u\|_{H^1(\partial\mathcal{C}_t)} + \int_0^t \|f(\tau)\|_{L^2(B_\tau)}\, d\tau\right). \tag{3.9}$$

At last, we observe that using the Weighted Sobolev norms defined in (1.24), the estimate (3.6) can be easily rewritten in the form

$$\|u\|_{X_T^1} \lesssim (1+T)\left[\|\varphi\|_{D_T^1} + T\,\|f\|_{X_T^0}\right]. \tag{3.10}$$

3.1.3 Non-reduced Energy Estimates

Let $k \in \mathbb{N}$ and $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^{n+1}$ be a multiindex such that $|\alpha| = k$, we introduce the multiindex notation to indicate the following family of derivatives

$$\partial^k u := \left\{\partial_t^{\alpha_0}\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} u \,|\, \alpha \in \mathbb{Z}_+^{n+1}, |\alpha| = k\right\}.$$

From Corollary 3.1 we easily derive the following *non-reduced Energy Estimates*

$$\left\|\partial^{k+1}u(t)\right\|_{L^2(B_t)} \leq \left\|\partial_+\partial^k u\right\|_{L^2(\partial\mathcal{C}_t)} + \int_0^t \left\|\partial^k f(\tau)\right\|_{L^2(B_\tau)} d\tau, \tag{3.11}$$

where ∂_+ indicates a generic tangential derivative, see (1.20).

We call (3.11) a "non-reduced" estimate since the term $\left\|\partial_+\partial^k u\right\|_{L^2(\partial\mathcal{C}_t)}$ still has to be reduced to the initial data φ and f. In fact it may contain derivatives that are not tangential to the cone, i.e. derivatives that can not be expressed in terms of derivatives of the initial data.

Nonetheless the first order transversal derivative solves the *Transport Equation* (1.23) , which does not hold on $\mathcal{C}$ but just on the characteristic boundary $\partial\mathcal{C}$. Similarly, the differential equations obtained by differentiating the Transport Equation are fulfilled by the transversal derivatives of order greater then one. For this hierarchy of equations we then calculate the Energy Inequalities; these estimates will allow us to control the L^2-norms of transversal derivatives in terms of norms of tangential derivatives of the data.

In order to simplify and better handle this hierarchy of Transport Equations, we use the characteristic coordinates defined in Section 1.2.1. Let's recall that, by using these coordinates, it is possible to clearly distinguish the tangential derivatives and the non-tangential ones.

3.2 Reduction to data in Classical Energy Estimates

Now we have to estimate the term $\left\|\partial_+\partial^k u\right\|_{L^2(\partial\mathcal{C}_t)}$ of the non-reduced Energy Estimate (3.11) by terms involving only the initial data. In terms of characteristic coordinates (1.16) we want to estimate any order $\partial_+\partial_-$ derivative of the function v on the hyperplane $\eta = 0$.

3.2.1 Tangential derivatives

In general, the spherical derivatives and the derivatives with respect to ξ are in a direction which is tangential to the boundary of the cone $\mathcal{C}$, in particular ∂_ξ is parallel to the cone directrix, while $\not\nabla$ follows the angular direction; so we can derive estimates for them by simply differentiating the initial data and eventually adding some coefficients which depends on n and on the derivative order.

Unfortunately, we cannot use the same procedure for estimating the non-tangential derivatives.

3.2.2 Non-tangential derivatives

Let's recall that the functions v, g and ψ are such that

$$u(t,x) = v(\xi,\eta,\omega)\ ,$$
$$f(t,x) = 4\,g(\xi,\eta,\omega)\ ,$$
$$\varphi(t,x) = \psi(\xi,\eta,\omega).$$

We consider now the non tangential derivative ∂_η. Fixed[1] $\omega \in S^{n-1}$, we set $X_1(\xi) = \partial_\eta v(\xi, 0, \omega)$, then the function X_1 solves the equation (1.18) on the hyperplane $\eta = 0$. In particular X_1 is such that

$$\partial_\xi X_1(\xi) + \frac{n-1}{2\xi} X_1(\xi) = G_1(\xi)\ , \tag{3.12}$$

where

$$G_1 = \frac{n-1}{2\xi}\,\partial_\xi\psi^* + \frac{1}{\xi^2}\Delta_{S^{n-1}}\psi^* + g^*\ . \tag{3.13}$$

Apart from the first order transversal derivative ∂_η which can be immediately estimated using (3.12), from equations (3.12) and (3.13) it is clear that, to estimate a transversal derivative ∂_η two tangential derivatives are needed; in fact at the right-hand term (3.13) we have the Laplace-Beltrami operator. In general, to estimate k transversal derivatives, at most $2k-1$ tangential derivatives could be needed. This iterative estimate generates a gap of regularity between the initial data and the solution.

[1]We omit the dependence on the variable $\omega \in S^{n-1}$.

We define $X_\ell(\xi) := \partial_\eta^\ell v(\xi, 0, \omega)$. When we apply ∂_η to (1.18) and we restrict to $\eta = 0$, we obtain a similar equation for X_2

$$\partial_\xi X_2(\xi) + \frac{n-1}{2\xi} X_2(\xi) = G_2(\xi) \ , \tag{3.14}$$

where

$$G_2 \simeq \frac{1}{\xi}\partial_\xi X_1 + \frac{1}{\xi^2}\partial_\xi \psi^* + \frac{1}{\xi^2} X_1 + \frac{1}{\xi^2}\nabla_\omega^2 X_1 + \frac{1}{\xi^3}\nabla_\omega^2 \psi^* + \partial_\eta g^* \simeq$$
$$\simeq \frac{1}{\xi^2}\partial_\xi \psi^* + \frac{1}{\xi^3}\nabla_\omega^2 \psi^* + \frac{1}{\xi^2} X_1 + \frac{1}{\xi^2}\nabla_\omega^2 X_1 + \frac{1}{\xi} g^* + \partial_\eta g^* \ . \tag{3.15}$$

By the symbol $\simeq$ we denote a generic linear combination of the terms that follow.

Remark 3.1. Observe that if the initial data have spherical symmetry (i.e. it does not depend on ω), we will not get the gap of differentiability. In fact just a ∂_ξ derivative would be sufficient to estimate a transversal derivative.

At first we obtain $\partial_\eta v$ on $\eta = 0$ by solving the differential equation (3.12), where the datum G_1 is given by (3.13). Then we obtain $\partial_\eta^2 v$ on $\eta = 0$ by the differential equation (3.14) inserting the known X_1 in the datum G_2. Then finally we get $\partial_\eta^\ell v$ on $\eta = 0$ by induction.

All the functions $\partial_\eta^\ell v$ fulfill a first order differential equation on $\eta = 0$ of the following type

$$\partial_\xi X + \frac{n-1}{2\xi} X = G \ , \tag{3.16}$$

where the datum G is given. In particular, we have the following Lemmas.

Lemma 3.1. *For any positive integer ℓ, we have*

$$G_{\ell+1} = \sum_{k=1}^{\ell}{}^{*} \xi^{-\ell-2+k} X_k + \sum_{k=0}^{\ell}{}^{*} \left[\xi^{-\ell-1+k}\partial_\xi X_k + \xi^{-\ell-2+k}\nabla_\omega^2 X_k\right] + g_\ell^* \ , \tag{3.17}$$

where $g_\ell^*(\xi) := \partial_\eta^\ell g(\xi, 0, \omega)$ and $\sum^*$ indicates a generic linear combination.

Proof. It suffices to derive ℓ times with respect to the variable η the equation (1.18) and evaluate it at $\eta = 0$. □

Lemma 3.2. *Assume*

$$\int_0^1 G(\xi'){\xi'}^{\frac{n-1}{2}} d\xi' < \infty \, , \tag{3.18}$$

uniformly with respect to $\omega \in S^{n-1}$; then the classical Cauchy problem

$$\begin{cases} \partial_\xi X + \frac{n-1}{2\xi} X = G \ , \\ X(\xi) = \circ\left(\xi^{-\frac{n-1}{2}}\right) \ \text{as } \xi \to 0^+ \end{cases}$$

admits a unique solution given by

$$X(\xi) = \xi^{-\frac{n-1}{2}} \int_0^\xi G(\xi')\xi'^{\frac{n-1}{2}} d\xi' \ .$$

Proof. For fixed $\lambda \in \mathbb{R}$, the explicit solution of the Cauchy problem

$$\begin{cases} X_\lambda'(\xi) + \dfrac{n-1}{2\xi} X_\lambda(\xi) = G(\xi) \\ X(1) = \lambda \end{cases}$$

is given by

$$X_\lambda(\xi) = \xi^{-\frac{n-1}{2}} \left[\lambda + \int_1^\xi G(\xi')\xi'^{\frac{n-1}{2}} d\xi' \right];$$

so

$$\lim_{\xi \to 0^+} \xi^{\frac{n-1}{2}} X_\lambda(\xi) = \lambda - \int_0^1 G(\xi')\xi'^{\frac{n-1}{2}} d\xi'.$$

Hence we have

$$\lim_{\xi \to 0^+} \xi^{\frac{n-1}{2}} X_\lambda(\xi) < \infty \tag{3.19}$$

provided (3.18), to have the previous condition fulfilled it suffices to assume

$$|G(\xi)| = \circ\left(\xi^{-\frac{n-1}{2}-1}\right) = \circ\left(\xi^{-\frac{n+1}{2}}\right) \quad \text{for } \xi \to 0^+.$$

Moreover we have

$$\lim_{\xi \to 0^+} \xi^{\frac{n-1}{2}} X_\lambda(\xi) = 0$$

for

$$\lambda = \int_0^1 G(\xi')\xi'^{\frac{n-1}{2}} d\xi'.$$

□

The expression of $G_{\ell+1}$ stated in Lemma 3.17 can be further simplified by iteratively replacing $\partial_\xi X_k$ with its representation in the ordinary differential equation

$$\partial_\xi X_k + \frac{n-1}{2\xi} X_k = G_k.$$

Lemma 3.3. *For any positive integer ℓ, we have*

$$G_{\ell+1} \simeq \xi^{-(\ell+1)}\partial_\xi \psi^* + \xi^{-(\ell+2)}\nabla_\omega^2 \psi^* + \\ + \sum_{k=1}^{\ell}{}^* \xi^{-\ell-2+k}\left(X_k + \nabla_\omega^2 X_k\right) + \sum_{k=0}^{\ell}{}^* \xi^{-\ell+k} g_k^* \ .$$

Proof. The Lemma follows immediately from Lemma 3.1, proceeding by induction on ℓ. □

Thus the usual Energy Inequality, which we exhibit in the Lemma below, holds.

Lemma 3.4. *Assume that in* (3.16) *we have*

$$G(\xi) = \circ\left(\xi^{-\frac{n+1}{2}}\right) \quad as\ \xi \to 0^+ \tag{3.20}$$

uniformly with respect to $\omega \in S^{n-1}$*; then*

$$X(\xi)\xi^{\frac{n-1}{2}} = \int_0^\xi G(\xi')\xi'^{\frac{n-1}{2}} d\xi' \,. \tag{3.21a}$$

Moreover, for any positive integer m *such that*

$$|\partial_\xi^m G(\xi)| = \circ\left(\xi^{-\frac{n+1}{2}}\right) \quad as\ \xi \to 0^+, \tag{3.21b}$$

we have the following Energy Inequality

$$\left\|\partial_\xi^m X(t)\right\|_{L^2(S_t)} \lesssim \sum_{h=0}^m \int_0^t \xi^{-m+h} \left\|\partial_\xi^h G(\xi)\right\|_{L^2(S_\xi)} d\xi \,. \tag{3.21c}$$

Proof. Applying the classical formula, we easily find out that the solution of (3.16) with initial datum

$$X(\xi_0) = X_0$$

is given by

$$X(\xi) = \left(\frac{\xi_0}{\xi}\right)^{\frac{n-1}{2}} \left[X_0 + \int_{\xi_0}^\xi G(\xi') \left(\frac{\xi'}{\xi_0}\right)^{\frac{n-1}{2}} d\xi'\right].$$

When ξ_0 goes to 0, this result together with the hypothesis (3.20) gives immediately (3.21a).

Now we apply the $L^2(S^{n-1})$ norm to the identity (3.21a),

$$\begin{aligned}\left(\int_{S^{n-1}} |X(t)|^2 t^{n-1} d\omega\right)^{\frac{1}{2}} &\leq \\ &\leq \left(\int_{S^{n-1}} \left(\int_0^\xi G(\xi)\xi^{\frac{n-1}{2}} d\xi\right)^2 d\omega\right)^{\frac{1}{2}} \overset{(*)}{\leq} \\ &\leq \int_0^t \left(\int_{S^{n-1}} |G(\xi)|^2 \xi^{n-1} d\omega\right)^{\frac{1}{2}} d\xi\end{aligned}$$

which gives immediately (3.21c) for $m = 0$. Note that in $(*)$ we applied the *Minkowski Integral Inequality.*

To prove (3.21c) for $m \geq 1$ we proceed by induction on m. Assume that (3.21c) holds for any $h < m$; then we derive the equation (3.16) m times with respect to the variable ξ, thus

$$\partial_\xi \left(\partial_\xi^m X\right) + \frac{n-1}{2\xi}\left(\partial_\xi^m X\right) = {\sum_{h=0}^{m-1}}^{*} \xi^{-m-1+h}\partial_\xi^h X + \partial_\xi^m G\ ,$$

which gives, together with hypothesis (3.21b),

$$\left\|\partial_\xi^m X\right\|_{L^2(S_t)} \lesssim \sum_{h=0}^{m-1}\int_0^t \xi^{-m-1+h}\left\|\partial_\xi^h X\right\|_{L^2(S_\xi)} d\xi + \int_0^t \left\|\partial_\xi^m G\right\|_{L^2(S_\xi)} d\xi\ .$$

Then the thesis follows immediately from the induction step. □

Remark 3.2. Note that hypothesis (3.21b) can be replaced with

$$|G(\xi)| = \circ\left(\xi^{-\frac{n+1}{2}+m}\right) \text{ as } \xi \to 0^+; \tag{3.22}$$

moreover, from Lemma 3.4 follows immediately that

$$|\partial_\xi^m X(\xi)| = \circ\left(\xi^{-\frac{n-1}{2}}\right) \text{ as } \xi \to 0^+.$$

Lemma 3.5. *Let ℓ and m be positive integers. Assume*

$$\psi^*(\xi) = \circ\left(\xi^{-\frac{n-1}{2}+m+\ell+1}\right) \text{ as } \xi \to 0^+, \tag{3.23a}$$

$$g^*(\xi) = \circ\left(\xi^{-\frac{n-1}{2}+m+\ell-1}\right) \text{ as } \xi \to 0^+, \tag{3.23b}$$

uniformly with respect to ω, then we have the following inequality

$$\left\|\partial_\xi^m X_{\ell+1}(t)\right\|_{L^2(S_t)} \lesssim \sum_{h=0}^{m}\sum_{k=0}^{2\ell+1}\int_0^t \xi^{-m-\ell-1+h+k}\left\|\partial_+^{h+k+1}\psi^*(\xi)\right\|_{L^2(S_\xi)} d\xi +$$
$$+\sum_{h=0}^{m}\sum_{k=0}^{\ell}\sum_{j=0}^{k}\int_0^t \xi^{-m+h-k+2j}\left\|\partial_+^{h+2j}\partial_\eta^{\ell-k} g^*(\xi)\right\|_{L^2(S_\xi)} d\xi\ . \tag{3.24}$$

where ∂_+ contains only derivatives tangential to the data surface. Moreover

$$X_{\ell+1}(\xi) = \circ\left(\xi^{-\frac{n-1}{2}+m}\right) \text{ as } \xi \to 0^+, \tag{3.25}$$

uniformly with respect to ω.

Proof. We prove the Lemma by induction on ℓ. First of all, we note that (3.23) implies

$$\psi^*(\xi) = \circ\left(\xi^{-\frac{n-1}{2}+m+k+1}\right) \text{ as } \xi \to 0^+,$$
$$g^*(\xi) = \circ\left(\xi^{-\frac{n-1}{2}+m+k-1}\right) \text{ as } \xi \to 0^+,$$

for all $k = 0, \ldots, \ell$ and uniformly with respect to ω, so at each iteration we can apply Lemma 3.4.

Consider the initial case $\ell = 0$, the Lemma 3.4 and the identity (3.13) give immediately

$$\begin{aligned}
\left\|\partial_\xi^m X_1\right\|_{L^2(S_t)} &\lesssim \sum_{h=0}^{m} \int_0^t \xi^{-m-1+h} \left\|\partial_\xi^{h+1}\psi^*(\xi)\right\|_{L^2(S_\xi)} + \\
&+ \xi^{-m-2+h} \left\|\partial_\omega^2 \partial_\xi^h \psi^*\right\|_{L^2(S_\xi)} + \xi^{-m+h} \left\|\partial_\xi^h g^*(\xi)\right\|_{L^2(S_\xi)} d\xi \lesssim \\
&\lesssim \sum_{h=0}^{m}\sum_{k=0}^{1} \int_0^t \xi^{-m-1+h+k} \left\|\partial_+^{h+k+1}\psi^*(\xi)\right\|_{L^2(S_\xi)} d\xi + \\
&+ \sum_{h=0}^{m} \int_0^t \xi^{-m+h} \left\|\partial_+^h g^*(\xi)\right\|_{L^2(S_\xi)} d\xi \,.
\end{aligned}$$

Assume now that the Lemma holds for a certain $\ell \geq 1$, from the Lemmas 3.4 and 3.3 we have

$$\begin{aligned}
\left\|\partial_\xi^m X_{\ell+1}(t)\right\|_{L^2(S_t)} &\lesssim \sum_{h=0}^{m}\sum_{k=0}^{\ell} \int_0^t \xi^{-m-\ell-2+h+k} \left\|\partial_\xi^h X_k(\xi)\right\|_{L^2(S_\xi)} + \\
&+ \xi^{-m-\ell-2+h+k} \left\|\nabla_\omega^2 \partial_\xi^h X_k(\xi)\right\|_{L^2(S_\xi)} + \xi^{-m-\ell+h+k} \left\|\partial_\xi^h g_k^*(\xi)\right\|_{L^2(S_\xi)} d\xi \,,
\end{aligned} \tag{3.26}$$

then the induction step gives

$$\begin{aligned}
\sum_{h=0}^{m}\sum_{k=0}^{\ell} \int_0^t \xi^{-m-\ell-2+h+k} \left[\left\|\partial_\xi^h X_k(\xi)\right\|_{L^2(S_\xi)} + \left\|\nabla_\omega^2 \partial_\xi^h X_k(\xi)\right\|_{L^2(S_\xi)}\right] d\xi &\lesssim \\
\lesssim \sum_{h=0}^{m}\sum_{k=0}^{2\ell+1} \int_0^t \xi^{-m-\ell-1+h+k} \left\|\partial_+^{h+k+1}\psi^*(\xi)\right\|_{L^2(S_\xi)} d\xi &+ \\
+ \sum_{h=0}^{m}\sum_{k=0}^{\ell-1}\sum_{j=0}^{\ell-k} \int_0^t \xi^{-m-\ell+h+k+2j} \left\|\partial_+^{h+2j}\partial_\eta^k g^*(\xi)\right\|_{L^2(S_\xi)} d\xi &\,,
\end{aligned}$$

plugging this result back into the (3.26) we obtain the thesis. □

We can generalize the previous result as follows.

Lemma 3.6. *Let $s > \frac{n-1}{2}$ be a positive integer. Assume*

$$\psi^*(\xi) = \circ\left(\xi^{-\frac{n-1}{2}+s}\right) \quad \textit{as } \xi \to 0^+, \tag{3.27a}$$

$$g^*(\xi) = \circ\left(\xi^{-\frac{n-1}{2}+s-2}\right) \quad \textit{as } \xi \to 0^+, \tag{3.27b}$$

uniformly with respect to ω, then we have the following inequality

$$
\begin{aligned}
\sum_{m+p+r+\ell=s-1} \left\| \frac{1}{t^p} \nabla\!\!\!/^r \partial_\xi^m X_{\ell+1}(t) \right\|_{L^2(S_t)} \lesssim \\
\lesssim \sum_{h=0}^{m} \sum_{k=0}^{2\ell+1} \int_0^t \xi^{-m-\ell-1+h+k-p} \left\| \partial_+^{h+k+r+1} \psi^*(\xi) \right\|_{L^2(S_\xi)} d\xi + \\
+ \sum_{h=0}^{m} \sum_{k=0}^{\ell} \sum_{j=0}^{k} \int_0^t \xi^{-m+h-k+2j-p} \left\| \partial_+^{h+2j+r} \partial_\eta^{\ell-k} g^*(\xi) \right\|_{L^2(S_\xi)} d\xi \, . \qquad (3.28)
\end{aligned}
$$

Proof. It suffices to to apply the previous Lemma, provided

$$
\begin{aligned}
\frac{1}{\xi^p} \nabla\!\!\!/^r \psi^*(\xi) &= \circ\left(\xi^{-\frac{n-1}{2}+m+\ell+1}\right) \text{ as } \xi \to 0^+, \\
\frac{1}{\xi^p} \nabla\!\!\!/^r g^*(\xi) &= \circ\left(\xi^{-\frac{n-1}{2}+m+\ell-1}\right) \text{ as } \xi \to 0^+
\end{aligned}
$$

uniformly with respect to ω, for all $m+p+r+\ell = s-1$, which gives

$$
\begin{aligned}
\psi^*(\xi) &= \circ\left(\xi^{-\frac{n-1}{2}+s}\right) \text{ as } \xi \to 0^+, \\
g^*(\xi) &= \circ\left(\xi^{-\frac{n-1}{2}+s-2}\right) \text{ as } \xi \to 0^+
\end{aligned}
$$

uniformly with respect to ω. □

Remark 3.3. To derive the Energy Estimate, we assumed that the initial data vanish of order high enough at the origin, in particular the conditions (3.23) must hold.

On the other hand, the vanishing order implicitly required by the norms we used (1.24) is greater than $-\frac{n-1}{2}+s$, in particular we have the following. Let f be a smooth function in (t,x) variables; if $\|f\|_{D_T^s} < +\infty$, then

$$
f(t,x) = \circ((t,x)^s) \text{ for } (t,x) \to 0,
$$

where by $\circ$ we denote the *little-o symbol of Landau.*

3.2.3 Reduction to data

In the non-reduced Energy Estimate (3.11) we used norms defined on the whole cone hypersurface.

On the contrary, to estimate the non-tangential derivatives X_ℓ on $\partial\mathcal{C}$ we used L^2-like norms on the spheres S_t, which are exactly the intersections of $\partial\mathcal{C}$ whit the hyperplanes t =const. Then if we introduce the weights $t^{-\alpha}$, which are due to the geometry of the cone, and the L^∞ norm with respect to t, we get nothing more than the norms (1.24) we introduced in Section 1.4.

Remark 3.4. Let $f \in \mathscr{C}_c^\infty(\mathbb{R}^{n+1})$, then the following equivalence between norms holds

$$\sup_{[0,T]} t^{-\frac{n}{2}} \|f\|_{L^2(\partial\mathcal{C}_t)} \lesssim \sup_{[0,T]} t^{-\frac{n-1}{2}} \|f(t)\|_{L^2(S_t)}\,.$$

Finally, we are able to state the following *Reduced Energy Estimates.* Note that it holds only for initial data which fulfill specific vanishing conditions in the origin.

Theorem 3.1 (Linear Reduced Energy Estimates)**.** *Let for the data hold* $\varphi \in D_T^s$ *and* $f \in Y_T^{s-1}$. *If* $u \in X_T^s$ *is a solution of the Characteristic Initial Value Problem* (1.15)*, then u fulfills the following* Linear Reduced Energy Estimates

$$\|u\|_{X_T^s} \le C_{n,s}(1+T)\left[\|\varphi\|_{D_T^s} + \sum_{k=0}^{s-2}\left\|\partial_-^k f\right\|_{D_T^{s-1-k}} + T\,\|f\|_{X_T^{s-1}}\right] \tag{3.29a}$$

that is equivalent to

$$\|u\|_{X_T^s} \le C_{n,s}(1+T)\left[\|\varphi\|_{D_T^s} + T\,\|f\|_{Y_T^{s-1}}\right]. \tag{3.29b}$$

Proof. Multiplying the non-reduced estimate (3.11) by $t^{-\frac{n}{2}}$, applying the sup with respect to $t \in [0,T]$ and Remark 3.4, we have

$$\begin{aligned}\sup_{[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|=s} \|\partial^\alpha u(t)\|_{L^2(B_t)} \lesssim \sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=s-1} \|\partial_+\partial^\alpha u(t)\|_{L^2(S_t)} +\\ + \sup_{[0,T]} t^{-\frac{n}{2}+1} \sum_{|\alpha|=s-1} \|\partial^\alpha f(t)\|_{L^2(B_t)}\,,\end{aligned} \tag{3.30}$$

where the first right-hand terms can be rewritten as

$$\begin{aligned}\sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=s-1} \|\partial_+\partial^\alpha u(t)\|_{L^2(S_t)} \simeq \sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=s} \left\|\partial_+^\alpha \varphi^*(t)\right\|_{L^2(S_t)} +\\ + \sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{\substack{p+r+m+\ell+1=s\\ p+m\ge 1}} \left\|\frac{1}{t^p}\nabla\!\!\!\!/^{\,r}\,\partial_\xi^m X_{\ell+1}(t)\right\|_{L^2(S_t)}.\end{aligned} \tag{3.31}$$

Using the Lemma 3.5 we can estimate the second term on the right-hand side of (3.31), as follows

$$\sum_{\substack{p+r+m+\ell+1=s\\ p+m\geq 1}} \left\| \frac{1}{t^p} \nabla^r \partial_\xi^m X_{\ell+1}(t) \right\|_{L^2(S_t)} \lesssim$$

$$\lesssim \sum_{\substack{p+r+m+\ell+1=s\\ p+m\geq 1}} \left[\sum_{h=0}^{m} \sum_{k=0}^{2\ell+1} \int_0^t \xi^{-m-\ell-1+h+k-p} \left\| \partial_+^{h+k+r+1} \psi^*(t) \right\|_{L^2(S_\xi)} d\xi + \right.$$

$$\left. + \sum_{h=0}^{m} \sum_{k=0}^{\ell} \sum_{j=0}^{\ell-k} \int_0^t \xi^{-m-\ell+h+k+2j-p} \left\| \partial_+^{h+r+2j} \partial_\eta^k g^*(t) \right\|_{L^2(S_\xi)} d\xi \right]. \quad (3.32)$$

Now we set $a := h+k+r+1$, then

$$1 \leq a \leq 2s-1$$
$$-m-\ell-1+h+k-p = -(s+1)+a\,,$$

analogously we set $b := h+r+2j$, then

$$0 \leq b \leq 2(s-1-k)-1$$
$$-m-\ell+h+k+2j-p = -(s-1)+b+k\,;$$

hence (3.32) becomes

$$\sum_{\substack{p+r+m+\ell+1=s\\ p+m\geq 1}} \left\| \frac{1}{t^p} \nabla^r \partial_\xi^m X_{\ell+1}(t) \right\|_{L^2(S_t)} \lesssim$$

$$\lesssim \sum_{a=1}^{2s-1} \int_0^t \xi^{-(s+1)+a} \left\| \partial_+^a \psi^*(t) \right\|_{L^2(S_\xi)} d\xi +$$

$$+ \sum_{k=0}^{s-2} \sum_{b=0}^{2(s-1-k)-1} \int_0^t \xi^{-(s-1)+k+b} \left\| \partial_+^b \partial_\eta^k g^*(t) \right\|_{L^2(S_\xi)} d\xi\,.$$

Thus

$$\sup_{[0,T]} \left[t^{-\frac{n-1}{2}} \sum_{\substack{p+r+m+\ell+1=s\\ p+m\geq 1}} \left\| \frac{1}{t^p} \nabla^r \partial_\xi^m X_{\ell+1}(t) \right\|_{L^2(S_t)} \right] \lesssim$$

$$\lesssim \sup_{[0,T]} \left[t^{-\frac{n-1}{2}} \sum_{a=1}^{2s-1} t^{-s+a} \left\| \partial_+^a \psi^*(t) \right\|_{L^2(S_t)} + \right.$$

$$\left. + t^{-\frac{n-1}{2}} \sum_{k=0}^{s-2} \sum_{b=0}^{2(s-1-k)-1} t^{-(s-1)+k+b+1} \left\| \partial_+^b \partial_\eta^k g^*(t) \right\|_{L^2(S_t)} \right] \lesssim$$

$$\lesssim \|\varphi\|_{D_T^s} + T \sum_{k=0}^{s-2} \left\| \partial_-^k f \right\|_{D_T^{s-1-k}}\,.$$

Plugging this result back into the estimate (3.30) we get

$$\sup_{[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|=s} \|\partial^\alpha u(t)\|_{L^2(B_t)} \lesssim$$

$$\lesssim \|\varphi\|_{D_T^s} + T \sum_{k=0}^{s-2} \left\| \partial_-^k f \right\|_{D_T^{s-1-k}} + T \|f\|_{X_T^{s-1}} \simeq$$

$$\simeq \|\varphi\|_{D_T^s} + T \|f\|_{Y_T^{s-1}} .$$

Retracing exactly the same steps, we have the same estimate for a generic derivative order $1 < \sigma < s$

$$\sup_{[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|=\sigma} \|\partial^\alpha u(t)\|_{L^2(B_t)} \lesssim \|\varphi\|_{D_T^s} + T \|f\|_{Y_T^{s-1}} ,$$

summing over all derivative orders $1 < \sigma \leq s$ we obtain the thesis. □

The definition of the norms and the relative Weighted Sobolev Spaces introduced in Section 1.4 take inspiration from the proof of Theorem 3.1.

3.3 Existence and uniqueness of a solution in X_T^s

We now state the following *Existence and Uniqueness Theorem*. Note that this is a partial result which assures the existence and the uniqueness of a regular solution in the space X_T^s. In other words, at this stage we can just prove the existence and the uniqueness provided a sufficiently high vanishing order of the solution in the origin.

Theorem 3.2. *Let for the data hold $\varphi \in D_T^s$ and $f \in Y_T^{s-1}$, with $s > \frac{n}{2}$ and $n \geq 1$. Then there exists a unique solution $u \in X_T^s$ of* (1.15), *which fulfills the Energy Estimate* (3.29b).

Proof. The existence of the solution in case of initial data with the required vanishing order at the origin immediately follows from the explicit formula we obtained in Chapter 2 (see Theorem 2.1). The uniqueness is implied by the H^1 Energy Estimate proved in Theorem 3.6. At last, Estimate (3.29b) ensures the H^s regularity of the solution in $\mathcal{C}$. □

3.4 Regularity of the Solution

By virtue of the Sobolev Imbedding Theorem (see Theorem B.1), the data $\varphi \in D_T^s$ and $f \in Y_T^{s-1}$ are regular functions[2] on $\overline{\mathcal{C}}$, provided $s > \frac{n}{2}$.

The Theorem 3.2 assures that the solution we found is regular in $\mathcal{C}$. Let us now analyze its behavior in terms of regularity in the closure $\overline{\mathcal{C}}$ of the cone and in particular in the origin (the tip of the cone).

[2]We have at least $\varphi(t,\cdot) \in \mathscr{C}^{s-1}(S_t)$ and $f(t,\cdot) \in \mathscr{C}^{s-3}(S_t) \cap \mathscr{C}^0(B_t)$ and the continuity with respect to the variable t.

3.4.1 Regularity

The Energy Estimate (3.29b) assures the H^s-regularity of the solution u in the cone $\mathcal{C}$. The Trace Theorem (see Theorem B.2) assures the H^s-regularity of the solution on the cone border $\partial\mathcal{C}$ and then in the closure $\overline{\mathcal{C}}$, excluding the tip of the cone.

Thanks to the vanishing conditions at the origin, the solution u is always at least continuous in the tip of the cone. We can gain more regularity if s is sufficiently large.

3.4.2 Gap of Differentiability Class

The Energy Estimate (3.29b) clearly highlight the *gap of differentiability class* between the solution $u \in X^s_T$ and the datum $\varphi^* \in H^{2s-1}_{\text{loc}}(\mathbb{R}^n)$ which amounts to $s-1$ differentiability orders. Without loss of generality we can consider data which vanish in a neighborhood of the origin, in fact the differentiability gap is not generated by the geometrical singularity and the Energy Estimates hold in the origin as well. In particular, we want to prove that this gap can not be reduced by more than half a differentiability order, this can be seen by the following simple computation.

Assume $u \in H^s_{\text{loc}}(\mathcal{C})$ to be the solution of the Characteristic Problem (1.15), this implies $v \in H^s_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^+ \times S^{n-1})$, where v fulfills the equation (1.18). Assume moreover that

$$\psi^*(\xi, \omega) = A(\xi)B(\omega), \tag{3.33}$$

where $A \in \mathscr{C}^\infty(\mathbb{R}^+)$ is identically null in a neighborhood of the origin and $B \in \mathscr{C}^\infty(S^{n-1})$.

The Trace Theorem (see Theorem B.2) gives

$$X_k \in H^{s-k-\frac{1}{2}}_{\text{loc}}(\mathbb{R}^+ \times S^{n-1}) \tag{3.34}$$

for all $k \in \mathbb{N}$ such that $s - k - \frac{1}{2} > 0$, which means for all $k = 0, \dots, s-1$.

Let's focus on X_1. It fulfills the equation (3.12), which using (3.33) simplifies to

$$\partial_\xi X_1(\xi,\omega) + \frac{n-1}{2\xi} X_1(\xi,\omega) = \frac{n-1}{2\xi} A'(\xi)B(\omega) + \frac{1}{\xi^2} A(\xi)\Delta_{S^{n-1}}B(\omega) + g^*(\xi,\omega)$$

thus, (3.21a) gives

$$\begin{aligned} X_1(\xi)\xi^{\frac{n-1}{2}} &= \frac{n-1}{2}\int_0^\xi \xi'^{\frac{n-1}{2}-1} A'(\xi')d\xi' B(\omega) + \\ &+ \int_0^\xi \xi'^{\frac{n-1}{2}-2} A(\xi')d\xi' \Delta_{S^{n-1}}B(\omega) + \int_0^\xi \xi'^{\frac{n-1}{2}} g^*(\xi',\omega)d\xi' = \\ &= A_1(\xi)\Delta_{S^{n-1}}B(\omega) + \text{lower order terms}, \end{aligned}$$

provided $X_1 = \circ(\xi^{-\frac{n-1}{2}})$ as $\xi \to 0^+$ uniformly with respect to ω. In general, by virtue of Lemma 3.1, if $X_k = \circ(\xi^{-\frac{n-1}{2}})$ as $\xi \to 0^+$ uniformly with respect to ω, then

$$X_k(\xi,\omega)\xi^{\frac{n-1}{2}} = A_k(\xi)\Delta^k_{S^{n-1}}B(\omega) + \text{more regular terms},$$

where by "more regular terms" we mean terms in ω which are more regular with respect to $\Delta^k_{S^{n-1}}B(\omega)$.

By virtue of (3.34), we have $A_k(\xi)\Delta^k_{S^{n-1}}B(\omega) \in H^{s-k-\frac{1}{2}}_{\text{loc}}(\mathbb{R}^+ \times S^{n-1})$. If we restrict X_k to the sphere S^{n-1}, the Trace Theorem B.2 assures $\Delta^k_{S^{n-1}}B \in H^{s-k-1}_{\text{loc}}(S^{n-1})$. Moreover the following implication holds

$$\Delta^k_{S^{n-1}}B \in H^{s-k-1}_{\text{loc}}(S^{n-1}) \implies B \in H^{s-k-1+2k}_{\text{loc}}(S^{n-1})\,.$$

In the particular case $k = s-1$ we get $B \in H^{2s-2}_{\text{loc}}(S^{n-1})$, which implies $\varphi \in H^{2s-1-\frac{1}{2}}_{\text{loc}}(\partial\mathcal{C})$.

If in the Energy Estimates obtained in Theorem 3.1 we lost something, it is no more than half a differentiability order, in fact the previous computation shows

$$\varphi \notin H^{2s-\frac{3}{2}}_{\text{loc}}(\partial\mathcal{C}) \implies u \notin H^s_{\text{loc}}(\mathcal{C}),$$

on the other hand by the Energy Estimates we have

$$\varphi \in H^{2s-1}_{\text{loc}}(\partial\mathcal{C}) \implies u \in H^s_{\text{loc}}(\mathcal{C}).$$

Chapter 4

Polynomial Solution of the Characteristic Initial Value Problem

Theorem 3.2 assures the existence and the uniqueness of the solution for the Linear Characteristic Initial Value Problem, but the solution obtained is still incomplete because of the particular vanishing hypotheses at the origin on the initial data, see (3.27). But this leaves room only for a finite number of degrees of freedom and the gap can be closed by solving the problem for polynomial data. Putting together the polynomial case with the result stated in Theorem 3.2, we have a complete solution of the linear problem.

To solve (1.15) with generic initial data, we split the functions φ and f in the sum of a polynomial and a remainder function with the required vanishing order.

In this Chapter we consider the linear wave equation

$$\Box u = f_p \ , \tag{4.1}$$

in $n \geq 1$ space dimensions, where f_p is a polynomial of degree $m-2$, $m \geq 2$, in the variables (t, x).

For the equation (4.1) we solve both the Classical Cauchy Problem and the Characteristic Initial Value Problem. For the Classic Problem we consider the initial conditions

$$\begin{cases} u(0,\cdot) = \varphi_p(0,\cdot) \\ \partial_t u(0,\cdot) = \psi_p(0,\cdot) \end{cases} , \tag{4.2}$$

while, for the Characteristic Initial Value Problem we consider the boundary condition

$$u|_{\partial C} = \varphi_p|_{\partial C} \ , \tag{4.3}$$

where φ_p and ψ_p are polynomials of degree m and $m-1$ respectively in the (t, x) variables.

For both problems the classical decomposition of homogeneous polynomials in harmonic polynomials will be used.

4.1 Polynomial Spaces

We start introducing the following spaces.

Definition 4.1 (Spaces $\mathcal{P}_m(x)$ and $\overset{\circ}{\mathcal{P}}_m(x)$)**.** We denote by $\mathcal{P}_m(x)$ the set of all polynomials in x of degree at most m with complex coefficients

$$\mathcal{P}_m = \left\{ p, \text{ s.t. } p(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha \, , a_\alpha \in \mathbb{C} \right\} ,$$

where α denotes an n-tuple $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers. Similarity, we define $\overset{\circ}{\mathcal{P}}_m$ as the set of all homogeneous polynomials in x of degree m with complex coefficients

$$\overset{\circ}{\mathcal{P}}_m = \left\{ p, \text{ s.t. } p(x) = \sum_{|\alpha| = m} a_\alpha x^\alpha \, , a_\alpha \in \mathbb{C} \right\} .$$

It is clear that the set of monomials x^α, $|\alpha| = m$ is a basis for $\overset{\circ}{\mathcal{P}}_m$. The number of such monomials is precisely the number d_m of ways an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers can be chosen so that $|\alpha| = m$. It is no hard to find out that

$$d_{m,n} = \binom{n+m-1}{n-1} = \binom{n+m-1}{m}. \tag{4.4a}$$

Moreover, we immediately have the following result.

Remark 4.1. Every $p \in \mathcal{P}_m(x)$ can be uniquely written as the sum of homogeneous polynomials, in other words

$$\mathcal{P}_m(x) = \bigoplus_{\mu=0}^{m} \overset{\circ}{\mathcal{P}}_\mu(x) \, .$$

Definition 4.2 (Space $\mathcal{H}_m(x)$)**.** We denote by $\mathcal{H}_m(x)$ the subset of $\overset{\circ}{\mathcal{P}}_m(x)$ consisting of harmonic homogeneous polynomials of degree m, in other words

$$\mathcal{H}_m(x) = \left\{ p \in \overset{\circ}{\mathcal{P}}_m(x), \text{ s.t. } \Delta p(x) = 0 \right\} ,$$

It can be proved (see [24])that

$$d^*_{m,n} = \dim \mathcal{H}_m = d_{m,n} - d_{m-2,n} \, . \tag{4.4b}$$

We state the following result which gives the well-known decomposition in harmonic polynomials (for the proof see [24]).

Theorem 4.1. *Every* $p \in \overset{\circ}{\mathcal{P}}_m(x)$ *can be uniquely written in the form*

$$p(x) = \sum_{k=0}^{[\frac{m}{2}]} |x|^{2k} q_{m-2k}(x)$$

where $q_j \in \mathcal{H}_j(x)$ *depend on* p.

The previous theorem leads immediately to the following corollary.

Corollary 4.1. *Let* $\left\{p_j^{(i)}\right\}_{j=1,\dots,d^*_{i,n}} \subset \mathcal{H}_i(x)$ *be a basis of* $\mathcal{H}_i(x)$, *then every* $p \in \overset{\circ}{\mathcal{P}}_m(x)$ *can be uniquely written in the form*

$$p(x) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{j=1}^{d^*_{m-2k,n}} |x|^{2k} c_{j,k}^{(m-2k)} p_j^{(m-2k)}(x) \; .$$

where the coefficients $\left\{c_{j,k}^{(m-2k)}\right\}_{j,k}$ *are uniquely determined.*

We list here some standard differentiation formulas that will be useful later.

$$\partial_j |x|^h = h|x|^{h-2} x_j$$
$$\partial_j^2 |x|^h = h|x|^{h-2} + h(h-2)|x|^{h-4} x_j^2$$

hence

$$\nabla |x|^h = h|x|^{h-2} x \; , \tag{4.5a}$$
$$\Delta |x|^h = h(n+h-2)|x|^{h-2} \; , \tag{4.5b}$$
$$\forall p \in \overset{\circ}{\mathcal{P}}_m(x) \, , \quad x \cdot \nabla p = mp \; ; \tag{4.5c}$$

where the last one is nothing more than the Euler Formula for the homogeneous functions. Moreover $\forall u, v \in \mathscr{C}^2$ we have

$$\Delta(uv) = v\Delta u + 2\nabla u \nabla v + u \Delta v \tag{4.5d}$$
$$\Box(uv) = v\Box u + 2\nabla u \nabla v - 2\partial_t u \partial_t v + u\Box v. \tag{4.5e}$$

4.2 Polynomial Solution of the Classical Cauchy Problem

Consider the Classical Cauchy problem for the Wave Equation (4.1)-(4.2), with polynomial data $\varphi_p \in \overset{\circ}{\mathcal{P}}_m(x)$, $f_p \in \overset{\circ}{\mathcal{P}}_{m-2}(t,x)$.

The Corollary 4.1 gives

$$\varphi_p(x) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{j=1}^{d^*_{m-2k,n}} |x|^{2k} a_{j,k}^{(m-2k)} p_j^{(m-2k)}(x), \tag{4.6}$$

$$\psi_p(x) = \sum_{k=0}^{[\frac{m-1}{2}]} \sum_{j=1}^{d^*_{m-1-2k,n}} |x|^{2k} e_{j,k}^{(m-1-2k)} p_j^{(m-1-2k)}(x) \tag{4.7}$$

and

$$f_p(t,x) = \sum_{\ell=0}^{m-2} \sum_{k=0}^{[\frac{m-\ell}{2}]-1} \sum_{j=1}^{d^*_{m-2-\ell-2k,n}} b_{j,\ell,k}^{(m-2-\ell-2k)} t^\ell |x|^{2k} p_j^{(m-2-\ell-2k)}(x) =$$

$$= \sum_{\ell=0}^{m-2} \sum_{k=1}^{[\frac{m-\ell}{2}]} \sum_{j=1}^{d^*_{m-\ell-2k,n}} b_{j,\ell,k-1}^{(m-\ell-2k)} t^\ell |x|^{2(k-1)} p_j^{(m-\ell-2k)}(x)\ ; \tag{4.8}$$

where the sequences of coefficients $\left\{a_{j,k}^{(m-2k)}\right\}_{j,k}$, $\left\{e_{j,k}^{(m-1-2k)}\right\}_{j,k}$ and $\left\{b_{j,\ell,k-1}^{(m-\ell-2k)}\right\}_{j,\ell,k}$ are known.

Analogously, the polynomial solution $u \in \mathring{\mathcal{P}}_m(t,x)$ can be written in the form

$$u(t,x) = \sum_{\ell=0}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]} \sum_{j=1}^{d^*_{m-\ell-2k,n}} c_{j,\ell,k}^{(m-\ell-2k)} t^\ell |x|^{2k} p_j^{(m-\ell-2k)}(x). \tag{4.9}$$

We can immediately retrieve from (4.9) the decomposition of $\partial_t u$

$$\partial_t u(t,x) = \sum_{\ell=1}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]} \sum_{j=1}^{d^*_{m-\ell-2k,n}} c_{j,\ell,k}^{(m-\ell-2k)} \ell t^{\ell-1} |x|^{2k} p_j^{(m-\ell-2k)}(x);$$

let's then compute $\Box\left(t^\ell |x|^{2k} p_j^{(m-\ell-2k)}\right)$ using the formulas (4.5)

$$\Box\left(t^\ell |x|^{2k} p_j^{(m-\ell-2k)}\right) = \ell(\ell-1) t^{\ell-2} |x|^{2k} p_j^{(m-\ell-2k)} + \\ - 2k(n+2m-2k-2\ell-2) t^\ell |x|^{2(k-1)} p_j^{(m-\ell-2k)}$$

hence

$$\Box u(t,x) = \sum_{\ell=0}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]} \sum_{j=1}^{d^*_{m-\ell-2k,n}} \Big[\ell(\ell-1) t^{\ell-2} |x|^{2k} + \\ -2k(n+2m-2k-2\ell-2) t^\ell |x|^{2(k-1)}\Big] c_{j,\ell,k}^{(m-\ell-2k)} p_j^{(m-\ell-2k)}, \tag{4.10}$$

shifting the indexes $\ell \mapsto \ell+2$, $k \mapsto k+1$ in (4.10), we get

$$\begin{aligned}
&\Box u(t,x) = \\
&= \sum_{\ell=0}^{m-2} \sum_{k=0}^{[\frac{m-\ell}{2}]-1} \sum_{j=1}^{d^*_{m-2-\ell-2k,n}} (\ell+1)(\ell+2) t^\ell c_{j,\ell+2,k}^{(m-2-\ell-2k)} |x|^{2k} p_j^{(m-2-\ell-2k)} + \\
&- \sum_{\ell=0}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]-1} \sum_{j=1}^{d^*_{m-\ell-2k-2,n}} 2(k+1)(n+2m-2k-2\ell-4) t^\ell |x|^{2k} c_{j,\ell,k+1}^{(m-2-\ell-2k)} p_j^{(m-2-\ell-2k)} \\
&= \sum_{\ell=0}^{m-2} \sum_{k=1}^{[\frac{m-\ell}{2}]} \sum_{j=1}^{d^*_{m-\ell-2k,n}} (\ell+1)(\ell+2) c_{j,\ell+2,k-1}^{(m-\ell-2k)} t^\ell |x|^{2(k-1)} p_j^{(m-\ell-2k)} + \\
&- \sum_{\ell=0}^{m} \sum_{k=1}^{[\frac{m-\ell}{2}]} \sum_{j=1}^{d^*_{m-\ell-2k,n}} 2k(n+2m-2k-2\ell-2) c_{j,\ell,k}^{(m-\ell-2k)} t^\ell |x|^{2(k-1)} p_j^{(m-\ell-2k)} .
\end{aligned} \tag{4.11}$$

From the relation (4.1) we obtain

$$(\ell+1)(\ell+2) c_{j,\ell+2,k-1}^{(m-\ell-2k)} - 4k\left(\frac{n}{2}+m-k-\ell-1\right) c_{j,\ell,k}^{(m-\ell-2k)} = b_{j,\ell,k-1}^{(m-\ell-2k)} \tag{4.12}$$

for every

$$\begin{cases} 0 \le \ell \le m-2 \\ 1 \le k \le [\frac{m-\ell}{2}] \\ 1 \le j \le d^*_{m-\ell-2k,n} \end{cases} .$$

Consider now the initial condition on u, then

$$\varphi_p(0,x) = u(0,x) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{j=1}^{d^*_{m-2k,n}} c_{j,0,k}^{(m-2k)} |x|^{2k} p_j^{(m-2k)}(x),$$

hence from (4.6) we must have

$$a_{j,k}^{(m-2k)} = c_{j,0,k}^{(m-2k)}, \tag{4.13}$$

for every

$$\begin{cases} 0 \le k \le [\frac{m}{2}] \\ 1 \le j \le d^*_{m-2k,n} \end{cases} .$$

Consider then the initial condition on $\partial_t u$, then

$$\psi_p(0,x) = \partial_t u(0,x) = \sum_{k=0}^{[\frac{m-1}{2}]} \sum_{j=1}^{d^*_{m-1-2k,n}} c_{j,1,k}^{(m-1-2k)} |x|^{2k} p_j^{(m-1-2k)}(x),$$

hence from (4.7) we must have

$$e_{j,k}^{(m-1-2k)} = c_{j,1,k}^{(m-1-2k)}, \tag{4.14}$$

for every

$$\begin{cases} 0 \leq k \leq [\frac{m}{2}] \\ 1 \leq j \leq d^*_{m-1-2k,n} \end{cases}.$$

The polynomial solution u is uniquely identified by the coefficients $\left\{c_{j,\ell,k}^{(m-\ell-2k)}\right\}_{j,\ell,k}$ that fulfill the conditions (4.12), (4.13) and (4.14).

It is easy to see that the sequence $\left\{c_{j,\ell,k}^{(m-\ell-2k)}\right\}_{j,\ell,k}$ consists of

$$\sum_{\ell=0}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]} d^*_{m-\ell-2k}$$

elements, while the conditions (4.12), (4.13) and (4.14) consist of

$$\sum_{\ell=0}^{m-2} \sum_{k=1}^{[\frac{m-\ell}{2}]} d^*_{m-\ell-2k,n}, \ \sum_{k=0}^{[\frac{m}{2}]} d^*_{m-2k,n} \text{ and } \sum_{k=0}^{[\frac{m-1}{2}]} d^*_{m-1-2k,n}$$

equations respectively, where

$$\sum_{\ell=0}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]} d^*_{m-\ell-2k,n} = \sum_{\ell=0}^{m-2} \sum_{k=1}^{[\frac{m-\ell}{2}]} d^*_{m-\ell-2k,n} + \sum_{k=0}^{[\frac{m-1}{2}]} d^*_{m-1-2k,n} + \sum_{k=0}^{[\frac{m}{2}]} d^*_{m-2k,n};$$

thus the problem of finding the polynomial solution u of the Classical Cauchy Problem for (4.1) is well-posed.

Finally, the formulas (4.4) give immediately

$$\sum_{\ell=0}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]} d^*_{m-\ell-2k,n} = \sum_{\ell=0}^{m} \sum_{k=0}^{[\frac{m-\ell}{2}]} d_{m-\ell-2k,n} - d_{m-\ell-2(k+1),n} = \sum_{\ell=0}^{m} d_{m-\ell,n}.$$

4.3 Polynomial Solution of the Characteristic Initial Value Problem

Consider the Characteristic Initial Value Problem for the Wave Equation (4.1)-(4.3) with polynomial data $\varphi_p \in \overset{\circ}{\mathcal{P}}_m(t,x)$, $f_p \in \overset{\circ}{\mathcal{P}}_{m-2}(t,x)$.

Remark 4.2. If $p \in \mathcal{H}_m(t,x)$, the polynomial $\widetilde{p}(t,x) = p(it,x)$ solves the homogeneous Wave Equation and vice versa.

Definition 4.3 (Space $\mathcal{W}_m(t,x)$)**.** We denote by $\mathcal{W}_m(t,x)$ the subset of $\overset{\circ}{\mathcal{P}}_m(t,x)$ consisting of homogeneous polynomials of degree m that solve the homogeneous Wave Equation, in other words

$$\mathcal{W}_m(t,x) = \left\{ p \in \overset{\circ}{\mathcal{P}}_m(t,x), \text{ s.t. } \Box p(t,x) = 0 \right\},$$

Remark 4.3. Let $\left\{ p_j^{(m)} \right\}_{j=1,\dots,d^*_{m,n+1}}$ be a basis of $\mathcal{H}_m(t,x)$, then it is easy to prove that $\left\{ \widetilde{p}_j^{(m)} \right\}_{j=1,\dots,d^*_{m,n+1}}$ is a basis of $\mathcal{W}_m(t,x)$.

Lemma 4.1. *Assume $w_j \in \mathcal{W}_j(t,x)$, then*

$$\Box((-t^2+|x|^2)^k w_j) = 4k\left(\frac{n-1}{2}+k+j\right)(-t^2+|x|^2)^{k-1} w_j$$

Proof. Formula (4.5) give

$$\begin{aligned}\Box((-t^2+|x|^2)^k w_j) &= \\ &= 4k\left(\frac{n-1}{2}+k\right)(-t^2+|x|^2)^{k-1} w_j + \\ &+ \sum_{h=1}^{n} 4k x_h (-t^2+|x|^2)^{k-1}\partial_h w_j + 4kt(-t^2+|x|^2)^{k-1}\partial_t w_j = \\ &= 4k\left(\frac{n-1}{2}+k+j\right)(-t^2+|x|^2)^{k-1} w_j\,.\end{aligned}$$

□

By Corollary 4.1 and Remark 4.2, every $u \in \overset{\circ}{\mathcal{P}}_m(t,x)$ can be uniquely rewritten in the form

$$u(t,x) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{j=1}^{d^*_{m-2k,n+1}} (-t^2+|x|^2)^k c_{j,k} \widetilde{p}_j^{(m-2k)}(t,x)\,. \tag{4.15}$$

Similarly, for the initial data $\varphi_p \in \overset{\circ}{\mathcal{P}}_m(t,x)$ and $f_p \in \overset{\circ}{\mathcal{P}}_{m-2}(t,x)$, we have

$$\varphi_p(t,x) = \sum_{k=0}^{[\frac{m}{2}]} \sum_{j=1}^{d^*_{m-2k,n+1}} (-t^2+|x|^2)^k a_{j,k} \widetilde{p}_j^{(m-2k)}(t,x)\,,$$

$$f_p(t,x) = \sum_{k=0}^{[\frac{m-2}{2}]} \sum_{j=1}^{d^*_{m-2-2k,n+1}} (-t^2+|x|^2)^k b_{j,k} \widetilde{p}_j^{(m-2-2k)}(t,x)\,,$$

where the sequences of coefficients[1] $\{a_{j,k}\}_{j,k}$ and $\{b_{j,k}\}_{j,k}$ are known.

Using the Lemma 4.3, we have

$$\Box\left((-t^2+|x|^2)^k\widetilde{p}_j^{(m-2k)}\right)=4k\left(\frac{n-1}{2}+m-k\right)(-t^2+|x|^2)^{k-1}\widetilde{p}_j^{(m-2k)}$$

hence

$$\begin{aligned}\Box u(t,x) &=\\ &=\sum_{k=1}^{[\frac{m}{2}]}\sum_{j=1}^{d^*_{m-2k,n+1}}4k\left(\frac{n-1}{2}+m-k\right)c_{j,k}(-t^2+|x|^2)^{k-1}\widetilde{p}_j^{(m-2k)}(t,x)=\\ &=\sum_{k=0}^{[\frac{m-2}{2}]}\sum_{j=1}^{d^*_{m-2-2k,n+1}}4(k+1)\left(\frac{n-1}{2}+m-k-1\right)\cdot\\ &\qquad\cdot c_{j,k+1}(-t^2+|x|^2)^k\widetilde{p}_j^{(m-2-2k)}(t,x)\,. \end{aligned}\tag{4.16}$$

From the relation (4.1) we obtain

$$4(k+1)\left(\frac{n-1}{2}+m-k-1\right)c_{j,k+1}=b_{j,k}\tag{4.17}$$

for every

$$\begin{cases}0\le k\le[\frac{m-2}{2}]\\ 1\le j\le d^*_{m-2-2k,n+1}\end{cases}.$$

Consider now the initial condition (4.3), then

$$\varphi_p(|x|,x)=u(|x|,x)$$

where

$$u(|x|,x)=\sum_{j=1}^{d^*_{m,n+1}}c_{j,0}\widetilde{p}_j^{(m)}(t,x)$$

$$\varphi_p(|x|,x)=\sum_{j=1}^{d^*_{m,n+1}}a_{j,0}\widetilde{p}_j^{(m)}(t,x)\,,$$

hence we must have

$$a_{j,0}=c_{j,0},\tag{4.18}$$

for every $1\le j\le d^*_{m,n+1}$.

The polynomial solution u is uniquely identified by the coefficients $\{c_{j,k}\}_{j,k}$ that fulfill the conditions (4.17) and (4.18).

[1] It would be sufficient to assign the function φ_p only on the cone $\partial\mathcal{C}$, i.e. to know the coefficients $\{a_{j,0}\}_{j=1,\dots,d^*_{m,n+1}}$.

The sequence $\{c_{j,k}\}_{j,k}$ consists of

$$\sum_{k=0}^{[\frac{m}{2}]} d^*_{m-2k,n+1}$$

elements, while the conditions (4.17) and (4.18) consist respectively of

$$\sum_{k=1}^{[\frac{m-2}{2}]} d^*_{m-2-2k,n+1} \quad \text{and} \quad d^*_{m,n+1}$$

equations, where

$$\sum_{k=0}^{[\frac{m}{2}]} d^*_{m-2k,n+1} = \sum_{k=0}^{[\frac{m-2}{2}]} d^*_{m-2-2k,n+1} + d^*_{m,n+1};$$

thus the problem of finding the polynomial solution u of the Classical Cauchy Problem for (4.1) is well-posed.

We summarize these results in the following theorem.

Theorem 4.2. *The Characteristic Initial Value problem for the Wave Equation* (4.1)-(4.3) *with polynomial data* $\varphi_p \in \overset{\circ}{\mathcal{P}}_m(t,x)$, $f_p \in \overset{\circ}{\mathcal{P}}_{m-2}(t,x)$ *admits a unique polynomial solution* $u \in \overset{\circ}{\mathcal{P}}_m(t,x)$ *of the form* (4.15), *where*

$$c_{j,k} = \begin{cases} a_{j,0} & \text{if } k = 0 \\ \frac{b_{j,k+1}}{4k\left(\frac{n-1}{2}+m-k\right)} & \text{if } k = 1, \ldots, [\frac{m}{2}] \end{cases}, \quad \forall j = 1, \ldots, d^*_{m-2k,n+1}. \tag{4.19}$$

The following Corollary extends Theorem 4.2 to non-homogeneous polynomial data.

Corollary 4.2. *The Characteristic Initial Value problem for the Wave Equation* (4.1)-(4.3) *with polynomial data* $\varphi_p \in \mathcal{P}_\mu(t,x)$, $f_p \in \mathcal{P}_{\mu-2}(t,x)$ *admits a unique polynomial solution* $u_p \in \mathcal{P}_\mu(t,x)$ *of the form*

$$u_p(t,x) = \sum_{m=0}^{\mu} u_m(t,x) = \sum_{m=0}^{\mu} \sum_{2k+j=m} (-t^2+|x|^2)^k u_{j,k}(t,x) \,,$$

where $u_m \in \overset{\circ}{\mathcal{P}}_m(t,x)$ *and* $u_{j,k} \in \mathcal{W}_j(t,x)$.

4.4 General Solution of the Linear Problem

In the previous chapter we proved Theorem 3.2. This is a partial result which assures the existence and the uniqueness of a regular solution in the space X_T^s for the Linear Characteristic Initial Value Problem (1.15), once assumed the initial data in D_T^s and Y_T^s respectively. In other words, up to now we just proved the existence and the uniqueness of the solution provided a sufficiently high vanishing order of the initial data, and then of the solution, in the origin.

On the other hand, in Section 4.3 we proved the existence and the uniqueness of a polynomial solution of the problem (1.15) for polynomial initial data.

The idea to prove the existence and the uniqueness of the solution in case of generic initial data is to rewrite them in the sum of a polynomial and a remainder function using the Taylor expansion. Thanks to the linearity, the original problem is so split in two separate Initial Value Problems: one with polynomial data, the other with data which have the vanishing order at the origin required by the hypotheses of Theorem 3.2.

4.4.1 Existence and Uniqueness of a solution in $\mathcal{X}$

The following Remark specifies the space of "generic" initial data for which we will state the general Existence and Uniqueness Theorem for the Linear Initial Value problem (1.15).

Remark 4.4. With reference to the vanishing order at the origin required by the hypotheses of Theorem 3.2 on the initial data, we set $\mu \geq s$ and we introduce the following spaces

$$\begin{aligned} \mathcal{X} &:= X_T^s \oplus \mathcal{P}_\mu(t,x), \\ \mathcal{D} &:= D_T^s \oplus \mathcal{P}_\mu(t,x), \\ \mathcal{Y} &:= Y_T^s \oplus \mathcal{P}_{\mu-2}(t,x). \end{aligned}$$

Consider the Linear Characteristic Initial Value Problem (1.15) with generic initial data $\varphi \in \mathcal{D}$ and $f \in \mathcal{Y}$. The functions φ and f are split in the sum of a polynomial of degree μ and $\mu - 2$ respectively, and a remainder function with the required vanishing order at the origin; in particular, in view of Remark 4.4, we have

$$\varphi = \varphi_p + \varphi_r, \tag{4.20}$$

$$f = f_p + f_r \tag{4.21}$$

where $\varphi_p \in \mathcal{P}_\mu(t,x)$, $f_p \in \mathcal{P}_{\mu-2}(t,x)$, $\varphi_r \in D_T^s$ and $f_r \in Y_T^s$.

Denote by $u_p \in \mathcal{P}_\mu(t,x)$ the solution of the polynomial problem

$$\begin{cases} \Box u = f_p \\ u|_{\partial C} = \varphi_p^* \end{cases},$$

then Corollary 4.2 assures existence and uniqueness of u_p. Analogously, we denote by $u_r \in X_T^s$ the solution of the problem

$$\begin{cases} \Box u = f_r \\ u|_{\partial C} = \varphi_r^* \end{cases},$$

then Theorem 3.2 assures existence and uniqueness of u_r.

By virtue of the linearity, the solution u of (1.15) is given by

$$u = u_p + u_r \, .$$

We have just proved the following Theorem.

Theorem 4.3. *Let $\varphi \in \mathcal{D}$ and $f \in \mathcal{Y}$. Assume $s > \frac{n-1}{2}$ and $\mu \geq s$, then there exists a unique solution $u \in \mathcal{X}$ of* (1.15).

4.5 Polynomial Solution of the Non-linear Problem

Our goal is to generalize the method applied to the linear problem to solve the following semi-linear problem. We start analyzing the polynomial part of the problem.

Consider the Characteristic Initial Value Problem for the Wave Equation with a quadratic non-linearity

$$\begin{cases} \Box u = Q(\partial u, \partial u) \\ u|_{\partial C} = \varphi_p|_{\partial C} \end{cases} \qquad t \geq 0, \; x \in \mathbb{R}^n, \tag{4.22}$$

where

$$Q(\partial u, \partial u) := \sum_{i,j=0}^{n} q_{ij} \partial_i u \partial_j u, \quad q_{ij} \in \mathbb{C} \tag{4.23}$$

is a generic quadratic form in the vector ∂u.

Assume $\varphi_p \in \mathcal{P}_\mu(t,x)$, then Theorem 4.1 and Remark 4.2 lead immediately to the following decomposition

$$\varphi_p(t,x) = \sum_{m=0}^{\mu} \Phi_m(t,x) = \sum_{m=0}^{\mu} \sum_{2k+j=m} (-t^2 + |x|^2)^k \Phi_{j,k}(t,x) \, , \tag{4.24}$$

where $\Phi_m \in \overset{\circ}{\mathcal{P}}_m(t,x)$ and $\Phi_{j,k} \in \mathcal{W}_j(t,x)$.

Since (4.22) is a non-linear problem, we expect not to find a polynomial solution. Assume that the solution of the problem (4.22) can be split in the sum of infinite terms as follows

$$u_p(t,x) = \sum_{m=0}^{+\infty} u_m(t,x) = \sum_{m=0}^{+\infty} \sum_{2k+j=m} (-t^2 + |x|^2)^k u_{j,k}(t,x) \, , \tag{4.25}$$

where $u_m \in \overset{\circ}{\mathcal{P}}_m(t,x)$ and $u_{j,k} \in \mathcal{W}_j(t,x)$. Clearly, each term contributes to the ones of greater order within the decomposition above. In order to avoid the problem of studying the convergence of the series (4.25), we truncate the solution u_p at the degree correspondent to the vanishing order of the solution required by the Energy Estimates, see hypotheses (3.27). This truncated function constitutes a good local approximation of u_p in a neighborhood of the origin.

Remark 4.5. Observe that decompositions (4.24) and (4.25) are unique.

Consider now the initial condition

$$u_p^* = \varphi_p^*$$

where

$$u_p^* = \sum_{m=0}^{+\infty} u_{m,0}^*$$
$$\varphi_p^* = \sum_{m=0}^{\mu} \Phi_{m,0}^* \,,$$

hence we must have

$$u_{m,0}^* = \begin{cases} \Phi_{m,0}^* & \text{if } m \leq \mu \\ 0 & \text{if } m > \mu \end{cases} . \tag{4.26}$$

By virtue of Theorem 4.2, for all $\mu = 0, \ldots, m$ there exists a unique polynomial $u_{m,0} \in \mathring{\mathcal{P}}_m$ that solves the Characteristic Initial Value Problem

$$\begin{cases} \Box u = 0 \\ u^* = u_{m,0}^* \end{cases} .$$

Recalling that $\Phi_{m,0} \in \mathcal{W}_m(t,x)$ and using the uniqueness of the solution we immediately arrive at

$$u_{m,0} = \begin{cases} \Phi_{m,0} & \text{if } m \leq \mu \\ 0 & \text{if } m > \mu \end{cases} . \tag{4.27}$$

Remark 4.6. In general, each polynomial $u_{j,k} \in \mathcal{W}_j(t,x)$ is uniquely determined by its restriction to the null cone $u_{j,k}^*$ as the solution of

$$\begin{cases} \Box u = 0 \\ u^* = u_{m,k}^* \end{cases} .$$

Let's now calculate $u_{j,k}$, for $k > 0$. For this purpose, we use Lemma 4.1 At the price of technical complications, but without any essential difficulty, Lemma 4.1 gives immediately the following Corollary.

Corollary 4.3. *Let u_p be an analytical function in $\mathbb{R}^{n+1}$, so u_p can be rewritten in the form* (4.25)*, then*

$$\Box u_p(t,x) = \sum_{m=0}^{+\infty} \sum_{2k+j=m} 4k \left(\frac{n-1}{2} + k + j \right) (-t^2 + |x|^2)^{k-1} u_{j,k}(t,x) \,,$$

in general, for all $h \in \mathbb{N}$ it results

$$\Box^h u_p(t,x) = \sum_{m=0}^{+\infty} \sum_{2k+j=m} 4^h k \cdots (k-h+1) \cdot$$
$$\cdot \left(\frac{n-1}{2}+k+j\right) \ldots \left(\frac{n-1}{2}+k-h+1+j\right) (-t^2+|x|^2)^{k-h} u_{j,k}(t,x) \; .$$

In view of Remark 4.6, it suffices to determine all the terms $u^*_{m,k}$.

Denote by P_m the projection operator which extracts the homogeneous component of degree m with respect to the origin. By virtue of Corollary 4.3 we have

$$P_m \Box^k u^*_p = 4^k k! \left(\frac{n-1}{2}+k+m\right) \ldots \left(\frac{n-1}{2}+1+m\right) u^*_{m,k}. \tag{4.28}$$

So $u^*_{m,k}$ is uniquely determined by the restriction to the null cone of the homogeneous component of degree m of $\Box^k u^*_p$. Let's prove now that the equation

$$P_m \Box^k u^*_p = P_m \Box^{k-1} \left(Q(\partial u_p, \partial u_p)\right)^* \tag{4.29}$$

uniquely identifies $u_{m,k}$, see Remark 4.6.

By distributing derivatives on the products, we have that

$$P_m \Box^{k-1} \left(Q(\partial u_p, \partial u_p)\right) \tag{4.30}$$

is a linear combination of terms of the form

$$\partial^{\alpha_1}(-t^2+|x|^2)^{k_1} \partial^{\beta_1} u_{m_1,k_1} \partial^{\alpha_2}(-t^2+|x|^2)^{k_2} \partial^{\beta_2} u_{m_2,k_2}, \tag{4.31}$$

where

$$\begin{cases} 2k_1 - |\alpha_1| + m_1 - |\beta_1| + 2k_2 - |\alpha_2| + m_2 - |\beta_2| = m \\ |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| = 2(k-1)+2 \end{cases} .$$

Remark 4.7. We can assume $\deg\left((-t^2+|x|^2)^{k_i} u_{m_i,k_i}\right) = 2k_i + m_i \neq 0$, for $i = 1,2$, in fact the case $2k_i + m_i = 0$ corresponds to a constant term that would be annihilated by the derivatives in front of u_p in (4.30).

In view of Remark 4.7, (4.31) holds for

$$\begin{cases} 2k_1 + m_1 + 2k_2 + m_2 = m + 2k \\ |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| = 2k \\ 2k_i + m_i > 0 \end{cases} .$$

Hence, the restriction

$$P_m \Box^{k-1} (\partial u_p \cdot \partial u_p)^*$$

is a linear combination of terms of the form

$$\partial^{\beta_1} u^*_{m_1,k_1} \partial^{\beta_2} u^*_{m_2,k_2}, \tag{4.32}$$

where

$$\begin{cases} 2k_1 + m_1 + 2k_2 + m_2 = m + 2k \\ 2k_1 + 2k_2 + |\beta_1| + |\beta_2| = 2k \\ m_i < m \\ k_i \leq k \end{cases} .$$

This, together with (4.29), implies that we can determine each term $u^*_{m,k}$ once all the terms

$$\{u^*_{m_i,k_i}\}_{\substack{m_i<m \\ k_i\leq k}}$$

are known, in particular

$$u^*_{m,k} = \sum^*_{\substack{2k_1+m_1+2k_2+m_2=m+2k \\ 2k_1+2k_2+|\beta_1|+|\beta_2|=2k \\ m_i<m,\ k_i\leq k}} \partial^{\beta_1} u^*_{m_1,k_1} \partial^{\beta_2} u^*_{m_2,k_2}, \tag{4.33}$$

where $\sum^*$ indicates a linear combination. Proceeding by induction on $2k+m$ and starting from (4.27), we can determine recursively all term $u_{m,k}$.

We summarize these results in the following theorem.

Theorem 4.4. *If the Characteristic Initial Value Problem for the non-linear Wave Equation* (4.22) *with polynomial initial datum $\varphi_p \in \mathcal{P}_\mu(t,x)$ admits an analytical solution u_p, then u_p is of the form* (4.25), *where*

$$u_{m,k}(t,x) = \sum^*_{\substack{2k_1+m_1+2k_2+m_2=m+2k \\ 2k_1+2k_2+|\beta_1|+|\beta_2|=2k \\ m_i<m,\ k_i\leq k}} \partial^{\beta_1} u_{m_1,k_1}(t,x) \partial^{\beta_2} u_{m_2,k_2}(t,x), \tag{4.34}$$

and $u_{m,0}$, $m = 0, \dots, \mu$ are given by (4.27).

Theorem 4.4 will play an important role in proving the existence and the uniqueness of the Non-linear problem for a generic initial datum.

Lemma 4.2. *Denote by $\mathbb{T}_\mu$ the operator that truncates the harmonic expansion of an analytic function at order μ.*

Let u_p be the analytical solution of (4.22), *so it is easy to prove*[2] *that the polynomial $\overline{u}_p = \mathbb{T}_\mu(u_p) \in \mathcal{P}_\mu(t,x)$ solves the problem*

$$\begin{cases} \Box u = \mathbb{T}_{\mu-2}\left(Q(\partial u, \partial u)\right) \\ u|_{\partial\mathcal{C}} = \varphi^*_p \end{cases} \quad \text{in } \mathcal{C}, \tag{4.35}$$

where $\varphi_p \in \mathcal{P}_\mu(t,x)$.

[2] We have

$$\Box\overline{u}_p = \Box\mathbb{T}_\mu(u_p) = \mathbb{T}_{\mu-2}(\Box u_p) = \mathbb{T}_{\mu-2}\left(Q(\partial u_p, \partial u_p)\right).$$

Remark 4.8. By virtue of the Lemma 4.2, we immediately get

$$\mathbb{T}_{\mu-2}\left(Q(\partial\overline{u}_p, \partial\overline{u}_p)\right) = \mathbb{T}_{\mu-2}\left(Q(\partial u_p, \partial u_p)\right).$$

Chapter 5

Non-Linear Characteristic Initial Value Problem

As anticipated in Section 4.5, we consider the Characteristic Initial Value Problem for the Wave Equation with a quadratic non-linearity

$$\begin{cases} \Box u = Q(\partial u, \partial u) \\ u|_{\partial \mathcal{C}} = \varphi^* \end{cases} \quad \text{in } \mathcal{C}, \tag{5.1}$$

where $\varphi \in \mathcal{D}$ and

$$Q(\partial u, \partial u) := \sum_{i,j=0}^{n} q_{ij} \partial_i u \partial_j u, \quad q_{ij} \in \mathbb{C}$$

is a generic quadratic form in the vector ∂u. Without loss of generality, we may assume Q symmetric.

5.1 A quadratic non linearity

By virtue of Remark 4.4, the initial datum φ can be split in the sum $\varphi = \varphi_p + \varphi_r$ where $\varphi_p \in \mathcal{P}_\mu(t,x)$ and $\varphi_r \in D^s_T$. Let $\overline{u}_p$ be the polynomial component of the solution u as calculated in Lemma 4.2, we set $u = \overline{u}_p + u_r$, then

$$\begin{aligned} \Box u_r &= \Box u - \Box \overline{u}_p = Q(\partial u, \partial u) - \mathbb{T}_{\mu-2}\left(Q(\partial \overline{u}_p, \partial \overline{u}_p)\right) = \\ &= Q(\partial u_r, \partial u_r) + 2Q(\partial u_r, \partial \overline{u}_p) + \left(\mathbb{I} - \mathbb{T}_{\mu-2}\right)\left(Q(\partial \overline{u}_p \cdot \partial \overline{u}_p)\right); \end{aligned} \tag{5.2}$$

in addition,

$$u_r|_{\partial \mathcal{C}} = u|_{\partial \mathcal{C}} - \overline{u}_p|_{\partial \mathcal{C}} = \varphi|_{\partial \mathcal{C}} - \varphi_p|_{\partial \mathcal{C}} = \varphi_r|_{\partial \mathcal{C}}. \tag{5.3}$$

Remark 5.1. Being $\overline{u}_p \in \mathcal{P}_\mu(t,x)$, it results

$$Q(\partial\overline{u}_p, \partial\overline{u}_p) \in \mathcal{P}_{2(\mu-1)},$$

thus, applying Remark 4.8

$$f_p := (\mathbb{I} - \mathbb{T}_{\mu-2})\,(Q(\partial\overline{u}_p, \partial\overline{u}_p)) \in \mathcal{P}_{2\mu-2}(t,x) \ominus \mathcal{P}_{\mu-2}(t,x), \tag{5.4}$$

where $\mathcal{P}_{2\mu-2}(t,x) \ominus \mathcal{P}_{\mu-2}(t,x) = \{p - \mathbb{T}_{\mu-2}(p),\, p \in \mathcal{P}_{2\mu-2}(t,x)\}$.

Finally, note that $f_p \in Y^s_T$.

From this point on, we will consider the following Non-linear Characteristic Problem

$$\begin{cases} \Box u = Q(\partial u, \partial u) + 2Q(\partial u, \partial\overline{u}_p) + f_p \\ u|_{\partial\mathcal{C}} = \varphi^*_r \end{cases} \quad \text{in } \mathcal{C}, \tag{5.5}$$

for which we have to generalize the Energy Estimates we get in Theorem 3.1.

5.2 Non-linear Energy Estimate

For the sake of simplicity, while building the Non-linear Energy Estimate for the problem (5.5) we will not involve in the calculation the linear term f_p. In fact its contribute to the final Energy Estimate can be easily retrieved from the Linear Reduced Energy Estimate (3.29a).

In other words, we will consider the "truncated" problem

$$\begin{cases} \Box u = Q(\partial u, \partial u) + 2Q(\partial u, \partial\overline{u}_p) \\ u|_{\partial\mathcal{C}} = \varphi^*_r \end{cases} \quad \text{in } \mathcal{C}. \tag{5.6}$$

5.2.1 Non-linear Energy Estimate on the cone hypersurface

At first we derive the Energy Estimate on the initial data hypersurface $\partial\mathcal{C}$. For this purpose we need to re-write the Wave Equation with the considered quadratic non-linearity (5.6) in characteristic coordinates, see Section 1.2.1.

Recalling that

$$\begin{aligned} \partial_t u &= \frac{1}{2}(\partial_\xi u + \partial_\eta u), \\ \partial_r u &= \frac{1}{2}(\partial_\xi u - \partial_\eta u), \\ \nabla u &= \not\nabla u + \partial_r u\ \omega \end{aligned}$$

we have

$$\begin{aligned} \partial u = (\partial_t u, \nabla u) &= \left(\frac{1}{2}(\partial_\xi u + \partial_\eta u), \not\nabla u + \frac{1}{2}(\partial_\xi u - \partial_\eta u)\omega\right) = \\ &= \frac{1}{2}(1,\omega)\partial_\xi u + \frac{1}{2}(1,-\omega)\partial_\eta u + (0,\not\nabla u) = \omega_+\partial_\xi u + \omega_-\partial_\eta u + \not\nabla u, \end{aligned} \tag{5.7}$$

where we put

$$\omega_\pm := \frac{1}{2}(1, \pm\omega) = \left(\frac{1}{2}, \pm\frac{x}{2|x|}\right).$$

Thus the problem (5.6) can be rewritten in characteristic coordinates as follows

$$Lv = \sum\nolimits^* c(\omega_\pm)\partial_\pm v \partial_\pm v + \sum\nolimits^* c(\omega_\pm)\partial_\pm v \partial_\pm v_p \tag{5.8a}$$

where the coefficients $c(\omega_\pm)$ depend on the variable ω and include the coefficients of the quadratic form Q.

The initial condition for (5.6) becomes

$$v(\xi, 0, \omega) = \psi_r(\xi, 0, \omega). \tag{5.8b}$$

For all $\ell = 1, \dots, s$, we define $X_\ell(\xi) := \partial_\eta^\ell v(\xi, 0, \omega)$, omitting the dependence on ω. Now, the function X_1 solves the equation (5.8a) on the hyperplane $\eta = 0$, in particular we have

$$\begin{aligned}\partial_\xi X_1(\xi) + \frac{n-1}{2\xi} X_1(\xi) \simeq \frac{1}{\xi}\, \partial_\xi \psi_r^* + \nabla\!\!\!/^2 \psi_r^* + \\ + \partial_+\psi_r^* \partial_+\psi_r^* + \partial_+\psi_r^* X_1 + X_1^2 + \\ + \partial_+\psi_r^* \partial_+\psi_p^* + \partial_+\psi_r^*(\partial_\eta v_p)^* + \partial_+\psi_p^* X_1 + X_1(\partial_\eta v_p)^*.\end{aligned} \tag{5.9}$$

Note that if the non-linearity consists of a quadratic form which is a null form for the wave equation, the term X_1^2 will not appear at the right-hand side of the equation (5.9).

Lemma 5.1. *Let $s > \frac{n+1}{2}$ be an integer. We set for all $\ell = 1, \dots, s-1$*

$$Y_\ell := \sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{p+r+m=s-\ell} \left\| \frac{1}{t^p} \nabla\!\!\!/^r \partial_\xi^m X_\ell(t) \right\|_{L^2(S_t)}, \tag{5.10}$$

$$\Phi_\ell := \sum_{k=0}^{\ell+s} \sup_{[0,T]} t^{-\frac{n-1}{2}-s+k} \sum_{|\alpha|\le k} \left\| \partial_+^\alpha \varphi_r^*(t) \right\|_{L^2(S_t)}, \tag{5.11}$$

$$U_p := \sup_{\xi\in[0,T]} \sum_{|\alpha|\le\mu} \|\partial^\alpha \overline{u}_p(\xi)\|_{L^\infty(S_\xi)}, \tag{5.12}$$

where $\mu \ge s$. Assume X_ℓ defined as before and T, $\varphi = \varphi_r + \varphi_p$ such that

(H1) $\varphi_r \in D_T^s$ and $\varphi_p \in \mathcal{P}_\mu(t, x)$;

(H2) $T\Phi_{[\frac{s}{2}]} < 1$;

(H3) $T(\Phi_1 + U_p) \ll 1$, where the constant U_p depends on the polynomial part φ_p of the initial datum.

Then, for all $\ell = 1, \dots, s-1$, we have

$$Y_\ell \lesssim \Phi_\ell. \tag{5.13}$$

Moreover, the constant in front of the right-hand side of (5.13) *is given by*

$$\frac{C(n,s,Q)}{[1 - T(\Phi_1 + U_p)]^\ell}.$$

Proof. First of all, we set for all $\ell = 1, \dots, s-1$

$$\phi_\ell(t) := \sum_{k=0}^{s+\ell} t^{-s+k} \sum_{|\alpha| \le k} \left\| \partial_+^\alpha \varphi_r^*(t) \right\|_{L^2(S_t)},$$
$$y_\ell(t) := \sum_{p+m+r=s-\ell} \left\| \frac{1}{\xi^p} \nabla\!\!\!/^r \partial_\xi^m X_\ell(t) \right\|_{L^2(S_t)}.$$

To prove (5.13) we proceed by induction on ℓ; we start considering the case $\ell = 1$. Consider the equation (5.9), the hypothesis (H1) assures that

$$\psi_r^* = \circ\left(\xi^{-\frac{n-1}{2}+s}\right) \text{ as } \xi \to 0^+,$$

uniformly with respect to ω, so the Lemma 3.4 gives

$$\begin{aligned}
&\sum_{p+m+r=s-1} \left\| \frac{1}{t^p} \nabla\!\!\!/^r \partial_\xi^m X_1(t) \right\|_{L^2(S_t)} \lesssim \\
&\lesssim \sum_{p+m+r=s-1} \sum_{h=0}^{m} \int_0^t \Bigg[\xi^{-m+h-p-1} \left\| \nabla\!\!\!/^r \partial_\xi^{h+1} \psi_r^* \right\|_{L^2(S_\xi)} + \\
&+ \xi^{-m+h-p} \Bigg(\left\| \nabla\!\!\!/^{r+2} \partial_\xi^h \psi_r^* \right\|_{L^2(S_\xi)} + \left\| \nabla\!\!\!/^r \partial_\xi^h (\partial_+ \psi_r^* \partial_+ \psi_r^*) \right\|_{L^2(S_\xi)} + \\
&+ \left\| \nabla\!\!\!/^r \partial_\xi^h (\partial_+ \psi_r^* X_1) \right\|_{L^2(S_\xi)} + \left\| \nabla\!\!\!/^r \partial_\xi^h (X_1)^2 \right\|_{L^2(S_\xi)} + \\
&+ \left\| \nabla\!\!\!/^r \partial_\xi^h (\partial_+ \psi_r^* \partial_+ \psi_p^*) \right\|_{L^2(S_\xi)} + \left\| \nabla\!\!\!/^r \partial_\xi^h [\partial_+ \psi_r^* (\partial_\eta v_p)^*] \right\|_{L^2(S_\xi)} + \\
&+ \left\| \nabla\!\!\!/^r \partial_\xi^h (\partial_+ \psi_p^* X_1) \right\|_{L^2(S_\xi)} + \left\| \nabla\!\!\!/^r \partial_\xi^h [X_1 (\partial_\eta v_p)^*] \right\|_{L^2(S_\xi)} \Bigg) \Bigg] d\xi
\end{aligned} \tag{5.14}$$

where

$$\left\|\nabla^r \partial_\xi^h (\partial_+\psi_r^* \partial_+\psi_r^*)\right\|_{L^2(S_\xi)} \le \sum_{\substack{r_1+r_2=r\\ h_1+h_2=h}}^{*} \left\|\nabla^{r_1} \partial_\xi^{h_1} \partial_+\psi_r^* \, \nabla^{r_2} \partial_\xi^{h_2} \partial_+\psi_r^*\right\|_{L^2(S_\xi)}, \tag{5.15a}$$

$$\left\|\nabla^r \partial_\xi^h (\partial_+\psi_r^* X_1)\right\|_{L^2(S_\xi)} \le \sum_{\substack{r_1+r_2=r\\ h_1+h_2=h}}^{*} \left\|\nabla^{r_1} \partial_\xi^{h_1} \partial_+\psi_r^* \, \nabla^{r_2} \partial_\xi^{h_2} X_1\right\|_{L^2(S_\xi)}, \tag{5.15b}$$

$$\left\|\nabla^r \partial_\xi^h (X_1)^2\right\|_{L^2(S_\xi)} \le \sum_{\substack{r_1+r_2=r\\ h_1+h_2=h}}^{*} \left\|\nabla^{r_1} \partial_\xi^{h_1} X_1 \, \nabla^{r_2} \partial_\xi^{h_2} X_1\right\|_{L^2(S_\xi)}, \tag{5.15c}$$

$$\left\|\nabla^r \partial_\xi^h (\partial_+\psi_r^* \partial_+\psi_p^*)\right\|_{L^2(S_\xi)} \le \sum_{\substack{r_1+r_2=r\\ h_1+h_2=h}}^{*} \left\|\nabla^{r_1} \partial_\xi^{h_1} \partial_+\psi_r^* \, \nabla^{r_2} \partial_\xi^{h_2} \partial_+\psi_p^*\right\|_{L^2(S_\xi)}, \tag{5.15d}$$

$$\left\|\nabla^r \partial_\xi^h [\partial_+\psi_r^* (\partial_\eta v_p)^*]\right\|_{L^2(S_\xi)} \le \sum_{\substack{r_1+r_2=r\\ h_1+h_2=h}}^{*} \left\|\nabla^{r_1} \partial_\xi^{h_1} \partial_+\psi_r^* \, \nabla^{r_2} \partial_\xi^{h_2} (\partial_\eta v_p)^*\right\|_{L^2(S_\xi)}, \tag{5.15e}$$

$$\left\|\nabla^r \partial_\xi^h (\partial_+\psi_p^* X_1)\right\|_{L^2(S_\xi)} \le \sum_{\substack{r_1+r_2=r\\ h_1+h_2=h}}^{*} \left\|\nabla^{r_1} \partial_\xi^{h_1} \partial_+\psi_p^* \, \nabla^{r_2+1} \partial_\xi^{h_2} X_1\right\|_{L^2(S_\xi)}, \tag{5.15f}$$

$$\left\|\nabla^r \partial_\xi^h [X_1 (\partial_\eta v_p)^*]\right\|_{L^2(S_\xi)} \le \sum_{\substack{r_1+r_2=r\\ h_1+h_2=h}}^{*} \left\|\nabla^{r_1} \partial_\xi^{h_1} (\partial_\eta v_p)^* \, \nabla^{r_2+1} \partial_\xi^{h_2} X_1\right\|_{L^2(S_\xi)}. \tag{5.15g}$$

Let's consider just one of the previous terms, since the other ones can be estimated applying exactly the same method. Consider (5.15b) assuming that $\partial_+ = \partial_\xi$; we retrace the step in the proof of Theorem 1.1. We set

$$\alpha_1 = s - r_1 - h_1 - 1 - m + h > 0,$$
$$\alpha_2 = s - r_2 - h_2 - 1 - p > 0,$$

then, only one of the following cases holds:

(a) $\alpha_1 > \frac{n-1}{2}$;

(b) $\alpha_2 > \frac{n-1}{2}$;

(c) $\alpha_1, \alpha_2 \le \frac{n-1}{2}$.

If (a) holds, the Theorem 1.5 gives

$$
\begin{aligned}
&\xi^{-m+h-p}\left\|\nabla\!\!\!/^{r_1}\partial_\xi^{h_1+1}\psi_r^*\,\nabla\!\!\!/^{r_2}\partial_\xi^{h_2}X_1\right\|_{L^2(S_\xi)}\lesssim\\
&\quad\lesssim\xi^{-m+h-p}\left\|\nabla\!\!\!/^{r_1}\partial_\xi^{h_1+1}\psi_r^*\right\|_{L^\infty(S_\xi)}\left\|\nabla\!\!\!/^{r_2}\partial_\xi^{h_2}X_1\right\|_{L^2(S_\xi)}\lesssim\\
&\lesssim\xi^{-\frac{n-1}{2}-m+h}\left\|\partial_\xi^{h_1+1}\psi_r^*\right\|_{H^{s-1-h_1-m+h}(S_\xi)}\xi^{-p}\left\|\partial_\xi^{h_2}X_1\right\|_{H^{s-1-h_2-p}(S_\xi)}\lesssim\\
&\qquad\lesssim\xi^{-\frac{n-1}{2}-m+h}\left\|\partial_+^{s-m+h}\psi_r^*\right\|_{L^2(S_\xi)}\xi^{-p}\left\|\partial_+^{s-1-p}X_1\right\|_{L^2(S_\xi)}
\end{aligned}
$$

hence

$$
\begin{aligned}
&\sum_{p+m+r=s-1}\sum_{h=0}^{m}\xi^{-m+h-p}\left\|\nabla\!\!\!/^{r}\partial_\xi^{h}(\partial_\xi\psi_r^*X_1)\right\|_{L^2(S_\xi)}\lesssim\\
&\quad\lesssim\xi^{-\frac{n-1}{2}}\sum_{p+m+r=s-1}\left(\sum_{h=0}^{m}\xi^{-m+h}\left\|\partial_+^{s-m+h}\psi_r^*\right\|_{L^2(S_\xi)}\right)\cdot\\
&\qquad\cdot\left(\xi^{-p}\left\|\partial_+^{s-1-p}X_1\right\|_{L^2(S_\xi)}\right)\overset{k=s-m+h}{\lesssim}\\
&\qquad\lesssim\xi^{-\frac{n-1}{2}}\left(\sum_{k=1}^{s}\xi^{-s+k}\left\|\partial_+^{k}\psi_r^*(\xi)\right\|_{L^2(S_\xi)}\right)\cdot\\
&\qquad\cdot\left(\sum_{p+m+r=s-1}\left\|\frac{1}{\xi^p}\nabla\!\!\!/^{r}\partial_\xi^{m}X_1\right\|_{L^2(S_\xi)}\right)\lesssim\\
&\qquad\qquad\lesssim\xi^{-\frac{n-1}{2}}\phi_0(\xi)y_1(\xi).
\end{aligned}
\tag{5.16}
$$

The case (b) is analogous.

Assume now that (c), then

$$
\frac{n-1-2\alpha_1}{n-1}+\frac{n-1-2\alpha_2}{n-1}=\frac{2(n-1)-2(\alpha_1+\alpha_2)}{n-1}\le 2-\frac{2s}{n-1}<1\,,
$$

in fact

$$
\begin{aligned}
\alpha_1+\alpha_2&=2(s-1)-(m+r_1+r_2+h_1+h_2+p)+h=\\
&=2(s-1)-(r+m+p)=s-1>\frac{n-1}{2}.
\end{aligned}
$$

Hence there exist positive numbers q, q', with $\frac{1}{q}+\frac{1}{q'}=1$, such that

$$
2\le 2q<\frac{n-1}{n-1-2\alpha_1}\,,\quad 2\le 2q'<\frac{n-1}{n-1-2\alpha_2}\,,
$$

thus by *Generalized Hölder's Inequality* and Theorem 1.4 we have

$$\xi^{-m+h-p}\left\|\nabla^{r_1}\partial_\xi^{h_1+1}\psi_r^*\,\nabla^{r_2}\partial_\xi^{h_2}X_1\right\|_{L^2(S_\xi)}\lesssim$$

$$\lesssim \xi^{-m+h-p}\left\|\nabla^{r_1}\partial_\xi^{h_1+1}\psi_r^*\right\|_{L^{2q}(S_\xi)}\left\|\nabla^{r_2}\partial_\xi^{h_2}X_1\right\|_{L^{2q'}(S_\xi)}\lesssim$$

$$\lesssim \xi^{-\frac{n-1}{2}+\frac{n-1}{2q}-m+h}\left\|\partial_\xi^{h_1+1}\psi_r^*\right\|_{H^{s-1-h_1-m+h}(S_\xi)}\cdot$$

$$\cdot\,\xi^{\frac{n-1}{2q'}-p}\left\|\partial_\xi^{h_2}X_1\right\|_{H^{s-1-h_2-p}(S_\xi)},$$

which immediately leads to (5.16).

An analogous trick applied to (5.15a), (5.15c), (5.15d), (5.15e), (5.15f) and (5.15g) gives, respectively

$$\xi^{-m+h-p}\left\|\nabla^{r}\partial_\xi^{h}(\partial_+\psi_r^*\partial_+\psi_r^*)\right\|_{L^2(S_\xi)}\lesssim$$

$$\lesssim \xi^{-\frac{n-1}{2}}\left(\sum_{k=1}^{s}\xi^{-s+k}\left\|\partial_+^k\psi_r^*(\xi)\right\|_{L^2(S_\xi)}\right)^2\lesssim \xi^{-\frac{n-1}{2}}\phi_0^2(\xi)$$

$$\xi^{-m+h-p}\left\|\nabla^{r}\partial_\xi^{h}(X_1)^2\right\|_{L^2(S_\xi)}\lesssim$$

$$\lesssim \xi^{-\frac{n-1}{2}}\left(\sum_{p+m+r=s-1}\left\|\frac{1}{\xi^p}\nabla^{r}\partial_\xi^{m}X_1\right\|_{L^2(S_\xi)}\right)^2\lesssim \xi^{-\frac{n-1}{2}}y_1^2(\xi)$$

$$\xi^{-m+h-p}\left\|\nabla^{r}\partial_\xi^{h}(\partial_+\psi_r^*\partial_+\psi_p^*)\right\|_{L^2(S_\xi)}\lesssim$$

$$\lesssim\left(\sum_{|\alpha|\le\mu}\left\|\partial_+^\alpha\psi_p^*(\xi)\right\|_{L^\infty(S_\xi)}\right)\left(\sum_{k=1}^{s}\xi^{-s+k}\left\|\partial_+^k\psi_r^*(\xi)\right\|_{L^2(S_\xi)}\right)\lesssim U_p\phi_0(\xi).$$

$$\xi^{-m+h-p}\left\|\nabla^{r}\partial_\xi^{h}[\partial_+\psi_r^*(\partial_\eta v_p)^*]\right\|_{L^2(S_\xi)}\lesssim$$

$$\lesssim\left(\sum_{|\alpha|\le\mu}\left\|\partial^\alpha\overline{u}_p^*(\xi)\right\|_{L^\infty(S_\xi)}\right)\left(\sum_{k=1}^{s}\xi^{-s+k}\left\|\partial_+^k\psi_r^*(\xi)\right\|_{L^2(S_\xi)}\right)\lesssim U_p\phi_0(\xi)$$

$$\xi^{-m+h-p}\left\|\nabla^{r}\partial_\xi^{h}(\partial_+\psi_p^*X_1)\right\|_{L^2(S_\xi)}\lesssim$$

$$\lesssim\left(\sum_{|\alpha|\le\mu}\left\|\partial_+^\alpha\psi_p^*(\xi)\right\|_{L^\infty(S_\xi)}\right)\left(\sum_{p+m+r=s-1}\left\|\frac{1}{\xi^p}\nabla^{r}\partial_\xi^{m}X_1\right\|_{L^2(S_\xi)}\right)$$

$$\lesssim U_p y_1(\xi)$$

$$\xi^{-m+h-p}\left\|\nabla\!\!\!/^r\partial_\xi^h[X_1(\partial_\eta v_p)^*]\right\|_{L^2(S_\xi)}\lesssim$$
$$\lesssim\left(\sum_{|\alpha|\le\mu}\left\|\partial^\alpha\overline{u}_p^*(\xi)\right\|_{L^\infty(S_\xi)}\right)\left(\sum_{p+m+r=s-1}\left\|\frac{1}{\xi^p}\nabla\!\!\!/^r\partial_\xi^m X_1\right\|_{L^2(S_\xi)}\right)$$
$$\lesssim U_p y_1(\xi)$$

Plugging these results back to (5.14) we have

$$y_1(t)\lesssim\phi_1(t)+\int_0^t\Big(\xi^{-\frac{n-1}{2}}\phi_0^2(\xi)+\xi^{-\frac{n-1}{2}}\phi_0(\xi)y_1(\xi)+$$
$$+\,\xi^{-\frac{n-1}{2}}y_1^2(\xi)+U_p\phi_0(\xi)+U_py_1(\xi)\Big)d\xi\lesssim$$
$$\lesssim\phi_1(t)+T(\Phi_0+U_p)y_1(t)+T(\Phi_0+U_p)\phi_0(t)+\int_0^t\xi^{-\frac{n-1}{2}}y_1^2(\xi)d\xi,$$

since (H3) holds, the term $T(\Phi_0+U_p)y_1(t)$ can be absorbed by the term on the left-hand side of the previous expression, so

$$y_1(t)\lesssim\phi_1(t)+\phi_0(t)+\int_0^t\xi^{-\frac{n-1}{2}}y_1^2(\xi)d\xi\lesssim\phi_1(t)+\int_0^t\xi^{-\frac{n-1}{2}}y_1^2(\xi)d\xi.$$

Multiplying the previous inequality by $t^{-\frac{n}{2}}$, we have

$$t^{-\frac{n-1}{2}}y_1(t)\lesssim t^{-\frac{n-1}{2}}\phi_1(t)+t^{-\frac{n-1}{2}}\int_0^t\xi^{-\frac{n-1}{2}}y_1^2(\xi)d\xi\lesssim$$
$$\lesssim t^{-\frac{n-1}{2}}\phi_1(t)+\int_0^t[\xi^{-\frac{n-1}{2}}y_1(\xi)]^2d\xi,\quad(5.17)$$

where the constant in front of the right-hand side term of the previous inequality is given by

$$\frac{C_1(n,s,Q)}{[1-T(\Phi_0+U_p)]}.$$

Applying Corollary A.1 to (5.17), we get

$$t^{-\frac{n-1}{2}}y_1(t)\le\frac{t^{-\frac{n-1}{2}}\phi_1(t)}{1-Ct^{-\frac{n-1}{2}+1}\phi_1(t)},\quad\forall t\in[0,T]\,;$$

then taking the sup with respect to $t\in[0,T]$ we have

$$Y_1\le\frac{\Phi_1}{1-CT\Phi_1}.$$

At last, by virtue of hypothesis (H3), we proved that (5.13) holds for $\ell=1$.

Assume now that (5.13) holds for any $k \leq \ell$; then we derive the equation (5.8a) ℓ times with respect to the variable η and we set $\eta = 0$

$$\partial_\xi X_{\ell+1} + \frac{n-1}{2\xi} X_{\ell+1} \simeq \xi^{-(\ell+1)} \partial_\xi \psi + \sum_{k=1}^{\ell} \xi^{-\ell-2+k} X_k + \sum_{k=0}^{\ell} \xi^{-\ell+k} \nabla\!\!\!/^2 X_k +$$
$$+ \sum_{k=0}^{\ell} \xi^{-\ell+k} \sum_{k_1+k_2=k}^{*} [\partial_+ X_{k_1} \partial_+ X_{k_2} + X_{k_1+1} \partial_+ X_{k_2} + X_{k_1+1} X_{k_2+1} +$$
$$+ \partial_+ X_{k_1} \partial_+ (\partial_\eta^{k_2} v_p)^* + \partial_+ X_{k_1} (\partial_\eta^{k_2+1} v_p)^* +$$
$$+ X_{k_1+1} \partial_+ (\partial_\eta^{k_2} v_p)^* + X_{k_1+1} (\partial_\eta^{k_2+1} v_p)^*] \,.$$

Lemma 3.5 assures that, for all $\ell = 1, \ldots s-1$, (3.25) holds; thus the Lemma 3.4 gives

$$\sum_{p+m+r=s-(\ell+1)} \left\| \frac{1}{t^p} \nabla\!\!\!/^r \partial_\xi^m X_{\ell+1}(t) \right\|_{L^2(S_t)} \lesssim \phi_{\ell+1}(t) +$$
$$+ \sum_{p+m+r=s-(\ell+1)} \sum_{k=0}^{\ell} \sum_{h=0}^{m} \int_0^t \xi^{-m-\ell+h+k-p} \sum_{k_1+k_2=k}^{*} \left(\left\| \nabla\!\!\!/^r \partial_\xi^h (\partial_+ X_{k_1} \partial_+ X_{k_2}) \right\|_{L^2(S_\xi)} + \right.$$
$$+ \left\| \nabla\!\!\!/^r \partial_\xi^h (X_{k_2} + X_{k_1+1} \partial_+ X_{k_2}) \right\|_{L^2(S_\xi)} + \left\| \nabla\!\!\!/^r \partial_\xi^h (X_{k_1+1} X_{k_2+1}) \right\|_{L^2(S_\xi)} +$$
$$+ \left\| \nabla\!\!\!/^r \partial_\xi^h [\partial_+ X_{k_1} \partial_+ (\partial_\eta^{k_2} v_p)^*] \right\|_{L^2(S_\xi)} + \left\| \nabla\!\!\!/^r \partial_\xi^h [\partial_+ X_{k_1} (\partial_\eta^{k_2+1} v_p)^*] \right\|_{L^2(S_\xi)} +$$
$$\left. + \left\| \nabla\!\!\!/^r \partial_\xi^h [X_{k_1+1} \partial_+ (\partial_\eta^{k_2} v_p)^*] \right\|_{L^2(S_\xi)} + \left\| \nabla\!\!\!/^r \partial_\xi^h [X_{k_1+1} (\partial_\eta^{k_2+1} v_p)^*] \right\|_{L^2(S_\xi)} \right) d\xi \,. \tag{5.18}$$

Proceeding as before, we get

$$y_{\ell+1}(t) \lesssim \phi_{\ell+1}(t) + \int_0^t \left(\xi^{-\frac{n-1}{2}} y_{k_1}(\xi) y_{k_2}(\xi) + \xi^{-\frac{n-1}{2}} y_{k_1+1}(\xi) y_{k_2}(\xi) + \right.$$
$$\left. + \xi^{-\frac{n-1}{2}} y_{k_1+1}(\xi) y_{k_2+1}(\xi) + U_p y_{k_1+1}(\xi) + U_p y_{k_1}(\xi) \right) d\xi \,,$$

that gives

$$Y_{\ell+1} \lesssim \Phi_{\ell+1} + T \sum_{k=0}^{\ell} \sum_{k_1+k_2=k}^{*} (Y_{k_1} Y_{k_2} + Y_{k_1+1} Y_{k_2} +$$
$$+ Y_{k_1+1} Y_{k_2+1} + U_p Y_{k_1+1} + U_p Y_{k_1})$$

and, by the induction step and the hypotheses (H2) and (H3),

$$\begin{aligned} Y_{\ell+1} \lesssim \Phi_{\ell+1} + T \sum_{k=0}^{\ell} \sum_{k_1+k_2=k}^{*} \Phi_{k_1}\Phi_{k_2} + U_p\Phi_{k_1} \\ + T \sum_{k=0}^{\ell-1} \sum_{k_1+k_2=k}^{*} (\Phi_{k_1+1}\Phi_{k_2} + U_p\Phi_{k_1+1} + \Phi_{k_1+1}\Phi_{k_2+1}) + \\ + T \sum_{\substack{k_1+k_2=\ell \\ k_i\neq 0}}^{*} (\Phi_{k_1+1}\Phi_{k_2} + U_p\Phi_{k_1+1} + \Phi_{k_1+1}\Phi_{k_2+1})) + \\ + T(\Phi_0 + \Phi_1 + U_p)Y_{\ell+1} \lesssim \Phi_{\ell+1} + T(\Phi_1 + U_p)Y_{\ell+1} \end{aligned}$$

which gives immediately (5.13) for $\ell + 1$. In addition, the constant in front of the right-hand side of the previous inequality is given by

$$\frac{C_{\ell+1}(n, s, Q)}{[1 - T(\Phi_1 + U_p)]^{\ell+1}} .$$

□

5.2.2 Non-linear Reduced Energy Estimate

Finally, we are able to state the following *Non-linear Reduced Energy Estimates*, where we express in terms of the initial datum φ the norm of the solution u and its derivatives on the hypersurface $\partial\mathcal{C}$.

Theorem 5.1 (Non-linear Reduced Energy Estimates)**.** *Assume $s > \frac{n+1}{2}$ a positive integer. Let for the datum φ_r hold the hypotheses es (H1), (H2) and (H3) stated in Lemma 5.1. If $u \in X_T^s$ is a solution of the Nonlinear Characteristic Problem* (5.6) *then u fulfills the following* Nonlinear Reduced Energy Estimate

$$\|u\|_{X_T^s} \lesssim \|\varphi_r\|_{D_T^s} + T\,\|Q(\partial u, \partial u)\|_{X_T^{s-1}} + T\,\|Q(\partial \overline{u}_p, \partial u)\|_{X_T^{s-1}} . \quad (5.19)$$

Proof. Multiplying the non-reduced estimate (3.11) for the Nonlinear Characteristic Problem (5.6) by $t^{-\frac{n}{2}}$, applying the sup with respect to $t \in [0, T]$ and Remark 3.4, we have

$$\begin{aligned} \sup_{[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|=s} \|\partial^\alpha u(t)\|_{L^2(B_t)} \lesssim \sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=s-1} \|\partial_+\partial^\alpha u(t)\|_{L^2(S_t)} + \\ + T \sup_{[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|=s-1} \|\partial^\alpha Q(\partial u, \partial u)(t)\|_{L^2(B_t)} + \\ + T \sup_{[0,T]} t^{-\frac{n}{2}} \sum_{|\alpha|=s-1} \|\partial^\alpha Q(\partial u, \partial \overline{u}_p)(t)\|_{L^2(B_t)} , \quad (5.20) \end{aligned}$$

where the first right-hand term can be rewritten as

$$\sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=s-1} \|\partial_+ \partial^\alpha u(t)\|_{L^2(S_t)} \lesssim \sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=s} \left\|\partial_+^\alpha \varphi_r(t)\right\|_{L^2(S_t)} + \\ + \sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{\substack{p+r+m+\ell+1=s \\ p+m\geq 1}} \left\| \frac{1}{t^p} \nabla\!\!\!/^r \partial_\xi^m X_{\ell+1}(t) \right\|_{L^2(S_t)} . \tag{5.21}$$

Using the Lemma 5.1 we can estimate the second term on the right-hand side of (5.21), as follows

$$\sum_{\substack{p+r+m+\ell+1=s \\ p+m\geq 1}} \left\| \frac{1}{t^p} \nabla\!\!\!/^r \partial_\xi^m X_{\ell+1}(t) \right\|_{L^2(S_t)} \lesssim \sum_{\ell=1}^{s-1} Y_\ell \lesssim \|\varphi_r\|_{D_T^s} ;$$

so (5.21) becomes

$$\sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=s-1} \|\partial_+ \partial^\alpha u(t)\|_{L^2(S_t)} \lesssim \|\varphi_r\|_{D_T^s} .$$

Retracing exactly the same steps, we have the same estimate for a generic derivative order $1 < \sigma < s$

$$\sup_{[0,T]} t^{-\frac{n-1}{2}} \sum_{|\alpha|=\sigma-1} \|\partial_+ \partial^\alpha u(t)\|_{L^2(S_t)} \lesssim \|\varphi\|_{D_T^s}$$

summing over all derivative orders $1 < \sigma \leq s$ we obtain the thesis. □

We are now able to extend the previous result to the problem (5.5), which contains also the non-homogeneous polynomial term f_p. The contribute to the estimate of this additional term will be exactly equal to the one in the Linear Reduced Energy Estimate (3.29b).

Corollary 5.1 (Non-linear Reduced Energy Estimate). *Assume $s > \frac{n+1}{2}$ a positive integer. Let for the datum hold φ_r hold the hypotheses (H1), (H2) and (H3) stated in Lemma 5.1. If $u \in X_T^s$ is a solution of the Nonlinear Characteristic Problem* (5.5) *then u fulfills the following* Nonlinear Reduced Energy Estimate

$$\|u\|_{X_T^s} \lesssim \|\varphi_r\|_{D_T^s} + T\, \|f_p\|_{Y_T^{s-1}} + T\, \|Q(\partial u, \partial u)\|_{X_T^{s-1}} + T\, \|Q(\partial \overline{u}_p, \partial u)\|_{X_T^{s-1}} . \tag{5.22}$$

Remark 5.2. the constant in front of the right-hand side of the inequality (5.22) is given by

$$C(n, s, T, \varphi, Q) = \frac{C(n, s, Q)(1+T)}{[1 - T(\Phi_1 + U_p)]^s} .$$

5.2.3 Existence and Uniqueness of the solution for the Non-linear problem

Theorem 5.2. *Assume $s > \frac{n}{2} + 1$ a positive integer and $\varphi_p \in \mathcal{P}_\mu$, where $\mu \geq s$. Let for the datum φ_r hold the following hypotheses*

(H1) $\varphi_r \in D^s_T$;

(H2) $T\Phi_{[\frac{s}{2}]} < 1$;

(H3) $T(\Phi_1 + U_p) \ll 1$, *where*

$$U_p := \sup_{\xi \in [0,T]} \sum_{|\alpha| \leq \mu} \left\| \partial^\alpha \overline{u}^*_p(\xi) \right\|_{L^\infty(S_\xi)};$$

(H4) $T(\|\varphi_r\|_{D^s_T} + T\|f_p\|_{Y^{s-1}_T}) \leq \dfrac{1 - 2TU_p}{4C^2}$, *where the constant C is given by*

$$C(n, s, T, \varphi, Q) = \frac{C_1(n, s, Q)(1 + T)}{[1 - T(\Phi_1 + U_p)]^s}.$$

Then exists $0 < \overline{T} < T$ such that the Nonlinear Characteristic Problem (5.5) *admits a unique solution $\overline{u}_r \in X^s_{\overline{T}}$, then u fulfills the following* Energy Estimate

$$\|u\|_{X^s_{\overline{T}}} \lesssim \|\varphi_r\|_{D^s_{\overline{T}}} + T\|f_p\|_{Y^{s-1}_{\overline{T}}}. \tag{5.23}$$

Proof. Let $R > 0$ be a positive real number that will be defined later. For all $u \in X^s_{T^*}$ such that $\|u\|_{X^s_T} \leq R$ we define a function F as follows

$$F(u) = v$$

where v is the unique solution of the problem

$$\begin{cases} \Box v = Q(\partial u, \partial u) + 2Q(\partial u, \partial \overline{u}_p) + f_p \\ v|_{\partial C} = \varphi_r|_{\partial C} \end{cases} \quad t \geq 0,\ x \in \mathbb{R}^n. \tag{5.24}$$

In view of Theorem 5.1, for v the following *Nonlinear Reduced Energy Estimate* holds

$$\|v\|_{X^s_T} \lesssim \|\varphi_r\|_{D^s_T} + T\|f_p\|_{Y^{s-1}_T} + T\|\partial u \cdot \partial u\|_{X^{s-1}_T} + T\|\partial \overline{u}_p \cdot \partial u\|_{X^{s-1}_T}. \tag{5.25}$$

Let's recall that the space X^σ_T forms an algebra under pointwise multiplication provided $\sigma > \frac{n}{2}$, see Lemma 1.1. So, assumed $s - 1 > \frac{n}{2}$, we have

$$\|v\|_{X^s_T} \lesssim \|\varphi_r\|_{D^s_T} + T\|f_p\|_{Y^{s-1}_T} + T\|\partial u\|^2_{X^{s-1}_T} + TU_p\|\partial u\|_{X^{s-1}_T} \lesssim$$
$$\lesssim \|\varphi_r\|_{D^s_T} + T\left(\|f_p\|_{Y^{s-1}_T} + \|u\|^2_{X^s_T} + U_p\|u\|_{X^s_T}\right).$$

We are going to prove that $\exists R > 0$ such that $\|u\|_{X_T^s} \leq R$ implies $\|v\|_{X_T^s} \leq R$. The previous estimate gives

$$\|v\|_{X_T^s} \leq C\left[\|\varphi_r\|_{D_T^s} + T\|f_p\|_{Y_T^{s-1}} + TR^2 + TU_pR\right]$$

so $\|v\|_{X_T^s} \leq R$ holds when $R > 0$ fulfills the following inequality

$$CD + CTR^2 + TU_pR \leq R, \tag{5.26}$$

where $D := \|\varphi_r\|_{D_T^s} + T\|f_p\|_{Y_T^{s-1}}$. It is easy to see[1] that such a $R > 0$ exists when

$$TD \leq \frac{1 - 2TU_p}{4C^2}$$

that is exactly hypothesis (H4).

Set $\mathcal{R} = \left\{u \in X_T^s \text{ s.t. } \|u\|_{X_T^s} \leq R\right\}$, we are going to show that the application $F : \mathcal{R} \to \mathcal{R}$ is a contraction in $\mathcal{R}$, which means that for all $u_1, u_2 \in \mathcal{R}$, $\exists \alpha \in (0,1)$ such that

$$\|F(u_1) - F(u_2)\|_{X_T^s} \leq \alpha \|u_1 - u_2\|_{X_T^s}.$$

It is easy to see that the function $v_1 - v_2$ solves the problem

$$\begin{cases} \Box(v_1 - v_2) = Q(\partial(u_1 - u_2), \partial(u_1 + u_2)) + 2Q(\partial\overline{u}_p, \partial(u_1 - u_2)) \\ (v_1 - v_2)|_{\partial C} = 0 \end{cases}, \tag{5.27}$$

where

$$|Q(\partial(u_1 - u_2), \partial(u_1 + u_2))| \leq C(n,Q)|\partial(u_1 - u_2)||\partial(u_1 + u_2)|.$$

In view of Theorem 5.1, for $v_1 - v_2$ the following *Non-linear Reduced Energy Estimate* holds

$$\begin{aligned} \|v_1 - v_2\|_{X_T^s} &\lesssim TC(n,Q)\,\||\partial(u_1 - u_2)||\partial(u_1 + u_2)|\|_{X_T^{s-1}} + \\ &\quad + 2TU_p\|\partial(u_1 - u_2)\|_{X_T^{s-1}} \lesssim \\ &\lesssim T\|u_1 - u_2\|_{X_T^s}\|u_1 + u_2\|_{X_T^s} + TU_p\|u_1 - u_2\|_{X_T^s} \lesssim \\ &\lesssim T\|u_1 - u_2\|_{X_T^s}(\|u_1\|_{X_T^s} + \|u_2\|_{X_T^s}) + TU_p\|u_1 - u_2\|_{X_T^s} \lesssim \\ &\lesssim T(2R + U_p)\|u_1 - u_2\|_{X_T^s}. \end{aligned} \tag{5.28}$$

[1]Note that the conditions

$$\begin{cases} CD \leq \frac{R}{2} \\ CTR^2 + TU_pR \leq \frac{R}{2} \\ R > 0 \end{cases} \Rightarrow \begin{cases} 2CD \leq R \\ R \leq \frac{1}{2CT} - \frac{U_p}{C} \\ R > 0 \end{cases} \Rightarrow 2CD \leq \frac{1}{2CT} - \frac{U_p}{C}$$

imply the inequality (5.26), provided T sufficiently small.

Let $0 < \overline{T} < T$ be such that

$$\alpha = C\overline{T}(2R + U_p) < 1\,,$$

so the application F is a contraction in $X^s_{\overline{T}}$. Since $X^s_{\overline{T}}$ is a Banach space, F has a unique fixed point, in other words, exists $\overline{u}_r \in X^s_{\overline{T}}$ such that

$$\overline{u}_r = F(\overline{u}_r)$$

so $\overline{u}_r$ solves the considered problem. □

5.2.4 Existence and Uniqueness of a solution for the general semi-linear Problem

To prove the existence and the uniqueness of the solution of the semi-linear problem (5.1) in case of generic initial datum φ, we generalized the method used in the linear case.

We started re-writing φ as the sum of a polynomial and a remainder function using the Taylor expansion and in Theorem 4.4 we proved the existence and the uniqueness of the solution u_p for the semi-linear problem (4.22) with polynomial data. This solution is known just as sum of a series. In order to avoid the problem of studying the convergence of the series, we truncate the solution u_p at the degree correspondent to the vanishing order required by the Energy Estimates.

When we removed the polynomial part from the problem (5.1) in terms of the truncation $\overline{u}_p$, we got the modified Characteristic Initial Value Problem (5.5) with an initial datum which has the required vanishing order at the origin. Due to the non-linearity, the "regular" problem (5.1) involves also the truncated solution $\overline{u}_p$ of the polynomial problem.

At last, by applying the classical fixed-point method, we solved the semi-linear problem (5.1) in Theorem 5.2, provided the the life-time $\overline{T}$ of the solution sufficiently small. Moreover $\overline{T}$ is proportional to the reciprocal of the "size" of the initial datum φ in the sense of the norms introduced in Theorem 5.2. In particular, the solution of the semi-linear problem (5.1) is given by

$$u = \overline{u}_p + \overline{u}_r\,.$$

We have just proved the following Theorem.

Theorem 5.3. *Assume $\varphi \in \mathcal{D}$ and $s > \frac{n}{2}+1$. In the hypotheses of Theorem 5.2, there exists $T > 0$ such that the Semi-linear Characteristic Problem* (5.1) *admits a unique solution $u \in \mathcal{X}$.*

Appendix A

Gronwall's Inequalities

A.1 Gronwall's Inequalites

Lemma A.1 (Gronwall's Inequality)**.** *If* y*,* f *and* g *are* $\mathscr{C}([0,T])$ *nonnegative functions,* g *is increasing and*

$$[y(t)]^2 \leq g(t) + \int_0^t f(\tau)y(\tau)\,d\tau\,, \quad \forall t \in [0,T]\,,$$

then

$$y(t) \leq \sqrt{g(t)} + \frac{1}{2}\int_0^t f(\tau)\,d\tau, \quad \forall t \in [0,T]\,. \tag{A.1}$$

Proof. It is enough to prove (A.1) when $t = T$ and g can be replaced by $g(T)$ then, so we may assume that g is constant. Writing

$$F(t) = g + \int_0^t f(\tau)y(\tau)\,d\tau$$

we have

$$F'(t) = f(t)y(t) \leq f(t)\sqrt{F(t)}\,,$$

since $[y(t)]^2 \leq F(t)$, hence

$$\frac{F'(t)}{\sqrt{F(t)}} \leq f(t)\,, \quad \forall t \in [0,T]\,.$$

Integrating the previous inequality on $[0,T]$, we get

$$\sqrt{F(T)} \leq \sqrt{F(0)} + \frac{1}{2}\int_0^T f(\tau)\,d\tau,$$

this result together with $y(t) \leq \sqrt{F(t)}$, allows us to conclude that

$$y(T) \leq \sqrt{g} + \frac{1}{2}\int_0^T f(\tau)\,d\tau.$$

□

Lemma A.2 (Generalized Gronwall's Inequality). *If y, f and g are $\mathscr{C}([0,T])$ nonnegative functions, $c \geq 0$ and*

$$[y(t)]^2 \leq c + 2\int_0^t f(\tau)y(\tau)\,d\tau + 2\int_0^t g(\tau)[y(\tau)]^2\,d\tau\,, \quad \forall t \in [0,T]\,,$$

then

$$y(t) \leq \sqrt{c}\left[\mathrm{e}^{\int_0^t g(\tau)\,d\tau}\right] + \int_0^t \mathrm{e}^{\int_\tau^t g(\tau')\,d\tau'} f(\tau)\,d\tau, \quad \forall t \in [0,T]\,. \tag{A.2}$$

Proof. Let Y be the solution of the following integral equation

$$Y(t) = c + 2\int_0^t f(\tau)\sqrt{Y(\tau)}\,d\tau + 2\int_0^t g(\tau)Y(\tau)\,d\tau\,, \quad \forall t \in [0,T]\,,$$

which is equivalent to the Cauchy problem

$$\begin{cases} \frac{d}{dt}\left[\sqrt{Y(t)}\right] = f(t) + g(t)\sqrt{Y(t)} \\ \sqrt{Y(0)} = \sqrt{c} \end{cases}, \quad t \geq 0. \tag{A.3}$$

The solution of (A.3) is given by

$$\sqrt{Y(t)} = \sqrt{c}\left[\mathrm{e}^{\int_0^t g(\tau)\,d\tau}\right] + \int_0^t \mathrm{e}^{\int_\tau^t g(\tau')\,d\tau'} f(\tau)\,d\tau;$$

it can be easily proved that $y(t) \leq Y(t)$ for all $t \geq 0$, this result, together with the previous one, gives immediately (A.2). □

At last, by applying a Gronwall-like method, we can prove the following result.

Corollary A.1. *If y and g are $\mathscr{C}([0,T])$ nonnegative functions, g is increasing and*

$$y(t) \leq g(t) + \int_0^t y^2(\tau)\,d\tau\,, \quad \forall t \in [0,T]\,,$$

then

$$y(t) \leq \frac{g(t)}{1 - tg(t)}, \quad \forall t \in [0,T]\,. \tag{A.4}$$

Proof. It is enough to prove (A.4) when $t = T$ and g can be replaced by $g(T)$ then, so we may assume that g is constant. Writing

$$F(t) = g + \int_0^t y(\tau)^2\,d\tau$$

we have

$$F'(t) = y^2(t)\,,$$

since $y(t) \leq F(t)$, hence

$$\frac{F'(t)}{F(t)^2} \leq 1\,, \quad \forall t \in [0,T]\,.$$

Integrating the previous inequality on $[0,T]$, we get

$$\frac{1}{F(0)} - \frac{1}{F(T)} \leq T,$$

this result, together with $y(t) \leq F(t)$, allows us to conclude that

$$y(T) \leq \frac{g}{1-Tg}.$$

□

Appendix B

Sobolev Spaces

B.1 Sobolev Spaces

Definition B.1 (Cone Property)**.** A domain Ω has the *Cone Property* if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.

Theorem B.1 (The Sobolev Imbedding Theorem)**.** *Let Ω be a domain in $\mathbb{R}^n$ having the cone property. Let j and m be nonnegative integers and let p satisfy $1 \leq p < \infty$ and $mp > n$, then*

$$W_p^{j+m}(\Omega) \hookrightarrow W_q^j(\Omega)$$

provided $p \leq q \leq \frac{np}{n-mp}$.

Lemma B.1. *Let Ω be a domain in $\mathbb{R}^n$ having the cone property. If $mp > n$, then $W_p^m(\Omega) \hookrightarrow \mathscr{C}(\Omega) \cap L^\infty(\Omega)$, the imbedding constant depending only on n, m, p and the finite cone determining the cone property for Ω.*

Theorem B.2 (Trace Theorem for $W_p^m(\Omega)$ Spaces)**.** *Let Ω be a domain in $\mathbb{R}^n$ having the* uniform $\mathscr{C}^m$-regularity *and a bounded domain. Let m be a nonnegative integer and let p satisfy $1 \leq p < n$. If $u \in W_p^m(\Omega)$ then the* trace $v := u|_{\partial\Omega}$ *belongs to $W_p^{m-\frac{1}{p}}(\partial\Omega)$, moreover*

$$\|v\|_{W_p^{m-\frac{1}{p}}(\partial\Omega)} \lesssim \|u\|_{W_p^m(\Omega)}.$$

Conversely, if $v \in W_p^{m-\frac{1}{p}}(\partial\Omega)$ then there exists $u \in W_p^m(\Omega)$ with $u|_{\partial\Omega} = v$ and

$$\|u\|_{W_p^m(\Omega)} \lesssim \|v\|_{W_p^{m-\frac{1}{p}}(\partial\Omega)}.$$

Theorem B.3 ($W_p^m(\Omega)$ as a Banach Algebra)**.** *Let Ω be a domain in $\mathbb{R}^n$ having the cone property. If $mp > n$, then there exists a constant C depending on n, m, p and the finite cone determining the cone property for Ω,* *such that for all* $u, v \in W_p^m(\Omega)$ *the product* uv*, defined pointwise a.e. in* Ω*, belongs to* $W_p^m(\Omega)$ *and satisfies*

$$\|uv\|_{W_p^m(\Omega)} \leq C \|u\|_{W_p^m(\Omega)} \|v\|_{W_p^m(\Omega)} .$$

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