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Microscopic and Kinetic Models in Financial Markets

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O s'io fossi povero come un miliardario... Che cos'¿ il denaro per l'anima? Un ladro insaziabile s'annida in essa: all'orda sfrenata di tutti i miei desideri non basta l'oro di tutte le Californie!

Contents

Introduction

The recent events of the world's financial crisis and its uncontrolled propagation across the global economic system, have opened the door to a deep rethink of some basic paradigms and fundamental believes in economic modelling. Already some efforts has been put in the understanding of stock price dynamics, and also in the attempt to derive useful models for risk estimation, price prediction, and taxation. Nevertheless the need to find a compromise between the extraordinary complexity of the systems and the request of tractable, simplified models from which some basic information can be derived, represents a big challenge, and it is one of the main difficulties one has to deal with in the construction of suitable models.

Any reasonable model needs to rely on some fundamental hypotheses and to rest on a theoretical framework, which should be able to provide some basic and universal principles. Unfortunately, this is not an easy task when we deal with economic and financial systems. Looking at stock markets in particular, it is not obvious to understand which are the fundamental dynamics to be considered and which aspects can be neglected in order to derive the basic issues.

One of the most classical approach has been to consider the *efficient market hypothesis* [50]. It relies on the belief that securities markets are extremely efficient in reflecting information about individual stocks and about the stock market as a whole. The efficient market hypothesis is associated with the idea of a *random walk* [36] , which is widely used in the finance literature to characterize a price series where all subsequent price changes represent independent, random departures from previous prices.

Strongly linked to the market efficiency hypothesis is the assumption of *rational behavior* among the traders. Rationality of traders can be basically assumed by two main features. First, when they receive new information, agents update their beliefs by correctly evaluating the probability of the hypotheses. Second, given their beliefs, agents make choices that are completely rational, in the sense that they arise from an optimization process of opportune subjective utility functions.

By the beginning of the twenty-first century, the intellectual dominance of the efficient market hypothesis had become far less universal. Many financial economists and statisticians began to believe that stock prices are at least partially predictable. A new breed of economists emphasized psychological and behavioral elements of stock-price determination. *Behavioral economics* and its related area of study, behavioral finance, have emerged in response to the difficulties faced by the traditional paradigm [49]. It relies in the fact that some financial phenomena can be better understood using models in which some agents are not fully rational.

A strong impact in the field of behavioral finance has been given by the introduction of the *prospect theory* by Kahneman and Tversky [75]. They present a critique of expected utility theory as a descriptive model of decision making under risk and develop an alternative model. Under prospect theory, value is assigned to gains and losses rather than to absolute wealth and probabilities are replaced by decision weights. The theory which they confirmed by experiments predicts a distinctive fourfold pattern of risk attitudes: risk aversion for gains of moderate to high probability and losses of low probability, and risk seeking for gains of low probability and losses of moderate to high probability.

Recently, *agent based modelling* methods have given an important contribute and provided a huge quantity of numerical simulations [86, 90]. The idea is to produce a big mass of artificial data and to observe how they can fit with empirical observations. This approach is now also supported by the availability of many recorded empirical data [131].The relevant part of physics that is used to build such models of financial markets consists in methods from statistical mechanics. This attempt by physicists to map out the statistical properties of financial markets considered as complex systems is usually referred to as *econophysics* [101].

The need to recover mathematical models which can display such scaling properties, but also capable to deal with systems of many interacting agents and to take into account the effects of collective endogenous dynamics, put the question on the choices of the most appropriate mathematical framework to use. The classical framework of stochastic differential equations which played a major rule in financial mathematics seems inadequate to describe the dynamics of such systems of interacting agents and their emerging collective behavior [105].

In the last years a new approach based on the use of *mean field models* and related mathematical tools has appeared in the mathematics and physics community [14, 25, 37, 38, 44, 85]. Mean field models were originally introduced in order to give a statistical description of systems with many interacting particles. Kinetic theory of rarefied gases can be thought as a paradigm of such complex systems, in which particles are described by random variables which represents their physical states, like position and velocity. A Boltzmann equation then prescribes the time evolution for the particles density probability function [29]. This seems to fit very well with the necessity to prescribe how the trading agents interacting in a stock market are leaded to form their expectations and revaluate their choices on the basis of the influence placed on the neighbor agents' behavior rather than the flux of news coming from some fundamental analysis or direct observations of the market dynamic.

Summay of the thesis

Our main goal of this PhD was to concentrate our attention on a number of stylized facts, like volatility clustering and fat tails of returns, that most speculative markets at national and international level share, for which a satisfactory explanation is still lacking in standard theories of financial markets [109]. Such stylized facts are now almost universally accepted among economists and physicists and it is now clear that financial markets dynamics give rise to some kind of universal scaling laws.

Showing similarities with scaling laws for other systems with many interacting particles, a description of financial markets as multi-agent interacting systems appeared to be a natural consequence [87, 89, 101, 131, 134]. This topic was pursued by quite a number of contributions appearing in both the physics and economics literature in recent years [14, 2, 26, 136, 59, 69, 86, 101, 112, 134]. This new research field borrows several methods and tools from classical statistical mechanics, where complex behavior arises from relatively simple rules due to the interaction of a large number of components.

Starting from the microscopic dynamics, kinetic models can be derived with the tools of classical kinetic theory of fluids [14, 37, 38, 136, 46, 69, 98, 103, 114, 125]. In contrast with microscopic dynamics, where behavior often can be studied only empirically through computer simulations, kinetic models based on PDEs allow us to derive analytically general information on the model and its asymptotic behavior. For example, the knowledge of the tails behavior for the distributions of returns is of primary importance, since it determines a posteriori whether the model can fit data of real financial markets.

The aim of this PHD thesis is to rewiew some of the more influential models of multi-agent interactive systems in financial markets and to present a new kinetic approach to the description of etherogeneous systems, where different populations of agents are involved and interact each others. In the first chapter, we present the Levy-Levy-Solomon model [86] and The Lux-Marchesi model [89] as microspic models. In the second chapter staring from the microsopic description we derive kinetics model for both Levy-Levy solomon and Lux-Marchesi models, furthermore through the introduction ok Fokker-Plank appoximation models, we are able yo illustrate some analitycal results and numerical simulations. [37, 98]. In the third chapter we present a more realistic model whic generalize the works of chapter two. For such model, starting from a mesoscopic decription an hydrodynamic model is derived and analytical and numerical results are provided.

We leave as appendix A and B full details of some technical proofs of the second chapter, in order to let it more readable. Appendix C contains a pubblication in the Esaim Proceedings where I'm co-author. It was the results of the CEMRACS summer school held in Marseille in the August 2010. Here a spatial coupling of an asymptotic preserving scheme with the asymptotic limit model, associated to a sin-

gularly perturbed, highly anisotropic, elliptic problem is investigated and compared with the numerical discretization of the initial singular perturbation model or the purely asymptotic preserving scheme.

Pubblication list

- Microsrcopic and kinetic models in financial markets. Autors: Stephane Cordier, Dario Maldarella, Lorenzo Pareschi and Crylle Piatecki

  Chapter of a Contribution Book :Matematical Modeling of Collective Behavior in Socio Economics and Life Science Birkhäuser, 2010

- Price Dynamics in Financial Markets : A Kinetic Approach. Autors : Lorenzo pareschi, Dario Maldarella
  Journal : Science and Culture Vol 76 9-10 October 2010

- Kinetic Models for socio-economics dynamicof speculative markets Autors : Lorenzo Pareschi, Dario Maldarella
  Journal : Pysica A Vol 391 February 2012

- Hybrid model for the coupling of an asymptotic preserving scheme with the asymptotic limit model:the one dimensional case Autors : Pierre Degond, Fabrice Deluzet, Dario Maldarella, Jacek Narski, Claudia Negulescu and Martin Parisot
  Journal : Esaim Proceedings, October 2011, Vol.32, p 23-30

Chapter 1

Microscopic models of financial markets

After the pioneering microscopic market models from economists like Nobel laureate Stigler [130], numerous microscopic models were published in the physics literature in the last two decades [23, 74, 86, 87, 95, 89, 90]. Here we do not aim at a comprehensive review of microscopic models in finance we refer to [87, 101, 131, 134] for a more detailed introduction to this topic. We mainly concentrate here on those models of speculative financial markets which are enough realistic to include some essential economic features like the notion of price, dividends and interest rates, and that at the same time, thanks to their particular structure, admit an interpretation as a kinetic model. These include the models of Levy-Levy-Solomon (LLS) [86, 87] and Lux-Marchesi (LM) [95, 89, 90].

From a mathematical viewpoint several of these models can be seen as generalizations of the the "Law of Proportionate Effect" introduced by Gibrat [63] according to which the expected value of the growth rate of a quantity is proportional to the current size of the quantity. It is well-known that this simple random multiplicative approach yields a lognormally distributed quantity whereas some seemingly trivial variations of the same process lead to power laws [97]. Microscopic and kinetic models that can be considered as variations to a random multiplicative process are the Cordier-Pareschi-Toscani model [38], the Bouchad-Mézard model [14] and the generalized Lotka-Volterra model by Solomon et al.[124, 92, 126]. All these models, although able to reproduce fat tails, are concerned with the distribution of wealth and the price formation dynamic is not considered.

1.1 The Levy-Levy-Solomon model

The LLS model considers a set of financial agents $i = 1, \ldots, N$ who can create their own portfolio between two alternative investments: a stock and a bond. Let us denote by w_i the wealth of agent i and by n_i the number of stocks of the agent. Additionally we use the notations S for the price of the stock and n for the total number of stocks.

1.1.1 The wealth dynamic

The essence of the dynamic is the choice of the agent's portfolio. More precisely, at each time step each agent selects which fraction of wealth to invest in bonds and which fraction in stocks. They indicate with r the (constant) interest rate of bonds. The bond is assumed to be a risk-less asset yielding a return at the end of each time period. The stock is a risky asset with overall returns rate x composed of two elements: a capital gain or loss and the distribution of dividends.

To simplify the description we omit the presence of dividends. Thus, if an agent has invested $\gamma_i w_i$ of its wealth in stocks and $(1-\gamma_i)w_i$ of its wealth in bonds, at the next time step in the dynamic he will achieve the new wealth value

$$w_i' = (1-\gamma_i)w_i(1+r) + \gamma_i w_i(1+x), \tag{1.1}$$

where the rate of return of the stock is given by

$$x = \frac{S'-S}{S}, \tag{1.2}$$

and S' is the new price of the stock.

The dynamic now is based on the agent choice of the new fraction of wealth he wants to invest in stocks at the next stage. According to the standard theory of investment each investor is characterized by a *utility function* (of its wealth) $U(w)$ that reflects the personal risk taking preference [70]. The optimal γ_i' is the one that maximizes the expected value of $U(w)$.

1.1.2 Utility function and optimal investments

Different models can be used for this (see [87, 134]), for example, maximizing a von Neumann-Morgenstern utility function with a constant risk aversion of the type

$$U(w) = \frac{w^{1-\alpha}}{1-\alpha}, \tag{1.3}$$

where α is the risk aversion parameter, or a logarithmic utility function

$$U(w) = \log(w). \tag{1.4}$$

As they don't know the future stock price S', the investors estimate the stock's next period return distribution and find an optimal mix of the stock and the bond that maximizes their expected utility $E[U]$. In practice, for any hypothetical price S^h, each investor finds the hypothetical optimal proportion $\gamma_i^h(S^h)$ which maximizes his/her expected utility evaluated at

$$w_i^h(S^h) = (1-\gamma_i^h)w_i'(1+r) + \gamma_i^h w_i'(1+x'(S^h)), \tag{1.5}$$

where $x'(S^h) = (S^h - S')/S'$ and S' is estimated in some way. For example in [87] the investors expectations for x' are based on extrapolating the past values thus originating a memory span for the trader.

Once each investor decides on the hypothetical optimal proportion of wealth γ_i^h that he/she wishes to invest in stocks, one can derive the number of stocks $n_i^h(S^h)$ he/she wishes to hold corresponding to each hypothetical stock price S^h. Since the total number of shares in the market n, is fixed there is a particular value of the price S' for which the sum of the $n_i^h(S^h)$ equals n. This value S' is the new market equilibrium price and the optimal proportion of wealth is $\gamma_i' = \gamma_i^h(S')$.

1.1.3 Market clearance and equilibrium price

More precisely, following [87], each agent formulates a *demand curve*

$$n_i^h = n_i^h(S^h) = \frac{\gamma^h(S^h)w_i^h(S^h)}{S^h}$$

characterizing the desired number of stocks as a function of the hypothetical stock price S^h. This number of share demands is a monotonically decreasing function of the hypothetical price S^h. As the total number of stocks

$$n = \sum_{i=1}^{N} n_i \tag{1.6}$$

is preserved, the new price of the stock at the next time level is given by the so-called *market clearance condition.* Thus the new stock price S' is the unique price at which the total demand equals the supply

$$\sum_{i=1}^{N} n_i^h(S') = n. \tag{1.7}$$

This will fix the value w' in (1.1) and the model can be advanced to the next time level. To make the model more realistic, typically a source of stochastic noise, which characterizes all factors causing the investor to deviate from his/her optimal portfolio, is introduced in the proportion of investments γ_i and in the rate of return of the stock x'.

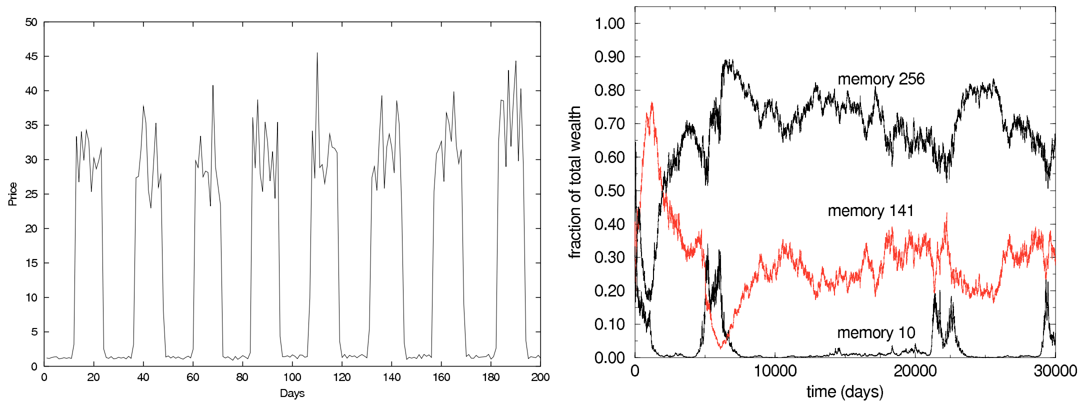

Figure 1.1: Cyclic behavior in LLS model with periodic booms and crashes using only one type of traders and a logarithmic utility function. Fraction of the total wealth as a function of time in LLS model with three equal investor population with different memory span.

As shown in [86, 87] the model is capable to provide realistic features of a stock market such as boom, crashes and cycles (see Fig. 1.1). It is not clear however if such model is able to reproduce fat tails for the price and/or the wealth distribution. Numerical simulations seem to exclude this possibility [131]. In particular, in Chapter 2, we give a mathematical proof of self-similar lognormal behavior for the corresponding kinetic model [37].

1.2 The Lux-Marchesi model

The LM model consider the behavior of an ensemble of N speculators. These traders may adhere to chartist or fundamentalist practices. The number of chartists at any point in time will be denoted by N_C, the number of fundamentalist is N_F ($N_C + N_F = N$). Furthermore, they distinguish two subgroups of chartists: those with an optimistic disposition and those who are pessimistic about the market's development in the near future. The number of individual in these groups is denoted by N_+ and N_-, respectively ($N_+ + N_- = N_C$). The dynamic of the model encapsulates the endogenous switching of agents between the groups defined above and the prices dynamics resulting from their market activities. So the entire dynamic is characterized by three elements, we discuss below.

1.2.1 Chartists switching between optimistic and pessimistic

An opinion index Y is introduced which is defined as the difference between optimistic and pessimistic chartists scaled by their total number

$$Y = \frac{N_+ - N_-}{N_C}, \quad Y \in [-1, 1] \tag{1.8}$$

denoting the price change in continuous time by $\dot{S} = dS/dt$ and following a convenient formalization for transition probabilities the probability of a formerly pessimistic individual to switch to the optimistic group (P_{-+}) and vice versa (P_{+-}) within some small time interval Δt may be written as

$$P_{-+} = \nu_1 \, (N_C/N) \exp(U_1), \quad P_{+-} = \nu_1 \, (N_C/N) \exp(-U_1), \tag{1.9}$$

where

$$U_1 = \alpha_1 y + \alpha_2 \dot{S}/(S\nu_1),$$

ν_1 is a parameter for the frequency of revaluation of opinion, and α_1 and α_2 are parameters measuring the importance the individuals place on majority opinion and actual price trend in forming expectation about future price changes.

1.2.2 Switching between chartist and fundamentalist strategy

Chartists are assumed to buy (sell) a fixed number of units if they are optimistic (pessimistic). Fundamentalists on the other hand are assumed to buy (sell) if the actual market price is below (above) the fundamental price S_F. These behavioral changes are modeled in the following way: agents meet individuals from the other groups, compare excess profits from both strategies and with a probability depending on the pay-off differential switch to the more successful strategy. Excess profits per unit (compared to alternative investments) gained by chartist are given by $(D+\dot{S})/S$. These are composed of nominal dividends (D) and capital gains due to the price change ($\dot{S}$). Dividing by the actual market price gives the revenue per unit of the asset. Excess returns compared to other investment opportunities are computed by subtracting the average real return R received by the holders of other assets in our economy.

In the case of fundamentalist traders excess profits per unit of the asset can be written as : $k|(S_F - S)/S|$. As the gains of chartists are immediately realized whereas those claimed by fundamentalists occur only in the future (and depend on the uncertain time for reversal to the fundamental value) the latter are discounted by a factor $k < 1$. Neglecting the dividend term in fundamentalists' profits is justified by assuming that they correctly perceive the (long-term) real returns to equal the average return of the economy (i.e. $D/S_F = R$) so that the only source of excess profits in their view is arbitrage when $S \neq S_F$.

According to above, transition probabilities for changes of strategies are formalized as follows

$$P_{F+} = \nu_2(N_+/N)\exp(U_{2,1}), \quad P_{+F} = \nu_2(N_F/N)\exp(-U_{2,1}), \tag{1.10}$$

with

$$U_{2,1} = \alpha_3 \left(\left(r + \frac{\dot{S}}{\nu_2} \right) \frac{1}{S} - R - k \left| \frac{S_F - S}{S} \right| \right),$$

and

$$P_{F-} = \nu_2(N_-/N)\exp(U_{2,2}), \quad P_{-F} = \nu_2(N_F/N)\exp(-U_{2,2}), \tag{1.11}$$

with

$$U_{2,2} = \alpha_3 \left(R - \left(r + \frac{\dot{S}}{\nu_2} \right) \frac{1}{S} - k \left| \frac{S_F - S}{S} \right| \right).$$

Here ν_2 is again a parameter for the frequency of the transition, while α_3 is a measure of the pressure exerted by profit differentials. As transition are governed by some type of pair interaction, the probability for an individual to change strategy also depend on the number of individuals pursuing other strategies at that time.

1.2.3 The Price formation process

The dynamic of the price is explained by the presence of an auctioneer which react with respect to the excess demand by adjusting the price to the next higher (lower) possible value within the next time increment with a certain probability depending on the extent of the balance between demand and supply. Assuming furthermore that there are additional liquidity traders in the market whose excess demand is stochastic or that the value of excess demand (ED) is perceived with some imprecision by the auctioneer, a small noise term μ is added, and arrive at transition probabilities for an increase or decrease of the market price by a fixed amount ΔS

$$P_{\uparrow S} = \max\{0, \beta(ED + \mu)\}, \quad P_{\downarrow S} = \min\{\beta(ED + \mu), 0\}.$$

Where β is a parameter for the reaction speed of the auctioneer. Hence, if for example, the perceived excess demand is positive, an increase of the price towards the next elementary unit occurs with probability $\beta(ED + \mu)\Delta t$ within an infinitesimal time increment. Aggregate excess demand ED is composed of excess demand of chartists and fundamentalists $ED = ED_C + ED_F$. The former is $ED_C = (N_+ - N_-)t_C$ since all chartists either buy or sell the same number t_C of units. Fundamentalists'excess demand is given by $ED_F = N_F\gamma(S_F - S)$, depending on the deviation from the fundamental value, reaction strength γ, and the number of individuals behaving this way at that time, N_F.

This probabilistic rule for price adjustments is, in fact, equivalent to the traditional Walrasian adjustment scheme. It can be shown that the mean value dynamics of the price is governed by the simple differential equation [95, 89]

$$\frac{dS}{dt} = \beta ED = \beta\left((N_+ - N_-)t_C + N_F\gamma(S_F - S)\right). \tag{1.12}$$

Typically in order to assure that none of the stylized facts of financial prices can be traced back to exogenous factors, one assumes that the log-changes of S_F in time are Gaussian random variables.

The overall results of this dynamics is easily understood by investigation of the properties of stationary states. Introducing the fraction of chartists $Z = N_C/N$ we have[95, 89]

Proposition 1.2.1

(a) The mean-value dynamics of Y, S and Z possesses the following stationary solutions:

- *(i) $Y = 0$, $S = S_F$ with arbitrary Z,*
- *(ii) $Y = 0$, $Z = 1$ with arbitrary S,*
- *(iii) $Z = 0$, $S = S_F$ with arbitrary Y;*

(b) no stationary states with both $Y \neq 0$ and $S \neq S_F$ exist.

The equilibria of major interest are those depicted in item (i) of the first part of the proposition. These stationary states are characterized by a balanced disposition among chartists and the price equal to the fundamental value. Situations (ii) and (iii) denotes the prevalence of one strategy over the other, since in case (ii) only chartists are present whereas in case (iii) only fundamentalists survive. It is remarkable that state (i) characterizes a stable point only under certain assumptions over the parameters otherwise it is unstable[95, 89].

In addition numerical results show the emergence of deviation from normal behavior with presence of fat tails for the distribution of the time series of returns[95]. Figure 1.2 illustrates the interplay between the dynamics of relative price changes and the fraction of chartists Z among traders. An increase of the number of chartists leads to intermittent fluctuations. Note that, thanks to the presence of fundamentalists, the model incorporates self-stabilizing forces leading to a reduction of the number of chartists after a period of severe fluctuations. In fact, large deviations of the price from its fundamental value lead to high potential profits of the fundamentalist strategy which induces a certain number of agents to switch away from

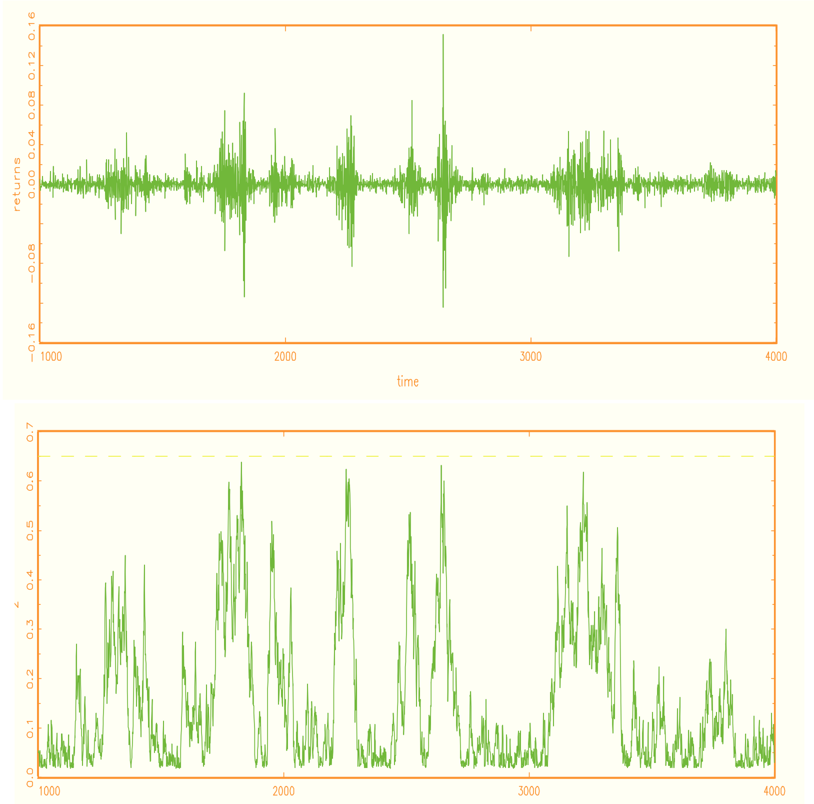

Figure 1.2: Time series of returns (top) and the fraction of chartists (bottom) from a typical simulation of the LM model.

chartism. The dashed line in the bottom picture is the critical threshold for Z leading to the loss of the self-stabilizing forces and to the extinction of fundamentalists.

As we will see in Chapter 2 using a related kinetic model we will determine an analogous of Proposition 1.2.1 and show how the presence of fundamentalists is essential in order to obtain fat tails in the price distribution.

Chapter 2

Kinetic models of financial markets

Starting from the microscopic dynamics one can aim at deriving the corresponding kinetic or mesoscopic models using the tools of classical kinetic theory of fluids [14, 38, 37, 136, 46, 69, 103, 98, 114, 125]. In contrast with microscopic dynamics, where behavior often can be studied only empirically through computer simulations, kinetic models based on partial differential equations allow to derive analytically general information on the model and its asymptotic behavior. For example, the knowledge of the tails behavior for the wealth/price is of primary importance, since it determines a posteriori whether the model can fit data of real markets. In the sequel we will consider two different kind of kinetic models for financial markets introduced in [37, 98] which are strictly related to the LLS and the LM model respectively. For standard Boltzmann-like models the determination of an explicit form of the asymptotic wealth/price distribution of the kinetic equation remains difficult and requires the use of suitable numerical methods. A complementary method to extract information on the tails is linked to the possibility to obtain particular asymptotics which maintain the characteristics of the solution to the original problem for large times. Following the analysis developed in [38], we shall prove that the Boltzmann models converge in a suitable asymptotic limit towards convection-diffusion equations of Fokker-Planck type. Other Fokker-Planck equations were obtained using different approaches in [14, 126, 127]. This permits to study the asymptotic behavior of the wealth and the price distributions and to characterize the regimes of lognormal behavior and the ones with power law tails.

2.1 Modeling speculative markets

One of the aims of kinetic modeling is motivated by the desire to have a more realistic description of the speculative dynamics in the multi-agents models. An interesting model in this direction has been obtained in [37], who derived a kinetic description of the behavior of a simple financial market where a population of homogeneous agents can create their own portfolio between two investment alternatives: a stock and a bond. The model is closely related to the Levy-Levy-Solomon (LLS) microscopic model in finance [86, 88], described in chapter 1

In this non-stationary financial market model, the average wealth is not conserved and this produces price variations. Let us point out that, even if the model is linear since no binary interaction dynamics between agents is present, the study of the large time behavior is not immediate. In fact, despite conservation of the total number of agents, there are no additional conservation equations, and the determination of an explicit form of the asymptotic wealth distribution of the kinetic equation remains difficult and requires the use of suitable numerical methods.

By resorting to a particular asymptotic which maintain the characteristics of the solution to the original problem for large times, one can prove also in this case that the Boltzmann model converges towards a Fokker-Planck type equation for the distribution of wealth among individuals.

In this case, however, due to the variation in time of the average wealth, no steady solutions exist, and one can only show that the Fokker-Planck equation admits self-similar solutions that can be computed explicitly and which are lognormal distributions.

2.1.1 Kinetic modeling

We define $f = f(w,t)$, $w \in \mathbb{R}_+$, $t > 0$ the distribution of wealth w, which represents the probability for an agent to have a wealth w. We assume that at time t the percentage of wealth invested is of the form $\gamma(\xi) = \mu(S) + \xi$, where ξ is a random variable in $[-z, z]$, and $z = \min\{-\mu(S), 1 - \mu(S)\}$ is distributed according to some probability density $\Phi(\mu(S), \xi)$ with zero mean and variance ζ^2. This probability density characterizes the individual strategy of an agent around the optimal choice $\mu(S)$. We assume Φ to be independent of the wealth of the agent. Here, the optimal demand curve $\mu(\cdot)$ is assumed to be a given monotonically non-increasing function of the price $S \geq 0$ such that $0 < \mu(0) < 1$.

Note that given $f(w,t)$ the actual stock price S satisfies the demand-supply relation

$$S = \frac{1}{n}\langle \gamma w \rangle, \tag{2.1}$$

and $f(w,t)$ has been normalized

$$\int_0^\infty f(w,t)dw = 1.$$

More precisely, since γ and w are independent, at each time t, the price $S(t)$ satisfies

$$S(t) = \frac{1}{n}\langle\gamma\rangle\langle w\rangle = \frac{1}{n}\mu(S(t))\bar{w}(t), \tag{2.2}$$

with

$$\bar{w}(t) := \langle w\rangle = \int_0^\infty f(w,t) w dw, \tag{2.3}$$

being the mean wealth and by construction,

$$\mu(S) = \int \Phi(\mu(S),\xi)\xi\, d\xi.$$

At the next round in the market, the new wealth of the investor will depend on the future price S' and the percentage γ of wealth invested according to

$$w'(S',\gamma,\eta) = (1-\gamma)w(1+r) + \gamma w(1 + x(S',\eta)), \tag{2.4}$$

where the expected rate of return of stocks is given by

$$x(S',\eta) = \frac{S' - S + D + \eta}{S}. \tag{2.5}$$

In the above relation, $D \geq 0$ represents a constant dividend paid by the company and η is a random variable distributed according to $\Theta(\eta)$ with zero mean and variance σ^2, which takes into account fluctuations due to price uncertainty and dividends [88, 66]. We assume η to take values in $[-d, d]$ with $0 < d \leq S' + D$ so that $w' \geq 0$ and thus negative wealths are not allowed in the model. Note that equation (2.5) requires estimation of the future price S', which is unknown.

The dynamics is then determined by the agent's new fraction of wealth invested in stocks, $\gamma'(\xi') = \mu(S') + \xi'$, where ξ' is a random variable in $[-z', z']$ and $z' = \min\{\mu(S'), 1-\mu(S')\}$ is distributed according to $\Phi(\mu(S'),\xi')$. We have the demand-supply relation

$$S' = \frac{1}{n}\langle\gamma' w'\rangle, \tag{2.6}$$

which permits us to write the following equation for the future price:

$$S' = \frac{1}{n}\langle\gamma'\rangle\langle w'\rangle = \frac{1}{n}\mu(S')\langle w'\rangle. \tag{2.7}$$

Now

$$w'(S',\gamma,\eta) = w(1+r) + \gamma w(x(S',\eta) - r), \tag{2.8}$$

thus

$$\begin{aligned} \langle w'\rangle &= \langle w\rangle(1+r) + \langle\gamma w\rangle(\langle x(S',\eta)\rangle - r) && (2.9)\\ &= \bar{w}(t)(1+r) + \mu(S)\bar{w}(t)\left(\frac{S' - S + D}{S} - r\right). && (2.10) \end{aligned}$$

This gives the identity

$$S' = \frac{1}{n}\mu(S')\bar{w}(t)\left[(1+r) + \mu(S)\left(\frac{S'-S+D}{S} - r\right)\right]. \tag{2.11}$$

Using equation (2.2) we can eliminate the dependence on the mean wealth and write

$$\begin{aligned} S' &= \frac{\mu(S')}{\mu(S)}\left[(1-\mu(S))S(1+r) + \mu(S)(S'+D)\right] \\ &= \frac{(1-\mu(S))\mu(S')}{(1-\mu(S'))\mu(S)}(1+r)S + \frac{\mu(S')}{1-\mu(S')}D. \end{aligned} \tag{2.12}$$

Equation (2.12) determines implicitly the future value of the stock price. Let us set

$$g(S) = \frac{1-\mu(S)}{\mu(S)}S. \tag{2.13}$$

Then the future price is given by the equation

$$g(S') = g(S)(1+r) + D \tag{2.14}$$

for a given S. Note that

$$\frac{dg(S)}{dS} = -\frac{d\mu(S)}{dS}\frac{S}{\mu(S)^2} + \frac{1-\mu(S)}{\mu(S)} > 0,$$

so the function $g(S)$ is strictly increasing with respect to S. This guarantees the existence of a unique solution

$$S' = g^{-1}\left(g(S)(1+r) + D\right) > S. \tag{2.15}$$

Moreover, if $r = 0$ and $D = 0$, the unique solution is $S' = S$ and the price remains unchanged in time.

For the average stock return, we have

$$\bar{x}(S') - r = \frac{(\mu(S') - \mu(S))(1+r)}{(1-\mu(S'))\mu(S)} + \frac{\mu(S')D}{S(1-\mu(S'))}, \tag{2.16}$$

where

$$\bar{x}(S') = E[x(S',\eta)] = \frac{S'-S+D}{S}. \tag{2.17}$$

Now the right hand side of (2.16) has non-constant sign since $\mu(S') \leq \mu(S)$. In particular, the average stock return is above the bonds rate r only if the (negative) rate of variation of the investments is above a certain threshold

$$\frac{\mu(S') - \mu(S)}{\mu(S)\mu(S')}S \geq -\frac{D}{(1+r)}.$$

In the constant investment case $\mu(\cdot) = C$, with $C \in (0,1)$ constant, then we have $g(S) = (1-C)S/C$ and

$$S' = (1+r)S + \frac{C}{1-C}D, \tag{2.18}$$

which corresponds to a dynamics of growth of the prices at rate r. As a consequence, the average stock return is always larger then the constant return of bonds:

$$\bar{x}(S') - r = \frac{D}{S(1-C)} \geq 0.$$

The previous dynamics originate the linear kinetic equation for the evolution of the wealth distribution, which is fruitfully written in weak form

$$\frac{d}{dt}\int \varphi(w) f(w,t)dw = \left\langle \int_{\mathbb{R}_+} \beta(w \to w')\left(\varphi(w') - \varphi(w)\right) f(w,t) \right\rangle. \tag{2.19}$$

The above equation takes into account all possible variations that can occur to the distribution of a given wealth w, according to (2.8) and equation (2.15) for the price. In (2.19), the kernel β takes the form

$$\beta(w \to w') = \Phi(\mu(S), \xi)\Theta(\eta). \tag{2.20}$$

The distribution function $\Phi(\mu(S), \xi)$, together with the function $\mu(\cdot)$, characterizes the behavior of the agents on the market (more precisely, they characterize the way the agents invest their wealth as a function of the actual price of the stock).

From (2.19) follows the conservation of the total number of investors if $\phi(w) = 1$. If we set $\phi(w) = w$ we obtain the time evolution of the average wealth which characterizes the price behavior. The mean wealth is not conserved since we have

$$\frac{d}{dt}\int_0^\infty f(w,t)w\,dw = \left(r + \mu(S)\left(\frac{S' - S + D}{S} - r\right)\right)\int_0^\infty f(w,t)w\,dw. \tag{2.21}$$

Note that since the sign of the right hand side is nonnegative, the mean wealth is nondecreasing in time. In particular, we can rewrite the equation as

$$\frac{d}{dt}\bar{w}(t) = \left((1-\mu(S))r + \mu(S)\bar{x}(S')\right)\bar{w}(t). \tag{2.22}$$

From this we get the equation for the price

$$\frac{d}{dt}S(t) = \frac{\mu(S(t))}{\mu(S(t)) - \dot{\mu}(S(t))S(t)}\left((1-\mu(S(t)))r + \mu(S(t))\bar{x}(S'(t))\right)S(t), \tag{2.23}$$

where S' is given by (2.12) and

$$\dot{\mu}(S) = \frac{d\mu(S)}{dS} \leq 0.$$

Now since from (2.16) it follows by the monotonicity of μ that

$$\bar{x}(S') \leq M{:=}r + \frac{D}{S(0)(1-\mu(S(0)))},$$

using (2.22) we have the bound

$$\bar{w}(t) \leq \bar{w}(0) \exp(Mt) . \tag{2.24}$$

From (2.2) we obtain immediately

$$\frac{S(t)}{\mu(S(t))} \leq \frac{S(0)}{\mu(S(0))} \exp(Mt) ,$$

which gives

$$S(t) \leq S(0) \exp(Mt) . \tag{2.25}$$

For a constant $\mu(\cdot) = C$, $C \in (0,1)$ we have the explicit expression for the growth of the wealth (and consequently of the price)

$$\bar{w}(t) = \bar{w}(0) \exp(rt) - (1-\exp(rt)) \frac{nD}{1-C}. \tag{2.26}$$

2.1.2 Fokker-Planck asymptotics and wealth distribution

As usual it is difficult to study in detail the large time behavior of the system. We can therefore apply a quasi-invariant limit technique described in the previous Chapters to derive simplified models whose behavior is easier to analyze. Here we consider the limit of large times in which the market originates a very small exchange of wealth (small rates of return r and x).

In order to study the asymptotic behavior of the distribution function $f(w,t)$, we set

$$\tau = rt, \quad \tilde{f}(w,\tau) = f(w,t), \quad \tilde{S}(\tau) = S(t), \quad \tilde{\mu}(\tilde{S}) = \mu(S),$$

which implies that $\tilde{f}(w,\tau)$ satisfies the weak form of the kinetic equation

$$\begin{aligned} &\frac{d}{d\tau}\int_0^\infty \tilde{f}(w,\tau)\phi(w)dw = \\ &\frac{1}{r}\int_0^\infty \int_{-d}^{d} \int_{-z}^{z} \Phi(\tilde{\mu}(\tilde{S}),\xi)\Theta(\eta)\tilde{f}(w,\tau)(\phi(w') - \phi(w)) d\xi\, d\eta\, dw \end{aligned} \tag{2.27}$$

and consider a second-order Taylor expansion of ϕ around w,

$$\phi(w') - \phi(w) = w(r + \gamma(x(S',\eta) - r))\phi'(w) + \frac{1}{2}w^2(r + \gamma(x(S',\eta)-r))^2\phi''(\tilde{w}),$$

where, for some $0 \leq \vartheta \leq 1$,

$$\tilde{w} = \vartheta w' + (1-\vartheta)w.$$

Now we insert this expansion into the right-end-side, and compute the limit of very small values of the constant rate r. In order for such a limit to make sense and preserve the characteristics of the model, we must assume that

$$\lim_{r\to 0} \frac{\sigma^2}{r} = \nu, \quad \lim_{r\to 0} \frac{D}{r} = \lambda. \tag{2.28}$$

Note that the above limits in (2.15) imply immediately that

$$\lim_{r\to 0} \tilde{S}' = \tilde{S}. \tag{2.29}$$

Omitting the details of the computations, sending $r \to 0$ under these assumptions, we obtain the weak form

$$\begin{aligned} &\frac{d}{d\tau}\int_0^\infty \tilde{f}(w,\tau)\phi(w)dw \\ &= \left(1+\tilde{\mu}(\tilde{S})\left((\kappa(\tilde{S})-1)+\frac{\tilde{\mu}(\tilde{S})(\kappa(\tilde{S})-1)+1}{1-\tilde{\mu}(\tilde{S})}\frac{\lambda}{\tilde{S}}\right)\right)\int_0^\infty \tilde{f}(w,\tau)w\phi'(w)\,dw \\ &+\frac{1}{2}\frac{(\tilde{\mu}(\tilde{S})^2+\zeta^2)}{\tilde{S}^2}\nu\int_0^\infty \tilde{f}(w,\tau)w^2\phi''(w)\,dw, \end{aligned}$$

with

$$0 < \kappa(\tilde{S}) := \frac{\tilde{\mu}(\tilde{S})(1-\tilde{\mu}(\tilde{S}))}{\tilde{\mu}(\tilde{S})(1-\tilde{\mu}(\tilde{S}))-\tilde{S}\dot{\tilde{\mu}}(\tilde{S})} \leq 1, \quad \dot{\tilde{\mu}}(\tilde{S}) = \frac{d\tilde{\mu}(\tilde{S})}{d\tilde{S}} \leq 0. \tag{2.30}$$

This corresponds to the Fokker-Planck equation

$$\frac{\partial}{\partial\tau}\tilde{f} = \frac{\partial}{\partial w}\left[-A(\tau)w\tilde{f}+\frac{1}{2}B(\tau)\frac{\partial}{\partial w}w^2\tilde{f}\right], \tag{2.31}$$

with

$$A(\tau) = 1+\tilde{\mu}(\tilde{S})\left((\kappa(\tilde{S})-1)+\frac{\tilde{\mu}(\tilde{S})(\kappa(\tilde{S})-1)+1}{1-\tilde{\mu}(\tilde{S})}\frac{\lambda}{\tilde{S}}\right) \tag{2.32}$$

$$B(\tau) = \frac{(\tilde{\mu}(\tilde{S})^2+\zeta^2)}{\tilde{S}^2}\nu. \tag{2.33}$$

In order to search for self-similar solutions, we consider the scaling

$$\tilde{f}(w,\tau) = \frac{1}{w}\tilde{g}(\chi,\tau), \quad \chi = \log(w). \tag{2.34}$$

Simple computations show that $\tilde{g}(\chi,\tau)$ satisfies the linear convection-diffusion equation

$$\frac{\partial}{\partial\tau}\tilde{g}(\chi,\tau) = \left(\frac{B(\tau)}{2} - A(\tau)\right)\frac{\partial}{\partial\chi}\tilde{g}(\chi,\tau) + \frac{B(\tau)}{2}\frac{\partial^2}{\partial\chi^2}\tilde{g}(\chi,\tau),$$

which admits the self-similar solution

$$\tilde{g}(\chi,\tau) = \frac{1}{(2b(\tau)\pi)^{1/2}}\exp\left(-\frac{(\chi + b(\tau)/2 - a(\tau))^2}{2b(\tau)}\right), \tag{2.35}$$

where

$$a(\tau) = \int_0^\tau A(s)\,ds + C_1, \qquad b(\tau) = \int_0^\tau B(s)\,ds + C_2.$$

Reverting to the original variables, we obtain the lognormal asymptotic behavior of the model,

$$\tilde{f}(w,\tau) = \frac{1}{w(2b(\tau)\pi)^{1/2}}\exp\left(-\frac{(\log(w) + b(\tau)/2 - a(\tau))^2}{2b(\tau)}\right). \tag{2.36}$$

The constants $C_1 = a(0)$ and $C_2 = b(0)$ can be determined from the initial data at $t = 0$. If we denote by $\bar{w}(0)$ and $\bar{e}(0)$ the initial values of the first two central moments, we get

$$C_1 = \log(\bar{w}(0)), \qquad C_2 = \log\left(\frac{\bar{e}(0)}{(\bar{w}(0))^2}\right).$$

2.1.3 Numerical examples

In this section we report the results of different numerical simulations of the proposed kinetic equations. In all the numerical tests, we use $N = 1000$ agents and $n = 10000$ shares. Initially, each investor has a total wealth of 1000 composed of 10 shares, at a value of 50 per share, and 500 in bonds. The random variables ξ and η are assumed distributed according to truncated normal distributions so that negative wealth values are avoided (no borrowing and no short selling). In the test case we consider the time-averaged Monte Carlo asymptotic behavior of the kinetic model and compare its numerical self-similar solution with the explicit one computed in the last section using the Fokker-Planck model.

To this end, we consider the self-similar scaling (2.34) and compute the solution for the values $r = 0.001$, $D = 0.0015$ with ξ and $\eta/S(0)$ distributed with standard

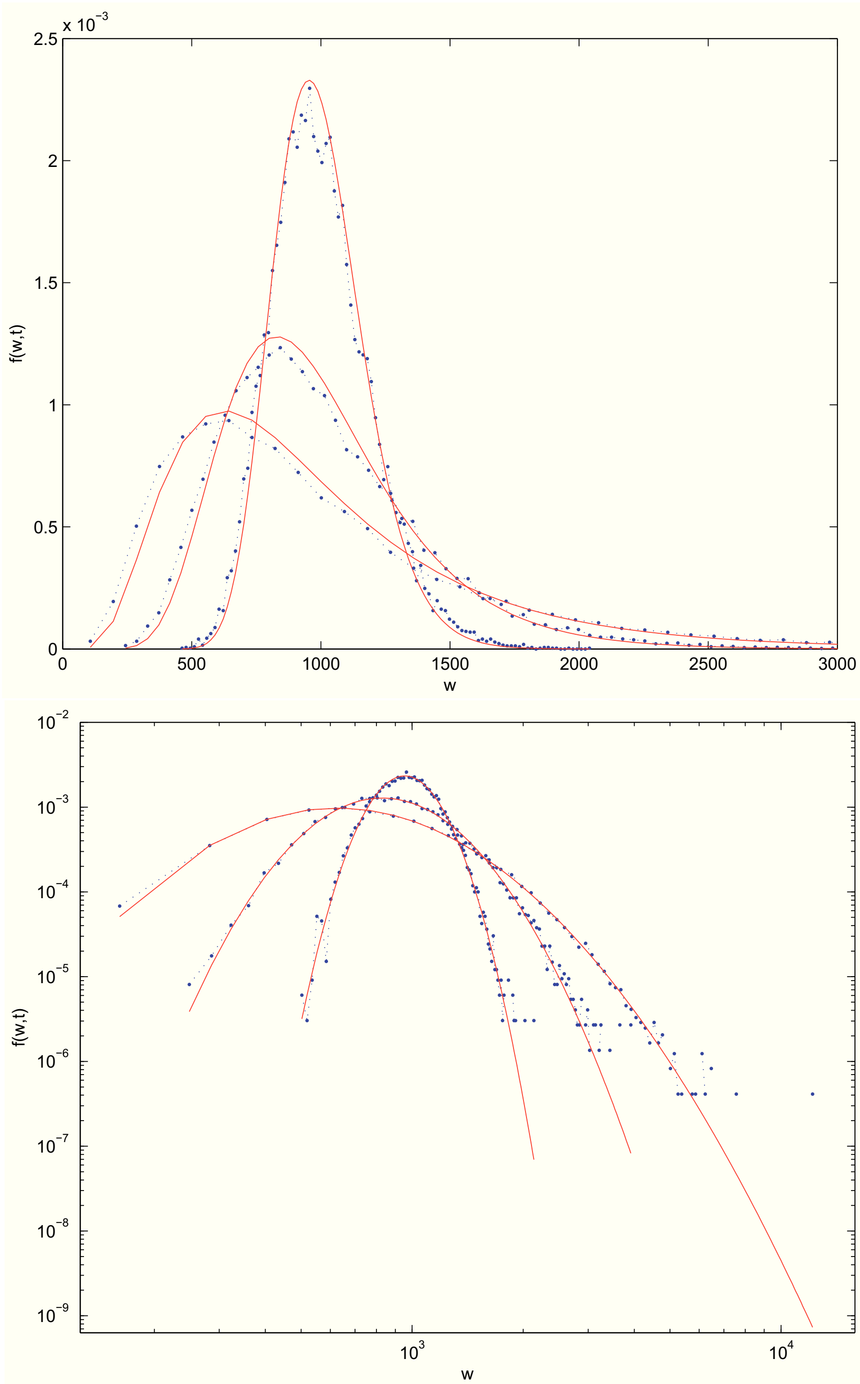

Figure 2.1: Test 3. Distribution function at $t = 50, 200, 500$. The continuous line is the lognormal Fokker-Planck solution. The right plot is in log-log scale.

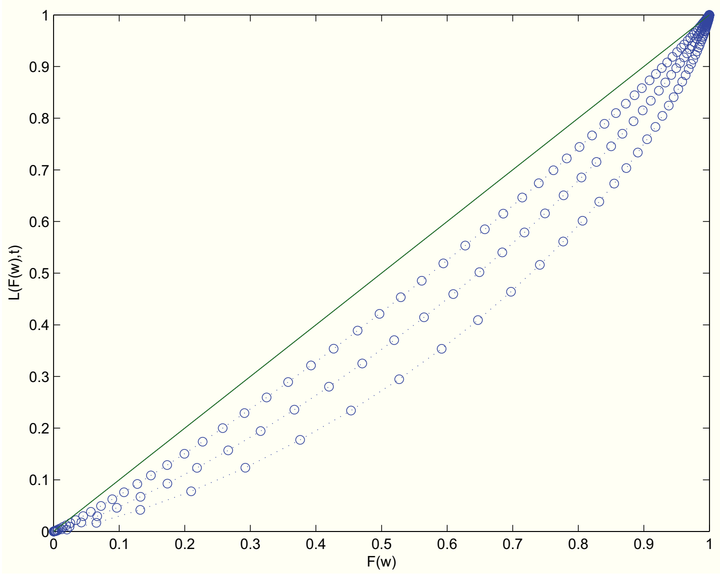

Figure 2.2: Test 3. The corresponding Lorentz curves. The Gini coefficients are $G = 0.1$, $G = 0.2$ and $G = 0.3$ respectively.

deviation 0.05. We report the numerical solution for a constant value of $\mu = 0.5$ at different times $t = 50, 200, 500$ in Figure 2.1. A very good agreement between the Boltzmann and the lognormal Fokker-Planck solutions is observed, as expected from the results of the last section. We also compute the corresponding Lorentz curve $L(F(w,t))$ defined as

$$L(F(w,t)) = \frac{\displaystyle\int_0^w f(v,t)v\,dv}{\displaystyle\int_0^\infty f(v,t)v\,dv}, \quad F(w,t) = \int_0^w f(v,t)\,dv,$$

and the Gini coefficient $G \in [0,1]$

$$G = 1 - 2\int_0^1 L(F(w,t))\,dw.$$

The Gini coefficient is a measure of the inequality in the wealth distribution [62]. A value of 0 corresponds to the line of perfect equality depicted in Figure 2.2 together with the different Lorentz curves. It is clear that inequalities grow in time due to the speculative dynamics.

2.2 A kinetic model for multiple agents interactions

In this chapter we introduce a simple Boltzmann-like model for a speculative market characterized by a single stock and an interplay between two different types of traders, chartists and fundamentalists. The model is strictly related to the microscopic Lux-Marchesi model [89].

Furthemore, the kinetic formalism allow us to introduce some psychological and behavioral components in the way the agents interact each other and percive the risk, giving the possibility to investigate non-rational dynamics, this can be done trough a suitable "value function" as introduced in the Prospect Theory by Kahneman and Tversky [75, 76], providing a bridge with some topics which have been deeply explored in the growing field of the Behavioral Finance.

Following the analysis developed in [38], we shall prove that the Boltzmann models converge in a suitable asymptotic limit towards convection-diffusion equations of Fokker-Planck type. Other Fokker-Planck equations were obtained using different approaches in [14, 126, 127]. This permits to study the asymptotic behavior of the investments and the price distributions and to characterize the regimes of lognormal behavior and the ones with power law tails.

We describe a simple financial market characterized by a single stock or good and an interplay between two different traders populations, chartists and fundamentalists, which determine the price dynamic of such stock (good). The aim is to introduce a kinetic description both for the behavior of the microscopic agents and for the price, and then to exploit the tools given by kinetic theory to get more insight about the way the microscopic dynamic of each trading agent can influence the evolution of the price, and be responsible of the appearance of 'stylized' fact like 'fat tails' and 'lognormal' behavior.

2.2.1 Kinetic setting

Similarly to Lux and Marchesi model [89], the starting point is a population of two different kind of traders, chartists and fundamentalists. Chartists are characterized by their number density ρ_C and the investment propensity (or opinion index) y of a single agent whereas fundamentalists appear only through their number density ρ_F. The value $\rho = \rho_F + \rho_C$ is invariant in time so that the total number of agents remains constant. In the sequel we will assume for simplicity $\rho = 1$.

2.2.2 Dynamic of investment propensity among chartists

Let us define $f(y, t)$, $y \in [-1, 1]$, the distribution function of chartists with investment propensity y at time t. Positive values of y represent buyers, negative values characterize sellers and close to $y = 0$ we have undecided agents. Clearly

$$\rho_C(t) = \int_{-1}^{1} f(y,t)\, dy.$$

Moreover we define the mean investment propensity

$$Y(t) = \frac{1}{\rho_C(t)} \int_{-1}^{1} f(y,t) y\, dy$$

.

For a given price $S(t)$ and price derivative $\dot{S}(t) = dS(t)/dt$ the microscopic dynamic of the investment propensity of chartists is characterized by the following binary interactions $(y, y_*) \to (y', y'_*)$ with

$$\begin{aligned}
y' &= (1 - \alpha_1 H(y) - \alpha_2) y + \alpha_1 H(y) y_* + \alpha_2 \Phi\left(\frac{\dot{S}(t)}{S(t)}\right) + D(y)\eta, \\
y'_* &= (1 - \alpha_1 H(y_*) - \alpha_2) y_* + \alpha_1 H(y_*) y + \alpha_2 \Phi\left(\frac{\dot{S}(t)}{S(t)}\right) + D(y_*)\eta_*.
\end{aligned}$$

Here $\alpha_1 \in [0,1]$ and $\alpha_2 \in [0,1]$, with $\alpha_1 + \alpha_2 \leq 1$, measure the importance the individuals place on others opinions and actual price trend in forming expectations about future price changes. The random variables η and η_* are assumed distributed accordingly to $\Theta(\eta)$ with zero mean and variance σ^2 and measure individual deviations from the average behavior. The function $H(y) \in [0,1]$ is taken symmetric on the interval I, and characterize the herding behavior, whereas $D(y)$ defines the diffusive behavior, and will be also taken symmetric on I. Simple examples of herding function and diffusion function are given by

$$H(y) = a + b(1 - |y|), \qquad D(y) = (1 - y^2)^\gamma,$$

with $0 \leq a + b \leq 1$, $a \geq 0, b > 0$, $\gamma > 0$ (see figure 2.2.2). Other choices are of course possible, note that in order to preserve the bounds for y it is essential that $D(y)$ vanishes in $y = \pm 1$. Both functions take into account that extremal positions suffer less herding and fluctuations. For $b = 0$, $H(y)$ is constant and no herding effect is present and the mean investment propensity is preserved when the market influence is neglected ($\alpha_2 = 0$) as in classical opinion models a model (see [132] at the reference therein).

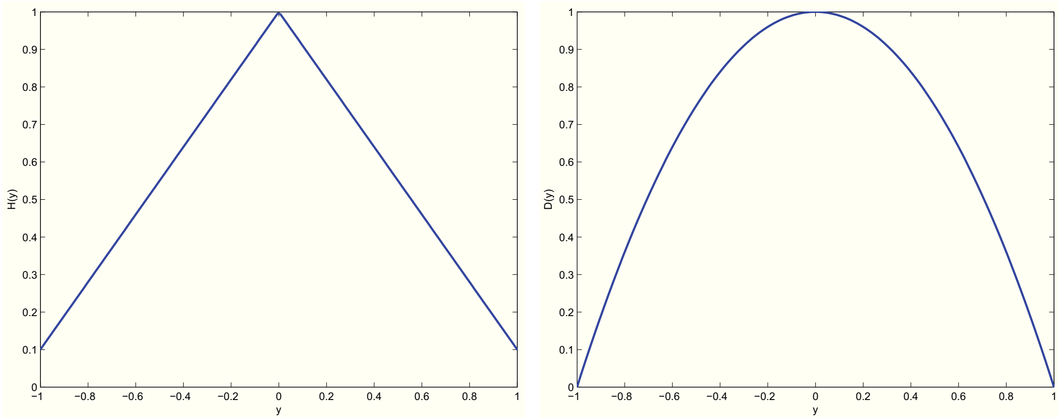

Figure 2.3: Typical examples of herding function $H(y)$ (left) and diffusion function $D(y)$ (right).

A remarkable feature of the above relations is the presence of the normalized value function $\Phi(\dot{S}(t)/S(t))$ in $[-1, 1]$ in the sense of Kahneman and Tversky [75, 76] that models the reaction of individuals towards potential gain and losses in the market [75]. This permits to introduce behavioral aspects in the market dynamic and to take into account the influence of psychology on the behavior of financial practitioners.

The value function is defined on deviations from a reference point, which is ususlly assumed equal to zero (but it can be considered also positive or negative), and is normally concave for gains (implying risk aversion), commonly convex for losses (risk seeking) and is generally steeper for losses than for gains (loss aversion)(see figure 2.2.2)

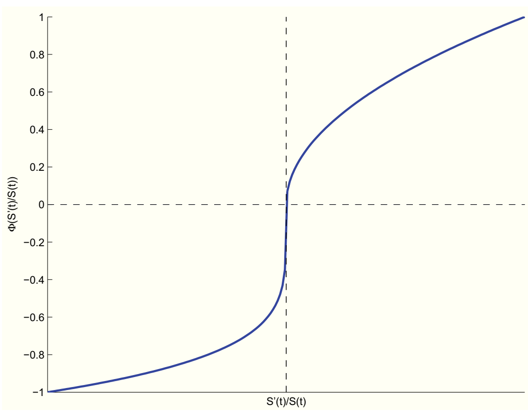

Figure 2.4: An example of value function $\Phi(\dot{S}(t)/S(t))$.

Let us ignore for the moment the price evolution. The above binary interaction gives the following kinetic equation for the time evolution of chartists

$$\frac{\partial f}{\partial t} = Q(f,f), \tag{2.37}$$

where for any test function φ the interaction operator Q can be conveniently written in weak form as

$$\int_{-1}^{1} Q\varphi(y)\,dy = \int_{[-1,1]^2}\int_{\mathbb{R}^2} B(y,y_*)f(y)f(y_*)(\varphi(y') - \varphi(y))d\eta\,d\eta_*\,dy_*\,dy$$

with the transition rate has given by

$$B(y,y_*) = \Theta(\eta)\Theta(\eta_*)\chi(|y'| \le 1)\chi(|y'_*| \le 1),$$

being $\chi(\cdot)$ the indicator function. Note that the mass density of chartists $\rho_C(t)$ is an invariant for the interaction, ($\varphi \equiv 1$).
It is worth to observe that for a given $D_C(y)$ a suitable choice of the support of the random variable η, avoids the dipendence of the collisional kernel $B(y,y_*)$ on the variables y,y^*.
As an example, if we take $D(y) = 1 - y^2$ we have

$$\begin{aligned} y' &= (1-\alpha_1 H(y) - \alpha_2)y + \alpha_1 H(y)y_* + \alpha_2\Phi\left(\frac{\dot{S(t)}}{S(t)}\right) + (1-y^2)\eta \\ &\le (1-\alpha_1 H(y) - \alpha_2)y + \alpha_1 H(y) + \alpha_2 + (1-y^2)\eta. \end{aligned}$$

Then to have $y' \le 1$ for any $y \in [-1,1]$, we have to chose η such that

$$(1-y^2)\eta \le (1-\alpha_1-\alpha_2)(1-y)$$

wich gives

$$\eta \le \frac{1}{2}(1-\alpha_1-\alpha_2)$$

Analogously we can ensure $y' \ge -1$, thus it is enough to take

$$\eta \in [-\frac{1}{2}(1-\alpha_1-\alpha_2), \frac{1}{2}(1-\alpha_1-\alpha_2)]$$

. to be $y \in [-1,1]$ at all times. For this reason, in the rest of the paper, we will consider only kernel of "maxwellian type"

$$B(y,y_*) = \Theta(\eta)\Theta(\eta_*)$$

.

2.2.3 Strategy exchange chartists-fundamentalists

In addition to the change of investment propensity due to a balance between herding behavior and the price followers nature of chartists, the model includes the possibility that an agent changes its strategy from chartist to fundamentalists and viceversa.

Agents meet individual from the other group, compare excess profits from both strategies and with a probability depending on the pay-off differential switch to the more successful strategy. When a chartist and a fundamentalist meet they characterize the success of a given strategy trough the profits earned by comparing

$$X_C(y,t) = \psi(y)\left(\frac{\dot{S}(t)/\mu + D}{S(t)} - r\right), \quad X_F(t) = k\frac{|S_F - S(t)|}{S(t)}. \tag{2.38}$$

Here $\psi(y) \in [-1,1]$ has the same sign of y and takes into account the change of sign in the profits accordingly to the actual behavior of the agent in the market which rely on his investment propensity y. The simplest choice is $\psi(y) = sign(y)$.

The value D is the nominal dividend and r the average real return of the market, such that $r = D/S_F$, i.e. evaluated at its fundamental value S_F in a state of stable price $\dot{S} = 0$ the asset yield the same returns of other investments, or equivalently $X_C = X_F = 0$. The discount factor $k < 1$ is justified by the observation that X_F is an expected gain realized only after reversal to the fundamental value. Finally $\mu > 0$ measures the frequency of the exchange rates.

A chartist characterized by an investment propensity y and a fundamentalist meet each other, and after comparing their strategies, they exchange strategies with a rate given by a suitable monotone function $B_{FC}(\cdot) \geq 0$. More precisely a chartist switch to fundamentalist with a rate $B_{FC}(X_F - X_C)$ and a fundamentalist switch to chartist at a rate $B_{FC}(X_C - X_F)$. A possible choice for the rate function is for example $B_{FC}(x) = e^x$.

For chartists we define the following linear strategy exchange operator

$$Q_{FC}(f) = \mu\rho_F(t)f(y)(B_{FC}(X_C - X_F) - B_{FC}(X_F - X_C))$$

where $\mu > 0$ measures the frequency of the exchange rates.

Taking into account such strategy exchanges we have the chartists-fundamentalists model

$$\begin{cases} \dfrac{\partial f}{\partial t} = Q(f,f) + \mu\rho_F(t)f(y)(B_{FC}(X_C - X_F) - B_{FC}(X_F - X_C)) \\ \dfrac{\partial \rho_F}{\partial t} = \mu\rho_F(t)\displaystyle\int_{-1}^{1} f(y)(B_{FC}(X_F - X_C) - B_{FC}(X_C - X_F))\,dy. \end{cases} \tag{2.39}$$

(It is immediate to verify that the total number density $\rho_C + \rho_F$ is conserved in time).

2.2.4 Price evolution

Finally we introduce the probability density $V(s,t)$ of a given price s at time t. The effective market price $S(t)$ is defined as the mean value

$$S(t) = \int_0^\infty V(s,t)s\,ds.$$

Following Lux and Marchesi [89] the microscopic dynamic of the price is given by

$$s' = s + \beta(\rho_C t_C Y(t)s + \rho_F \gamma(S_F - s)) + \eta s$$

where the parameters β, represent the price speed evaluation, η is a random variable with zero mean and variance ζ^2, distributed accordingly to $\Psi(\eta)$. In the above relation chartists either buy or sell the same number t_C of units and γ is the reaction strength of fundamentalists to deviations from the fundamental value.

Thus the chartists-fundamentalists system of equations (2.39) is complemented with the equation for the price distribution

$$\frac{\partial V}{\partial t} = L(V), \tag{2.40}$$

where the operator L, is linear, and in weak form it reads

$$\int_0^\infty L\varphi(s)\,ds = \int_0^\infty \int_{\mathbb{R}} b(s)V(s)(\varphi(s') - \varphi(s))d\eta\,ds \tag{2.41}$$

with the transition rate $b(s) = \Psi(\eta)\chi(s' \geq 0)$.
As before, a suitable choice of the domain for the support of variable η ensures $s' \geq 0$, and permit to express the transition rate in the simpler form

$$b(s) = \Psi(\eta).$$

Note that the expected value for the stock price satisfies the same differential equation as in[95, 89]

$$\frac{dS(t)}{dt} = \beta\rho_C t_C Y(t)S(t) + \beta\rho_F\gamma(S_F - S(t)). \tag{2.42}$$

2.2.5 Booms, crashes and macroscopic stationary states

In order to study the macroscopic steady states of the system let us start by observing that the equilibrium states for the price satisfy

$$\rho_C t_C Y S + \rho_F \gamma(S_F - S) = 0$$

and thus fall in one of the following categories

(i) $\rho_F \neq 0, \quad S = \frac{\rho_F \gamma S_F}{\rho_F \gamma - \rho_C t_C Y}, \qquad \rho_F \gamma S_F - \rho_C t_C Y \geq 0.$

(ii) $\quad \rho_F = 0, \quad Y = 0, \quad S$ arbitrary,

(iii) $\quad \rho_F = 0, \quad S = 0, \quad Y$ arbitrary.

At equilibrium we require ρ_F, ρ_C and Y to be constants. In order for the number densities to be constants we require $Q_{FC} = 0$. For $\rho_F \neq 0$ and $\rho_C \neq 0$, thanks to monotonicity of B_{FC}, we have $X_C = X_F$ or equivalently $S = S_F$. Note that Q_{FC} vanishes also when $\rho_F = 0$ or $\rho_C = 0$. These considerations reduce the set of possible equilibrium configurations to

(i) $\quad \rho_F \neq 0, \quad S = S_F, \quad Y = 0,$

(ii) $\quad \rho_F = 0, \quad Y = 0, \quad S$ arbitrary,

(iii) $\quad \rho_F = 0, \quad S = 0, \quad Y$ arbitrary.

Finally we consider the requirements for Y to be constant. In the case $Q_{FC} = 0$ the first moment equation reads

$$\begin{aligned} \frac{d}{dt} Y(t) = & - \alpha_1 \int_{-1}^{1} H(y) y f(y) dy - \alpha_2 \rho_C Y(t) \\ & + \alpha_1 Y(t) \int_{-1}^{1} H(y) f(y) dy + \alpha_2 \rho_C \Phi \left(\frac{\dot{S}(t)}{S(t)} \right), \end{aligned}$$

which gives the steady state condition

$$-\alpha_1 \int_{-1}^{1} H(y) y f(y) dy - \alpha_2 \rho_C Y + \alpha_1 Y \int_{-1}^{1} H(y) f(y) dy + \alpha_2 \rho_C \Phi \left(\frac{\dot{S}(t)}{S(t)} \right) = 0.$$

This gives a constraint for the value function Φ, precisely

$$\alpha_2 \rho_C \Phi \left(\frac{\dot{S}(t)}{S(t)} \right) = \alpha_1 \int_{-1}^{1} H(y) y f(y) dy + \alpha_2 \rho_C Y - \alpha_1 Y \int_{-1}^{1} H(y) f(y) dy$$

which in the simple case of H constant reduces to

$$\alpha_2 \rho_C \left(\Phi \left(\frac{\dot{S}(t)}{S(t)} \right) - Y \right) = 0.$$

Now using the fact that

$$\frac{\dot{S}(t)}{S(t)} = \beta \rho_C t_C Y(t) + \beta \rho_F \gamma \frac{(S_F - S(t))}{S(t)},$$

we can state

PROPOSITION 2.2.1 *The system of equations (2.39) in the case of H constant admits the following possible equilibrium configurations*

$$(i) \quad \rho_F \neq 0, \quad S = S_F, \quad Y = 0, \quad \Phi(0) = 0,$$

$$(ii) \quad \rho_F = 0, \quad Y = 0, \quad \Phi(0) = 0, \quad S \text{ arbitrary},$$

$$(iii) \quad \rho_F = 0, \quad Y = Y_*, \text{ with } Y_* = \Phi(\beta t_C Y_*), \quad S = 0.$$

Note that if the reference point for the value function $\Phi(0) \neq 0$ configuration (i) and (ii) are not possible for a constant H. This is in good agreement with the fact that an emotional perception of the market from the chartists acts as a source of instability for the market itself. In contrast configuration (iii), corresponding to a market crash, can be achieved also for $\Phi(0) \neq 0$. The existence of a unique fixed point Y_* has to be guaranteed by the choice of Φ, β and t_C. Of course if the reference point is set to zero, $\Phi(0) = 0$, we have $Y_* = 0$. It is easy to verify that these possible equilibrium configurations includes the ones in the original Lux-Marchesi model [**?**].

In addition to the above equilibrium configurations the model admits several other possible asymptotic behavior in the form of booms and cycles. Some of the fundamental features of the model are summarized in the following.

REMARK 2.2.2

- *Chartists alone ($\rho_F = 0, \rho_C = 1$) influence the price through their mean propensity to invest $Y(t)$ and at the same time the price trend influences their mean propensity through the value function $\Phi(\dot{S}(t)/S(t))$, since $\dot{S}(t)/S(t) = \beta Y(t) t_C$. Thus, except for the particular shape of the value function, if the mean propensity is initially (sufficiently) positive then it will continue to grow together with the price and the opposite occurs if it is initially (sufficiently) negative.*

  The market goes towards a boom (exponential grow of the price) or a crash (exponential decay of the price) with

  $$S(0)e^{-\beta t_C} \leq S(t) \leq S(0)e^{\beta t_C},$$

  and agents tend to concentrate in $y = 1$ and $y = -1$ respectively depending on the choices of H and Φ. This is in good agreement with the price followers nature of chartists.

- *Fundamentalists alone ($\rho_F = 1, \rho_C = 0$) influence the price through their expectation of the fundamental price. So their effect is to drive the price towards the fundamental price. For a constant fundamental price S_F the equilibrium state reached is characterized by $S = S_F$ and the trend is exponential.*

- *The presence of fundamentalists acts in contrast to the chartists pressure towards market booms or crashes. If their number is large enough they are capable to drive the price towards the fundamental value otherwise the chartists dynamic may dominate. In addition to booms and crashes, we have now the possibility of price cycles/oscillations around the fundamental value.*

2.2.6 Fokker-Plank approximation and kinetic asymptotic behaviour

Now we consider what happens at the kinetic scale. Due to the extreme difficulty to get detailed information on the asymptotic behavior of the kinetic coupled system, we will recover for both distribution functions f, and V, simplified Fokker-Planck models which preserve the main features of the original kinetic model. To keep notations simple, since we are mostly interested in the study of the equilibrium states we ignore the presence of the terms describing the change of strategy. However they can be easily included in the scaling described below.

For this purpose we introduce a time scaling parameter ξ and define

$$\tau = \xi t, \quad \tilde{f}(y, \tau) = f(y, t), \quad \tilde{V}(s, \tau) = V(s, t).$$

To preserve the chartists dynamic in the limit, we must require that

$$\lim_{\alpha_1, \xi \to 0} \frac{\alpha_1}{\xi} = \tilde{\alpha_1}, \quad \lim_{\alpha_2, \xi \to 0} \frac{\alpha_2}{\xi} = \tilde{\alpha_2}, \quad \lim_{\sigma, \xi \to 0} \frac{\sigma^2}{\xi} = \lambda,$$

where λ is a positive constant.

Similarly for the price dynamic, we assume

$$\lim_{\beta, \xi \to 0} \frac{\beta}{\xi} = \tilde{\beta}, \quad \lim_{\zeta, \xi \to 0} \frac{\zeta^2}{\xi} = \nu.$$

Performing similar computations as in [37] (see Appendix A and B for details) we recover the following Fokker-Plank system

$$\frac{\partial \tilde{f}}{\partial \tau} + \frac{\partial}{\partial y} \left[\left(\rho_C \tilde{\alpha_1} H(y)(\tilde{Y} - y) + \rho_C \tilde{\alpha_2} \left(\Phi \left(\frac{\dot{\tilde{S}}}{\tilde{S}} \right) - y \right) \right) \tilde{f} \right]$$
$$= \frac{\lambda \rho_C}{2} \frac{\partial^2}{\partial y^2} [(D^2(y)) \tilde{f}], \tag{2.43}$$

$$\frac{\partial}{\partial \tau} \tilde{V} + \frac{\partial}{\partial s} \left[\tilde{\beta} \left(\rho_C \tilde{Y} t_C s + \rho_F \gamma (S_F - s) \right) \tilde{V} \right] = \frac{\nu}{2} \frac{\partial^2}{\partial s^2} \left(s^2 \tilde{V} \right). \tag{2.44}$$

For notation simplicity in the sequel we will omit the tildes in the variables f, V, Y and S.

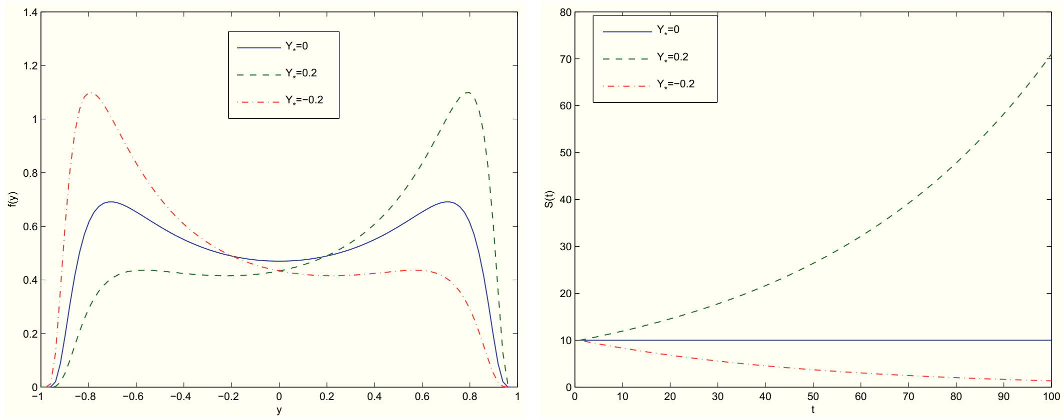

Figure 2.5: Equilibrium distribution function of the chartist investment propensity for different values of $Y_* = 0, 0.2, -0.2$ (left) and corresponding behavior of the price S (right). Exact solutions with $\rho_C = 1$, $\beta = 0.1$, $t_C = 1$, $\lambda/(\tilde{\alpha}_1 + \tilde{\alpha}_2) = 1$ and $f(y, 0) = f^\infty(y)$.

If we now take $D(y) = 1 - y^2$, and $H(y) = 1$ we can compute explicitly the equilibrium state for chartists with a constant mean investment propensity $Y = Y_*$ as

$$\begin{aligned} f^\infty(y) &= C_0(1+y)^{-2+Y_*\frac{(\tilde{\alpha}_1+\tilde{\alpha}_2)}{2\lambda}}(1-y)^{-2-Y_*\frac{(\tilde{\alpha}_1+\tilde{\alpha}_2)}{2\lambda}} \\ &\quad \exp\left(-\frac{(1-Y_*y)(\tilde{\alpha}_1+\tilde{\alpha}_2)}{\lambda(1-y^2)}\right) \end{aligned} \tag{2.45}$$

where $C_0 = C_0(Y_*, \lambda/(\tilde{\alpha}_1 + \tilde{\alpha}_2))$ is such that the mass of f^∞ is equal to ρ_C. Other choices of the diffusion function originate different steady states (see [132]).

Observe that, in the case $Y_* \neq 0$, the distribution is not symmetric and in the chartist population a predominant behavior arise. Otherwise when the reference point of the value function is set to zero we have a symmetric distribution with two peaks and mean value zero, and the macroscopic state of indecision is given, microscopically, by a polarization of the chartist population among two opposite kind of behaviors (see Figure 2.5).

In order to study the asymptotic behavior for the price we must distinguish between the case $\rho_F \neq 0$ and $\rho_F = 0$.

Let us consider first the situation in which $\rho_F = 0$ (or equivalently $\rho_C = 1$). For this purpose, we introduce the scaling

$$V(s, \tau) = \frac{1}{s} v(\chi, \tau), \quad \chi = \log(s).$$

It is straightforward to show that $v(\chi,\tau)$ satisfies the following linear convection diffusion equation

$$\frac{\partial}{\partial\tau}v(\chi,\tau)=\left[\frac{\nu}{2}-\tilde{\beta}Yt_C\right]\frac{\partial}{\partial\chi}v(\chi,\tau)+\frac{\nu}{2}\frac{\partial^2}{\partial\chi^2}v(\chi,\tau),$$

which admits the self-similar solution [37]

$$v(\chi,\tau)=\frac{1}{(2\log(E(\tau)/S(\tau)^2)\pi)^{\frac{1}{2}}}\exp\left(-\frac{(\chi+\log(\sqrt{E(\tau)}/S(\tau))-\log(S(\tau)))^2}{2\log(E(\tau)/S(\tau)^2)}\right),$$

with

$$E(\tau)=\int_0^\infty V(s,\tau)s^2\,ds.$$

Then reverting to the original variables it gives the lognormal behavior

$$V(s,\tau)=\frac{1}{s(2\log(E(\tau)/S(\tau)^2)\pi)^{\frac{1}{2}}}\exp\left(-\frac{(\log(s\sqrt{E(\tau)}/S(\tau)^2)^2}{2\log(E(\tau)/S(\tau)^2)}\right). \tag{2.46}$$

where $E(\tau)$ satisfies the differential equation

$$\frac{dE}{d\tau}=(2\tilde{\beta}Yt_C+\nu)E(\tau).$$

Thus for a steady state characterized by (ii) in Proposition 2.2.1 we have $S(\tau)=S_0$, $Y=0$ and $E(\tau)=e^{\nu\tau}E_0$.

Besides the above equilibrium state, equation (2.46) characterizes also the self-similar behavior of the price distribution in the case of booms and crashes, when the price $S(\tau)$ grows arbitrary or decays to zero. In particular in the limit $S(\tau)\to 0$, point (iii) in Proposition 2.2.1, the distribution function $V(s,\tau)$ concentrates near zero.

Finally we consider the microscopic behavior of the model where both $\rho_C\neq 0$ and $\rho_F\neq 0$.

Recall now the Fokker-Plank equation for the price in (2.43) and consider the stationary case (i) in Proposition 2.2.1. The Fokker-Planck equation in such case reads

$$\frac{\partial}{\partial\tau}V+\frac{\partial}{\partial s}\left[\left(\tilde{\beta}\rho_F\gamma(S_F-s)\right)V\right]=\frac{\nu}{2}\frac{\partial^2}{\partial s^2}\left(s^2V\right).$$

In this case the steady state can be computed as [14, 38] and yelds

$$V^\infty(s)=C_1(\mu)\frac{1}{s^{1+\mu}}e^{-\frac{(\mu-1)S_F}{s}}, \tag{2.47}$$

where $\mu = 1 + 2\tilde{\beta}\rho_F\gamma/\nu$ and $C_1(\mu) = ((\mu-1)S_F)^\mu/\Gamma(\mu)$ with $\Gamma(\cdot)$ being the usual Gamma function. Therefore the stationary state is described by a Gamma-like distribution with Pareto power law tails.

Remark 2.2.3 *1. The presence of fundamentalists is then essential in order to obtain fat tails in the price distribution. Their presence force the price to approach the mean value S_F in a way similar to the redistribution of wealth in the models proposed in [14, 38]. This feature seems to be essential for the development of power law behaviors. The stationary state for the price (2.47) has in fact the same structure of the stationary states for the wealth in [14, 38].*

2. we have considered in our description a constant value for the fundamental price. This may look as an unrealistic choice, according to the economic literature where the fundamental price is often treated like a temporal series with a stationary lognormal distribution. This reflect the facts that the returns in logaritmic form are gaussian distributed with zero mean and a fixed variance, i.e big jumps between two successive realizations are rarely verified. It is worth to observe that by introducing a time independent distribution function, $V_{SF}(q,t)$ such that

$$S_F(t) = \int_0^{+\infty} V_{SF}(q,t)dq$$

and considering the following dynamic in the kinetic evolution of the price

$$s' = s + \beta(\rho_C t_C Y(t)s + \rho_F\gamma(q-s)) + \eta s$$

we get

$$\int_0^\infty L\varphi(s)\,ds = \int_0^\infty\int_0^\infty\int_{\mathbb{R}} b(s)V_{SF}(s*)V(s)(\varphi(s')-\varphi(s))d\eta\,ds\,ds*$$

which is the analogous of 2.41 which permit to recover the same Fokker-Plank equation 2.43 for the asymptotic behaviour. We omit the details.

2.2.7 Numerical tests

We considered a Monte Carlo simulation of the kinetic system using $N = 50000$ chartists agents and no averages. In order to simulate the kinetic behavior of the price, we use a set of $N_S = 50000$ samples which can be though as possible realizations of the random variable s denoting the price. Since at the initial time the stock price S_0 is supposed to be known, all samples are initialized at the same value initially. In all our computations we take the value function

$$\Phi(x) = \begin{cases} \left(\frac{x-R_0}{L-R_0}\right)^r, & L > x > R_0, \\ -\left(\frac{R_0-x}{R_0+L}\right)^l, & -L < x \le R_0, \end{cases}$$

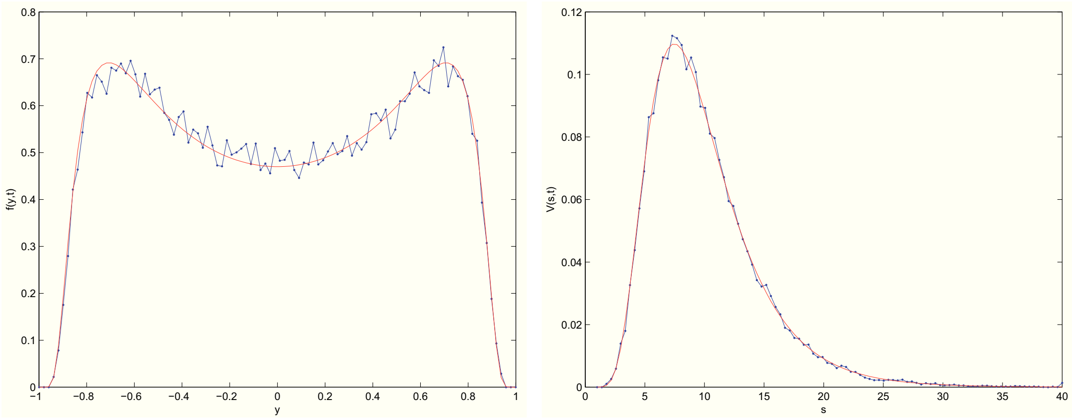

Figure 2.6: Equilibrium distribution function of the chartist investment propensity, with $\Phi(0) = 0$ (left) and log-normal distribution for the price (right) at $t = 1500$. The continuous line is the solution of the corresponding Fokker-Plank equations.

where $x \in [-L, L]$, R_0 is the reference point and $0 < l \leq r < 1$. For example we choose $r = 1/2$ and $l = 1/4$.

Test 1 In the first test we consider the case with $\rho_F = 0$ i.e only chartists are present in the model. We computed the equilibrium distribution for $\Phi(0) = 0$ the investment propensity. We take $\beta = 0.1$, $t_C = 1$, a constant herding function $H(y) = 1$ and the coefficients $\alpha_1 = \alpha_2 = 0.01$. The initial data for the chartists is perfectly symmetric with $Y = 0$, so the price remains constant $S = S_0$ with $S_0 = 10$. A particular care is required in the simulation to keep $Y = 0$ since the equilibrium point is unstable and as soon as $Y \neq 0$ the results deviate towards a market boom or crash.

After $T = 1500$ iteration the solution for the investment propensity has reached a stationary state and is plotted together with the solution of the Fokker Plank-limit in Figure 2.6. In the same figure we report also the computed solution for the price distribution and the self-similar lognormal solution of the corresponding Fokker-Plank equation. A very good agreement between the computed Boltzman solution and the Fokker-Plank solution is obseved.

Test 2 In the second test case we considered the most interesting situation with the presence of fundamentalists, i.e both chartists and fundamentalists interact in the stock market. We compute an equilibrium situation where $\rho_F = \rho_C = 1/2$ and the price is stationary at the fundamental value $S_F = 20$. We take $\beta = 0.1$, $tC = 1$, $\gamma = 1.3$, $\alpha_1 = \alpha_2 = 0.01$,and $\tilde{\beta} = 0.4$, $\nu = 0.5$ for the analytical steady state. We report the result of the simulation for the price distribution at the stationary state.

In Figure 2.7 we show the price distribution together with the steady state of the corresponding Fokker-Planck equation. The emergence of a power law is clear also for the Boltzmann model, and deviations of the two models is observed for small values of the price.

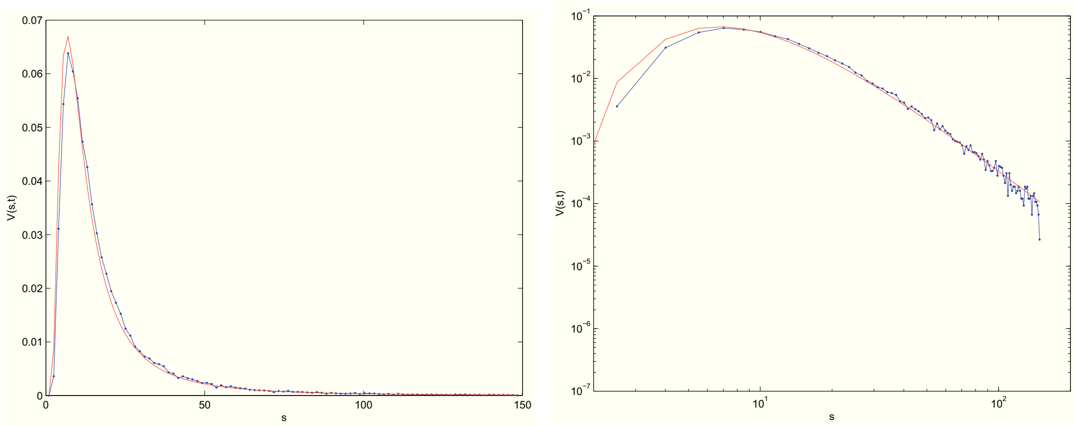

Figure 2.7: Stationary price distribution for the price with $\rho_F = \rho_C = 0.5$. Figure on the right is in log-log scale. The continuous line is the Fokker-Planck solution.

Test 3 In the third test we consider the case with strategy exchange between the two populations of interacting agents. The swithcing rates used to run the simulation are of the following form:

$$B_{FC}(XC - XF) = e^{(\sigma(XC-XF))}, \quad B_{FC}(XF - XC) = e^{(\sigma(XF-XC))}$$

Where XC and XF represent the profits realized by chartists and fundamentalists respectively who pursue their own strategies, their expression are given by 2.38 and σ represent the inertia of the reaction to profit differentials. We start the simulation considering $N_C = 2500$ chartists and $N_F = 2500$ fundamentalists, futhemore in order to make simulations more realistic we take a time varing fundamental price, by defining the following sequence $SF^{n+1} = SF^n + \eta$, where η it'a a random number sampeld from a truncated gaussian distribution with 0 mean and variance 0.1, at initial time we have $SF^0 = 20$. For the others parameter we take $\beta = 6$, $Tc = 0.02$, $\gamma = 0.1$, $\sigma = 0.8$, $\mu = 0.2$, $D = 0.004$, $k = 0.75$, and $\psi(y) = sign(y)$. We run different simulations for different values of α_1, and α_2, which measures respectively the herding and the market influence on the chartists. We note that the price seems to follow the fundamental price, but when the herding effects become predominant with respect to the direct influences of the market, stronger deviations can be observed, up to produce a situations of boom or chras of the market itself.

In particular three fundamental behaviours can be highlighted. The predominance of chartists, which leads the market thowards a crash or a boom (see figure 2.8), the predominance of fundamentalist, which originates damped oscillation of the price towards the fundamental value (see figure 2.9,2.10).

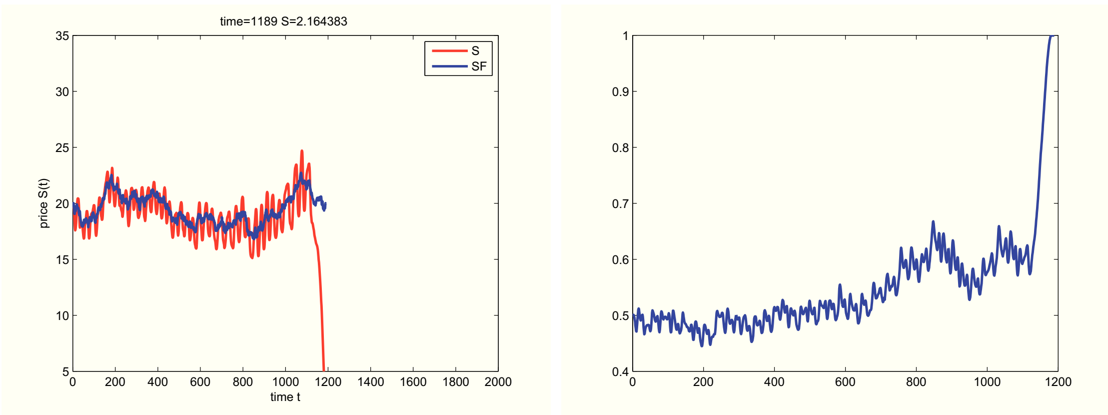

Figure 2.8: Crash market due to a chartist predominance. The chartist population is characterized by the parameters $\alpha_1 = 0.5$, and $\alpha_2 = 0.35$. Figure on the right represent the variation of the chartists's fraction among the entire population of agents.

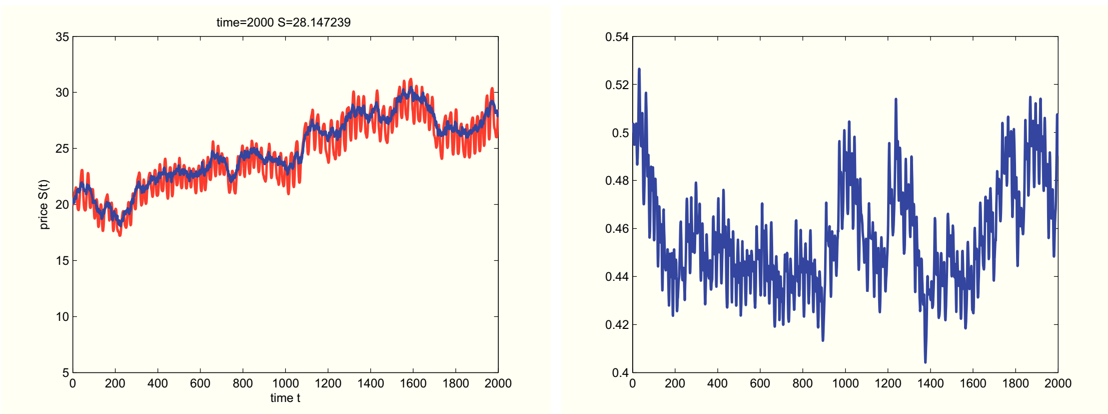

Figure 2.9: The price follow the fundamental price but strong oscillations appears. The chartist population is characterized by the parameters $\alpha_1 = 0.3$, and $\alpha_1 = 0.45$. Figure on the right represent the variation of the chartists's fraction.

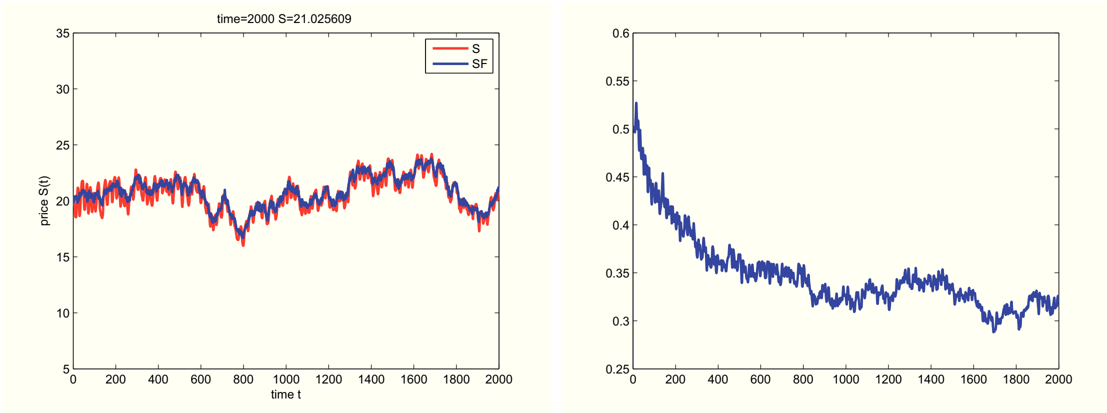

Figure 2.10: A strong fundamentalist predominance force the price to follow the fundamental on. Again the chartist population is characterized by the parameters $\alpha_1 = 0.2$, and $\alpha_2 = 0.5$. Figure on the right represent the variation of the chartists's fraction.

Remark 2.2.4 *The picture in the Test* 3 *seems to confirm the belive that the market efficent hypotesys does not hold, while the fundamental price react the flux of the incoming news with gaussian variations, the price'behaviour can deviate from it considerabily, due to the non rational pressure coming from the chartist's action.*

2.2.8 Conclusion

We derived an interacting agent model for a simple stock market, characterized by two different strategies that can be pursued by the agents, a chartist strategy an a fundamentalist strategy. We have introduced a mesoscopic description for the opinion formation of a class of chartist agents, and also for the price formation mechanism, finally a coupled system of kinetic equations has been derived. The model is able to describe both phenomena of boom or chrash, and cyclic oscillations of the market'sprice. The long time behavior has been studied in a suitable asymptotic regimes and a pair of Fokker-Planck equation has been recovered, for the chartist's opinion dynamics and the price formation. We found that in a system of agents acting only in a chartist way the distribution of price converges towards a lognormal distribution, while when the fundamentalist strategy is also allowed price's distributions display a power low with fat tails, wich is in accordance to what observed in the reality. In the description of charist's behavior we also introduce a value function, this tales into account some psychological factors in the opinion formation dynamic, which is relate to the way investors percive the risk at the time of buy or sell in the stock market. We preserve for future works a deeper investigation of the roole of suchs ingredient.

Chapter 3

Towards hydrodynamic models of financial markets

The goal of the models introduced in Chapter 2 is to follow the evolution of wealth distribution in a multi-agent society, and overall to be able to describe its large-time behavior, which in most cases is represented by an equilibrium distribution with a certain number of properties to be interpreted in terms of economic relevance.

As a matter of fact, in statistical physics, the knowledge of the large time behavior of the density of energy, and its approach to a universal profile are subsequently used to construct equations for hydrodynamics, which describe the space-time evolution of macroscopic observables.

A similar procedure could be in principle applied to the evolution of the density of wealth, in case this density depends on other important parameters (typically the space variable in a physical system).

The analogy with the evolution of a gas density depending of both the space and velocity variable has been recently developed in [46], in the case in which the behavior of each agent in the system is identified by two main variables, the first given by his wealth, the second by his propensity to invest (in the hope of making a profit).

Similarly to what happens in a physical system, where each particle is identified by its position and velocity, and the position depends on the velocity through the classical equations of motion, in [46] it has been assumed that the propensity still depends on the wealth through a suitable *equation of motion*. This relationship between propensity and wealth is largely formal, and could be modified in many ways. Nevertheless, it is quite interesting to observe how the behavior of the hydrodynamics equations depends of the assumption of this *equation of motion*.

One important aspect of this passage is to justify it by solid arguments. In statistical mechanics the validity of the fluid dynamics is related to the gas density, or, what is the same, to the mean time between subsequent collisions [30]. In the present situation this validity is linked to the mean number of trades which occur in a fixed interval of time. This aspect of the matter has been recently discussed in a paper of Y. Wang, N. Ding, L. Zhang [135], who introduced the concept of the statistical description of the velocity of money circulation. This concept is based on *holding time of money* which is defined as time interval between two transactions. Although this concept is kept in mind when economists think of the velocity, even if the term referring to this kind of time interval has been mentioned in several cases, it is somewhat new to them since there has been no explicit specification of it in economics. Recently, several efforts have been devoted to measure the waiting time distributions in financial markets, see e.g. [99, 120]. In the process of money circulation, not only the amount of money each agent holds can be considered as random variable, but also the holding time between two transactions varies randomly. The theoretical investigation and the numerical simulations in [135] led to the conclusion that the velocity of money is proportional to the share for exchange, and, most important, reversely proportional to the number of agents, and independent of the average amount of money.

Using this result in a kinetic model of Boltzmann type, shows that the velocity of relaxation to the steady distribution of wealth is inversely proportional to the velocity of money circulation, which justifies an hydrodynamical description when the same velocity is sufficiently high.

3.1 Inhomogeneous kinetic models

Taking into account the results of Chapter 2, collision-like kinetic models seem to share some common features. Almost all of these models identify a very important variable for the shape of the wealth distribution, which is usually called the saving propensity to trade or the saving rate, respectively. This parameter can both enter into the collision rule as a constant factor [27], or it can be chosen as a random quantity [26]. Other studies include the saving propensity as an independent variable [32], without questioning on the relationship between wealth and saving.

These approaches to study both wealth and saving distributions show that in any case it could be reasonable to introduce other types of propensities into the game, which are not directly connected to the microscopic binary trade, while they could be important in the evolution of wealth in a market of agents. Among others, one can assume that the evolution of the density of wealth is heavily dependent on the propensity to invest, and at the same time that this propensity is closely related to the amount of money one agents has to deal with.

3.1.1 Models including propensity and wealth

If this is the case, one is led to study the evolution of the distribution function as a function depending on the propensity $x \in [0,1]$, wealth $w \in \mathbb{R}_+$ and time $t \in \mathbb{R}_+$, $f = f(x, w, t)$. In analogy with the classical kinetic theory of rarefied gases, it is useful to emphasize the role of the different parameters by identifying the velocity with the wealth, and the position with the saving propensity. By doing this, one assumes at once that the variation of the distribution $f(x, w, t)$ with respect to the wealth parameter w will depend on *collisions* between agents, while the change of distributions in terms of the propensity x depends on the *transport* term, which contains the equation of motion, namely the law of variation of x with respect to time,

$$\frac{dx}{dt} = \Phi(x, w). \tag{3.1}$$

The time-evolution of the distribution will obey a non-homogenous Boltzmann-like equation, given by

$$\frac{\partial}{\partial t} f(x, w, t) + \Phi(x, w) \frac{\partial}{\partial x} f(x, w, t) = \frac{1}{\tau} \mathcal{Q}(f)(x, w, t). \tag{3.2}$$

In (3.2) Φ is the law of variation of the propensity to invest given in (3.1), while $\mathcal{Q}$ represents as usual the collision operator which describes the change of density due to binary trades. Finally τ is a suitable relaxation time, depending on the velocity of money circulation [135]. Note that in physical applications where no forces are present, the transport term is simply $\Phi(x, w) = w$.

In any trading, savings come naturally [121]. In a real society or economy, the saving propensity is a very inhomogeneous parameter, and the interest of saving varies from person to person, according to their wealths. To move a step closer to the real situation, one has to introduce a saving factor widely distributed within the population [26, 32], and responsible of different outcomes into binary trades. The evolution of money in such a trading can be written as

$$\begin{aligned} v^* &= \gamma v + \epsilon(\gamma, \mu) \left[(1-\gamma) v + (1-\mu) w \right], \\ w^* &= \mu w + (1 - \epsilon(\gamma, \mu)) \left[(1-\gamma) v + (1-\mu) w \right]. \end{aligned} \tag{3.3}$$

Here (γ, w) and (μ, v) denote the saving propensities and wealths of agents before collisions. In a single collision it is assumed that the agents maintain their saving propensities fixed, so that the post-collision parameters are (γ, w^*) and (μ, v^*). Moreover $\epsilon(\gamma, \mu)$ denotes a random fraction, coming from the stochastic nature of the trading. Also, one can assume, like in [38] the trade

$$\begin{aligned} v^* &= (1-\gamma) v + \gamma w + \eta v, \\ w^* &= \gamma v + (1-\gamma) w + \tilde{\eta} w. \end{aligned} \tag{3.4}$$

In (3.4) the trade depends on a single saving rate $\gamma \in (0,1)$, while the risks of the market are described by η and $\tilde{\eta}$, equally distributed random variables with zero mean and variance σ^2.

In all cases, the operator which describes the binary interaction is the usual bilinear operator , we rewrite below by convenience

$$\int_{\mathbb{R}_+} \mathcal{Q}(f,f)(w)\phi(w)\,dw =$$

$$\frac{1}{2}\left\langle \int_0^1 dy \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} dv\,dw\,(\phi(v^*)+\phi(w^*)-\phi(v)-\phi(w))f(y,v)f(x,w)\right\rangle. \qquad (3.5)$$

In the continuous trading limit ($\gamma \to 0, \sigma^2/\gamma \to \lambda$), it has been shown in [38], that the collision operator (3.5) is well described by the Fokker-Planck collision operator

$$\mathcal{P}(w) = \frac{\lambda}{2}\frac{\partial^2}{\partial w^2}(w^2 f(w)) + \frac{\partial}{\partial w}(w - m(f))f(w), \qquad (3.6)$$

where $m(f)$ is the mean wealth of $f(w)$. The key parameter λ is obtained as the limit of the quotient of the variance and the saving rate. The homogenous kinetic equation

$$\frac{\partial}{\partial t} f(w,t) = \mathcal{P}(w,t), \qquad (3.7)$$

is such that both the mass and the mean wealth $m(f)$ are conserved in time. Moreover, for any initial density $f(w,t=0) = f_0(w)$ with mass ρ and mean m, equation (3.7) has a unique stationary state, the so-called Maxwellian state $M_{\rho,m}(w)$ given by

$$M_{\rho,m}(w) = \rho\frac{((\mu-1)m)^\mu}{\Gamma(\mu-1)}\frac{1}{w^{1+\mu}}\exp\left(-\frac{(\mu-1)m}{w}\right), \qquad (3.8)$$

where

$$\mu = 1 + \frac{2}{\lambda} > 1.$$

Therefore the Maxwellian distribution exhibits a Pareto power law tail for large w's. In particular, higher moments of the equilibrium Maxwellian are given in terms of mass ρ and mean m. The second moment can be easily evaluated considering that in equilibrium, i.e. as $t \to \infty$, one has

$$\begin{aligned}
0 =& \frac{\lambda}{2} \int_{\mathbb{R}_+} w^2 \frac{\partial^2}{\partial w^2} (w^2 M_{\rho,m}(w))\, dw + \int_{\mathbb{R}_+} w^2 \frac{\partial}{\partial w} [M_{\rho,m}(w)(w-m)]\, dw \\
=& \lambda \int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\, dw - 2 \int_{\mathbb{R}_+} w(w-m) M_{\rho,m}(w)\, dw \\
=& (\lambda - 2) \int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\, dw + 2m \int_{\mathbb{R}_+} w M_{\rho,m}(w)\, dw \\
=& (\lambda - 2) \int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\, dw + 2\rho m^2.
\end{aligned}$$

Thus, if $\lambda < 2$, the second moment of the Maxwellian is bounded, and

$$\int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\, dw = \frac{2}{2-\lambda} \rho m^2. \tag{3.9}$$

In the rest of this Chapter, we will assume that in a closed economy the Maxwellian distribution $M_{\rho,m}$, equilibrium solution of the Fokker-Planck equation (3.7), plays the same role as played by the Maxwell distribution in kinetic theory of rarefied gases. However, on the contrary to what happens in classical kinetic theory, where the equilibrium Maxwellian has all moments bounded, in this case the number of moments bounded in the equilibrium depends on the parameter λ in front of the second-order term in (3.6).

3.2 Hydrodynamic modelling

Section 3.1 enlightened the main properties of the collision operator (3.6), like the existence of a unique Maxwellian equilibrium with tails, and the consequent possibility to obtain higher order moments from the first two (mass and mean wealth). Like in classical kinetic theory of rarefied gases, these properties are the basis of the construction of a reasonable hydrodynamics for the evolution of the propensity. The underlying kinetic model is obtained by substituting the Fokker-Planck operator into the Boltzmann equation (3.2)

$$\frac{\partial}{\partial t} f(x,w,t) + \Phi(x,w) \frac{\partial}{\partial x} f(x,w,t) = \frac{1}{\tau} \mathcal{P}(f)(x,w,t). \tag{3.10}$$

The τ-parameter (the analogous of the Knudsen number in kinetic theory of gases [29]) represents a suitable relaxation time, and has to be assumed small in fluid dynamical regimes. A clear understanding of the derivation of macroscopic equations in kinetic theory can by obtained through the use of the splitting method, very popular in the numerical approach to the Boltzmann equation [64, 113]. If at each time step we consider sequentially the transport and relaxation operators in the Boltzmann equation (3.10), during this short time interval we recover the evolution of the density from the joint action of the relaxation

$$\frac{\partial f}{\partial t} = \frac{1}{\tau}\, \mathcal{P}(x, w, t), \tag{3.11}$$

and transport

$$\frac{\partial f}{\partial t} + \Phi(x, w)\frac{\partial}{\partial x} f(x, w, t) = 0. \tag{3.12}$$

As in classical kinetic theory, where the energy is conserved in collisions, the conservation of the mean wealth in the relaxation step is enough to guarantee that (3.11) pushes the solution towards the Maxwellian equilibrium with the same mass and mean of the initial datum. Then, if τ is sufficiently small, one can easily argue that the solution to (3.11) is *sufficiently close* to the Maxwellian, and this Maxwellian can be used into the transport step (3.12) to close the equations. In more details, since the Fokker-Planck operator (3.6) is both mass and momentum preserving, integrating equation (3.10) with respect to the wealth velocity w, using as test functions $\phi(w) = 1, w$ respectively we obtain

$$\int_{\mathbb{R}_+} \left(\frac{\partial f}{\partial t} + \Phi(x, w)\frac{\partial}{\partial x} f(x, w, t) \right) dw = 0, \tag{3.13}$$

and

$$\int_{\mathbb{R}_+} w \left(\frac{\partial f}{\partial t} + \Phi(x, w)\frac{\partial}{\partial x} f(x, w, t) \right) dw = 0, \tag{3.14}$$

Let us fix the law Φ to be linearly dependent on w,

$$\Phi(x, w) = (w - \chi\bar{w})\mu(x), \tag{3.15}$$

where χ is a positive constant and $\bar{w}$ represent a suitable fixed value of the wealth. Then, one obtains from (3.13), (3.14) the equations

$$\frac{\partial \rho}{\partial t} + \mu(x)\frac{\partial}{\partial x}\Big[\rho\big(m - \chi\bar{w}\big)\Big] = 0, \tag{3.16}$$

$$\frac{\partial(\rho m)}{\partial t} + \mu(x)\frac{\partial}{\partial x}\left[\int_{\mathbb{R}_+} w^2 f(x, w, t)\, dw - \chi\bar{w}\rho m\right] = 0. \tag{3.17}$$

In (3.16), (3.17) $\rho(x, t), m(x, t)$ are the macroscopic variables, the local density of agents with propensity x at time t, given by

$$\rho(x, t) = \int_{\mathbb{R}_+} f(x, w, t)\, dw, \tag{3.18}$$

and the local mean

$$m(x,t) = \frac{1}{\rho(x,t)} \int_{\mathbb{R}_+} w f(x,w,t)\, dw. \tag{3.19}$$

Equation (3.17) depends on the second moment of the density. Using the equilibrium Maxwellian (3.9), however, we can express this second moment in terms of the first two. By this relationship we finally obtain the following system of equations

$$\frac{\partial \rho}{\partial t} + \mu(x) \frac{\partial}{\partial x}\Big[\rho\big(m - \chi\bar{w}\big)\Big] = 0, \tag{3.20}$$

$$\frac{\partial(\rho m)}{\partial t} + \mu(x) \frac{\partial}{\partial x}\Big[\rho m\Big(\frac{2}{2-\lambda} m - \chi\bar{w}\Big)\Big] = 0, \tag{3.21}$$

which have to be solved on $(0,1) \times (0,T)$ with appropriate boundary and initial conditions. Using (3.20) we can rewrite the second equation as

$$\frac{\partial m}{\partial t} + \mu(x)(m - \chi\bar{w})\frac{\partial m}{\partial x} + \frac{\lambda}{2-\lambda}\frac{1}{\rho}\frac{\partial}{\partial x}\big[\rho m^2\big] = 0. \tag{3.22}$$

The hydrodynamic equations (3.20), (3.21) can be written in symmetric hyperbolic form [**?**]. Multiplying (3.20) by m and subtracting it from (3.21) we obtain

$$\begin{aligned} \rho m_t + \mu(x)\Big[&-(\rho_x m + \rho m_x)m \\ &+ \chi\bar{w} m \rho_x + (\rho_x m + 2\rho m_x)\frac{2}{2-\lambda} m - (\rho_x m + \rho m_x)\chi\bar{w}\Big] = 0. \end{aligned} \tag{3.23}$$

Define $u = (\rho, m)$. Then a direct computation shows that the system (3.20), (3.23) can be written in compact form as

$$A^0(t,x,u)\frac{\partial u}{\partial t} + A^1(t,x,u)\frac{\partial u}{\partial x} = 0. \tag{3.24}$$

In (3.24) A^0 and A^1 denote the with symmetric matrices

$$A^0(t,x,u) = \begin{pmatrix} \frac{1}{\rho}\frac{\lambda}{2-\lambda} m^2 & 0 \\ 0 & \rho \end{pmatrix}, \tag{3.25}$$

$$A^1(t,x,u) = \begin{pmatrix} \rho(m-\chi\bar{w})\frac{\lambda}{2-\lambda} m^2 & \frac{\lambda}{2-\lambda} m^2 \\ \frac{\lambda}{2-\lambda} m^2 & \big(1 + \frac{4}{2-\lambda}\big)\rho m - \rho\chi\bar{w} \end{pmatrix}. \tag{3.26}$$

Note that, for all $\rho, m \in G$ belonging to a suitable set G, $A^0(t,x,u)$ is uniformly positive definite provided $\lambda < 2$. Due to their structure, suitable numerical methods are available [83].

3.2.1 Universality of the closure

One of the main advantages linked to the use of the Fokker-Planck collision operator (3.6) is that it is immediate to recover its steady state, namely the Maxwellian (3.8). Unlikely, if we use a different collision operator like the Boltzmann operator (3.5), the explicit form of this Maxwellian is unknown. This problem is not present in classical elastic kinetic theory of rarefied gases, where the Maxwellian is uniquely defined independently of the choice of the binary collision operator [30]. This fact causes a first serious problem in the justification of the validity of the closure, which in principle has to be independent of the choice of the underlying microscopic model of collisions, except, eventually, for constant parameters. Using the results of [104], however, we can easily conclude that the closure law

$$\int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\, dw = C\rho m^2. \tag{3.27}$$

where $C = 2/(2-\lambda)$ in (3.9), has a universal validity, and the type of binary trade used into the collision operator (3.5) is reflected only through the precise value of the constant C. Let us suppose once more that the (conservative) binary interactions are described by the rules

$$v^* = p_1 v + q_1 w, \qquad w^* = p_2 v + q_2 w, \tag{3.28}$$

where

$$\langle p_1 + p_2 \rangle = 1, \quad \langle q_1 + q_2 \rangle = 1. \tag{3.29}$$

We remark that both trades (3.3) and (3.4) satisfy assumption (3.29). In this case, application of formula (3.5) with $\phi(w) = w^n$ allows to compute recursively the evolution of the principal moments

$$M_n(t) = \int_{\mathbb{R}_+} w^n f(w,t)\, dw$$

with $n \geq 2$ (see [104] for details). One obtains

$$\begin{aligned} \frac{d}{dt} M_n(t) &= \frac{1}{2} \left\langle (p_1^n + p_2^n - 1) + (q_1^n + q_2^n - 1) \right\rangle M_n(t) + \\ &\frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} \left\langle p_1^k q_1^{n-k} + p_2^k q_2^{n-k} \right\rangle M_k(t) M_{n-k}(t). \end{aligned} \tag{3.30}$$

Considering that the first moment is conserved, $M_1(t) = m$, equation (3.30) also furnishes a recursive computation of the principal moments of the stationary solution

$$M_n = \rho \frac{\sum_{k=1}^{n-1} \binom{n}{k} \left\langle p_1^k q_1^{n-k} + p_2^k q_2^{n-k} \right\rangle M_k M_{n-k}}{2 - \left\langle p_1^n + p_2^n + q_1^n + q_2^n \right\rangle}. \tag{3.31}$$

In particular,

$$M_2 = \int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\, dw = \rho m^2 \frac{2\,\langle p_1 q_1 + p_2 q_2\rangle}{2 - \langle p_1^2 + p_2^2 + q_1^2 + q_2^2\rangle}, \tag{3.32}$$

that coincides with the law (3.27), in which

$$C = \frac{2\,\langle p_1 q_1 + p_2 q_2\rangle}{2 - \langle p_1^2 + p_2^2 + q_1^2 + q_2^2\rangle}. \tag{3.33}$$

If we consider the trade (3.4), where

$$\begin{aligned} p_1 &= 1 - \gamma + \eta, \quad q_1 = \gamma, \\ p_2 &= \gamma, \quad q_2 = 1 - \gamma + \tilde{\eta}. \end{aligned} \tag{3.34}$$

we obtain for C the value

$$C = \frac{2\gamma(1-\gamma)}{2\gamma(1-\gamma) - \sigma^2}. \tag{3.35}$$

Note that this value corresponds to the choice

$$\lambda = \frac{\sigma^2}{\gamma(1-\gamma)}$$

in (3.9). This result enlightens the meaning of the constant λ appearing in the Fokker-Planck equation (3.6) in terms of the underlying binary trade (3.4). We remark that also in this case the stationary state of the Boltzmann equation possesses tails. Consider now the trade (3.3), where for simplicity $\epsilon(\gamma,\mu) = 1/2$,

$$\begin{aligned} p_1 &= (1+\gamma)/2, \quad q_1 = (1-\mu)/2, \\ p_2 &= (1-\gamma)/2, \quad q_2 = (1+\mu)/2. \end{aligned} \tag{3.36}$$

In this case

$$C = \frac{2(1-\gamma\mu)}{2 - (\gamma^2 + \mu^2)}. \tag{3.37}$$

This corresponds to the choice

$$\lambda = \frac{(\gamma-\mu)^2}{1-\gamma\mu}$$

in (3.9). Here, however, the constant λ is always strictly less than 2. This is related to the fact that, for pointwise collisions like the one defined by (3.36), the stationary solution has all moments bounded [104]. In conclusion, there is a universal validity of the closure of hydrodynamic equations, at least in the well-defined case of conservative economies.

3.2.2 Which law for the propensity to trade?

To proceed and to obtain (at least numerical) results on the time-evolution of the macroscopic quantities, it is necessary to set the two variables x and w into relation. In classical hydrodynamics, where the variables are position and velocity, this relation is obvious, since velocity is the time derivative of position. In absence of forces, it corresponds to choose $\Phi(x, w) = w$. To find an analogue for our economic setting, namely a law for the propensity to trade, we resort to some arguments within the concepts of opinion formation.

There, a class of kinetic models of opinion formation, based on two-body interactions involving both compromise and diffusion properties in exchanges between individuals have been introduced. In the quasi-invariant opinion limit, these models are described by partial differential equation of Fokker-Planck type

The equilibrium state of the Fokker-Planck equation can be computed explicitly and, in absence of internal points in which diffusion is missing, is in most cases well represented by a Beta distribution

$$B(x; a, b) = \frac{x^a(1-x)^b}{\int_0^1 u^a(1-u)^b\,du} = \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)}\,x^a(1-x)^b \tag{3.38}$$

where $a, b > -1$, and Γ is the gamma function. In what follows, we assume that the stationary profile for the distribution of our propensity to trade follows a law of type (3.38). Taking into account that this stationary profile is stable, a highly reasonable hypothesis is to assume that the rate of variation of the propensity is proportional to the density of people having that propensity. In this case, in order to maintain the lower and upper bounds of $x(t)$ we assume the law

$$\Phi(x, w) = x^a(1-x)^b H(w), \tag{3.39}$$

where $a, b > 0$, and the coefficient $H(w)$ takes into account the dependence of the law of variation on the (relative) wealth. We remark that, thanks to the positivity of a, b, people with propensity to trade close to zero or one are more stable in their propensity, while people with intermediate propensity have more inclination to change their idea.

Last, the form of $H(w)$ can be deduced from an argument similar to that introduced in Chapter2, Section 2.2.

Consider the case of a rational investor seeking to maximize his utility from wealth after each trade. He can choose a combination of saving his current wealth and the return from a trade, thus, since η has zero mean, his expected post-trade wealth is

$$\langle w^* \rangle = w + x(v - w) = (1-x)w + xv.$$

The investor's choice of his propensity to invest can be interpreted as the choice of combinations of two options, w and v. The investor tries to maximize his expected utility from post-trade wealth $\langle u(w^*)\rangle$, where u is a utility function characterizing the investor's satisfaction from wealth, e.g. the Cobb-Douglas function

$$u(w, v) = w^\alpha v^\beta,\ \alpha + \beta = 1.$$

The projection of this function onto the w-v-plane gives level curves of constant utility, the so-called *indifference curves*. The investor is indifferent to the different combinations of v and w on such a curve, or in other words, at each point on an indifference curve he has no preference for one combination over another. A rational investor will choose a value for x that maximizes his utility from post-trade wealth. Therefore it is clear, that x should be a function of the investor's wealth w or his relative wealth $w - \chi\bar{w}$, if utility from wealth is measured with respect to the mean wealth $\chi\bar{w}$ as a reference value (ξ denotes here a suitable constant). In other words, the fundamental law of physics about position and velocity and their relation is replaced here by an economic law that relates a rational investor's propensity to invest and his wealth based on the principle of utility maximization.

In general, the optimal choice of x depends on the underlying utility function and this can lead to quite complex and non-linear relationships between propensity to invest x and wealth w. However, recall that we are considering a regime where the time between trades τ is very small. If we assume that Φ is smooth enough, we can approximate it by a linear relation and ignore terms of higher order. Furthermore, empirical results [121] from economic literature suggest, that typically individuals have decreasing absolute risk aversion.

A simple way to take into account these facts is to relate the time variation of the propensity to the relative (with respect to the mean) wealth. A way to cope with these demands is to introduce the following law

$$\Phi(x, w) = \pm\vartheta x^a (1 - x)^b \big(w - \chi\bar{m}(t)\big), \tag{3.40}$$

where $\bar{m}(t) = \int_0^1 m(x, t)\, dx$ denotes the mean wealth at time t, and ϑ is a positive constant. Let us remark that the choice $a = b = 1$ implies an exponential decay of $x(t)$ towards one of the two extremal points. Since $x(t) < 1$, its variation with respect to time can be controlled by a suitable choice of these parameters. Let us note further that the choice of a positive (negative) sign into (3.40) implies that individuals with a higher (lower) wealth will be more (less) willing to trade than individuals with lower (higher) wealth. Clearly, this is only one choice among many possibilities. However, it seems a promising, quite natural approach, which is at the same time flexible enough, and sufficiently easy to be tractable from a numerical point of view. Using this choice in (3.20) and (3.21), and absorbing ϑ into time, we arrive to the following system

$$\frac{\partial \rho}{\partial t} + x^a(1-x)^b \frac{\partial}{\partial x}\Big[\rho\big(m - \chi \bar{m}\big)\Big] = 0, \tag{3.41}$$

$$\frac{\partial \rho m}{\partial t} + x^a(1-x)^b \frac{\partial}{\partial x}\Big[\rho m\Big(\frac{2}{2-\lambda} m - \chi \bar{m}\Big)\Big] = 0. \tag{3.42}$$

As before, using (3.41) we can rewrite the second equation as

$$\frac{\partial m}{\partial t} + x^a(1-x)^b(m - \chi\bar{m})\frac{\partial m}{\partial x} + \frac{\lambda}{2-\lambda}\frac{1}{\rho}\frac{\partial}{\partial x}\big[\rho m^2\big] = 0. \tag{3.43}$$

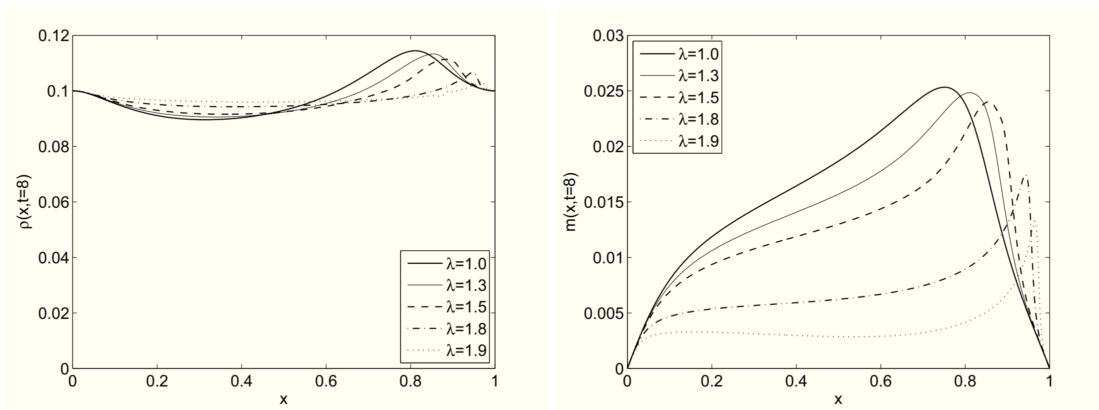

Figure 3.1: Influence of different values for λ ($\lambda = 1, 1.3, 1.5, 1.8, 1.9$)

3.2.3 Numerical examples

To solve (3.41), (3.42) numerically, we use a standard finite element method. We choose quadratic Lagrangian elements on a uniform grid with 480 nodes. We use the initial conditions

$$\rho_0(x) = 0.1, \quad m_0(x) = x(1-x), \quad x \in (0,1).$$

At the boundaries we use homogenous Neumann conditions for ρ and homogenous Dirichlet conditions for m. If not mentioned otherwise, we choose $a = b = 2$, $\chi = 1$, $\lambda = 1$ and the final time is $T = 15$.

Figure 3.1 displays the numerical solution for different values of λ. Recall that a higher λ corresponds to tails with a lower Pareto index, and this corresponds to a society with a strong economy. We can observe the same effect here at the macroscopic level. As λ increases, the density of agents with a high propensity to

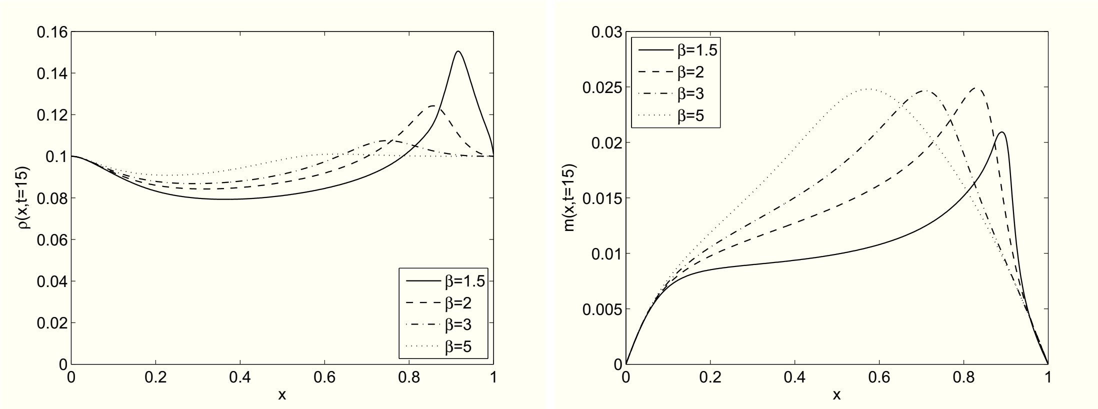

Figure 3.2: Influence of different values for b ($b = 1.5, 2, 3, 5$)

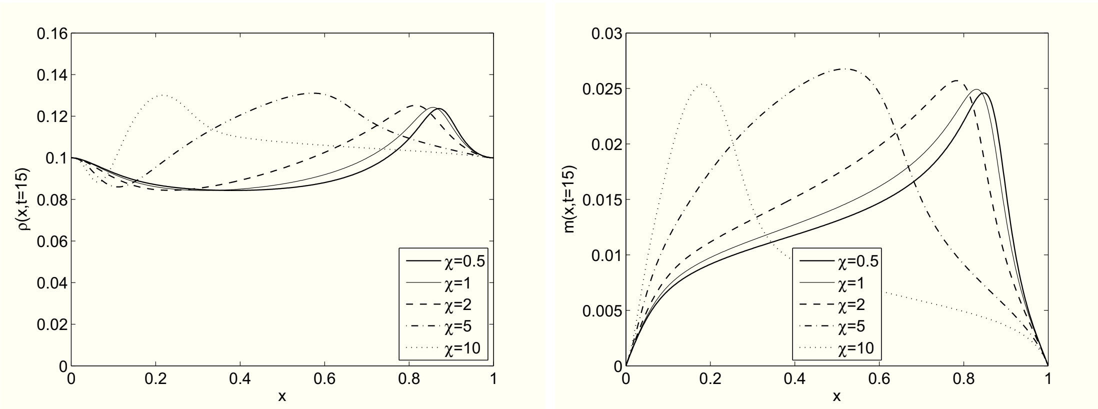

Figure 3.3: Influence of different values for χ ($\chi = 0.5, 1, 2, 5, 10$)

trade increases while the wealth density is increasing for larger propensities and decreasing for smaller propensities, i.e. a large fraction of the total wealth is owned by a small group of agents.

Figure 3.2 shows the influence of different variants of law (3.40). For a value of $b = 1.5$ agents with wealth above the mean wealth increase their propensity to trade which leads to a peak formation in the density ρ close to $x = 1$. For higher values of b the propensity to invest grows slower when above the average wealth, therefore the peak is less pronounced.

The influence of the parameter χ can be observed in Figure 3.3. For high values of χ only very wealthy agents increase their propensity to trade, all other decrease it. This results in a peak formation at lower levels of x with agents saving most of their wealth.

The previous examples enlighten the influence of the law (3.1) in the evolution

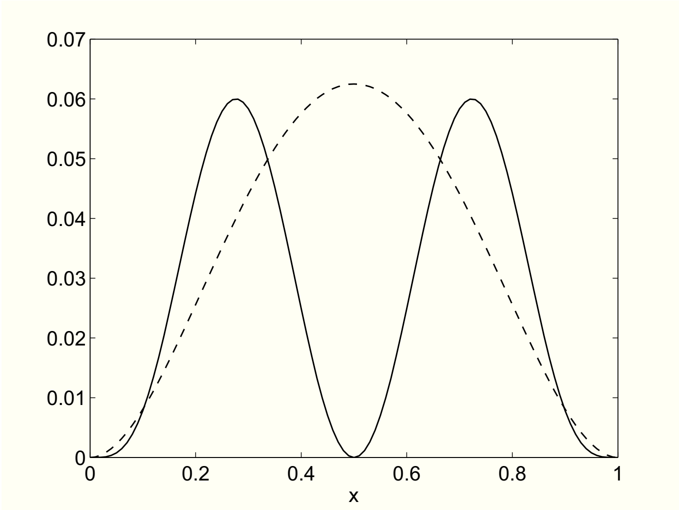

Figure 3.4: Functions $b(x)$ and $\nu(x)$ used in the numerical illustration.

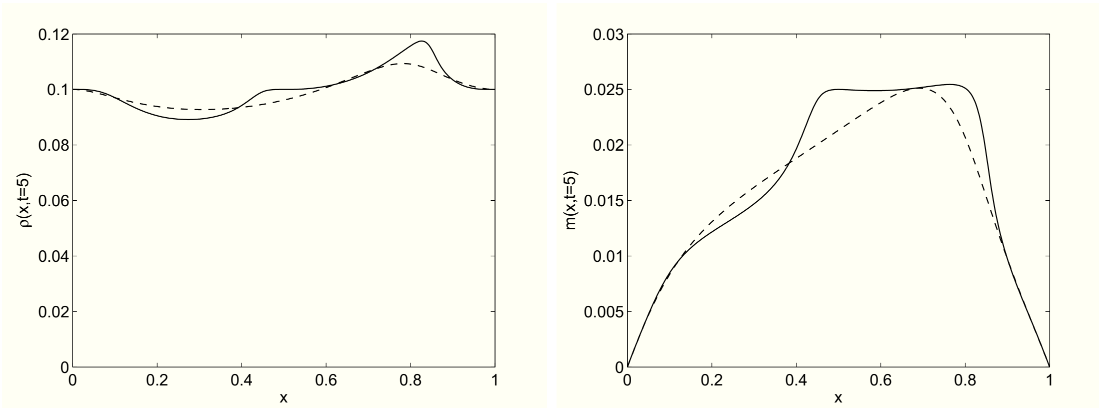

Figure 3.5: Influence of different laws for propensity: solid lines correspond to law (3.44), broken lines to (3.40).

of the macroscopic quantities. Clearly, the proposed law (3.40) constitutes only a prototype towards a better understanding of the whole matter. Changing the law (3.40) allows to clarify the role of the various parameters involved. By maintaining the linearity with respect to the wealth parameter w, we consider in what follows the law

$$\tilde{\Phi}(x, w) = \nu(x)(w - \bar{m}(t)), \tag{3.44}$$

with $\nu(x) = cx^a(1/2 - x)^\gamma(1 - x)^b$. Law (3.44) assumes as hypothesis the (reasonable?) fact that in correspondence to some point (in this case $x = 1/2$) the propensity tends to stabilize. Figure 3.4 shows a comparison of $\mu(x)$, $\nu(x)$ corresponding to the same values of a and b.

For numerical simulation we choose $a = b = 4$, $\gamma = 2$, $c = 750$, $T = 5$ and $\lambda = 1$ and the same initial and boundary conditions as above. Figure 3.5 shows the plot of the densities ρ and m at time $T = 5$ resulting from the computations with the different laws.

Appendix A

Derivation of the Fokker-Plank approximation for the chartist distribution

First we recall the definition of weak solution for kinetic equations of the form (2.37). Let $M_p(I) = \left\{ \Theta \in M_p : \int_I |y|^p d\Theta(y) < +\infty \right\}$ be the space of all Borel measure of finite p-th order momentum, equipped with the topology of weak convergence of the measures. Let $F_s(I)$ be the class of all real functions h on I such that $h(1) = h(-1) = h'(1) = h'(-1) = 0$ and $h^m(y)$ is holder continuous of order δ

$$\|h^{(m)}\|_\delta = \sup_{y_1 \neq y_2} \frac{|h^{(m)}(y_1) - h^m(y_2)|}{|y_1 - y_2|^\delta} < \infty$$

where $0 < \delta \leq 1$ and $m + \delta = s$

Definition A.0.1 *Let $f^0(y) \in M_p(I)$ with $p > 1$ an initial probability density, a weak solution for (2.37) is any probability density $f \in C^1(\mathbb{R}^+, M_p(I))$ satisfying*

$$\frac{d}{dt}\int_I f(y,t)\phi(y)dy = (Q(f,f),\phi) =$$
$$\int_{I^2}\int_B B(y,y_*)f(y,t)f(y_*,t)(\phi(y') - \phi(y))dy_* dy d\eta d\eta_*$$

for $t > 0$ and all $\phi \in F_p(I)$, and such that

$$\lim_{t\to 0}\int_I f(y,t)\phi(y)dy = \int_I f^0(y)\phi(y)dy$$

Similarly

Definition A.0.2 *Given an initial price distribution $V_0(s) \in M_p(\mathbb{R}^+)$ with $p > 1$ a weak solution to (2.40) is any probability density $V \in C^1(\mathbb{R}^+, M_p(\mathbb{R}^+))$ satisfying*

$$\frac{d}{dt}\int_{\mathbb{R}^+} V(s,t)\phi(y)dy = L(V,\phi) =$$
$$\int_{\mathbb{R}^+}\int_B b(s)V(s,t)(\phi(s') - \phi(s))dsd\eta$$

for $t > 0$ and all $\phi \in F_p(K)$, with K compact $\in \mathbb{R}^+$ and such that

$$\lim_{t\to 0}\int_I V(s,t)\phi(s)ds = \int_I \phi(s)V_0(s)ds$$

We start analyzing the Boltzman equation for the investment propensity distribution of chartists. The scaled density $\tilde{f}(y,\tau)$ satisfies the equation in weak form

$$\frac{d}{d\tau}\int_I \tilde{f}(y,\tau)\phi(y)dy = \frac{1}{\xi}(Q(\tilde{f},\tilde{f}),\phi(y)) =$$
$$\frac{1}{\xi}\int_{I^2}\int_{B^2}\Theta(\eta)\Theta(\eta_*)\tilde{f}(y)\tilde{f}(y_*)(\phi(y') - \phi(y))dy_*dyd\eta d\eta_*$$

Given $\delta \geq 0$ let us take $\phi \in F_{2+\delta}(I)$.
From the microscopic dynamic of the Chartists we have

$$y' - y = -(\alpha_1 H(y) + \alpha_2)y + \alpha_1 H(y)y_* + \alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right) + D(y)\eta$$

. In the asymptotic limit $\xi \to 0$, $\sigma^2 \to 0$, we have $y - y' \sim 0$ and we can use the Taylor expansion

$$\begin{aligned}\phi(y') - \phi(y) &= \left(-(\alpha_1 H(y) + \alpha_2)y + \alpha_1 H(y)y_* + \alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right) + D(y)\eta\right)\phi'(y)\\ &+\frac{1}{2}\left(-(\alpha_1 H(y) + \alpha_2)y + \alpha_1 H(y)y_* + \alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right) + D(y)\eta\right)^2\phi''(\tilde{y}),\end{aligned}$$

where, for some $0 \leq \theta \leq 1$

$$\tilde{y} = \theta y' + (1-\theta)y.$$

Inserting this expansion in the weak formulation of the Boltzman equation, we get

$$\begin{aligned}&\frac{d}{d\tau}\int_I \tilde{f}(y,\tau)\phi(y)dy =\\ &\frac{1}{\xi}\int_{I^2}\int_{B^2}\Theta(\eta)\Theta(\eta_*)[\left(-(\alpha_1 H(y) + \alpha_2)y + \alpha_1 H(y)y_* + \alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right) + D(y)\eta\right)\phi'(y)+\\ &+\frac{1}{2}\left(-(\alpha_1 H(y) + \alpha_2)y + \alpha_1 H(y)y_* + \alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right) + D(y)\eta\right)^2\phi''(y)]\tilde{f}(y,\tau)\tilde{f}(y_*,\tau)dy_*dyd\eta d\eta_*\\ &+R(\xi,\sigma)\end{aligned} \tag{A.1}$$

where

$$R(\xi,\sigma) = \frac{1}{2\xi}\int_{I^2}\int_{B^2}\Theta(\eta)\Theta(\eta_*)\left(-(\alpha_1 H(y)+\alpha_2)y+\alpha_1 H(y)y_*+\alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right)+D(y)\eta\right)^2 \cdot$$
$$\cdot \quad (\phi''(\tilde{y})-\phi''(y))\tilde{f}(y,\tau)\tilde{f}(y_*,\tau))dy_*dyd\eta_*d\eta$$

To get a rigorous derivation of the Fokker-Plank limit we need to show that the above expression goes to zero in the asymptotic limit. In order to prove that the remainder in (A.1), goes to zero we start observing that, being $\phi \in F_{2+\delta}(I)$, and $|\tilde{y}-y| = \theta|y'-y|$

$$|\phi''(\tilde{y})-\phi''(y)| \le \|\phi''\||\tilde{y}-y|^\delta \le \|\phi''\|_\delta |y'-y|^\delta$$

Hence

$$|R(\xi,\sigma)| \le \frac{\|\phi''\|_\delta}{2\xi}\int_{I^2}\int_{B^2}\Theta(\eta)\Theta(\eta_*)\cdot$$
$$\cdot\left|-(\alpha_1 H(y)+\alpha_2)y+\alpha_1 H(y)y_*+\alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right)+D(y)\eta\right|^{2+\delta}\tilde{f}(y,\tau)\tilde{f}(y_*,\tau))dy_*dyd\eta_*d\eta.$$

Using the fact that $|H(y)| \le 1$ and $|\Phi\left(\frac{S'(t)}{S(t)}\right)| \le 1$, and applying the following inequality

$$\begin{aligned}
&\left|-(\alpha_1 H(y)+\alpha_2)y+\alpha_1 H(y)y_*+\alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right)+D(y)\eta\right|^{2+\delta} \le \\
&2^{1+\delta}\left\{[\alpha_1 H(y)y_*-(\alpha_1 H(y)+\alpha_2)y]^{2+\delta}+[\alpha_2\Phi\left(\tfrac{S'(t)}{S(t)}\right)+D(y)\eta]^{2+\delta}\right\} \\
&\le 2^{1+\delta}\left\{2^{1+\delta}[\alpha_1^{2+\delta}+2^{1+\delta}(\alpha_1^{2+\delta}+\alpha_2^{2+\delta})]+2^{1+\delta}\alpha_2^{2+\delta}+2^{1+\delta}\eta^{2+\delta}\right\} \\
&\le 2^{2+\delta}\left[K(\delta)(\alpha_1^{2+\delta}+\alpha_2^{2+\delta})+\eta^{2+\delta}\right]
\end{aligned}$$

with $K(\delta)$ a suitable constant, we finally obtain

$$|R(\xi,\sigma)| \le 2^{1+\delta}\|\phi''\|_\delta\left(K(\delta)(\frac{\alpha_1^{2+\delta}}{\xi}+\frac{\alpha_2^{2+\delta}}{\xi}+\frac{1}{2\xi}\int_B\Theta(\eta)|\eta|^{2+\delta}d\eta\right)$$

To simplify computations, we assume that Θ, with zero mean and variance $\lambda\xi$ is the density of $\sqrt{\lambda\xi}W$, where W is a random variable with zero mean and unit variance, that belongs to $M_{2+\alpha}$, for $\alpha > \delta$, so we have

$$\int_B\Theta(\eta)|\eta|^{2+\delta}d\eta = E\left(\left|\sqrt{\lambda\xi}W\right|^{2+\delta}\right) = (\lambda\xi)^{1+\frac{\delta}{2}}E\left(|W|^{2+\delta}\right),$$

and $E\left(|W|^{2+\delta}\right)$ is bounded. This is enough to show that when both α_i, $i = 1, 2$, ξ and σ goes to zero with $\sigma^2 = \lambda\xi$ the quantity $R(\xi,\sigma)$ tends to zero.

Taking the limit in the weak formulation we obtain now

$$\lim_{\xi\to 0}\frac{1}{\xi}\int_{I^2}\int_{B^2}\Theta(\eta)\Theta(\eta_*)\left[\left(-(\alpha_1 H(y)+\alpha_2)y+\alpha_1 H(y)y_*+\alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right)+D(y)\eta\right)\phi'(y)+\right.$$
$$\left.+\frac{1}{2}\left(-(\alpha_1 H(y)+\alpha_2)y+\alpha_1 H(y)y_*+\alpha_2\Phi\left(\frac{S'(t)}{S(t)}\right)+D(y)\eta\right)^2\phi''(y)\right]\tilde{f}(y,\tau)\tilde{f}(y_*,\tau)dy_*dyd\eta d\eta_*$$
$$=\int_I\left[\left(\rho_C\tilde{\alpha_1}H(y)(Y-y)+\rho_C\tilde{\alpha_2}\left(\Phi\left(\frac{S'(t)}{S(t)}\right)-y\right)\right)\phi'(y)+\frac{\lambda}{2}(\rho_C D^2(y))\phi''(y)\right]\tilde{f}(y,\tau)dy$$

Which is nothing but the weak form of the Fokker-Plank equation

$$\frac{\partial\tilde{f}}{\partial\tau}+\frac{\partial}{\partial y}\left[\left(\rho_C\tilde{\alpha_1}H(y)(Y-y)+\rho_C\tilde{\alpha_2}\left(\Phi\left(\frac{S'(t)}{S(t)}\right)-y\right)\right)\tilde{f}_C\right]=\frac{\lambda\rho_C}{2}\frac{\partial^2}{\partial y^2}[(D^2(y))\tilde{f}]. \tag{A.2}$$

We can then state the following theorem

THEOREM A.0.3 *Let the probability density $f^0\in\boldsymbol{M}_0(I)$, and let the symmetric density Θ be in $M_{2+\alpha}$ with $\alpha>\delta$. Then, as $\beta\to 0$, $\sigma\to 0$ in such a way that $\sigma^2=\lambda\beta$ the weak solution to the Boltzmann equation for the scaled density $\tilde{f}(y,\tau)$ with $\tau=\beta t$ converges, up to extraction of a subsequence, to the weak solution of the Fokker-Plank equation (A.2).*

Appendix B

Derivation of the Fokker-Plank limit for the price distribution

In this appendix we recover the Fokker-Plank limit for the Boltzman equation of the scaled density distribution of the price.
Again we start with the weak formulation which now is

$$\frac{d}{d\tau}\int_0^{+\infty}\tilde{V}(s,\tau)\phi(s)ds = \frac{1}{\xi}\int_0^{+\infty}\int_B \Psi(\eta)\tilde{V}(s,\tau)(\phi(s')-\phi(s))dsd\eta \tag{B.1}$$

for all $\phi \in F_{2+\delta}(K)$, with $\delta > 0$ and for any compact interval $I = [0, a] \subset [0, +\infty)$
Using a Taylor expansion of ϕ around s

$$\begin{aligned}\phi(s')-\phi(s) &= (\beta(\rho_C t_C Y(t)s + \rho_F\gamma(S_F - s)) + \eta s)\,\phi'(s) \\ &+ \frac{1}{2}(\beta(\rho_C t_C Y(t)s + \rho_F\gamma(S_F - s)) + \eta s)^2\,\phi''(\tilde{s}),\end{aligned}$$

where for some $0 \le \theta \le 1$

$$\tilde{s} = \theta s' + (1-\theta)s$$

and substituting into (B.1) we have

$$\begin{aligned}\frac{d}{d\tau}\int_0^{+\infty}\tilde{V}(s,\tau)\phi(s)ds &= \frac{1}{\xi}\int_0^{+\infty}\int_B \Psi(\eta)[(\beta(\rho_C t_C Y(t)s + \rho_F\gamma(S_F - s)) + \eta s)\,\phi'(s) \\ &+ \frac{1}{2}(\beta(\rho_C t_C Y(t)s + \rho_F\gamma(S_F - s)) + \eta s)^2\,\phi''(s)]\tilde{V}(s,\tau)dsd\eta \\ &+\frac{1}{\xi}R(\beta,\zeta)\end{aligned}$$

where

$$\frac{1}{\xi}R(\beta,\zeta) = \frac{1}{2\xi}\int_0^{+\infty}\int_B \Psi(\eta)\,(\beta(\rho_C t_C Y_C(t)s + \rho_F\gamma(S_F - s)) + \eta s)^2\cdot(\phi''(\tilde{s}) - \phi''(s))\tilde{V}(s,\tau)dsd\eta$$

Analogously as before, in order to perform the asymptotic limit we need to show that the quantity $R(\beta,\zeta)$ approach to zero. We observe that being $\phi \in F_{2+\delta}(\mathbb{R}+)$ and $|\tilde{s}-s| = \theta|s'-s|$ we have

$$|\phi''(\tilde{s}) - \phi''(s)| \leq \|\phi''\|_\delta |s'-s|^\delta$$

hence

$$|\frac{1}{\xi}R(\beta,\zeta)| \leq \frac{\|\phi''\|_\delta}{2\xi}\int_0^{+\infty}\int_B \Psi(\eta)\left|\beta(\rho_C t_C Y(t) + \rho_F\gamma\frac{(S_F-s)}{s}) + \eta\right|^{2+\delta} s^{2+\delta}\tilde{V}(s,\tau)dsd\eta$$

we observe that

$$\left|\beta(\rho_C t_C Y(t) + \rho_F\gamma\frac{(S_F-s)}{s}) + \eta\right|^{2+\delta} \leq$$

$$2^{2(1+\delta)}\beta^{2+\delta}|\rho_C t_C Y(t)|^{2+\delta} + 2^{2(1+\delta)}\beta^{2+\delta}(\rho_F\gamma)^{2+\delta}\left|\frac{S_F-s}{s}\right|^{2+\delta} + 2^{1+\delta}|\eta|^{2+\delta}$$

As in appendix A we assume that Ψ, with zero mean and variance $\nu\zeta$ is the density of $\sqrt{\nu\zeta}W$, where W is a random variable with zero mean and unit variance, that belongs to $M_{2+\alpha}$, for $\alpha > \delta$, so we have

$$\int_B \Psi(\eta)|\eta|^{2+\delta}d\eta = E\left(\left|\sqrt{\nu\zeta}W\right|^{2+\delta}\right) = (\nu\zeta)^{1+\frac{\delta}{2}}E\left(|W|^{2+\delta}\right), \tag{B.2}$$

and $E\left(|W|^{2+\delta}\right)$ is bounded.

Then we can see that

$$\begin{aligned}|\frac{1}{\xi}R(\beta,\zeta)| \;&\leq\; \frac{\|\phi''\|_\delta}{2\xi}\left[2^{2(1+\delta)}\beta^{2+\delta}|\rho_C t_C Y_C(t)|^{2+\delta} + 2^{3(1+\delta)}(\beta\rho_F\gamma)^{2+\delta} + (\nu\xi)^{1+\frac{\delta}{2}}K\right]\int_0^{+\infty} s^{2+\delta}\tilde{V}(s)ds \\ &+\; 2^{3(2+\delta)}(\beta\rho_F\gamma)^{2+\delta}S_F^{2+\delta},\end{aligned}$$

where the constant K is a bound for $E\left(|W|^{2+\delta}\right)$. From this inequality it follows that $\frac{1}{\xi}R(\beta,\zeta)$ tends to zero as $\xi \to 0$ if

$$\int_0^{+\infty}\tilde{V}(s,\tau)s^{2+\delta}ds$$

is bounded at any fixed time $\tau > 0$, provided that the same bound holds at time $\tau = 0$.

To show this we start from the weak formulation of the Boltzman equation associated to $V(s)$ given by (2.40-2.41).

The choice $\phi(y) = y^p$ give us the following

$$\frac{d}{dt}\int_0^{+\infty} V(s,t)s^p ds = \int_0^{+\infty}\int_B \Psi(\eta)V(s,t)(s'^p - s^p))dsd\eta.$$

Now

$$s'^p - s^p = ps^{p-1}(s'-s) + \frac{1}{2}p(p-1)\tilde{s}^{p-2}(s'-s)^2$$

where for some $0 \leq \theta \leq 1$,

$$\tilde{s} = \theta s' + (1-\theta)s.$$

Recalling the microscopic dynamic for the evolution of the price variable s we can write

$$\frac{d}{dt}\int_0^{+\infty} V(s,t)s^p ds = \int_0^{+\infty}\int_B \Psi(\eta)V(s)\left[ps^{p-1}(s'-s) + \frac{1}{2}p(p-1)\tilde{s}^{p-2}(s'-s)^2\right]dsd\eta =$$
$$\int_0^{+\infty}\int_B \Psi(\eta)V(s,t)\left[\beta ps^{p-1}\left(\beta(\rho_C t_C Y(t)s + \rho_F\gamma(S_F - s)) + \eta s\right)\right]dsd\eta +$$
$$\frac{p(p-1)}{2}\int_0^{+\infty}\int_B \Psi(\eta)V(s,t)\tilde{s}^{p-2}\left\{s^2\left[\beta\left(\rho_C t_C Y(t) + \rho_F\gamma\frac{(S_F - s)}{s}\right) + \eta\right]^2\right\}dsd\eta$$

by virtue of the fact that the random variable η has zero mean value, the first therm of the last inequality reduces to

$$p\beta\left(\rho_C t_C Y(t) - \rho_F\gamma\right)\int_0^{+\infty} V(s,t)s^p ds + \beta\rho_F\gamma\int_0^{+\infty} V(s,t)s^{p-1}ds.$$

and so both of the two coefficent goes to 0 when $\beta \to 0$. For the second therm, we know that

$$\tilde{s} = \theta(s + \beta(\rho_C t_C Y(t)s + \rho_F\gamma(S_F - s)) + \eta s) + (1-\theta)s =$$
$$s[\theta\beta(\rho_C t_C Y(t) + \rho_F\gamma\frac{(S_F - s)}{s}) + \theta\eta + 1].$$

which implies the following estimation

$$\tilde{s}^{p-2} \leq C_p\left[|\beta\rho_C t_C Y(t)|^{p-2} + |\eta|^{p-2} + \frac{\bar{C}_p}{C_p}|\beta\rho_F\gamma|^{p-2} + 1\right]s^{p-2} + \bar{C}_p|\beta\rho_F\gamma|^{p-2}S_F^{p-2}$$

with C_p and $\bar{C}_p$ suitable constants.

Recalling now that

$$s^2\left[\beta\left(\rho_C t_C Y(t)+\rho_F\gamma\frac{(S_F-s)}{s}\right)+\eta\right]^2 \le C_2\left[(\beta\rho_C t_C Y(t))^2+\eta^2+\frac{\bar{C}_2}{C_2}(\beta\rho_F\gamma)^{p-2}\right]s^2+\bar{C}_2(\beta\rho_F\gamma)^2 S_F^2.$$

Gathering all this and substituting in the weak formulation gives

$$\begin{aligned}
&\frac{d}{dt}\int_0^{+\infty} V(s)s^p ds \le \\
&p\beta\left(\rho_C t_C Y(t)-\rho_F\gamma\right)\int_0^{+\infty} V(s,t)s^p ds + \\
&\frac{p(p-1)}{2}\int_0^{+\infty}\int_B \psi(\eta)V(s,t)\left\{C_p\left[|\beta\rho_C t_C Y(t)|^{p-2}+|\eta|^{p-2}+\frac{\bar{C}_p}{C_p}|\beta\rho_F\gamma|^{p-2}+1\right]s^{p-2}+\bar{C}_p|\beta\rho_F\gamma|^{p-2}S_F^{p}\right. \\
&\left.\left\{C_2\left[(\beta\rho_C t_C Y(t))^2+\eta^2+\tfrac{\bar{C}_2}{C_2}(\beta\rho_F\gamma)^{p-2}\right]s^2+\bar{C}_2(\beta\rho_F\gamma)^2 S_F^2\right\}dsd\eta\right.
\end{aligned}$$

Now if we consider in the symptotic limit $\beta,\zeta\to 0$ and recalling B.2 for the high order moments of η, the above expression goes to zero.

Coming back to the asymptotic expansion we can finally perform the limit

$$\begin{aligned}
&\lim_{\xi\to 0}\frac{1}{\xi}\int_0^{+\infty}\int_B \Theta_V(\eta)[(\beta(\rho_C(t)Y(t)t_C s+\rho_F\gamma(S_F-s))+\eta s)\,\phi'(s) \\
&\qquad +\frac{1}{2}\left(\beta(\rho_C(t)Y(t)t_C s+\rho_F\gamma(S_F-s))+\eta s\right)^2\phi''(s)]\tilde{V}(s,\tau)dsd\eta \\
&=\int_0^{+\infty}\left[\tilde{\beta}(\rho_C(t)Y(t)t_C s\rho_F\gamma(S_F-s))\phi'(s)+\frac{\nu}{2}s^2\phi''(s)\right]\tilde{V}(s,\tau)ds.
\end{aligned}$$

Which is the weak form of the Fokker-Plank equation

$$\frac{\partial}{\partial\tau}\tilde{V}+\frac{\partial}{\partial s}\left[\tilde{\beta}\left(\rho_C(t)Y(t)t_C)s+\rho_F\gamma(S_F-s)\right)\tilde{V}\right]=\frac{\nu}{2}\frac{\partial^2}{\partial s^2}\left(s^2\tilde{V}\right). \tag{B.3}$$

So we proved the following

THEOREM B.0.4 *Let the probability density* $V_0\in\mathbf{M_0}(\mathbb{R}^+)$*. Then as* $\beta\to 0$*, and* $\zeta\to 0$ *in such a way that* $\zeta^2=\nu\beta$ *the weak solution to the Boltzman equation for the scaled density* $\tilde{V}(s,\tau)=V(s,t)$*, with* $\tau=\beta t$ *converges up to extraction of a subsequence, to a weak solution of* $B.3$*.*

Appendix C

Hybrid model for the coupling of an asymptotic preserving scheme with the asymptotic limit model: the one dimensional case

In the present appendix I show the results of an investigation work produced duringthe CEMRACS summer school held in Marseille in the August 2010. Here a spatial coupling of an asymptotic preserving scheme with the asymptotic limit model, associated to a singularly perturbed, highly anisotropic, elliptic problem is investigated and compared with the numerical discretization of the initial singular perturbation model or the purely asymptotic preserving scheme

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