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On Plane Cremona Transformations of Fixed Degree

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Abstract We study the quasi-projective variety Bir_d of plane Cremona transformations defined by three polynomials of fixed degree d and its subvariety Bir_d° where the three polynomials have no common factor. We compute their dimension and the decomposition in irreducible components. We prove that Bir_d is connected for each d and Bir_d° is connected when $d < 7$.

Keywords Plane Cremona transformations · Homaloidal nets · De Jonquières transformations

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1 Introduction

The birational geometry of algebraic varieties is governed by the group of birational self-maps. In general, it is very difficult to determine this group for an arbitrary variety, and as a matter of fact only few examples are completely understood. The special case of the projective plane has attracted lots of attention since the 19-th century. The pioneering work of Cremona and then the classical geometers of the Italian and German school were able to give partial descriptions of it, but it was only after Noether and Castelnuovo that generators of the group were described. The

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Noether–Castelnuovo theorem (see [1], for instance) states that the group of birational self-maps of $\mathbb{P}^2$, usually called the *plane Cremona group* and denoted by Cr(2), is generated by linear automorphisms of $\mathbb{P}^2$ and a single birational non-biregular map, the so-called *elementary quadratic* transformation $\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $\sigma([x:y:z]) = [yz:xz:xy]$.

Even if the generators of Cr(2) have been known for a century now, many other properties of this group are still mysterious. Only after decades, see [25], a complete set of relations has been described, and more recently the non-simplicity of Cr(2) has been shown [9], and a good understanding of its finite subgroups has been achieved; see [16] and [5]. This brief and fairly incomplete list is only meant to stress the difficulties and the large unknown parts in the study of Cr(2); for a more complete picture the interested reader should refer to [15] and [14]. Amid all its subgroups the one associated with polynomial automorphisms of the plane, $\mathrm{Aut}(\mathbb{C}^2)$, attracted even more attention than Cr(2) itself [3]. The generators of $\mathrm{Aut}(\mathbb{C}^2)$ have been known since 1942 [28], and later on [31] it has been proved that $\mathrm{Aut}(\mathbb{C}^2)$ is the amalgamated product of two of its subgroups, more precisely of the affine and elementary ones. Nevertheless, this group is not less mysterious and challenging than the entire Cremona group. Jung's description yields a natural decomposition

$$\mathrm{Aut}(\mathbb{C}^2) = A \cup G[2] \cup G[3] \cup G[2,2] \cup G[4] \cup G[5] \cdots$$

into sets of polynomial automorphisms of multidegree $(d_1, \ldots, d_m)$ and the affine subgroup A. In [21], Friedland and Milnor proved that $G[d_1, \ldots, d_m]$ is a smooth analytic manifold of dimension $(d_1 + d_2 + \cdots + d_m + 6)$. Later on, Furter [22] computed the number of irreducible components of polynomial automorphisms of $\mathbb{C}^2$ with fixed degree less than or equal to 9 and proved that the variety of polynomial automorphisms of the plane with degree bounded by the positive integer n is reducible when $n \geqslant 4$. Edo and Furter [17] studied some degenerations of the multidegrees; for example, they were able to show that $G[3] \cap \overline{G[2,2]} \neq \emptyset$ using the lower semicontinuity of the length of a plane polynomial automorphism as a word of the amalgamated product [23]. Contrary to what happens in the Cremona group Cr(2), see [6], the group of polynomial automorphisms G of the plane can be endowed with the structure of an infinite-dimensional algebraic group. Denoted by G_d the set of polynomial automorphisms of fixed degree d, Furter proved in [24] that G_d is a smooth and locally closed subset of G.

Inspired by these works on $\mathrm{Aut}(\mathbb{C}^2)$ and by [10], we approach the study of Cr(2) by forgetting its group structure. Our aim is to study an explicit set of functions generating a birational map. In this way the same birational map is associated with different triples of homogeneous polynomials.

Let f_1, f_2, f_3 be homogeneous polynomials of degree d. Whenever f_1, f_2, f_3 are not all zero, let us denote by $[f_1 : f_2 : f_3]$ the equivalence class of (f_1, f_2, f_3) with respect to the relation $(f_1, f_2, f_3) \sim (\lambda f_1, \lambda f_2, \lambda f_3)$, for $\lambda \in \mathbb{C}^*$. Consider $[f_1 : f_2 : f_3]$ as an element of $\mathbb{P}^{3N-1=3\binom{d+2}{2}-1=3d(d+3)/2+2}$, where the homogeneous coordinates in $\mathbb{P}^{3N-1}$ are the coefficients of the f_i's, up to multiplication by the same nonzero scalar. Setting

$$\gamma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad \gamma([x:y:z]) = [f_1(x,y,z) : f_2(x,y,z) : f_3(x,y,z)], \tag{1}$$

let us define

$$\mathrm{Bir}_d = \{[f_1 : f_2 : f_3] \mid \gamma \text{ is birational}\} \subset \mathbb{P}^{3N-1}.$$

The natural subset to consider in Bir_d is the one corresponding to triplets without common factors. The latter called Bir_d°, which parameterizes birational maps of degree d, will be the main actor of this paper; see Sect. 2 for all the relevant definitions.

Among other things, in [10] the authors describe Bir_d° and Bir_d for $d \leqslant 3$; see Sect. 2 for some of their results. Their description is essentially based on a set-theoretic analysis of the plane curves contracted by a Cremona transformation of degree $\leqslant 3$.

In this paper, we describe Cremona transformations $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in the usual classical way, i.e., we consider the so-called *homaloidal net* of plane curves in the source plane corresponding to the net of lines in the target plane. Roughly speaking, this means studying the span $\langle f_1, f_2, f_3\rangle \subset \mathbb{C}^{N=\binom{d+2}{2}}$ in the above setting.

By looking at base points of homaloidal nets, we prove the following; cf. Sect. 3.

Theorem 1 *For each $d \geqslant 2$, Bir_d° is a quasi-projective variety of dimension $4d+6$ in $\mathbb{P}^{3d(d+3)/2+2}$ and, if $d \geqslant 4$, it is reducible.*

Each irreducible component of Bir_d° is rational.

Each element of the maximal dimension component of Bir_d° is a de Jonquières transformation, *i.e., it is defined by a homaloidal net of plane curves of degree d with a base point of multiplicity $d-1$ and $2d-2$ simple base points.*

If $d \geqslant 4$, there are further irreducible components of Bir_d° having dimension at most $2d+12$. Indeed, each irreducible component of Bir_d° is determined by a set of $(d-1)$-tuples of non-negative integers $(\nu_1, \nu_2, \dots, \nu_{d-2}, \nu_{d-1})$ such that $\sum_{i=1}^{d-1} i\nu_i = 3d-3$ and $\sum_{i=1}^{d-1} i\nu_i^2 = d^2-1$. Each one of these components of Bir_d° has dimension $8+2\sum_{i=1}^{d-1}\nu_i$ and its general element is a Cremona transformation given by a homaloidal net of plane curves of degree d with ν_i base points of multiplicity i, in general position.

Quite surprisingly, in this setup, we may say that the general Cremona transformation in Bir_d° is *de Jonquières*. For small values of the degree d something more can be said. Our approach yields an explicit description of Bir_d° and allows us to prove its connectedness, for $d \leqslant 6$.

Theorem 2 *For each $d \leqslant 6$, Bir_d° is connected.*

Even if, from our point of view, Bir_d° is the right variety to consider, we complete the study extending the results to the whole Bir_d.

Theorem 3 *For each $d \geqslant 2$, Bir_d is a connected quasi-projective variety of dimension*

$$\max\left\{4d+6, \frac{d(d+1)}{2}+7\right\} = \begin{cases} 4d+6 & \text{if } d \leqslant 6, \\ \dfrac{(d+1)d}{2}+7 & \text{if } d \geqslant 7 \end{cases}$$

in $\mathbb{P}^{3d(d+3)/2+2}$. If $d \geqslant 3$, Bir_d is reducible.

Each irreducible component of Bir_d *is rational.*
If $d \geqslant 7$, *each element of the maximal dimension component of* Bir_d *is* $[hl_1 : hl_2 : hl_3]$, *where* $\deg(h) = d - 1$ *and* $\deg(l_i) = 1$, $i = 1, 2, 3$.

Also in this case our approach allows a more precise representation of Bir_d and an explicit description of it for low d; see Sect. 3. This also corrects an imprecision in [10] about Bir_3. In principle, one could describe the irreducible components of Bir_d° and of Bir_d, for a fixed degree d. As a matter of fact, the combinatorial and computations become wild when the degree increases. For this reason we think that it would be interesting to find closed formulas for the numbers $N^\circ(d)$ and $N(d)$ of irreducible components of Bir_d° and Bir_d and to study the connectedness of Bir_d° for each d.

This study of Bir_d°, as pointed out by Cerveau and Déserti in [10], also allows one to build a bridge between the classical algebraic geometry of Cr(2) and foliations on $\mathbb{P}^k$, $k \geqslant 2$. The reconstruction of a foliation from its singular set [7, 8], the study of irreducible components of foliations on $\mathbb{P}^k$, $k \geqslant 3$ [12, 13], and the description of the orbits by the action of $\mathrm{PGL}(3, \mathbb{C})$ on the foliations of fixed degree 2 on $\mathbb{P}^2$ [11] have been extensively studied and suggest the possibility of using our methods also in this context. We will not dwell on this here. However, we investigate the dynamical behavior of plane Cremona transformations in a work in progress; see [4].

When this work was finished we were informed by J. Blanc of D. Nguyen Dat's PhD thesis, [30]. Some of the results, and the techniques, in this paper overlap the content of his work. In particular, he was able to compute the dimension of Bir_d°. Our approach allows one to prove his Conjecture 3 about the irreducible components of Bir_d° and corrects the wrong statement on the non-connectedness of $\mathcal{H}_d$ and Bir_d°, in [30, Théorème 16, Sect. 6.2], for $d \geqslant 4$.

2 Notation, Definitions, and Known Results

We work over the complex field. Let $\mathbb{C}[x, y, z]_d$ be the set of homogeneous polynomials of degree d in the variables x, y, z with coefficients in $\mathbb{C}$, including the null polynomial. In particular, $\mathbb{C}[x, y, z]_d \cong \mathbb{C}^N$ as $\mathbb{C}$-vector spaces, where

$$N = \binom{d+2}{2} = \frac{d(d+3)}{2} + 1$$

is the number of coefficients of homogeneous polynomials of degree d in 3 unknowns.

Whenever $f_1, f_2, f_3 \in \mathbb{C}[x, y, z]_d$ are not all zero, let us denote by $[f_1 : f_2 : f_3]$ the equivalence class of the triplets (f_1, f_2, f_3) with respect to the relation $(f_1, f_2, f_3) \sim (\lambda f_1, \lambda f_2, \lambda f_3)$, for $\lambda \in \mathbb{C}^*$. Consider $[f_1 : f_2 : f_3]$ as an element of $\mathbb{P}^{3N-1=3d(d+3)/2+2}$, where the homogeneous coordinates are all coefficients of the three polynomials f_1, f_2, f_3, up to multiplication by the same nonzero scalar for all of them. Setting

$$\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad \gamma([x : y : z]) = [f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z)],$$

let us define, according to [10],

$$\mathrm{Bir}_d = \{[f_1 : f_2 : f_3] \mid \gamma \text{ is birational}\} \subset \mathbb{P}^{3N-1}.$$

In [10], the integer d is called the *degree* of the map γ. However, in algebraic geometry, if f_1, f_2, f_3 have a common factor, i.e., their greatest common divisor $\gcd(f_1, f_2, f_3) = h$ is a polynomial of positive degree d', then the degree of the map γ is usually considered as $d - d'$. If $\gcd(f_1, f_2, f_3) = 1$, the authors of [10] say that γ has *pure degree* d and they define

$$\mathrm{Bir}_d^\circ = \{[f_1 : f_2 : f_3] \in \mathrm{Bir}_d \mid \gamma \text{ has pure degree } d\} \subset \mathrm{Bir}_d \subset \mathbb{P}^{3N-1}.$$

If $[f_1 : f_2 : f_3] \in \mathrm{Bir}_d^\circ$, let us identify it with the birational map γ.

One may check that the subset $\{[f_1 : f_2 : f_3] \mid \deg(\gcd(f_1, f_2, f_3)) > 0\}$ of $\mathbb{P}^{3N-1}$ is Zariski closed, e.g., by using resultants.

Remark 4 When $d = 1$, write the 3×3 nonzero matrix (a_{ij}), whose rows are the coefficients of f_i, $i = 1, 2, 3$, thus $\mathrm{Bir}_1^\circ = \mathrm{Bir}_1 = \mathrm{PGL}(3)$ is the Zariski open subset of $\mathbb{P}^8$ where the determinant $\det(a_{ij})$ is nonzero and its Zariski closure is $\overline{\mathrm{Bir}_1} = \mathbb{P}^8$.

Recall that $\mathbb{P}^8 \setminus \mathrm{Bir}_1 = Z(\det(a_{ij}))$ is a cubic, irreducible hypersurface, which is singular along the locus where $\mathrm{rank}(a_{ij}) = 1$.

In [10], Cerveau and Déserti are interested in Bir_d also for applications to the study of *foliations*. Among other things, they prove the following.

Theorem 5 (Cerveau–Déserti) *If $d = 2$, the subset Bir_2° [Bir_2, resp.] in $\mathbb{P}^{17}$ is a smooth [singular, resp.], unirational, irreducible, quasi-projective variety of dimension 14. Their Zariski closures in $\mathbb{P}^{17}$ coincide.*

If $d = 3$, the subset $\mathrm{Bir}_3^\circ \subset \mathbb{P}^{29}$ is an irreducible, rationally connected, quasi-projective variety of dimension 18. Furthermore, the subset $\mathrm{Bir}_3 \subset \mathbb{P}^{29}$ is singular and contains at least two irreducible components.

Actually, in [10], the authors state that Bir_3 has exactly two irreducible components. We will see later, in Example 44, that Bir_3 has exactly three irreducible components.

Now we recall the classical notions which will be used later.

Definition 6 Let $\gamma = [f_1 : f_2 : f_3] \in \mathrm{Bir}_d^\circ$, $d \geqslant 2$. Setting W to be the linear span $\langle f_1, f_2, f_3\rangle$ of the polynomials f_1, f_2, f_3 in $\mathbb{C}^N$, the plane $\mathbb{P}(W) \subset \mathbb{P}^{N-1}$ is called the *homaloidal net* associated with γ, and we denote it by $\mathcal{L}_\gamma$.

The general element of $\mathcal{L}_\gamma$ defines an irreducible rational plane curve of degree d passing through some fixed points $p_1, \ldots, p_r$ in $\mathbb{P}^2$, called *set-theoretic base points* of $\mathcal{L}_\gamma$, with certain multiplicities. Let $\phi\colon S \to \mathbb{P}^2$ be the blowing up at $p_1, \ldots, p_r$. The strict transform of the general element of $\mathcal{L}_\gamma$ may have further base points on the exceptional curves in S, classically called *infinitely near* base points; cf. Sect. 2 in [1]. Equivalently, $\phi \circ \gamma$ may not be a morphism yet. Eventually, there is a birational morphism $\phi'\colon S' \to \mathbb{P}^2$ such that the strict transform of the net $\mathcal{L}_\gamma$ is a complete 2-dimensional linear system on S', or equivalently $\phi' \circ \gamma$ is a morphism.

Set ν_i to be the number of base points of $\mathcal{L}_\gamma$ with multiplicity i, $i = 1, \dots, d-1$, including infinitely near base points. The *multi-index* of the homaloidal net $\mathcal{L}_\gamma$ is

$$\nu_I = (\nu_1, \nu_2, \dots, \nu_{d-2}, \nu_{d-1}). \tag{2}$$

It is an easy consequence of the definition of homaloidal net (cf., e.g., Sect. 2.5 in [1]) that

$$\sum_{i=1}^{d-1} i^2 \nu_i = d^2 - 1, \qquad \sum_{i=1}^{d-1} i\nu_i = 3(d-1), \tag{3}$$

and $\nu_i = 0$, $i \geqslant d$. Formulas (3) are usually called Noether's equations.

We say that the multi-index ν_I is the *H-type* of the homaloidal net $\mathcal{L}_\gamma$ and of the Cremona transformation γ.

One usually says that γ is *symmetric* if its H-type has a unique nonzero ν_i. By (3), γ is symmetric if and only if either $d = 2$ and $\nu_I = (3)$, or $d = 5$ and $\nu_I = (0, 6, 0, 0)$, or $d = 8$ and $\nu_I = (0, 0, 7, 0, \dots)$, or $d = 17$ and the unique nonzero term in ν_I is $\nu_6 = 8$ (cf., e.g., Lemma 2.5.5 in [1]).

Finally, let us define the *index* $i(\gamma)$ of γ as follows. We set $i(\gamma) = 0$ if and only if there is no infinitely near base point of $\mathcal{L}_\gamma$, i.e., all base points of $\mathcal{L}_\gamma$ belong to $\mathbb{P}^2$. If $\mathcal{L}_\gamma$ has infinitely near base points, consider birational morphisms $\psi\colon S \to \mathbb{P}^2$ such that the strict transform $\mathcal{L}'$ of $\mathcal{L}_\gamma$ in S is a net with all base points belonging to S, i.e., $\mathcal{L}'$ has no infinitely near base point. Recalling that ψ is the composition of finitely many blowing ups, each one at the maximal ideal of a single point, define $i(\gamma)$ as the minimum number of such blowing ups among all birational morphisms ψ with the above property.

Remark 7 If $d \geqslant 3$, the irreducibility of the general member of $\mathcal{L}_\gamma$ implies that $\sum_{i>d/2} \nu_i \leqslant 1$, i.e., $\mathcal{L}_\gamma$ has at most one base point of multiplicity $> d/2$.

Remark 8 Usually, the H-type of a linear system $\mathcal{L}$ is encoded by listing the multiplicities m_i of the base points p_i of $\mathcal{L}$, which are commonly written in nonincreasing order, say $m_1 \geqslant m_2 \geqslant m_3 \geqslant \cdots$, instead of a multi-index like ν_I. Given a multi-index $\nu_I = (\nu_1, \dots, \nu_{d-1})$ of a homaloidal net $\mathcal{L}_\gamma$, the multiplicities of the base points of $\mathcal{L}_\gamma$, in nonincreasing order, can be computed as follows:

$$m_i = \max\left\{ j : \sum_{k \geqslant j} \nu_k > i - 1 \right\}, \quad \text{for each } i \geqslant 1. \tag{4}$$

Equivalently, set $m_1 = \max\{j : \nu_j \neq 0\}$ and then

$$m_1 = m_2 = \cdots = m_{R_1} > m_{R_1+1} = \max\{j < m_1 : \nu_j \neq 0\}, \quad R_1 = \nu_{m_1},$$
$$m_{R_1+1} = \cdots = m_{R_1+R_2} > m_{R_1+R_2+1} = \max\{j < m_{R_1+1} : \nu_j \neq 0\}, \quad R_2 = \nu_{m_{R_1+1}},$$

and so on, until $m_R = \min\{j : \nu_j \neq 0\}$ with $R = \sum_{i=1}^{d-1} \nu_i = R_1 + R_2 + \cdots + R_s$, $s = |\{j : \nu_j \neq 0\}|$.

Definition 9 Let $\nu_I = (\nu_1, \ldots, \nu_{d-1})$ be a multi-index with non-negative entries. We say that the positive numbers $m_1, \ldots, m_r$, with $r = \sum_{i=1}^{d-1} \nu_i$, computed in Remark 8 are the *multiplicities associated with* ν_I.

Definition 10 One says that $\gamma \in \mathrm{Bir}_2^\circ$ is a *quadratic* transformation. By (3), its H-type is $\nu_I = (3)$, namely, γ is defined by a homaloidal net of generically irreducible conics passing through three simple base points p_1, p_2, p_3. We say that γ is a quadratic transformation *centered at* p_1, p_2, p_3.

Definition 11 The map $\gamma \in \mathrm{Bir}_d^\circ$ is called a *de Jonquières transformation* if there exists a base point of $\mathcal{L}_\gamma$ with multiplicity $d-1$. If $d \geqslant 3$, irreducibility and (3) force $\nu_{d-1} = 1$, $\nu_i = 0$, $2 \leqslant i \leqslant d-2$, and $\nu_1 = 2d-2$, hence the H-type of a de Jonquières transformation of degree $d \geqslant 3$ is $(2d-2, 0, \ldots, 0, 1)$.

A little work on (3) proves the following (cf., e.g., Proposition 2.6.4 in [1]).

Theorem 12 (Noether's Inequality) *Let* $\gamma \in \mathrm{Bir}_d^\circ$, $d \geqslant 2$, *and let* $m_1 \geqslant m_2 \geqslant m_3$ *be the maximal multiplicities of the base points of the homaloidal net* $\mathcal{L}_\gamma$. *Then*

$$m_1 + m_2 + m_3 \geqslant d + 1. \tag{5}$$

Furthermore, equality holds if and only if γ *is either symmetric or* de Jonquières.

The next, classical, lemma is very important for our purposes, and we give here a proof only for the reader's convenience.

Lemma 13 (cf. [26, p. 72]) *If* $d \geqslant 3$, *the maximal number of base points of a homaloidal net* $\mathcal{L}_\gamma$ *is* $\sum_{i=1}^{d-1} \nu_i = 2d-1$ *and, in that case,* γ *is a* de Jonquières *transformation.*

Proof In (3), multiply the second equation by d and then subtract the first equation. Thus,

$$(d-1)(2d-1) = \sum_{i=1}^{d-1} i(d-i)\nu_i. \tag{6}$$

On the real interval $[1, d-1]$, the function $g(x) = x(d-x)$ has a maximum at $x = d/2$ and a minimum at $x = 1$ and at $x = d-1$, so $1(d-1) \leqslant g(x) \leqslant d^2/4$. Hence, (6) implies $2d-1 \geqslant \sum_{i=1}^{d-1} \nu_i$. Equality holds if and only if $\nu_1 + \nu_{d-1} = 2d-1$ and $\nu_i = 0$, $i = 2, \ldots, d-2$. Remark 7, formulas (3), and the hypothesis $d \geqslant 3$ force $\nu_{d-1} = 1$. □

Remark 14 In algebraic geometry, when a family of objects $\{X_q\}_{q \in \Sigma}$ like varieties, maps, etc., is parameterized by the points of an *irreducible* algebraic variety Σ, one usually says that the *general* object X has a certain property P if the subset of points $q \in \Sigma$, such that X_q has the property P, contains a Zariski open dense subset of Σ.

Later we will use finite sets of points in $\mathbb{P}^2$ and in its symmetric products.

Definition 15 (cf. [29]) The *r-th symmetric product* of $\mathbb{P}^2$, denoted by $\mathrm{Sym}^r(\mathbb{P}^2)$, is the quotient $(\mathbb{P}^2)/\mathfrak{S}_r$ of the Cartesian product $(\mathbb{P}^2)^r$ under the action of the symmetric group $\mathfrak{S}_r$ permuting the factors. One has that $\mathrm{Sym}^1(\mathbb{P}^2) = \mathbb{P}^2$, if $r = 1$; otherwise, $\mathrm{Sym}^r(\mathbb{P}^2)$ is a rational, singular, irreducible, projective variety of dimension $2r$. If $p_1, \dots, p_r \in \mathbb{P}^2$ are distinct points, then $\mathrm{Sym}^r(\mathbb{P}^2)$ is smooth at the class of $(p_1, \dots, p_r)$.

A useful tool to take care of infinitely near points is the Hilbert scheme.

Definition 16 (cf. [19, 20, 27]) The Hilbert scheme $\mathrm{Hilb}^r(\mathbb{P}^2)$ parameterizes zero-dimensional subschemes of $\mathbb{P}^2$ with *length* r, i.e., whose Hilbert polynomial has degree 0 and is equal to r. It is a rational, smooth, irreducible, projective variety of dimension $2r$ and is a desingularization of $\mathrm{Sym}^r(\mathbb{P}^2)$.

Recall that r points in general position in $\mathbb{P}^2$ determine an open dense subset of either $\mathrm{Hilb}^r(\mathbb{P}^2)$ or $\mathrm{Sym}^r(\mathbb{P}^2)$.

Definition 17 Let d be a positive integer. A multi-index $\nu_I := (\nu_1, \nu_2, \dots, \nu_{d-2}, \nu_{d-1})$ with *length* ρ and *reduced length* r is a $(d-1)$-tuple of non-negative integers with $r = \sum_{i=1}^{d-1} \nu_i$ and $\rho = \sum_{i=1}^{d-1} i(i+1)\nu_i/2$. Setting $I^* = \{i \mid \nu_i > 0\}$, let us define

$$\mathrm{Hilb}^{\nu_I}(\mathbb{P}^2) = \prod_{i \in I^*} \mathrm{Hilb}^{\nu_i}(\mathbb{P}^2).$$

Thus, a point $Z \in \mathrm{Hilb}^{\nu_I}(\mathbb{P}^2)$ is $Z = ([Z_i])_{i \in I^*}$ with $[Z_i] \in \mathrm{Hilb}^{\nu_i}(\mathbb{P}^2)$, for each $i \in I^*$.

Let us denote by $\mathrm{Hilb}_{\bullet}^{\nu_I}(\mathbb{P}^2)$ the dense open subset of $\mathrm{Hilb}^{\nu_I}(\mathbb{P}^2)$ whose elements are $Z = ([Z_i])_{i \in I^*}$ such that the zero-dimensional scheme $Z^r = \bigcup_{i \in I^*} Z_i$ is supported on r distinct points in $\mathbb{P}^2$; in particular, Z_i is a collection of points $p_{i,1}, \dots, p_{i,\nu_i}$ in $\mathbb{P}^2$. To a point $Z \in \mathrm{Hilb}_{\bullet}^{\nu_I}(\mathbb{P}^2)$, we associate the zero scheme $[Z_{\nu_I}]$ in $\mathrm{Hilb}^{\rho}(\mathbb{P}^2)$ with $(Z_{\nu_I})_{\mathrm{red}} = Z^r$ and Z_{ν_I} given by the union of the points $p_{i,j}$ with multiplicity i, for each $i \in I^*$ and $j = 1, \dots, \nu_i$.

Remark 18 Note that $[Z_{\nu_I}]$ is not a general point of $\mathrm{Hilb}^{\rho}(\mathbb{P}^2)$. This is the main motivation for introducing both length and reduced length. Equations (3) force the length ρ of ν_I to be uniquely determined by d, namely, $\rho = (d+4)(d-1)/2$.

Definition 19 We say that a multi-index $\nu_I = (\nu_1, \nu_2, \dots, \nu_{d-2}, \nu_{d-1})$ is *admissible* if there is an element $Z \in \mathrm{Hilb}_{\bullet}^{\nu_I}(\mathbb{P}^2)$ such that the linear system $\Lambda_Z := |\mathcal{I}_{Z_{\nu_I}}(d)|$ is non-empty, of the expected dimension

$$\dim \Lambda_Z = \binom{d+2}{2} - \sum_{i=1}^{d-1} \frac{i(i+1)\nu_i}{2},$$

and the general element of Λ_Z is an irreducible curve. In such a case we say that Z is an *admissible cycle* associated with ν_I.

The following theorem is classical; cf., e.g., [18, V.II.20] and [1, Theorem 5.1.1].

Theorem 20 *Fix a positive integer d and a multi-index ν_I. Assume that ν_I satisfies* (3). *Let $Z \in \mathrm{Hilb}_{\bullet}^{\nu_I}(\mathbb{P}^2)$ be a point. Then ν_I and Z are admissible if and only if Λ_Z is a homaloidal net.*

Remark 21 There are finitely many multi-indexes ν_I satisfying (3), when d is fixed, but not all such multi-indexes are H-types, i.e., do give rise to homaloidal nets in Bir_d°. The first, classical, example is $(6, 0, 2, 0)$, $d = 5$. In this case the line through the two triple points is a fixed component of the linear system of 5-ics with the assigned multiplicities. The movable part of the net is given by elliptic quartic curves. In particular, the point representing it in $\mathbb{P}^{62}$ is not in Bir_5.

To get rid of this behavior we proceed as follows; cf. Hudson's test in [26] and in [1].

Definition 22 Let $\nu_I = (\nu_1, \dots, \nu_{d-1})$, $d \geqslant 2$, be a multi-index satisfying (3). When $d = 2$, we say that $\nu_I = (3)$ is *1-irreducible*. Suppose then that $d \geqslant 3$, and let $m_1 \geqslant m_2 \geqslant m_3$ be the maximal multiplicities associated with ν_I; cf. Definition 9.

We say that ν_I is *1-irreducible* if $m_1 + m_2 \leqslant d$. Let $d' = 2d - m_1 - m_2 - m_3$. By (3) and $d \geqslant 3$, the same proof of Noether's inequality (5) in [1] shows that $d > d' \geqslant 2$.

Now define a new multi-index $q(\nu_I) = (\nu'_1, \nu'_2, \dots, \nu'_{d'-2}, \nu'_{d'-1})$ by the following steps:

- for each $j = 1, 2, 3$, decrease ν_{m_j} by 1 (if $m_3 = m_2 = m_1$, this means decrease ν_{m_1} by 3);
- set $\varepsilon = d - m_1 - m_2 - m_3$;
- for each $j = 1, 2, 3$, set $k = m_j + \varepsilon$ and, if $k > 0$, increase ν_k by 1;
- finally, for each $i = 1, \dots, d' - 1$, set $\nu'_i = \nu_i$.

We say that ν_I is *irreducible* if ν_I is 1-irreducible, $q(\nu_I)$ is 1-irreducible, $q(q(\nu_I))$ is 1-irreducible, and so on, for all new multi-indexes until one stops, when d becomes 2. A script, which runs this irreducibility test, is listed in the Appendix.

Remark 23 If ν_I is the H-type of a Cremona transformation $\gamma \in \mathrm{Bir}_d^{\circ}$, and the maximal multiplicities $m_1 \geqslant m_2 \geqslant m_3$ of the base points of the homaloidal net $\mathcal{L}_\gamma$ occur at three points p_1, p_2, p_3 such that a quadratic transformation ω centered at p_1, p_2, p_3 is well defined, then $q(\nu_I)$ is just the H-type of $\omega \circ \gamma \in \mathrm{Bir}_{d'}^{\circ}$.

Setting $p_4, \dots, p_r$ the other base points of $\mathcal{L}_\gamma$, with respective multiplicities $m_4 \geqslant \cdots \geqslant m_r$, one has (cf., e.g., Corollary 4.2.6 in [1]) that the multiplicities of the homaloidal net of $\omega \circ \gamma$ at the points corresponding via ω to $p_1, \dots, p_r$ are respectively $m'_1, m'_2, \dots, m'_r$ where

$$m'_i = m_i - \varepsilon, \quad i = 1, 2, 3, \qquad \varepsilon = m_1 + m_2 + m_3 - d, \qquad m'_j = m_j, \quad j \geqslant 4.$$

Note that $d' = d - \varepsilon$, $m'_1 = d - m_2 - m_3$, $m'_2 = d - m_1 - m_3$, and $m'_3 = d - m_1 - m_2$.

The same formulas hold even if m_1, m_2, m_3 are not the maximal multiplicities of the base points of $\mathcal{L}_\gamma$. Note that $\varepsilon > 0$ if and only if $m_1 + m_2 + m_3 > d$.

The next theorem appears to be classical, and it has been implicitly used by Hudson, but it had probably fallen into oblivion; cf. historical remark 5.3.6 in [1] and the references therein. For a modern proof, see Theorems 5.2.19 and 5.3.4 in [1].

Theorem 24 *Fix an integer d and a multi-index ν_I which satisfies (3) and is irreducible, according to Definition 22. Setting $r = \sum_{i=1}^{d-1} \nu_i$, let $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r$ be the multiplicities associated with ν_I; cf. Definition 9.*

Then ν_I is admissible, and there exists a non-empty Zariski open subset U of $(\mathbb{P}^2)^r$ (that is, $\mathbb{P}^2 \times \cdots \times \mathbb{P}^2$, r times) such that for each $(p_1, \ldots, p_r) \in U$ there exists a Cremona transformation of degree d which has p_i as the base point of multiplicity m_i, $i = 1, \ldots, r$, and has no other base points.

In our notation, this translates as follows.

Corollary 25 *In the hypothesis of the previous theorem, there exists an open dense subset $U_{\nu_I} \subset \mathrm{Hilb}^{\nu_I}(\mathbb{P}^2)$ such that for any point $Z \in U_{\nu_I}$, Z is associated with ν_I and the linear system Λ_Z is a homaloidal net.*

The algebraic structure of Bir_d and Bir_d° is already known; cf. [6, Lemma 2.4].

Lemma 26 *The subsets Bir_d and Bir_d° in $\mathbb{P}^{3N-1}$ are quasi-projective varieties.*

We will use also planes, i.e., linear subspaces of dimension 2, in $\mathbb{P}^{N-1}$, or equivalently three-dimensional vector subspaces of $\mathbb{C}^N$. A convenient setting is Grassmannians.

Definition 27 Denote by $\mathrm{Gr}(3, N)$ the Grassmannian variety parameterizing three-dimensional vector subspaces of $\mathbb{C}^N$, i.e., planes in $\mathbb{P}^{N-1}$. It is a smooth, irreducible, rational, projective variety of dimension $3(N - 3)$.

3 Cremona Transformations of Fixed Degree

In this section we use notation introduced in Sect. 2. Fix d a positive integer.

Remark 28 For each $d \geqslant 2$, there is a one-to-one map

$$\mathrm{Bir}_d \setminus \mathrm{Bir}_d^\circ \to \coprod_{a=1}^{d-1} \left(\mathbb{P}(\mathbb{C}[x, y, z]_a) \times \mathrm{Bir}_{d-a}^\circ \right),$$

$$[f_1 : f_2 : f_3] \mapsto \left(h, \left[\frac{f_1}{h} : \frac{f_2}{h} : \frac{f_3}{h} \right] \right),$$

where $h = \gcd(f_1, f_2, f_3)$. The inverse is the collection of the maps

$$\tau_a \colon \mathbb{P}(\mathbb{C}[x, y, z]_a) \times \mathrm{Bir}^\circ_{d-a} \to \mathrm{Bir}_d, \qquad \tau_a(h, [f_1 : f_2 : f_3]) = [hf_1 : hf_2 : hf_3], \tag{7}$$

which are injective, for each $a = 1, \dots, d-1$.

Let us focus on $\gamma = [f_1 : f_2 : f_3] \in \mathrm{Bir}^\circ_d$; in particular, $\gcd(f_1, f_2, f_3) = 1$. Let $\mathcal{L}_\gamma = \mathbb{P}(\langle f_1, f_2, f_3\rangle)$ be the homaloidal net associated with γ, recalled in Sect. 2.

Definition 29 We denote by "Bs" the map which sends a Cremona transformation $\gamma \in \mathrm{Bir}^\circ_d$ to its base locus:

$$\mathrm{Bs} \colon \mathrm{Bir}^\circ_d \to \mathrm{Hilb}(\mathbb{P}^2), \qquad \gamma = [f_1 : f_2 : f_3] \mapsto \mathrm{Bs}(\gamma) = Z(f_1, f_2, f_3), \tag{8}$$

where $\mathrm{Bs}(\gamma) = Z(f_1, f_2, f_3)$ is a 0-dimensional subscheme of $\mathbb{P}^2$.

Remark 30 The map Bs is algebraic. To see this, fix an irreducible component $A \subset \mathrm{Bir}^\circ_d$, take

$$I_c = \{([f_1 : f_2 : f_3], [Z]) \mid Z \subset Z(f_1, f_2, f_3)\} \subset A \times \mathrm{Hilb}^c(\mathbb{P}^2),$$

and let p_1, p_2 be the projections on the two factors. By (3), each homaloidal net in Bir°_d has the expected dimension. Hence, for each A there is a unique c such that $p_1^{-1}([f_1 : f_2 : f_3]) \cap I_c$ is a point. In this setting, $\mathrm{Bs} = p_2 \circ p_1^{-1}$.

There is no inverse map to Bs in (8), because the map Bs is not injective. The definition of Bir°_d is such that two different bases of the same homaloidal net gives different elements in Bir°_d while having the same base locus. This is quite awkward, at least from the algebraic geometry point of view, but can be, somehow, settled as follows.

Lemma 31 *Let* $\gamma = [f_1 : f_2 : f_3]$ *and* $\delta = [g_1 : g_2 : g_3]$ *be in* Bir°_d *such that* $\mathrm{Bs}(\gamma) = \mathrm{Bs}(\delta)$. *Then* $\langle f_1, f_2, f_3\rangle = \langle g_1, g_2, g_3\rangle = W \subset \mathbb{C}^{d(d+3)/2+1}$, *and there exists a unique change of basis matrix* $\omega \in \mathrm{PGL}(3)$ *such that the triplet* $(g_1, g_2, g_3) = (f_1, f_2, f_3)\omega$ *in* $W \cong \mathbb{C}^3$.

Proof Let $Z_\gamma = (\mathrm{Bs}(\gamma))_{\mathrm{red}} = (\mathrm{Bs}(\delta))_{\mathrm{red}}$ be the reduced base locus of γ and δ. Theorem 20 says that $\dim \Lambda_{Z_\gamma} = 2$ and thus $W = \langle f_1, f_2, f_3\rangle \ni g_i$, $i = 1, 2, 3$. Hence, by definition, the f_i's and the g_i's are two bases of the 3-dimensional vector space $W \subset \mathbb{C}^{d(d+3)+1}$, and there is a unique change of basis ω sending one to the other. To conclude, observe that $[f_1 : f_2 : f_3]$ and $[\lambda f_1 : \lambda f_2 : \lambda f_3]$, $\lambda \in \mathbb{C}^*$, represent the same element in Bir°_d. □

The previous lemma suggests how to change the target space in (8) in order to get a birational map. For this purpose, we construct suitable morphisms.

Fix a positive integer d and an irreducible (according to Definition 22) multi-index ν_I, satisfying (3). By Corollary 25 there is a dense open subset $U_{\nu_I} \subset \mathrm{Hilb}^{\nu_I}(\mathbb{P}^2)$

made of admissible cycles associated with ν_I. Recall that $Z = ([Z_i])_{i \in I^*} \in U_{\nu_I}$ is such that $[Z_i] \in \mathrm{Hilb}^{\nu_i}(\mathbb{P}^2)$ and Z_i is a collection of ν_i points $p_{i,1}, \dots, p_{i,\nu_i}$ in $\mathbb{P}^2$. Let us define the map

$$\beta_{\nu_I} : U_{\nu_I} \to \mathrm{Gr}(3, N), \qquad \beta_{\nu_I}(Z) = [H^0(\mathcal{I}_{Z_{\nu_I}}(d))], \tag{9}$$

which sends Z to the homaloidal net of plane curves of degree d with multiplicity i at the points $p_{i,j}$, for each $i \in I^*$ and $j = 1, \dots, \nu_i$; cf. Definition 17.

Remark 32 It is important to stress that the base locus in $\mathrm{Hilb}^{\rho}(\mathbb{P}^2)$ of a homaloidal net associated with a multi-index ν_I and supported on $r = \sum_{i=1}^{d-1} \nu_i$ distinct points determines a unique admissible cycle in $\mathrm{Hilb}^{\nu_I}(\mathbb{P}^2)$ associated with ν_I. In that case, we say that such a base locus in $\mathrm{Hilb}^{\rho}(\mathbb{P}^2)$ is admissible. Vice versa, an admissible cycle determines uniquely the homaloidal net.

This, together with Lemma 31, yields that the morphism β_{ν_I} is a birational map onto its image in the Grassmannian $\mathrm{Gr}(3, N)$ of planes in $\mathbb{P}^{N-1} = \mathbb{P}(\mathbb{C}[x, y, z]_d)$.

Next we want to go from the Grassmannian to Bir_d°. This is done by distinguishing a basis in the general point of the image of β_{ν_I}. To do this, choose three general $N-3$ planes H_1, H_2, H_3 in $\mathbb{P}(\mathbb{C}[x, y, z]_d) = \mathbb{P}^{N-1}$, e.g., we may choose

$$H_1 = \langle y^d, y^{d-1}z \rangle^{\perp}, \qquad H_2 = \langle z^d, z^{d-1}x \rangle^{\perp}, \qquad H_3 = \langle x^d, x^{d-1}y \rangle^{\perp}.$$

This allows us to universally choose the basis $W \cap H_1$, $W \cap H_2$, $W \cap H_3$ for a general 3-dimensional linear vector subspace $W \subset \mathbb{C}^N$. In other words, we have chosen three sections $\sigma_i : \mathrm{Gr}(3, N) \to U$ of the universal bundle over the Grassmannian $\mathrm{Gr}(3, N)$.

Definition 33 In the above setting, define the rational map

$$\alpha_{\nu_I} : \mathrm{PGL}(3) \times U_{\nu_I} \dashrightarrow \mathrm{Bir}_d^\circ,$$
$$(\omega, Z) \mapsto [\omega(\sigma_1(\beta_{\nu_I}(Z))) : \omega(\sigma_2(\beta_{\nu_I}(Z))) : \omega(\sigma_3(\beta_{\nu_I}(Z)))],$$

where $\omega \in \mathrm{PGL}(3)$ is acting on the 3-dimensional vector subspace $W \subset \mathbb{C}^N$ as described in Lemma 31. The map α_{ν_I} is well defined; in fact, $\alpha_{\nu_I}(\lambda\omega, Z) = \alpha_{\nu_I}(\omega, Z)$, for any $\omega \in \mathrm{PGL}(3)$ and $\lambda \in \mathbb{C}^*$.

Lemma 34 *The map α_{ν_I} is birational onto its image.*

Proof It is enough to prove that α_{ν_I} is generically injective. But this follows immediately by Lemma 31 and Remark 32. □

Lemma 35 *Let $\nu_I = (\nu_1, \dots, \nu_{d-1})$ and $\mu_I = (\mu_1, \dots, \mu_{d-1})$ be two distinct admissible multi-indexes. Then $\mathrm{Im}(\alpha_{\nu_I})$ and $\mathrm{Im}(\alpha_{\mu_I})$ lie in two different components of Bir_d°.*

Proof Let r and m be the two reduced lengths of ν_I and μ_I, respectively. If $r = m$ we conclude again by Remark 32. Assume that $r > m$. Then we have to prove that $\mathrm{Im}(\alpha_{\nu_I}) \not\supset \mathrm{Im}(\alpha_{\mu_I})$. This is equivalent to saying that the base locus of a general element in $\mathrm{Im}(\alpha_{\mu_I})$ cannot be obtained as the limit of base loci of general elements in $\mathrm{Im}(\alpha_{\nu_I})$.

Assume that this is not the case. Then in $\mathrm{Hilb}^{\rho}(\mathbb{P}^2)$ there is a curve whose general point represents the base locus of an element in $\mathrm{Im}(\alpha_{\nu_I})$ and with a special point associated with Z_{μ_I}. In other words, we are saying that a bunch of points of multiplicity $m_{i,1}, \ldots, m_{i,h_i}$, with ordinary singularities, limits to a point of some multiplicity m_i, with ordinary singularity, and this is done in such a way that Noether's equations (3) are always satisfied.

Fix one point in the limit, say p_1 of multiplicity m_1, and assume that p_1 is the limit of $\{q_1, \ldots, q_{h_1}\}$ of respective multiplicity $m_{1,j}$. The existence of the limit forces

$$\frac{m_1(m_1+1)}{2} = \sum_j \frac{m_{1,j}(m_{1,j}+1)}{2}; \quad \text{in particular, } m_1 \leqslant \sum_j m_{1,j},$$

with strict inequality if $h_1 > 1$. This, together with (3) yields

$$d^2 - 1 = \sum_i m_i^2 = \sum_i \left(\sum_j (m_{i,j}^2 + m_{i,j}) - m_i \right) = d^2 - 1 + \sum_i (\sum_j m_{i,j}) - m_i .$$

Hence we have the contradiction $\sum_i (\sum_j m_{i,j}) - m_i = 0$. □

In the previous lemmas, we found an irreducible component of Bir_d° for each admissible multi-index ν_I. In the next lemma, we show that each Cremona transformation $\gamma \in \mathrm{Bir}_d^{\circ}$ belongs to one of these irreducible components.

Lemma 36 *Let $\gamma \in \mathrm{Bir}_d^{\circ}$ be a birational transformation, $Z^{\gamma} = \mathrm{Bs}(\gamma)$ its base locus, and $\nu_I = (\nu_1, \ldots, \nu_{d-1})$ the corresponding H-type. Setting ρ to be the length of ν_I and r its reduced length, there is a curve $C \in \mathrm{Hilb}^{\rho}(\mathbb{P}^2)$ such that*

(i) $[Z^{\gamma}] \in C$,
(ii) *the general point $[Z_t] \in C$ is a zero-dimensional scheme supported on r distinct points, with ordinary singularities, in $\mathbb{P}^2$,*
(iii) *Z_t is admissible.*

Proof Let $\mathcal{L}_{\gamma} \subset |\mathcal{O}(d)|$ be the homaloidal net associated with Z^{γ}. Consider the index $i(\gamma)$ introduced in Definition 6. If $i(\gamma) = 0$, the assertion is immediate, for any degree d. To conclude, we argue by induction on $i(\gamma)$. Assume that $i(\gamma) = M > 0$, and let $p \in Z_{\mathrm{red}}$ be a point of multiplicity m with infinitely near other base points. Let $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a quadratic transformation centered in p and two general points q_1 and q_2, and $\mathcal{L}' = \omega_* \mathcal{L}_{\gamma}$ the strict transform linear system. Then $\mathcal{L}' \subset |\mathcal{O}(2d - m)|$ is a homaloidal net defining $\gamma' = \omega \circ \gamma$. By construction, $i(\gamma') \leqslant i(\gamma) - 1$ and by inductive hypothesis we may describe its base locus $Z^{\gamma'} = \mathrm{Bs}\,\mathcal{L}'$ as the limit of admissible cycles with two ordinary points of multiplicity d. Then applying ω^{-1} we get the desired curve in $\mathrm{Hilb}^{\rho}(\mathbb{P}^2)$. □

Definition 37 Let $\nu_I = (\nu_1, \ldots, \nu_{d-1})$ be an admissible multi-index. We denote by $\mathrm{Bir}^{\circ}_{\nu_I}$ the irreducible component of Bir°_d whose general element is defined by a homaloidal net of H-type ν_I. Moreover, we denote by Bir_{ν_I} the intersection of the Zariski closure of $\mathrm{Bir}^{\circ}_{\nu_I}$ in $\mathbb{P}^{3N-1}$ with Bir_d.

Remark 38 It is important to stress that a homaloidal net may degenerate to a linear system with a fixed component in such a way that the residual part is not a homaloidal net. In particular, the Zariski closure of $\mathrm{Bir}^{\circ}_{\nu_I}$ in $\mathbb{P}^{3N-1}$ is not contained in Bir_d. The easiest occurrences of this behavior are as follows.

Let $\mathcal{L}$ be a general homaloidal net in $\mathrm{Bir}^{\circ}_{(4,1)}$ given by plane cubics with a double point p_0 and four simple base points $p_1, \ldots, p_4$. If we let $p_1, \ldots, p_4$ become aligned, then $\mathcal{L}$ degenerates to a net with a fixed line and the residual part is composed with the pencil of lines through p_0.

Let $\mathcal{L}$ be a general homaloidal net in $\mathrm{Bir}^{\circ}_{(3,3,0)}$ given by plane quartics with three double points p_1, p_2, p_3 and three simple base points. If we let p_1, p_2, p_3 become aligned, then $\mathcal{L}$ degenerates to a linear system with a fixed line and the residual part is a 3-dimensional linear system of cubics with six simple base points.

The next lemma is classical; see, e.g., [26, p. 73], but the proof therein is not complete.

Lemma 39 *If $d \geqslant 4$ and $\gamma \in \mathrm{Bir}^{\circ}_d$ is not a de Jonquières transformation, then the number of base points of the homaloidal net $\mathcal{L}_\gamma$ is at most $d+2$.*

Proof By Lemma 36 (cf. also Theorems 20 and 24), the H-type ν_I of γ, and hence the number $r = \sum_{i=1}^{d-1} \nu_i$ of base points $\mathcal{L}_\gamma$, does not depend on the position of the points in $\mathbb{P}^2$. Therefore, we may and will assume that these base points are in general position, in such a way that γ can be factored as a composition of quadratic transformations, centered at base points of $\mathcal{L}$ only, each one decreasing the degree of γ.

Now we proceed by induction on the degree d of γ.

The base of induction is $d = 4$. In this case there is only one possible H-type, which is $(3, 3, 0)$, i.e., six base points and the assertion is trivially true.

Suppose then that $d > 4$. We set $m_1, m_2, m_3, \tilde{d}, \varepsilon, \mu_i$ according to Definition 22; in particular, $\tilde{d} = 2d - m_1 - m_2 - m_3 = d + \varepsilon$. Recall that $r = \sum_{i=1}^{d-1} \nu_i$ and set $\tilde{r} = \sum_{i=1}^{\tilde{d}-1} \mu_i$.

Assume first that μ_I is the H-type of a de Jonquières transformation, namely, $\mu_I = (2\tilde{d} - 2, 0, \ldots, 0, 1)$ and $\tilde{r} = 2\tilde{d} - 1$. Since ν_I is not the H-type of a de Jonquières transformation and $\tilde{d} < d$, it follows that ν_I is obtained from μ_I by performing a quadratic transformation centered at points of multiplicity $e_j \leqslant 1$, say $1 \geqslant e_1 \geqslant e_2 \geqslant e_3 \geqslant 0$. On the other hand, the point with multiplicity e_3 for μ_I would have multiplicity $m_3 = e_3 + c = \tilde{d} - e_1 - e_2$ for ν_I, that still have the point of multiplicity $\tilde{d} - 1$ corresponding to the highest multiplicity point of μ_I. Since m_1, m_2, m_3 are chosen to be the highest multiplicities, this is possible only if $e_2 = e_3 = 0$. It follows that $d = 2\tilde{d} - e_1$, $m_1 = \tilde{d}$, and $m_2 = m_3 = \tilde{d} - e_1$; hence, $\tilde{r} = 2\tilde{d} - 1 = d + e_1 - 1$

and $r = \tilde{r} + 3 - e_1 = d + 2$, where r is the number of base points of $\mathcal{L}_\gamma$, which is the assertion.

Now we may and will assume that μ_I is not the H-type of a de Jonquières transformation, so the inductive hypothesis says that $\tilde{r} \leqslant \tilde{d} + 2$.

The construction of the μ_i implies that $r \leqslant \tilde{r} + 3$, and one sees that equality holds if and only if $m_1 = m_2 = m_3 = -\varepsilon = d/2 = \tilde{d}$. Since $d > 4$, one has $\tilde{r} \leqslant \tilde{d} + 2 = d/2 + 2 \leqslant d - 1$ and therefore $r = \tilde{r} + 3 \leqslant d + 2$, which is the assertion.

Assume then $r \leqslant \tilde{r} + 2$: One sees that equality holds if and only if $m_1 = \tilde{d} > m_2 = m_3 = -\epsilon$ and $d = m_1 + m_2$. Since μ_I is not a de Jonquières transformation, one has $\tilde{d} = m_1 \leqslant d - 2$ and therefore $r \leqslant \tilde{r} + 2 \leqslant \tilde{d} + 4 \leqslant d + 2$, which is the assertion.

Finally, the remaining case is $r \leqslant \tilde{r} + 1$. Since $\tilde{d} \leqslant d - 1$, one has

$$r \leqslant \tilde{r} + 1 \leqslant \tilde{d} + 3 \leqslant d + 2,$$

which concludes the proof of this lemma. □

We are finally ready for the proof of Theorem 1 in the Introduction.

Proof of Theorem 1 Fix an integer $d \geqslant 2$. By Lemma 26, Bir_d° is a quasi-projective variety. By Lemma 36, each irreducible component $\mathrm{Bir}_{\nu_I}^\circ$ of Bir_d° is determined by an admissible and irreducible multi-index $\nu_I = (\nu_1, \dots, \nu_{d-1})$ satisfying (3). By Lemma 34, $\mathrm{Bir}_{\nu_I}^\circ$ is rational and has dimension $8 + \dim U_{\nu_I} = 8 + 2r$, where $r = \sum_{i=1}^{d-1} \nu_i$ is the reduced length of ν_I. By Lemma 13, the maximum r is $2d - 1$ and occurs for $\nu_I = (2d - 2, 0, \dots, 0, 1)$, which gives an irreducible component of dimension $8 + 2(2d - 1) = 4d + 6$, whose elements are de Jonquières transformations. By Lemma 39, when $d \geqslant 4$, the other irreducible components have dimension at most $8 + 2(d + 2) = 2d + 12$. □

Definition 40 For any fixed positive integers d and $a < d$ let

$$\mathrm{Bir}_d^a = \mathbb{P}^{\binom{a+2}{2}-1} \times \mathrm{Bir}_{d-a}\,.$$

As already observed in Remark 28, there is a natural inclusion of Bir_d^a into Bir_d. For this we often identify Bir_d^a with its image in Bir_d.

Remark 41 For integers $a < b < d$ we have, with natural identifications,

$$\mathrm{Bir}_d^a \cap \mathrm{Bir}_d^b = \mathbb{P}^{\binom{a+2}{2}-1} \times \mathrm{Bir}_{d-a}^{b-a}.$$

Remark 42 When $0 < a < b < d$, the general element $[f_1 : f_2 : f_3]$ of $\tau_b(\mathrm{Bir}_{d-b}^\circ) \subset \mathrm{Bir}_d$ cannot be the limit of elements in $\tau_a(\mathrm{Bir}_{d-a}^\circ) \subset \mathrm{Bir}_d$ because $\gcd(f_1, f_2, f_3)$ is an irreducible polynomial of degree $b > a$.

We are able to completely describe the behavior of these varieties in low degrees.

Example 43 By Theorem 1, $\mathrm{Bir}_2^\circ = \mathrm{Bir}_{(3)}^\circ$ is irreducible of dimension 14. By Remark 28, $\mathrm{Bir}_2 = \mathrm{Bir}_{(3)}^\circ \cup \mathrm{Bir}_2^1$. A general element γ in $\mathrm{Bir}_{(3)}^\circ$ is given by a homaloidal net of conics with three distinct base points p_1, p_2, p_3. If we let p_3 move to a general point of the line through p_1 and p_2, the line splits from the net and we get a degeneration of γ to an element in Bir_2^1. Since each element in Bir_2^1 can be obtained in this way, Bir_2 is irreducible.

Example 44 Again by Theorem 1 and Remark 28, $\mathrm{Bir}_3^\circ = \mathrm{Bir}_{(4,1)}^\circ$ is irreducible of dimension 18 and $\mathrm{Bir}_3 = \mathrm{Bir}_{(4,1)} \cup \mathrm{Bir}_3^1 \cup \mathrm{Bir}_3^2$, where $\mathrm{Bir}_3^2 = \tau_2(\mathrm{Bir}_1^\circ)$ has dimension 13 and Bir_3^1 contains $\tau_1(\mathrm{Bir}_2^\circ)$, which has dimension 16.

We claim that Bir_3 has three irreducible components and is connected.

By Remark 41, $\mathrm{Bir}_3^1 \cap \mathrm{Bir}_3^2 = \mathbb{P}^2 \times \mathrm{Bir}_2^1$ has dimension 12. On the other hand, a general element γ in $\mathrm{Bir}_{(4,1)}^\circ$ is given by a homaloidal net of cubics with a double point p and four simple base points $q_1, \ldots, q_4$. If we let q_2 move to a general point of the line through p and q_1, the line splits from the net and we get a degeneration of γ to an element in Bir_3^1. This shows that Bir_3 is connected.

Note that any degeneration of an element in Bir_3° has to contain a double point. Linear systems of conics with a double point are homaloidal only in the presence of a fixed component. Then the only possible degenerations of Bir_3° are either a pair of fixed lines together with the linear system of lines or a fixed line, say l, and a linear system of conics with a base point on l. However, the general element in either Bir_3^1 or Bir_3^2 is such that there is no base point, of the mobile part, lying on the fixed component, and therefore it cannot be obtained as a limit of elements in Bir_3°. This means that Bir_3^1 and Bir_3^2 give two further irreducible components, other than $\mathrm{Bir}_{(4,1)}$. In [10], it seems that the authors missed the component Bir_3^2 in Bir_3.

Example 45 By Theorem 1, Bir_4° has two irreducible components $\mathrm{Bir}_{(6,0,1)}^\circ$ and $\mathrm{Bir}_{(3,3,0)}^\circ$, having respective dimensions 22 and 20.

Reasoning as in the previous examples, one may check that Bir_4 is connected (we will prove it in Theorem 55 later) and its decomposition into irreducible components is

$$\mathrm{Bir}_4 = \mathrm{Bir}_{(3,3,0)} \cup \mathrm{Bir}_{(6,0,1)} \cup \mathrm{Bir}_4^1 \cup \mathrm{Bir}_4^2 \cup \mathrm{Bir}_4^3,$$

where the last three components have respective dimensions 17, 19, and 20.

To study the connectedness of Bir_d°, we need to understand degenerations of base loci of homaloidal systems. This is a hard task, and almost nothing is known. The only example we are aware of is the one stated in [18, V.III.25, p. 231] of a quartic curve with three double points degenerating to a quartic with a triple point. This suggests that the component of Bir_4° associated with the multi-index $(3,3,0)$ intersects the component of a de Jonquières transformation.

Example 46 Take the linear system of quartic curves with three double base points at $p_1 = [0,0,1]$, $p_2 = [t,0,1]$, $p_3 = [0,t,1]$, $t \neq 0$. Its affine equation is

$$a_0x^4 + a_1x^3y + a_2x^2y^2 + a_4xy^3 + a_5y^4 - 2a_0tx^3 + a_3x^2y + (-a_4t + a_1t + a_3)xy^2 \\ - 2a_5ty^3 + a_0t^2x^2 + (-a_1t^2 - ta_3)xy + a_5t^2y^2 = 0.$$

For $t = 0$, this is a linear system of quartics, whose general member is irreducible, with a triple base point at $[0, 0, 1]$ and three infinitely near simple base points in the direction of the lines $x = 0$, $y = 0$, and $x + y = 0$, which are the limits of the lines $\overline{p_1 p_3} : x = 0$, $\overline{p_1 p_2} : y = 0$, and $\overline{p_2 p_3} : x + y - tz = 0$.

By imposing further three simple base points p_4, p_5, p_6 in general position, one gets a homaloidal net of type $(3, 3, 0)$ for a general $t \neq 0$ and a homaloidal net of type $(6, 0, 1)$ for $t = 0$. E.g., we choose $p_4 = [1, -2, 1]$, $p_5 = [-2, 1, 1]$, and $p_6 = [2, 3, 1]$ and we get the following Cremona transformation, for $t = 1$,

$$\Big[\left(3x^3 - 6x^2z + 80xy^2 - 107xyz + 3xz^2 - 9y^3 - 98y^2z + 107yz^2\right)x$$
$$: -3\left(-x^2 + 10xy - 12xz - y^2 - 12yz + 13z^2\right)xy$$
$$: 3(-y + z)\left(12x^2 - 3xy - 12xz - y^2 + yz\right)y\Big],$$

which has the inverse map

$$\Big[-\left(-36x^2 - 243xy + 42xz - 396y^2 + 116yz\right)\left(-36xy + 39xz - 99y^2 + 107yz\right)$$
$$: \left(-36xy + 39xz - 99y^2 + 107yz\right)\left(36xy - 42xz + 99y^2 - 125yz + 10z^2\right)$$
$$: -108x^3z - 1296x^2y^2 + 1539x^2yz - 1152x^2z^2 - 7128xy^3 + 10809xy^2z$$
$$- 6195xyz^2 - 30xz^3 - 9801y^4 + 15840y^3z - 8317y^2z^2 - 90yz^3\Big],$$

while, for $t = 0$, we get the de Jonquières transformation

$$\Big[-\left(12x^3 - 217xy^2 + 308xyz - 30y^3 + 308y^2z\right)x : 6(-2x^2 - 19xy + 28xz$$
$$- 2y^2 + 28yz)xy : 6\left(-23x^2y + 42x^2z - 5xy^2 + 42xyz - 2y^3\right)y\Big],$$

which has the inverse map

$$\Big[-7(6x + 11y)^2(-3y + 2z)(-6x - 17y + 4z)$$
$$: 14(-6x - 17y + 4z)(6x + 11y)(-3y + 2z)^2$$
$$: 216x^3z - 2484x^2y^2 + 4896x^2yz - 1368x^2z^2 - 9648xy^3$$
$$+ 16710xy^2z - 5544xyz^2 + 96xz^3 - 9555y^4$$
$$+ 15942y^3z - 5854y^2z^2 + 240yz^3\Big].$$

These computations have been performed by using Maple.

Proposition 47 $\mathrm{Bir}_4^\circ = \mathrm{Bir}^\circ_{(6,0,1)} \cup \mathrm{Bir}^\circ_{(3,3,0)}$ *is connected*, $\dim(\mathrm{Bir}^\circ_{(6,0,1)} \cap \mathrm{Bir}^\circ_{(3,3,0)}) = 19$.

Proof The connectedness follows from Example 46, which shows that the general element in $\mathrm{Bir}^{\circ}_{(3,3,0)}$ may degenerate to a special element in the component $\mathrm{Bir}^{\circ}_{(6,0,1)}$ of a de Jonquières transformation. The general choice of the base points of such de Jonquières transformations is one triple point in the plane, three infinitely near simple points, and further three simple points in the plane. Thus it lies in an open subset of $\mathbb{P}^2 \times \mathrm{Sym}^3 \mathbb{P}^1 \times \mathrm{Sym}^3 \mathbb{P}^2$, which has dimension 11, plus 8 dimensions for the action of PGL(3) on the homaloidal net; cf. Lemma 31. □

Proposition 48 $\mathrm{Bir}^{\circ}_5 = \mathrm{Bir}^{\circ}_{(8,0,0,1)} \cup \mathrm{Bir}^{\circ}_{(3,3,1,0)} \cup \mathrm{Bir}^{\circ}_{(0,6,0,0)}$ *is connected.*

Proof The decomposition of Bir°_5 into three irreducible components $\mathrm{Bir}^{\circ}_{(8,0,0,1)}$, $\mathrm{Bir}^{\circ}_{(3,3,1,0)}$, and $\mathrm{Bir}^{\circ}_{(0,6,0,0)}$, having respective dimensions 26, 22, and 20, follows from Theorem 1.

One has $\mathrm{Bir}^{\circ}_{(3,3,1,0)} \cap \mathrm{Bir}^{\circ}_{(0,6,0,0)} \neq \emptyset$ for the same reason of the previous proposition, namely, that a linear system of quintics with three double base points may degenerate to a linear system of quintics with a triple base point and three infinitely near simple base points. Using the notation of Example 46, one gets such a degeneration just by applying a quadratic transformation centered at p_4, p_5, p_6 to the degeneration of the linear system of quartics.

In order to prove the connectedness of Bir°_5, it is enough to show, with an example, that $\mathrm{Bir}^{\circ}_{(8,0,0,1)} \cap \mathrm{Bir}^{\circ}_{(3,3,1,0)} \neq \emptyset$. We remark that, if we make collide a triple point and three double points, in general we get a quintuple point, not a quadruple one. Thus we perform a special degeneration: We take the linear system $\mathcal{L}$ of quintics with an *oscnode* at $[0, 0, 1]$, along the direction of the conic $xz + y^2 = 0$, and a triple point at $[t, 0, 1]$, when $t \neq 0$. The affine equation of $\mathcal{L}$ is

$$\begin{aligned}
&a_0x^5 + a_1x^4y + a_2x^3y^2 + a_3x^2y^3 + a_4xy^4 + a_5y^5 \\
&\quad - 3\,a_0tx^4 - 2\,a_1tx^3y + \left(2\,a_0t^2 - a_2t\right)x^2y^2 + \left(a_1t^2 + a_5\right)xy^3 \\
&\quad - a_0t^3y^4 + 3\,a_0t^2x^3 + a_1t^2x^2y - 2\,a_0t^3xy^2 - a_0t^3x^2 = 0.
\end{aligned}$$

When $t = 0$, we get a linear system of quintics with a quadruple point whose general member is irreducible. By imposing further three simple base points in general position, we get a homaloidal net $\subset \mathcal{L}$ of type $(3, 3, 1, 0)$ which degenerates to a homaloidal net defining a de Jonquières transformation. E.g., if we choose $[1, 1, 1]$, $[1, -1, 1]$, and $[2, 1, 1]$ as simple base points, we get the following Cremona transformation, for $t = 1$,

$$\begin{aligned}
&\Big[10\,x^2y^2z + 9\,xy^3z - 18\,y^3x^2 + 5\,y^4x + 9\,y^5 + 5\,x^5 \\
&\quad - 10\,xy^2z^2 + 15\,x^3z^2 - 15\,x^4z + -5\,x^2z^3 - 5\,y^4z \\
&\quad : y\left(7\,y^2xz + 2\,y^4 - 9\,y^2x^2 + 5\,x^4 - 10\,x^3z + 5\,x^2z^2\right) \\
&\quad : y^2\left(4\,y^3 + 4\,yxz - 5\,x^2z - 8\,yx^2 + 5\,x^3\right)\Big]
\end{aligned}$$

while, for $t = 0$, we get the de Jonquières transformation

$$\big[-60\,y^3x^2 - 5\,y^4x + 30\,y^5 + 5\,x^5 + 30\,xy^3z : 12\,y^5 - 29\,y^3x^2 + 5\,yx^4 + 12\,xy^3z$$
$$: 6\,y^5 - 5\,y^4x - 12\,y^3x^2 + 5\,y^2x^3 + 6\,xy^3z\big].$$

We are able to check the properties of these maps and to find their inverse maps by using Maple, as we did in Example 46. □

Proposition 49 $\mathrm{Bir}_6^\circ = \mathrm{Bir}^\circ_{(10,0,0,0,1)} \cup \mathrm{Bir}^\circ_{(3,4,0,1,0)} \cup \mathrm{Bir}^\circ_{(4,1,3,0,0)} \cup \mathrm{Bir}^\circ_{(1,4,2,0,0)}$ *is connected.*

Proof The decomposition of Bir_6° into four irreducible components follows from Theorem 1. The usual degeneration of three double base points to one triple point with infinitely near three simple base points implies that $\mathrm{Bir}^\circ_{(4,1,3,0,0)} \cap \mathrm{Bir}^\circ_{(1,4,2,0,0)} \neq \emptyset$. It can be obtained by that of Example 46 by applying a quadratic transformation centered at p_4, p_5, and at a general point in the plane.

To conclude we show, with two examples, that

$$\mathrm{Bir}^\circ_{(3,4,0,1,0)} \cap \mathrm{Bir}^\circ_{(1,4,2,0,0)} \neq \emptyset \quad \text{and} \quad \mathrm{Bir}^\circ_{(3,4,0,1,0)} \cap \mathrm{Bir}^\circ_{(10,0,0,0,1)} \neq \emptyset.$$

First example: Take the linear system of sextics with a triple base point at $[0, 0, 1]$, with an infinitely near double base point in the direction of the line $x = 0$, and another triple base point at $[t, 0, 1]$, $t \neq 0$. When $t = 0$, we get a linear system of sextics with a quadruple base point with an infinitely near double base point. By imposing further three double base points and a simple base point, e.g., we choose $[1, 1, 1]$, $[-1, 1, 1]$, $[2, 1, 1]$, and $[2, -3, 1]$, respectively, we get a homaloidal net of type $(1, 4, 2, 0, 0)$ for a general $t \neq 0$ and a homaloidal net of type $(3, 4, 0, 1, 0)$ for $t = 0$. In particular, for $t = 1$, we get the map

$$\Big[27\,x^6 - 216\,y^6 - 81\,x^2yz^3 + 135\,x^3yz^2 - 108\,x^2y^2z^2$$
$$- 368\,xy^3z^2 - 27\,x^4yz + +108\,x^3y^2z + 520\,xy^4z$$
$$- 27\,x^3z^3 + 81\,x^4z^2 - 81\,x^5z + 324\,y^5z - 27\,x^5y - 260\,xy^5$$
$$: 3\,xy\,(-z+y)\left(-4\,y^3 + 22\,y^2z - 18\,xy^2 + 9\,xz^2 - 18\,x^2z + 9\,x^3\right)$$
$$: 9\,y^2\,(-z+y)\left(6\,y^3 + 4\,xy^2 - 7\,yxz - 3\,x^2y + 3\,x^3 - 3\,x^2z\right)\Big]$$

and, for $t = 0$, we get the map

$$\Big[27\,x^6 - 540\,y^6 - 430\,xy^3z^2 + 54\,x^4yz + 216\,x^3y^2z - 702\,x^2y^3z + 644\,xy^4z$$
$$+ 648\,y^5z - 108\,x^5y + 513\,x^2y^4 - 322\,xy^5$$
$$: 3\,y\,(-z+y)\left(36\,y^4 + 20\,xy^3 + -20\,y^2xz - 45\,x^2y^2 + 9\,x^4\right)$$
$$: 9\,y^2\,(-z+y)\left(6\,y^3 + 2\,xy^2 - 5\,yxz - 6\,x^2y + 3\,x^3\right)\Big].$$

Second example: Take the linear system of sextics with a double base point at [0, 0, 1] with three other infinitely near double base points, each one infinitely near to the previous one, along the conic $xz + y^2 = 0$, and a quadruple base point at $[t, 0, 1]$, $t \neq 0$. When $t = 0$, we get a linear system of sextics with a base point of multiplicity 5. By imposing further three simple base points, e.g., we choose [1, 1, 1], [−1, 1, 1], and [2, 1, 1], we get a homaloidal net of type $(3, 4, 0, 1, 0)$ for a general $t \neq 0$ which degenerates to a homaloidal net of de Jonquières type $(10, 0, 0, 0, 1)$. In particular, for $t = 1$, we get the map

$$\Big[-2x^6 - 8y^6 - 4xy^2z^3 + 8x^2y^2z^2 - 4x^3y^2z + 5x^2y^3z - 8xy^4z - 2x^2z^4 + 8x^3z^3 - 12x^4z^2 - 2y^4z^2 + 8x^5z - 5x^3y^3 + 13x^2y^4 + 5xy^5 : -2y\left(-xz + x^2 + xy - y^2\right)\left(-xz + x^2 - y^2\right)(-z + x - y) : -y^2\left(-2xz + 2x^2 - xy - 2y^2\right)\left(-xz + x^2 + xy - y^2\right)\Big]$$

and, for $t = 0$, we get the de Jonquières transformation

$$\Big[-3x^6 - 20y^6 - 20xy^4z + 20x^3y^3 + 23x^2y^4 : -y\left(8y^5 + 8y^3xz + 3x^5 - 11x^3y^2 - 8x^2y^3\right) : -y^2\left(4y^4 + 4xzy^2 + 3x^4 - 4x^3y - 7x^2y^2\right)\Big].$$

We again checked the properties of these maps and found their inverse maps by using Maple, as we did in Example 46. □

Remark 50 We want to stress a difference between Bir_5° and Bir_6°. It is not difficult to prove that any pair of components in Bir_5° intersects. The situation for Bir_6° is, quite unexpectedly, different. We claim that $\mathrm{Bir}^\circ_{(4,1,3,0,0)} \cap \mathrm{Bir}^\circ_{(10,0,0,0,1)} = \emptyset$. Let $\mathbb{P}^2 \times \Delta \to \Delta$ be a degeneration, over a complex disk Δ, with a linear system $\mathcal{L}$ such that the map induced by $\mathcal{L}_0$ is in $\mathrm{Bir}_{(10,0,0,0,1)}$ and the map induced by $\mathcal{L}_t$ is in $\mathrm{Bir}^\circ_{(4,1,3,0,0)}$, for $t \in \Delta \setminus \{0\}$. Let $\mu : Y \to \mathbb{P}^2 \times \Delta$ be the blow up of the unique singular point p in $\mathcal{L}_0$, with exceptional divisor $E \simeq \mathbb{P}^2$. Note that the sections associated with the singular points of $\mathcal{L}_t$ have to intersect in p, because p is the unique singular point in $\mathcal{L}_0$. Hence, the strict transform $\mathcal{L}^Y$ is such that $\mathcal{L}^Y_{0|E}$ is a curve with degree equal to the multiplicity of $\mathcal{L}_0$ at p, which is 5, and with three triple points and a double point. This forces, by a direct computation, $\mathcal{L}^Y_{0|E}$ to be non-reduced and therefore introduces a fixed component in $\mathcal{L}_0$. Hence, the map induced by $\mathcal{L}_0$ is in $\mathrm{Bir}_{(10,0,0,0,1)} \setminus \mathrm{Bir}^\circ_{(10,0,0,0,1)}$ and $\mathrm{Bir}^\circ_{(4,1,3,0,0)} \cap \mathrm{Bir}^\circ_{(10,0,0,0,1)} = \emptyset$.

The connectedness of Bir_d is considerably simpler even if not all irreducible components intersect each other.

Remark 51 If $\nu_I = (2d-2, 0, \ldots, 0, 1)$, i.e., if ν_I is the H-type of de Jonquières transformations, then it is easy to check that Bir_{ν_I} meets Bir_d^a, for each $a = 1, \ldots, d-1$.

However, the same statement does not hold for each admissible multi-index ν_I.

For example, one may check that the minimum d such that there exists an admissible $\nu_I = (\nu_1, \ldots, \nu_{d-1})$ with $\mathrm{Bir}_{\nu_I} \cap \mathrm{Bir}_d^1 = \emptyset$ is $d = 10$. Moreover, there are exactly two such admissible ν_I of degree $d = 10$, namely, $(0, 0, 7, 0, 0, 1, 0, 0, 0)$ and $(3, 0, 0, 6, 0, 0, 0, 0, 0)$.

We already recognized that $\mathrm{Bir}_d^a \cap \mathrm{Bir}_d^b \neq \emptyset$. Hence, to conclude the connectedness it is enough to show that for any admissible index ν_I there is a degeneration with a fixed component. The following lemmas allow us to produce these degenerations.

Lemma 52 *Let $\chi : \mathbb{P}^2 \times \Delta \to \Delta$ be a one-dimensional family over a complex disk Δ. Let F_t be the fiber over the point $t \in \Delta$. Let C_1, C_2, and C_3 be three disjoint sections of χ. Let $p_t^i := C_i \cap F_t$ be the intersection of the i-th section with the fiber F_t, $i = 1, 2, 3$. Assume that the points p_t^i are in general position for any t. Then there is a birational modification $\Omega : \mathbb{P}^2 \times \Delta \dashrightarrow \mathbb{P}^2 \times \Delta$ such that $\Omega_t := \Omega_{|F_t}$ is the quadratic Cremona transformation centered at the points $\{p_t^i\}$. Assume that there is a linear system $\mathcal{H} \in Pic(\mathbb{P}^2 \times \Delta)$ such that $\mathcal{H}_t$ is a homaloidal net associated with a multi-index ν_I and $\mathcal{H}_0$ is a homaloidal net associated with a multi-index μ_I and*

$$\sum_i \mathrm{mult}_{Z_{\nu_I}} p_t^i < \sum_i \mathrm{mult}_{Z_{\mu_J}} p_0^i.$$

Let $\mathcal{H}' := \Omega_* \mathcal{H}$ be the transformed linear system. Then $\mathcal{H}'_{|F_0}$ has a fixed component.

Proof Let D_{ij} be the divisor covered by lines spanning the points p_t^i and p_t^j inside F_t. The general position assumption ensures that D_{ij} is a smooth minimally ruled surface. Let $\phi : Y \to \mathbb{P}^2 \times \Delta$ be the blow up of $\mathbb{P}^2 \times \Delta$ along the disjoint sections C_i with exceptional divisors E_i. Let D_{ij}^Y be the strict transform of D_{ij} on Y and l_t^{ij} the strict transform of the line $\langle p_t^i, p_t^j \rangle \subset F_t$, and F_t^Y the strict transform of the fiber F_t. Then we have the following intersection numbers.

$$K_Y \cdot l_t^{ij} = K_{F_t^Y} \cdot l_t^{ij} = -1$$

and

$$D_{ij}^Y \cdot l_t^{ij} = D_{ij}^Y \cdot D_{ij}^Y \cdot F_t^Y = (l_t^{ij} \cdot l_t^{ij})_{F_t^Y} = -1.$$

Moreover, D_{ij}^Y is ruled by l_t^{ij} and all fibers are irreducible and reduced. This shows, by Mori theory (see, for instance, [2, Theorem 4.1.2]), that l_t^{ij} spans an extremal ray and the extremal ray can be contracted to a smooth curve Z_{ij} in a smooth 3-fold. Let ψ be the blow down of the three disjoint divisors D_{ij}. Then ψ is a morphism from Y to $\mathbb{P}^2 \times \Delta$. The required map Ω is just $\psi \circ \phi^{-1}$. To conclude, observe that for the general fiber $\Omega_t(\mathcal{H}_t) = \mathcal{H}'_{|F_t}$ and the $\deg \Omega_t(\mathcal{H}_t) = \deg \mathcal{H}' = 2d - \sum \mathrm{mult}_{Z_{\nu_I}} p_i^t$.

The numerical assumption on the multiplicities forces $\deg \Omega_0(\mathcal{H}_0) < \deg \mathcal{H}'$. This yields a fixed component in $\mathcal{H}'_{|F_0}$. □

Remark 53 The usage of the above lemma is to produce degenerations with fixed components starting from a known degeneration in a different pure degree.

To apply the above lemma we have to construct degenerations. This is the aim of the next lemma.

Lemma 54 *Let ν_I be an admissible multi-index in degree d. Let Z_{ν_I} be a base locus of a general homaloidal net associated with the multi-index ν_I, and $p_1, p_2 \in Z_{\nu_I}$ two points. Assume that*

$$m_1 := \operatorname{mult}_{Z_{\nu_I}} p_1 \leqslant \operatorname{mult}_{Z_{\nu_I}} p_2 =: m_2.$$

Then there is a degeneration $\chi : \mathbb{P}^2 \times \Delta \to \Delta$ and a base scheme $\mathcal{Z}$ such that $Z_t := \mathcal{Z}_{|F_t}$ is associated with the multi-index ν_I and $Z_0 := \mathcal{Z}_{|F_0}$ has the point p_1 infinitely near to p_2.

Proof The multiplicity of p_2 is at least the one of p_1 and we may degenerate p_1 into p_2. Assume that $\operatorname{mult}_{p_1} Z_{\nu_I} = m_1$ and $\operatorname{mult}_{p_2} Z_{\nu_I} = m_2$. Then a local equation of such a degeneration can be

$$tx_0^{d-m_1} p + x_1^{d-m_2}(x_2^{m_1} h + tg) + x_2^d = 0,$$

where $g \in \mathbb{C}[x_0, x_1, x_2]$ is such that for $t \neq 0$ the points $[1, 0, 0]$ and $[0, 1, 0]$ are ordinary points of multiplicities m_1 and m_2, respectively, and for $t = 0$ the point $[0, 1, 0]$ is of multiplicity m_2 with an infinitely near point of multiplicity m_1. □

Theorem 55 *The quasi-projective variety* Bir_d *is connected.*

Proof As already observed, we have only to prove that for any admissible index ν_I the general element admits a degeneration with a fixed component. Let $\mathcal{L}$ be a general homaloidal net associated with ν_I and ω a standard Cremona transformation centered in three points of $\operatorname{Bs}\mathcal{L}$ that lowers the degree. Let $\mathcal{L}'$ be the transformed homaloidal net and q_1, q_2, q_3 the three points of indeterminacy of ω^{-1}. Then by the Noether–Castelnuovo theorem the q_i's are not of maximal multiplicity. That is, we may assume that there is a point $x \in \operatorname{Bs}\mathcal{L}'$ with $\operatorname{mult}_x \mathcal{L}' > \operatorname{mult}_{q_1} \mathcal{L}'$. Then by Lemma 54 there is a degeneration $\chi : \mathbb{P}^2 \times \Delta \to \Delta$ and a base scheme $\mathcal{Z}$ such that $Z_t := \mathcal{Z}_{|F_t}$ is associated with the multi-index ν_I and $Z_0 := \mathcal{Z}_{|F_0}$ has the point q_1 infinitely near to x.

Let C_i be the section of χ associated with the point q_i and Ω, $\mathcal{H}'$ the birational modification and linear system on $\mathbb{P}^2 \times \Delta$ as in Lemma 52. Then we may apply Lemma 52 to produce a Cremona transformation $\Omega^{-1} : \mathbb{P}^2 \times \Delta \dashrightarrow \mathbb{P}^2 \times \Delta$ that induces ω^{-1} on the general fiber and produces a fixed component in the special linear system $\mathcal{H}_{|F_0}$. In particular, this produces a degeneration of $\mathcal{L}$ to a homaloidal net with a fixed component. □

We are now ready to complete the proof of Theorem 3.

Proof of Theorem 3 By Remark 28, Bir_d is the union of Bir_d° and Bir_d^a, for each $a = 1, \dots, d-1$. Note that Bir_d^{d-1}, that is, $\tau_{d-1}(\mathrm{Bir}_1^\circ)$, has dimension $8 + \binom{d+1}{2} - 1 = d(d+1)/2 + 7$, for $d \geqslant 2$. By Theorem 1, irreducible components of Bir_d coming from irreducible components of Bir_{d-a}°, $a \leqslant d-2$, have dimension at most

$$4(d-a) + 6 + a(a+3)/2 = 4d + 6 + a(a-5)/2 \leqslant d(d-1)/2 + 13.$$

When $d \geqslant 7$, this implies that Bir_d^a, $a \leqslant d-2$, has components of smaller dimension. □

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Appendix: Test if a Multi-Index is Irreducible, i.e., Admissible

The following script defines a function "adm" in PARI/GP that checks if a multi-index ν_I is irreducible, or equivalently admissible; cf. Definition 22 and Theorem 24.

The input is a vector $\nu_I = [\nu_1, \nu_2, \dots, \nu_{d-1}]$, where the square brackets are used in GP for denoting row vectors. The output of $\mathrm{adm}(\nu_I)$ is either $1 = \text{true}$, i.e., ν_I is admissible, or $0 = \text{false}$, i.e., ν_I is not admissible.

```
adm(v) = { local( d=1+#v , s=1-(#v+1)^2 ,
                  m=vector(3) , t=0 , e=0 );
  for( i=1,#v , s = s+i^2*v[i] );
  if( s , print("ERROR: the self-intersection is not 1");
      return(0) );
  s = -#v*3;
  for( i=1,#v , s = s+i*v[i] );
  if( s , print("ERROR: the genus is not 0");
      return(0) );
  while( d>2,
    s = 0; t = d; e = d;
    for( i=1,3 , while( !s, t=t-1; s=s+v[t] );
       m[i]=t; e=e-t; v[t]=v[t]-1 ; s=s-1 );
    if( m[1]+m[2]>d, print("The net is reducible");
        return(0) );
    for( j=1,3 , t=m[j]+e; if( t , v[t] = v[t] +1 ) );
  d = d+e );
  1}
```

For example, after defining this function "adm" in GP, the command "adm([0, 6, 0, 0])" returns "1", while the command "adm([6, 0, 2, 0])" prints "The net is reducible" and returns "0"; cf. Remark 21.

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