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Milnor–Moore categories and monadic decomposition ☆

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ABSTRACT

In this paper monoidal Hom-Lie algebras, Lie color algebras, Lie superalgebras and other type of generalized Lie algebras are recovered by means of an iterated construction, known as monadic decomposition of functors, which is based on Eilenberg–Moore categories. To this aim we introduce the notion of Milnor–Moore category as a monoidal category for which a Milnor–Moore type Theorem holds. We also show how to lift the property of being a Milnor–Moore category whenever a suitable monoidal functor is given and we apply this technique to provide examples.

Contents

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Introduction

The celebrated Milnor–Moore Theorem [32, Theorem 5.18] establishes, in characteristic zero, an equivalence between the category of primitively generated braided bialgebras and the category of Lie algebras. The functors giving the equivalence are the universal enveloping algebra functor $\mathcal{U}$, associating the universal enveloping algebra $\mathcal{U}(L)$ to a Lie algebra L, and the primitive functor $\mathcal{P}$ which gives the primitive part $\mathcal{P}(B)$ of a given bialgebra B. The fact that the counit $\mathcal{U}\mathcal{P} \to \mathrm{Id}$ of the adjunction involved is an isomorphism just encodes the fact that the bialgebras considered are primitively generated. On the other hand the crucial point in the proof of the theorem is that the maps $\eta L : L \to \mathcal{P}(\mathcal{U}(L))$ giving the unit of the adjunction $(\mathcal{U}, \mathcal{P})$ are isomorphisms. Now observe that the tensor algebra $T(V)$, defined for any vector space V, yields a functor T from the category of vector spaces to the category of bialgebras which is a left adjoint of the functor P obtained from $\mathcal{P}$ forgetting the Lie algebra structure. In this case the unit $V \to P(T(V))$ fails to be an isomorphism in general. Note also that $\mathcal{U}(L)$ is a quotient of $T(L)$. Thus we could say that $(\mathcal{U}, \mathcal{P})$ is a refinement of the adjunction (T, P) obtained by restricting the codomain of P and changing the left adjoint in order to obtain a new adjunction with invertible unit.

Considering the wider context of a monoidal category $\mathcal{M}$, bialgebras and Lie algebras are substituted by their symmetric braided analogue (for instance a braided symmetric bialgebra in $\mathcal{M}$ is an object equipped with an algebra structure, a coalgebra structure and a symmetric Yang–Baxter operator satisfying the expected compatibility axioms); the same happens for the primitive functor and the enveloping functor. Such a category $\mathcal{M}$ is called Milnor–Moore (MM) exactly when the unit of this adjunction is an isomorphism. In this case, the category of symmetric braided Lie algebras can be described by the so-called monadic decomposition of the primitive functor (see below). When the category $\mathcal{M}$ is also symmetric, then we can consider bialgebras and Lie algebras in $\mathcal{M}$ as in the case of vector spaces. In this case we prove that if $\mathcal{M}$ is a MM-category then the unit $\eta : \mathrm{Id} \to \mathcal{P}\mathcal{U}$ of the corresponding adjunction $(\mathcal{U}, \mathcal{P})$ is an isomorphism too and that the category of Lie algebras can be recovered from the starting category $\mathcal{M}$ by means of the same iterated procedure. For this reason, the remaining part of our investigation focuses on giving examples of MM-categories. The first example is the category $\mathfrak{M}$ of vector spaces in characteristic zero. Then, under mild conditions, we find that a monoidal category $\mathcal{M}$ endowed with a conservative and exact monoidal functor $\mathcal{M} \to \mathfrak{M}$ preserving denumerable coproducts is still MM. As a consequence we can prove that monoidal Hom-Lie algebras, Lie color algebras, Lie superalgebras and other type of generalized Lie algebras are recovered by means of the same iterated construction based on Eilenberg–Moore categories.

In order to explain our results more precisely, we need now to enter into the technical details of our setting. Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction with unit η and counit ϵ. Then RL is a monad on $\mathcal{B}$ (with multiplication $R\epsilon L$ and unit η) and one can consider the Eilenberg–Moore category ${}_{RL}\mathcal{B}$ associated to this monad and the so-called *comparison functor* $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ which is defined by $KX := (RX, R\epsilon X)$ and $Kf := Rf$. This gives the diagram

$$\begin{array}{ccc}
\mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} \\
{\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle K} \\
\mathcal{B} & \xleftarrow{{}_{RL}U} & {}_{RL}\mathcal{B}
\end{array}$$

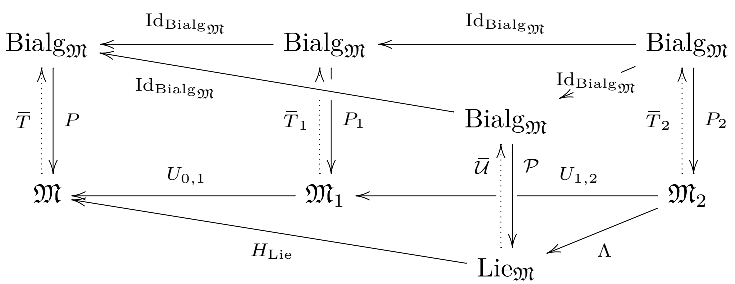
where the undashed part commutes. In the case when K itself has a left adjoint Λ, one can repeat this construction starting from the new adjunction (Λ, K). Going on this way one possibly obtains a diagram of the form

$$\begin{array}{ccccccc}
\mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \cdots \\
{\scriptstyle L_0}\uparrow\downarrow{\scriptstyle R_0} & & {\scriptstyle L_1}\uparrow\downarrow{\scriptstyle R_1} & & {\scriptstyle L_2}\uparrow\downarrow{\scriptstyle R_2} & & \\
\mathcal{B}_0 & \xleftarrow{U_{0,1}} & \mathcal{B}_1 & \xleftarrow{U_{1,2}} & \mathcal{B}_2 & \xleftarrow{U_{2,3}} & \cdots
\end{array}$$

where it is more convenient to relabel (L, R) and (Λ, K) as (L_0, R_0) and (L_1, R_1) respectively. If there is a minimal $N \in \mathbb{N}$ such that L_N is full and faithful, then R is said to have *monadic decomposition of monadic length* N. This is equivalent to requiring that the forgetful functor $U_{N,N+1}$ is a category isomorphism and no $U_{n,n+1}$ has this property for $0 \leq n \leq N - 1$ (see e.g. [4, Remark 2.4]). In [4, Theorem 3.4], we investigated the particular case

$$\begin{array}{ccccc}
\mathrm{Bialg}_{\mathfrak{M}} & \xleftarrow{\mathrm{Id}_{\mathrm{Bialg}_{\mathfrak{M}}}} & \mathrm{Bialg}_{\mathfrak{M}} & \xleftarrow{\mathrm{Id}_{\mathrm{Bialg}_{\mathfrak{M}}}} & \mathrm{Bialg}_{\mathfrak{M}} \\
{\scriptstyle \overline{T}}\uparrow\downarrow{\scriptstyle P} & & {\scriptstyle \overline{T}_1}\uparrow\downarrow{\scriptstyle P_1} & & {\scriptstyle \overline{T}_2}\uparrow\downarrow{\scriptstyle P_2} \\
\mathfrak{M} & \xleftarrow{U_{0,1}} & \mathfrak{M}_1 & \xleftarrow{U_{1,2}} & \mathfrak{M}_2
\end{array}$$

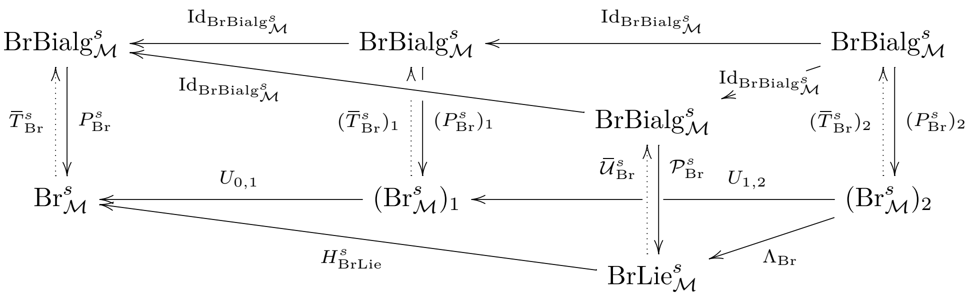
where $\mathfrak{M}$ denotes the category of vector spaces over a fixed base field $\Bbbk$, $\mathrm{Bialg}_{\mathfrak{M}}$ is the category of $\Bbbk$-bialgebras, $\overline{T}$ is the tensor bialgebra functor (the barred notation serves to distinguish this functor from the tensor algebra functor $T : \mathfrak{M} \to \mathrm{Alg}_{\mathfrak{M}}$ which goes into $\Bbbk$-algebras) and P is the primitive functor which assigns to each $\Bbbk$-bialgebra its space of primitive elements. We proved that this P has a monadic decomposition of monadic length at most 2. Moreover, when $\mathrm{char}(\Bbbk) = 0$, for every $V_2 = ((V, \mu), \mu_1) \in \mathfrak{M}_2$ one can define $[x, y] := \mu(xy - yx)$ for every $x, y \in V$. Then $(V, [-, -])$ is an ordinary Lie algebra and $\overline{T}_2 V_2 = TV/(xy - yx - [x, y] \mid x, y \in V)$ is the corresponding universal enveloping algebra. This suggests a connection between the category $\mathfrak{M}_2$ and the category $\mathrm{Lie}_{\mathfrak{M}}$ of Lie $\Bbbk$-algebras. It is then natural to expect the existence of a category equivalence Λ such that the following diagram



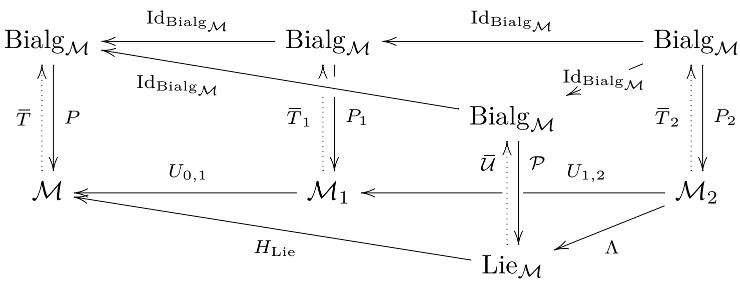
commutes in its undashed part, where H_{Lie} denotes the forgetful functor, $\overline{\mathcal{U}}$ the universal enveloping bialgebra functor and $\mathcal{P}$ the corresponding primitive functor.

A first investigation showed that, in order to solve the problem above, it is more natural to work with braided $\Bbbk$-vector spaces instead of ordinary $\Bbbk$-vector spaces and to replace the categories $\mathfrak{M}$, $\mathrm{Bialg}_{\mathfrak{M}}$ and $\mathrm{Lie}_{\mathfrak{M}}$ with their braided analogues $\mathrm{Br}_{\mathfrak{M}}$, $\mathrm{BrBialg}_{\mathfrak{M}}$ and $\mathrm{BrLie}_{\mathfrak{M}}$ consisting of braided vector spaces, braided bialgebras and braided Lie algebras respectively. We were further led to substitute $\mathfrak{M}$ with an arbitrary monoidal category $\mathcal{M}$. We point out that, in order to produce a braided analogue of the universal enveloping algebra which further carries a braided bialgebra structure, the assumption that the underlying Yang–Baxter operator is symmetric is also needed. Thus let $\mathrm{Br}^s_{\mathcal{M}}$, $\mathrm{BrBialg}^s_{\mathcal{M}}$ and $\mathrm{BrLie}^s_{\mathcal{M}}$ be the analogue of $\mathrm{Br}_{\mathcal{M}}$, $\mathrm{BrBialg}_{\mathcal{M}}$ and $\mathrm{BrLie}_{\mathcal{M}}$ consisting of objects with symmetric Yang–Baxter operator. Let $\overline{T}^s_{\mathrm{Br}} : \mathrm{Br}^s_{\mathcal{M}} \to \mathrm{BrBialg}^s_{\mathcal{M}}$ be the symmetric braided tensor bialgebra functor and let P^s_{Br} be its right adjoint, the primitive functor. We look for a condition for P^s_{Br} to have monadic decomposition of monadic length at most two. On the other hand the functor P^s_{Br} induces a functor $\mathcal{P}^s_{\mathrm{Br}} : \mathrm{BrBialg}^s_{\mathcal{M}} \to \mathrm{BrLie}^s_{\mathcal{M}}$ which turns out to have a left adjoint $\overline{\mathcal{U}}^s_{\mathrm{Br}}$, the universal enveloping bialgebra functor.

In view of the celebrated Milnor–Moore Theorem, see Remark 7.5, we say that a category $\mathcal{M}$ is a Milnor–Moore category (MM-category for short) whenever the unit of the adjunction $(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}})$ is a functorial isomorphism (plus other conditions required for the existence of the functors involved). One of the main results in the paper is Theorem 7.1, which ensures that, for a MM-category $\mathcal{M}$, the functor P^s_{Br} has a monadic decomposition of monadic length at most two. Moreover, in this case, we can identify the category $(\mathrm{Br}^s_{\mathcal{M}})_2$ with $\mathrm{BrLie}^s_{\mathcal{M}}$ through an equivalence $\Lambda_{\mathrm{Br}} : (\mathrm{Br}^s_{\mathcal{M}})_2 \to \mathrm{BrLie}^s_{\mathcal{M}}$.



Hence MM-categories, besides having an interest in their own, give us an environment where the functor P^s_{Br} has a behavior completely analogous to the classical vector space situation we investigated in [4, Theorem 3.4]. In the case of a symmetric MM-category $\mathcal{M}$ the connection with Milnor–Moore Theorem becomes more evident. In fact, in this case, we can apply Theorem 7.2 to obtain that the unit of the adjunction $(\bar{\mathcal{U}}, \mathcal{P})$ is a functorial isomorphism.



The next step is to provide meaningful examples of MM-categories. A first result in this direction is Theorem 8.1, based on a result by Kharchenko, which states that the category $\mathfrak{M}$ of vector spaces over a field of characteristic 0 is a MM-category. Note that the Lie algebras involved are not ordinary ones but they depend on a symmetric Yang–Baxter operator.

Much of the material developed in the paper (see e.g. Proposition 3.7, Theorem 8.3 and the construction of the adjunctions used therein) is devoted to the proof of our central result namely Theorem 8.4 which allows us to lift the property of being a MM-category whenever a suitable monoidal functor is given. A main tool in this proof is the concept of commutation datum which we introduce and investigate in Section 2. We use this Theorem 8.4 in the case of the forgetful functor $F : \mathcal{M} \to \mathfrak{M}$ where $\mathcal{M}$ is a subcategory of $\mathfrak{M}$. The goal is to provide, in this way, meaningful examples of MM-categories $\mathcal{M}$ and, in the case when $\mathcal{M}$ is symmetric, to recognize the corresponding type of Lie algebras. A first example of MM-category obtained in this way is the category of Yetter–Drinfeld modules, over a Hopf algebra over a field of characteristic zero, which is considered in Example 9.1. Subsection 9.1 (resp. 9.2) deals with the case when $\mathcal{M}$ is the category of modules (resp. comodules) over a quasi-bialgebra (resp. over a dual quasi-bialgebra). We prove that the forgetful functor satisfies the assumptions of Theorem 8.4 if and only if the quasi-bialgebra (resp. the dual quasi-bialgebra) is a deformation of a usual bialgebra, see Lemma 9.4 (resp. Lemma 9.13). As particular cases of this situation we prove that the category $\widetilde{\mathcal{H}}(\mathfrak{M})$ of [16, Proposition 1.1] is an MM-category, see Remark 9.10. Note that an object in $\mathrm{Lie}_{\mathcal{M}}$, for $\mathcal{M} = \widetilde{\mathcal{H}}(\mathfrak{M})$, is nothing but a monoidal Hom-Lie algebra. In Remark 9.17, we recover (H, R)-Lie algebras in the sense of [13, Definition 4.1] by considering the category of comodules over a co-triangular bialgebra (H, R) regarded as a co-triangular dual quasi-bialgebra with trivial reassociator. In particular, let G be an abelian group endowed with an anti-symmetric bicharacter $\chi : G \times G \to \Bbbk \setminus \{0\}$ and extend χ by linearity to a $\Bbbk$-linear map $R : \Bbbk[G] \otimes \Bbbk[G] \to \Bbbk$, where $\Bbbk[G]$ denotes the group algebra. Then $(\Bbbk[G], R)$ is a co-triangular bialgebra and, as a consequence, we recover (G, χ)-Lie color algebras in the sense of [33, Example 10.5.14], in Example 9.18, and in particular Lie superalgebras in Example 9.19.

The appendices contain general results regarding the existence of (co)equalizers in the category of (co)algebras, bialgebras and their braided analogue over a monoidal category. These results are applied to obtain Proposition B.11, which is used in the proof of Theorem 7.1.

1. Preliminaries

In this section, we shall fix some basic notation and terminology.

Notation 1.1. *Throughout this paper $\Bbbk$ will denote a field. All vector spaces will be defined over $\Bbbk$. The unadorned tensor product $\otimes$ will denote the tensor product over $\Bbbk$ if not stated otherwise.*

1.2. Monoidal Categories. Recall that (see [26, Chap. XI]) a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \to X$, $r_X : X \otimes \mathbf{1} \to X$, for every X, Y, Z in $\mathcal{M}$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom*, that is the equality

$$(U \otimes a_{V,W,X}) \circ a_{U,V \otimes W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W \otimes X} \circ a_{U \otimes V,W,X}$$

holds true, for every U, V, W, X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom*, that is $(V \otimes l_W) \circ a_{V,\mathbf{1},W} = r_V \otimes W$, for every V, W in $\mathcal{M}$.

A *monoidal functor* (also called strong monoidal in the literature)

$$(F, \phi_0, \phi_2) : (\mathcal{M}, \otimes, \mathbf{1}, a, l, r) \to (\mathcal{M}', \otimes', \mathbf{1}', a', l', r')$$

between two monoidal categories consists of a functor $F : \mathcal{M} \to \mathcal{M}'$, an isomorphism $\phi_2(U,V) : F(U) \otimes' F(V) \to F(U \otimes V)$, natural in $U, V \in \mathcal{M}$, and an isomorphism $\phi_0 : \mathbf{1}' \to F(\mathbf{1})$ such that the diagram

$$\begin{array}{ccccc} (F(U) \otimes' F(V)) \otimes' F(W) & \xrightarrow{\phi_2(U,V)\otimes' F(W)} & F(U \otimes V) \otimes' F(W) & \xrightarrow{\phi_2(U\otimes V,W)} & F((U \otimes V) \otimes W) \\ \downarrow{\scriptstyle a'_{F(U),F(V),F(W)}} & & & & \downarrow{\scriptstyle F(a_{U,V,W})} \\ F(U) \otimes' (F(V) \otimes' F(W)) & \xrightarrow{F(U)\otimes'\phi_2(V,W)} & F(U) \otimes' F(V \otimes W) & \xrightarrow{\phi_2(U,V\otimes W)} & F(U \otimes (V \otimes W)) \end{array}$$

is commutative, and the following conditions are satisfied:

$$F(l_U) \circ \phi_2(\mathbf{1}, U) \circ (\phi_0 \otimes' F(U)) = l'_{F(U)}, \quad F(r_U) \circ \phi_2(U, \mathbf{1}) \circ (F(U) \otimes' \phi_0) = r'_{F(U)}.$$

The monoidal functor is called *strict* if the isomorphisms ϕ_0, ϕ_2 are identities of $\mathcal{M}'$.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories.

As it is noticed in [28, p. 420], the Pentagon Axiom solves the consistency problem that appears because there are two ways to go from $((U \otimes V) \otimes W) \otimes X$ to $U \otimes (V \otimes (W \otimes X))$. The coherence theorem, due to S. Mac Lane [31, Chapter VII, Section 2], solves the similar problem for the tensor product of an arbitrary number of objects in $\mathcal{M}$. Accordingly with this theorem, we can always omit all brackets and simply write $X_1 \otimes \cdots \otimes X_n$ for any object obtained from $X_1, \ldots, X_n$ by using $\otimes$ and brackets. Also as a consequence of the coherence theorem, the morphisms a, l, r take care of themselves, so they can be omitted in any computation involving morphisms in $\mathcal{M}$. Thus, for sake of simplicity, from now on we will omit the associativity and unit constraints unless needed to clarify the context.

Let V be an object in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. Define iteratively $V^{\otimes n}$ for all $n \in \mathbb{N}$ by setting $V^{\otimes 0} := \mathbf{1}$ for $n = 0$ and $V^{\otimes n} := V^{\otimes (n-1)} \otimes V$ for $n > 0$.

Remark 1.3. Let $\mathcal{M}$ be a monoidal category. Denote by $\mathrm{Alg}_{\mathcal{M}}$ the category of algebras in $\mathcal{M}$ and their morphisms. Assume that $\mathcal{M}$ has denumerable coproducts and that the tensor products (i.e. $M \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes M : \mathcal{M} \to \mathcal{M}$, for every object M in $\mathcal{M}$) preserve such coproducts. By [31, Theorem 2, page 172], the forgetful functor

$$\Omega : \mathrm{Alg}_{\mathcal{M}} \to \mathcal{M}$$

has a left adjoint $T : \mathcal{M} \to \mathrm{Alg}_{\mathcal{M}}$. By construction $\Omega TV = \oplus_{n \in \mathbb{N}} V^{\otimes n}$ for every $V \in \mathcal{M}$. For every $n \in \mathbb{N}$, we will denote by

$$\alpha_n V : V^{\otimes n} \to \Omega TV$$

the canonical injection. Given a morphism $f : V \to W$ in $\mathcal{M}$, we have that Tf is uniquely determined by the following equality

$$\Omega Tf \circ \alpha_n V = \alpha_n W \circ f^{\otimes n}, \text{ for every } n \in \mathbb{N}. \tag{1}$$

The multiplication $m_{\Omega TV}$ and the unit $u_{\Omega TV}$ are uniquely determined by

$$m_{\Omega TV} \circ (\alpha_m V \otimes \alpha_n V) = \alpha_{m+n} V, \text{ for every } m, n \in \mathbb{N}, \tag{2}$$

$$u_{\Omega TV} = \alpha_0 V. \tag{3}$$

Note that (2) should be integrated with the proper unit constrains when m or n is zero.

The unit η and the counit ϵ of the adjunction (T, Ω) are uniquely determined, for all $V \in \mathcal{M}$ and $(A, m_A, u_A) \in \mathrm{Alg}_{\mathcal{M}}$ by the following equalities

$$\eta V := \alpha_1 V \qquad \text{and} \qquad \Omega\epsilon\,(A, m_A, u_A) \circ \alpha_n A := m_A^{n-1} \text{ for every } n \in \mathbb{N} \tag{4}$$

where $m_A^{n-1} : A^{\otimes n} \to A$ is the iterated multiplication of A defined by $m_A^{-1} := u_A, m_A^0 := \mathrm{Id}_A$ and, for $n \geq 2$, $m_A^{n-1} = m_A(m_A^{n-2} \otimes A)$.

Definition 1.4. Recall that a *monad* on a category $\mathcal{A}$ is a triple $\mathbb{Q} := (Q, m, u)$, where $Q : \mathcal{A} \to \mathcal{A}$ is a functor, $m : QQ \to Q$ and $u : \mathcal{A} \to Q$ are functorial morphisms satisfying the associativity and the unitality conditions $m \circ mQ = m \circ Qm$ and $m \circ Qu = \mathrm{Id}_Q = m \circ uQ$. An *algebra* over a monad $\mathbb{Q}$ on $\mathcal{A}$ (or simply a $\mathbb{Q}$-*algebra*) is a pair (X, μ) where $X \in \mathcal{A}$ and $\mu : QX \to X$ is a morphism in $\mathcal{A}$ such that $\mu \circ Q\mu = \mu \circ mX$ and $\mu \circ uX = \mathrm{Id}_X$. A *morphism between two* $\mathbb{Q}$-*algebras* (X, μ) and (X', μ') is a morphism $f : X \to X'$ in $\mathcal{A}$ such that $\mu' \circ Qf = f \circ \mu$. We will denote by ${}_{\mathbb{Q}}\mathcal{A}$ the category of $\mathbb{Q}$-algebras and their morphisms. This is the so-called *Eilenberg–Moore category* of the monad $\mathbb{Q}$ (which is sometimes also denoted by $\mathcal{A}^{\mathbb{Q}}$ in the literature). When the multiplication and unit of the monad are clear from the context, we will just write Q instead of $\mathbb{Q}$.

A monad $\mathbb{Q}$ on $\mathcal{A}$ gives rise to an adjunction $(F, U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathcal{A} \to A$ is the forgetful functor and $F : \mathcal{A} \to {}_{\mathbb{Q}}\mathcal{A}$ is the free functor. Explicitly:

$$U\,(X, \mu) := X, \quad Uf := f \qquad \text{and} \qquad FX := (QX, mX)\,, \quad Ff := Qf.$$

Note that $UF = Q$. The unit of the adjunction (F, U) is given by the unit $u : \mathcal{A} \to UF = Q$ of the monad $\mathbb{Q}$. The counit $\lambda : FU \to {}_{\mathbb{Q}}\mathcal{A}$ of this adjunction is uniquely determined by the equality $U\,(\lambda\,(X, \mu)) = \mu$ for every $(X, \mu) \in {}_{\mathbb{Q}}\mathcal{A}$. It is well-known that the forgetful functor $U : {}_{\mathbb{Q}}\mathcal{A} \to \mathcal{A}$ is faithful and reflects isomorphisms (see e.g. [12, Proposition 4.1.4]).

Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction with unit η and counit ϵ. Then $(RL, R\epsilon L, \eta)$ is a monad on $\mathcal{B}$ and we can consider the so-called *comparison functor*

$K : \mathcal{A} \to {}_{RL}\mathcal{B}$ of the adjunction (L, R) which is defined by $KX := (RX, R\epsilon X)$ and $Kf := Rf$. Note that ${}_{RL}U \circ K = R$.

Definition 1.5. An adjunction $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ is called *monadic* (tripleable in Beck's terminology [10, Definition 3, page 8]) whenever the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ is an equivalence of categories. A functor R is called *monadic* if it has a left adjoint L such that the adjunction (L, R) is monadic, see [10, Definition 3', page 8]. In a similar way one defines *comonadic* adjunctions and functors using the Eilenberg–Moore category ${}^{LR}\mathcal{A}$ of coalgebras over the comonad induced by (L, R).

The notion of an idempotent monad is tightly connected with the monadic length of a functor.

Definition 1.6. (See [8, page 231].) A monad (Q, m, u) is called *idempotent* whenever m is an isomorphism. An adjunction (L, R) is called *idempotent* whenever the associated monad is idempotent.

The interested reader can find results on idempotent monads in [8,34]. Here we just note that (L, R) is idempotent if and only if ηR is a functorial isomorphism.

Definition 1.7. (See [4, Definition 2.7], [5, Definition 2.1] and [34, Definitions 2.10 and 2.14].) Fix a $N \in \mathbb{N}$. We say that a functor R has a *monadic decomposition of monadic length* N whenever there exists a sequence $(R_n)_{n \leq N}$ of functors R_n such that

1) $R_0 = R$;

2) for $0 \leq n \leq N$, the functor R_n has a left adjoint functor L_n;

3) for $0 \leq n \leq N - 1$, the functor R_{n+1} is the comparison functor induced by the adjunction (L_n, R_n) with respect to its associated monad;

4) L_N is full and faithful while L_n is not full and faithful for $0 \leq n \leq N - 1$.

Compare with the construction performed in [29, 1.5.5, page 49].

Note that for functor $R : \mathcal{A} \to \mathcal{B}$ having a monadic decomposition of monadic length N, we have a diagram

$$\begin{array}{ccccccccc}
\mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} & \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \cdots & \cdots \xleftarrow{\mathrm{Id}_{\mathcal{A}}} & \mathcal{A} \\
{\scriptstyle L_0}\uparrow\downarrow{\scriptstyle R_0} & & {\scriptstyle L_1}\uparrow\downarrow{\scriptstyle R_1} & & {\scriptstyle L_2}\uparrow\downarrow{\scriptstyle R_2} & & & & {\scriptstyle L_N}\uparrow\downarrow{\scriptstyle R_N} \\
\mathcal{B}_0 & \xleftarrow{U_{0,1}} & \mathcal{B}_1 & \xleftarrow{U_{1,2}} & \mathcal{B}_2 & \xleftarrow{U_{2,3}} & \cdots & \cdots \xleftarrow{U_{N-1,N}} & \mathcal{B}_N
\end{array} \tag{5}$$

where $\mathcal{B}_0 = \mathcal{B}$ and, for $1 \leq n \leq N$,

- $\mathcal{B}_n$ is the category of $(R_{n-1}L_{n-1})$-algebras ${}_{R_{n-1}L_{n-1}}\mathcal{B}_{n-1}$;
- $U_{n-1,n} : \mathcal{B}_n \to \mathcal{B}_{n-1}$ is the forgetful functor ${}_{R_{n-1}L_{n-1}}U$.

We will denote by $\eta_n : \mathrm{Id}_{\mathcal{B}_n} \to R_n L_n$ and $\epsilon_n : L_n R_n \to \mathrm{Id}_{\mathcal{A}}$ the unit and counit of the adjunction (L_n, R_n) respectively for $0 \leq n \leq N$. Note that one can introduce the forgetful functor $U_{m,n} : \mathcal{B}_n \to \mathcal{B}_m$ for all $m \leq n$ with $0 \leq m, n \leq N$.

Proposition 1.8. *(See [4, Proposition 2.9].) Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an idempotent adjunction. Then $R : \mathcal{A} \to \mathcal{B}$ has a monadic decomposition of monadic length at most* 1*.*

We refer to [4, Remarks 2.8 and 2.10] for further comments on monadic decompositions.

Definition 1.9. We say that a functor R is *comparable* whenever there exists a sequence $(R_n)_{n\in\mathbb{N}}$ of functors R_n such that $R_0 = R$ and, for $n \in \mathbb{N}$,

1) the functor R_n has a left adjoint functor L_n;

2) the functor R_{n+1} is the comparison functor induced by the adjunction (L_n, R_n) with respect to its associated monad.

In this case we have a diagram as (5) but not necessarily stationary. Hence we can consider the forgetful functors $U_{m,n} : \mathcal{B}_n \to \mathcal{B}_m$ for all $m \leq n$ with $m, n \in \mathbb{N}$.

Remark 1.10. Fix a $N \in \mathbb{N}$. A functor R having a monadic decomposition of monadic length N is comparable, see [4, Remark 2.10].

By the proof of Beck's Theorem [10, Proof of Theorem 1] one gets the following result.

Lemma 1.11. *Let $\mathcal{A}$ be a category such that, for any (reflexive) pair (f, g) [15, 3.6, page 98] where $f, g : X \to Y$ are morphisms in $\mathcal{A}$, one can choose a specific coequalizer. Then the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ of an adjunction (L, R) is a right adjoint. Thus any right adjoint $R : \mathcal{A} \to \mathcal{B}$ is comparable.*

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We denote by $\mathrm{Im}(F)$, the image of F, the full subcategory of $\mathcal{B}$ whose objects are those of the form FA for some $A \in \mathcal{A}$.

Lemma 1.12. *Let $F : \mathcal{C} \to \mathcal{B}$ be a full and faithful functor which is also injective on objects.*

1) Let $G : \mathcal{A} \to \mathcal{B}$ be a functor such that $\mathrm{Im}(G) \subseteq \mathrm{Im}(F)$. Then there is a unique functor $\widehat{G} : \mathcal{A} \to \mathcal{C}$ such that $F\widehat{G} = G$.

2) Let $G, G' : \mathcal{A} \to \mathcal{B}$ be functors as in 1). For any natural transformation $\gamma : G \to G'$ there is a unique natural transformation $\widehat{\gamma} : \widehat{G} \to \widehat{G'}$ such that $F\widehat{\gamma} = \gamma$.

2. Commutation data

Definition 2.1. A functor is called *conservative* if it reflects isomorphisms.

Lemma 2.2. *Let (L, R) and (L', R') be adjunctions that fit into the following commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \tag{6}$$

Then there is a unique natural transformation $\zeta : L'G \longrightarrow FL$ such that

$$R'\zeta \circ \eta' G = G\eta \tag{7}$$

holds, namely

$$\zeta := \left(L'G \overset{L'G\eta}{\longrightarrow} L'GRL = L'R'FL \overset{\epsilon' FL}{\longrightarrow} FL \right). \tag{8}$$

Moreover we have that

$$\epsilon' F = F\epsilon \circ \zeta R. \tag{9}$$

Definition 2.3. We will say that $(F, G) : (L, R) \to (L', R')$ is a *commutation datum* if

1) (L, R) and (L', R') are adjunctions that fit into the commutative diagram (6).

2) The natural transformation $\zeta : L'G \longrightarrow FL$ of Lemma 2.2 is a functorial isomorphism.

The map ζ will be called the *canonical transformation* of the datum.

Proposition 2.4. *Let $(F, G) : (L, R) \to (L', R')$ and $(F', G') : (L', R') \to (L'', R'')$ be a commutation data. Then $(F'F, G'G) : (L, R) \to (L'', R'')$ is a commutation datum.*

In the following result we will adopt the notations of Definition 1.7 for L_1, R_1, $\mathcal{B}_1$ and their analogue with primes.

Proposition 2.5. *Let $(F, G) : (L, R) \to (L', R')$ be a commutation datum of functors as in (6). Assume also that F preserves coequalizers of reflexive pairs of morphisms in $\mathcal{A}$ and that the comparison functors R'_1 and R_1 have left adjoints L'_1 and L_1 respectively. Then G lifts to a functor $G_1 : \mathcal{B}_1 \to \mathcal{B}'_1$ such that $G_1(B, \mu) := (GB, G\mu \circ R'\zeta B)$, $G_1(f) = Gf$ and the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{G_1} & \mathcal{B}'_1 \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ {\scriptstyle R_1}\downarrow & & \downarrow{\scriptstyle R'_1} \\ \mathcal{B}_1 & \xrightarrow{G_1} & \mathcal{B}'_1 \end{array}$$

Moreover $(F, G_1) : (L_1, R_1) \to (L'_1, R'_1)$ is a commutation datum.

Furthermore the functor G_1 is conservative (resp. faithful) whenever G is. If G is faithful then G_1 is full (resp. injective on objects) whenever G is.

Proof. Denote by ζ the canonical map of the datum $(F,G):(L,R)\to(L',R')$. Set $\lambda := R'\zeta : (R'L')\,G \to R'FL = G\,(RL)$. By Lemma 2.2, ζ fulfills (9). By (7), we have $\lambda\circ\eta' G = G\eta$ and

$$GR\epsilon L\circ\lambda RL\circ R'L'\lambda = GR\epsilon L\circ R'\zeta RL\circ R'L'R'\zeta = R'\left[F\epsilon L\circ\zeta RL\circ L'R'\zeta\right]$$
$$\overset{(9)}{=} R'\left[\epsilon' FL\circ L'R'\zeta\right] = R'\left[\zeta\circ\epsilon' L'G\right] = \lambda\circ R'\epsilon' L'G$$

Hence we can apply [23, Lemma 1] to the case "K" $= R'L'$, "H" $= RL$ and "T" $= G$. Thus we get a functor $G_1 : \mathcal{B}_1\to\mathcal{B}_1'$ such that $U'G_1 = GU$. Explicitly $G_1\,(B,\mu) := (GB, G\mu\circ R'\zeta B)$, $G_1\,(f) = Gf$. We have

$$G_1R_1A = G_1\,(RA, R\epsilon A) = (GRA, GR\epsilon A\circ R'\zeta RA)$$
$$= (R'FA, R'\left[F\epsilon A\circ\zeta RA\right]) \overset{(9)}{=} (R'FA, R'\epsilon' FA) = R_1'FA$$

and $G_1R_1f = GRf = R'Ff = R_1'Ff$ so that $G_1R_1 = R_1'F$. By the proof of [10, Theorem 1], if we set $\pi := \epsilon L_1\circ LU\eta_1$, we get the following coequalizer of a reflexive pair of morphisms in $\mathcal{A}$.

$$LRLB \underset{\epsilon LB}{\overset{L\mu}{\rightrightarrows}} LB = LU\,(B,\mu) \xrightarrow{\pi(B,\mu)} L_1(B,\mu)$$

Since F preserves coequalizers of reflexive pairs of morphisms in $\mathcal{A}$, we get the bottom fork in the diagram below is a coequalizer.

$$\begin{array}{ccccc} L'R'L'GB & \underset{\epsilon' L'GB}{\overset{L'(G\mu\circ R'\zeta B)}{\rightrightarrows}} & L'GB & \xrightarrow{F\pi(B,\mu)\circ\zeta B} & FL_1\,(B,\mu) \\ {\scriptstyle \zeta RLB\circ L'R'\zeta B}\downarrow & & \downarrow{\scriptstyle \zeta B} & & \downarrow{\scriptstyle \mathrm{Id}_{FL_1(B,\mu)}} \\ FLRLB & \underset{F\epsilon LB}{\overset{FL\mu}{\rightrightarrows}} & FLB & \xrightarrow{F\pi(B,\mu)} & FL_1(B,\mu) \end{array} \tag{10}$$

We compute

$$FL\mu\circ(\zeta RLB\circ L'R'\zeta B) = \zeta B\circ L'G\mu\circ L'R'\zeta B = \zeta B\circ L'\left(G\mu\circ R'\zeta B\right),$$
$$F\epsilon LB\circ(\zeta RLB\circ L'R'\zeta B) \overset{(9)}{=} \epsilon' FLB\circ L'R'\zeta B = \zeta B\circ\epsilon' L'GB$$

so that diagram (10) serially commutes. Since, in this diagram, the vertical arrows are isomorphisms, the upper fork is a coequalizer too. In a similar way, if we set $\pi' := \epsilon' L_1'\circ L'U'\eta_1'$ we have the coequalizer

$$L'R'L'B' \underset{\epsilon' L'B'}{\overset{L'\mu'}{\rightrightarrows}} L'B' \xrightarrow{\pi'(B',\mu')} L'_1(B',\mu')$$

For $(B',\mu') := G_1(B,\mu)$ we get the coequalizer

$$L'R'L'GB \underset{\epsilon' L'GB}{\overset{L'(G\mu\circ R'\zeta B)}{\rightrightarrows}} L'GB \xrightarrow{\pi' G_1(B,\mu)} L'_1 G_1(B,\mu)$$

By the foregoing, $F\pi(B,\mu)\circ\zeta B$ coequalizes the pair $(L'(G\mu\circ R'\zeta B), \epsilon' L'GB)$. By the universal property of coequalizers, there is a unique morphism $\zeta_1(B,\mu) : L'_1 G_1(B,\mu) \longrightarrow FL_1(B,\mu)$ such that $\zeta_1(B,\mu)\circ\pi' G_1(B,\mu) = F\pi(B,\mu)\circ\zeta B$. By the uniqueness of the coequalizers, $\zeta_1(B,\mu)$ is an isomorphism.

Let us check that $\zeta_1(B,\mu)$ is natural. Let $f : (B,\mu)\to(B',\mu')$ in $\mathcal{B}_1$. Then

$$\begin{aligned} FL_1 f\circ\zeta_1(B,\mu)\circ\pi' G_1(B,\mu) &= FL_1 f\circ F\pi(B,\mu)\circ\zeta B = F\pi(B',\mu')\circ FLUf\circ\zeta B \\ &= F\pi(B',\mu')\circ\zeta B'\circ L'GUf = \zeta_1(B',\mu')\circ\pi' G_1(B',\mu')\circ L'U'G_1 f \\ &= \zeta_1(B',\mu')\circ L_1 G_1 f\circ\pi' G_1(B,\mu) \end{aligned}$$

so that $FL_1 f\circ\zeta_1(B,\mu) = \zeta_1(B',\mu')\circ L_1 G_1 f$ and hence we get a functorial isomorphism $\zeta_1 : L'_1 G_1 \longrightarrow FL_1$. We have

$$\epsilon_1\circ\pi R_1 = \epsilon_1\circ\epsilon L_1 R_1\circ LU\eta_1 R_1 = \epsilon\circ LR\epsilon_1\circ LU\eta_1 R_1 = \epsilon\circ LU[R_1\epsilon_1\circ\eta_1 R_1] = \epsilon,$$

$$R\pi\circ\eta U = R\epsilon L_1\circ RLU\eta_1\circ\eta U = R\epsilon L_1\circ\eta U R_1 L_1\circ U\eta_1 = R\epsilon L_1\circ\eta R L_1\circ U\eta_1 = U\eta_1$$

so that, we obtain that $\epsilon_1\circ\pi R_1 = \epsilon$ and $R\pi\circ\eta U = U\eta_1$ and similar equations for (L', R'). We compute

$$\begin{aligned} U'(R'_1\zeta_1\circ\eta'_1 G_1) &= R'\zeta_1\circ R'\pi' G_1\circ\eta' U' G_1 \overset{\text{def. }\zeta_1}{=} R'F\pi\circ R'\zeta U\circ\eta' GU \\ &\overset{(7)}{=} R'F\pi\circ G\eta U = G[R\pi\circ\eta U] = GU\eta_1 = U'G_1\eta_1 \end{aligned}$$

so that $R'_1\zeta_1\circ\eta'_1 G_1 = G_1\eta_1$. Let us check that G_1 is conservative whenever G is. Let $f : (B,\mu)\to(B',\mu')$ in $\mathcal{B}_1$ be such that $G_1 f$ is an isomorphism. Then $U'G_1 f = GUf$ is an isomorphism. Since G and U are conservative (see [12, Proposition 4.1.4, page 189]), we get that f is an isomorphism.

If G is faithful, from $U'G_1 = GU$ and the fact that U is faithful, we deduce that G_1 is faithful.

Assume G is faithful and full. Let $f \in \mathcal{B}'_1(G_1(B,\mu), G_1(B',\mu'))$. Then $U'f \in \mathcal{B}'(GB, GB')$ so that there is $h\in\mathcal{B}(B,B')$ such that $Gh = U'f$. We have

$$\begin{aligned} G\left(\mu' \circ RLh\right) \circ R'\zeta B &= G\mu' \circ GRLh \circ R'\zeta B = G\mu' \circ R'FLh \circ R'\zeta B \\ &= G\mu' \circ R'\zeta B' \circ R'L'Gh = G\mu' \circ R'\zeta B' \circ R'L'U'f \\ &= U'f \circ G\mu \circ R'\zeta B = Gh \circ G\mu \circ R'\zeta B = G\left(h \circ \mu\right) \circ R'\zeta B. \end{aligned}$$

Since ζ is an isomorphism and G is faithful, we get that $\mu' \circ RLh = h \circ \mu$ so that there is a unique morphism $k \in \mathcal{B}_1\left(\left(B,\mu\right),\left(B',\mu'\right)\right)$ such that $Uk = h$. Hence $U'f = Gh = GUk = U'G_1k$ and hence $f = G_1k$. Thus G_1 is faithful and full.

Assume G is faithful and injective on objects. If $G_1\left(B,\mu\right) = G_1\left(B',\mu'\right)$ i.e. $(GB, G\mu \circ R'\zeta B) = (GB', G\mu' \circ R'\zeta B')$ then $GB = GB'$ and $G\mu \circ R'\zeta B = G\mu' \circ R'\zeta B'$. In view of the assumptions on G and since ζ is an isomorphism, we get $(B,\mu) = (B',\mu')$ so that G_1 is faithful and injective on objects. □

Lemma 2.6. *Let (L,R) and (L',R') be adjunctions of functors as in (6). Assume that $R'\zeta R$ is a functorial isomorphism where $\zeta : L'G \longrightarrow FL$ is the natural transformation of Lemma 2.2. Assume also that G is conservative.*

1) Let $A \in \mathcal{A}$ be such that $\eta' R'FA$ is an isomorphism. Then ηRA is an isomorphism.

2) If the adjunction (L',R') is idempotent then (L,R) is idempotent.

Proof. 1) Since $\eta' R'FA = \eta' GRA$ is an isomorphism and $R'\zeta R$ is an isomorphism, we get that $R'\zeta RA \circ \eta' GRA$ is an isomorphism. By (7) this means that $G\eta RA$ is an isomorphism. Since G is conservative, we conclude.

2) (L,R) is idempotent if and only if ηR is a functorial isomorphism and similarly for (L',R'). Thus (L',R') is idempotent if and only if $\eta' R'$ is a functorial isomorphism. If the latter condition holds then $\eta' R'F$ is a functorial isomorphism and, by 1), so is ηR and hence (L,R) is idempotent. □

Lemma 2.7. *Let $(F,G) : (L,R) \to (L',R')$ be a commutation datum. If G is conservative and η' is an isomorphism so is η.*

Proof. By (7), we have $R'\zeta \circ \eta' G = G\eta$. □

Corollary 2.8. *Let $(F,G) : (L,R) \to (L',R')$ be a commutation datum. Assume also that F preserves coequalizers of reflexive pairs of morphisms in $\mathcal{A}$ and that G is conservative. Assume that both R and R' are comparable. Let $N \in \mathbb{N}$.*

1) Let $A \in \mathcal{A}$ be such that $\eta'_N R'_N FA$ is an isomorphism. Then $\eta_N R_N A$ is an isomorphism.

2) If (L'_N, R'_N) is idempotent so is (L_N, R_N).

Proof. Apply Proposition 2.5 and Lemma 2.6. □

Next lemma will be a useful tool to construct new commutation data.

Lemma 2.9. *Let (L', R') be an adjunction and let F and G be full and faithful functors which are also injective on objects and have domain and codomain as in the following diagrams. Assume that $\mathrm{Im}(L'G) \subseteq \mathrm{Im}(F)$ and that $\mathrm{Im}(R'F) \subseteq \mathrm{Im}(G)$. Set $L := \widehat{L'G}$ and $R := \widehat{R'F}$ with notation as in Lemma 1.12 so that L and R are the unique functors which make the following diagrams commute*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ {\scriptstyle L}\uparrow & & \uparrow{\scriptstyle L'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{B}' \end{array}$$

Then (L, R) is an adjunction with unit $\eta : \mathrm{Id}_{\mathcal{B}} \to RL$ and counit $\epsilon : LR \to \mathrm{Id}_{\mathcal{A}}$ which satisfy

$$G\eta = \eta' G \qquad \text{and} \qquad F\epsilon = \epsilon' F \tag{11}$$

where η' and ϵ' are the corresponding unit and counit of (L', R'). Moreover $(F, G) : (L, R) \to (L', R')$ is a commutation datum and the canonical transformation $\zeta : L'G \to FL$ is $\mathrm{Id}_{L'G}$.

Proof. Apply Lemma 1.12 once observed that $RL = \widehat{R'L'G}$, $LR = \widehat{L'R'F}$, $\widehat{G} = \mathrm{Id}_{\mathcal{B}}$ and $\widehat{F} = \mathrm{Id}_{\mathcal{A}}$. Then define $\eta := \widehat{\eta' G}$ and $\epsilon := \widehat{\epsilon' F}$. $\square$

3. Braided objects and adjunctions

Definition 3.1. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category (as usual we omit the brackets although we are not assuming the constraints are trivial).

1) Let V be an object in $\mathcal{M}$. A morphism $c = c_V : V \otimes V \to V \otimes V$ is called a *Yang–Baxter operator* (see [26, Definition XIII.3.1]) if it satisfies the quantum Yang–Baxter equation

$$(c \otimes V)(V \otimes c)(c \otimes V) = (V \otimes c)(c \otimes V)(V \otimes c) \tag{12}$$

on $V \otimes V \otimes V$. **We further assume that c is invertible**. The pair (V, c) will be called a *braided object in $\mathcal{M}$*. A morphism of braided objects (V, c_V) and (W, c_W) in $\mathcal{M}$ is a morphism $f : V \to W$ such that $c_W(f \otimes f) = (f \otimes f)c_V$. This defines the category $\mathrm{Br}_{\mathcal{M}}$ of braided objects and their morphisms.

2) [9] A quadruple (A, m, u, c) is called a *braided algebra* if

- (A, m, u) is an algebra in $\mathcal{M}$;
- (A, c) is a braided object in $\mathcal{M}$;
- m and u commute with c, that is the following conditions hold:

$$c(m \otimes A) = (A \otimes m)(c \otimes A)(A \otimes c), \tag{13}$$

$$c(A \otimes m) = (m \otimes A)(A \otimes c)(c \otimes A), \tag{14}$$

$$c(u \otimes A)l_A^{-1} = (A \otimes u) r_A^{-1}, \qquad c(A \otimes u)r_A^{-1} = (u \otimes A) l_A^{-1}. \tag{15}$$

A morphism of braided algebras is, by definition, a morphism of algebras which, in addition, is a morphism of braided objects. This defines the category $\mathrm{BrAlg}_{\mathcal{M}}$ of braided algebras and their morphisms.

3) Dually one introduces the category $\mathrm{BrCoalg}_{\mathcal{M}}$ of braided coalgebras and their morphisms.

4) [39, Definition 5.1] A sextuple $(B, m, u, \Delta, \varepsilon, c)$ is a called a *braided bialgebra* if

- (B, m, u, c) is a braided algebra;
- $(B, \Delta, \varepsilon, c)$ is a braided coalgebra;
- the following relations hold:

$$\Delta m = (m \otimes m)(B \otimes c \otimes B)(\Delta \otimes \Delta), \qquad \Delta u = (u \otimes u)\Delta_{\mathbf{1}}, \tag{16}$$

$$\varepsilon m = m_{\mathbf{1}} (\varepsilon \otimes \varepsilon), \qquad \varepsilon u = \mathrm{Id}_{\mathbf{1}}. \tag{17}$$

A morphism of braided bialgebras is both a morphism of braided algebras and coalgebras. This defines the category $\mathrm{BrBialg}_{\mathcal{M}}$ of braided bialgebras.

Recall that a Yang–Baxter operator c is called symmetric or a symmetry whenever $c^2 = \mathrm{Id}$. Denote by $\mathrm{Br}^s_{\mathcal{M}}$, $\mathrm{BrAlg}^s_{\mathcal{M}}$, $\mathrm{BrCoalg}^s_{\mathcal{M}}$ and $\mathrm{BrBialg}^s_{\mathcal{M}}$ the full subcategories of the respective categories above consisting of objects with symmetric Yang–Baxter operator. Denote by

$$\mathbb{I}^s_{\mathrm{Br}} : \mathrm{Br}^s_{\mathcal{M}} \to \mathrm{Br}_{\mathcal{M}}, \qquad \mathbb{I}^s_{\mathrm{BrAlg}} : \mathrm{BrAlg}^s_{\mathcal{M}} \to \mathrm{BrAlg}_{\mathcal{M}},$$

$$\mathbb{I}^s_{\mathrm{BrCoalg}} : \mathrm{BrCoalg}^s_{\mathcal{M}} \to \mathrm{BrCoalg}_{\mathcal{M}}, \qquad \mathbb{I}^s_{\mathrm{BrBialg}} : \mathrm{BrBialg}^s_{\mathcal{M}} \to \mathrm{BrBialg}_{\mathcal{M}}$$

the obvious inclusion functors. Note that they are full, faithful, injective on objects and conservative.

Remark 3.2. Let $\mathcal{M}$ be a monoidal category. Let $\mathcal{A}$ be one of the following categories $\mathrm{Br}_{\mathcal{M}}$, $\mathrm{BrAlg}_{\mathcal{M}}$, $\mathrm{BrCoalg}_{\mathcal{M}}$ and $\mathrm{BrBialg}_{\mathcal{M}}$, let $\mathcal{A}^s$ be the corresponding full subcategory of objects with symmetric Yang–Baxter operator and denote by $\mathbb{I}^s_{\mathcal{A}} : \mathcal{A}^s \to \mathcal{A}$ the obvious inclusion functor. Let $\mathbb{D}_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}$ be the forgetful functor.

1) Let $\overline{X} \in \mathcal{A}, \overline{Y}^s \in \mathcal{A}^s$ and let $\overline{\alpha} : \overline{X} \to \mathbb{I}^s_{\mathcal{A}} \overline{Y}^s$ be a morphism in $\mathcal{A}$ such that $\alpha := \mathbb{D}_{\mathcal{A}} \overline{\alpha}$ is a monomorphism. Set $X := \mathbb{D}_{\mathcal{A}} \overline{X}$ and $Y := \mathbb{D}_{\mathcal{A}} \mathbb{I}^s_{\mathcal{A}} \overline{Y}^s$. Since α is braided we have $(\alpha \otimes \alpha) c_X^2 = c_Y^2 (\alpha \otimes \alpha) = \alpha \otimes \alpha$ where c_X and c_Y are the Yang–Baxter operators of X and Y respectively. Assume that $\alpha \otimes \alpha$ is a monomorphism. Then we obtain $c_X^2 = \mathrm{Id}_{X \otimes X}$ so that we can write $\overline{X} = \mathbb{I}^s_{\mathcal{A}} \overline{X}^s$ for some $\overline{X}^s \in \mathcal{A}^s$ and $\overline{\alpha}$ is a morphism in $\mathcal{A}^s$. Since $\mathbb{D}_{\mathcal{A}}$

reflects monomorphisms, we have proved that $\mathcal{A}^s$ is closed in $\mathcal{A}$ for those subobjects in $\mathcal{A}$ which are preserved by $\mathbb{D}_{\mathcal{A}}$ and by $(-)^{\otimes 2} \circ \mathbb{D}_{\mathcal{A}}$ where $(-)^{\otimes 2} : \mathcal{M} \to \mathcal{M} : V \mapsto V \otimes V$.

2) Dually $\mathcal{A}^s$ is closed in $\mathcal{A}$ for those quotients in $\mathcal{A}$ which are preserved by $\mathbb{D}_{\mathcal{A}}$ and by $(-)^{\otimes 2} \circ \mathbb{D}_{\mathcal{A}}$.

3.3. Let $\mathcal{M}$ and $\mathcal{M}'$ be monoidal categories. Following [6, Proposition 2.5], every monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{M}'$ induces in a natural way suitable functors $\mathrm{Br}F$, $\mathrm{Alg}F$, $\mathrm{BrAlg}F$ and $\mathrm{BrBialg}F$ such that the following diagrams commute

$$\begin{array}{ccc} \mathrm{Br}_{\mathcal{M}} & \xrightarrow{\mathrm{Br}F} & \mathrm{Br}_{\mathcal{M}'} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H'} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{\mathrm{Alg}F} & \mathrm{Alg}_{\mathcal{M}'} \\ {\scriptstyle \Omega}\downarrow & & \downarrow{\scriptstyle \Omega'} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \end{array} \qquad \begin{array}{ccc} \mathrm{BrAlg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrAlg}F} & \mathrm{BrAlg}_{\mathcal{M}'} \\ {\scriptstyle H_{\mathrm{Alg}}}\downarrow & & \downarrow{\scriptstyle H'_{\mathrm{Alg}}} \\ \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{\mathrm{Alg}F} & \mathrm{Alg}_{\mathcal{M}'} \end{array}$$

$$\begin{array}{ccc} \mathrm{BrAlg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrAlg}F} & \mathrm{BrAlg}_{\mathcal{M}'} \\ {\scriptstyle \Omega_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \Omega'_{\mathrm{Br}}} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{\mathrm{Br}F} & \mathrm{Br}_{\mathcal{M}'} \end{array} \qquad \begin{array}{ccc} \mathrm{BrBialg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrBialg}F} & \mathrm{BrBialg}_{\mathcal{M}'} \\ {\scriptstyle \mho_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \mho'_{\mathrm{Br}}} \\ \mathrm{BrAlg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrAlg}F} & \mathrm{BrAlg}_{\mathcal{M}'} \end{array}$$

where the vertical arrows denote the obvious forgetful functors. Moreover

(1) The functors H, Ω, H_{Alg}, Ω_{Br}, $\mho_{\mathrm{Br}}$ are conservative.
(2) $\mathrm{Br}F$, $\mathrm{Alg}F$, $\mathrm{BrAlg}F$ and $\mathrm{BrBialg}F$ are equivalences (resp. isomorphisms or conservative) whenever F is.
(3) F preserves symmetric objects (this follows by definition of the Yang–Baxter operator induced by F). Thus we can define $\mathrm{Br}^s F$, $\mathrm{BrAlg}^s F$ and $\mathrm{BrBialg}^s F$ such that

$$\begin{array}{ccc} \mathrm{Br}^s_{\mathcal{M}} & \xrightarrow{\mathrm{Br}^s F} & \mathrm{Br}^s_{\mathcal{M}'} \\ {\scriptstyle \mathbb{I}^s_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \mathbb{I}^s_{\mathrm{Br}}} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{\mathrm{Br}F} & \mathrm{Br}_{\mathcal{M}'} \end{array} \qquad \begin{array}{ccc} \mathrm{BrAlg}^s_{\mathcal{M}} & \xrightarrow{\mathrm{BrAlg}^s F} & \mathrm{BrAlg}^s_{\mathcal{M}'} \\ {\scriptstyle \mathbb{I}^s_{\mathrm{BrAlg}}}\downarrow & & \downarrow{\scriptstyle \mathbb{I}^s_{\mathrm{BrAlg}}} \\ \mathrm{BrAlg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrAlg}F} & \mathrm{BrAlg}_{\mathcal{M}'} \end{array} \qquad \begin{array}{ccc} \mathrm{BrBialg}^s_{\mathcal{M}} & \xrightarrow{\mathrm{BrBialg}^s F} & \mathrm{BrBialg}^s_{\mathcal{M}'} \\ {\scriptstyle \mathbb{I}^s_{\mathrm{BrBialg}}}\downarrow & & \downarrow{\scriptstyle \mathbb{I}^s_{\mathrm{BrBialg}}} \\ \mathrm{BrBialg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrBialg}F} & \mathrm{BrBialg}_{\mathcal{M}'} \end{array} \tag{18}$$

Next aim is to recall some meaningful adjunctions that will be investigated in the paper.

3.4. Let $\mathcal{M}$ be a monoidal category. Assume that $\mathcal{M}$ has denumerable coproducts and that the tensor products preserve such coproducts. In view of [6, Proposition 3.1], the functor Ω_{Br} has a left adjoint T_{Br} and the following diagrams commute.

$$\begin{array}{ccc} \mathrm{BrAlg}_{\mathcal{M}} & \xrightarrow{H_{\mathrm{Alg}}} & \mathrm{Alg}_{\mathcal{M}} \\ {\scriptstyle T_{\mathrm{Br}}}\uparrow & & \uparrow{\scriptstyle T} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{H} & \mathcal{M} \end{array} \qquad \begin{array}{ccc} \mathrm{BrAlg}_{\mathcal{M}} & \xrightarrow{H_{\mathrm{Alg}}} & \mathrm{Alg}_{\mathcal{M}} \\ {\scriptstyle \Omega_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \Omega} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{H} & \mathcal{M} \end{array} \tag{19}$$

The unit η_{Br} and the counit ϵ_{Br} are uniquely determined by the following equations

$$H\eta_{\mathrm{Br}} = \eta H, \qquad H_{\mathrm{Alg}}\epsilon_{\mathrm{Br}} = \epsilon H_{\mathrm{Alg}}, \tag{20}$$

where η and ϵ denote the unit and counit of the adjunction (T, Ω) of Remark 1.3. Using Lemma 2.9, one shows that the adjunction $(T_{\mathrm{Br}}, \Omega_{\mathrm{Br}})$ induces an adjunction $(T^s_{\mathrm{Br}}, \Omega^s_{\mathrm{Br}})$ such that the following diagrams commute.

$$\begin{array}{ccc} \mathrm{BrAlg}^s_{\mathcal{M}} & \xrightarrow{\mathbb{I}^s_{\mathrm{BrAlg}}} & \mathrm{BrAlg}_{\mathcal{M}} \\ {\scriptstyle T^s_{\mathrm{Br}}}\uparrow & & \uparrow{\scriptstyle T_{\mathrm{Br}}} \\ \mathrm{Br}^s_{\mathcal{M}} & \xrightarrow{\mathbb{I}^s_{\mathrm{Br}}} & \mathrm{Br}_{\mathcal{M}} \end{array} \qquad \begin{array}{ccc} \mathrm{BrAlg}^s_{\mathcal{M}} & \xrightarrow{\mathbb{I}^s_{\mathrm{BrAlg}}} & \mathrm{BrAlg}_{\mathcal{M}} \\ {\scriptstyle \Omega^s_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \Omega_{\mathrm{Br}}} \\ \mathrm{Br}^s_{\mathcal{M}} & \xrightarrow{\mathbb{I}^s_{\mathrm{Br}}} & \mathrm{Br}_{\mathcal{M}} \end{array} \tag{21}$$

The lemma can be applied by the following argument. It is clear that $\mathrm{Im}(\Omega_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{BrAlg}}) \subseteq \mathrm{Im}(\mathbb{I}^s_{\mathrm{Br}})$. Let $(M, c) \in \mathrm{Br}^s_{\mathcal{M}}$ and set $(A, m_A, u_A, c_A) := T_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}(M, c)$.

Using [6, (42)], we have $c_A(\alpha_m M \otimes \alpha_n M) = (\alpha_n V \otimes \alpha_m M)\, c_A^{m,n}$ so that

$$c_A^2(\alpha_m M \otimes \alpha_n M) = c_A(\alpha_n V \otimes \alpha_m M)\, c_A^{m,n} = (\alpha_m M \otimes \alpha_n M)\, c_A^{n,m} c_A^{m,n}$$

and $c_A^{n,m} c_A^{m,n} = \mathrm{Id}_{M^{\otimes(m+n)}}$. The latter is proved by induction on $t = m + n \in \mathbb{N}$ using [6, Proposition 2.7].

Thus $c_A^2(\alpha_m M \otimes \alpha_n M) = (\alpha_m M \otimes \alpha_n M)$ for every $m, n \in \mathbb{N}$ and hence $c_A^2 = \mathrm{Id}_{A\otimes A}$. Therefore $(A, m_A, u_A, c_A) \in \mathrm{BrAlg}^s_{\mathcal{M}}$ and $T_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}(M, c) = \mathbb{I}^s_{\mathrm{BrAlg}}(A, m_A, u_A, c_A)$. Hence $\mathrm{Im}(T_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}) \subseteq \mathrm{Im}(\mathbb{I}^s_{\mathrm{BrAlg}})$. Thus, by Lemma 2.9 we have the desired adjunction with unit $\eta^s_{\mathrm{Br}} : \mathrm{Id}_{\mathrm{Br}^s_{\mathcal{M}}} \to \Omega^s_{\mathrm{Br}} T^s_{\mathrm{Br}}$ and counit $\epsilon^s_{\mathrm{Br}} : T^s_{\mathrm{Br}}\Omega^s_{\mathrm{Br}} \to \mathrm{Id}_{\mathrm{BrAlg}^s_{\mathcal{M}}}$ which are uniquely defined by

$$\mathbb{I}^s_{\mathrm{BrAlg}}\epsilon^s_{\mathrm{Br}} = \epsilon_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{BrAlg}} \qquad \text{and} \qquad \mathbb{I}^s_{\mathrm{Br}}\eta^s_{\mathrm{Br}} = \eta_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}. \tag{22}$$

Furthermore $\left(\mathbb{I}^s_{\mathrm{BrAlg}}, \mathbb{I}^s_{\mathrm{Br}}\right) : (T^s_{\mathrm{Br}}, \Omega^s_{\mathrm{Br}}) \to (T_{\mathrm{Br}}, \Omega_{\mathrm{Br}})$ is a commutation datum with canonical transformation given by the identity.

Definition 3.5. Let $\mathcal{M}$ be a preadditive monoidal category with equalizers. Assume that the tensor products are additive. Let $\mathbb{C} := (C, \Delta_C, \varepsilon_C, u_C)$ be a coalgebra $(C, \Delta_C, \varepsilon_C)$ endowed with a coalgebra morphism $u_C : \mathbf{1} \to C$. In this setting we always implicitly assume that we can choose a specific equalizer

$$P(\mathbb{C}) \xrightarrow{\xi\mathbb{C}} C \underset{(C\otimes u_C)r_C^{-1} + (u_C\otimes C)l_C^{-1}}{\overset{\Delta_C}{\rightrightarrows}} C \otimes C \tag{23}$$

We will use the same symbol when $\mathbb{C}$ comes out to be enriched with an extra structure such us when $\mathbb{C}$ will denote a bialgebra or a braided bialgebra.

We now investigate some properties of T_{Br}.

3.6. Let $\mathcal{M}$ be a preadditive monoidal category with equalizers and denumerable coproducts. Assume that the tensor products are additive and preserve equalizers and denumerable coproducts. By 3.4, the forgetful functor $\Omega_{\mathrm{Br}} : \mathrm{BrAlg}_{\mathcal{M}} \to \mathrm{Br}_{\mathcal{M}}$ has a left adjoint $T_{\mathrm{Br}} : \mathrm{Br}_{\mathcal{M}} \to \mathrm{BrAlg}_{\mathcal{M}}$. In view of [6, Lemma 3.4], T_{Br} induces a functor $\overline{T}_{\mathrm{Br}}$ such that

$$\begin{array}{ccc} \mathrm{BrBialg}_{\mathcal{M}} & \xrightarrow{\mho_{\mathrm{Br}}} & \mathrm{BrAlg}_{\mathcal{M}} \\ & {}_{\overline{T}_{\mathrm{Br}}}\nwarrow \quad \nearrow_{T_{\mathrm{Br}}} & \\ & \mathrm{Br}_{\mathcal{M}} & \end{array} \tag{24}$$

Explicitly, for all $(V,c) \in \mathrm{Br}_{\mathcal{M}}$, we can write $\overline{T}_{\mathrm{Br}}(V,c)$ in the form $(A, m_A, u_A, \Delta_A, \varepsilon_A, c_A)$ where $\Delta_A : A \to A \otimes A$ and $\varepsilon_A : A \to \mathbf{1}$ are unique algebra morphisms such that

$$\Delta_A \circ \alpha_1 V = \delta_V^l + \delta_V^r, \tag{25}$$

$$\varepsilon_A \circ \alpha_1 V = 0, \tag{26}$$

where $\delta_V^l := (u_A \otimes \alpha_1 V) \circ l_V^{-1}$ and $\delta_V^r := (\alpha_1 V \otimes u_A) \circ r_V^{-1}$. Moreover

$$\varepsilon_A \circ \alpha_n V = \delta_{n,0} \mathrm{Id}_{\mathbf{1}}, \text{ for every } n \in \mathbb{N}. \tag{27}$$

In view of [6, Theorem 3.5], the functor $\overline{T}_{\mathrm{Br}}$ has a right adjoint $P_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{Br}_{\mathcal{M}}$, which is constructed in [6, Lemma 3.3]. The unit $\bar{\eta}_{\mathrm{Br}}$ and the counit $\bar{\epsilon}_{\mathrm{Br}}$ are uniquely determined by the following equalities

$$\xi \overline{T}_{\mathrm{Br}} \circ \bar{\eta}_{\mathrm{Br}} = \eta_{\mathrm{Br}}, \tag{28}$$

$$\epsilon_{\mathrm{Br}} \mho_{\mathrm{Br}} \circ T_{\mathrm{Br}} \xi = \mho_{\mathrm{Br}} \bar{\epsilon}_{\mathrm{Br}}, \tag{29}$$

where $(V,c) \in \mathrm{Br}_{\mathcal{M}}, \mathbb{B} \in \mathrm{BrBialg}_{\mathcal{M}}$ while η_{Br} and ϵ_{Br} denote the unit and counit of the adjunction $(T_{\mathrm{Br}}, \Omega_{\mathrm{Br}})$ respectively. Moreover $\xi : P_{\mathrm{Br}} \to \Omega_{\mathrm{Br}} \mho_{\mathrm{Br}}$ is a natural transformation induced by the canonical morphism in (23).

Note that from 3.4 it is clear that $\mathrm{Im}(\overline{T}_{\mathrm{Br}} \mathbb{I}_{\mathrm{Br}}^s) \subseteq \mathrm{Im}(\mathbb{I}_{\mathrm{BrBialg}}^s)$. Let $\mathbb{B} \in \mathrm{BrBialg}_{\mathcal{M}}^s$ and set $(P, c_P) := P_{\mathrm{Br}} \mathbb{I}_{\mathrm{BrBialg}}^s \mathbb{B}$. Since the tensor products preserve equalizers, we have that $\xi\mathbb{B} \otimes \xi\mathbb{B}$ is a monomorphism so that we can apply 1) in Remark 3.2 to get that $(P, c_P) \in \mathrm{Br}_{\mathcal{M}}^s$. Thus $\mathrm{Im}(P_{\mathrm{Br}} \mathbb{I}_{\mathrm{BrBialg}}^s) \subseteq \mathrm{Im}(\mathbb{I}_{\mathrm{Br}}^s)$. Hence, by Lemma 2.9 we have an adjunction $\left(\overline{T}_{\mathrm{Br}}^s, P_{\mathrm{Br}}^s\right)$ such that the diagrams

$$\begin{array}{ccc} \mathrm{BrBialg}_{\mathcal{M}}^s & \xrightarrow{\mathbb{I}_{\mathrm{BrBialg}}^s} & \mathrm{BrBialg}_{\mathcal{M}} \\ {\scriptstyle \overline{T}_{\mathrm{Br}}^s}\uparrow & & \uparrow{\scriptstyle \overline{T}_{\mathrm{Br}}} \\ \mathrm{Br}_{\mathcal{M}}^s & \xrightarrow{\mathbb{I}_{\mathrm{Br}}^s} & \mathrm{Br}_{\mathcal{M}} \end{array} \qquad \begin{array}{ccc} \mathrm{BrBialg}_{\mathcal{M}}^s & \xrightarrow{\mathbb{I}_{\mathrm{BrBialg}}^s} & \mathrm{BrBialg}_{\mathcal{M}} \\ {\scriptstyle P_{\mathrm{Br}}^s}\downarrow & & \downarrow{\scriptstyle P_{\mathrm{Br}}} \\ \mathrm{Br}_{\mathcal{M}}^s & \xrightarrow{\mathbb{I}_{\mathrm{Br}}^s} & \mathrm{Br}_{\mathcal{M}} \end{array} \tag{30}$$

commute and the unit $\overline{\eta}^s_{\mathrm{Br}} : \mathrm{Id}_{\mathrm{Br}^s_{\mathcal{M}}} \to P^s_{\mathrm{Br}} \overline{T}^s_{\mathrm{Br}}$ and the counit $\overline{\epsilon}^s_{\mathrm{Br}} : \overline{T}^s_{\mathrm{Br}} P^s_{\mathrm{Br}} \to \mathrm{Id}_{\mathrm{BrBialg}^s_{\mathcal{M}}}$ are uniquely defined by

$$\mathbb{I}^s_{\mathrm{BrBialg}} \overline{\epsilon}^s_{\mathrm{Br}} = \overline{\epsilon}_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrBialg}} \qquad \text{and} \qquad \mathbb{I}^s_{\mathrm{Br}} \overline{\eta}^s_{\mathrm{Br}} = \overline{\eta}_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{Br}}. \tag{31}$$

Moreover $\left(\mathbb{I}^s_{\mathrm{BrBialg}}, \mathbb{I}^s_{\mathrm{Br}}\right) : \left(\overline{T}^s_{\mathrm{Br}}, P^s_{\mathrm{Br}}\right) \to \left(\overline{T}_{\mathrm{Br}}, P_{\mathrm{Br}}\right)$ is a commutation datum with canonical transformation given by the identity. Note that the functor $\mho_{\mathrm{Br}}$ induces a functor $\mho^s_{\mathrm{Br}}$ such that the following diagrams commute.

$$\begin{array}{ccc} \mathrm{BrBialg}^s_{\mathcal{M}} & \xrightarrow{\mho^s_{\mathrm{Br}}} & \mathrm{BrAlg}^s_{\mathcal{M}} \\ {\scriptstyle \mathbb{I}^s_{\mathrm{BrBialg}}} \downarrow & & \downarrow {\scriptstyle \mathbb{I}^s_{\mathrm{BrAlg}}} \\ \mathrm{BrBialg}_{\mathcal{M}} & \xrightarrow{\mho_{\mathrm{Br}}} & \mathrm{BrAlg}_{\mathcal{M}} \end{array} \qquad \begin{array}{ccccc} \mathrm{Br}^s_{\mathcal{M}} & & \xrightarrow{\overline{T}^s_{\mathrm{Br}}} & & \mathrm{BrBialg}^s_{\mathcal{M}} \\ & {\scriptstyle T^s_{\mathrm{Br}}} \searrow & & \swarrow {\scriptstyle \mho^s_{\mathrm{Br}}} & \\ & & \mathrm{BrAlg}^s_{\mathcal{M}} & & \end{array} \tag{32}$$

Furthermore, by Lemma 1.12, the natural transformation $\xi : P_{\mathrm{Br}} \to \Omega_{\mathrm{Br}} \mho_{\mathrm{Br}}$ induces a natural transformation $\xi := \widehat{\xi \mathbb{I}^s_{\mathrm{BrBialg}}} : P^s_{\mathrm{Br}} \to \Omega^s_{\mathrm{Br}} \mho^s_{\mathrm{Br}}$ such that $\mathbb{I}^s_{\mathrm{Br}} \xi = \xi \mathbb{I}^s_{\mathrm{BrBialg}}$.

Proposition 3.7. *Let $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{M}'$ be a monoidal functor between monoidal categories. Assume that $\mathcal{M}$ and $\mathcal{M}'$ have denumerable coproducts and that F and the tensor products preserve such coproducts. Then both*

$$(\mathrm{Alg}F, F) : (T, \Omega) \to (T', \Omega') \qquad \textit{and} \qquad (\mathrm{BrAlg}F, \mathrm{Br}F) : (T_{\mathrm{Br}}, \Omega_{\mathrm{Br}}) \to (T'_{\mathrm{Br}}, \Omega'_{\mathrm{Br}})$$

are commutation data.

Proof. First we deal with $(\mathrm{Alg}F, F) : (T, \Omega) \to (T', \Omega')$. By 3.3, we have that $\Omega' \circ \mathrm{Alg}F = F \circ \Omega$. By Remark 1.3, we have that Ω and Ω' have left adjoints T and T' respectively. The structure morphisms ϕ_0, ϕ_2 induce, for every $n \in \mathbb{N}$, the isomorphism $\widehat{\phi}_n V : (FV)^{\otimes n} \to F\left(V^{\otimes n}\right)$ given by

$$\widehat{\phi}_0 V := \phi_0, \qquad \widehat{\phi}_1 V := \mathrm{Id}_{FV}, \qquad \widehat{\phi}_2 V := \phi_2 (V, V), \qquad \text{and, for } n > 2$$
$$\widehat{\phi}_n V := \phi_2 \left(V^{\otimes(n-1)}, V\right) \circ \left(\widehat{\phi}_{n-1} \otimes FV\right).$$

Using the naturality of ϕ_2 and (2) it is straightforward to check, by induction on $n \in \mathbb{N}$, that

$$m^{n-1}_{(\mathrm{Alg}F)TV} \circ (F\alpha_1 V)^{\otimes n} = F\alpha_n V \circ \widehat{\phi}_n V. \tag{33}$$

Let ζ be the map of Lemma 2.2 i.e. $\zeta = \epsilon' (\mathrm{Alg}F) T \circ T' F \eta$. We compute

$$\Omega' \zeta V \circ \alpha_n FV = \Omega' \epsilon' (\mathrm{Alg}F) TV \circ \Omega' T' F \eta V \circ \alpha_n FV$$
$$\overset{(4)}{=} \Omega' \epsilon' (\mathrm{Alg}F) TV \circ \Omega' T' F \alpha_1 V \circ \alpha_n FV = \Omega' \epsilon' (\mathrm{Alg}F) TV \circ \alpha_n F\Omega TV \circ (F\alpha_1 V)^{\otimes n}$$

$$\overset{(4)}{=} m^{n-1}_{(\mathrm{Alg}F)TV} \circ (F\alpha_1 V)^{\otimes n} \overset{(33)}{=} F\alpha_n V \circ \widehat{\phi}_n V$$

$$= (\nabla_{t\in\mathbb{N}} F\alpha_t V) \circ j_n V \circ \widehat{\phi}_n V = (\nabla_{t\in\mathbb{N}} F\alpha_t V) \circ \left(\oplus_{t\in\mathbb{N}} \widehat{\phi}_t V\right) \circ \alpha_n FV$$

where $j_n V : F\left(V^{\otimes n}\right) \to \oplus_{t\in\mathbb{N}} F\left(V^{\otimes t}\right)$ denotes the canonical morphism. Since this equality holds for an arbitrary $n \in \mathbb{N}$, we obtain $\Omega'\zeta V = (\nabla_{n\in\mathbb{N}} F\alpha_n V) \circ \left(\oplus_{n\in\mathbb{N}} \widehat{\phi}_n V\right)$. Now $\widehat{\phi}_n$ is an isomorphism by construction and $\nabla_{n\in\mathbb{N}} F\alpha_n V : \oplus_{n\in\mathbb{N}} F\left(V^{\otimes n}\right) \to F\left(\oplus_{n\in\mathbb{N}} V^{\otimes n}\right)$ is an isomorphism as F preserves denumerable coproducts. Hence $\Omega'\zeta V$ is an isomorphism. This clearly implies ζV is an isomorphism and hence $(\mathrm{Alg}F, F) : (T, \Omega) \to (T', \Omega')$ is a commutation datum.

Now, let us consider $(\mathrm{BrAlg}F, \mathrm{Br}F) : (T_{\mathrm{Br}}, \Omega_{\mathrm{Br}}) \to (T'_{\mathrm{Br}}, \Omega'_{\mathrm{Br}})$. By 3.4, the functor $\Omega_{\mathrm{Br}} : \mathrm{BrAlg}_{\mathcal{M}} \to \mathrm{Br}_{\mathcal{M}}$ has a left adjoint $T_{\mathrm{Br}} : \mathrm{Br}_{\mathcal{M}} \to \mathrm{BrAlg}_{\mathcal{M}}$ and the (co)unit of the adjunction obeys (20). Moreover $H_{\mathrm{Alg}} T_{\mathrm{Br}} = TH$. By 3.3, we have $H'(\mathrm{Br}F) = FH$, $\Omega'(\mathrm{Alg}F) = F\Omega$, $H'_{\mathrm{Alg}}(\mathrm{BrAlg}F) = (\mathrm{Alg}F) H_{\mathrm{Alg}}$ and $\Omega'_{\mathrm{Br}}(\mathrm{BrAlg}F) = (\mathrm{Br}F)\Omega_{\mathrm{Br}}$. In view of Lemma 2.2 the diagrams

$$\begin{array}{ccc} \mathrm{BrAlg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrAlg}F} & \mathrm{BrAlg}_{\mathcal{M}'} \\ {\scriptstyle \Omega_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \Omega'_{\mathrm{Br}}} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{\mathrm{Br}F} & \mathrm{Br}_{\mathcal{M}'} \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{\mathrm{Alg}F} & \mathrm{Alg}_{\mathcal{M}'} \\ {\scriptstyle \Omega}\downarrow & & \downarrow{\scriptstyle \Omega'} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \end{array} \tag{34}$$

induce the maps $\zeta_{\mathrm{Br}} : T'_{\mathrm{Br}}(\mathrm{Br}F) \to (\mathrm{BrAlg}F) T_{\mathrm{Br}}$ and $\zeta : T'F \to (\mathrm{Alg}F) T$ defined by

$$\zeta_{\mathrm{Br}} = \epsilon'_{\mathrm{Br}}(\mathrm{BrAlg}F) T_{\mathrm{Br}} \circ T'_{\mathrm{Br}}(\mathrm{Br}F)\eta_{\mathrm{Br}} \qquad \text{and} \qquad \zeta = \epsilon'(\mathrm{Alg}F) T \circ T'F\eta. \tag{35}$$

One easily checks that

$$H'_{\mathrm{Alg}}\zeta_{\mathrm{Br}} = \zeta H. \tag{36}$$

By the first part of the proof, ζ is a functorial isomorphism so that we get that $H'_{\mathrm{Alg}}\zeta_{\mathrm{Br}}$ is a functorial isomorphism too. Since H'_{Alg} trivially reflects isomorphisms, we get that ζ_{Br} is a functorial isomorphism. □

Proposition 3.8. *Let $\mathcal{M}$ and $\mathcal{M}'$ be preadditive monoidal categories with equalizers. Assume that the tensor functors are additive and preserve equalizers in both categories. For any monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{M}'$ which preserves equalizers, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{BrBialg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrBialg}F} & \mathrm{BrBialg}_{\mathcal{M}'} \\ {\scriptstyle P_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle P'_{\mathrm{Br}}} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{\mathrm{Br}F} & \mathrm{Br}_{\mathcal{M}'} \end{array} \tag{37}$$

where BrBialg$F$ *and* BrF *are the functors of 3.3. Moreover we have*

$$\xi'\left(\mathrm{BrBialg}F\right)=\left(\mathrm{Br}F\right)\xi. \tag{38}$$

Assume also that the categories $\mathcal{M}$ *and* $\mathcal{M}'$ *have denumerable coproducts and that* F *and the tensor products preserve such coproducts. Then* $(\mathrm{BrBialg}F,\mathrm{Br}F):\left(\overline{T}_{\mathrm{Br}},P_{\mathrm{Br}}\right)\to\left(\overline{T}'_{\mathrm{Br}},P'_{\mathrm{Br}}\right)$ *is a commutation datum.*

Proof. The first part is [6, Proposition 3.6]. Let us prove the last assertion. Assume that the monoidal category $\mathcal{M}$ has denumerable coproducts and that the tensor products preserve such coproducts. By 3.6, we have that P_{Br} and P'_{Br} have left adjoints $\overline{T}_{\mathrm{Br}}$ and $\overline{T}'_{\mathrm{Br}}$ respectively. By 3.3, we have $\mho'_{\mathrm{Br}}\left(\mathrm{BrBialg}F\right)=\left(\mathrm{BrAlg}F\right)\mho_{\mathrm{Br}}$ and $\Omega'_{\mathrm{Br}}\left(\mathrm{BrAlg}F\right)=\left(\mathrm{Br}F\right)\Omega_{\mathrm{Br}}$. By (24), we have $\mho_{\mathrm{Br}}\overline{T}_{\mathrm{Br}}=T_{\mathrm{Br}}$. The commutative diagrams (37) and (34)-left induce the natural transformations $\bar{\zeta}_{\mathrm{Br}}:\overline{T}'_{\mathrm{Br}}\left(\mathrm{Br}F\right)\to\left(\mathrm{BrBialg}F\right)\overline{T}_{\mathrm{Br}}$ and $\zeta_{\mathrm{Br}}:T'_{\mathrm{Br}}\left(\mathrm{Br}F\right)\to\left(\mathrm{BrAlg}F\right)T_{\mathrm{Br}}$ of Lemma 2.2 i.e.

$$\bar{\zeta}_{\mathrm{Br}}=\bar{\epsilon}'_{\mathrm{Br}}\left(\mathrm{BrBialg}F\right)\overline{T}_{\mathrm{Br}}\circ\overline{T}'_{\mathrm{Br}}\left(\mathrm{Br}F\right)\bar{\eta}_{\mathrm{Br}}\qquad\text{and}$$
$$\zeta_{\mathrm{Br}}=\epsilon'_{\mathrm{Br}}\left(\mathrm{BrAlg}F\right)T_{\mathrm{Br}}\circ T'_{\mathrm{Br}}\left(\mathrm{Br}F\right)\eta_{\mathrm{Br}}.$$

Using (29), (38) and (28), one easily checks that $\mho'_{\mathrm{Br}}\bar{\zeta}_{\mathrm{Br}}=\zeta_{\mathrm{Br}}$. By Proposition 3.7, we know that ζ_{Br} is a functorial isomorphism. Since $\mho'_{\mathrm{Br}}$ is trivially conservative, we deduce that $\bar{\zeta}_{\mathrm{Br}}$ is a functorial isomorphism too. □

4. Braided categories

4.1. A *braided monoidal category* $(\mathcal{M},\otimes,\mathbf{1},c)$ is a monoidal category $(\mathcal{M},\otimes,\mathbf{1})$ equipped with a *braiding* c, that is an isomorphism $c_{U,V}:U\otimes V\to V\otimes U$, natural in $U,V\in\mathcal{M}$, satisfying, for all $U,V,W\in\mathcal{M}$,

$$c_{U,V\otimes W}=(V\otimes c_{U,W})\circ(c_{U,V}\otimes W)\qquad\text{and}\qquad c_{U\otimes V,W}=(c_{U,W}\otimes V)\circ(U\otimes c_{V,W}).$$

A braided monoidal category is called *symmetric* if we further have $c_{V,U}\circ c_{U,V}=\mathrm{Id}_{U\otimes V}$ for every $U,V\in\mathcal{M}$.

A *(symmetric) braided monoidal functor* is a monoidal functor $F:\mathcal{M}\to\mathcal{M}'$ such that $F\left(c_{U,V}\right)\circ\phi_2(U,V)=\phi_2(V,U)\circ c'_{F(U),F(V)}$. More details on these topics can be found in [26, Chapter XIII].

Remark 4.2. Given a braided monoidal category $(\mathcal{M},\otimes,\mathbf{1},c)$ the category $\mathrm{Alg}_{\mathcal{M}}$ becomes monoidal where, for every $A,B\in\mathrm{Alg}_{\mathcal{M}}$ the multiplication and unit of $A\otimes B$ are given by

$$m_{A\otimes B}:=(m_A\otimes m_B)\circ(A\otimes c_{B,A}\otimes B):(A\otimes B)\otimes(A\otimes B)\to A\otimes B,$$
$$u_{A\otimes B}:=(u_A\otimes u_B)\circ l_{\mathbf{1}}^{-1}:\mathbf{1}\to A\otimes B.$$

Moreover the forgetful functor $\mathrm{Alg}_{\mathcal{M}} \to \mathcal{M}$ is a strict monoidal functor, cf. [25, page 60].

Definition 4.3. A *bialgebra* in a braided monoidal category $(\mathcal{M}, \otimes, \mathbf{1}, c)$ is a coalgebra (B, Δ, ε) in the monoidal category $\mathrm{Alg}_{\mathcal{M}}$. Equivalently a bialgebra is a quintuple $(A, m, u, \Delta, \varepsilon)$ where (A, m, u) is an algebra in $\mathcal{M}$ and (A, Δ, ε) is a coalgebra in $\mathcal{M}$ such that Δ and ε are morphisms of algebras where $A \otimes A$ is an algebra as in the previous remark. Denote by $\mathrm{Bialg}_{\mathcal{M}}$ the category of bialgebras in $\mathcal{M}$ and their morphisms, defined in the expected way.

4.4. Let $\mathcal{M}$ be a braided monoidal category. In view of [6, Proposition 4.4], there are obvious functors J, J_{Alg} and J_{Bialg} such that the diagrams

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{J_{\mathrm{Bialg}}} & \mathrm{BrBialg}_{\mathcal{M}} \\ {\scriptstyle \mho}\downarrow & & \downarrow{\scriptstyle \mho_{\mathrm{Br}}} \\ \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{J_{\mathrm{Alg}}} & \mathrm{BrAlg}_{\mathcal{M}} \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{J_{\mathrm{Alg}}} & \mathrm{BrAlg}_{\mathcal{M}} \\ {\scriptstyle \Omega}\downarrow & & \downarrow{\scriptstyle \Omega_{\mathrm{Br}}} \\ \mathcal{M} & \xrightarrow{J} & \mathrm{Br}_{\mathcal{M}} \end{array} \tag{39}$$

commute. In fact the functors J, J_{Alg} and J_{Bialg} add the evaluation of the braiding of $\mathcal{M}$ on the object on which they act. Moreover they are full, faithful, injective on objects and conservative.

Assume that $\mathcal{M}$ has denumerable coproducts and that the tensor functors preserve such coproducts. Then, by [6, Proposition 4.5], the following diagram

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{J_{\mathrm{Alg}}} & \mathrm{BrAlg}_{\mathcal{M}} \\ {\scriptstyle T}\uparrow & & \uparrow{\scriptstyle T_{\mathrm{Br}}} \\ \mathcal{M} & \xrightarrow{J} & \mathrm{Br}_{\mathcal{M}} \end{array} \tag{40}$$

is commutative. When $\mathcal{M}$ is symmetric the functors J, J_{Alg} and J_{Bialg} factor through functors J^s, J^s_{Alg} and J^s_{Bialg} i.e. the following diagrams commute (apply Lemma 1.12).

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{J^s} & \mathrm{Br}^s_{\mathcal{M}} \\ & {\scriptstyle J}\searrow \quad \swarrow{\scriptstyle \mathbb{I}^s_{\mathrm{Br}}} & \\ & \mathrm{Br}_{\mathcal{M}} & \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Alg}}} & \mathrm{BrAlg}^s_{\mathcal{M}} \\ & {\scriptstyle J_{\mathrm{Alg}}}\searrow \quad \swarrow{\scriptstyle \mathbb{I}^s_{\mathrm{BrAlg}}} & \\ & \mathrm{BrAlg}_{\mathcal{M}} & \end{array}$$

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Bialg}}} & \mathrm{BrBialg}^s_{\mathcal{M}} \\ & {\scriptstyle J_{\mathrm{Bialg}}}\searrow \quad \swarrow{\scriptstyle \mathbb{I}^s_{\mathrm{BrBialg}}} & \\ & \mathrm{BrBialg}_{\mathcal{M}} & \end{array} \tag{41}$$

Note that they are full, faithful, injective on objects and conservative and the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Bialg}}} & \mathrm{BrBialg}^s_{\mathcal{M}} \\ \mho \downarrow & & \downarrow \mho^s_{\mathrm{Br}} \\ \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Alg}}} & \mathrm{BrBialg}^s_{\mathcal{M}} \end{array} \tag{42}$$

4.5. Let $\mathcal{M}$ be a preadditive braided monoidal category with equalizers. Assume that the tensor products are additive and preserve equalizers. Define the functor

$$P := H \circ P_{\mathrm{Br}} \circ J_{\mathrm{Bialg}} : \mathrm{Bialg}_{\mathcal{M}} \to \mathcal{M}$$

For any $\mathbb{B} := (B, m_B, u_B, \Delta_B, \varepsilon_B) \in \mathrm{Bialg}_{\mathcal{M}}$ one easily gets that $P(\mathbb{B}) = P(B, \Delta_B, \varepsilon_B, u_B)$, see [6, 4.6]. The canonical inclusion $\xi P(B, \Delta_B, \varepsilon_B, u_B) : P(B, \Delta_B, \varepsilon_B, u_B) \to B$ will be denoted by $\xi\mathbb{B}$. Thus we have the equalizer

$$P(\mathbb{B}) \xrightarrow{\xi\mathbb{B}} B \underset{(B\otimes u_B)r_B^{-1}+(u_B\otimes B)l_B^{-1}}{\overset{\Delta_B}{\rightrightarrows}} B \otimes B$$

By [6, Proposition 4.7], we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{J_{\mathrm{Bialg}}} & \mathrm{BrBialg}_{\mathcal{M}} \\ P \downarrow & & \downarrow P_{\mathrm{Br}} \\ \mathcal{M} & \xrightarrow{J} & \mathrm{Br}_{\mathcal{M}} \end{array} \tag{43}$$

where the horizontal arrows are the functors of 4.4. Furthermore

$$\xi J_{\mathrm{Bialg}} = J\xi. \tag{44}$$

Assume further that $\mathcal{M}$ has denumerable coproducts and that the tensor products preserve such coproducts. By Remark 1.3, the forgetful functor $\Omega : \mathrm{Alg}_{\mathcal{M}} \to \mathcal{M}$ has a left adjoint $T : \mathcal{M} \to \mathrm{Alg}_{\mathcal{M}}$. Note that

$$\mathbb{I}^s_{\mathrm{BrAlg}} T^s_{\mathrm{Br}} J^s \overset{(21)}{=} T_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{Br}} J^s \overset{(41)}{=} T_{\mathrm{Br}} J \overset{(40)}{=} J_{\mathrm{Alg}} T \overset{(41)}{=} \mathbb{I}^s_{\mathrm{BrAlg}} J^s_{\mathrm{Alg}} T$$

and hence, since $\mathbb{I}^s_{\mathrm{BrAlg}}$ is both injective on morphisms and objects, we get that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Alg}}} & \mathrm{BrAlg}^s_{\mathcal{M}} \\ T \uparrow & & \uparrow T^s_{\mathrm{Br}} \\ \mathcal{M} & \xrightarrow{J^s} & \mathrm{Br}^s_{\mathcal{M}} \end{array} \tag{45}$$

In view of [6, 4.8], there is a functor

$$\overline{T} : \mathcal{M} \to \mathrm{Bialg}_{\mathcal{M}}$$

such that the following diagrams commute.

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{J_{\mathrm{Bialg}}} & \mathrm{BrBialg}_{\mathcal{M}} \\ {\scriptstyle \overline{T}}\uparrow & & \uparrow{\scriptstyle \overline{T}_{\mathrm{Br}}} \\ \mathcal{M} & \xrightarrow{J} & \mathrm{Br}_{\mathcal{M}} \end{array} \tag{46}$$

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{\mho} & \mathrm{Alg}_{\mathcal{M}} \\ {\scriptstyle \overline{T}}\nwarrow & & \nearrow{\scriptstyle T} \\ & \mathcal{M} & \end{array} \tag{47}$$

By [6, Theorem 4.9], the functor $\overline{T}$ is a left adjoint of the functor $P : \mathrm{Bialg}_{\mathcal{M}} \to \mathcal{M}$. The unit $\bar{\eta}$ and counit $\bar{\epsilon}$ of the adjunction are uniquely determined by the following equalities

$$\xi\overline{T} \circ \bar{\eta} = \eta, \qquad \epsilon\mho \circ T\xi = \mho\bar{\epsilon}, \tag{48}$$

where η and ϵ denote the unit and counit of the adjunction (T, Ω) respectively. We have that

$$\mathbb{I}^s_{\mathrm{BrBialg}} \overline{T}^s_{\mathrm{Br}} J^s \overset{(30)}{=} \overline{T}_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{Br}} J^s \overset{(41)}{=} \overline{T}_{\mathrm{Br}} J \overset{(46)}{=} J_{\mathrm{Bialg}} \overline{T} \overset{(41)}{=} \mathbb{I}^s_{\mathrm{BrBialg}} J^s_{\mathrm{Bialg}} \overline{T}$$

and that

$$\mathbb{I}^s_{\mathrm{Br}} P^s_{\mathrm{Br}} J^s_{\mathrm{Bialg}} \overset{(30)}{=} P_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrBialg}} J^s_{\mathrm{Bialg}} \overset{(41)}{=} P_{\mathrm{Br}} J_{\mathrm{Bialg}} \overset{(43)}{=} JP \overset{(41)}{=} \mathbb{I}^s_{\mathrm{Br}} J^s P$$

so that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Bialg}_{M} & \xrightarrow{J^s_{\mathrm{Bialg}}} & \mathrm{BrBialg}^s_{M} \\ {\scriptstyle \overline{T}}\uparrow & & \uparrow{\scriptstyle \overline{T}^s_{\mathrm{Br}}} \\ \mathcal{M} & \xrightarrow{J^s} & \mathrm{Br}^s_{\mathcal{M}} \end{array} \qquad \begin{array}{ccc} \mathrm{Bialg}_{M} & \xrightarrow{J^s_{\mathrm{Bialg}}} & \mathrm{BrBialg}^s_{M} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P^s_{\mathrm{Br}}} \\ \mathcal{M} & \xrightarrow{J^s} & \mathrm{Br}^s_{\mathcal{M}} \end{array} \tag{49}$$

Proposition 4.6. *Let $\mathcal{M}$ be a preadditive braided monoidal category with equalizers. Assume that the tensor products are additive and preserve equalizers. Assume further that $\mathcal{M}$ has denumerable coproducts and that the tensor products preserve such coproducts. Then the morphism $\zeta : \overline{T}_{\mathrm{Br}} J \longrightarrow J_{\mathrm{Bialg}} \overline{T}$ of Lemma 2.2 is $\mathrm{Id}_{\overline{T}_{\mathrm{Br}} J}$. In particular $(J_{\mathrm{Bialg}}, J) : (\overline{T}, P) \to (\overline{T}_{\mathrm{Br}}, P_{\mathrm{Br}})$ is a commutation datum.*

Proof. Consider the commutative diagram (43). By Lemma 2.2, then there is a unique natural transformation $\zeta : \overline{T}_{\mathrm{Br}} J \longrightarrow J_{\mathrm{Bialg}} \overline{T}$ such that $P_{\mathrm{Br}}\zeta \circ \bar{\eta}_{\mathrm{Br}} J = J\bar{\eta}$. By [6, Equality (75)], we also have $\bar{\eta}_{\mathrm{Br}} J = J\bar{\eta}$. By uniqueness of ζ, we have $\zeta = \mathrm{Id}_{\overline{T}_{\mathrm{Br}} J}$. □

Proposition 4.7. *Let $\mathcal{M}$ be a preadditive symmetric monoidal category with equalizers. Assume that the tensor products are additive and preserve equalizers. Assume further that $\mathcal{M}$ has denumerable coproducts and that the tensor products preserve such coproducts. Then the morphism $\zeta^s : \overline{T}^s_{\mathrm{Br}} J^s \longrightarrow J^s_{\mathrm{Bialg}} \overline{T}$ of Lemma 2.2 is $\mathrm{Id}_{\overline{T}^s_{\mathrm{Br}} J^s}$. In particular $\left(J^s_{\mathrm{Bialg}}, J^s\right) : \left(\overline{T}, P\right) \to \left(\overline{T}^s_{\mathrm{Br}}, P^s_{\mathrm{Br}}\right)$ is a commutation datum.*

Proof. Consider the commutative diagram (49). By Lemma 2.2, then there is a unique natural transformation $\zeta^s : \overline{T}^s_{\mathrm{Br}} J^s \longrightarrow J^s_{\mathrm{Bialg}} \overline{T}$ such that $P^s_{\mathrm{Br}} \zeta^s \circ \overline{\eta}^s_{\mathrm{Br}} J^s = J^s \overline{\eta}$. Now

$$\mathbb{I}^s_{\mathrm{Br}} \overline{\eta}^s_{\mathrm{Br}} J^s \overset{(31)}{=} \overline{\eta}_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{Br}} J^s \overset{(41)}{=} \overline{\eta}_{\mathrm{Br}} J \overset{(*)}{=} J \overline{\eta} \overset{(41)}{=} \mathbb{I}^s_{\mathrm{Br}} J^s \overline{\eta}$$

where in $(*)$ we used [6, Equality (75)]. Thus $\overline{\eta}^s_{\mathrm{Br}} J^s = J^s \overline{\eta}$. By uniqueness of ζ^s, we have $\zeta^s = \mathrm{Id}_{\overline{T}^s_{\mathrm{Br}} J^s}$ (note that we are using that the domain and codomain of ζ^s coincide by (49)). $\square$

4.8. Let $\mathcal{M}$ and $\mathcal{M}'$ be braided monoidal categories. Following [6, Proposition 4.10], every braided monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{M}'$ induces in a natural way a functor $\mathrm{Bialg}F$ and the following diagrams commute.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J'} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{\mathrm{Br}F} & \mathrm{Br}_{\mathcal{M}'} \end{array} \qquad \begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{\mathrm{Bialg}F} & \mathrm{Bialg}_{\mathcal{M}'} \\ {\scriptstyle J_{\mathrm{Bialg}}}\downarrow & & \downarrow{\scriptstyle J'_{\mathrm{Bialg}}} \\ \mathrm{BrBialg}_{\mathcal{M}} & \xrightarrow{\mathrm{BrBialg}F} & \mathrm{BrBialg}_{\mathcal{M}'} \end{array} \qquad \begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{\mathrm{Bialg}F} & \mathrm{Bialg}_{\mathcal{M}'} \\ {\scriptstyle \mho}\downarrow & & \downarrow{\scriptstyle \mho'} \\ \mathrm{Alg}_{\mathcal{M}} & \xrightarrow{\mathrm{Alg}F} & \mathrm{Alg}_{\mathcal{M}'} \end{array}$$

Moreover

1) $\mathrm{Bialg}F$ is an equivalence (resp. category isomorphism or conservative) whenever F is.
2) If F preserves equalizers, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{\mathrm{Bialg}F} & \mathrm{Bialg}_{\mathcal{M}'} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \end{array}$$

5. Lie algebras

The following definition extends the classical notion of Lie algebra to a monoidal category which is not necessarily braided. We expected this notion to be well-known, but we could not find any reference. We point out that in the following definition we should more properly speak of "right braided Lie" algebra as condition (51) and its left analogue (56) seem not to be equivalent in general, see Lemma 5.3.

Definition 5.1. 1) Given an abelian monoidal category $\mathcal{M}$ a *braided Lie algebra* in $\mathcal{M}$ consists of a tern $(M, c, [-] : M \otimes M \to M)$ where $(M, c) \in \mathrm{Br}_{\mathcal{M}}$ and the following equalities hold true:

$$[-] = -[-] \circ c \text{ (skew-symmetry);} \tag{50}$$

$$\begin{aligned}&[-] \circ (M \otimes [-]) \circ \left[\mathrm{Id}_{(M\otimes M)\otimes M} + (M \otimes c)(c \otimes M) + (c \otimes M)(M \otimes c)\right] \\ &\quad = 0 \quad \text{(Jacobi condition);}\end{aligned} \tag{51}$$

$$c \circ (M \otimes [-]) = ([-] \otimes M) \circ (M \otimes c) \circ (c \otimes M); \tag{52}$$

$$c \circ ([-] \otimes M) = (M \otimes [-]) \circ (c \otimes M) \circ (M \otimes c). \tag{53}$$

Of course one should take care of the associativity constraints, but as we did before, we continue to omit them. A morphism of braided Lie algebras $(M, c, [-])$ and $(M', c', [-]')$ in $\mathcal{M}$ is a morphism $f : (M, c) \to (M', c')$ of braided objects such that $f \circ [-] = [-]' \circ (f \otimes f)$. This defines the category $\mathrm{BrLie}_{\mathcal{M}}$ of braided Lie algebras in $\mathcal{M}$ and their morphisms. Denote by

$$H_{\mathrm{BrLie}} : \mathrm{BrLie}_{\mathcal{M}} \to \mathrm{Br}_{\mathcal{M}} : (M, c, [-]) \mapsto (M, c)$$

the obvious functor forgetting the bracket and acting as the identity on morphisms. Note that H_{BrLie} is faithful and conservative.

Denote by $\mathrm{BrLie}^s_{\mathcal{M}}$ the full subcategory $\mathrm{BrLie}_{\mathcal{M}}$ consisting of braided Lie algebras with symmetric Yang–Baxter operator. Denote by

$$\mathbb{I}^s_{\mathrm{BrLie}} : \mathrm{BrLie}^s_{\mathcal{M}} \to \mathrm{BrLie}_{\mathcal{M}}$$

the inclusion functor. It is clear that, by Lemma 1.12, the functor H_{BrLie} induces a functor H^s_{BrLie} such that the diagram

$$\begin{array}{ccc} \mathrm{BrLie}^s_{\mathcal{M}} & \xrightarrow{H^s_{\mathrm{BrLie}}} & \mathrm{Br}^s_{\mathcal{M}} \\ {\scriptstyle \mathbb{I}^s_{\mathrm{BrLie}}}\downarrow & & \downarrow{\scriptstyle \mathbb{I}^s_{\mathrm{Br}}} \\ \mathrm{BrLie}_{\mathcal{M}} & \xrightarrow{H_{\mathrm{BrLie}}} & \mathrm{Br}_{\mathcal{M}} \end{array} \tag{54}$$

commutes. Since H_{BrLie} and both vertical arrows are faithful and conservative, the same is true for H^s_{BrLie}.

2) Let $\mathcal{M}$ be an abelian braided monoidal category. A *Lie algebra* in $\mathcal{M}$ consists of a pair $(M, [-] : M \otimes M \to M)$ such that $(M, c_{M,M}, [-]) \in \mathrm{BrLie}_{\mathcal{M}}$, where $c_{M,M}$ is the braiding c of $\mathcal{M}$ evaluated on M. A morphism of Lie algebras $(M, [-])$ and $(M', [-]')$ in $\mathcal{M}$ is a morphism $f : M \to M'$ in $\mathcal{M}$ such that $f \circ [-] = [-]' \circ (f \otimes f)$. This defines the category $\mathrm{Lie}_{\mathcal{M}}$ of Lie algebras in $\mathcal{M}$ and their morphisms. Note that there is a full, faithful, injective on objects and conservative functor

$$J_{\mathrm{Lie}} : \mathrm{Lie}_{\mathcal{M}} \to \mathrm{BrLie}_{\mathcal{M}} : (M, [-]) \mapsto (M, c_{M,M}, [-])$$

which acts as the identity on morphisms. This notion already appeared in [30, c) page 82], where a Lie algebra in $\mathcal{M}$ is called an $\mathcal{M}$-Lie algebra. Denote by

$$H_{\mathrm{Lie}} : \mathrm{Lie}_{\mathcal{M}} \to \mathcal{M} : (M, [-]) \mapsto M$$

the obvious functor forgetting the bracket and acting as the identity on morphisms. Note that $H_{\mathrm{BrLie}} J_{\mathrm{Lie}} = J H_{\mathrm{Lie}}$.

3) Let $\mathcal{M}$ be an abelian symmetric monoidal category. Given $(M, [-]) \in \mathrm{Lie}_{\mathcal{M}}$ it is clear that $(M, c_{M,M}, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$ so that J_{Lie} factors through a functor J^s_{Lie} such that the following diagrams commute.

$$\begin{array}{ccc} \mathrm{Lie}_{\mathcal{M}} & \xrightarrow{J_{\mathrm{Lie}}} & \mathrm{BrLie}_{\mathcal{M}} \\ & {\scriptstyle J^s_{\mathrm{Lie}}}\searrow \quad \nearrow{\scriptstyle \mathbb{I}^s_{\mathrm{BrLie}}} & \\ & \mathrm{BrLie}^s_{\mathcal{M}} & \end{array} \qquad \begin{array}{ccc} \mathrm{BrLie}^s_{\mathcal{M}} & \xrightarrow{H^s_{\mathrm{BrLie}}} & \mathrm{Br}^s_{\mathcal{M}} \\ {\scriptstyle J^s_{\mathrm{Lie}}}\uparrow & & \uparrow{\scriptstyle J^s} \\ \mathrm{Lie}^s_{\mathcal{M}} & \xrightarrow{H_{\mathrm{Lie}}} & \mathcal{M} \end{array} \tag{55}$$

Remark 5.2. We point out that $\mathrm{BrLie}^s_{\mathcal{M}} = \mathrm{YBLieAlg}(\mathcal{M})$ with the notations of [21, Definition 2.5] (note that (52) follows from (53) as we are in the symmetric case).

Lemma 5.3. *Let $\mathcal{M}$ be an abelian monoidal category. Consider a tern $(M, c, [-] : M \otimes M \to M)$ where $(M, c) \in \mathrm{Br}_{\mathcal{M}}$. If $c^2 = \mathrm{Id}$ and (50) holds, then we have that (51) is equivalent to*

$$[-] \circ ([-] \otimes M) \circ \left[\mathrm{Id}_{(V \otimes V) \otimes V} + (M \otimes c)(c \otimes M) + (c \otimes M)(M \otimes c)\right] = 0. \tag{56}$$

Proof. This proof is essentially the same as [20, Lemma 2.9]. □

Remark 5.4. In view of Lemma 5.3, in the particular case when $\mathcal{M}$ is the category of vector spaces and $(M, c) \in \mathrm{Br}_{\mathcal{M}}$, conditions (50) and (56) encode the notion of Lie algebra in the sense of Gurevich's [19].

Definition 5.5. Let $\mathcal{M}$ a preadditive monoidal category with equalizers and denumerable coproducts. Let $(M, c) \in \mathrm{Br}_{\mathcal{M}}$. For $\alpha_2 M$ as in of Remark 1.3, we set

$$\theta_{(M,c)} := \alpha_2 M \circ (\mathrm{Id}_{M \otimes M} - c) : M \otimes M \to \Omega T M. \tag{57}$$

When $\mathcal{M}$ is braided and its braiding on M is $c_{M,M}$ we will simply write θ_M for $\theta_{(M, c_{M,M})}$.

Definition 5.6. Let $\mathcal{M}$ be a monoidal category. Let (A, m_A, u_A) be an algebra in $\mathcal{M}$ and let $f : X \to A$ be a morphism in $\mathcal{M}$. We set

$$\Lambda_f := m_A \circ (m_A \otimes A) \circ (A \otimes f \otimes A) : A \otimes X \otimes A \to A. \tag{58}$$

When the category $\mathcal{M}$ is also abelian we can consider the two-sided ideal of A *generated by* f which is defined by $(\langle f\rangle, i_f) := \mathrm{Im}(\Lambda_f)$ and it has the following property (see e.g. [7, Lemma 3.18]): for every algebra morphism $g : A \to B$ one has that $g \circ i_f = 0$ if and only if $g \circ f = 0$.

Remark 5.7. Let $\mathcal{M}$ be an abelian monoidal category. Let (A, m_A, u_A) be an algebra in $\mathcal{M}$.

1) Note that $\Lambda_f - \Lambda_g = \Lambda_{f-g}$ for every $f, g : X \to A$.

2) Assume that the tensor products preserve epimorphisms. Let $f : X \to A$ be a morphism in $\mathcal{M}$ and set $(S, j : S \to A) := \mathrm{Im}(f)$. Define the ideal (S) generated by S by setting $((S), i) := \mathrm{Im}(\Lambda_j)$. Write $f = j \circ p$ where $p : X \to S$ is an epimorphism. We compute $\mathrm{Im}(\Lambda_f) = \mathrm{Im}(\Lambda_{j\circ p}) = \mathrm{Im}(\Lambda_{j\circ p}) = \mathrm{Im}(\Lambda_j \circ (A \otimes p \otimes A)) = \mathrm{Im}(\Lambda_j)$ so that $(\langle f\rangle, i_f) := \mathrm{Im}(\Lambda_f) = ((S), i)$. Therefore $\langle f\rangle = (\mathrm{Im}(f))$.

Next aim is to construct suitable universal enveloping algebra type functors.

Remark 5.8. Let $\mathcal{M}$ an abelian monoidal category with denumerable coproducts. Assume that the tensor products preserve denumerable coproducts. Note that $\mathcal{M}$ has also finite coproducts as it has a zero object and denumerable coproduct. Thus, by [37, Proposition 3.3] the tensor products are additive as they preserve denumerable coproducts.

Proposition 5.9. *Let $\mathcal{M}$ an abelian monoidal category with denumerable coproducts. Assume that the tensor products are right exact and preserve denumerable coproducts.*

Let $(M, c, [-] : M \otimes M \to M) \in \mathrm{BrLie}_{\mathcal{M}}$ and set

$$f := f_{(M,c,[-])} := \alpha_1 M \circ [-] - \theta_{(M,c)} : M \otimes M \to \Omega TM.$$

Let $\mathcal{U}_{\mathrm{Br}}(M, c, [-]) := R := \Omega TM / \langle f\rangle$ and let $p_R : \Omega TM \to R$ denote the canonical projection. Then there are morphisms m_R, u_R, c_R such that $(R, m_R, u_R, c_R) \in \mathrm{BrAlg}_{\mathcal{M}}$ and p_R is a morphism of braided algebras. This way we get a functor

$$\mathcal{U}_{\mathrm{Br}} : \mathrm{BrLie}_{\mathcal{M}} \to \mathrm{BrAlg}_{\mathcal{M}},$$

and the projections p_R define a natural transformation $p : T_{\mathrm{Br}} H_{\mathrm{BrLie}} \to \mathcal{U}_{\mathrm{Br}}$. Moreover there is a functor $\mathcal{U}^s_{\mathrm{Br}} : \mathrm{BrLie}^s_{\mathcal{M}} \to \mathrm{BrAlg}^s_{\mathcal{M}}$ such that the diagram

$$\begin{array}{ccc} \mathrm{BrLie}^s_M & \xrightarrow{\mathbb{I}^s_{\mathrm{BrLie}}} & \mathrm{BrLie}_M \\ {\scriptstyle \mathcal{U}^s_{\mathrm{Br}}}\downarrow & & \downarrow{\scriptstyle \mathcal{U}_{\mathrm{Br}}} \\ \mathrm{BrAlg}^s_M & \xrightarrow{\mathbb{I}^s_{\mathrm{BrAlg}}} & \mathrm{Alg}_M \end{array} \tag{59}$$

commutes and there is a natural transformation $p^s : T^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \to \mathcal{U}^s_{\mathrm{Br}}$ uniquely defined by

$$\mathbb{I}^s_{\mathrm{BrAlg}} p^s = p \mathbb{I}^s_{\mathrm{BrLie}}. \tag{60}$$

Proof. Set $(A, m_A, u_A, c_A) := T_{\mathrm{Br}}(M, c)$. We will use the equalities for the graded part $c_A^{m,n}$ of the Yang–Baxter operator c_A which are in [6, Proposition 2.7]. Note that, by [6, (42)], we have that $c_A \circ (\alpha_m M \otimes \alpha_n M) = (\alpha_n M \otimes \alpha_m M) \circ c_A^{m,n}$ for every $m, n \in \mathbb{N}$. By induction on $n \in \mathbb{N}$, using (52), one checks that

$$c_A^{n,1} \circ \left(M^{\otimes n} \otimes [-]\right) = \left([-] \otimes M^{\otimes n}\right) \circ c_A^{n,2}. \tag{61}$$

If we apply [6, (32) and (34)], we get

$$c_A^{l,n+m}\left(M^{\otimes l} \otimes c_A^{m,n}\right) = \left(c_A^{m,n} \otimes M^{\otimes l}\right) c_A^{l,m+n}.$$

If we apply this equality to the case "(l, m, n)" $= (n, 1, 1)$, we obtain

$$c_A^{n,2}\left(M^{\otimes n} \otimes c\right) = \left(c \otimes M^{\otimes n}\right) c_A^{n,2}, \text{ for every } n \in \mathbb{N}. \tag{62}$$

Since $\langle f \rangle$ is an ideal of TM, it is clear that R is an algebra and p_R is an algebra morphism. Consider the exact sequence

$$0 \to \langle f \rangle \overset{i_f}{\to} A \overset{p_R}{\to} R \to 0$$

If we apply to it the functor $A \otimes (-)$, we obtain the exact sequence

$$A \otimes \langle f \rangle \overset{A \otimes i_f}{\to} A \otimes A \overset{A \otimes p_R}{\to} A \otimes R \to 0$$

We have that $(\langle f \rangle, i_f) := \mathrm{Im}(\Lambda_f)$ so that we can write $\Lambda_f = i_f \circ p_f$ where $p_f : A \otimes X \otimes A \to \langle f \rangle$ is an epimorphism. Since the tensor products preserve epimorphisms, we have that $A \otimes p_f$ is an epimorphism so that $(p_R \otimes A) c_A (A \otimes i_f) = 0$ if and only if $(p_R \otimes A) c_A (A \otimes \Lambda_f) = 0$. Using the definition of c_A, (61) and (62) one checks that $(p_R \otimes A) c_A (A \otimes f)(\alpha_n M \otimes M \otimes M) = 0$. Since this holds for every $n \in \mathbb{N}$ and the tensor products preserve the denumerable coproducts, we get

$$(p_R \otimes A) c_A (A \otimes f) = 0. \tag{63}$$

Now using (14) and (63) one gets $(p_R \otimes A) c_A (A \otimes \Lambda_f) = 0$. Hence, by the foregoing, we get $(p_R \otimes A) c_A (A \otimes i_f) = 0$. Thus there is a unique morphism $c_{A,R} : A \otimes R \to R \otimes A$ such that $c_{A,R} \circ (A \otimes p_R) = (p_R \otimes A) \circ c_A$. Consider now the exact sequence

$$\langle f \rangle \otimes R \overset{i_f \otimes R}{\to} A \otimes R \overset{p_R \otimes R}{\to} R \otimes R \to 0.$$

We will prove that $(R \otimes p_R) c_{A,R} (i_f \otimes R) = 0$.

This equality is equivalent to prove $(R \otimes p_R) c_{A,R} (\Lambda_f \otimes R) = 0$. We have

$$\begin{aligned}&(R\otimes p_R)\,c_{A,R}\,(\Lambda_f\otimes R)\,(A\otimes M\otimes M\otimes A\otimes p_R)\\&\quad=(R\otimes p_R)\,c_{A,R}\,(A\otimes p_R)\,(\Lambda_f\otimes A)=(R\otimes p_R)\,(p_R\otimes A)\,c_A\,(\Lambda_f\otimes A)\\&\quad=(p_R\otimes R)\,(A\otimes p_R)\,c_A\,(\Lambda_f\otimes A)\,.\end{aligned}$$

Note that the latter term vanishes as $(A\otimes p_R)\,c_A\,(\Lambda_f\otimes A)=0$ by a similar argument to the one used to prove $(p_R\otimes A)\,c_A\,(A\otimes\Lambda_f)=0$ and using (53). Since $A\otimes M\otimes M\otimes A\otimes p_R$ is an epimorphism, we get that $(R\otimes p_R)\,c_{A,R}\,(\Lambda_f\otimes R)=0$ and hence there is a unique morphism $c_R:R\otimes R\to R\otimes R$ such that $c_R\circ(p_R\otimes R)=(R\otimes p_R)\circ c_{A,R}$. We get

$$\begin{aligned}c_R\,(p_R\otimes p_R)&=c_R\,(p_R\otimes R)\,(A\otimes p_R)=(R\otimes p_R)\,c_{A,R}\,(A\otimes p_R)\\&=(R\otimes p_R)\,(p_R\otimes A)\,c_A=(p_R\otimes p_R)\,c_A.\end{aligned}$$

If we rewrite (52) and (53) in terms of c^{-1} we get that $\left(M,c^{-1}\right)$ fulfills (52) and (53). Thus we can repeat the argument above obtaining a morphism c'_R such that $c'_R\,(p_R\otimes p_R)=(p_R\otimes p_R)\,c_A^{-1}$. It is easy to check that c'_R is an inverse for c_R. By Lemma B.4, we get that (R,c_R) is an object in $\mathrm{Br}_{\mathcal{M}}$ and p_R becomes a morphism in $\mathrm{Br}_{\mathcal{M}}$ from (A,c_A) to this object. We have

$$\begin{aligned}c_R(m_R\otimes R)\,(p_R\otimes p_R\otimes p_R)&=c_R\,(p_R\otimes p_R)\,(m_A\otimes A)=(p_R\otimes p_R)\,c_A(m_A\otimes A)\\&\overset{(13)}{=}(p_R\otimes p_R)\,(A\otimes m)(c\otimes A)(A\otimes c)\\&=(R\otimes m_R)(c_R\otimes R)(R\otimes c_R)\,(p_R\otimes p_R\otimes p_R)\end{aligned}$$

so that (13) holds for (R,m_R,c_R). Similarly one proves (14). Moreover

$$\begin{aligned}c_R(u_R\otimes R)l_R^{-1}p_R&=c_R(u_R\otimes R)\,(\mathbf{1}\otimes p_R)\,l_A^{-1}=c_R(p_Ru_A\otimes p_R)l_A^{-1}\\&=(p_R\otimes p_R)\,c_A(u_A\otimes A)l_A^{-1}\overset{(15)}{=}(p_R\otimes p_R)\,(A\otimes u_A)\,r_A^{-1}\\&=(p_R\otimes u_R)\,r_A^{-1}=(R\otimes u_R)\,r_R^{-1}p_R\end{aligned}$$

and hence $c_R(u_R\otimes R)l_R^{-1}=(R\otimes u_R)\,r_R^{-1}$. Similarly one gets $c_R(R\otimes u_R)r_R^{-1}=(u_R\otimes R)\,l_R^{-1}$. We have so proved that $(R,m_R,u_R,c_R)\in\mathrm{BrAlg}_{\mathcal{M}}$. It is clear that p_R is a morphism of braided algebras.

Let $\nu:(M,c,[-])\to\left(M',c',[-]'\right)$ be a morphism of braided Lie algebras. Consider the morphism of braided algebras $T_{\mathrm{Br}}\nu:T_{\mathrm{Br}}\,(M,c)\to T_{\mathrm{Br}}\,(M',c')$. Set $R':=\mathcal{U}_{\mathrm{Br}}\left(M',c',[-]'\right)$ and denote by $p_{R'}$ the corresponding projection and set $f':=f_{(M',c',[-]')}$. We have

$$\begin{aligned}&p_{R'}\circ\Omega H_{\mathrm{Alg}}T_{\mathrm{Br}}H_{\mathrm{BrLie}}\nu\circ f\overset{(19)}{=}p_{R'}\circ\Omega THH_{\mathrm{BrLie}}\nu\circ\left(\alpha_1M\circ[-]-\theta_{(M,c)}\right)\\&\quad=p_{R'}\circ\Omega THH_{\mathrm{BrLie}}\nu\circ\alpha_1M\circ[-]-p_{R'}\circ\Omega THH_{\mathrm{BrLie}}\nu\circ\alpha_2M\circ(\mathrm{Id}_{M\otimes M}-c)\\&\quad\overset{(1)}{=}p_{R'}\circ\alpha_1M'\circ HH_{\mathrm{BrLie}}\nu\circ[-]-p_{R'}\circ\alpha_2M'\circ(HH_{\mathrm{BrLie}}\nu\otimes HH_{\mathrm{BrLie}}\nu)\end{aligned}$$

$$
\begin{aligned}
&\circ (\mathrm{Id}_{M\otimes M} - c)\\
&= p_{R'} \circ \alpha_1 M' \circ HH_{\mathrm{BrLie}}\nu \circ [-] - p_{R'} \circ \alpha_2 M' \circ (\mathrm{Id}_{M'\otimes M'} - c')\\
&\quad \circ (HH_{\mathrm{BrLie}}\nu \otimes HH_{\mathrm{BrLie}}\nu)\\
&= p_{R'} \circ \alpha_1 M' \circ [-]' \circ (HH_{\mathrm{BrLie}}\nu \otimes HH_{\mathrm{BrLie}}\nu) - p_{R'} \circ \theta_{(M',c')}\\
&\quad \circ (HH_{\mathrm{BrLie}}\nu \otimes HH_{\mathrm{BrLie}}\nu)\\
&= p_{R'} \circ f' \circ (HH_{\mathrm{BrLie}}\nu \otimes HH_{\mathrm{BrLie}}\nu) = 0.
\end{aligned}
$$

Since $p_{R'} \circ \Omega H_{\mathrm{Alg}} T_{\mathrm{Br}} H_{\mathrm{BrLie}}\nu \circ$ is an algebra morphism we get $p_{R'} \circ \Omega H_{\mathrm{Alg}} T_{\mathrm{Br}} H_{\mathrm{BrLie}}\nu \circ i_f = 0$ so that there is a unique morphism $\mathcal{U}_{\mathrm{Br}}\nu : \mathcal{U}_{\mathrm{Br}}(M, c, [-]) \to \mathcal{U}_{\mathrm{Br}}(M', c', [-]')$ such that $\mathcal{U}_{\mathrm{Br}}\nu \circ p_R = p_{R'} \circ T_{\mathrm{Br}} H_{\mathrm{BrLie}}\nu$. It is easy to check that $\mathcal{U}_{\mathrm{Br}}\nu$ is a morphism of braided bialgebras. Since T_{Br} is a functor it is then clear that $\mathcal{U}_{\mathrm{Br}}$ becomes a functor as well and that the projections define a natural transformation $p : T_{\mathrm{Br}} H_{\mathrm{BrLie}} \to \mathcal{U}_{\mathrm{Br}}$.

Let us construct $\mathcal{U}^s_{\mathrm{Br}}$. We already observed that the functor $\mathbb{I}^s_{\mathrm{BrAlg}}$ is full, faithful and injective on objects.

Let $(M, c, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$. Then, by Remark 3.2-2), we get that $R = \mathcal{U}_{\mathrm{Br}}(M, c, [-]) \in \mathrm{BrAlg}^s_{\mathcal{M}}$ as R is a quotient of $T_{\mathrm{Br}} H_{\mathrm{BrLie}}(M, c, [-])$ which is preserved by the required functors. Hence $\mathrm{Im}(\mathcal{U}_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrLie}}) \subseteq \mathrm{Im}(\mathbb{I}^s_{\mathrm{BrAlg}})$. By Lemma 1.12, there is a unique functor $\mathcal{U}^s_{\mathrm{Br}} := \widehat{\mathcal{U}_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrLie}}}$ such that (59) commutes. We have

$$
T_{\mathrm{Br}} H_{\mathrm{BrLie}} \mathbb{I}^s_{\mathrm{BrLie}} \overset{(54)}{=} T_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \overset{(21)}{=} \mathbb{I}^s_{\mathrm{BrAlg}} T^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}}. \tag{64}
$$

By Lemma 1.12, we have $T_{\mathrm{Br}} \widehat{H_{\mathrm{BrLie}} \mathbb{I}^s_{\mathrm{BrLie}}} = T^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}}$, $\widehat{\mathcal{U}_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrLie}}} = \mathcal{U}^s_{\mathrm{Br}}$ and there is a unique natural transformation $p^s := \widehat{p \mathbb{I}^s_{\mathrm{BrLie}}} : T^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \to \mathcal{U}^s_{\mathrm{Br}}$ such that (60) holds. □

Lemma 5.10. *Let $\mathcal{M}$ a preadditive monoidal category with denumerable coproducts. Assume that the tensor products are additive and preserve such coproducts. Let $(M, c, [-] : M \otimes M \to M) \in \mathrm{BrLie}_{\mathcal{M}}$, set $(A, m_A, u_A, \Delta_A, \varepsilon_A, c_A) := \overline{T}_{\mathrm{Br}}(M, c)$ and use the notations of 3.6. Then,*

$$
\Delta_A \circ \theta_{(M,c)} = \left[(u_A \otimes A) \circ l_A^{-1} + (A \otimes u_A) \circ r_A^{-1}\right] \circ \theta_{(M,c)} \text{ if } c^2 = \mathrm{Id}_{M\otimes M}; \tag{65}
$$

$$
\Delta_A \circ \alpha_1 M = \left[(u_A \otimes A) \circ l_A^{-1} + (A \otimes u_A) \circ r_A^{-1}\right] \circ \alpha_1 M. \tag{66}
$$

Proof. Using, in the given order, (2), the multiplicativity of Δ_A, (25), the definitions of δ^l_M and δ^r_M, the equalities $c_A \circ (\alpha_i M \otimes \alpha_j M) = (\alpha_j M \otimes \alpha_j M) \circ c_A^{i,j}$ for $i, j \in \{1, 2\}$, the equalities $c_A^{1,0} = l_M^{-1} r_M, c_A^{1,1} = c, c_A^{0,0} = l_{\mathbf{1}}^{-1} r_{\mathbf{1}}$ and $c_A^{0,1} = r_M^{-1} l_M$, the equalities (3) and (2), the equalities $r_M \otimes M = M \otimes l_M$ and $r_{\mathbf{1}} \otimes M = \mathbf{1} \otimes l_M$, the equalities $l_M^{-1} \otimes M = l_{M\otimes M}^{-1}$, $M \otimes l_{\mathbf{1}}^{-1} = r_M^{-1} \otimes \mathbf{1}$ and $M \otimes r_M^{-1} = r_{M\otimes M}^{-1}$, the equalities $m_{\mathbf{1}} = r_{\mathbf{1}} = l_{\mathbf{1}}$, the naturality of the unit constraints, $l_M^{-1} \otimes M = l_{M\otimes M}^{-1}$, $M \otimes r_M^{-1} = r_{M\otimes M}^{-1}$ and $r_M \otimes M = M \otimes l_M$, the equality (3) and the naturality of the unit constraints one proves that

$$\Delta_A \circ \alpha_2 M = \left[(u_A \otimes A) \circ l_A^{-1} + (A \otimes u_A) \circ r_A^{-1}\right] \circ \alpha_2 M \\ + (\alpha_1 M \otimes \alpha_1 M) \circ (\mathrm{Id}_{M\otimes M} + c)\,.$$

From this equality, composing with $\mathrm{Id}_{M\otimes M} - c$ on both sides, we get (65) holds true when $c^2 = \mathrm{Id}_{M\otimes M}$.

On the other hand, (66) follows by (25), the definitions of δ^l_M and δ^r_M, the naturality of the unit constraints. □

Proposition 5.11. *Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \in \mathrm{BrBialg}_{\mathcal{M}}$ be a bialgebra in a monoidal category $\mathcal{M}$. Assume that the category $\mathcal{M}$ is abelian and the tensor products are additive and right exact. Let $(R, m_R, u_R, c_R) \in \mathrm{BrAlg}_{\mathcal{M}}$ and let $p_R : B \to R$ be an epimorphism which is a morphism of braided algebras. Set $(I, i_I : I \to B) := \mathrm{Ker}\,(p_R)$. Assume that*

$$(p_R \otimes p_R) \circ \Delta_B \circ i_I = 0, \tag{67}$$

$$\varepsilon_B \circ i_I = 0. \tag{68}$$

Then there are morphisms Δ_R, ε_R such that $(R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) \in \mathrm{BrBialg}_{\mathcal{M}}$ and p_R is a morphism of braided bialgebras.

Proof. Since $(R, p_R) = \mathrm{Coker}\,(i_I)$, by (67), there is a unique morphism $\Delta_R : R \to R\otimes R$ such that $\Delta_R \circ p_R = (p_R \otimes p_R) \circ \Delta_B$ and, by (68), there is a unique morphism $\varepsilon_R : R \to \mathbf{1}$ such that $\varepsilon_R \circ p_R = \varepsilon_B$. The rest of the proof is straightforward and relies on the fact that $p_R \otimes p_R = (p_R \otimes R)\,(A \otimes p_R)$ is an epimorphism by exactness of the tensor functors. □

Theorem 5.12. *Let $\mathcal{M}$ an abelian monoidal category with denumerable coproducts. Assume that the tensor products are right exact and preserve denumerable coproducts. Then there is a functor $\overline{\mathcal{U}}^s_{\mathrm{Br}} : \mathrm{BrLie}^s_{\mathcal{M}} \to \mathrm{BrBialg}^s_{\mathcal{M}}$ such that*

$$\begin{array}{ccc} \mathrm{BrLie}^s_{\mathcal{M}} & \xrightarrow{\overline{\mathcal{U}}^s_{\mathrm{Br}}} & \mathrm{BrBialg}^s_{\mathcal{M}} \\ & {\scriptstyle \mathcal{U}^s_{\mathrm{Br}}}\searrow \quad \swarrow{\scriptstyle \mho^s_{\mathrm{Br}}} & \\ & \mathrm{BrAlg}^s_{\mathcal{M}} & \end{array} \tag{69}$$

Moreover there is a natural transformation $\bar{p}^s : \overline{T}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \to \overline{\mathcal{U}}^s_{\mathrm{Br}}$ uniquely defined by

$$\mho_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrBialg}} \bar{p}^s = p \mathbb{I}^s_{\mathrm{BrLie}} \qquad \text{and} \qquad \mho^s_{\mathrm{Br}} \bar{p}^s = p^s \tag{70}$$

where $p : T_{\mathrm{Br}} H_{\mathrm{BrLie}} \to \mathcal{U}_{\mathrm{Br}}$ and $p^s : T^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \to \mathcal{U}^s_{\mathrm{Br}}$ are the natural transformations of Proposition 5.9.

Proof. Let $(M, c, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$ and set $(A, m_A, u_A, \Delta_A, \varepsilon_A, c_A) := \overline{T}^s_{\mathrm{Br}}\,(M, c)$ and $f := f_{(M,c,[-])}$. Set $(R, m_R, u_R, c_R) := \mathcal{U}_{\mathrm{Br}}\,(M, c, [-])$ and let p_R be the morphism in

$\mathcal{M}$ underlying the canonical projection $p(M,c,[-]) : T_{\mathrm{Br}}(M,c) \to \mathcal{U}_{\mathrm{Br}}(M,c,[-])$. By Proposition 5.9, we know that $p_R : A \to R$ is a morphism of braided algebras. Using (65) and (66), we get

$$\Delta_A \circ f = \left[(u_A \otimes A) \circ l_A^{-1} + (A \otimes u_A) \circ r_A^{-1}\right] \circ f \tag{71}$$

Since p_R is an algebra morphism and $p_R \circ i_f = 0$, we get that $p_R \circ f = 0$. We want to apply Proposition 5.11 to the case $(I, i_I) = (\langle f \rangle, i_f)$. Since $(p_R \otimes p_R) \circ \Delta_A$ is an algebra morphism as a composition of algebra morphisms (use e.g. [6, Proposition 2.2-3)] to prove that $p_R \otimes p_R$ is an algebra morphism and use (16) to have that Δ_A is an algebra morphism), we have that (67) is equivalent to $(p_R \otimes p_R) \circ \Delta_A \circ f = 0$ and the latter holds by (71), unitality of p_R, naturality of the unit constraints, and the equality $p_R \circ f = 0$.

Since ε_A is an algebra morphism, we have that (68) if and only if $\varepsilon_A \circ f = 0$ and the latter holds by definition of f and (27). Then, by Proposition 5.11, there are morphisms Δ_R, ε_R such that $(R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) \in \mathrm{BrBialg}_{\mathcal{M}}$ and p_R is a morphism of braided bialgebras. By Remark 3.2-2) one easily checks that $(R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) \in \mathrm{BrBialg}^s_{\mathcal{M}}$. We denote this datum by $\overline{\mathcal{U}}^s_{\mathrm{Br}}(M,c,[-])$. Let $\nu : (M,c,[-]) \to (M',c',[-]')$ be a morphism in $\mathrm{BrLie}^s_{\mathcal{M}}$. We know that $\widetilde{v} := \Omega H_{\mathrm{Alg}} \mathcal{U}_{\mathrm{Br}} \nu : R \to R'$ is a morphism in $\mathrm{BrAlg}_{\mathcal{M}}$. Using that p_R is comultiplicative and natural, and that $\Omega H_{\mathrm{Alg}} \mho_{\mathrm{Br}} \overline{T}_{\mathrm{Br}} H_{\mathrm{BrLie}} v$ is a coalgebra morphism one easily gets that $(\widetilde{v} \otimes \widetilde{v}) \circ \Delta_R \circ p_R = \Delta_{R'} \circ \widetilde{v} \circ p_R$ and hence $\widetilde{v}$ is comultiplicative. A similar argument shows that $\widetilde{v}$ is also counitary and hence $\mathcal{U}_{\mathrm{Br}}\nu$ is a morphism in $\mathrm{BrBialg}^s_{\mathcal{M}}$. This defines a functor $\overline{\mathcal{U}}^s_{\mathrm{Br}} : \mathrm{BrLie}^s_{\mathcal{M}} \to \mathrm{BrBialg}^s_{\mathcal{M}}$ such that $\mho^s_{\mathrm{Br}} \circ \overline{\mathcal{U}}^s_{\mathrm{Br}} = \mathcal{U}^s_{\mathrm{Br}}$. Since p_R is a morphism of braided bialgebras and it is natural in R at the level of $\mathrm{BrAlg}_{\mathcal{M}}$, it is clear that $\bar{p}^s$ such that $\mho_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrBialg}} \bar{p}^s = p \mathbb{I}^s_{\mathrm{BrLie}}$ exists. Moreover we have

$$\mathbb{I}^s_{\mathrm{BrAlg}} p^s \overset{(60)}{=} p \mathbb{I}^s_{\mathrm{BrLie}} = \mho_{\mathrm{Br}} \mathbb{I}^s_{\mathrm{BrBialg}} \bar{p}^s \overset{(32)}{=} \mathbb{I}^s_{\mathrm{BrAlg}} \mho^s_{\mathrm{Br}} \bar{p}^s$$

and hence $p^s = \mho^s_{\mathrm{Br}} \bar{p}^s$. $\square$

6. Adjunctions for enveloping functors

Given a braided algebra B, in general it is not true that the commutator bracket $[-]_B := m_B \circ (\mathrm{Id}_{B \otimes B} - c_B)$ defines a braided Lie algebra structure on B as in the classic case unless c is a symmetry. For example, let (V,c) be a braided vector space and consider the braided tensor algebra $B := T_{\mathrm{Br}}(V,c)$. Assume that $[-]_B$ fulfills (50). An easy calculation shows that the restriction of this equality to $V \otimes V$ forces c (whence also c_B) to be a symmetry. For this reason in this section we restrict to symmetries in order to construct an adjoint for $\mathcal{U}_{\mathrm{Br}}$.

Proposition 6.1. *Let $\mathcal{M}$ an abelian monoidal category with denumerable coproducts. Assume that the tensor products are right exact and preserve denumerable coproducts. Then* *the functor* $\mathcal{U}_{\mathrm{Br}}^{s}:\mathrm{BrLie}_{\mathcal{M}}^{s}\rightarrow\mathrm{BrAlg}_{\mathcal{M}}^{s}$ *has a right adjoint* $\mathcal{L}_{\mathrm{Br}}^{s}:\mathrm{BrAlg}_{\mathcal{M}}^{s}\rightarrow\mathrm{BrLie}_{\mathcal{M}}^{s}$ *acting as the identity on morphisms and defined on objects by* $\mathcal{L}_{\mathrm{Br}}^{s}(B,m_B,u_B,c_B):=(B,c_B,[-]_B)$*, where* $[-]_B:=m_B\circ(\mathrm{Id}_{B\otimes B}-c_B)$*. The unit* $\eta_{\mathrm{BrL}}^{s}:\mathrm{Id}_{\mathrm{BrLie}_{\mathcal{M}}^{s}}\rightarrow\mathcal{L}_{\mathrm{Br}}^{s}\mathcal{U}_{\mathrm{Br}}^{s}$ *and the counit* $\epsilon_{\mathrm{BrL}}^{s}:\mathcal{U}_{\mathrm{Br}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}\rightarrow\mathrm{Id}_{\mathrm{BrAlg}_{\mathcal{M}}^{s}}$ *of the adjunction fulfill*

$$\epsilon_{\mathrm{BrL}}^{s}\circ p^{s}\mathcal{L}_{\mathrm{Br}}^{s}=\epsilon_{\mathrm{Br}}^{s}\qquad\text{and}\qquad H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}p^{s}\circ\eta_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}=H_{\mathrm{BrLie}}^{s}\eta_{\mathrm{BrL}}^{s}.\tag{72}$$

Proof. The construction of the functor $\mathcal{L}_{\mathrm{Br}}^{s}$ is given in [21, Construction 2.16] where $\mathrm{BrAlg}_{\mathcal{M}}^{s}$ plays the role of $\mathrm{YBAlg}(\mathcal{M})$ therein. Let us check that $(\mathcal{U}_{\mathrm{Br}}^{s},\mathcal{L}_{\mathrm{Br}}^{s})$ is an adjunction.

Consider the natural transformation $p^{s}:T_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}\rightarrow\mathcal{U}_{\mathrm{Br}}^{s}$ of Proposition 5.9.

Note that $H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}(B,m_B,u_B,c_B)=H_{\mathrm{BrLie}}^{s}(B,c_B,[-]_B)=(B,c_B)=\Omega_{\mathrm{Br}}^{s}(B,m_B,u_B,c_B)$ and $H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}$ and Ω_{Br}^{s} both act as the identity on morphisms so that $H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}=\Omega_{\mathrm{Br}}^{s}$. Then we have $p^{s}\mathcal{L}_{\mathrm{Br}}^{s}:T_{\mathrm{Br}}^{s}\Omega_{\mathrm{Br}}^{s}\rightarrow\mathcal{U}_{\mathrm{Br}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}$. Consider $\epsilon_{\mathrm{Br}}^{s}:T_{\mathrm{Br}}^{s}\Omega_{\mathrm{Br}}^{s}\rightarrow\mathrm{Id}_{\mathrm{BrAlg}_{\mathcal{M}}^{s}}$. Using the notation of Proposition 5.9, by means of (22), (20), (57) and (4) we get

$$\Omega H_{\mathrm{Alg}}\mathbb{I}_{\mathrm{BrAlg}}^{s}\epsilon_{\mathrm{Br}}^{s}(B,m_B,u_B,c_B)\circ f_{\mathcal{L}_{\mathrm{Br}}^{s}(B,m_B,u_B,c_B)}=0.$$

Since $\epsilon_{\mathrm{Br}}^{s}$ is a morphism of braided algebras, by construction of $\mathcal{U}_{\mathrm{Br}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}$, the latter equality implies there is a unique morphism $\epsilon_{\mathrm{BrL}}^{s}:\mathcal{U}_{\mathrm{Br}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}\rightarrow\mathrm{Id}_{\mathrm{BrAlg}_{\mathcal{M}}^{s}}$ such that $\epsilon_{\mathrm{BrL}}^{s}\circ p^{s}\mathcal{L}_{\mathrm{Br}}^{s}=\epsilon_{\mathrm{Br}}^{s}$.

Consider the morphism $H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}p^{s}\circ\eta_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}:H_{\mathrm{BrLie}}^{s}\rightarrow H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}\mathcal{U}_{\mathrm{Br}}^{s}$. Let $(M,c_M,[-])\in\mathrm{BrLie}_{\mathcal{M}}^{s}$ and set $\nu:=H\mathbb{I}_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}p^{s}(M,c_M,[-])\circ H\mathbb{I}_{\mathrm{Br}}^{s}\eta_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}(M,c_M,[-])$, $(R,m_R,u_R,c_R):=\mathcal{U}_{\mathrm{Br}}^{s}(M,c_M,[-])$ and $(A,m_A,u_A,c_A):=T_{\mathrm{Br}}^{s}(M,c_M)$. Clearly $\nu:(M,c_M)\rightarrow(R,c_R)$ is a morphism of braided objects. Using (54), (22), (60), (20), (4) and the equality $p_R=H\Omega_{\mathrm{Br}}p\mathbb{I}_{\mathrm{BrLie}}^{s}(M,c_M,[-])$ (which follows by definition of p in Proposition 5.9), we obtain that $\nu=p_R\circ\alpha_1M$. By the latter formula, the fact that p_R is a braided morphisms, the definition of c_A given by [6, (42)], the multiplicativity of p_R, using (2), (57) and the formula $p_R\circ f_{(M,c_M,[-])}=0$, we obtain $[-]_R\circ(\nu\otimes\nu)=\nu\circ[-]$. Since ν is the morphism in $\mathcal{M}$ defining $H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}p^{s}\circ\eta_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}:H_{\mathrm{BrLie}}^{s}\rightarrow H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}\mathcal{U}_{\mathrm{Br}}^{s}$, we get that there is a unique natural transformation $\eta_{\mathrm{BrL}}^{s}:\mathrm{Id}_{\mathrm{BrLie}_{\mathcal{M}}^{s}}\rightarrow\mathcal{L}_{\mathrm{Br}}^{s}\mathcal{U}_{\mathrm{Br}}^{s}$ such that $H_{\mathrm{BrLie}}^{s}\mathcal{L}_{\mathrm{Br}}^{s}p^{s}\circ\eta_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}=H_{\mathrm{BrLie}}^{s}\eta_{\mathrm{BrL}}^{s}$. It is straightforward to check that this gives rise to the claimed adjunction. Note that

$$H\mathbb{I}_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}\eta_{\mathrm{BrL}}^{s}(M,c_M,[-])=v=p_R\circ\alpha_1M.\tag{73}$$

The latter equality will be used elsewhere. □

As a consequence of the construction of $\mathcal{U}_{\mathrm{Br}}$ we can introduce an enveloping algebra functor $\mathcal{U}$ in the braided case. We remark that in [22, 2.2] such a functor is just assumed to exist and the functor $\mathcal{L}:\mathrm{Alg}_{\mathcal{M}}\rightarrow\mathrm{Lie}_{\mathcal{M}}$ in the following result is also considered.

Theorem 6.2. *Let $\mathcal{M}$ be an abelian symmetric monoidal category with denumerable coproducts. Assume that the tensor products are right exact and preserve denumerable coproducts. There are unique functors $\mathcal{U}$ and $\mathcal{L}$ such that the following diagrams commute.*

$$\begin{array}{ccc} \mathrm{Alg}_M & \xrightarrow{J^s_{\mathrm{Alg}}} & \mathrm{BrAlg}^s_M \\ \mathcal{U}\uparrow & & \uparrow \mathcal{U}^s_{\mathrm{Br}} \\ \mathrm{Lie}_M & \xrightarrow{J^s_{\mathrm{Lie}}} & \mathrm{BrLie}^s_M \end{array} \qquad \begin{array}{ccc} \mathrm{Alg}_M & \xrightarrow{J^s_{\mathrm{Alg}}} & \mathrm{BrAlg}^s_M \\ \mathcal{L}\downarrow & & \downarrow \mathcal{L}^s_{\mathrm{Br}} \\ \mathrm{Lie}_M & \xrightarrow{J^s_{\mathrm{Lie}}} & \mathrm{BrLie}^s_M \end{array} \tag{74}$$

Moreover $(\mathcal{U},\mathcal{L})$ is an adjunction with unit $\eta_{\mathrm{L}} : \mathrm{Id}_{\mathrm{Lie}_{\mathcal{M}}} \to \mathcal{L}\mathcal{U}$ and counit $\epsilon_{\mathrm{L}} : \mathcal{U}\mathcal{L} \to \mathrm{Id}_{\mathrm{Alg}_{\mathcal{M}}}$ defined by

$$J^s_{\mathrm{Alg}}\epsilon_{\mathrm{L}} = \epsilon^s_{\mathrm{BrL}} J^s_{\mathrm{Alg}} \qquad \text{and} \qquad J^s_{\mathrm{Lie}}\eta_{\mathrm{L}} = \eta^s_{\mathrm{BrL}} J^s_{\mathrm{Lie}}, \tag{75}$$

and $\left(J^s_{\mathrm{Alg}}, J^s_{\mathrm{Lie}}\right) : (\mathcal{U},\mathcal{L}) \to (\mathcal{U}^s_{\mathrm{Br}}, \mathcal{L}^s_{\mathrm{Br}})$ is a commutation datum with canonical transformation given by the identity. The functor $\mathcal{U}$ can be described explicitly by $\mathcal{U} := H_{\mathrm{Alg}}\mathcal{U}_{\mathrm{Br}}J_{\mathrm{Lie}}$ while $\mathcal{L} : \mathrm{Alg}_{\mathcal{M}} \to \mathrm{Lie}_{\mathcal{M}}$ acts as the identity on morphisms and is defined on objects by $\mathcal{L}(B, m_B, u_B) := (B, [-]_B)$, where $[-]_B := m_B \circ (\mathrm{Id}_{B\otimes B} - c_{B,B})$.

Proof. The existence and uniqueness of $\mathcal{U}$ and $\mathcal{L}$ as in the statement follows by Lemma 2.9. It remains to prove the last sentence. The equality $\mathcal{U} = H_{\mathrm{Alg}}\mathcal{U}_{\mathrm{Br}}J_{\mathrm{Lie}}$ follows by (74), (59) and (55). For $(B, m_B, u_B) \in \mathrm{Alg}_{\mathcal{M}}$, by the foregoing, we have

$$J^s_{\mathrm{Lie}}\mathcal{L}(B, m_B, u_B) \overset{(74)}{=} \mathcal{L}^s_{\mathrm{Br}} J^s_{\mathrm{Alg}}(B, m_B, u_B) = (B, [-]_B, c_{B,B})$$

so that $\mathcal{L}(B, m_B, u_B) = (B, [-]_B)$. Since J^s_{Lie}, $\mathcal{L}^s_{\mathrm{Br}}$ and J^s_{Alg} act as the identity on morphisms so does $\mathcal{L}$. □

Proposition 6.3. *Let $\mathcal{M}$ be an abelian monoidal category with denumerable coproducts. Assume that the tensor products are right exact and preserve denumerable coproducts. Then the functor $\overline{\mathcal{U}}^s_{\mathrm{Br}} : \mathrm{BrLie}^s_{\mathcal{M}} \to \mathrm{BrBialg}^s_{\mathcal{M}}$ has a right adjoint $\mathcal{P}^s_{\mathrm{Br}} : \mathrm{BrBialg}^s_{\mathcal{M}} \to \mathrm{BrLie}^s_{\mathcal{M}}$ such that the following diagram commutes*

$$\begin{array}{ccccc} & & \mathrm{BrBialg}^s_{\mathcal{M}} & & \\ & \swarrow^{\mathcal{P}^s_{\mathrm{Br}}} & & \searrow^{P^s_{\mathrm{Br}}} & \\ \mathrm{BrLie}^s_{\mathcal{M}} & & \xrightarrow{H^s_{\mathrm{BrLie}}} & & \mathrm{Br}^s_{\mathcal{M}} \end{array} \tag{76}$$

and the natural transformation $\xi : P^s_{\mathrm{Br}} \to \Omega^s_{\mathrm{Br}}\mho^s_{\mathrm{Br}}$ induces a natural transformation $\xi : \mathcal{P}^s_{\mathrm{Br}} \to \mathcal{L}^s_{\mathrm{Br}}\mho^s_{\mathrm{Br}}$ such that $H^s_{\mathrm{BrLie}}\xi = \xi$. The unit $\bar{\eta}^s_{\mathrm{BrL}} : \mathrm{Id}_{\mathrm{BrLie}^s_{\mathcal{M}}} \to \mathcal{P}^s_{\mathrm{Br}}\overline{\mathcal{U}}^s_{\mathrm{Br}}$ and the counit $\bar{\epsilon}^s_{\mathrm{BrL}} : \overline{\mathcal{U}}^s_{\mathrm{Br}}\mathcal{P}^s_{\mathrm{Br}} \to \mathrm{Id}_{\mathrm{BrBialg}^s_{\mathcal{M}}}$ of the adjunction satisfy

$$\xi\overline{\mathcal{U}}^s_{\mathrm{Br}} \circ \bar{\eta}^s_{\mathrm{BrL}} = \eta^s_{\mathrm{BrL}} \qquad \text{and} \qquad \epsilon^s_{\mathrm{BrL}}\mho^s_{\mathrm{Br}} \circ \mathcal{U}^s_{\mathrm{Br}}\xi = \mho^s_{\mathrm{Br}}\bar{\epsilon}^s_{\mathrm{BrL}}. \tag{77}$$

Proof. Let $\mathbb{B} := (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \in \mathrm{BrBialg}^s_{\mathcal{M}}$. Write $P^s_{\mathrm{Br}}\mathbb{B} = (P, c_P)$. By [21, Proposition 6.6(i)], there is a morphism $[-]_P := P \otimes P \to P$ such that $\mathcal{P}^s_{\mathrm{Br}}\mathbb{B} := (P, c_P, [-]_P) \in \mathrm{BrLie}^s_{\mathcal{M}}$ and $\xi\mathbb{B} : (P, c_P, [-]_P) \to (B, c_B, [-]_B)$ is a morphism in $\mathrm{BrLie}^s_{\mathcal{M}}$ where $[-]_B := m_B \circ (\mathrm{Id}_{B\otimes B} - c_B)$. Clearly $[-]_P$ is uniquely determined by the compatibility with $\xi\mathbb{B}$. In this way we get a functor $\mathcal{P}^s_{\mathrm{Br}} : \mathrm{BrBialg}^s_{\mathcal{M}} \to \mathrm{BrLie}^s_{\mathcal{M}}$ which acts as P^s_{Br} on morphisms. Let us check that there is a unique morphism $\bar{\eta}^s_{\mathrm{BrL}} : \mathrm{Id}_{\mathrm{BrLie}^s_{\mathcal{M}}} \to \mathcal{P}^s_{\mathrm{Br}}\, \overline{\mathcal{U}}^s_{\mathrm{Br}}$ such that $\xi\overline{\mathcal{U}}^s_{\mathrm{Br}} \circ \bar{\eta}^s_{\mathrm{BrL}} = \eta^s_{\mathrm{BrL}}$. Let $(M, c, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$, set $(R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) := \overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-])$ and set also $(A, m_A, u_A, \Delta_A, \varepsilon_A, c_A) := \overline{T}^s_{\mathrm{Br}}(M, c)$. Using that p_R is comultiplicative, the equality (25), unitality of p_R and the naturality of the unit constraints, one easily checks that

$$\nu := H\mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \eta^s_{\mathrm{BrL}}(M, c, [-]) \overset{(73)}{=} p_R \circ \alpha_1 M : M \to R$$

is equalized by the fork in (23). Hence ν induces a morphism $\nu' : M \to P\left(\overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-])\right)$ $=: P$ such that $\xi\overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-]) \circ \nu' = \nu$. One easily proves that ν' defines a natural transformation $\bar{\eta}^s_{\mathrm{BrL}} : \mathrm{Id}_{\mathrm{BrLie}^s_{\mathcal{M}}} \to \mathcal{P}^s_{\mathrm{Br}}\, \overline{\mathcal{U}}^s_{\mathrm{Br}}$ such that $\xi\overline{\mathcal{U}}^s_{\mathrm{Br}} \circ \bar{\eta}^s_{\mathrm{BrL}} = \eta^s_{\mathrm{BrL}}$. Let us check there is a natural transformation $\bar{\epsilon}^s_{\mathrm{BrL}} : \overline{\mathcal{U}}^s_{\mathrm{Br}}\mathcal{P}^s_{\mathrm{Br}} \to \mathrm{Id}_{\mathrm{BrBialg}^s_{\mathcal{M}}}$ such that $\epsilon^s_{\mathrm{BrL}}\mho^s_{\mathrm{Br}} \circ \mathcal{U}^s_{\mathrm{Br}}\xi = \mho^s_{\mathrm{Br}}\bar{\epsilon}^s_{\mathrm{BrL}}$.

Let $\mathbb{B} := (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \in \mathrm{BrBialg}^s_{\mathcal{M}}$ and consider

$$\gamma := H\Omega_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{BrAlg}}(\epsilon^s_{\mathrm{BrL}}\mho^s_{\mathrm{Br}}\mathbb{B} \circ \mathcal{U}^s_{\mathrm{Br}}\xi\mathbb{B}) : R \to B$$

where $(R, m_R, u_R, \Delta_R, \varepsilon_R, c_R) := \overline{\mathcal{U}}^s_{\mathrm{Br}}\mathcal{P}^s_{\mathrm{Br}}\mathbb{B}$. By definition γ is a morphism of braided algebras and a direct computation shows that $\gamma \circ p_R = H\Omega_{\mathrm{Br}}\mho_{\mathrm{Br}}\bar{\epsilon}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{BrBialg}}\mathbb{B}$, using the equality $p_R = H\Omega_{\mathrm{Br}} I^s_{\mathrm{BrAlg}} p^s P^s_{\mathrm{Br}} B$ and the equalities (72), (22), (21), (32), (29). Since $\bar{\epsilon}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{BrBialg}}\mathbb{B}$ is a morphism of braided bialgebras and p_R is an epimorphism and a morphism of braided bialgebras, it is straightforward to prove that also γ is. Hence there is a unique morphism $\bar{\epsilon}^s_{\mathrm{BrL}}\mathbb{B} : \overline{\mathcal{U}}^s_{\mathrm{Br}}\mathcal{P}^s_{\mathrm{Br}}\mathbb{B} \to \mathbb{B}$ such that $H\Omega_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{BrAlg}}\bar{\epsilon}^s_{\mathrm{BrL}}\mathbb{B} = \gamma$. From the definition of γ and the fact that $H\Omega_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{BrAlg}}$ is faithful, we deduce $\epsilon^s_{\mathrm{BrL}}\mho^s_{\mathrm{Br}}\mathbb{B} \circ \mathcal{U}^s_{\mathrm{Br}}\xi\mathbb{B} = \mho^s_{\mathrm{Br}}\bar{\epsilon}^s_{\mathrm{BrL}}\mathbb{B}$. The naturality of the left-hand side of the latter equality and the faithfulness of $\mho^s_{\mathrm{Br}}$ yield the naturality of $\bar{\epsilon}^s_{\mathrm{BrL}}\mathbb{B}$. One easily checks that the $\bar{\eta}^s_{\mathrm{BrL}}$ and $\bar{\epsilon}^s_{\mathrm{BrL}}$ make $(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}})$ an adjunction. □

Next aim is to prove that, in the symmetric case, the functor $\mathcal{U}$ factors through a functor $\overline{\mathcal{U}} : \mathrm{Lie}_{\mathcal{M}} \to \mathrm{Bialg}_{\mathcal{M}}$ such that $\mho \circ \overline{\mathcal{U}} = \mathcal{U}$.

Theorem 6.4. *Let $\mathcal{M}$ an abelian symmetric monoidal category with denumerable coproducts. Assume that the tensor products are right exact and preserve denumerable coproducts. Then there are unique functors $\overline{\mathcal{U}}$ and $\mathcal{P}$ such that the following diagrams commute*

$$\begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Bialg}}} & \mathrm{BrBialg}^s_{\mathcal{M}} \\ {\scriptstyle \overline{\mathcal{U}}}\uparrow & & \uparrow{\scriptstyle \overline{\mathcal{U}}^s_{\mathrm{Br}}} \\ \mathrm{Lie}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Lie}}} & \mathrm{BrLie}^s_{\mathcal{M}} \end{array} \qquad \begin{array}{ccc} \mathrm{Bialg}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Bialg}}} & \mathrm{BrBialg}^s_{\mathcal{M}} \\ {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \mathcal{P}^s_{\mathrm{Br}}} \\ \mathrm{Lie}_{\mathcal{M}} & \xrightarrow{J^s_{\mathrm{Lie}}} & \mathrm{BrLie}^s_{\mathcal{M}} \end{array}$$

$$\begin{array}{ccc} \mathrm{Lie}_{\mathcal{M}} & \xrightarrow{\overline{\mathcal{U}}} & \mathrm{Bialg}_{\mathcal{M}} \\ & {\scriptstyle \mathcal{U}}\searrow \quad \swarrow{\scriptstyle \mho} & \\ & \mathrm{Alg}_{\mathcal{M}} & \end{array} \tag{78}$$

where $\mathcal{U}$ is the functor of Theorem 6.2. Moreover $\left(\overline{\mathcal{U}},\mathcal{P}\right)$ is an adjunction with unit $\bar{\eta}_{\mathrm{L}}:\mathrm{Id}_{\mathrm{Lie}_{\mathcal{M}}}\to\mathcal{P}\overline{\mathcal{U}}$ and counit $\bar{\epsilon}_{\mathrm{L}}:\overline{\mathcal{U}}\mathcal{P}\to\mathrm{Id}_{\mathrm{Bialg}_{\mathcal{M}}}$ uniquely determined by

$$J^s_{\mathrm{Lie}}\bar{\eta}_{\mathrm{L}}=\bar{\eta}^s_{\mathrm{BrL}}J^s_{\mathrm{Lie}} \qquad \text{and} \qquad J^s_{\mathrm{Bialg}}\bar{\epsilon}_{\mathrm{L}}=\bar{\epsilon}^s_{\mathrm{BrL}}J^s_{\mathrm{Bialg}}, \tag{79}$$

and $(J^s_{\mathrm{Bialg}},J^s_{\mathrm{Lie}}):(\overline{\mathcal{U}},\mathcal{P})\to(\overline{\mathcal{U}}^s_{\mathrm{Br}},\mathcal{P}^s_{\mathrm{Br}})$ is a commutation datum with canonical transformation given by the identity. Furthermore there is a natural transformation $\bar{p}:\overline{T}H_{\mathrm{Lie}}\to\overline{\mathcal{U}}$ such that

$$\bar{p}^sJ^s_{\mathrm{Lie}}=J^s_{\mathrm{Bialg}}\bar{p} \qquad \text{and} \qquad \mho_{\mathrm{Br}}J_{\mathrm{Bialg}}\bar{p}=pJ_{\mathrm{Lie}} \tag{80}$$

where $\bar{p}^s:\overline{T}^s_{\mathrm{Br}}H^s_{\mathrm{BrLie}}\to\overline{\mathcal{U}}^s_{\mathrm{Br}}$ is the natural transformation of Theorem 5.12 and $p:T_{\mathrm{Br}}H_{\mathrm{BrLie}}\to\mathcal{U}_{\mathrm{Br}}$ is the natural transformation of Proposition 5.9. The natural transformation $\xi:\mathcal{P}^s_{\mathrm{Br}}\to\mathcal{L}^s_{\mathrm{Br}}\mho^s_{\mathrm{Br}}$ induces a natural transformation $\xi:\mathcal{P}\to\mathcal{L}\mho$ such that $J^s_{\mathrm{Lie}}\xi=\xi J^s_{\mathrm{Bialg}}$.

Proof. The first part is a consequence of Lemma 2.9. The commutativity of the third diagram of (78) follows by (42), (78), (69) and (74). By Lemma 1.12, there is a natural transformation $\bar{p}:=\widehat{\bar{p}^sJ^s_{\mathrm{Lie}}}:\overline{T}H_{\mathrm{Lie}}\to\overline{\mathcal{U}}$ such that $J^s_{\mathrm{Bialg}}\bar{p}=\bar{p}^sJ^s_{\mathrm{Lie}}$. Using (41) (80), (70) and (55) we get $\mho_{\mathrm{Br}}J_{\mathrm{Bialg}}\bar{p}\overset{(55)}{=}pJ_{\mathrm{Lie}}$. By Lemma 1.12, there is a natural transformation $\xi:=\widehat{\xi J^s_{\mathrm{Bialg}}}:\mathcal{P}\to\mathcal{L}\mho$ such that $J^s_{\mathrm{Lie}}\xi=\xi J^s_{\mathrm{Bialg}}$. □

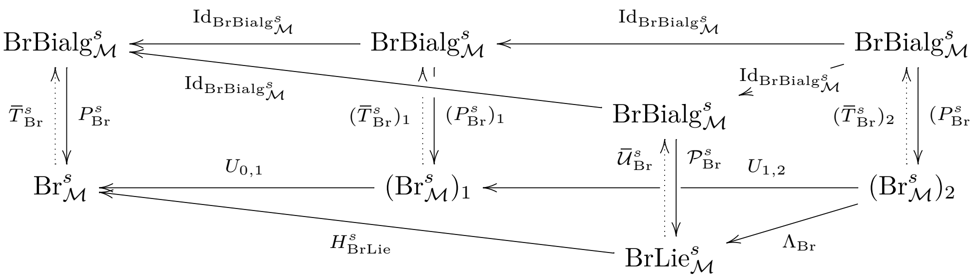
Remark 6.5. By Lemma 1.12, there is a natural transformation $\bar{q}:=\widehat{\bar{p}^sJ^s_{\mathrm{Lie}}}:\overline{T}H_{\mathrm{Lie}}\to\overline{\mathcal{U}}$ such that

$$J^s_{\mathrm{Bialg}}\bar{q}=\bar{p}^sJ^s_{\mathrm{Lie}}. \tag{81}$$

Using (81), (70) and (55) one checks that $\mho\bar{q}=H_{\mathrm{Alg}}pJ_{\mathrm{Lie}}$ where p is the morphism of Proposition 5.9. This means that for every $(M,[-])\in\mathrm{Lie}_{\mathcal{M}}$ the morphism $\bar{q}\,(M,[-])$ is really induced by the canonical projection $p_R:\Omega TM\to R:=\mathcal{U}^s_{\mathrm{Br}}J_{\mathrm{Lie}}\,(M,[-])$ defining in this lemma the universal enveloping algebra. Summing up, as a bialgebra in $\mathcal{M}$ we have that $\overline{\mathcal{U}}\,(M,[-])$ is a quotient of $\overline{T}H_{\mathrm{Lie}}\,(M,[-])=\overline{T}M$ via $\bar{q}\,(M,[-])$ and the underlying algebra structure is the original one underlying $\mathcal{U}^s_{\mathrm{Br}}J_{\mathrm{Lie}}\,(M,[-])$.

7. Stationary monadic decomposition

Theorem 7.1. *Let $\mathcal{M}$ be an abelian monoidal category with denumerable coproducts. Assume that the tensor products are exact and preserve denumerable coproducts.*



(82)

The functor P^s_{Br} is comparable so that we can use the notation of Definition 1.9. There is a functor $\Lambda_{\mathrm{Br}} : (\mathrm{Br}^s_{\mathcal{M}})_2 \to \mathrm{BrLie}^s_{\mathcal{M}}$ such that $\Lambda_{\mathrm{Br}} \circ (P^s_{\mathrm{Br}})_2 = \mathcal{P}^s_{\mathrm{Br}}$ and $H^s_{\mathrm{BrLie}} \circ \Lambda_{\mathrm{Br}} = U_{0,2}$. Moreover there exists a natural transformation $\overline{\chi}^s_{\mathrm{Br}} : \overline{\mathcal{U}}^s_{\mathrm{Br}} \Lambda_{\mathrm{Br}} \to (\overline{T}^s_{\mathrm{Br}})_1 U_{1,2}$ such that

$$\overline{\chi}^s_{\mathrm{Br}} \circ \bar{p}^s \Lambda_{\mathrm{Br}} = \pi^s_1 U_{1,2} \tag{83}$$

where $\bar{p}^s$ is the natural transformation of Theorem 5.12 and $\pi^s_1 : \overline{T}^s_{\mathrm{Br}} U_{0,1} \to (\overline{T}^s_{\mathrm{Br}})_1$ is the canonical natural transformation defining $(\overline{T}^s_{\mathrm{Br}})_1$.

Assume $\overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}$ is an isomorphism.

1) *The adjunction $\left(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}}\right)$ is idempotent.*
2) *The adjunction $\left((\overline{T}^s_{\mathrm{Br}})_1, (P^s_{\mathrm{Br}})_1\right)$ is idempotent, we can choose $(\overline{T}^s_{\mathrm{Br}})_2 := (\overline{T}^s_{\mathrm{Br}})_1 U_{1,2}$, $\pi^s_2 = \mathrm{Id}_{(\overline{T}^s_{\mathrm{Br}})_2}$ and $(\overline{T}^s_{\mathrm{Br}})_2$ is full and faithful i.e. $(\overline{\eta}^s_{\mathrm{Br}})_2$ is an isomorphism.*
3) *The functor P^s_{Br} has a monadic decomposition of monadic length at most two.*
4) *$(\mathrm{Id}_{\mathrm{BrBialg}^s_{\mathcal{M}}}, \Lambda_{\mathrm{Br}}) : ((\overline{T}^s_{\mathrm{Br}})_2, (P^s_{\mathrm{Br}})_2) \to (\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}})$ is a commutation datum whose canonical transformation is $\overline{\chi}^s_{\mathrm{Br}}$.*
5) *The pair $\left((P^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}}, \Lambda_{\mathrm{Br}}\right)$ is an adjunction with unit $\overline{\eta}^s_{\mathrm{BrL}}$ and counit $(\overline{\eta}^s_{\mathrm{Br}})_2^{-1} \circ (P^s_{\mathrm{Br}})_2 \overline{\chi}^s_{\mathrm{Br}}$ so that Λ_{Br} is full and faithful. Hence $\overline{\eta}^s_{\mathrm{BrL}}$ is an isomorphism if and only if $\left((P^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}}, \Lambda_{\mathrm{Br}}\right)$ is an equivalence of categories. In this case $\left((\overline{T}^s_{\mathrm{Br}})_2, (P^s_{\mathrm{Br}})_2\right)$ identifies with $\left(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}}\right)$ via Λ_{Br}.*

Proof. By 3.6 we have an adjunction $\left(\overline{T}^s_{\mathrm{Br}}, P^s_{\mathrm{Br}}\right)$. By Proposition B.11, the right adjoint functor $R = P^s_{\mathrm{Br}}$ is comparable and we can use the notation of Definition 1.9.

Let $M_2 = (M_1, \mu_1) \in (\mathrm{Br}^s_{\mathcal{M}})_2$. Then we can write $M_1 = (M_0, \mu_0) \in (\mathrm{Br}^s_{\mathcal{M}})_1$ and $M_0 = (M, c) \in \mathrm{Br}^s_{\mathcal{M}}$. Let $\theta_{(M,c)} := \theta_{\mathbb{I}_{\mathrm{Br}^s}(M,c)} : M \otimes M \to \Omega T(M)$ be defined as in (57) and set $\mathbb{A} := (A, m_A, u_A, \Delta_A, \varepsilon_A, c_A) := \overline{T}_{\mathrm{Br}} M_0 = \overline{T}_{\mathrm{Br}}(M, c)$. Since $c^2 = \mathrm{Id}_{M \otimes M}$

we have that $\theta_{(M,c)}$ fulfills (65). Thus there is a unique morphism $\bar{\theta}_{(M,c)} := \bar{\theta}_{\mathbb{I}^s_{\mathrm{Br}}(M,c)} : M \otimes M \to P\left(\overline{T}_{\mathrm{Br}}(M,c)\right)$ such that

$$\xi\mathbb{A} \circ \bar{\theta}_{(M,c)} = \theta_{(M,c)}. \tag{84}$$

Set

$$[-] := H\mathbb{I}^s_{\mathrm{Br}}\mu_0 \circ \bar{\theta}_{(M,c)} : M \otimes M \to M.$$

Let us check that $(M, c, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$. Now $\mu_1 \circ (\bar{\eta}^s_{\mathrm{Br}})_1 M_1 = \mathrm{Id}_{M_1}$ so that $(\bar{\eta}^s_{\mathrm{Br}})_1 M_1$ is a split monomorphism. Set $\mathbb{S} := (S, m_S, u_S, \Delta_S, \varepsilon_S, c_S) := \left(\overline{T}^s_{\mathrm{Br}}\right)_1 M_1$. Thus $H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1 : M \to P(\mathbb{S})$ is a split monomorphism too. Let $\pi^s_1 : \overline{T}^s_{\mathrm{Br}}U_{0,1} \to \left(\overline{T}^s_{\mathrm{Br}}\right)_1$ be the canonical natural transformation defining $\left(\overline{T}^s_{\mathrm{Br}}\right)_1$. By construction one has

$$P^s_{\mathrm{Br}}\pi^s_1 \circ \bar{\eta}^s_{\mathrm{Br}}U_{0,1} = U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1. \tag{85}$$

We have

$$\begin{aligned} &H\xi\overline{T}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}} \circ H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}} = H\left(\xi\overline{T}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}} \circ \mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}\right) \\ &\overset{(31)}{=} H\left(\xi\overline{T}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}} \circ \bar{\eta}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}\right) \overset{(28)}{=} H\eta_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}} \overset{(20)}{=} \eta H\mathbb{I}^s_{\mathrm{Br}}. \end{aligned} \tag{86}$$

In particular, we have

$$\begin{aligned} &\xi\mathbb{A} \circ H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 = \xi\mathbb{A} \circ H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 = H\xi\overline{T}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}M_0 \circ H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \\ &\overset{(86)}{=} \eta H\mathbb{I}^s_{\mathrm{Br}}M_0 = \eta M \overset{(4)}{=} \alpha_1 M \end{aligned}$$

so that

$$\xi\mathbb{A} \circ H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 = \alpha_1 M \tag{87}$$

We compute

$$\begin{aligned} &\xi\mathbb{A} \circ [-]_{P(\mathbb{A})} \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0) \\ &= [-]_A \circ (\xi\mathbb{A} \otimes \xi\mathbb{A}) \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0) \\ &= m_A \circ (\mathrm{Id}_{A\otimes A} - c_A) \circ (\xi\mathbb{A} \otimes \xi\mathbb{A}) \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0) \\ &\overset{(87)}{=} m_A \circ (\mathrm{Id}_{A\otimes A} - c_A) \circ (\alpha_1 M \otimes \alpha_1 M) = m_A \circ (\alpha_1 M \otimes \alpha_1 M) \circ \left(\mathrm{Id}_{M\otimes M} - c_A^{1,1}\right) \\ &\overset{(2)}{=} \alpha_2 M \circ (\mathrm{Id}_{M\otimes M} - c) \overset{(57)}{=} \theta_{(M,c)} \overset{(84)}{=} \xi\mathbb{A} \circ \bar{\theta}_{(M,c)}. \end{aligned}$$

Since $\xi\mathbb{A}$ is a monomorphism we get

$$\bar{\theta}_{(M,c)} = [-]_{P(\mathbb{A})} \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0). \tag{88}$$

Moreover since $\pi^s_1 M_1 : \mathbb{A} = \overline{T}^s_{\mathrm{Br}}U_{0,1}M_1 \to \left(\overline{T}^s_{\mathrm{Br}}\right)_1 M_1 = \mathbb{S}$ is a morphism in $\mathrm{BrBialg}^s_{\mathcal{M}}$, we have that $H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1 \overset{(76)}{=} H\mathbb{I}^s_{\mathrm{Br}}H^s_{\mathrm{BrLie}}\mathcal{P}^s_{\mathrm{Br}}\pi^s_1 M_1$ commutes with Lie brackets i.e.

$$[-]_{P(\mathbb{S})} \circ (H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1 \otimes H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1) = H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1 \circ [-]_{P(\mathbb{A})} \tag{89}$$

Hence we get

$$\begin{aligned}
&[-]_{P(\mathbb{S})} \circ (H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1 \otimes H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1)\\
&\overset{(85)}{=} [-]_{P(\mathbb{S})} \circ (H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1 \otimes H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1) \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0)\\
&\overset{(89)}{=} H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1 \circ [-]_{P(\mathbb{A})} \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0)\\
&\overset{(88)}{=} H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1 \circ \bar{\theta}_{(M,c)} \overset{(*)}{=} H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\pi^s_1 M_1 \circ H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}U_{0,1}M_1 \circ H\mathbb{I}^s_{\mathrm{Br}}\mu_0 \circ \bar{\theta}_{(M,c)}\\
&\overset{(85)}{=} H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1 \circ [-]
\end{aligned}$$

where in $(*)$ we used that $P^s_{\mathrm{Br}}\pi^s_1 M_1 \circ \bar{\eta}^s_{\mathrm{Br}}U_{0,1}M_1 \circ \mu_0 = P^s_{\mathrm{Br}}\pi^s_1 M_1$ which follows from $\pi^s_1 M_1 \circ \overline{T}^s_{\mathrm{Br}}\mu_0 = \pi^s_1 M_1 \circ \bar{\epsilon}^s_{\mathrm{Br}}\overline{T}^s_{\mathrm{Br}}M_0$ (true by definition of π_1) and [4, Lemma 3.3]. We have so proved

$$[-]_{P(\mathbb{S})} \circ (H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1 \otimes H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1) = H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1 \circ [-]. \tag{90}$$

Using the fact that $H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1$ is a monomorphism in $\mathcal{M}$ and

$$\left(P(\mathbb{S}), c_{P(\mathbb{S})}, [-]_{P(\mathbb{S})}\right) = \mathcal{P}^s_{\mathrm{Br}}\mathbb{S} \in \mathrm{BrLie}^s_{\mathcal{M}},$$

one easily checks that $\Lambda_{\mathrm{Br}}(M_2) := (M, c, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$ and that $H\mathbb{I}^s_{\mathrm{Br}}U_{0,1}(\bar{\eta}^s_{\mathrm{Br}})_1 M_1 : M \to P(\mathbb{S})$ is a morphism in $\mathrm{BrLie}^s_{\mathcal{M}}$. Let $\nu : M_2 \to M_2'$ be a morphism in $(\mathrm{Br}^s_{\mathcal{M}})_2$. It is clearly a morphism of braided objects. Since, by (76), we have $H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}} = H\mathbb{I}^s_{\mathrm{Br}}H^s_{\mathrm{BrLie}}\mathcal{P}^s_{\mathrm{Br}}$, then $H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\overline{T}^s_{\mathrm{Br}}U_{0,2}\nu$ commutes with Lie brackets and hence

$$\begin{aligned}
&\bar{\theta}_{(M',c')} \circ (H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu \otimes H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu)\\
&\overset{(88)}{=} [-]_{P(\mathbb{A}')} \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0' \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0') \circ (H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu \otimes H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu)\\
&= [-]_{P(\mathbb{A}')} \circ \left(H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\overline{T}^s_{\mathrm{Br}}U_{0,2}\nu \otimes H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\overline{T}^s_{\mathrm{Br}}U_{0,2}\nu\right) \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0)\\
&= H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\overline{T}^s_{\mathrm{Br}}U_{0,2}\nu \circ [-]_{P(\mathbb{A})} \circ (H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0 \otimes H\mathbb{I}^s_{\mathrm{Br}}\bar{\eta}^s_{\mathrm{Br}}M_0)\\
&\overset{(88)}{=} H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\overline{T}^s_{\mathrm{Br}}U_{0,2}\nu \circ \bar{\theta}_{(M,c)}
\end{aligned}$$

so that $\bar{\theta}_{(M',c')} \circ (H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu \otimes H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu) = HP_{\mathrm{Br}}\overline{T}_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu \circ \bar{\theta}_{(M,c)}$. Using the latter equality, (30) and that ν is a morphism in $(\mathrm{Br}^s_{\mathcal{M}})_2$ we obtain that $[-]' \circ (H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu \otimes H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu) = H\mathbb{I}^s_{\mathrm{Br}}U_{0,2}\nu \circ [-]$. Thus ν induces a morphism $\Lambda_{\mathrm{Br}}\upsilon \in \mathrm{BrLie}^s_{\mathcal{M}}$. It is clear that this defines a functor $\Lambda_{\mathrm{Br}} : (\mathrm{Br}^s_{\mathcal{M}})_2 \to \mathrm{BrLie}^s_{\mathcal{M}}$ acting as the identity on morphisms. Let $\mathbb{B} := (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \in \mathrm{BrBialg}^s_{\mathcal{M}}$. Set $M_2 := (P^s_{\mathrm{Br}})_2\,\mathbb{B}$. Then

$$\begin{aligned}(M_1,\mu_1) :&= M_2 = ((P^s_{\mathrm{Br}})_1\,\mathbb{B}, (P^s_{\mathrm{Br}})_1\,(\bar{\varepsilon}^s_{\mathrm{Br}})_1\,\mathbb{B}),\\ (M_0,\mu_0) :&= M_1 = (P^s_{\mathrm{Br}})_1\,\mathbb{B} = (P^s_{\mathrm{Br}}\mathbb{B}, P^s_{\mathrm{Br}}\bar{\varepsilon}^s_{\mathrm{Br}}\mathbb{B}),\\ (M,c) :&= M_0 = P^s_{\mathrm{Br}}\mathbb{B}\end{aligned}$$

The bracket for this specific M is

$$[-] := H\mathbb{I}^s_{\mathrm{Br}}\mu_0 \circ \bar{\theta}_{(M,c)} = H\mathbb{I}^s_{\mathrm{Br}}P^s_{\mathrm{Br}}\bar{\varepsilon}^s_{\mathrm{Br}}\mathbb{B} \circ \bar{\theta}_{P^s_{\mathrm{Br}}\mathbb{B}}.$$

It is straightforward to prove that $\xi\mathbb{B} \circ [-] = [-]_B \circ (\xi\mathbb{B} \otimes \xi\mathbb{B}) = \xi\mathbb{B} \circ [-]_{P(\mathbb{B})}$ so that $[-] = [-]_{P(\mathbb{B})}$ and hence

$$\Lambda_{\mathrm{Br}}\,(P^s_{\mathrm{Br}})_2\,\mathbb{B} = \left(P^s_{\mathrm{Br}}\mathbb{B}, [-]_{P(\mathbb{B})}\right) = \mathcal{P}^s_{\mathrm{Br}}\mathbb{B}.$$

It is clear that the functors $\Lambda_{\mathrm{Br}}\,(P^s_{\mathrm{Br}})_2$ and $\mathcal{P}^s_{\mathrm{Br}}$ coincide also on morphisms so that we obtain $\Lambda_{\mathrm{Br}} \circ (P^s_{\mathrm{Br}})_2 = \mathcal{P}^s_{\mathrm{Br}}$. Let $M_2 \in (\mathrm{Br}^s_{\mathcal{M}})_2$. Then

$$H^s_{\mathrm{BrLie}}\Lambda_{\mathrm{Br}}M_2 = H^s_{\mathrm{BrLie}}\,(M, c, [-]) = (M, c) = U_{0,2}M_2.$$

Since H^s_{BrLie}, Λ_{Br} and $U_{0,2}$ act as the identity on morphisms, we get $H^s_{\mathrm{BrLie}} \circ \Lambda_{\mathrm{Br}} = U_{0,2}$.

In view (87), naturality of ξ, the equality $(*)$ used above and (84) we obtain

$$H\mathbb{I}^s_{\mathrm{Br}}\Omega^s_{\mathrm{Br}}\mho^s_{\mathrm{Br}}\pi^s_1M_1 \circ \alpha_1 M \circ [-] = H\mathbb{I}^s_{\mathrm{Br}}\Omega^s_{\mathrm{Br}}\mho^s_{\mathrm{Br}}\pi^s_1M_1 \circ \theta_{(M,c)}.$$

Thus we get $H\mathbb{I}^s_{\mathrm{Br}}\Omega^s_{\mathrm{Br}}\mho^s_{\mathrm{Br}}\pi^s_1M_1 \circ f_{\Lambda_{\mathrm{Br}}M_2} = 0$.

Since $\pi^s_1M_1$ is an algebra map, we have $H\mathbb{I}^s_{\mathrm{Br}}\Omega^s_{\mathrm{Br}}\mho^s_{\mathrm{Br}}\pi^s_1M_1 \circ i_{f_{\Lambda_{\mathrm{Br}}M_2}} = 0$ so that, by construction of $\mathcal{U}^s_{\mathrm{Br}}$ there is a braided algebra morphism $\chi^s_{\mathrm{Br}}M_2 : \mathcal{U}^s_{\mathrm{Br}}\Lambda_{\mathrm{Br}}M_2 \to \mho^s_{\mathrm{Br}}\left(\overline{T}^s_{\mathrm{Br}}\right)_1 M_1$ such

$$\chi^s_{\mathrm{Br}}M_2 \circ p^s\Lambda_{\mathrm{Br}}M_2 = \mho^s_{\mathrm{Br}}\pi^s_1M_1 = \mho^s_{\mathrm{Br}}\pi^s_1U_{1,2}M_2.$$

By naturality of the other terms we obtain that also $\chi^s_{\mathrm{Br}}M_2$ is natural in M_2 so that we get

$$\chi^s_{\mathrm{Br}} \circ p^s\Lambda_{\mathrm{Br}} = \mho^s_{\mathrm{Br}}\pi^s_1U_{1,2}.$$

By (70) we get $\chi^s_{\mathrm{Br}} \circ \mho^s_{\mathrm{Br}} \bar{p}^s \Lambda_{\mathrm{Br}} = \mho^s_{\mathrm{Br}} \pi^s_1 U_{1,2}$. Since both $\bar{p}^s \Lambda_{\mathrm{Br}}$ and $\pi^s_1 U_{1,2}$ are morphism of braided bialgebras and the underlying morphism in $\mathcal{M}$ of $\bar{p}^s$ is p which is an epimorphism, one gets that χ^s_{Br} is a morphism of braided bialgebras too that will be denoted by $\overline{\chi}^s_{\mathrm{Br}} : \overline{\mathcal{U}}^s_{\mathrm{Br}} \Lambda_{\mathrm{Br}} \to \left(\overline{T}^s_{\mathrm{Br}}\right)_1 U_{1,2}$. Thus $\mho^s_{\mathrm{Br}} \overline{\chi}^s_{\mathrm{Br}} = \chi^s_{\mathrm{Br}}$ and hence

$$\overline{\chi}^s_{\mathrm{Br}} \circ \bar{p}^s \Lambda_{\mathrm{Br}} = \pi^s_1 U_{1,2}. \tag{91}$$

A direct computation, shows that $\mathbb{I}^s_{\mathrm{Br}} \eta^s_{\mathrm{Br}} = \mathbb{I}^s_{\mathrm{Br}} \xi \overline{T}^s_{\mathrm{Br}} \circ \mathbb{I}^s_{\mathrm{Br}} \overline{\eta}^s_{\mathrm{Br}}$ and hence

$$\eta^s_{\mathrm{Br}} = \xi \overline{T}^s_{\mathrm{Br}} \circ \overline{\eta}^s_{\mathrm{Br}}.$$

Thus, using (76), naturality of ξ, (77), (72), (70), (91), again naturality of ξ and (85) in the given order, we get

$$\xi \left(\overline{T}^s_{\mathrm{Br}}\right)_1 U_{1,2} \circ H^s_{\mathrm{BrLie}} \left(\mathcal{P}^s_{\mathrm{Br}} \overline{\chi}^s_{\mathrm{Br}} \circ \overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}\right) = \xi \left(\overline{T}^s_{\mathrm{Br}}\right)_1 U_{1,2} \circ U_{0,1} \left(\overline{\eta}^s_{\mathrm{Br}}\right)_1 U_{1,2}.$$

Therefore, we obtain

$$H^s_{\mathrm{BrLie}} \left(\mathcal{P}^s_{\mathrm{Br}} \overline{\chi}^s_{\mathrm{Br}} \circ \overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}\right) = U_{0,1} \left(\overline{\eta}^s_{\mathrm{Br}}\right)_1 U_{1,2}. \tag{92}$$

The latter is a split monomorphism. Since H^s_{BrLie} is faithful, we get that the evaluation on objects of $\mathcal{P}^s_{\mathrm{Br}} \overline{\chi}^s_{\mathrm{Br}} \circ \overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}$ is a monomorphism.

Assume that $\overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}$ is an isomorphism. Note that $\overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}$ isomorphism implies $\overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}} \left(P^s_{\mathrm{Br}}\right)_2$ isomorphism. Since $\mathcal{P}^s_{\mathrm{Br}} = \Lambda_{\mathrm{Br}} \left(P^s_{\mathrm{Br}}\right)_2$ this means that $\overline{\eta}^s_{\mathrm{BrL}} \mathcal{P}^s_{\mathrm{Br}}$ is an isomorphism and hence the adjunction $\left(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}}\right)$ is idempotent, cf. [34, Proposition 2.8]. Moreover, since $\overline{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}$ is an isomorphism, then the evaluation of $\mathcal{P}^s_{\mathrm{Br}} \overline{\chi}^s_{\mathrm{Br}} : \mathcal{P}^s_{\mathrm{Br}} \overline{\mathcal{U}}^s_{\mathrm{Br}} \Lambda_{\mathrm{Br}} \to \mathcal{P}^s_{\mathrm{Br}} \left(\overline{T}^s_{\mathrm{Br}}\right)_1 U_{1,2}$ is a monomorphism. Let $M_2 \in (\mathrm{Br}^s_{\mathcal{M}})_2$ and consider the coequalizer

$$\overline{T}^s_{\mathrm{Br}} P^s_{\mathrm{Br}} \overline{T}^s_{\mathrm{Br}} M_0 \underset{\bar{\epsilon}^s_{\mathrm{Br}} \overline{T}^s_{\mathrm{Br}} M_0}{\overset{\overline{T}^s_{\mathrm{Br}} \mu_0}{\rightrightarrows}} \overline{T}^s_{\mathrm{Br}} M_0 \xrightarrow{\pi^s_1 M_1} (\overline{T}^s_{\mathrm{Br}})_1 M_1$$

Then, from $\overline{\chi}^s_{\mathrm{Br}} \circ \bar{p}^s \Lambda_{\mathrm{Br}} = \pi^s_1 U_{1,2}$, we get $\overline{\chi}^s_{\mathrm{Br}} M_2 \circ \bar{p}^s \Lambda_{\mathrm{Br}} M_2 \circ \overline{T}^s_{\mathrm{Br}} \mu_0 = \overline{\chi}^s_{\mathrm{Br}} M_2 \circ \bar{p}^s \Lambda_{\mathrm{Br}} M_2 \circ \bar{\epsilon}^s_{\mathrm{Br}} \overline{T}^s_{\mathrm{Br}} M_0$. If we apply $\mathcal{P}^s_{\mathrm{Br}}$, from the fact that $\mathcal{P}^s_{\mathrm{Br}} \overline{\chi}^s_{\mathrm{Br}} M_2$ is a monomorphism, we obtain

$$\mathcal{P}^s_{\mathrm{Br}} \left(\bar{p}^s \Lambda_{\mathrm{Br}} M_2 \circ \overline{T}^s_{\mathrm{Br}} \mu_0\right) = \mathcal{P}^s_{\mathrm{Br}} \left(\bar{p}^s \Lambda_{\mathrm{Br}} M_2 \circ \bar{\epsilon}^s_{\mathrm{Br}} \overline{T}^s_{\mathrm{Br}} M_0\right).$$

If we apply on both sides H^s_{BrLie}, by (76), we obtain

$$P^s_{\mathrm{Br}} \left(\bar{p}^s \Lambda_{\mathrm{Br}} M_2 \circ \overline{T}^s_{\mathrm{Br}} \mu_0\right) = P^s_{\mathrm{Br}} \left(\bar{p}^s \Lambda_{\mathrm{Br}} M_2 \circ \bar{\epsilon}^s_{\mathrm{Br}} \overline{T}^s_{\mathrm{Br}} M_0\right).$$

Since $\left(\overline{\mathcal{U}}_{\mathrm{Br}}^{s},\mathcal{P}_{\mathrm{Br}}^{s}\right)$ is idempotent, by [34, Proposition 2.8], we also have that $\bar{\epsilon}_{\mathrm{BrL}}^{s}\overline{\mathcal{U}}_{\mathrm{Br}}^{s}$ is an isomorphism. Note that the arguments of P_{Br}^{s} in the above displayed equality are morphisms of the form $\overline{T}_{\mathrm{Br}}^{s}X\rightarrow Y$ for some objects X, Y. Given two such morphisms $f,\,g:\overline{T}_{\mathrm{Br}}^{s}X\rightarrow Y$ with $P_{\mathrm{Br}}^{s}f=P_{\mathrm{Br}}^{s}g$ we have

$$\begin{aligned} f &= f\circ\bar{\epsilon}_{\mathrm{Br}}^{s}\overline{T}_{\mathrm{Br}}^{s}X\circ\overline{T}_{\mathrm{Br}}^{s}\bar{\eta}_{\mathrm{Br}}^{s}X=\bar{\epsilon}_{\mathrm{Br}}^{s}Y\circ\overline{T}_{\mathrm{Br}}^{s}P_{\mathrm{Br}}^{s}f\circ\overline{T}_{\mathrm{Br}}^{s}\bar{\eta}_{\mathrm{Br}}^{s}X\\ &=\bar{\epsilon}_{\mathrm{Br}}^{s}Y\circ\overline{T}_{\mathrm{Br}}^{s}P_{\mathrm{Br}}^{s}g\circ\overline{T}_{\mathrm{Br}}^{s}\bar{\eta}_{\mathrm{Br}}^{s}X=g\circ\bar{\epsilon}_{\mathrm{Br}}^{s}\overline{T}_{\mathrm{Br}}^{s}X\circ\overline{T}_{\mathrm{Br}}^{s}\bar{\eta}_{\mathrm{Br}}^{s}X=g.\end{aligned}$$

In our case we get $\bar{p}^{s}\Lambda_{\mathrm{Br}}M_{2}\circ\overline{T}_{\mathrm{Br}}^{s}\mu_{0}=\bar{p}^{s}\Lambda_{\mathrm{Br}}M_{2}\circ\bar{\epsilon}_{\mathrm{Br}}^{s}\overline{T}_{\mathrm{Br}}^{s}M_{0}$. By the universal property of the coequalizer above, there is a braided bialgebra morphism $\tau M_{2}:\left(\overline{T}_{\mathrm{Br}}^{s}\right)_{1}M_{1}\rightarrow\overline{\mathcal{U}}_{\mathrm{Br}}^{s}\Lambda_{\mathrm{Br}}M_{2}$ such that

$$\tau M_{2}\circ\pi_{1}^{s}M_{1}=\bar{p}^{s}\Lambda_{\mathrm{Br}}M_{2}.$$

Note that, by Proposition B.11, the morphism $\pi_{1}^{s}M_{1}$ can be chosen in such a way to be a coequalizer when regarded as a morphism in $\mathcal{M}$. We already observed that $\bar{p}^{s}$ is also an epimorphism in $\mathcal{M}$. Using these facts one easily checks that $\overline{\chi}_{\mathrm{Br}}^{s}M_{2}$ and τM_{2} are mutual inverses and hence $\overline{\chi}_{\mathrm{Br}}^{s}:\overline{\mathcal{U}}_{\mathrm{Br}}^{s}\Lambda_{\mathrm{Br}}\rightarrow\left(\overline{T}_{\mathrm{Br}}^{s}\right)_{1}U_{1,2}$ is an isomorphism.

Therefore $U_{0,1}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{1}U_{1,2}=H_{\mathrm{BrLie}}^{s}\left(\mathcal{P}_{\mathrm{Br}}^{s}\overline{\chi}_{\mathrm{Br}}^{s}\circ\bar{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}\right)$ is an isomorphism. Since $U_{0,1}$ reflects isomorphisms, we conclude that $\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{1}U_{1,2}$ is an isomorphism. We have so proved that the adjunction $\left(\left(\overline{T}_{\mathrm{Br}}^{s}\right)_{1},\left(P_{\mathrm{Br}}^{s}\right)_{1}\right)$ is idempotent. Note that in this case we can choose $\left(\overline{T}_{\mathrm{Br}}^{s}\right)_{2}:=\left(\overline{T}_{\mathrm{Br}}^{s}\right)_{1}U_{1,2}$ (and π_{2}^{s} to be the identity) and it is full and faithful (cf. [4, Proposition 2.3]) i.e. $\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}$ is an isomorphism. By the quoted result we also have $\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{1}U_{1,2}=U_{1,2}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}$ so that

$$H_{\mathrm{BrLie}}^{s}\left(\mathcal{P}_{\mathrm{Br}}^{s}\overline{\chi}_{\mathrm{Br}}^{s}\circ\bar{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}\right)\overset{(92)}{=}U_{0,1}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{1}U_{1,2}=U_{0,1}U_{1,2}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}=H_{\mathrm{BrLie}}^{s}\Lambda_{\mathrm{Br}}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}$$

and hence $\mathcal{P}_{\mathrm{Br}}^{s}\overline{\chi}_{\mathrm{Br}}^{s}\circ\bar{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}=\Lambda_{\mathrm{Br}}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}$. This proves (7) holds i.e. that $(\mathrm{Id}_{\mathrm{BrBialg}_{\mathcal{M}}^{s}},\Lambda_{\mathrm{Br}}):(\left(\overline{T}_{\mathrm{Br}}^{s}\right)_{2},\left(P_{\mathrm{Br}}^{s}\right)_{2})\rightarrow(\overline{\mathcal{U}}_{\mathrm{Br}}^{s},\mathcal{P}_{\mathrm{Br}}^{s})$ is a commutation datum whose canonical transformation is $\overline{\chi}_{\mathrm{Br}}^{s}$. Let us check that $\left(\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\mathcal{U}}_{\mathrm{Br}}^{s},\Lambda_{\mathrm{Br}}\right)$ is an adjunction with unit and counit as in the statement. We have

$$\begin{aligned}&\Lambda_{\mathrm{Br}}\left(\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}^{-1}\circ\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\chi}_{\mathrm{Br}}^{s}\right)\circ\bar{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}=\Lambda_{\mathrm{Br}}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}^{-1}\circ\Lambda_{\mathrm{Br}}\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\chi}_{\mathrm{Br}}^{s}\circ\bar{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}\\ &\quad=\Lambda_{\mathrm{Br}}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}^{-1}\circ\mathcal{P}_{\mathrm{Br}}^{s}\overline{\chi}_{\mathrm{Br}}^{s}\circ\bar{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}=\Lambda_{\mathrm{Br}}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}^{-1}\circ\Lambda_{\mathrm{Br}}\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}=\Lambda_{\mathrm{Br}}.\end{aligned}$$

Moreover, by (9) applied to our commutation datum, we have $\left(\bar{\epsilon}_{\mathrm{Br}}^{s}\right)_{2}\circ\overline{\chi}_{\mathrm{Br}}^{s}\left(P_{\mathrm{Br}}^{s}\right)_{2}=\bar{\epsilon}_{\mathrm{BrL}}^{s}$ so that

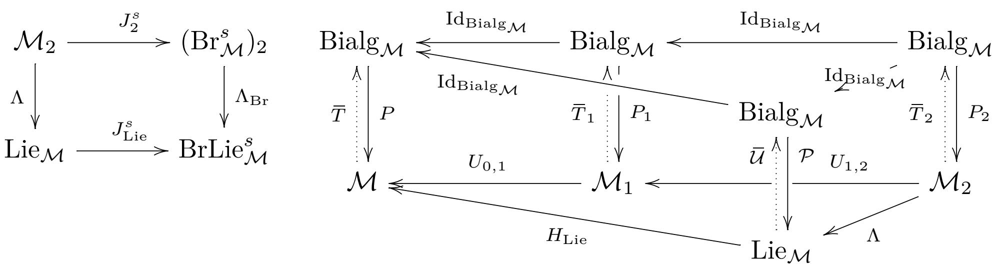
$$\begin{aligned}&\left(\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}^{-1}\circ\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\chi}_{\mathrm{Br}}^{s}\right)\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\mathcal{U}}_{\mathrm{Br}}^{s}\circ\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\mathcal{U}}_{\mathrm{Br}}^{s}\bar{\eta}_{\mathrm{BrL}}^{s}\\ &=\left(\bar{\eta}_{\mathrm{Br}}^{s}\right)_{2}^{-1}\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\mathcal{U}}_{\mathrm{Br}}^{s}\circ\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\chi}_{\mathrm{Br}}^{s}\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\mathcal{U}}_{\mathrm{Br}}^{s}\circ\left(P_{\mathrm{Br}}^{s}\right)_{2}\overline{\mathcal{U}}_{\mathrm{Br}}^{s}\bar{\eta}_{\mathrm{BrL}}^{s}\end{aligned}$$

$$
\begin{aligned}
&= (P^s_{\mathrm{Br}})_2 \, (\bar{\epsilon}^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}} \circ (P^s_{\mathrm{Br}})_2 \, \overline{\chi}^s_{\mathrm{Br}} \, (P^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}} \circ (P^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}} \bar{\eta}^s_{\mathrm{BrL}} \\
&= (P^s_{\mathrm{Br}})_2 \left[(\bar{\epsilon}^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}} \circ \overline{\chi}^s_{\mathrm{Br}} \, (P^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}} \circ \overline{\mathcal{U}}^s_{\mathrm{Br}} \bar{\eta}^s_{\mathrm{BrL}} \right] \\
&= (P^s_{\mathrm{Br}})_2 \left[\bar{\epsilon}^s_{\mathrm{BrL}} \overline{\mathcal{U}}^s_{\mathrm{Br}} \circ \overline{\mathcal{U}}^s_{\mathrm{Br}} \bar{\eta}^s_{\mathrm{BrL}} \right] = (P^s_{\mathrm{Br}})_2 \, .
\end{aligned}
$$

Note that the counit is an isomorphism so that Λ_{Br} is full and faithful.

It is then clear that $\left((P^s_{\mathrm{Br}})_2 \overline{\mathcal{U}}^s_{\mathrm{Br}}, \Lambda_{\mathrm{Br}} \right)$ is an equivalence of categories if and only if $\bar{\eta}^s_{\mathrm{BrL}}$ is an isomorphism (see e.g. [11, Proposition 3.4.3]). □

Theorem 7.2. *Let $\mathcal{M}$ be an abelian symmetric monoidal category with denumerable coproducts. Assume that the tensor products are exact and preserve denumerable coproducts.*



(93)

The functor P is comparable so that we can use the notation of Definition 1.9. We have $H_{\mathrm{Lie}}\mathcal{P} = P$ and there is a functor $\Lambda : \mathcal{M}_2 \to \mathrm{Lie}_{\mathcal{M}}$ such that $\Lambda_{\mathrm{Br}} J^s_2 = J^s_{\mathrm{Lie}} \Lambda$, $\Lambda \circ P_2 = \mathcal{P}$ and $H_{\mathrm{Lie}} \circ \Lambda = U_{0,2}$. Moreover there exists a natural transformation $\overline{\chi} : \overline{\mathcal{U}}\Lambda \to \overline{T}_1 U_{1,2}$ such that

$$
J^s_{\mathrm{Bialg}} \overline{\chi} = \zeta^s_1 U_{1,2} \circ \overline{\chi}^s_{\mathrm{Br}} J^s_2, \qquad \overline{\chi} \circ \bar{p}\Lambda = \pi_1 U_{1,2}
$$

where $\bar{p}$ is the natural transformation of Theorem 6.4 and $\pi_1 : \overline{T} U_{0,1} \to \overline{T}_1$ is the canonical natural transformation defining $\overline{T}_1$.

Assume $\bar{\eta}^s_{\mathrm{BrL}} \Lambda_{\mathrm{Br}}$ is an isomorphism.

1) *The adjunction $(\overline{\mathcal{U}}, \mathcal{P})$ is idempotent.*
2) *The adjunction $\left(\overline{T}_1, P_1\right)$ is idempotent, we can choose $\overline{T}_2 := \overline{T}_1 U_{1,2}$, $\pi_2 = \mathrm{Id}_{\overline{T}_2}$ and $\overline{T}_2$ is full and faithful i.e. $\bar{\eta}_2$ is an isomorphism.*
3) *The functor P has a monadic decomposition of monadic length at most two.*
4) *$(\mathrm{Id}_{\mathrm{Bialg}_{\mathcal{M}}}, \Lambda) : (\overline{T}_2, P_2) \to (\overline{\mathcal{U}}, \mathcal{P})$ is a commutation datum whose canonical transformation is $\overline{\chi}$.*
5) *The pair $\left(P_2 \overline{\mathcal{U}}, \Lambda\right)$ is an adjunction with unit $\bar{\eta}_{\mathrm{L}}$ and counit $(\bar{\eta}_2)^{-1} \circ P_2 \overline{\chi}$ so that Λ is full and faithful. Hence $\bar{\eta}_{\mathrm{L}}$ is an isomorphism if and only if $\left(P_2 \overline{\mathcal{U}}, \Lambda\right)$ is an equivalence of categories. In this case $\left(\overline{T}_2, P_2\right)$ identifies with $\left(\overline{\mathcal{U}}, \mathcal{P}\right)$ via Λ.*
6) *If $\bar{\eta}^s_{\mathrm{BrL}}$ is an isomorphism so is $\bar{\eta}_{\mathrm{L}}$.*

Proof. We have

$$J^s H_{\mathrm{Lie}}\mathcal{P} \overset{(55)}{=} H^s_{\mathrm{BrLie}} J^s_{\mathrm{Lie}}\mathcal{P} \overset{(78)}{=} H^s_{\mathrm{BrLie}}\mathcal{P}^s_{\mathrm{Br}} J^s_{\mathrm{Bialg}} \overset{(76)}{=} P^s_{\mathrm{Br}} J^s_{\mathrm{Bialg}} \overset{(49)}{=} J^s P$$

so that $H_{\mathrm{Lie}}\mathcal{P} = P$. By Proposition 4.7, $\left(J^s_{\mathrm{Bialg}}, J^s\right) : \left(\overline{T}, P\right) \to \left(\overline{T}^s_{\mathrm{Br}}, P^s_{\mathrm{Br}}\right)$ is a commutation datum. Moreover, by Lemma A.5, $J^s_{\mathrm{Bialg}} : \mathrm{Bialg}_{\mathcal{M}} \to \mathrm{BrBialg}^s_{\mathcal{M}}$ preserves coequalizers. By Proposition B.11, the right adjoint functor $R = P^s_{\mathrm{Br}}$ is comparable and we can use the notation of Definition 1.9. By Lemma A.4 and Lemma 1.11 we have that P is also comparable. Applying iteratively Proposition 2.5, we get functors $J^s_n : \mathcal{M}_n \to (\mathrm{Br}^s_{\mathcal{M}})_n$, for all $n \in \mathbb{N}$, such that $J^s_n \circ P_n = (P^s_{\mathrm{Br}})_n \circ J^s_{\mathrm{Bialg}}$. Let $M_2 \in \mathcal{M}_2$ and consider $\Lambda_{\mathrm{Br}} J^s_2 M_2$. Note that, by construction we have

$$J^s_2 M_2 = (J^s_1 M_1, J^s_1 \mu_1 \circ (P^s_{\mathrm{Br}})_1 \zeta^s_1 M_1) \qquad \text{and} \qquad J^s_1 M_1 = (J^s M_0, J^s \mu_0 \circ P^s_{\mathrm{Br}} \zeta^s_0 M_0)$$

where $\zeta^s_i : (\overline{T}^s_{\mathrm{Br}})_i J^s_i \to J^s_{\mathrm{Bialg}} \overline{T}_i$ for $i = 0, 1$ are the canonical transformations of the respective commutation data. By construction we have $\Lambda_{\mathrm{Br}} J^s_2 M_2 = (M_0, c_{M_0,M_0}, [-])$ where

$$[-] := H\mathbb{I}^s_{\mathrm{Br}} J^s \mu_0 \circ H\mathbb{I}^s_{\mathrm{Br}} P^s_{\mathrm{Br}} \zeta^s_0 M_0 \circ \bar{\theta}_{(M_0, c_{M_0,M_0})} = \mu_0 \circ H\mathbb{I}^s_{\mathrm{Br}} P^s_{\mathrm{Br}} \zeta^s_0 M_0 \circ \bar{\theta}_{M_0}.$$

Now $\Lambda_{\mathrm{Br}} J^s_2 M_2 \in \mathrm{BrLie}^s_{\mathcal{M}}$ so that $(M_0, c_{M_0,M_0}, [-]) \in \mathrm{BrLie}_{\mathcal{M}}$ i.e. $(M_0, [-]) \in \mathrm{Lie}_{\mathcal{M}}$ and $\Lambda_{\mathrm{Br}} J^s_2 M_2 = J^s_{\mathrm{Lie}}(M_0, [-])$. Thus $\mathrm{Im}(\Lambda_{\mathrm{Br}} J^s_2) \subseteq \mathrm{Im}(J^s_{\mathrm{Lie}})$. Hence, by Lemma 1.12, there is a unique functor $\Lambda : \mathcal{M}_2 \to \mathrm{Lie}_{\mathcal{M}}$ such that $\Lambda_{\mathrm{Br}} J^s_2 = J^s_{\mathrm{Lie}}\Lambda$. This equality implies that Λ acts as the identity on morphisms and that

$$\Lambda M_2 = (M_0, [-]).$$

Note that, by Proposition 4.7, we have $\zeta^s_0 = \mathrm{Id}_{\overline{T}^s_{\mathrm{Br}} J^s}$ so that we obtain

$$[-] := \mu_0 \circ \bar{\theta}_{M_0}.$$

We have

$$J^s_{\mathrm{Lie}} \Lambda P_2 = \Lambda_{\mathrm{Br}} J^s_2 P_2 = \Lambda_{\mathrm{Br}} (P^s_{\mathrm{Br}})_2 J^s_{\mathrm{Bialg}} \overset{(82)}{=} \mathcal{P}^s_{\mathrm{Br}} J^s_{\mathrm{Bialg}} \overset{(78)}{=} J^s_{\mathrm{Lie}}\mathcal{P}.$$

Since J^s_{Lie} is both injective on morphisms and objects, we get $\Lambda P_2 = \mathcal{P}$. It is clear that $H_{\mathrm{Lie}}\Lambda = U_{0,2}$. We have

$$J^s_{\mathrm{Bialg}} \overline{\mathcal{U}} \Lambda \overset{(78)}{=} \overline{\mathcal{U}}^s_{\mathrm{Br}} J^s_{\mathrm{Lie}} \Lambda = \overline{\mathcal{U}}^s_{\mathrm{Br}} \Lambda_{\mathrm{Br}} J^s_2$$

so that $\widehat{\overline{\mathcal{U}}^s_{\mathrm{Br}} \Lambda_{\mathrm{Br}} J^s_2} = \overline{\mathcal{U}}\Lambda$. Thus, by Lemma 1.12, there is a natural transformation $\overline{\chi} := \widehat{\zeta^s_1 U_{1,2} \circ \overline{\chi}^s_{\mathrm{Br}} J^s_2} : \overline{\mathcal{U}}\Lambda \to \overline{T}_1 U_{1,2}$ such that $J^s_{\mathrm{Bialg}} \overline{\chi} = \zeta^s_1 U_{1,2} \circ \overline{\chi}^s_{\mathrm{Br}} J^s_2$. We compute

$$J_{\mathrm{Lie}}^{s}\overline{\eta}_{\mathrm{L}}\Lambda \overset{(79)}{=} \overline{\eta}_{\mathrm{BrL}}^{s}J_{\mathrm{Lie}}^{s}\Lambda = \overline{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}J_{2}^{s} \tag{94}$$

so that

$$\begin{aligned}
&J^{s}H_{\mathrm{Lie}}\left(\mathcal{P}\overline{\chi}\circ\overline{\eta}_{\mathrm{L}}\Lambda\right) = J^{s}H_{\mathrm{Lie}}\mathcal{P}\overline{\chi}\circ J^{s}H_{\mathrm{Lie}}\overline{\eta}_{\mathrm{L}}\Lambda\\
&\overset{(55)}{=} J^{s}P\overline{\chi}\circ H_{\mathrm{BrLie}}^{s}J_{\mathrm{Lie}}^{s}\overline{\eta}_{\mathrm{L}}\Lambda \overset{(49),(94)}{=} P_{\mathrm{Br}}^{s}J_{\mathrm{Bialg}}^{s}\overline{\chi}\circ H_{\mathrm{BrLie}}^{s}\overline{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}J_{2}^{s}\\
&= P_{\mathrm{Br}}^{s}\zeta_{1}^{s}U_{1,2}\circ P_{\mathrm{Br}}^{s}\overline{\chi}_{\mathrm{Br}}^{s}J_{2}^{s}\circ H_{\mathrm{BrLie}}^{s}\overline{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}J_{2}^{s}\\
&\overset{(76)}{=} P_{\mathrm{Br}}^{s}\zeta_{1}^{s}U_{1,2}\circ H_{\mathrm{BrLie}}^{s}\mathcal{P}_{\mathrm{Br}}^{s}\overline{\chi}_{\mathrm{Br}}^{s}J_{2}^{s}\circ H_{\mathrm{BrLie}}^{s}\overline{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}J_{2}^{s}\\
&= P_{\mathrm{Br}}^{s}\zeta_{1}^{s}U_{1,2}\circ H_{\mathrm{BrLie}}^{s}\left(\mathcal{P}_{\mathrm{Br}}^{s}\overline{\chi}_{\mathrm{Br}}^{s}\circ\overline{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}\right)J_{2}^{s}\\
&\overset{(92)}{=} U_{0,1}\left(P_{\mathrm{Br}}^{s}\right)_{1}\zeta_{1}^{s}U_{1,2}\circ U_{0,1}\left(\overline{\eta}_{\mathrm{Br}}^{s}\right)_{1}U_{1,2}J_{2}^{s}\\
&\overset{(92)}{=} U_{0,1}\left(P_{\mathrm{Br}}^{s}\right)_{1}\zeta_{1}^{s}U_{1,2}\circ U_{0,1}\left(\overline{\eta}_{\mathrm{Br}}^{s}\right)_{1}J_{1}^{s}U_{1,2} = U_{0,1}\left(\left(P_{\mathrm{Br}}^{s}\right)_{1}\zeta_{1}^{s}\circ\left(\overline{\eta}_{\mathrm{Br}}^{s}\right)_{1}J_{1}^{s}\right)U_{1,2}\\
&= U_{0,1}J_{1}^{s}\overline{\eta}_{1}U_{1,2} = J^{s}U_{0,1}\overline{\eta}_{1}U_{1,2}
\end{aligned}$$

so that

$$H_{\mathrm{Lie}}\left(\mathcal{P}\overline{\chi}\circ\overline{\eta}_{\mathrm{L}}\Lambda\right) = U_{0,1}\overline{\eta}_{1}U_{1,2}. \tag{95}$$

We have

$$\begin{aligned}
J_{\mathrm{Bialg}}^{s}\left(\overline{\chi}\circ\overline{p}\Lambda\right) &= J_{\mathrm{Bialg}}^{s}\overline{\chi}\circ J_{\mathrm{Bialg}}^{s}\overline{p}\Lambda \overset{(80)}{=} \zeta_{1}^{s}U_{1,2}\circ\overline{\chi}_{\mathrm{Br}}^{s}J_{2}^{s}\circ\overline{p}^{s}J_{\mathrm{Lie}}^{s}\Lambda\\
&= \zeta_{1}^{s}U_{1,2}\circ\overline{\chi}_{\mathrm{Br}}^{s}J_{2}^{s}\circ\overline{p}^{s}\Lambda_{\mathrm{Br}}J_{2}^{s} \overset{(83)}{=} \zeta_{1}^{s}U_{1,2}\circ\pi_{1}^{s}U_{1,2}J_{2}^{s}\\
&= \zeta_{1}^{s}U_{1,2}\circ\pi_{1}^{s}J_{1}^{s}U_{1,2} = \left(\zeta_{1}^{s}\circ\pi_{1}^{s}J_{1}^{s}\right)U_{1,2} \overset{(*)}{=} \left(J_{\mathrm{Bialg}}^{s}\pi_{1}\circ\zeta_{0}^{s}\right)U_{1,2}\\
&= J_{\mathrm{Bialg}}^{s}\pi_{1}U_{1,2}
\end{aligned}$$

where $(*)$ follows by construction of ζ_{1}^{s} (see the proof of Proposition 2.5). Thus we obtain $\overline{\chi}\circ\overline{p}\Lambda = \pi_{1}U_{1,2}$.

Assume $\overline{\eta}_{\mathrm{BrL}}^{s}\Lambda_{\mathrm{Br}}$ is an isomorphism. By Theorem 7.1, we have that $\overline{\chi}_{\mathrm{Br}}^{s}$ is an isomorphism. Thus, from $J_{\mathrm{Bialg}}^{s}\overline{\chi} = \zeta_{1}^{s}U_{1,2}\circ\overline{\chi}_{\mathrm{Br}}^{s}J_{2}^{s}$ and the fact that ζ_{1}^{s} is an isomorphism, we deduce that $\overline{\chi}$ is an isomorphism too. Moreover, by (94), we also have that $\overline{\eta}_{\mathrm{L}}\Lambda$ is an isomorphism. From this we get that $\overline{\eta}_{\mathrm{L}}\Lambda P_{2}$ is an isomorphism. Since $\Lambda P_{2} = \mathcal{P}$ we have that $\overline{\eta}_{\mathrm{L}}\mathcal{P}$ is an isomorphism. By [34, Proposition 2.8], this means that the adjunction $(\overline{\mathcal{U}}, \mathcal{P})$ is idempotent.

Moreover, since $\overline{\eta}_{\mathrm{L}}\Lambda$ is an isomorphism, by (95), we deduce that $\overline{\eta}_{1}U_{1,2}$ is an isomorphism i.e. $\left(\overline{T}_{1}, P_{1}\right)$ is idempotent (cf. [4, Remark 2.2]). Note that in this case we can choose $\overline{T}_{2} := \overline{T}_{1}U_{1,2}$ and it is full and faithful (cf. [4, Proposition 2.3]) i.e. $\overline{\eta}_{2}$ is an isomorphism. The choice $\overline{T}_{2} := \overline{T}_{1}U_{1,2}$ implies we can choose the canonical projection $\pi_{2} : \overline{T}_{1}U_{1,2} \to \overline{T}_{2}$ to be the identity. In this case by definition, $\overline{\eta}_{1}$ is given by the formula $\overline{\eta}_{1}U_{1,2} = U_{1,2}\overline{\eta}_{2}$. Thus the second term of 95 becomes $U_{0,1}\overline{\eta}_{1}U_{1,2} = U_{0,1}U_{1,2}\overline{\eta}_{2} =$

$U_{0,2}\overline{\eta}_2 = H_{\mathrm{Lie}}\Lambda\overline{\eta}_2$. Since H_{Lie} is faithful, by (95) we obtain $\mathcal{P}\overline{\chi} \circ \overline{\eta}_{\mathrm{L}}\Lambda = \Lambda\overline{\eta}_2$ which means that $(\mathrm{Id}_{\mathrm{Bialg}_{\mathcal{M}}}, \Lambda) : (\overline{T}_2, P_2) \to (\overline{\mathcal{U}}, \mathcal{P})$ is a commutation datum whose canonical transformation is $\overline{\chi}$.

We already observed that $\Lambda_{\mathrm{Br}} J_2^s = J_{\mathrm{Lie}}^s \Lambda$. Moreover, from $J_n^s \circ P_n = (P_{\mathrm{Br}}^s)_n \circ J_{\mathrm{Bialg}}^s$, we deduce

$$J_2^s \left(P_2\overline{\mathcal{U}}\right) = (P_{\mathrm{Br}}^s)_2 J_{\mathrm{Bialg}}^s \overline{\mathcal{U}} \overset{(78)}{=} \left((P_{\mathrm{Br}}^s)_2 \overline{\mathcal{U}}_{\mathrm{Br}}^s\right) J_{\mathrm{Lie}}^s.$$

We know that J_{Lie}^s is full, faithful and injective on objects. Since J^s fulfills the same properties, by Proposition 2.5 applied to the commutation datum $\left(J_{\mathrm{Bialg}}^s, J^s\right) : \left(\overline{T}, P\right) \to \left(\overline{T}_{\mathrm{Br}}^s, P_{\mathrm{Br}}^s\right)$, we deduce that the same is true for J_1^s and hence, by same argument, also for J_2^s. Thus we can apply Lemma 2.9 to the case $L' = \left((P_{\mathrm{Br}}^s)_2 \overline{\mathcal{U}}_{\mathrm{Br}}^s\right), R' = \Lambda_{\mathrm{Br}}, F = J_2^s, G = J_{\mathrm{Lie}}^s$. Then $L = P_2\overline{\mathcal{U}}$ and $R = \Lambda$, the pair (L, R) is an adjunction and the unit and counit of (L, R) and (L', R') are related by (11). Since F and G are both conservative, we get that ϵ and η are an isomorphism whenever ϵ' and $\eta' = \overline{\eta}_{\mathrm{BrL}}^s$ are. By Theorem 7.1, we know that Λ_{Br} is full and faithful i.e. ϵ' is an isomorphism and hence ϵ is an isomorphism i.e. Λ is full and faithful. It is clear that $\left(P_2\overline{\mathcal{U}}, \Lambda\right)$ is an equivalence if and only if η is an isomorphism. By (79), we have $J_{\mathrm{Lie}}^s \overline{\eta}_{\mathrm{L}} = \overline{\eta}_{\mathrm{BrL}}^s J_{\mathrm{Lie}}^s$ i.e. $G\overline{\eta}_{\mathrm{L}} = \eta' G$. Thus, since G is faithful, (11) implies $\eta = \overline{\eta}_{\mathrm{L}}$. If we write (7) for the commutation datum $\left(J_{\mathrm{Bialg}}^s, J_2^s\right) : \left(\overline{T}_2, P_2\right) \to \left((\overline{T}_{\mathrm{Br}}^s)_2, (P_{\mathrm{Br}}^s)_2\right)$, we get $(P_{\mathrm{Br}}^s)_2 \zeta_2^s \circ (\overline{\eta}_{\mathrm{Br}}^s)_2 J_2^s = J_2^s \overline{\eta}_2$ (note that $(\overline{\eta}_{\mathrm{Br}}^s)_2$ is an isomorphism by Theorem 7.1-2)). Using this equality we compute

$$\begin{aligned}
J_2^s \left((\overline{\eta}_2)^{-1} \circ P_2\overline{\chi}\right) &= J_2^s (\overline{\eta}_2)^{-1} \circ J_2^s P_2 \overline{\chi} = J_2^s (\overline{\eta}_2)^{-1} \circ (P_{\mathrm{Br}}^s)_2 J_{\mathrm{Bialg}}^s \overline{\chi} \\
&= J_2^s (\overline{\eta}_2)^{-1} \circ (P_{\mathrm{Br}}^s)_2 \left(\zeta_1^s U_{1,2} \circ \overline{\chi}_{\mathrm{Br}}^s J_2^s\right) \\
&= \left[(P_{\mathrm{Br}}^s)_2 \zeta_2^s \circ (\overline{\eta}_{\mathrm{Br}}^s)_2 J_2^s\right]^{-1} \circ (P_{\mathrm{Br}}^s)_2 \zeta_1^s U_{1,2} \circ (P_{\mathrm{Br}}^s)_2 \overline{\chi}_{\mathrm{Br}}^s J_2^s \\
&= (\overline{\eta}_{\mathrm{Br}}^s)_2^{-1} J_2^s \circ (P_{\mathrm{Br}}^s)_2 (\zeta_2^s)^{-1} \circ (P_{\mathrm{Br}}^s)_2 \zeta_1^s U_{1,2} \circ (P_{\mathrm{Br}}^s)_2 \overline{\chi}_{\mathrm{Br}}^s J_2^s.
\end{aligned}$$

Now, by construction of ζ_2^s (see the proof of Proposition 2.5), the fact that $\pi_2 : \overline{T}_1 U_{1,2} \to \overline{T}_2$ is the identity and that also π_2^s is the identity (see Theorem 7.1-2)), we have that $\zeta_2^s = \zeta_1^s U_{1,2}$ and hence

$$\begin{aligned}
F\left((\overline{\eta}_2)^{-1} \circ P_2\overline{\chi}\right) &= J_2^s \left((\overline{\eta}_2)^{-1} \circ P_2\overline{\chi}\right) = (\overline{\eta}_{\mathrm{Br}}^s)_2^{-1} J_2^s \circ (P_{\mathrm{Br}}^s)_2 \overline{\chi}_{\mathrm{Br}}^s J_2^s \\
&= \left((\overline{\eta}_{\mathrm{Br}}^s)_2^{-1} \circ (P_{\mathrm{Br}}^s)_2 \overline{\chi}_{\mathrm{Br}}^s\right) J_2^s = \epsilon' F.
\end{aligned}$$

Thus, by (11), we get $\epsilon = (\overline{\eta}_2)^{-1} \circ P_2\overline{\chi}$. □

The following remark was inspired by the comments of the Referee.

Remark 7.3. 1) Theorem 7.1 establishes that the functor P_{Br}^s has a monadic decomposition of monadic length at most two whenever $\overline{\eta}_{\mathrm{BrL}}^s \Lambda_{\mathrm{Br}}$ is invertible. In this case note that this monadic decomposition cannot have monadic length 0 as $\overline{T}^s_{\mathrm{Br}}$ is not full and faithful in general: the unit $\overline{\eta}^s_{\mathrm{Br}} : \mathrm{Id} \to P^s_{\mathrm{Br}}\overline{T}^s_{\mathrm{Br}}$ needs not to be invertible; for instance, if $\mathcal{M}$ is the category of vector spaces over a field $\Bbbk$ and V is a vector space endowed with the canonical flip τ, then $\overline{\eta}^s_{\mathrm{Br}}(V,\tau)$ is not surjective. It is not clear to us if the length above can be exactly 2. A similar argument applies to the setting of Theorem 7.2.

2) It is natural to wonder if there is a primitive type functor, similar to the one in Theorem 7.1, but having monadic decomposition of monadic length strictly greater than two. In this direction, consider the adjunction $(\overline{T}_{\mathrm{Br}}, P_{\mathrm{Br}})$ of 3.6 in the case when $\mathcal{M}$ is the category of vector spaces over a field $\Bbbk$. Then we expect that the functor P_{Br} has a monadic decomposition of monadic length strictly greater than two. Analogously the functor P of Theorem 7.2 is expected to have a monadic decomposition of monadic length strictly greater than two if we drop the assumption that $\mathcal{M}$ is symmetric. These facts will be hopefully investigated in a different paper.

Definition 7.4. An **MM-category** (Milnor–Moore-category) is an abelian monoidal category $\mathcal{M}$ with denumerable coproducts such that the tensor products are exact and preserve denumerable coproducts and such that the unit $\overline{\eta}^s_{\mathrm{BrL}} : \mathrm{Id}_{\mathrm{BrLie}^s_{\mathcal{M}}} \to \mathcal{P}^s_{\mathrm{Br}}\overline{\mathcal{U}}^s_{\mathrm{Br}}$ of the adjunction $(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}})$ is a functorial isomorphism i.e. the functor $\overline{\mathcal{U}}^s_{\mathrm{Br}} : \mathrm{BrLie}^s_{\mathcal{M}} \to \mathrm{BrBialg}^s_{\mathcal{M}}$ is full and faithful (see e.g. [11, dual of Proposition 3.4.1, page 114]).

Remark 7.5. 1) The celebrated Milnor–Moore Theorem, cf. [32, Theorem 5.18] states that, in characteristic zero, there is a category equivalence between the category of Lie algebras and the category of primitively generated bialgebras. The fact that the counit of the adjunction involved is an isomorphism just encodes the fact that the bialgebras considered are primitively generated. On the other hand the crucial point in the proof is that the unit of the adjunction is an isomorphism.

In our wider context this translates to the unit of the adjunction $(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}})$ being a functorial isomorphism. From this the definition of MM-category stems. Note that for a MM-category $\mathcal{M}$ we can apply Theorem 7.1 to obtain that the functor P^s_{Br} has a monadic decomposition of monadic length at most two. Moreover we can identify the category $(\mathrm{Br}^s_{\mathcal{M}})_2$ with $\mathrm{BrLie}^s_{\mathcal{M}}$.

2) In the case of a symmetric MM-category $\mathcal{M}$ the connection with Milnor–Moore Theorem becomes more evident. In fact, in this case, we can apply Theorem 7.2 to obtain that the unit of the adjunction $(\overline{\mathcal{U}}, \mathcal{P})$ is a functorial isomorphism.

8. Lifting the structure of MM-category

We first prove a crucial result for braided vector spaces.

Theorem 8.1. *The category of vector spaces over a fixed field $\Bbbk$ of characteristic zero is a MM-category.*

Proof. Let $\mathcal{M} = \mathfrak{M}$ be the category of vector spaces over $\Bbbk$. We have just to prove that $\bar{\eta}^s_{\mathrm{BrL}}$ is an isomorphism. Let $(M, c, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$. Since we are working on vector spaces, we can express explicitly the universal enveloping algebra $\overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-])$ with elements as follows

$$\overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-]) = \frac{\overline{T}^s_{\mathrm{Br}}(M, c)}{([x \otimes y] - x \otimes y + c(x \otimes y) \mid x, y \in M)}.$$

By Lemma 5.3, $(M, [-])$ is a Lie c-algebra and $\overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-])$ coincides with the corresponding universal enveloping algebra in the sense of [27, Section 2.5]. Hence we can apply [27, Lemma 6.2] to conclude that the canonical map from M into the primitive part of $\overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-])$ is an isomorphism. In our notation this means that

$$H\mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \bar{\eta}^s_{\mathrm{BrL}}(M, c, [-]) : M \to H\mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \mathcal{P}^s_{\mathrm{Br}} \overline{\mathcal{U}}^s_{\mathrm{Br}}(M, c, [-])$$

is bijective. Note that H, $\mathbb{I}^s_{\mathrm{Br}}$ and H^s_{BrLie} are conservative by 3.3, Definition 3.1 and Definition 5.1 respectively. Thus $H\mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}}$ is conservative and hence we get that $\bar{\eta}^s_{\mathrm{BrL}}(M, c, [-])$ is an isomorphism for all $(M, c, [-]) \in \mathrm{BrLie}^s_{\mathcal{M}}$. We have so proved that $\bar{\eta}^s_{\mathrm{BrL}}$ is an isomorphism. $\square$

In the rest of this section we will deal with symmetric braided monoidal categories $\mathcal{M}$ endowed with a faithful monoidal functor $W : \mathcal{M} \to \mathfrak{M}$ which is not necessarily braided. The examples we will treat take $\mathcal{M} = \mathfrak{M}^H$ for a dual quasi-bialgebra H or $\mathcal{M} = {}_H\mathfrak{M}$ for a quasi-bialgebra case. Note that in general the obvious forgetful functors need not to be monoidal, see e.g. [28, Example 9.1.4] so that further conditions will be required on H. Note that the results on $\mathfrak{M}^H$ and ${}_H\mathfrak{M}$ are not dual each other, unless H is finite-dimensional.

Lemma 8.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories. Any monoidal functor (F, ϕ_0, ϕ_2) : $\mathcal{M} \to \mathcal{N}$ induces a functor $\mathrm{BrLie}F : \mathrm{BrLie}_{\mathcal{M}} \to \mathrm{BrLie}_{\mathcal{N}}$ which acts as F on morphisms and such that $\mathrm{BrLie}F(M, c_M, [-]_M) := (FM, c_{FM}, [-]_{FM})$ where $(FM, c_{FM}) = \mathrm{Br}F(M, c_M)$ and*

$$[-]_{FM} := F[-]_M \circ \phi_2(M, M) : FM \otimes FM \to F(M).$$

Moreover the first diagram below commutes and there is a unique functor $\mathrm{BrLie}^s F$ such that the second diagram commutes.

$$\begin{array}{ccc} \mathrm{BrLie}_{\mathcal{M}} & \xrightarrow{\mathrm{BrLie}F} & \mathrm{BrLie}_{\mathcal{N}} \\ {\scriptstyle H_{\mathrm{BrLie}}}\downarrow & & \downarrow{\scriptstyle H_{\mathrm{BrLie}}} \\ \mathrm{Br}_{\mathcal{M}} & \xrightarrow{\mathrm{Br}F} & \mathrm{Br}_{\mathcal{N}} \end{array} \qquad \begin{array}{ccc} \mathrm{BrLie}^s_{\mathcal{M}} & \xrightarrow{\mathrm{BrLie}^s F} & \mathrm{BrLie}^s_{\mathcal{N}} \\ {\scriptstyle \mathbb{I}^s_{\mathrm{BrLie}}}\downarrow & & \downarrow{\scriptstyle \mathbb{I}^s_{\mathrm{BrLie}}} \\ \mathrm{BrLie}_{\mathcal{M}} & \xrightarrow{\mathrm{BrLie}F} & \mathrm{BrLie}_{\mathcal{N}} \end{array}$$

$$\begin{array}{ccc} \mathrm{BrLie}^s_{\mathcal{M}} & \xrightarrow{\mathrm{BrLie}^s F} & \mathrm{BrLie}^s_{\mathcal{N}} \\ {\scriptstyle H^s_{\mathrm{BrLie}}}\downarrow & & \downarrow {\scriptstyle H^s_{\mathrm{BrLie}}} \\ \mathrm{Br}^s_{\mathcal{M}} & \xrightarrow{\mathrm{Br}^s F} & \mathrm{Br}^s_{\mathcal{N}} \end{array} \tag{96}$$

Furthermore the functors BrLie$F$ *and* BrLie$^s F$ *are conservative whenever* F *is.*

Proof. It is straightforward. □

Theorem 8.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories. Assume that both $\mathcal{M}$ and $\mathcal{N}$ are abelian with denumerable coproducts, and that the tensor products are exact and preserves denumerable coproducts. Assume that there exists an exact monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{N}$ which preserves denumerable coproducts. Then we have the following commutation data with the respective canonical transformations*

$$(\mathrm{BrAlg}^s F, \mathrm{BrLie}^s F) : (\mathcal{U}^s_{\mathrm{Br}}, \mathcal{L}^s_{\mathrm{Br}}) \to (\mathcal{U}^s_{\mathrm{Br}}, \mathcal{L}^s_{\mathrm{Br}}),$$
$$\zeta^s_{\mathrm{BrL}} : \mathcal{U}^s_{\mathrm{Br}} (\mathrm{BrLie}^s F) \to (\mathrm{BrAlg}^s F) \mathcal{U}^s_{\mathrm{Br}},$$
$$(\mathrm{BrBialg}^s F, \mathrm{BrLie}^s F) : \left(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}}\right) \to \left(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}}\right),$$
$$\bar{\zeta}^s_{\mathrm{BrL}} : \overline{\mathcal{U}}^s_{\mathrm{Br}} (\mathrm{BrLie}^s F) \to (\mathrm{BrBialg}^s F) \overline{\mathcal{U}}^s_{\mathrm{Br}}.$$

Proof. A direct computation using (96) shows that

$$\mathbb{I}^s_{\mathrm{BrLie}} (\mathrm{BrLie}^s F) \mathcal{L}^s_{\mathrm{Br}} (B, m_B, u_B, c_B) = \mathbb{I}^s_{\mathrm{BrLie}} \mathcal{L}^s_{\mathrm{Br}} (\mathrm{BrAlg}^s F) (B, m_B, u_B, c_B).$$

Since both functors act as F on morphisms, we get $\mathbb{I}^s_{\mathrm{BrLie}} (\mathrm{BrLie}^s F) \mathcal{L}^s_{\mathrm{Br}} = \mathbb{I}^s_{\mathrm{BrLie}} \mathcal{L}^s_{\mathrm{Br}} (\mathrm{BrAlg}^s F)$. Since $\mathbb{I}^s_{\mathrm{BrLie}}$ is both injective on morphisms and objects we obtain

$$(\mathrm{BrLie}^s F) \mathcal{L}^s_{\mathrm{Br}} = \mathcal{L}^s_{\mathrm{Br}} (\mathrm{BrAlg}^s F).$$

Now, using in the given order (54), (96), again (54), (38) and again (96), we get the equality $\mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} (\mathrm{BrLie}^s F) \xi = \mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \xi (\mathrm{BrBialg}^s F)$. Then one shows that $(\mathrm{BrLie}^s F) \xi$ and $\xi (\mathrm{BrBialg}^s F)$ have the same domain and codomain. Thus, from $\mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} (\mathrm{BrLie}^s F) \xi = \mathbb{I}^s_{\mathrm{Br}} H^s_{\mathrm{BrLie}} \xi (\mathrm{BrBialg}^s F)$ we deduce that

$$(\mathrm{BrLie}^s F) \xi = \xi (\mathrm{BrBialg}^s F).$$

Consider the natural transformation $\bar{\zeta}^s_{\mathrm{BrL}} : \overline{\mathcal{U}}^s_{\mathrm{Br}} (\mathrm{BrLie}^s F) \to (\mathrm{BrBialg}^s F) \overline{\mathcal{U}}^s_{\mathrm{Br}}$ of Lemma 2.2. By definition

$$\bar{\zeta}^s_{\mathrm{BrL}} := \bar{\epsilon}^s_{\mathrm{BrL}} (\mathrm{BrBialg}^s F) \overline{\mathcal{U}}^s_{\mathrm{Br}} \circ \overline{\mathcal{U}}^s_{\mathrm{Br}} (\mathrm{BrLie}^s F) \bar{\eta}^s_{\mathrm{BrL}}.$$

It is straightforward to check that

$$\mho_{\mathrm{Br}}^{s}\bar{\zeta}_{\mathrm{BrL}}^{s}=\zeta_{\mathrm{BrL}}^{s} \tag{97}$$

where $\zeta_{\mathrm{BrL}}^{s}:\mathcal{U}_{\mathrm{Br}}^{s}\left(\mathrm{BrLie}^{s}F\right)\rightarrow\left(\mathrm{BrAlg}^{s}F\right)\mathcal{U}_{\mathrm{Br}}^{s}$ is the canonical morphism of Lemma 2.2, and also

$$\zeta_{\mathrm{BrL}}^{s}\circ p^{s}\left(\mathrm{BrLie}^{s}F\right)=\left(\mathrm{BrAlg}^{s}F\right)p^{s}\circ\zeta_{\mathrm{Br}}^{s}H_{\mathrm{BrLie}}^{s}.$$

Let $(M,c_M,[-]_M)\in\mathrm{BrLie}_{\mathcal{M}}^{s}$. Then we have that $(M\otimes M,c_{M\otimes M})\in\mathrm{BrLie}_{\mathcal{M}}^{s}$ where $c_{M\otimes M}:=(M\otimes c_M\otimes M)(c_M\otimes c_M)(M\otimes c_M\otimes M)$. It is easy to check that $[-]:M\otimes M\rightarrow M$ and $\theta_{(M,c_M)}:M\otimes M\rightarrow\Omega TM$ induce morphisms of braided objects

$$[-]^{s}:(M\otimes M,c_{M\otimes M})\rightarrow(M,c_M)\qquad\text{and}$$
$$\theta_{(M,c_M)}^{s}:(M\otimes M,c_{M\otimes M})\rightarrow\Omega_{\mathrm{Br}}^{s}T_{\mathrm{Br}}^{s}(M,c_M)$$

such that

$$H\mathbb{I}_{\mathrm{Br}}^{s}[-]^{s}=[-]\qquad\text{and}\qquad H\mathbb{I}_{\mathrm{Br}}^{s}\theta_{(M,c_M)}^{s}=\theta_{(M,c_M)}.$$

Let us check that the following is a coequalizer in $\mathrm{BrAlg}_{\mathcal{M}}^{s}$

$$T_{\mathrm{Br}}^{s}(M\otimes M,c_{M\otimes M})\underset{\epsilon_{\mathrm{Br}}^{s}T_{\mathrm{Br}}^{s}(M,c_M)\circ T_{\mathrm{Br}}^{s}\theta_{(M,c_M)}^{s}}{\overset{T_{\mathrm{Br}}^{s}[-]^{s}}{\rightrightarrows}}T_{\mathrm{Br}}^{s}(M,c_M)\xrightarrow{p^{s}(M,c_M,[-]_M)}\mathcal{U}_{\mathrm{Br}}^{s}(M,c_M,[-])_M. \tag{98}$$

By applying $H_{\mathrm{Alg}}\mathbb{I}_{\mathrm{BrAlg}}^{s}$ to this diagram we get the diagram

$$T(M\otimes M)\underset{\epsilon TM\circ T\theta_{(M,c_M)}}{\overset{T[-]}{\rightrightarrows}}T(M)\xrightarrow{H_{\mathrm{Alg}}p\mathbb{I}_{\mathrm{BrLie}}^{s}(M,c_M,[-]_M)}H_{\mathrm{Alg}}\mathcal{U}_{\mathrm{Br}}\mathbb{I}_{\mathrm{BrLie}}^{s}(M,c,[-]), \tag{99}$$

which can be checked to be a coequalizer in $\mathrm{Alg}_{\mathcal{M}}$. By Lemma B.6 we have that H_{Alg} reflects coequalizers and by [11, Proposition 2.9.9], we have that $\mathbb{I}_{\mathrm{BrAlg}}^{s}$ reflects coequalizers. Thus (98) is also a coequalizer. By Lemma B.10, since F preserves coequalizers, we get that $\mathrm{Alg}F$ preserves the coequalizer (99). Denote by $\mathrm{Alg}F(99)$ the coequalizer obtained in this way. Now, with the same notation, $\mathrm{Alg}F(99)$ can also be obtained as $H_{\mathrm{Alg}}\mathbb{I}_{\mathrm{BrAlg}}^{s}(\mathrm{BrAlg}^{s}F)(98)$ (this is straightforward). Since we already observed that both H_{Alg} and $\mathbb{I}_{\mathrm{BrAlg}}^{s}$ reflect coequalizers, we deduce that $(\mathrm{BrAlg}^{s}F)(98)$ is a coequalizer too. This coequalizer appears in the second line of the diagram

$$\begin{array}{ccccc}
T^s_{\mathrm{Br}}\left(FM\otimes FM, c_{FM\otimes FM}\right) & \overset{T^s_{\mathrm{Br}}[-]^s_{FM}}{\underset{\epsilon^s_{\mathrm{Br}}T^s_{\mathrm{Br}}(FM,c_{FM})\circ T^s_{\mathrm{Br}}\theta^s_{(FM,c_{FM})}}{\rightrightarrows}} & T^s_{\mathrm{Br}}H^s_{\mathrm{BrLie}}\left(\mathrm{BrLie}^s F\right)\overline{M} & \xrightarrow{p^s\left(\mathrm{BrLie}^s F\right)\overline{M}} & \mathcal{U}^s_{\mathrm{Br}}\left(\mathrm{BrLie}^s F\right)\overline{M} \\
\vdots & & \downarrow{\scriptstyle \zeta^s_{\mathrm{Br}}H^s_{\mathrm{BrLie}}\overline{M}} & & \downarrow{\scriptstyle \zeta^s_{\mathrm{BrL}}\overline{M}} \\
\overline{F}T^s_{\mathrm{Br}}\left(M\otimes M, c_{M\otimes M}\right) & \overset{\overline{F}T^s_{\mathrm{Br}}[-]^s}{\underset{\overline{F}\left(\epsilon^s_{\mathrm{Br}}T^s_{\mathrm{Br}}(M,c_M)\circ T^s_{\mathrm{Br}}\theta^s_{(M,c_M)}\right)}{\rightrightarrows}} & \overline{F}T^s_{\mathrm{Br}}\left(M, c_M\right) & \xrightarrow{\overline{F}p^s\overline{M}} & \overline{F}\mathcal{U}^s_{\mathrm{Br}}\overline{M}
\end{array}$$

where, for sake of shortness, we set $\overline{M} := (M, c_M, [-]_M)$ and $\overline{F} := \mathrm{BrAlg}^s F$. One proves that the morphism

$$T^s_{\mathrm{Br}}\left(FM\otimes FM, c_{FM\otimes FM}\right) \overset{T^s_{\mathrm{Br}}\phi_2(M,M)}{\longrightarrow} T^s_{\mathrm{Br}}\left(F\left(M\otimes M\right), c_{F(M\otimes M)}\right)$$
$$= T^s_{\mathrm{Br}}\left(\mathrm{Br}^s F\right)\left(M\otimes M, c_{M\otimes M}\right) \overset{\zeta^s_{\mathrm{Br}}(M\otimes M, c_{M\otimes M})}{\longrightarrow} \left(\mathrm{BrAlg}^s F\right) T^s_{\mathrm{Br}}\left(M\otimes M, c_{M\otimes M}\right)$$

is an isomorphism (we just point out that, as one easily checks, the morphism $\phi_2(M, M)$ is a braided morphism so that the morphism above is well-defined) and it completes the diagram above on the left making it a serially commutative diagram. The fact it is serially commutative depends on the following equality that can be easily checked

$$\mathbb{I}^s_{\mathrm{BrAlg}}\zeta^s_{\mathrm{Br}} = \zeta_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}. \tag{100}$$

Now, by (100) we have $\mathbb{I}^s_{\mathrm{BrAlg}}\zeta^s_{\mathrm{Br}} = \zeta_{\mathrm{Br}}\mathbb{I}^s_{\mathrm{Br}}$. On the other hand, by Proposition 3.7 (here we use the fact that F preserves denumerable coproducts), we know that ζ_{Br} is a functorial isomorphism. Since $\mathbb{I}^s_{\mathrm{BrAlg}}$ is conservative, we deduce that ζ^s_{Br} is a functorial isomorphism. Thus, by the uniqueness of coequalizers (note that the first line in the diagram above is just (98) applied to $(\mathrm{BrLie}^s F)(M, c_M, [-]_M) = (FM, c_{FM}, [-]_{FM})$ instead of $(M, c_M, [-]_M)$), we get that $\zeta^s_{\mathrm{BrL}}(M, c, [-])$ is an isomorphism too. Thus ζ^s_{BrL} is a functorial isomorphism.

By (97) we have $\mho^s_{\mathrm{Br}}\bar{\zeta}^s_{\mathrm{BrL}} = \zeta^s_{\mathrm{BrL}}$ so that $\bar{\zeta}^s_{\mathrm{BrL}}$ is a functorial isomorphism too. $\square$

Theorem 8.4. *Let $\mathcal{M}$ be an abelian monoidal category with denumerable coproducts and such that the tensor products are exact and preserve denumerable coproducts. Let $\mathcal{N}$ be a MM-category and assume that there exists a conservative (see 2.1) and exact monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{N}$ which preserves denumerable coproducts. Then $\mathcal{M}$ is a MM-category.*

Proof. By Theorem 8.3, we have the following commutation datum

$$\left(\mathrm{BrBialg}^s F, \mathrm{BrLie}^s F\right) : \left(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}}\right) \to \left(\overline{\mathcal{U}}^s_{\mathrm{Br}}, \mathcal{P}^s_{\mathrm{Br}}\right).$$

By Lemma 8.2, we know that $\mathrm{BrLie}^s F$ is conservative as F is. By Lemma 2.7, we have that the unit $\bar{\eta}^s_{\mathrm{BrL}} : \mathrm{Id}_{\mathrm{BrLie}^s_{\mathcal{M}}} \to \mathcal{P}^s_{\mathrm{Br}}\overline{\mathcal{U}}^s_{\mathrm{Br}}$ is a functorial isomorphism. $\square$

Theorem 8.5. *Let $\mathfrak{M}$ be the category of vector spaces over a field $\Bbbk$ with $\mathrm{char}\Bbbk = 0$. Let $\mathcal{M}$ be an abelian monoidal category with denumerable coproducts, such that the tensor* *functors are exact and preserve denumerable coproducts. Assume that there exists a conservative and exact monoidal functor* $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathfrak{M}$ *which preserves denumerable coproducts. Then* $\mathcal{M}$ *is a MM-category.*

Proof. By Theorem 8.1 we have $\mathfrak{M}$ is a MM-category. We conclude by Theorem 8.4. □

9. Examples of MM-categories

Example 9.1. Let $\Bbbk$ be a field with $\mathrm{char}\,(\Bbbk) = 0$. Let H be any Hopf algebra over $\Bbbk$ and consider the monoidal category of Yetter–Drinfeld modules $\left({}_H^H\mathcal{YD}, \otimes_\Bbbk, \Bbbk\right)$. Then the forgetful functor $F : \left({}_H^H\mathcal{YD}, \otimes_\Bbbk, \Bbbk\right) \to (\mathfrak{M}, \otimes_\Bbbk, \Bbbk)$ is monoidal. One can prove by hand that ${}_H^H\mathcal{YD}$ is abelian with denumerable coproducts. The tensor products are clearly exact and preserve denumerable coproducts in ${}_H^H\mathcal{YD}$ as this is the case in $\mathfrak{M}$. Furthermore F is clearly conservative and exact and preserves denumerable coproducts. By Theorem 8.5, we conclude that $\left({}_H^H\mathcal{YD}, \otimes_\Bbbk, \Bbbk\right)$ is a MM-category. Note that, by Theorem [35], this category, with respect to its standard pre-braiding, is not symmetric unless $H = \Bbbk$.

9.1. Quasi-bialgebras

The following definition is not the original one given in [17, page 1421]. We adopt the more general form of [17, Remark 1, page 1423] (see also [26, Proposition XV.1.2]) in order to comprise the case of monoidal Hom-Lie algebras. Later on, for dual quasi-bialgebras, we will take the simplified respective definition from the very beginning having no meaningful example to treat in the full generality.

Definition 9.2. A *quasi-bialgebra* is a datum $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ where (H, m, u) is an associative algebra, $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \Bbbk$ are algebra maps, $\lambda, \rho \in H$ are invertible elements, $\phi \in H \otimes H \otimes H$ is a counital 3-cocycle i.e. it is an invertible element and satisfies

$$(H \otimes H \otimes \Delta)(\phi) \cdot (\Delta \otimes H \otimes H)(\phi) = (1_H \otimes \phi) \cdot (H \otimes \Delta \otimes H)(\phi) \cdot (\phi \otimes 1_H),$$
$$(H \otimes \varepsilon \otimes H)(\phi) = \rho \otimes \lambda^{-1}.$$

Moreover Δ is required to be quasi-coassociative and counitary i.e. to satisfy

$$(H \otimes \Delta)\Delta(h) = \phi \cdot (\Delta \otimes H)\Delta(h) \cdot \phi^{-1},$$
$$(\varepsilon \otimes H)\Delta(h) = \lambda^{-1} h \lambda, \qquad (H \otimes \varepsilon)\Delta(h) = \rho^{-1} h \rho.$$

A morphism of quasi-bialgebras $\Xi : (H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho) \to (H', m', u', \Delta', \varepsilon', \phi', \lambda', \rho')$ (see [26, page 371]) is an algebra homomorphism $\Xi : (H, m, u) \to (H', m', u')$ such that $(\Xi \otimes \Xi)\Delta = \Delta'\Xi$, $\varepsilon'\Xi = \varepsilon$, $(\Xi \otimes \Xi \otimes \Xi)(\phi) = \phi'$, $\Xi(\lambda) = \lambda'$ and $\Xi(\rho) = \rho'$. It is an isomorphism of quasi-bialgebras if, in addition, it is invertible. We will adopt the standard notation

$$\phi^1 \otimes \phi^2 \otimes \phi^3 := \phi \quad \text{(summation understood)}.$$

In the case when ϕ is not trivial and $\lambda = \rho = 1_H$, we call H an *ordinary quasi-bialgebra*. If further ϕ is trivial we then land at the classical concept of bialgebra.

A quasi-subbialgebra of a quasi-bialgebra H' is a quasi-bialgebra H such that H is a vector subspace of H' and the canonical inclusion is a morphism of dual quasi-bialgebras.

Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra. It is well-known, see [26, page 285 and Proposition XV.1.2], that the category ${}_H\mathfrak{M}$ of left H-modules becomes a monoidal category as follows. Given a left H-module V, we denote by $\mu = \mu_V^l : H \otimes V \to V$, $\mu(h \otimes v) = hv$, its left H-action. The tensor product of two left H-modules V and W is a module via diagonal action i.e. $h(v \otimes w) = h_1 v \otimes h_2 w$. The unit is $\Bbbk$, which is regarded as a left H-module via the trivial action i.e. $hk = \varepsilon(h) k$, for all $h \in H, k \in \Bbbk$. The associativity and unit constraints are defined, for all $V, W, Z \in {}_H\mathfrak{M}$ and $v \in V, w \in W, z \in Z$, by $a_{V,W,Z}((v \otimes w) \otimes z) := \phi^1 v \otimes (\phi^2 w \otimes \phi^3 z)$, $l_V(1 \otimes v) := \lambda v$ and $r_V(v \otimes 1) := \rho v$. This monoidal category will be denoted by $({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r)$. Given an invertible element $\alpha \in H \otimes H$, we can construct a new quasi-bialgebra $H_\alpha = (H, m, u, \Delta_\alpha, \varepsilon, \phi_\alpha, \lambda_\alpha, \rho_\alpha)$ where

$$\Delta_\alpha(h) = \alpha \cdot \Delta(h) \cdot \alpha^{-1}, \quad \lambda_\alpha = \lambda \cdot (\varepsilon_H \otimes H)\left(\alpha^{-1}\right), \quad \rho_\alpha = \rho \cdot (H \otimes \varepsilon_H)\left(\alpha^{-1}\right),$$
$$\phi_\alpha = (1_H \otimes \alpha) \cdot (H \otimes \Delta)(\alpha) \cdot \phi \cdot (\Delta \otimes H)\left(\alpha^{-1}\right) \cdot \left(\alpha^{-1} \otimes 1_H\right).$$

Definition 9.3. We refer to [26, Proposition XV.2.2] but with a different terminology (cf. [17, page 1439]). A quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ is called *quasi-triangular* whenever there exists an invertible element $R \in H \otimes H$ such that, for every $h \in H$, one has

$$(\Delta \otimes H)(R) = \begin{bmatrix} \left(\phi^2 \otimes \phi^3 \otimes \phi^1\right)\left(R^1 \otimes 1 \otimes R^2\right)\left(\phi^1 \otimes \phi^3 \otimes \phi^2\right)^{-1} \\ \left(1 \otimes R^1 \otimes R^2\right)\left(\phi^1 \otimes \phi^2 \otimes \phi^3\right) \end{bmatrix}$$
$$(H \otimes \Delta)(R) = \begin{bmatrix} \left(\phi^3 \otimes \phi^1 \otimes \phi^2\right)^{-1}\left(R^1 \otimes 1 \otimes R^2\right)\left(\phi^2 \otimes \phi^1 \otimes \phi^3\right) \\ \left(R^1 \otimes R^2 \otimes 1\right)\left(\phi^1 \otimes \phi^2 \otimes \phi^3\right)^{-1} \end{bmatrix}$$
$$\Delta^{cop}(h) = R\Delta(h) R^{-1}$$

where $\phi := \phi^1 \otimes \phi^2 \otimes \phi^3$, $R = R^1 \otimes R^2$. A *morphism of quasi-triangular quasi-bialgebras* is a morphism $\Xi : H \to H'$ of quasi-bialgebras such that $(\Xi \otimes \Xi)(R) = R'$.

By [26, Proposition XV.2.2], ${}_H\mathfrak{M} = ({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r)$ is braided if and only if there is an invertible element $R \in H \otimes H$ such that $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho, R)$ is quasi-triangular. Note that the braiding is given, for all $X, Y \in {}_H\mathfrak{M}$, by

$$c_{X,Y} : X \otimes Y \to Y \otimes X : x \otimes y \mapsto R^2 y \otimes R^1 x.$$

Moreover ${}_H\mathfrak{M}$ is symmetric if and only if we further assume $R^2 \otimes R^1 = R^{-1}$. Such a quasi-bialgebra will be called a *triangular quasi-bialgebra*. A morphism of triangular quasi-bialgebras is just a morphism of the underlying quasi-triangular quasi-bialgebras structures.

Given an invertible element $\alpha \in H \otimes H$, if H is (quasi-)triangular so is H_α with respect to $R_\alpha = \left(\alpha^2 \otimes \alpha^1\right) R\alpha^{-1}$, where $\alpha := \alpha^1 \otimes \alpha^2$.

Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra. We want to apply Theorem 8.5 to the case $\mathcal{M} = {}_H\mathfrak{M}$. Let $F : {}_H\mathfrak{M} \to \mathfrak{M}$ be the forgetful functor. We need a monoidal $(F, \psi_0, \psi_2) : ({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r) \to \mathfrak{M}$.

Lemma 9.4. *Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra. Let $F : {}_H\mathfrak{M} \to \mathfrak{M}$ be the forgetful functor. The following are equivalent.*

(1) *There is a natural transformation ψ_2 such that $(F, \mathrm{Id}_\Bbbk, \psi_2) : ({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r) \to \mathfrak{M}$ is monoidal.*

(2) *There is an invertible element $\alpha \in H \otimes H$ such that H_α is an ordinary bialgebra.*

(3) *There is an invertible element $\alpha \in H \otimes H$ such that*

$$\phi = (H \otimes \Delta)\left(\alpha^{-1}\right) \cdot \left(1_H \otimes \alpha^{-1}\right) \cdot (\alpha \otimes 1_H) \cdot (\Delta \otimes H)(\alpha), \tag{101}$$

$$(\varepsilon_H \otimes H)(\alpha) = \lambda, \qquad (H \otimes \varepsilon_H)(\alpha) = \rho. \tag{102}$$

Moreover, if (2) holds, we can choose $\psi_2(V, W)(v \otimes w) := \alpha^{-1}(v \otimes w)$.

Proof. (1) $\Leftrightarrow$ (2) Cf. [2, Proposition 1]. (2) $\Leftrightarrow$ (3) We have that H_α is an ordinary bialgebra if and only if $\phi_\alpha = 1_H \otimes 1_H \otimes 1_H, \lambda_\alpha = 1_H$ and $\rho_\alpha = 1_H$, if and only if α fulfills the equations in (3). □

Note that $F : {}_H\mathfrak{M} \to \mathfrak{M}$ is clearly conservative and preserves equalizers, epimorphisms and coequalizers. Furthermore we need ${}_H\mathfrak{M}$ to be braided.

Theorem 9.5. *Let $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ be a quasi-bialgebra such that (101) and (102) hold for some invertible element $\gamma \in H \otimes H$. Let $\mathcal{M}$ be the monoidal category $({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r)$ of left modules over H. Assume* $\mathrm{char}\Bbbk = 0$*. Then $\mathcal{M}$ is a MM-category. In particular, if $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ is endowed with a triangular structure, then $\mathcal{M}$ is a symmetric MM-category.*

Proof. First note that $\mathcal{M}$ is a Grothendieck category. In $\mathcal{M}$ the tensor products are exact and preserve denumerable coproducts. We can apply Theorem 8.5 to the monoidal functor $(F, \mathrm{Id}_\Bbbk, \psi_2) : ({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r) \to \mathfrak{M}$ of Lemma 9.4. Then $\mathcal{M}$ is a MM-category.

If $(H, m, u, \Delta, \varepsilon, \phi, \lambda, \rho)$ is endowed with a triangular structure, by the foregoing $\mathcal{M}$ is also symmetric monoidal. □

Example 9.6. Let H be a bialgebra over a field $\Bbbk$ of characteristic zero. Then H is a quasi-bialgebra with ϕ, λ, ρ trivial. Note that (101) and (102) hold for $\gamma = \varepsilon_H \otimes \varepsilon_H$. Thus, by Theorem 9.5, the monoidal category ${}_H\mathfrak{M}$ of left modules over H is an MM-category.

Example 9.7. Examples of triangular quasi-bialgebra structures on the group algebra $\Bbbk[G]$ over a torsion-free abelian group G are investigated in [2, Proposition 3]. Consider the particular case when $G = \langle g \rangle$ is the group $\mathbb{Z}$ in multiplicative notation, where g is a generator. Let $(q, a, b) \in (\Bbbk \backslash \{0\}) \times \mathbb{Z} \times \mathbb{Z}$. In view of [2, Proposition 3], we have the triangular quasi-bialgebra

$$\Bbbk[\langle g \rangle]_q^{a,b} = (\Bbbk[\langle g \rangle], \Delta, \varepsilon, \phi, \lambda, \rho, R)$$

on the group algebra $\Bbbk[\langle g \rangle]$ which is defined by

$$\begin{aligned} \Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad \phi = g^a \otimes 1_H \otimes g^b \\ \lambda = q g^{-b}, \qquad \rho = q g^a, \qquad R = g^{a+b} \otimes g^{-a-b}. \end{aligned}$$

In order to apply Theorem 9.5 in case $H = \Bbbk[\langle g \rangle]_q^{a,b}$, we must check that (101) and (102) hold for some invertible element $\gamma \in H \otimes H$. By [2, Theorem 2], one has $\Bbbk[\langle g \rangle]_q^{a,b} = \Bbbk[\langle g \rangle]_\alpha$ where $\alpha := q^{-1} g^{-a} \otimes g^b$ and $\Bbbk[\langle g \rangle]$ is the usual bialgebra structure on the group algebra regarded as a trivial triangular quasi-bialgebras (i.e. ϕ, λ, ρ, R are all trivial). Set $\gamma := \alpha^{-1} = q g^a \otimes g^{-b}$. Then $H_\gamma = \Bbbk \langle g \rangle$ which is an ordinary bialgebra so that, by Lemma 9.4 we have that (101) and (102) hold for our γ. Hence by Theorem 9.5 the symmetric monoidal category $({}_H\mathfrak{M}, \otimes, \Bbbk, a, l, r)$ of left modules over H is a MM-category.

Definition 9.8. Let $\mathcal{C}$ be an ordinary category. Following [16, Section 1], we associate to $\mathcal{C}$ a new category $\mathcal{H}(\mathcal{C})$ as follows. Objects are pairs (M, f_M) with $M \in \mathcal{C}$ and $f_M \in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $\xi : (M, f_M) \to (N, f_N)$ is a morphism $\xi : M \to N$ in $\mathcal{C}$ such that $f_N \circ \xi = \xi \circ f_M$. The category $\mathcal{H}(\mathcal{C})$ is called the *Hom-category* associated to $\mathcal{C}$.

Example 9.9. Take $\mathcal{C} := \mathfrak{M}$. In view of [2, Theorem 4], to each datum $(q, a, b) \in (\Bbbk \backslash \{0\}) \times \mathbb{Z} \times \mathbb{Z}$ one associates a monoidal category

$$\mathcal{H}_q^{a,b}(\mathfrak{M}) = (\mathcal{H}(\mathfrak{M}), \otimes, (\Bbbk, f_\Bbbk), a, l, r)$$

which consists of the category $\mathcal{H}(\mathfrak{M})$ equipped with a suitable braided (actually symmetric) monoidal structure. By [2, Theorem 4] there is a strict symmetric monoidal category isomorphism

$$(W, w_0, w_2) : {}_{\Bbbk[\langle g \rangle]_q^{a,b}}\mathfrak{M} \to \mathcal{H}_q^{a,b}(\mathfrak{M}).$$

The underlying functor $W : {}_{\Bbbk[\langle g \rangle]}\mathfrak{M} \to \mathcal{H}(\mathfrak{M})$ is given on objects by

$$W\left(X,\mu_{X}:\Bbbk[\langle g\rangle]\otimes X\rightarrow X\right)=\left(X,f_{X}:X\rightarrow X\right),$$

where $f_X(x) := \mu_X(g \otimes x)$, for all $x \in X$, and on morphisms by $W\xi = \xi$.

Composing W^{-1} with the forgetful functor ${}_{\Bbbk[\langle g\rangle]^{a,b}_{q}}\mathfrak{M} \to \mathfrak{M}$ we get a monoidal functor $\mathcal{H}^{a,b}_q(\mathfrak{M}) \to \mathfrak{M}$ to which we can apply Theorem 8.5 to get that $\mathcal{H}^{a,b}_q(\mathfrak{M})$ is an MM-category.

Remark 9.10. By [2, Proposition 5], $\mathcal{M} := \mathcal{H}^{1,-1}_1(\mathfrak{M})$ is the symmetric braided monoidal category $\widetilde{\mathcal{H}}(\mathfrak{M})$ of [16, Proposition 1.1]. Thus, by the foregoing, $\widetilde{\mathcal{H}}(\mathfrak{M})$ is a MM-category. By [16, page 2236], an object in $(M,[-]) \in \mathrm{Lie}_{\mathcal{M}}$ is nothing but a monoidal Hom-Lie algebra. By Remark 6.5, $\overline{\mathcal{U}}(M,[-])$ as a bialgebra is a quotient of $\overline{T}M$. The morphism giving the projection is induced by the canonical projection $p_R : \Omega TM \to R := \mathcal{U}^s_{\mathrm{Br}}J_{\mathrm{Lie}}(M,[-])$ defining the universal enveloping algebra. At algebra level we have

$$\begin{aligned}\mho\overline{\mathcal{U}}(M,[-]) &= \frac{TM}{\left(f_{J^s_{\mathrm{Lie}}(M,[-])}(x\otimes y)\,|x,y\in M\right)}\\ &= \frac{TM}{\left([x,y]-\theta_{(M,c_{M,M})}(x\otimes y)\,|x,y\in M\right)}\\ &= \frac{TM}{\left([x,y]-x\otimes y+c_{M,M}(x\otimes y)\,|x,y\in M\right)}\\ &= \frac{TM}{\left([x,y]-x\otimes y+y\otimes x|x,y\in M\right)}\end{aligned}$$

which is the Hom-version of the universal enveloping algebra, see [16, Section 8]. Note that, as a by-product, we have that $\overline{\eta}_{\mathrm{L}} : \mathrm{Id}_{\mathrm{Lie}_{\mathcal{M}}} \to \mathcal{P}\overline{\mathcal{U}}$ is an isomorphism so that $(M,[-]) \cong \mathcal{P}\overline{\mathcal{U}}(M,[-])$.

9.2. Dual quasi-bialgebras

First, observe that dual quasi-bialgebras can be understood as a dual version of quasi-bialgebras just in the finite-dimensional case. In fact, for an infinite-dimensional quasi-bialgebra H (as in the case for $H = \Bbbk\mathbb{Z}$ considered above) it is not true that the dual is a dual quasi-bialgebra so that the results in the two settings are independent, in general.

Definition 9.11. A *dual quasi-bialgebra* is a datum $(H,m,u,\Delta,\varepsilon,\omega)$ where (H,Δ,ε) is a coassociative coalgebra, $m : H \otimes H \to H$ and $u : \Bbbk \to H$ are coalgebra maps called multiplication and unit respectively, we set $1_H := u(1_\Bbbk)$, $\omega : H \otimes H \otimes H \to \Bbbk$ is a unital 3-cocycle i.e. it is convolution invertible and satisfies

$$\omega(H\otimes H\otimes m)*\omega(m\otimes H\otimes H)=(\varepsilon\otimes\omega)*\omega(H\otimes m\otimes H)*(\omega\otimes\varepsilon) \tag{103}$$

$$\text{and}\quad \omega(h\otimes k\otimes l)=\varepsilon(h)\varepsilon(k)\varepsilon(l)\quad\text{whenever } 1_H\in\{h,k,l\}. \tag{104}$$

Moreover m is quasi-associative and unitary i.e. it satisfies

$$m(H \otimes m) * \omega = \omega * m(m \otimes H),$$
$$m(1_H \otimes h) = h \qquad \text{and} \qquad m(h \otimes 1_H) = h, \qquad \text{for all } h \in H.$$

The map ω is called *the reassociator* of the dual quasi-bialgebra.

A morphism of dual quasi-bialgebras $\Xi : (H, m, u, \Delta, \varepsilon, \omega) \to (H', m', u', \Delta', \varepsilon', \omega')$ is a coalgebra homomorphism $\Xi : (H, \Delta, \varepsilon) \to (H', \Delta', \varepsilon')$ such that $m'(\Xi \otimes \Xi) = \Xi m$, $\Xi u = u'$ and $\omega'(\Xi \otimes \Xi \otimes \Xi) = \omega$. It is an isomorphism of dual quasi-bialgebras if, in addition, it is invertible.

A dual quasi-subbialgebra of a dual quasi-bialgebra H' is a quasi-bialgebra H such that H is a vector subspace of H' and the canonical inclusion is a morphism of dual quasi-bialgebras.

Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. It is well-known that the category $\mathfrak{M}^H$ of right H-comodules becomes a monoidal category as follows. Given a right H-comodule V, we denote by $\rho = \rho_V^r : V \to V \otimes H, \rho(v) = v_0 \otimes v_1$, its right H-coaction. The tensor product of two right H-comodules V and W is a comodule via diagonal coaction i.e. $\rho(v \otimes w) = v_0 \otimes w_0 \otimes v_1 w_1$. The unit is $\Bbbk$, which is regarded as a right H-comodule via the trivial coaction i.e. $\rho(k) = k \otimes 1_H$. The associativity and unit constraints are defined, for all $U, V, W \in \mathfrak{M}^H$ and $u \in U, v \in V, w \in W, k \in \Bbbk$, by $a_{U,V,W}((u \otimes v) \otimes w) := u_0 \otimes (v_0 \otimes w_0)\omega(u_1 \otimes v_1 \otimes w_1)$, $l_U(k \otimes u) := ku$ and $r_U(u \otimes k) := uk$. This monoidal category will be denoted by $(\mathfrak{M}^H, \otimes, \Bbbk, a, l, r)$.

Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. Let $v : H \otimes H \to \Bbbk$ be a *gauge transformation* i.e. a convolution invertible map such that $v(1_H \otimes h) = \varepsilon(h) = v(h \otimes 1_H)$ for all $h \in H$. Then $H^v := (H, m^v, u, \Delta, \varepsilon, \omega^v)$ is also a dual quasi-bialgebra where

$$m^v := v * m * v^{-1} \tag{105}$$
$$\omega^v := (\varepsilon \otimes v) * v(H \otimes m) * \omega * v^{-1}(m \otimes H) * \left(v^{-1} \otimes \varepsilon\right). \tag{106}$$

Definition 9.12. A dual quasi-bialgebra $(H, m, u, \Delta, \varepsilon, \omega)$ is called *quasi-co-triangular* whenever there exists $R \in \mathrm{Reg}\left(H^{\otimes 2}, \Bbbk\right)$ such that

$$R(m \otimes H) = \begin{bmatrix} \omega\tau_{H\otimes H,H} * R(H \otimes l_H)(H \otimes \varepsilon \otimes H) \\ *\omega^{-1}(H \otimes \tau_{H,H}) * m_\Bbbk(\varepsilon \otimes R) * \omega \end{bmatrix},$$
$$R(H \otimes m) = \begin{bmatrix} \omega^{-1}\tau_{H,H\otimes H} * R(H \otimes l_H)(H \otimes \varepsilon \otimes H) \\ *\omega(\tau_{H,H} \otimes H) * m_\Bbbk(R \otimes \varepsilon) * \omega^{-1} \end{bmatrix},$$
$$m\tau_{H,H} * R = R * m.$$

By [28, Exercice 9.2.9, page 437], [26, dual to Proposition XIII.1.4, page 318], $\mathfrak{M}^H = \left(\mathfrak{M}^H, \otimes_\Bbbk, \Bbbk, a, l, r\right)$ is braided if and only if there is a map $R \in \mathrm{Reg}\left(H^{\otimes 2}, \Bbbk\right)$ such that

$(H, m, u, \Delta, \varepsilon, \omega, R)$ is quasi-co-triangular. Note that the braiding is given, for all $X, Y \in \mathfrak{M}^H$, by

$$c_{X,Y} : X \otimes Y \to Y \otimes X : x \otimes y \mapsto \sum y_{\langle 0 \rangle} \otimes x_{\langle 0 \rangle} R\left(x_{\langle 1 \rangle} \otimes y_{\langle 1 \rangle}\right).$$

Moreover $\mathfrak{M}^H$ is symmetric if and only if $c_{Y,X} \circ c_{X,Y} = \mathrm{Id}_{X \otimes Y}$ for all $X, Y \in \mathfrak{M}^H$ i.e. if and only if

$$\sum x_{\langle 0 \rangle} \otimes y_{\langle 0 \rangle} R\left(y_{\langle 1 \rangle} \otimes x_{\langle 1 \rangle}\right) R\left(x_{\langle 2 \rangle} \otimes y_{\langle 2 \rangle}\right) = x \otimes y.$$

This is equivalent to requiring that

$$R\left(h_{\langle 1 \rangle} \otimes l_{\langle 1 \rangle}\right) R\left(l_{\langle 2 \rangle} \otimes h_{\langle 2 \rangle}\right) = \varepsilon_H(h) \varepsilon_H(l), \text{ for every } h, l \in H. \tag{107}$$

Such a dual quasi-bialgebra will be called a *co-triangular dual quasi-bialgebra.*

Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. We want to apply Theorem 8.5 to the case $\mathcal{M} = \mathfrak{M}^H$. We need a monoidal functor $(F, \phi_0, \phi_2) : (\mathfrak{M}^H, \otimes, \Bbbk, a, l, r) \to \mathfrak{M}$. Take $F : \mathfrak{M}^H \to \mathfrak{M}$ to be the forgetful functor. Note that F is clearly conservative and preserves equalizers, epimorphisms and coequalizers. Note also that we will further need $\mathfrak{M}^H$ to be braided.

Lemma 9.13. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra. Let $F : \mathfrak{M}^H \to \mathfrak{M}$ be the forgetful functor. The following are equivalent.*

(1) *There is a natural transformation ψ_2 such that $(F, \mathrm{Id}_\Bbbk, \phi_2) : (\mathfrak{M}^H, \otimes, \Bbbk, a, l, r) \to \mathfrak{M}$ is monoidal.*

(2) *There is a gauge transformation $v : H \otimes H \to \Bbbk$ such that H^v is an ordinary bialgebra.*

(3) *There is a gauge transformation $v : H \otimes H \to \Bbbk$ such that*

$$\omega = v^{-1}(H \otimes m) * \left(\varepsilon \otimes v^{-1}\right) * (v \otimes \varepsilon) * v(m \otimes H) \tag{108}$$

Moreover, if (2) *holds, we can choose $\phi_2(V, W)(x \otimes y) = x_0 \otimes y_0 v^{-1}(x_1 \otimes y_1)$.*

Proof. It is similar to the one of Lemma 9.4. □

Lemma 9.14. *(Cf. [28, Lemma 2.2.2].) Let $(H, m, u, \Delta, \varepsilon, \omega, R)$ be a quasi-co-triangular dual quasi-bialgebra. Then R is unital i.e. $R(1_H \otimes h) = \varepsilon(h) = R(h \otimes 1_H)$ for all $h \in H$.*

Theorem 9.15. *Let $(H, m, u, \Delta, \varepsilon, \omega)$ be a dual quasi-bialgebra such that ω fulfills (108) for some gauge transformation $\gamma : H \otimes H \to \Bbbk$. Let $\mathcal{M}$ be the monoidal category $\left(\mathfrak{M}^H, \otimes_\Bbbk, \Bbbk, a, l, r\right)$ of right comodules over H. Assume* $\mathrm{char}\Bbbk = 0$. *Then $\mathcal{M}$ is a MM-category. In particular, if $(H, m, u, \Delta, \varepsilon, \omega)$ is endowed with a co-triangular structure, then $\mathcal{M}$ is a symmetric MM-category.*

Proof. It is analogous to the proof of Theorem 9.5 but using, from Lemma 9.13, the functor $(F, \mathrm{Id}_{\Bbbk}, \phi_2) : (\mathfrak{M}^H, \otimes, \Bbbk, a, l, r) \to \mathfrak{M}$. $\square$

Example 9.16. Let H be a bialgebra over a field $\Bbbk$ of characteristic zero. Then H is a dual quasi-bialgebra with reassociator $\omega = \varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H$. Note that ω fulfills (108) for $\gamma = \varepsilon_H \otimes \varepsilon_H$. Thus, by Theorem 9.15, the monoidal category $\mathfrak{M}^H$ of right comodules over H is a MM-category. In particular, for $H = \Bbbk[\mathbb{N}]$, the monoid bialgebra over the naturals, defined by taking $\Delta n = n \otimes n$ and $\varepsilon(n) = 1$ for every $n \in \mathbb{N}$, then the category $\mathfrak{M}^H$ is the category of $\mathbb{N}$-graded vector spaces $V = \oplus_{n\in\mathbb{N}} V_n$ with monoidal structure having tensor product given by $(V \otimes W)_n = \oplus_{i=0}^{n} (V_i \otimes W_{n-i})$ and unit $\Bbbk$ concentrated in degree 0. The constraints are the same of vector spaces. The category $\mathfrak{M}^H$ is braided with respect to the canonical flip (this can be seen by showing that $R = \varepsilon_H \otimes \varepsilon_H$ turns H into a co-triangular bialgebra, see remark below).

Remark 9.17. Let $(H, m, u, \Delta, \varepsilon, \omega, R)$ be a co-triangular dual quasi-bialgebra. Assume that ω fulfills (108) for $\gamma = \varepsilon_H \otimes \varepsilon_H$. This means $\omega = \varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H$ and $(H, m, u, \Delta, \varepsilon, R)$ is a co-triangular bialgebra i.e. for every $x, y, z \in H$ we have

$$R(xy \otimes z) = R(x \otimes z_1) R(y \otimes z_2), \quad R(x \otimes yz) = R(x_1 \otimes z) R(x_2 \otimes y),$$
$$y_1 x_1 R(x_2 \otimes y_2) = R(x_1 \otimes y_1) x_2 y_2.$$

Let $(M, [-]) \in \mathrm{Lie}_{\mathcal{M}}$. Then (50) and (56) become

$$[x, y] = -\sum \left[y_{\langle 0\rangle}, x_{\langle 0\rangle}\right] R\left(x_{\langle 1\rangle} \otimes y_{\langle 1\rangle}\right),$$
$$\sum [[x, y], z] + \sum \left[\left[y_{\langle 0\rangle}, z_{\langle 0\rangle}\right], x_{\langle 0\rangle}\right] R\left(x_{\langle 1\rangle} \otimes y_{\langle 1\rangle} z_{\langle 1\rangle}\right)$$
$$+ \sum \left[\left[z_{\langle 0\rangle}, x_{\langle 0\rangle}\right], y_{\langle 0\rangle}\right] R\left(x_{\langle 1\rangle} y_{\langle 1\rangle} \otimes z_{\langle 1\rangle}\right) = 0.$$

This means that $(M, [-])$ is an (H, R)-Lie algebra in the sense of [13, Definition 4.1]. By Remark 6.5, $\overline{\mathcal{U}}(M, [-])$ as a bialgebra is a quotient of $\overline{T}M$. The morphism giving the projection is induced by the canonical projection $p_R : \Omega TM \to R := \mathcal{U}^s_{\mathrm{Br}} J_{\mathrm{Lie}}(M, [-])$ defining the universal enveloping algebra. At algebra level we have

$$\mathcal{U}(M, [-]) \overset{(78)}{=} \mho\overline{\mathcal{U}}(M, [-]) = \frac{TM}{\left(f_{J^s_{\mathrm{Lie}}(M,[-])}(x \otimes y) \,|\, x, y \in M\right)}$$
$$= \frac{TM}{\left([x, y] - \theta_{(M, c_{M,M})}(x \otimes y) \,|\, x, y \in M\right)} = \frac{TM}{\left([x, y] - x \otimes y + c_{M,M}(x \otimes y) \,|\, x, y \in M\right)}$$
$$= \frac{TM}{\left([x, y] - x \otimes y + \sum y_{\langle 0\rangle} \otimes x_{\langle 0\rangle} R\left(x_{\langle 1\rangle} \otimes y_{\langle 1\rangle}\right) \,|\, x, y \in M\right)}$$

which is the universal enveloping algebra of our (H, R)-Lie algebra, see e.g. [18, (2.6)]. Note that, as a by-product, we have that $\bar{\eta}_{\mathrm{L}} : \mathrm{Id}_{\mathrm{Lie}_{\mathcal{M}}} \to \mathcal{P}\overline{\mathcal{U}}$ is an isomorphism so that $(M, [-]) \cong \mathcal{P}\overline{\mathcal{U}}(M, [-])$.

Example 9.18. Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) = 0$ and let G be an abelian group endowed with an anti-symmetric bicharacter $\chi : G \times G \to \Bbbk \setminus \{0\}$, i.e. for all $g, h, k \in G$, we have:

$$\chi(g, hk) = \chi(g, h)\chi(g, k), \quad \chi(gh, k) = \chi(g, k)\chi(h, k), \quad \chi(g, h)\chi(h, g) = 1.$$

Extend χ by linearity to a $\Bbbk$-linear map $R : \Bbbk[G] \otimes \Bbbk[G] \to \Bbbk$, where $\Bbbk[G]$ denotes the group algebra. Then $(\Bbbk[G], R)$ is a co-triangular bialgebra, cf. [28, Example 2.2.5]. Hence, we can apply Theorem 9.15 and Remark 9.17 to $H = \Bbbk[G]$. Note that the category $\left(\mathfrak{M}^H, \otimes_\Bbbk, \Bbbk, a, l, r, c\right)$ consists of G-graded modules $V = \oplus_{g\in G} V_g$. Given G-graded modules V and W, their tensor product $V \otimes W$ is graded with $(V \otimes W)_g := \oplus_{hl=g} (V_h \otimes W_l)$. The braiding is given on homogeneous elements by

$$c_{V,W} : V \otimes W \to W \otimes V, \quad c_{V,W}(v \otimes w) = w \otimes v\chi(|v|, |w|),$$

where $|v|$ denotes the degree of v. In this case a (H, R)-Lie algebra $(V, [-, -])$ in the sense of [13, Definition 4.1] means

$$[x, y] = -[y, x]\,\chi(|x|, |y|),$$
$$[[x, y], z] + \sum [[y, z], x]\,\chi(|x|, |y|\,|z|) + \sum [[z, x], y]\,\chi(|x|\,|y|, |z|) = 0.$$

Multiplying by $\chi(|z|, |x|)$ the two sides of the second equality, we get the equivalent

$$[[x, y], z]\,\chi(|z|, |x|) + \sum [[y, z], x]\,\chi(|x|, |y|) + \sum [[z, x], y]\,\chi(|y|, |z|) = 0.$$

This means that $(V, [-, -])$ is a (G, χ)-Lie color algebra in the sense of [33, Example 10.5.14]. Note that the braiding defined in [33, page 200] is $c'_{V,W}(v \otimes w) = w \otimes v\chi(|w|, |v|) = w \otimes v\chi^{-1}(|v|, |w|)$ so that we should say more precisely that $(V, [-, -])$ is a $\left(G, \chi^{-1}\right)$-Lie color algebra. The corresponding enveloping algebra is

$$\mathcal{U}(V, [-]) = \frac{TV}{([x, y] - x \otimes y + y \otimes x\chi(|x|, |y|) \mid x, y \in V \text{ homogeneous})}.$$

Example 9.19. Lie superalgebras are a particular instance of the construction above. One has to take $G = \mathbb{Z}_2$ and consider the anti-symmetric bicharacter $\chi : G \times G \to \Bbbk \setminus \{0\}$ defined by $\chi\left(\bar{a}, \bar{b}\right) := (-1)^{ab}$ for all $a, b \in \mathbb{Z}$.

Example 9.20. Let $G := (\mathbb{Z}, +, 0)$. Let $\Bbbk$ be a field and let $q \in \Bbbk \setminus \{0\}$. Then it is easy to check that $\langle -, - \rangle : G \times G \to \Bbbk$, $\langle a, b \rangle := q^{ab}$ is a bicharacter of G.

Remark 9.21. Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) = 0$. Let H be a finite-dimensional Hopf algebra. By [36, Proposition 6], the category of Yetter–Drinfeld modules ${}^H_H\mathcal{YD}$ and ${}_H\mathcal{YD}^H$ are isomorphic. Moreover, by [33, Proposition 10.6.16], the ${}_H\mathcal{YD}^H$ can be identified with the category ${}_{D(H)}\mathfrak{M}$ of left modules over the Drinfeld double $D(H)$. Now ${}_{D(H)}\mathfrak{M} \cong \mathfrak{M}^{D(H)^*}$

and $D(H)^*$ is a quasi-co-triangular bialgebra. Thus we can identify ${}_H^H\mathcal{YD}$ with $\mathfrak{M}^{D(H)^*}$. One is tempted to apply Theorem 9.15. Unfortunately, $D(H)$ is never triangular (whence $D(H)^*$ is never co-triangular) in view of [35], unless $H = \Bbbk$.

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Appendix A. (Co)equalizers and (co)monadicity

Definition A.1. (See [31, page 112].) Let $\mathcal{I}$ be a small category. Recall that a functor $V : \mathcal{A} \to \mathcal{B}$ **creates limits for a functor** $F : \mathcal{I} \to \mathcal{A}$ if in case VF has a limit $\left(X, (\tau_I : X \to VFI)_{I\in\mathcal{I}}\right)$, then there is exactly one pair $\left(L, (\sigma_I : L \to FI)_{I\in\mathcal{I}}\right)$ which is a limit of F and such that $VL = X$, $V\sigma_I = \tau_I$ for every $I \in \mathcal{I}$. We just say that $V : \mathcal{A} \to \mathcal{B}$ **creates limits** if it creates limits for all functors $F : \mathcal{I} \to \mathcal{A}$ and for all small category $\mathcal{I}$. Similarly one defines creation of colimits.

Lemma A.2. *Let $\mathcal{M}$ be a monoidal category. Then the functor $\Omega : \mathrm{Alg}_{\mathcal{M}} \to \mathcal{M}$ creates limits and the functor $\mho : \mathrm{Coalg}_{\mathcal{M}} \to \mathcal{M}$ creates colimits.*

Proof. It is straightforward. □

A.3. Let $\mathcal{M}$ be a monoidal category. Assume that $\mathcal{M}$ has coequalizers and that the tensor products preserve them. It is well-known that $\mathrm{Alg}_{\mathcal{M}}$ has coequalizers, see e.g. [3, Proposition 2.1.5]. Given an algebra morphism $\alpha : E \to A$, consider $\Lambda_\alpha := m_A^2 \circ (A \otimes \alpha \otimes A)$ of (58) where $m_A^2 : A \otimes A \otimes A \to A$ is the iterated multiplication. The coequalizer of algebra morphisms $\alpha, \beta : E \to A$ is, as an object in $\mathcal{M}$, the coequalizer $(B, \pi : A \to B)$ of $(\Lambda_\alpha, \Lambda_\beta)$ in $\mathcal{M}$ and the algebra structure is the unique one making π an algebra morphism.

Lemma A.4. *Let $\mathcal{M}$ be a monoidal category.*

1) If $\mathcal{M}$ has coequalizers then $\mathrm{Coalg}_{\mathcal{M}}$ has coequalizers, and $\mho : \mathrm{Coalg}_{\mathcal{M}} \to \mathcal{M}$ preserves coequalizers. Moreover if the tensor products preserve the coequalizers in $\mathcal{M}$, then $\mathrm{Alg}_{\mathcal{M}}$ has coequalizers.

2) If $\mathcal{M}$ has equalizers then $\mathrm{Alg}_{\mathcal{M}}$ has equalizers, and $\Omega : \mathrm{Alg}_{\mathcal{M}} \to \mathcal{M}$ preserves equalizers. Moreover if the tensor products preserve the equalizers in $\mathcal{M}$, then $\mathrm{Coalg}_{\mathcal{M}}$ has equalizers.

3) If $\mathcal{M}$ is braided, it has coequalizers and the tensor products preserve them, then $\mathrm{Bialg}_{\mathcal{M}}$ has coequalizers and $\mho : \mathrm{Bialg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{M}}$ preserves coequalizers.

4) If $\mathcal{M}$ is braided, it has equalizers and the tensor products preserve them, then $\mathrm{Bialg}_{\mathcal{M}}$ has equalizers and $\Omega : \mathrm{Bialg}_{\mathcal{M}} \to \mathrm{Coalg}_{\mathcal{M}}$ preserves equalizers.

Proof. 1) The first part follows by Lemma A.2 and uniqueness of coequalizers in $\mathrm{Coalg}_{\mathcal{M}}$. By A.3, $\mathrm{Alg}_{\mathcal{M}}$ has coequalizers. 2) is dual to 1).

3) Note that $\mathrm{Bialg}_{\mathcal{M}} = \mathrm{Coalg}_{\mathcal{N}}$ for $\mathcal{N} := \mathrm{Alg}_{\mathcal{M}}$. By 1) we have that $\mathcal{N}$ has coequalizers and then $\mathrm{Coalg}_{\mathcal{N}}$ has coequalizers, and $\mho : \mathrm{Coalg}_{\mathcal{N}} \to \mathcal{N}$ preserves coequalizers. 4) is dual to 3). □

Lemma A.5. *Let $\mathcal{M}$ be a braided monoidal category. Assume that $\mathcal{M}$ is abelian and that the tensor products preserve equalizers, coequalizers.*

1) Let $\alpha : J_{\mathrm{Bialg}}D \to E$ be a morphism in $\mathrm{BrBialg}_{\mathcal{M}}$. Then there is a bialgebra $Q \in \mathrm{Bialg}_{\mathcal{M}}$ a morphism $\pi : D \to Q$ in $\mathrm{Bialg}_{\mathcal{M}}$ and a morphism $\sigma : J_{\mathrm{Bialg}}Q \to E$ in $\mathrm{BrBialg}_{\mathcal{M}}$ such that $\alpha = \sigma \circ J_{\mathrm{Bialg}}(\pi)$ and σ and π are a monomorphism and an epimorphism respectively when regarded as morphism in $\mathcal{M}$.

2) The functor $J_{\mathrm{Bialg}} : \mathrm{Bialg}_{\mathcal{M}} \to \mathrm{BrBialg}_{\mathcal{M}}$ preserves coequalizers.

3) Assume that $\mathcal{M}$ is symmetric. Then $J^s_{\mathrm{Bialg}} : \mathrm{Bialg}_{\mathcal{M}} \to \mathrm{BrBialg}^s_{\mathcal{M}}$ preserves coequalizers.

Proof. 1) Denote by D and E the underlying objects in $\mathcal{M}$ of D and E. Since $\mathcal{M}$ is abelian we can factor $\alpha : D \to E$ as the composition of a monomorphism $\sigma : Q \to E$ and an epimorphism $\pi : D \to Q$ in $\mathcal{M}$ where Q is the image of α in $\mathcal{M}$.

It is straightforward to check that Q fulfills the required properties.

2) By 4.4, we have $J_{\mathrm{Bialg}}(B, m_B, u_B, \Delta_B, \varepsilon_B) = (B, m_B, u_B, \Delta_B, \varepsilon_B, c_{B,B})$ and $J_{\mathrm{Bialg}}(f) = f$.

Let (e_0, e_1) from $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ to $(D, m_D, u_D, \Delta_D, \varepsilon_D)$ be a pair of morphisms in $\mathrm{Bialg}_{\mathcal{M}}$. Assume that this pair has coequalizer (E, p) in $\mathrm{Bialg}_{\mathcal{M}}$

$$B \underset{e_1}{\overset{e_0}{\rightrightarrows}} D \xrightarrow{p} E$$

Let us check that J_{Bialg} preserves this coequalizer. Let $\alpha : J_{\mathrm{Bialg}}D \to Z$ be a morphism in $\mathrm{BrBialg}_{\mathcal{M}}$ such that $\alpha e_0 = \alpha e_1$. By 1) we write $\alpha = \sigma \circ J_{\mathrm{Bialg}}(\pi)$. Since σ is a monomorphism in $\mathcal{M}$, we have that $\pi e_0 = \pi e_1$. Since the coequalizer (E, p) is in $\mathrm{Bialg}_{\mathcal{M}}$, there is a unique morphism $\overline{\pi} : E \to Q$ in $\mathrm{Bialg}_{\mathcal{M}}$ such that $\overline{\pi} \circ p = \pi$. Set $\overline{\alpha} := \sigma\overline{\pi} : E \to Z$ as morphisms in $\mathcal{M}$. Then $\overline{\alpha}p = \sigma\overline{\pi}p = \sigma\pi = \alpha$. Moreover σ and $\overline{\pi}$ commute with (co)multiplications and (co)units and

$$\begin{aligned}(\overline{\alpha} \otimes \overline{\alpha})\, c_{E,E} &= (\sigma \otimes \sigma)(\overline{\pi} \otimes \overline{\pi})\, c_{E,E} \\ &= (\sigma \otimes \sigma)\, c_{Q,Q}(\overline{\pi} \otimes \overline{\pi}) = c_Z(\sigma \otimes \sigma)(\overline{\pi} \otimes \overline{\pi}) = c_Z(\overline{\alpha} \otimes \overline{\alpha}).\end{aligned}$$

We have so proved that $\overline{\alpha}$ is a morphism in $\mathrm{BrBialg}_{\mathcal{M}}$ from $J_{\mathrm{Bialg}}E$ to Z.

Let $\beta : J_{\mathrm{Bialg}}E \to Z$ in $\mathrm{BrBialg}_{\mathcal{M}}$ be such that $\beta p = \alpha$ as morphisms in $\mathrm{BrBialg}_{\mathcal{M}}$. Then $\beta p = \overline{\alpha}p$ as morphisms in $\mathcal{M}$. Since (E, p) is a coequalizer in $\mathrm{Bialg}_{\mathcal{M}}$ and $\mathcal{M}$ has coequalizers (it is abelian) we have that (E, p) can be constructed as a suitable coequalizer in $\mathcal{M}$ (cf. the proof of Lemma A.4) so that p is an epimorphism in $\mathcal{M}$. Hence we get $\beta = \overline{\alpha}$ as morphisms in $\mathcal{M}$ whence in $\mathrm{BrBialg}_{\mathcal{M}}$.

3) By 2) $J_{\mathrm{Bialg}} : \mathrm{Bialg}_{\mathcal{M}} \to \mathrm{BrBialg}_{\mathcal{M}}$ preserves the coequalizers. Since $J_{\mathrm{Bialg}} = \mathbb{I}^s_{\mathrm{BrBialg}} \circ J^s_{\mathrm{Bialg}}$ we get that $\mathbb{I}^s_{\mathrm{BrBialg}} \circ J^s_{\mathrm{Bialg}}$ preserves coequalizers. Since $\mathbb{I}^s_{\mathrm{BrBialg}}$ is both full and faithful, it reflects colimits (see the dual of [11, Proposition 2.9.9]) so that J^s_{Bialg} preserves coequalizers. □

The following result can be obtained mimicking the proof of (1) ⇒ (2) in [12, Theorem 4.6.2]. For the reader's sake we write here a proof in the specific case we are concerned.

Theorem A.6. *Let $\mathcal{M}$ be a monoidal category.*

1) *If the forgetful functor $\Omega : \mathrm{Alg}_{\mathcal{M}} \to \mathcal{M}$ has a left adjoint, then Ω is monadic. In fact the comparison functor is a category isomorphism.*
2) *If the forgetful functor $\mho : \mathrm{Coalg}_{\mathcal{M}} \to \mathcal{M}$ has a right adjoint, then $\mho$ is comonadic. In fact the comparison functor is a category isomorphism.*

Proof. 1) We will apply Theorem [14, Theorem 2.1] (which is a form of Beck's Theorem). First, in order to prove that Ω is monadic, we have to check that Ω is conservative and that for any reflexive pair of morphisms in $\mathrm{Alg}_{\mathcal{M}}$ whose image by Ω has a split coequalizer has a coequalizer which is preserved by Ω. Clearly if f is a morphism in $\mathrm{Alg}_{\mathcal{M}}$ such that Ωf is an isomorphism then the inverse of Ωf is a morphism of monoids whence it gives rise to an inverse of f in $\mathrm{Alg}_{\mathcal{M}}$. Thus Ω is conservative.

Let (d_0, d_1) from A to A' be a reflexive pair as above. Then there exists $C \in \mathcal{M}$ and a morphism $c : \Omega A' \to C$ such that

$$\Omega A \underset{\Omega d_1}{\overset{\Omega d_0}{\rightrightarrows}} \Omega A' \xrightarrow{c} C$$

is a split coequalizer, whence preserved by any functor in particular by $F_n : \mathcal{M} \to \mathcal{M}$, the functor defined by $F_n := (-)^{\otimes n}$ i.e. the nth tensor power functor. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccc} \Omega A \otimes \Omega A & \underset{\Omega d_1 \otimes \Omega d_1}{\overset{\Omega d_0 \otimes \Omega d_0}{\rightrightarrows}} & \Omega A' \otimes \Omega A' & \xrightarrow{c \otimes c} & C \otimes C \\ \downarrow m_{\Omega A} & & \downarrow m_{\Omega A'} & & \\ \Omega A & \underset{\Omega d_1}{\overset{\Omega d_0}{\rightrightarrows}} & \Omega A' & \xrightarrow{c} & C \end{array}$$

By the universal property of coequalizers there is a unique morphism $m_C : C \otimes C \to C$ in $\mathcal{M}$ such that $m_C \circ (c \otimes c) = c \circ m_{\Omega A'}$. One easily checks that $Q := (C, m_C, u_C) \in \mathrm{Alg}_{\mathcal{M}}$

where $u_C := c \circ u_{\Omega A'}$. Moreover c gives rise to a morphism $q : A' \to Q$ in $\mathrm{Alg}_{\mathcal{M}}$ such that $\Omega q = c$. Since Ω is faithful, it is straightforward to check that (Q, q) is the coequalizer of (d_0, d_1) in $\mathcal{M}_m$. Thus Ω is monadic.

Let us check that the comparison functor is indeed a category isomorphism. It suffices to check that for any isomorphism $f : \Omega X \to B$ in the category $\mathcal{M}$ there exists a unique pair $(A, g : X \to A)$, where A is an object in $\mathrm{Alg}_{\mathcal{M}}$ and g a morphism in $\mathrm{Alg}_{\mathcal{M}}$, such that $\Omega A = B$ and $\Omega g = f$. This is trivial (just induce on B the monoid structure of X via f).

2) It is dual to 1). □

Example A.7. Let $\Bbbk$ be a field. Let $\mathfrak{M}$ be the category of vector spaces over $\Bbbk$.

1) By [38, Theorem 6.4.1], the forgetful functor $\mho : \mathrm{Coalg}_{\mathfrak{M}} \to \mathfrak{M}$ has a right adjoint given by the cofree coalgebra functor.

2) By [1, Theorem 2.3], the forgetful functor $\mho : \mathrm{Bialg}_{\mathfrak{M}} \to \mathrm{Alg}_{\mathfrak{M}}$ has a right adjoint.

In both cases, by Theorem A.62), we have that $\mho$ is comonadic and that the comparison functor is a category isomorphism.

Lemma A.8. *Let $\mathcal{M}$ be a monoidal category. Assume that the tensor products preserve coequalizers of reflexive pairs in $\mathcal{M}$. Given two coequalizers*

$$X_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} Y_1 \xrightarrow{p_1} Z_1 \qquad X_2 \underset{g_2}{\overset{f_2}{\rightrightarrows}} Y_2 \xrightarrow{p_2} Z_2$$

in $\mathcal{M}$, where (f_1, g_1) and (f_2, g_2) are reflexive pairs of morphisms in $\mathcal{M}$, we have that

$$X_1 \otimes X_2 \underset{g_1 \otimes g_2}{\overset{f_1 \otimes f_2}{\rightrightarrows}} Y_1 \otimes Y_2 \xrightarrow{p_1 \otimes p_2} Z_1 \otimes Z_2$$

is a coequalizer too.

Proof. See [40, Proposition 2] (where we can drop the assumption on abelianity as the result follows by [24, Lemma 0.17] where this condition is not used). □

Proposition A.9. *Let $\mathcal{M}$ be a monoidal category. Assume that the tensor products preserve coequalizers of reflexive pairs in $\mathcal{M}$. Then the forgetful functor $\Omega : \mathrm{Alg}\mathcal{M} \to \mathcal{M}$ creates coequalizers of those pairs (f, g) in $\mathrm{Alg}\mathcal{M}$ for which $(\Omega f, \Omega g)$ is a reflexive pair.*

Proof. Let f, $g : (A, m_A, u_A) \to (B, m_B, u_B)$ be a pair of morphism in $\mathrm{Alg}\mathcal{M}$ that fits into a coequalizer

$$A \underset{\Omega g}{\overset{\Omega f}{\rightrightarrows}} B \xrightarrow{p} C$$

in $\mathcal{M}$ such that $(\Omega f, \Omega g)$ is a reflexive pair. By Lemma A.8, we have the following coequalizer

$$A \otimes A \overset{\Omega f \otimes \Omega f}{\underset{\Omega g \otimes \Omega g}{\rightrightarrows}} B \otimes B \xrightarrow{p \otimes p} C \otimes C$$

We have

$$p \circ m_B \circ (\Omega f \otimes \Omega f) = p \circ \Omega f \circ m_A = p \circ \Omega g \circ m_A = p \circ m_B \circ (\Omega g \otimes \Omega g)\,.$$

The universal property of the latter coequalizer entails there is a unique morphism $m_C : C \otimes C \to C$ such that $m_C \circ (p \otimes p) = p \circ m_B$. Set $u_C := p \circ u_B$. It is easy to check that $(C, m_C, u_C) \in \mathrm{Alg}\mathcal{M}$, that p becomes an algebra morphism from (B, m_B, u_B) to (C, m_C, u_C) and that

$$(A, m_A, u_A) \overset{f}{\underset{g}{\rightrightarrows}} (B, m_B, u_B) \xrightarrow{p} (C, m_C, u_C)$$

is a coequalizer in $\mathrm{Alg}\mathcal{M}$. □

Corollary A.10. *Let $\mathcal{M}$ be a monoidal category with coequalizers of reflexive pairs. Assume these coequalizers are preserved by the tensor products in $\mathcal{M}$. Then $\mathrm{Alg}\mathcal{M}$ has coequalizers of reflexive pairs and they are preserved by the forgetful functor $\Omega : \mathrm{Alg}\mathcal{M} \to \mathcal{M}$.*

Proof. It follows by Proposition A.9 and uniqueness of coequalizers in $\mathrm{Alg}\mathcal{M}$. □

Appendix B. Braided (co)equalizers

Lemma B.1. *Let $\mathcal{M}$ be a monoidal category. We have functors*

$$\mathrm{Br}_{\mathcal{M}} \longrightarrow \mathrm{Br}_{\mathcal{M}} : (V, c) \to \left(V, c^{-1}\right), f \mapsto f$$
$$\mathrm{BrAlg}_{\mathcal{M}} \to \mathrm{BrAlg}_{\mathcal{M}} : (A, m, u, c) \to \left(A, m, u, c^{-1}\right), f \mapsto f$$

Proof. It is straightforward. □

Lemma B.2. *Let $\mathcal{M}$ be a monoidal category and let (V, c_V) be an object in $\mathrm{Br}_{\mathcal{M}}$. Assume there is a morphism $d : D \to V$ in $\mathcal{M}$ and a morphism $c_D : D \otimes D \to D \otimes D$ such that $(d \otimes d)\, c_D = c_V\, (d \otimes d)$ and $d \otimes d \otimes d$ is a monomorphism.*

1) Assume that c_D is an isomorphism. Then (D, c_D) is an object in $\mathrm{Br}_{\mathcal{M}}$ and d becomes a morphism in $\mathrm{Br}_{\mathcal{M}}$ from (D, c_D) to (V, c_V).

2) Assume that $d \otimes d$ is a monomorphism. If $(V, c_V) \in \mathrm{Br}^s_{\mathcal{M}}$ then $(D, c_D) \in \mathrm{Br}^s_{\mathcal{M}}$.

Proof. Using $(d \otimes d)\, c_D = c_V\, (d \otimes d)$ and the quantum Yang–Baxter equation for c_V one gets

$$(d \otimes d \otimes d)\, (c_D \otimes D)\, (D \otimes c_D)\, (c_D \otimes D) = (d \otimes d \otimes d)\, (D \otimes c_D)\, (c_D \otimes D)\, (D \otimes c_D)\, .$$

Since $d \otimes d \otimes d$ is a monomorphism we get that c_D satisfies the quantum Yang–Baxter equation.

1) Since c_D is an isomorphism it is clear that $(D, c_D) \in \mathrm{Br}_{\mathcal{M}}$ and that $d : (D, c_D) \to (V, c_V)$ is a morphism in $\mathrm{Br}_{\mathcal{M}}$.

2) Since $(d \otimes d)\, c_D^2 = c_V^2\, (d \otimes d) = d \otimes d$ and $d \otimes d$ is a monomorphism we get $c_D^2 = \mathrm{Id}_{D \otimes D}$ so that we can apply 1). □

Lemma B.3. *Let $\mathcal{M}$ be a monoidal category and let $H : \mathrm{Br}_{\mathcal{M}} \longrightarrow \mathcal{M}$ be the forgetful functor. Let (e_0, e_1) be a pair of morphisms in $\mathrm{Br}_{\mathcal{M}}$ such (He_0, He_1) is a coreflexive pair of morphisms in $\mathcal{M}$. Assume that (He_0, He_1) has an equalizer which is preserved by the tensor products. Then (e_0, e_1) has an equalizer in $\mathrm{Br}_{\mathcal{M}}$ which is preserved by H. The same statement holds when we replace $\mathrm{Br}_{\mathcal{M}}$ by $\mathrm{Br}^s_{\mathcal{M}}$ and H by the corresponding forgetful functor.*

Proof. Let (e_0, e_1) from (V, c_V) to (W, c_W) a coreflexive pair of morphisms in $\mathrm{Br}_{\mathcal{M}}$. We denote (He_0, He_1) by (e_0, e_1) to simplify the notation. By definition, there exists a morphism $p : W \to V$ in $\mathcal{M}$ such that $p \circ e_0 = \mathrm{Id}_V = p \circ e_1$. Consider the equalizer

$$D \xrightarrow{d} V \underset{e_1}{\overset{e_0}{\rightrightarrows}} W$$

By the dual version of Lemma A.8, we have the following equalizer

$$D \otimes D \xrightarrow{d \otimes d} V \otimes V \underset{e_1 \otimes e_1}{\overset{e_0 \otimes e_0}{\rightrightarrows}} W \otimes W$$

We have

$$(e_0 \otimes e_0)\, c_V\, (d \otimes d) = c_W\, (e_0 \otimes e_0)\, (d \otimes d) = c_W\, (e_1 \otimes e_1)\, (d \otimes d) = (e_1 \otimes e_1)\, c_V\, (d \otimes d)\, .$$

Hence there is a unique morphism $c_D : D \otimes D \to D \otimes D$ such that $(d \otimes d)\, c_D = c_V\, (d \otimes d)$.

Since $\left(V, c_V^{-1}\right)$ and $\left(W, c_W^{-1}\right)$ are also braided objects, and e_0, e_1 are also morphisms from $\left(V, c_V^{-1}\right)$ to $\left(W, c_W^{-1}\right)$, as above we can construct a morphism $\gamma_D : D \otimes D \to D \otimes D$ such that $(d \otimes d)\, \gamma_D = c_V^{-1}\, (d \otimes d)$. We have $(d \otimes d)\, c_D \gamma_D = c_V\, (d \otimes d)\, \gamma_D = c_V c_V^{-1}\, (d \otimes d) = d \otimes d$ and hence $c_D \gamma_D = \mathrm{Id}_{D \otimes D}$. Similarly $\gamma_D c_D = \mathrm{Id}_{D \otimes D}$. Thus c_D is invertible. Since $d \otimes d \otimes d = (d \otimes V \otimes V)\, (D \otimes d \otimes V)\, (D \otimes D \otimes d)$ we have that $d \otimes d \otimes d$ is a monomorphism. Thus we can apply Lemma B.2 to get that (D, c_D) is an object in

$\mathrm{Br}_{\mathcal{M}}$ and $d:(D,c_D)\to(V,c_V)$ is a morphism in $\mathrm{Br}_{\mathcal{M}}$. It is straightforward to check that

$$(D,c_D)\xrightarrow{d}(V,c_V)\overset{e_0}{\underset{e_1}{\rightrightarrows}}(W,c_W)$$

is an equalizer in $\mathrm{Br}_{\mathcal{M}}$. Consider now the case of $\mathrm{Br}^s_{\mathcal{M}}$ so that (e_0,e_1) as above is a pair in $\mathrm{Br}^s_{\mathcal{M}}$. Since d is a monomorphism, by Remark 3.2, we get that $(D,c_D)\in\mathrm{Br}^s_{\mathcal{M}}$ and d becomes a morphism in this category. Since $\mathrm{Br}^s_{\mathcal{M}}$ is a full subcategory of $\mathrm{Br}_{\mathcal{M}}$ we have that $\mathbb{I}^s_{\mathrm{Br}}:\mathrm{Br}^s_{\mathcal{M}}\to\mathrm{Br}_{\mathcal{M}}$ is full and faithful and hence it reflects equalizers (see [11, Proposition 2.9.9]) so that the above equalizer obtained in $\mathrm{Br}_{\mathcal{M}}$ is indeed an equalizer in $\mathrm{Br}^s_{\mathcal{M}}$. □

Lemma B.4. *Let $\mathcal{M}$ be a monoidal category and let (W,c_W) be an object in $\mathrm{Br}_{\mathcal{M}}$. Assume there is a morphism $d:W\to D$ in $\mathcal{M}$ and a morphism $c_D:D\otimes D\to D\otimes D$ such that $c_D(d\otimes d)=(d\otimes d)c_W$ and $d\otimes d\otimes d$ is an epimorphism.*

1) Assume that c_D is an isomorphism. Then (D,c_D) is an object in $\mathrm{Br}_{\mathcal{M}}$ and d becomes a morphism in $\mathrm{Br}_{\mathcal{M}}$ from (W,c_W) to this object.

2) Assume that $d\otimes d$ is an epimorphism. If $(V,c_V)\in\mathrm{Br}^s_{\mathcal{M}}$ then $(D,c_D)\in\mathrm{Br}^s_{\mathcal{M}}$.

Proof. It is dual to Lemma B.2. □

Lemma B.5. *Let $\mathcal{M}$ be a monoidal category and let $H:\mathrm{Br}_{\mathcal{M}}\longrightarrow\mathcal{M}$ be the forgetful functor. Let (e_0,e_1) be a pair of morphisms in $\mathrm{Br}_{\mathcal{M}}$ such (He_0,He_1) is a reflexive pair of morphisms in $\mathcal{M}$. Assume that (He_0,He_1) has a coequalizer which is preserved by the tensor products. Then (e_0,e_1) has a coequalizer in $\mathrm{Br}_{\mathcal{M}}$ which is preserved by H.*

The same statement holds when we replace $\mathrm{Br}_{\mathcal{M}}$ by $\mathrm{Br}^s_{\mathcal{M}}$ and H by the corresponding forgetful functor.

Proof. It is dual to Lemma B.3. □

Lemma B.6. *Let $\mathcal{M}$ be a monoidal category. Assume that $\mathcal{M}$ has coequalizers and that the tensor products preserve them. Then the functor $H_{\mathrm{Alg}}:\mathrm{BrAlg}_{\mathcal{M}}\to\mathrm{Alg}_{\mathcal{M}}$ reflects coequalizers.*

Proof. Let

$$(A,c_A)\overset{\alpha}{\underset{\beta}{\rightrightarrows}}(B,c_B)\xrightarrow{p}(D,c_D)$$

be a diagram of morphisms and objects in $\mathrm{BrAlg}_{\mathcal{M}}$ which is sent by H_{Alg} to a coequalizer in $\mathrm{Alg}_{\mathcal{M}}$. Since H_{Alg} is faithful we have that $p\alpha=p\beta$ as morphisms in $\mathrm{BrAlg}_{\mathcal{M}}$. Let $\lambda:(B,c_B)\to(E,c_E)$ be a morphism in $\mathrm{BrAlg}_{\mathcal{M}}$ such that $\lambda\alpha=\lambda\beta$. Then $H_{\mathrm{Alg}}\lambda\circ$

$H_{\mathrm{Alg}}\alpha = H_{\mathrm{Alg}}\lambda \circ H_{\mathrm{Alg}}\beta$ so that there is a unique algebra morphism $\lambda' : D \to E$ such that $\lambda' \circ H_{\mathrm{Alg}}p = H_{\mathrm{Alg}}\lambda$. We have

$$\begin{aligned} c_E\left(\Omega\lambda' \otimes \Omega\lambda'\right)\left(\Omega H_{\mathrm{Alg}}p \otimes \Omega H_{\mathrm{Alg}}p\right) &= c_E\left(\Omega H_{\mathrm{Alg}}\lambda \otimes \Omega H_{\mathrm{Alg}}\lambda\right) = \left(\Omega H_{\mathrm{Alg}}\lambda \otimes \Omega H_{\mathrm{Alg}}\lambda\right) c_B \\ &= \left(\Omega\lambda' \otimes \Omega\lambda'\right)\left(\Omega H_{\mathrm{Alg}}p \otimes \Omega H_{\mathrm{Alg}}p\right) c_B = \left(\Omega\lambda' \otimes \Omega\lambda'\right) c_D \left(\Omega H_{\mathrm{Alg}}p \otimes \Omega H_{\mathrm{Alg}}p\right). \end{aligned}$$

By A.3, we have that $(H_{\mathrm{Alg}}\alpha, H_{\mathrm{Alg}}\beta)$ has a coequalizer in $\mathrm{Alg}_{\mathcal{M}}$ which is a regular epimorphism in $\mathcal{M}$. By the uniqueness of coequalizers in $\mathrm{Alg}_{\mathcal{M}}$, we get that $\Omega H_{\mathrm{Alg}}p$ is also regular epimorphism in $\mathcal{M}$. By the assumption on the tensor products, we get that $\Omega H_{\mathrm{Alg}}p \otimes \Omega H_{\mathrm{Alg}}p$ is an epimorphism in $\mathcal{M}$. Thus the computation above implies $c_E\left(\Omega\lambda' \otimes \Omega\lambda'\right) = \left(\Omega\lambda' \otimes \Omega\lambda'\right) c_D$ so that there is a morphism $\lambda'' : (D, c_D) \to (E, c_E)$ in $\mathrm{BrAlg}_{\mathcal{M}}$ such that $H_{\mathrm{Alg}}\lambda'' = \lambda'$. Since H_{Alg} is faithful we get $\lambda'' \circ p = \lambda$. Also the uniqueness follows by the fact that H_{Alg} is faithful. □

Lemma B.7. *Let $\mathcal{M}$ be a monoidal category. Let $(e_0, e_1) : \mathbb{A} \to \mathbb{B}$ be a pair of morphisms in $\mathrm{BrAlg}_{\mathcal{M}}$ such $(\Omega H_{\mathrm{Alg}}e_0, \Omega H_{\mathrm{Alg}}e_1)$ is a reflexive pair of morphisms in $\mathcal{M}$. Assume that $\mathcal{M}$ has coequalizers and that the tensor products preserve them. Then (e_0, e_1) has a coequalizer $(\mathbb{C}, p : \mathbb{B} \to \mathbb{C})$ in $\mathrm{BrAlg}_{\mathcal{M}}$ which is preserved both by the functor $H_{\mathrm{Alg}} : \mathrm{BrAlg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{M}}$ and the functor ΩH_{Alg} (in particular $\Omega H_{\mathrm{Alg}}p$ is a regular epimorphism in $\mathcal{M}$ in the sense of [11, Definition 4.3.1]).*

The same statement holds when we replace $\mathrm{BrAlg}_{\mathcal{M}}$ by $\mathrm{BrAlg}^s_{\mathcal{M}}$ and H_{Alg} by the corresponding forgetful functor.

Proof. Let $(A, m_A, u_A, c_A) := \mathbb{A}$ and let $(B, m_B, u_B, c_B) := \mathbb{B}$. By Proposition A.9, we have that $(H_{\mathrm{Alg}}e_0, H_{\mathrm{Alg}}e_1)$ has a coequalizer $((C, m_C, u_C), p : (B, m_B, u_B) \to (C, m_C, u_C))$ in $\mathrm{Alg}_{\mathcal{M}}$ and it is preserved by Ω. Thus, we have the following coequalizer in $\mathcal{M}$

$$A \underset{e_1}{\overset{e_0}{\rightrightarrows}} B \xrightarrow{p} C$$

where e_0, e_1 and p denotes the same morphisms regarded as morphisms in $\mathcal{M}$ (hence, by construction, p is a regular epimorphism in $\mathcal{M}$). By Lemma A.8, we have the following coequalizer

$$A \otimes A \underset{e_1 \otimes e_1}{\overset{e_0 \otimes e_0}{\rightrightarrows}} B \otimes B \xrightarrow{p \otimes p} C \otimes C$$

We have

$$(p \otimes p)\, c_B\, (e_0 \otimes e_0) = (p \otimes p)\,(e_0 \otimes e_0)\, c_A = (p \otimes p)\,(e_1 \otimes e_1)\, c_A = (p \otimes p)\, c_B\, (e_1 \otimes e_1)$$

so that there is a unique morphism $c_C : C \otimes C \to C \otimes C$ such that

$$c_C\,(p\otimes p)=(p\otimes p)\,c_B.$$

Now, by Lemma B.1, we have that $e_0,\ e_1:\left(A,m_A,u_A,c_A^{-1}\right)\to\left(B,m_B,u_B,c_B^{-1}\right)$ are morphisms of braided objects. Hence the same argument we used above proves that there is a unique morphism $\widetilde{c}_C:C\otimes C\to C\otimes C$ such that $\widetilde{c}_C\,(p\otimes p)=(p\otimes p)\,c_B^{-1}$. We have $\widetilde{c}_C c_C\,(p\otimes p)=\widetilde{c}_C\,(p\otimes p)\,c_B=(p\otimes p)\,c_B^{-1}c_B=p\otimes p$ and hence $\widetilde{c}_C c_C=\mathrm{Id}_{C\otimes C}$. Similarly $c_C\widetilde{c}_C=\mathrm{Id}_{C\otimes C}$ so that c_C is invertible. By Lemma B.4, (C,c_C) is an object in $\mathrm{Br}_{\mathcal{M}}$ and $p:(B,c_B)\to(C,c_C)$ is a morphism in $\mathrm{Br}_{\mathcal{M}}$. It is straightforward to check that (C,m_C,u_C,c_C) is a braided algebra (whence p is a braided algebra morphism) and that $((C,m_C,u_C,c_C)\,,p:(B,m_B,u_B,c_B)\to(C,m_C,u_C,c_C))$ is the coequalizer of (e_0,e_1).

Consider now the case of $\mathrm{BrAlg}^s_{\mathcal{M}}$ so that (e_0,e_1) as above is a pair in $\mathrm{BrAlg}^s_{\mathcal{M}}$. Since p is an epimorphism, by Remark 3.2, we get that $(C,m_C,u_C,c_C)\in\mathrm{BrAlg}^s_{\mathcal{M}}$ and p becomes a morphism in this category. Since $\mathrm{BrAlg}^s_{\mathcal{M}}$ is a full subcategory of $\mathrm{BrAlg}_{\mathcal{M}}$ we have that $\mathbb{I}^s_{\mathrm{BrAlg}}:\mathrm{BrAlg}^s_{\mathcal{M}}\to\mathrm{BrAlg}_{\mathcal{M}}$ is full and faithful and hence it reflects coequalizers (dual to [11, Proposition 2.9.9]) so that the above coequalizer obtained in $\mathrm{BrAlg}_{\mathcal{M}}$ is indeed a coequalizer in $\mathrm{BrAlg}^s_{\mathcal{M}}$. □

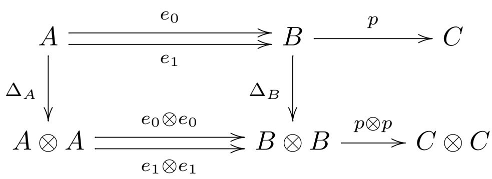
Proposition B.8. *Let $\mathcal{M}$ be a monoidal category such that the tensor products preserve coequalizers. Let $(e_0,e_1):\mathbb{A}\to\mathbb{B}$ be a pair of morphisms in $\mathrm{BrBialg}_{\mathcal{M}}$ such $(\mho_{\mathrm{Br}}e_0,\mho_{\mathrm{Br}}e_1)$ has a coequalizer in $\mathrm{BrAlg}_{\mathcal{M}}$ which is preserved by the functor $H_{\mathrm{Alg}}:\mathrm{BrAlg}_{\mathcal{M}}\to\mathrm{Alg}_{\mathcal{M}}$ and which is a regular epimorphism when regarded in $\mathcal{M}$. Then (e_0,e_1) has a coequalizer in $\mathrm{BrBialg}_{\mathcal{M}}$ which is preserved by the functor $\mho_{\mathrm{Br}}:\mathrm{BrBialg}_{\mathcal{M}}\to\mathrm{BrAlg}_{\mathcal{M}}$.*

The same statement holds when we replace $\mathrm{BrBialg}_{\mathcal{M}}$, $\mathrm{BrAlg}_{\mathcal{M}}$ and $\mho_{\mathrm{Br}}$ by $\mathrm{BrAlg}^s_{\mathcal{M}}$, $\mathrm{BrAlg}^s_{\mathcal{M}}$ and $\mho^s_{\mathrm{Br}}$ respectively and we replace H_{Alg} by the corresponding forgetful functor.

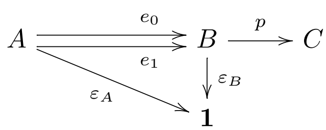
Proof. Let $(A,m_A,u_A,\Delta_A,\varepsilon_A,c_A)$ be the domain of e_0 and let $(B,m_B,u_B,\Delta_B,\varepsilon_B,c_B)$ be its codomain. Now, the pair $(\mho_{\mathrm{Br}}e_0,\mho_{\mathrm{Br}}e_1)$ has a coequalizer in $\mathrm{BrAlg}_{\mathcal{M}}$, say

$$((C,m_C,u_C,c_C)\,,p:(B,m_B,u_B,c_B)\to(C,m_C,u_C,c_C))\,,$$

which is preserved by the functor $H_{\mathrm{Alg}}:\mathrm{BrAlg}_{\mathcal{M}}\to\mathrm{Alg}_{\mathcal{M}}$ and such that p is a regular epimorphism in $\mathcal{M}$. Denote by e_0, e_1 and p the same morphisms regarded as morphisms in $\mathrm{Alg}_{\mathcal{M}}$. By [6, Lemma 2.3], $(A\otimes A,m_{A\otimes A},u_{A\otimes A}),\in\mathrm{Alg}_{\mathcal{M}}$, where $m_{A\otimes A}:=(m_A\otimes m_A)\,(A\otimes c_A\otimes B)$ and $u_{A\otimes A}:=(u_A\otimes u_A)\,\Delta_{\mathbf{1}}$. Similarly $(C\otimes C,m_{C\otimes C},u_{C\otimes C})\in\mathrm{Alg}_{\mathcal{M}}$. Since $(B,m_B,u_B,\Delta_B,\varepsilon_B,c_B)$ is a braided bialgebra, we have that $\Delta_B:(B,m_B,u_B)\to(A\otimes A,m_{A\otimes A},u_{A\otimes A})$ is an algebra map. Moreover, by Proposition [6, 3) of Proposition 2.2], we have that $p\otimes p$ is a morphism in $\mathrm{Alg}_{\mathcal{M}}$. Thus $(p\otimes p)\,\Delta_B$ is an algebra map. Since $H_{\mathrm{Alg}}:\mathrm{BrAlg}_{\mathcal{M}}\to\mathrm{Alg}_{\mathcal{M}}$ preserves the coequalizer of $(\mho_{\mathrm{Br}}e_0,\mho_{\mathrm{Br}}e_1)$ the first row in the following diagram is a coequalizer in $\mathrm{Alg}_{\mathcal{M}}$.



Since the same diagram serially commutes, by the universal property of the coequalizer in $\mathrm{Alg}_{\mathcal{M}}$, we get that there is a unique algebra morphism $\Delta_C : (C, m_C, u_C) \to (C \otimes C, m_{C\otimes C}, u_{C\otimes C})$ such that $\Delta_C p = (p \otimes p)\, \Delta_B$. Denote by Δ_C the same morphism regarded as a morphism in $\mathcal{M}$. Since p is an epimorphism in $\mathcal{M}$, one easily checks that Δ_C is coassociative using coassociativity of Δ_B. Since the diagram



serially commutes, we get that there is a unique algebra morphism $\varepsilon_C : (C, m_C, u_C) \to (\mathbf{1}, m_{\mathbf{1}}, u_{\mathbf{1}})$ such that $\varepsilon_C p = \varepsilon_B$. Denote by ε_C the same morphism regarded as a morphism in $\mathcal{M}$. Since p is an epimorphism in $\mathcal{M}$, one easily checks that Δ_C is counitary using counitarity of Δ_B. Hence $(C, \Delta_C, \varepsilon_C)$ is a coalgebra in $\mathcal{M}$ and $p : (B, \Delta_B, \varepsilon_B) \to (C, \Delta_C, \varepsilon_C)$ is a coalgebra map.

Since p is a regular epimorphism in $\mathcal{M}$ we have $p \otimes p$ is an epimorphism too by the assumption on the tensor products. Using this fact, that $(B, \Delta_B, \varepsilon_B, c_B)$ is a braided coalgebra and that p is a coalgebra morphism compatible with the Yang–Baxter operator, one easily checks that $(C, \Delta_C, \varepsilon_C, c_C)$ is a braided coalgebra too and hence p a morphism of these braided coalgebras. Summing up $p : (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \to (C, m_C, u_C, \Delta_C, \varepsilon_C, c_C)$ is a morphism of braided bialgebras in $\mathcal{M}$. Using the fact that p is an epimorphism in $\mathcal{M}$, one easily checks it is the searched coequalizer. The symmetric case can be treated analogously. □

Corollary B.9. *Let $\mathcal{M}$ be a monoidal category. Let (e_0, e_1) be a pair of morphisms in $\mathrm{BrBialg}_{\mathcal{M}}$ such $(\Gamma e_0, \Gamma e_1)$ is a reflexive pair of morphisms in $\mathcal{M}$ where $\Gamma := \Omega H_{\mathrm{Alg}} \mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathcal{M}$ denotes the forgetful functor. Assume that $\mathcal{M}$ has coequalizers and that the tensor products preserve them. Then (e_0, e_1) has a coequalizer in $\mathrm{BrBialg}_{\mathcal{M}}$ which is preserved by the functors $\mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{BrAlg}_{\mathcal{M}}$, $H_{\mathrm{Alg}} \mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{M}}$ and Γ, and which is a regular epimorphism when regarded in $\mathcal{M}$.*

The same statement holds when we replace $\mathrm{BrBialg}_{\mathcal{M}}$, $\mathrm{BrAlg}_{\mathcal{M}}$ and $\mho_{\mathrm{Br}}$ by $\mathrm{BrAlg}^s_{\mathcal{M}}$, $\mathrm{BrAlg}^s_{\mathcal{M}}$ and $\mho^s_{\mathrm{Br}}$ respectively and we replace H_{Alg} by the corresponding forgetful functor.

Proof. The pair $(\mho_{\mathrm{Br}} e_0, \mho_{\mathrm{Br}} e_1)$ fulfills the requirements of Lemma B.7 so that $(\mho_{\mathrm{Br}} e_0, \mho_{\mathrm{Br}} e_1)$ has a coequalizer in $\mathrm{BrAlg}_{\mathcal{M}}$ which is preserved by the functors $H_{\mathrm{Alg}} : \mathrm{BrAlg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{M}}$ and ΩH_{Alg} (and which is a regular epimorphism when regarded in $\mathcal{M}$). Hence we can apply Proposition B.8 to conclude. □

Lemma B.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories. Assume that both $\mathcal{M}$ and $\mathcal{N}$ have coequalizers and that the tensor products preserve them. Assume that there exists a monoidal functor $(F, \phi_0, \phi_2) : \mathcal{M} \to \mathcal{N}$ which preserves coequalizers. Then*

1) $\mathrm{Alg}F : \mathrm{Alg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{N}}$ *preserves coequalizers.*

2) $\mathrm{BrBialg}F : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{BrBialg}_{\mathcal{N}}$ *preserves coequalizers of reflexive pairs of morphisms. The same statement holds when we replace* BrBialg *by* $\mathrm{BrBialg}^s$ *everywhere.*

Proof. 1) In view of A.3, the coequalizer of the pair (α, β) of algebra morphisms $E \to A$ is, as an object in $\mathcal{M}$, the coequalizer $(B, \pi : A \to B)$ of $(\Lambda_\alpha, \Lambda_\beta)$ in $\mathcal{M}$ and the algebra structure is the unique one making π an algebra morphism. Since F preserves coequalizers, we get the coequalizer in $\mathcal{N}$

$$F(A \otimes E \otimes A) \underset{F(\Lambda_\beta)}{\overset{F(\Lambda_\alpha)}{\rightrightarrows}} FA \xrightarrow{F\pi} FB$$

Note that, since $\mathrm{Alg}F$ is a functor, we have that FA, FB are algebras and $F\pi$ is an algebra morphism.

Using the definition of Λ_α, the naturality of ϕ_2, the equality $m_{FA} = Fm_A \circ \phi_2(A, A)$ and the definition of $\Lambda_{F\alpha}$ one proves that $F(\Lambda_\alpha) \circ \phi_2(A \otimes E, A) \circ (\phi_2(A, E) \otimes FA) = \Lambda_{F\alpha}$ and similarly with β in place of α. Since $\phi_2(A \otimes E, A) \circ (\phi_2(A, E) \otimes FA)$ is an isomorphism, we get the coequalizer

$$FA \otimes FE \otimes FA \underset{\Lambda_{F\beta}}{\overset{\Lambda_{F\alpha}}{\rightrightarrows}} FA \xrightarrow{F\pi} FB\ .$$

By construction we get that $(FB, F\pi)$ is the coequalizer of $(\Lambda_{F\alpha}, \Lambda_{F\beta})$ in $\mathcal{N}$. Since, as observed, FA and FB are algebras and $F\pi$ is an algebra morphism, we conclude that $(FB, F\pi)$ is the coequalizer of $(F\alpha, F\beta)$ in $\mathcal{N}$ (apply A.3 again).

2) Consider a coequalizer of a reflexive pair in $\mathrm{BrBialg}_{\mathcal{M}}$

$$B \underset{e_1}{\overset{e_0}{\rightrightarrows}} D \xrightarrow{d} E \tag{109}$$

If we apply the forgetful functor $\Gamma := \Omega H_{\mathrm{Alg}} \mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathcal{M}$ to the pair, we get a reflexive pair in $\mathcal{M}$. By Corollary B.9, (e_0, e_1) has a coequalizer in $\mathrm{BrBialg}_{\mathcal{M}}$ (different from (109), in principle) which is preserved by the functor $H_{\mathrm{Alg}} \mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{M}}$. By uniqueness of coequalizers, we get that the coequalizer (109) is preserved by $H_{\mathrm{Alg}} \mho_{\mathrm{Br}}$ and hence, by 1), it is preserved by $(\mathrm{Alg}F)\, H_{\mathrm{Alg}} \mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{N}}$. Hence (FE, Fd) is the coequalizer of (Fe_0, Fe_1) in $\mathrm{Alg}_{\mathcal{N}}$.

Note that (Fe_0, Fe_1) is a reflexive pair of morphisms in $\mathrm{BrBialg}_{\mathcal{N}}$. By Corollary B.9, (Fe_0, Fe_1) has a coequalizer $(E', \pi : FD \to E')$ in $\mathrm{BrBialg}_{\mathcal{N}}$ which is preserved by the functor $H'_{\mathrm{Alg}} \mho'_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{N}} \to \mathrm{Alg}_{\mathcal{N}}$. By uniqueness of coequalizers in $\mathrm{Alg}_{\mathcal{N}}$, there is an algebra isomorphism $\xi : E' \to FE$ such that $\xi \circ \pi = Fd$. Since $\mathrm{BrBialg}F$ is a functor we have that FE is a braided bialgebra and Fd is a morphism in $\mathrm{BrBialg}_{\mathcal{N}}$. Now, by construction π is a suitable coequalizer in $\mathcal{N}$ (which further inherits a proper braided bialgebra structure) so that, by assumption it is preserved by the tensor products. Hence both π and $\pi \otimes \pi$ are epimorphisms in $\mathcal{N}$. Using these properties one proves that $\xi : E' \to FE$ is a morphism in $\mathrm{BrBialg}_{\mathcal{N}}$.

Since ξ is invertible, we obtain that (FE, Fd) is the coequalizer of (Fe_0, Fe_1) in $\mathrm{BrBialg}_{\mathcal{N}}$ i.e. $\mathrm{BrBialg}F : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{BrBialg}_{\mathcal{N}}$ preserves coequalizers of reflexive pairs of morphisms. The symmetric case follows analogously once observed that F preserves symmetric objects, see 3.3. □

Proposition B.11. *Let $\mathcal{M}$ be a monoidal category. Assume that $\mathcal{M}$ has a coequalizers and that the tensor products preserve them. Consider a right adjoint functor $R : \mathrm{BrBialg}_{\mathcal{M}} \to \mathcal{B}$ into an arbitrary category $\mathcal{B}$. Then the comparison functor R_1 has a left adjoint L_1 which is uniquely determined by the following properties.*

1) For every object $(B, \mu) \in {}_{RL}\mathcal{B}$, there is a morphism $\pi(B, \mu) : LB \to L_1(B, \mu)$ such that

$$\Gamma LRLB \underset{\Gamma \epsilon LB}{\overset{\Gamma L\mu}{\rightrightarrows}} \Gamma LB \xrightarrow{\Gamma \pi(B,\mu)} \Gamma L_1(B, \mu) \tag{110}$$

is a coequalizer in $\mathcal{M}$, where $\Gamma := \Omega H_{\mathrm{Alg}} \mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathcal{M}$ denotes the forgetful functor.

2) The bialgebra structure of $\Gamma L_1(B, \mu)$ is uniquely determined by the fact that $\Gamma\pi(B, \mu)$ is a morphism of braided bialgebras in $\mathcal{M}$.

3) R is comparable.

4) The statements above still hold true when $\mathrm{BrBialg}^s_{\mathcal{M}}$ replaces $\mathrm{BrBialg}_{\mathcal{M}}$.

Proof. By Beck's Theorem, it suffices to check that for every $(B, \mu) \in {}_{RL}\mathcal{B}$ the fork $(L\mu, \epsilon LB)$ has a coequalizer in $\mathrm{BrBialg}_{\mathcal{M}}$, where L denotes the left adjoint of R and $\epsilon : LR \to \mathrm{Id}_{\mathrm{BrBialg}_{\mathcal{M}}}$ the counit of the adjunction. Now $L\mu \circ L\eta B = \mathrm{Id}_{LB} = \epsilon LB \circ L\eta B$ where $\eta : \mathrm{Id}_{\mathcal{B}} \to RL$ is the unit of the adjunction. Thus $(L\mu, \epsilon LB)$ is a reflexive pair of morphisms in $\mathrm{BrBialg}_{\mathcal{M}}$. Therefore $(\Gamma L\mu, \Gamma\epsilon LB)$ is a reflexive pair of morphisms in $\mathcal{M}$. By Corollary B.9, the pair $(L\mu, \epsilon LB)$ has a coequalizer in $\mathrm{BrBialg}_{\mathcal{M}}$ which is preserved both by the functors $\mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{BrAlg}_{\mathcal{M}}$, $H_{\mathrm{Alg}}\mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathrm{Alg}_{\mathcal{M}}$ and $\Gamma := \Omega H_{\mathrm{Alg}}\mho_{\mathrm{Br}} : \mathrm{BrBialg}_{\mathcal{M}} \to \mathcal{M}$. By construction the coequalizer of $(L\mu, \epsilon LB)$ is $(L_1(B, \mu), \pi(B, \mu) : LB \to L_1(B, \mu))$. Furthermore (110) is a coequalizer in $\mathcal{M}$ and the bialgebra structure of $\Gamma L_1(B, \mu)$ is uniquely determined by the fact that $\Gamma\pi(B, \mu)$ is a morphism of braided bialgebras in $\mathcal{M}$. By Lemma 1.11, R is comparable.

The symmetric case follows analogously. □

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