ON DEGENERATIONS OF PLANE CREMONA TRANSFORMATIONS

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Abstract. This article studies the possible degenerations of Cremona transformations of the plane of some degree into maps of smaller degree.

1. Introduction

Let us fix $\mathbf{k}$ to be the ground field, which will be algebraically closed of characteristic 0. The Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ is the group of birational transformations of the plane.

There is a natural Zariski topology on $\mathrm{Bir}(\mathbb{P}^2)$, introduced in [Ser10] (see §3.1) and studied then in many recent texts: [Bla10], [Pop13], [BF13], [PR13], [BCM13], [Can13]. For each integer d, the subset $\mathrm{Bir}(\mathbb{P}^2)_d$ of elements of $\mathrm{Bir}(\mathbb{P}^2)$ of degree d is locally closed, and has a natural structure of algebraic variety, compatible with the Zariski topology of $\mathrm{Bir}(\mathbb{P}^2)$. However, neither the group $\mathrm{Bir}(\mathbb{P}^2)$ nor the subset $\mathrm{Bir}(\mathbb{P}^2)_{\le d}$ of maps of degree at most d (for $d \geq 2$) have a structure of an (ind)-algebraic variety [BF13, Proposition 3.4], and the bad structure comes from the degeneration of maps of degree d into maps of smaller degree.

The aim of this article consists exactly in trying to understand this degeneration, and more precisely to describe the closure $\overline{\mathrm{Bir}(\mathbb{P}^2)_d}$ of $\mathrm{Bir}(\mathbb{P}^2)_d$ in $\mathrm{Bir}(\mathbb{P}^2)$, which is a subset of $\mathrm{Bir}(\mathbb{P}^2)_{\le d}$. In particular, the first question one can ask is to understand for which d we have an equality $\overline{\mathrm{Bir}(\mathbb{P}^2)_d} = \mathrm{Bir}(\mathbb{P}^2)_{\le d}$ (already asked in [BF13, Remark 2.16]). The complete answer is the following:

Theorem 1. *Let $d \geq 1$ be an integer. Then $\overline{\mathrm{Bir}(\mathbb{P}^2)_d} = \mathrm{Bir}(\mathbb{P}^2)_{\le d}$ if and only if $d \le 8$ or $d \in \{10, 12\}$.*

This theorem shows that $\mathrm{Bir}(\mathbb{P}^2)_8$ is not contained in the closure of $\mathrm{Bir}(\mathbb{P}^2)_9$, but is in the closure of $\mathrm{Bir}(\mathbb{P}^2)_{10}$, and the same holds replacing $8, 9, 10$ with $10, 11, 12$. One can then ask for a birational map of degree d what is the minimum k needed such that it belongs to the closure of $\mathrm{Bir}(\mathbb{P}^2)_{d+k}$. As we will show, there is an upper bound for k, depending on d, but no universal bound:

Theorem 2. *For each integer $k \geq 1$ there exists an integer d and a birational map φ of degree d such that φ does not belong to $\overline{\mathrm{Bir}(\mathbb{P}^2)_{d+i}}$ for each i with $1 \le i \le k$.*

Every birational map of degree $d \geq 1$ is contained in $\overline{\mathrm{Bir}(\mathbb{P}^2)_{d+i}}$ for some i with $1 \le i \le \max\{1, \frac{d}{3}\}$.

The two theorems are obtained by a detailed study of the possible degenerations of birational maps and of the relation with their base-points. For example, we give a criterion that determines whether a birational map φ of degree d with only proper base-points, no three of them collinear, belongs to $\overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$: it is the case if and only if φ has multiplicity m_1 and m_2 at two points of $\mathbb{P}^2$ such that $m_1 + m_2 = d - 1$ (Corollary 5.2). We also give three propositions that provide existence of degenerations associated to the base-points of a birational map (Propositions 4.3, 4.9 and 4.13).

The authors acknowledge support by the Swiss National Science Foundation Grant "Birational Geometry" PP00P2_128422 /1. The second author is member of G.N.S.A.G.A. group of Italian I.N.d.A.M. and warmly thanks Massimiliano Mella for useful discussions.

Let us finish this introduction by describing the situation for the subgroup $\mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Bir}(\mathbb{P}^2)$ consisting of automorphisms of the affine plane. The question of degeneration was already studied in this case, see for example [Fur97], [EF04], [Fur13]. Because of Jung's theorem, every element $f \in \mathrm{Aut}(\mathbb{A}^2)$ has a multidegree $(d_1, \ldots, d_k)$ which satisfies $\deg(f) = \prod_{i=1}^{k} d_i$, and its length is defined to be the integer k. By [Fur02], the length of an automorphism is lower semicontinuous. In particular, elements of multidegree $(2, 2)$ are not in the closure of $\mathrm{Aut}(\mathbb{A}^2)_5$, even if they are in the closure of $\mathrm{Bir}(\mathbb{P}^2)_5$, and one can construct many of such examples, using the rigidity of $\mathrm{Aut}(\mathbb{A}^2)$. The closure of the subvarieties of some given multidegrees are however not well understood, and quite hard to describe. See [Fur13] for some descriptions and conjectures in the cases of length ≤ 2 and [EF04] for a recent work in case of length 3.

2. Degree, multiplicities of base-points and homaloidal types

Definition 2.1. Let

$$\varphi\colon [x : y : z] \dashrightarrow [f_0(x, y, z) : f_1(x, y, z) : f_2(x, y, z)]$$

be a birational map of $\mathbb{P}^2$, where f_0, f_1, f_2 are homogeneous polynomials of degree d without common factor (of degree ≥ 1). We will say that the *degree of* φ *is* d and that the *homaloidal type* of φ is

$$(d; m_1, m_2, \ldots, m_r)$$

if the linear system of φ, which is given by the set of curves of equation

$$\sum_{i=0}^{2} \lambda_i f_i(x, y, z) = 0,$$

for $\lambda_0, \lambda_1, \lambda_2 \in \mathbf{k}$, has base-points $p_1, \ldots, p_r$ of multiplicity $m_1, \ldots, m_r$. (Here by base-points we include all such, including infinitely near base-points).

Remark 2.2. If $(d; m_1, m_2, \ldots, m_r)$ is the homaloidal type of a birational map of $\mathbb{P}^2$, the integers m_i and d satisfy the following equations

$$\sum_{i=1}^{r} m_i = 3(d-1), \qquad \sum_{i=1}^{r} (m_i)^2 = d^2 - 1, \tag{1}$$

that are the classical *Noether equalities*, and directly follow from the fact that the map is birational (see for instance [A-C02, §2.5, page 51]).

We will use the following notation of [A-C02, Definition 5.2.1, page 130]:

Definition 2.3. Let $d, m_1, \ldots, m_r$ be integers. We will say that $T = (d; m_1, m_2, \ldots, m_r)$ is a *homaloidal type of degree* d if it satisfies the Noether equalities (1).

If there exists moreover a birational map $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ of homaloidal type T, then we say that T is *proper*, and otherwise we say that T is *improper*.

Note that the type $(3; 1, 1, 1, 1, 1, 1, 1, -1)$ is improper, as it contains a negative integer. Similarly, $(5; 3, 3, 1, 1, 1, 1, 1, 1)$ is another improper type, as the linear system associated to such a type would be reducible (the line through the two points of multiplicity 3 would be a fixed component).

In order to decide whether a homaloidal type is proper or improper, there is what is called the Hudson's test (see [A-C02, Definition 5.2.15, page 134] and [BCM13, Definition 25 and the appendix]). We will explain why this algorithm works, using a modern language, which is a simplified version of the action of $\mathrm{Bir}(\mathbb{P}^2)$ on a hyperbolic space of infinite dimension given by the Picard-Manin space (the interest reader can have a look at [Can11, Section 3] or [BC13, Section 5]).

2.1. **Hudson's test.** Let us consider the free $\mathbb{Z}$-module V of infinite countable rank, whose basis is $\{e_i\}_{i\in\mathbb{N}}$. Each homaloidal type $(d; m_1, \ldots, m_r)$ corresponds to the element $de_0 - \sum_{i=1}^{r} m_i e_i \in V$. We then consider the scalar product on V given by $(e_0)^2 = 1$, $(e_i)^2 = -1$ for $i \geq 1$ and $e_i \cdot e_j = 0$ for $i \neq j$. This corresponds to the intersection of divisors on the blow-ups of $\mathbb{P}^2$ associated to the base-points of the corresponding maps.

We denote by σ_0 the automorphism of V given by the reflection by the root $e_0 - e_1 - e_2 - e_3$:

$$\begin{array}{lll} \sigma_0(e_0) = 2e_0 - e_1 - e_2 - e_3, & \sigma_0(e_i) = e_i \text{ for } i \geq 4, & \\ \sigma_0(e_1) = e_0 - e_2 - e_3, & \sigma_0(e_2) = e_0 - e_1 - e_3, & \sigma_0(e_3) = e_0 - e_1 - e_2. \end{array}$$

This corresponds exactly to the action of the standard quadratic map

$$\sigma\colon [x:y:z] \dashmapsto [yz:xz:xy]$$

on the blow-up of the three points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$.

We then define W as the group of automorphisms of V generated by σ_0 and by the permutations of the e_i fixing e_0. A simple calculation shows that W preserve the intersection form and the canonical form given by $de_0 - \sum m_i e_i \to 3d - \sum m_i$. Hence, the group W preserves the set of homaloidal types. We then have the following:

Proposition 2.4. *Let $T = (d; m_1, m_2, \ldots, m_r)$ be a homaloidal type.*

(i) The type T is proper if and only if it belongs to the orbit $W(e_0)$.

(ii) If T is proper, there is a dense open subset in $U \subset (\mathbb{P}^2)^r$ such that for each $(p_1, \ldots, p_r) \in U$, there exists a birational transformation φ having degree d and which base-points are the p_i with multiplicity m_i.

Proof. Suppose first that T is proper, which corresponds to saying that there exists a map $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ of type T. By Noether-Castelnuovo theorem, φ can be written as

$$\varphi = \alpha_k \sigma \alpha_{k-1} \ldots \alpha_2 \sigma \alpha_1,$$

where σ is the standard quadratic involution and $\alpha_1, \ldots, \alpha_k \in \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3, \mathbf{k})$ (See for example [Sha67, Chapter V, §5, Theorem 2, p. 100] or [A-C02, Chapter 8]).

For each i, we write $\varphi_i = \alpha_i \sigma \alpha_{i-1} \ldots \alpha_2 \sigma \alpha_1$, and observe that φ_1 is linear and $\varphi_k = \varphi$. The homaloidal type of φ_i is obtained from the one of φ_{i-1} by applying σ_0 and a permutation of coordinates. Hence, the homaloidal type of φ belongs to $W(e_0)$.

We now take an element $f \in W$ that we can write as

$$f = a_k \sigma_0 a_{k-1} \ldots a_2 \sigma_0 a_1,$$

where σ_0 is the automorphism of V defined before and $a_1, \ldots, a_k$ are permutations of the e_i fixing e_0. We prove by induction on k that the type of $f(e_0)$ satifies assertion (ii), which will achieve the proof.

If $k = 1$ or $k = 2$, the result is obvious, as the type is of degree 1 or of degree 2 with three simple base-points. We can then assume $k > 2$ and show the result using induction hypothesis.

In order to simplify the proof, we will assume that a_k is the identity, since permutations of points does not change the result of (ii). We then write $f(e_0) = (d, m_1, \ldots, m_r)$ and $f' = a_{k-1}\sigma_0 \ldots a_2 \sigma_0 a_1$, which implies that

$$f'(e_0) = (2d - m_1 - m_2 - m_3, d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, \ldots, m_r).$$

Using induction hypothesis, we obtain a dense open subset $U \subset (\mathbb{P}^2)^r$ such that for each $(p_1, \ldots, p_r) \in U$, there exists a birational transformation φ having degree $2d - m_1 - m_2 - m_3$ and which base-points are the p_i with multiplicities given by $f'(e_0)$. This follows from the induction hypothesis and the fact that we can add general points if one of the multiplicities $d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2$ is zero.

We denote by $V \subset U$ the open subset where no three of the points p_i are collinear. For each $(p_1,\dots,p_r) \in V$ we take the map φ associated to these points and define $\hat{\varphi} = \varphi\psi$ where $\psi = \tau\sigma\tau^{-1}$ and $\tau \in \mathrm{Aut}(\mathbb{P}^2)$ sends the three base-points of σ onto p_1, p_2, p_3. Then $\hat{\varphi}$ is a birational map of degree d with multiplicities $m_1,\dots,m_r$ at $p_1,p_2,p_3,\psi(p_4),\dots,\psi(p_r)$ respectively. We can the define $\hat{V} \subset (\mathbb{P}^2)^r$ as the open set

$$\hat{V} = \{(q_1,\dots,q_r) \mid (q_1,q_2,q_3,\psi(q_4),\dots,\psi(q_r)) \in V\};$$

that concludes the proof. □

Remark 2.5. Following Proposition 2.4, we can associate to each element $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ an element $g \in W$, unique up to permutations at source and target, such that $g(e_0)$ corresponds to the type of φ. The matrix g corresponds to the characteristic matrix studied in [A-C02, Chapter 5] and gives the curves contracted by g and its inverse. In particular, $g^{-1} \in W$ is the map associated to φ^{-1}, so the homaloidal type of φ^{-1} is obtained by computing $g^{-1}(e_0)$.

Using Proposition 2.4, one obtain the classical algorithm (Hudson's test) that decides whether a homaloidal type is proper or improper. Let us recall how it works:

(1) Taking a homaloidal type $(d; m_1,\dots,m_r)$ with $d \geq 2$, and all integers m_i non-negative and order them so that $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_r$.
(2) We then replace $(d; m_1,\dots,m_r)$ with $(d-\epsilon, m_1-\epsilon, m_2-\epsilon, m_3-\epsilon, m_4,\dots,m_r)$, where $\epsilon = m_1+m_2+m_3-d$, and then go back an apply the first step.
(3) We end when we reach $(1; 0,\dots,0)$, in which case the test is fullfilled, or when at least one m_i is negative, in which case the test is not fullfilled.

Then, we recall the following result.

Lemma 2.6. *A homaloidal type is proper if and only if it satisfies Hudson's test.*

Proof. We observe first that if $(d; m_1,\dots,m_r)$ is a homaloidal type with $d \geq 2$, $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$, then $m_1+m_2+m_3 \geq d+1$. This was already observed by Noether and is a direct consequence of the Noether equalities (see for example [A-C02, Corollary 2.6.7, page 55]). Hence, the integer ϵ in the test above is always non-negative.

If $d = 1$, the only possibilities for the m_i is to be zero. Hence, the algorithm above always has a end: either we reach $d = 1$ with all m_i being zero or at some point one m_i is negative.

Note that Hudson's test consists of applying elements of W to $de_0 - \sum m_i e_i$. If the test is fulfilled, then the type is in the image of W, and is thus a proper homaloidal type by Proposition 2.4. If the test is not fullfilled, then we finish with a type which is improper as it contains a negative integer. The type from which we started is then in the same orbit as this one by W and is thus improper by Proposition 2.4. □

Remark 2.7. One may stop Hudson's test as soon as one reaches in step (1) a homaloidal type which is already known to be either proper or improper, and accordingly the test is either fulfilled or not fulfilled.

Remark 2.8. Applying Hudson's test to the type of a birational map φ we also obtain the matrix of φ, which is the element of W corresponding to it (Remark 2.5). This shows in particular that the homaloidal type of φ^{-1} only depends on the homaloidal type of φ and not of the position of the base-points, and also provides a method to compute the type of the inverse (already explained in [A-C02, Definition 5.4.24, page 156]).

Example 2.9. Applying the algorithm corresponding to Hudson's test, one easily finds all linear systems of small degree. We use the notation m^r to write $m,\dots,m$ (r times).

For each degree $d \geq 2$, the type $(d; d-1, 1^{2d-2})$ is a de Jonquières homaloidal type. For $d \geq 4$, we also have another type, which is $(d; d-2, 2^{d-2}, 1^3)$. In degree $d \leq 11$, all

$(5; 2^6)$			
$(6; 3^3, 2, 1^4)$	$(6; 3^2, 2^4, 1)$		
$(7; 4, 3^3, 1^5)$	$(7; 4, 3^2, 2^3, 1^2)$	$(7; 3^4, 2^3)$	
$(8; 5, 3^3, 2^2, 1^3)$	$(8; 5, 3^2, 2^5)$	$(8; 4^3, 3, 1^6)$	$(8; 4^3, 2^3, 1^3)$
$(8; 4^2, 3^2, 2^3, 1)$	$(8; 4, 3^5, 1^2)$	$(8; 3^7)$	
$(9; 6, 3^4, 2, 1^4)$	$(9; 6, 3^3, 2^4, 1)$	$(9; 5, 4^3, 1^7)$	$(9; 5, 4^2, 3, 2^3, 1^2)$
$(9; 5, 4, 3^4, 1^3)$	$(9; 5, 4, 3^3, 2^3)$	$(9; 4^4, 2^4)$	$(9; 4^3, 3^3, 2, 1)$
$(10; 7, 3^5, 1^5)$	$(10; 7, 3^4, 2^3, 1^2)$	$(10; 6, 4^3, 2^3, 1^3)$	$(10; 6, 4^2, 3^3, 1^4)$
$(10; 6, 4^2, 3^2, 2^3, 1)$	$(10; 6, 3^7)$	$(10; 5^3, 4, 1^8)$	$(10; 5^3, 3, 2^3, 1^3)$
$(10; 5^3, 2^6)$	$(10; 5^2, 4^2, 2^4, 1)$	$(10; 5^2, 4, 3^3, 2, 1^2)$	$(10; 5^2, 3^5, 2)$
$(10; 5, 4^3, 3^2, 2^2)$	$(10; 4^6, 1^3)$	$(10; 4^5, 3^2, 1)$	
$(11; 8, 3^5, 2^2, 1^3)$	$(11; 8, 3^4, 2^5)$	$(11; 7, 4^3, 3^2, 1^5)$	$(11; 7, 4^3, 3, 2^3, 1^2)$
$(11; 7, 4^2, 3^3, 2^3)$	$(11; 7, 4, 3^6, 1)$	$(11; 6, 5^3, 1^9)$	$(11; 6, 5^2, 4, 2^4, 1^2)$
$(11; 6, 5^2, 3^3, 2, 1^3)$	$(11; 6, 5^2, 3^2, 2^4)$	$(11; 6, 5, 4^2, 3^2, 2^2, 1)$	$(11; 6, 4^5, 1^4)$
$(11; 6, 4^3, 3^4)$	$(11; 5^4, 2^5)$	$(11; 5^3, 4, 3^3, 1^2)$	$(11; 5^2, 4^4, 2, 1^2)$
$(11; 5^2, 4^3, 3^2, 2)$			

Table 1. Proper homaloidal types of degree $d \leq 11$ which are not of type $(d; d-1, 1^{2d-2})$ or $(d; d-2, 2^{d-2}, 1^3)$.

other proper homaloidal types are given in Table 1. It can also be checked that these types are the same as in [Hud27, Table I, pages 437–438].

In Section 5 we will need other proper homaloidal types in each degree.

Example 2.10. For each integer $m \geq 3$, the following homaloidal types

$$(3m; 3m-6, 6^{m-3}, 4^3, 3^2, 2, 1), \tag{2}$$

$$(3m+1; 3m-5, 6^{m-2}, 4, 3^3, 1^4), \tag{3}$$

$$(3m+2; 3m-4, 6^{m-2}, 4^2, 3^2, 2^2, 1), \tag{4}$$

are proper [Hud27, Table II]. This can also be shown by applying the Hudson's test for $m = 3$ and $m = 4$ and then apply induction on m: running once step (2) of the algorithm, the properness of the homaloidal types (2), (3), (4) for m proves the properness of these types for $m + 2$.

3. Varieties parametrising birational maps of small degree

3.1. **The topology on** $\mathrm{Bir}(\mathbb{P}^2)$**.** We recall the notion of families of birational maps, introduced by M. Demazure in [Dem70] (see also [Ser10], [Bla10]).

Definition 3.1. Let A, X be irreducible algebraic varieties, and let f be a A-birational map of the A-variety $A \times X$, inducing an isomorphism $U \to V$, where U, V are open subsets of $A \times X$, whose projections on A are surjective.

The rational map f is given by $(a, x) \dashrightarrow (a, p_2(f(a, x)))$, where p_2 is the second projection, and for each $\mathbf{k}$-point $a \in A$, the birational map $x \dashrightarrow p_2(f(a, x))$ corresponds to an element $f_a \in \mathrm{Bir}(X)$. The map $a \mapsto f_a$ represents a map from A (more precisely from the $A(k)$-points of A) to $\mathrm{Bir}(X)$, and will be called a *morphism* from A to $\mathrm{Bir}(X)$.

These notions yield the natural Zariski topology on $\mathrm{Bir}(X)$, introduced implicitly by M. Demazure [Dem70] and explicitly by J.-P. Serre [Ser10]:

Definition 3.2. A subset $F \subseteq \mathrm{Bir}(X)$ is closed in the Zariski topology if for any algebraic variety A and any morphism $A \to \mathrm{Bir}(X)$ the preimage of F is closed.

Remark 3.3. Any birational map $X \dashrightarrow Y$ yields a homeomorphism between $\mathrm{Bir}(X)$ and $\mathrm{Bir}(Y)$, and for any $\varphi \in \mathrm{Bir}(X)$ the maps $\mathrm{Bir}(X) \to \mathrm{Bir}(X)$ given by $\psi \mapsto \psi \circ \varphi$, $\psi \mapsto \varphi \circ \psi$ and $\psi \mapsto \psi^{-1}$ are homeomorphisms.

Remark 3.4. In the sequel, the topology on $\mathrm{Bir}(\mathbb{P}^2)$ and its subsets will always be the Zariski topology given in Definition 3.2.

3.2. **The varieties W_d, Bir_d and Bir_d°.** Let us recall the following notation, which is taken from [BF13, Definition 2.3] and [BCM13, p. 1112].

Definition 3.5. Let d be a positive integer.

(1) We define W_d to be the set of equivalence classes of non-zero triples (h_0, h_1, h_2) of homogeneous polynomials $h_i \in \mathbf{k}[x, y, z]$ of degree d, where (h_0, h_1, h_2) is equivalent to $(\lambda h_0, \lambda h_1, \lambda h_2)$ for any $\lambda \in \mathbf{k}^*$. The equivalence class of (h_0, h_1, h_2) will be denoted by $[h_0 : h_1 : h_2]$.
(2) We define $\mathrm{Bir}_d \subseteq W_d$ to be the set of elements $h = [h_0 : h_1 : h_2] \in W_d$ such that the rational map $\psi_h \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by
$$[x : y : z] \mapstochar\dashrightarrow [h_0(x, y, z) : h_1(x, y, z) : h_2(x, y, z)]$$
is birational. We denote by π_d the map $\mathrm{Bir}_d \to \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ which sends h onto ψ_h.
(3) We define by $\mathrm{Bir}_d^\circ \subseteq \mathrm{Bir}_d$ the subset of elements $[h_0 : h_1 : h_2] \in \mathrm{Bir}_d$ such that the polynomials h_0, h_1, h_2 have no common factor of degree ≥ 1.

Remark 3.6. Note that Bir_d is the notation of [BCM13] and was called H_d in [BF13].

Remark 3.7. The map π_d is not injective for $d \geq 2$ but restricts to a natural bijection between Bir_d° and the set $\mathrm{Bir}(\mathbb{P}^2)_d$ of maps of degree d.

Lemma 3.8. *Let W_d, Bir_d be as in Definition 3.5. Then, the following holds:*

(1) *The set W_d is isomorphic to $\mathbb{P}^r$, where $r = 3\binom{d+2}{2} - 1 = 3d(d+3)/2 + 2$.*
(2) *The set Bir_d is locally closed in W_d, and thus inherits from W_d the structure of an algebraic variety.*
(3) *The map $\pi_d \colon \mathrm{Bir}_d \to \mathrm{Bir}(\mathbb{P}^2)$ is a morphism, which is continuous and closed. Its image is the set $\mathrm{Bir}(\mathbb{P}^2)_{\leq d}$ of birational transformations of degree $\leq d$.*
(4) *For any $\varphi \in \mathrm{Bir}(\mathbb{P}^n)_{\leq d}$, the set $(\pi_d)^{-1}(\varphi)$ is closed in W_d (hence in Bir_d).*
(5) *The set Bir_d° is open in Bir_d.*

Proof. Follows from [BF13, Lemma 2.4, Corollary 2.9 and Proposition 2.15]. □

Corollary 3.9. *Let $d \geq 1$. We have an equality*
$$\pi_d(\overline{\mathrm{Bir}_d^\circ}) = \overline{\mathrm{Bir}(\mathbb{P}^2)_d},$$
where the closure of Bir_d° is taken in Bir_d and the closure of $\mathrm{Bir}(\mathbb{P}^2)_d$ is taken in $\mathrm{Bir}(\mathbb{P}^2)$.

Proof. Follows from the fact that $\pi_d(\mathrm{Bir}_d^\circ) = \mathrm{Bir}(\mathbb{P}^2)_d$ and that $\pi_d \colon \mathrm{Bir}_d \to \mathrm{Bir}(\mathbb{P}^2)$ is closed and continuous. □

Lemma 3.10. *The map*

$$\begin{array}{rcl} \mathrm{Bir}(\mathbb{P}^2) & \to & \mathrm{Bir}(\mathbb{P}^2) \\ \varphi & \mapsto & \varphi^{-1} \end{array}$$

is a homeomorphism, which sends $\mathrm{Bir}(\mathbb{P}^2)_d$ *onto itself for each* d.

In particular, an element $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ *belongs to the closure of* $\mathrm{Bir}(\mathbb{P}^2)_d$ *if and only if* φ^{-1} *belongs to the closure of* $\mathrm{Bir}(\mathbb{P}^2)_d$.

Proof. Follows from the definition of the topology of $\mathrm{Bir}(\mathbb{P}^2)$ (see Remark 3.3), and from the fact that the degree of an element and its inverse are the same. □

Proposition 3.11. *Let* $d \geq 2$.

(1) *If* $I \subset \mathrm{Bir}^\circ_d$ *is an irreducible component, there exists a proper homaloidal type* $\Lambda = (d; m_1, \ldots, m_r)$ *such that a general element of* I *is a birational map* φ *of* $\mathbb{P}^2$ *of type* Λ*, such that neither* φ *nor* φ^{-1} *have infinitely near base-points. Moreover, all birational maps of type* Λ *are contained in* I.
(2) *The association in* (1) *yields a one-to-one correspondence between the irreducible components of* Bir°_d *and the proper homaloidal types of degree* d.

Proof. It follows from [BCM13, Theorem 1 and Lemma 36]. □

Notation 3.12. If $\Lambda = (d; m_1, \ldots, m_r)$ is a proper homaloidal type, we will follow [BCM13, Definition 37] and denote by $\mathrm{Bir}^\circ_\Lambda \subset \mathrm{Bir}^\circ_d$ the irreducible component of Bir°_d whose general element is of type Λ.

Even if all birational maps of type Λ belong to $\mathrm{Bir}^\circ_\Lambda$, not all elements of $\mathrm{Bir}^\circ_\Lambda$ have homaloidal type Λ. Indeed, some map can belong to two or more different irreducible components. In particular, Bir°_d is connected if $d \leq 6$ [BCM13, Theorem 2].

Notation 3.13. If $\Lambda = (d; m_1, \ldots, m_r)$ is a proper homaloidal type, we will denote by Λ^* the homaloidal type such that the inverse of a map of type Λ has type Λ^* (as observed in Remark 2.8, the homaloidal type of the inverse of φ^{-1} only depends on the homaloidal type of $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$).

Remark 3.14. The map $\varphi \mapsto \varphi^{-1}$ yields a homeomorphism of Bir°_d (see Lemma 3.10). In particular, it sends an irreducible component $\mathrm{Bir}^\circ_\Lambda$ onto $\mathrm{Bir}^\circ_{\Lambda^*}$.

Remark 3.15. In degree $d \leq 5$, every proper homaloidal type Λ satisfies $\Lambda = \Lambda^*$, but this is not true in degree 6, where $(6; 4, 2^4, 1^3)^* = (6; 3^3, 2, 1^4)$ (see [Hud27, Table I, pages 437–443] for the description in each degree $d \leq 16$).

One can moreover observe that for each $d \geq 6$ there are types which are self-dual and types which are not:

(1) $(d; d-1, 1^{2d-2})^* = (d; d-1, 1^{2d-2})$ for $d \geq 2$;
(2) $(d; d-2, 2^{d-2}, 1^3)^* = (d; \lfloor \frac{d}{2} \rfloor^3, \lceil \frac{d-1}{2} \rceil, 1^{d-2}) \neq (d; d-2, 2^{d-2}, 1^3)$ for $d \geq 6$.

3.3. **Jacobians and curves contracted.** The curves contracted by birational maps and the Jacobian of the maps will be useful to parametrise subvarieties of Bir_d and Bir°_d, and to obtain results on possible degenerations.

Definition 3.16. If $f = [f_0 : f_1 : f_2] \in \mathrm{Bir}_d$, we denote by $J(f)$ the polynomial, defined up to multiple by a constant of $\mathbf{k}^*$, which is the determinant of the matrix of partial derivatives of f_0, f_1, f_2 with respect to x, y, z. It is the *Jacobian* of f. This gives a morphism $J\colon \mathrm{Bir}_d \to \mathbb{P}(\mathbf{k}[x, y, z]_{3(d-1)})$.

We now introduce a new definition, that we will then use to study degenerations of maps (together with Definition 3.20).

Definition 3.17. Let $f = [f_0 : f_1 : f_2] \in \mathrm{Bir}_d$, let $h \in \mathbf{k}[x, y, z]$ be a homogeneous polynomial and let $q = [q_0 : q_1 : q_2] \in \mathbb{P}^2$.

(i) We say that f *contracts* h *onto* $q \in \mathbb{P}^2$ if $q_i f_j - q_j f_i$ is a multiple of h for each $i, j \in \{0, 1, 2\}$.

(ii) We say that q *is a base-point of* f *of multiplicity* k if a general linear combination of f_0, f_1, f_2 has multiplicity k at q.

Remark 3.18. If h is a factor of $\gcd(f_0, f_1, f_2)$, then h is contracted onto any point of $\mathbb{P}^2$. But otherwise, there is only one possible point where h can be contracted.

If $f \in \mathrm{Bir}_d$ and $\varphi = \pi_d(f) \in \mathrm{Bir}(\mathbb{P}^2)_{\leq d}$ is the corresponding birational map, every base-point of φ is a base-point of f. But if φ has degree $< d$, then f has infinitely many base-points, which correspond to the points of the common factor of f_0, f_1, f_2.

Let us recall the following classical result.

Lemma 3.19. *Let $f = [f_0 : f_1 : f_2] \in \mathrm{Bir}_d^\circ$ be an element which corresponds to the birational map $\pi_d(f) = \varphi \in \mathrm{Bir}(\mathbb{P}^2)_d$ and denote by $(d; m_1, \ldots, m_r)$ the homaloidal type of φ^{-1}. We also denote by $g = [g_0 : g_1 : g_2] \in \mathrm{Bir}_d^\circ$ the element corresponding to φ^{-1}, and assume that φ^{-1} has no infinitely near base-point.*

(1) *If $h \in \mathbf{k}[x, y, z]$ is a homogeneous polynomial which is contracted by f onto a point $q \in \mathbb{P}^2$, then each point of the curve of $\mathbb{P}^2$ given by $h = 0$ which is not a base-point of φ is sent by φ onto q. Moreover, h is a divisor of the Jacobian $J(f)$.*

(2) *The Jacobian $J(f)$ admits a decomposition into $J(f) = \prod_{i=1}^r p_i$, where $p_1, \ldots, p_r$ are homogeneous polynomials of degree $m_1, \ldots, m_r$ respectively, each of them contracted by f onto points $q_1, \ldots, q_r \in \mathbb{P}^2$ respectively, all being base-points of φ^{-1} of multiplicity equal to $m_1, \ldots, m_r$ respectively.*

   Moreover, the following hold:

   (a) *The points q_i are pairwise distinct.*

   (b) *The points $q_1, \ldots, q_r$ are the base-points of φ^{-1}.*

   (c) *Each p_i is an irreducible polynomial.*

   (d) *The decomposition $J(f) = \prod_{i=1}^r p_i$ corresponds to the decomposition of $J(f)$ into irreducible polynomials.*

Proof. Follows from [A-C02, Proposition 3.5.3 and Theorem 3.5.6]. □

Definition 3.20. Let Λ be a homaloidal type, such that $\Lambda^* = (d; m_1, \ldots, m_r)$. We denote by

$$\mathrm{Bir}_\Lambda \subset \mathrm{Bir}_d$$

the set of elements $f = [f_0 : f_1 : f_2] \in \mathrm{Bir}_d$ such that there exist $g = [g_0 : g_1 : g_2] \in \mathrm{Bir}_d$ and homogeneous polynomials $p_1, \ldots, p_r$ of degree $m_1, \ldots, m_r$ respectively, each of them contracted by f onto points $q_1, \ldots, q_r$, being base-points of g of multiplicity at least $m_1, \ldots, m_r$, and such that $J(f) = \prod\limits_{i=1}^{r} p_i$ and $\pi_d(g) \circ \pi_d(f)$ is the identity.

The following result shows that this definition is consistent with Notation 3.12.

Proposition 3.21. *Let Λ be a proper homaloidal type of degree $d \geq 2$. Then, the following hold:*

(1) *The set Bir_Λ is closed in Bir_d.*

(2) *$\mathrm{Bir}_\Lambda^\circ$ is the unique irreducible component of Bir_d° contained in $\mathrm{Bir}_\Lambda \cap \mathrm{Bir}_d^\circ$.*

Proof. (1) We write $\Lambda^* = (d; m_1, \ldots, m_r)$ and prove that Bir_Λ is closed in Bir_d. To do this, we denote by X_Λ the subset of

$$\mathrm{Bir}_d \times W_d \times \mathbb{P}(\mathbf{k}[x, y, z]_{m_1}) \times \cdots \times \mathbb{P}(\mathbf{k}[x, y, z]_{m_r}) \times (\mathbb{P}^2)^r$$

consisting of elements

$$(f, g, p_1, \dots, p_r, q_1, \dots, q_r)$$

such that

- The equality $J(f) = \prod_{i=1}^{r} p_i$ holds in $\mathbb{P}(\mathbf{k}[x, y, z]_{3d-3})$.
- The element $[g_0(f_0, f_1, f_2) : g_1(f_0, f_1, f_2) : g_2(f_0, f_1, f_2)] \in W_{d^2}$ is equal to $[xh : yh : zh]$ for some element $h \in \mathbf{k}[x, y, z]_{d^2-1}$; this corresponds to ask that $g_i(f_0, f_1, f_2)x_j = g_j(f_0, f_1, f_2)x_i$ for each i, j, where $x_0 = x$, $x_1 = y$, $x_2 = z$.
- For each j, the polynomial p_j is contracted by f onto q_j.
- For each j, the point q_j is a base-point of h of multiplicity $\geq m_j$.

Since all above conditions are closed, the set X_Λ is closed in $\mathrm{Bir}_d \times W_d \times \mathbb{P}(\mathbf{k}[x, y, z]_{m_1}) \times \dots \times \mathbb{P}(\mathbf{k}[x, y, z]_{m_r}) \times (\mathbb{P}^2)^r$. Moreover, the second condition is equivalent to ask that $g \in \mathrm{Bir}_d$ and that $\pi_d(f) \circ \pi_d(g)$ is the identity. Hence, the set Bir_Λ is the projection of X_Λ onto Bir_d and is thus closed in Bir_d, because $W_d \times \mathbb{P}(\mathbf{k}[x, y, z]_{m_1}) \times \dots \times \mathbb{P}(\mathbf{k}[x, y, z]_{m_r}) \times (\mathbb{P}^2)^r$ is projective.

If $f = [f_0 : f_1 : f_2] \in \mathrm{Bir}_d^\circ$ is of homaloidal type Λ such that $(\pi_d(f))^{-1}$ has no infinitely near base-points, then f belongs to Bir_Λ and not to any other $\mathrm{Bir}_{\Lambda'}$ (Lemma 3.19). This shows, together with Proposition 3.11, that $\mathrm{Bir}_\Lambda^\circ \subset \mathrm{Bir}_\Lambda \cap \mathrm{Bir}_d^\circ$ and that $\mathrm{Bir}_\Lambda \cap \mathrm{Bir}_d^\circ$ does not contain any other irreducible component. □

Remark 3.22. Each element of $\overline{\mathrm{Bir}_d^\circ} \subset \mathrm{Bir}_d$ is contained in a Bir_Λ, for some homaloidal type Λ of degree d. This will give some conditions on the elements of $\overline{\mathrm{Bir}_d^\circ}$, and thus on the set $\overline{\mathrm{Bir}(\mathbb{P}^2)_d} = \pi_d(\overline{\mathrm{Bir}_d^\circ})$ (Corollary 3.9).

The set $\overline{\mathrm{Bir}_\Lambda^\circ}$ is contained in Bir_Λ, but we do not know if equality holds. We also do not know if $\mathrm{Bir}_\Lambda^\circ = \mathrm{Bir}_\Lambda \cap \mathrm{Bir}_d^\circ$.

Example 3.23. Let us consider

$$\begin{array}{rcll} f &=& [(x+y-z)y(3y-z) : (2y-z)x(3y-z) : (2y-z)xy] & \in \mathrm{Bir}_3^\circ \\ g &=& [(y-2z)yz : (xy-xz-yz)z : (xy-xz-yz)(y-3z)] & \in \mathrm{Bir}_3^\circ \end{array}$$

which are such that $\pi_3(f) \circ \pi_3(g)$ is the identity and $J(f) = 3xy(3y-z)(z-y)(2y-z)^2$. The polynomials x, y, $3y-z$, $z-y$ and $2y-z$ are contracted respectively onto $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[2 : 2 : 1]$, $[1 : 0 : 0]$.

There are multiple ways to choose the polynomials $p_1, \dots, p_5$ and the points $q_1, \dots, q_5$ and in each way there is a polynomial p_i of degree 1 contracted onto $q_i = [1 : 0 : 0]$, which is a base-point of g of multiplicity 2. The fact that the multiplicity is higher than the degree of the polynomial is because this polynomial corresponds in fact to a base-point of g infinitely near to $[1 : 0 : 0]$, having multiplicity 1.

4. Existence of degenerations

4.1. **Degeneration associated to two base-points.** We first prove the following simple degeneration lemma.

Lemma 4.1. *Let $p_1 \in \mathbb{P}^2$ and let p_2 be a point which is either in the first neighbourhood of p_1 or a distinct point of $\mathbb{P}^2$.*

Then, there exists a morphism $\nu\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ and a morphism $p_3\colon \mathbb{A}^1 \to \mathbb{P}^2$ such that the following hold:

(1) *For $t \neq 0$, $\nu(t)$ is a quadratic map with base-points $p_1, p_2, p_3(t)$;*
(2) *The map $\nu(0)$ is the identity and $p_3(0)$ is collinear with p_1 and p_2.*

Remark 4.2. In this degeneration, the linear system of conics through $p_1, p_2, p_3(t)$ degenerates to a system of conics through three collinear points, which is the union of the line through the points and the system of lines of $\mathbb{P}^2$.

Proof. We first assume that p_1, p_2 are proper points of $\mathbb{P}^2$ and can then assume that $p_1 = [1:0:0]$ and $p_2 = [0:1:0]$. Then, we consider the morphism $\kappa\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ given by

$$\kappa(t)\colon [x:y:z] \dashrightarrow [(ty-z)x:(tx-z)y:(tx-z)(ty-z)].$$

For $t \neq 0$, $\kappa(t)$ is a quadratic birational involution with base-points $p_1, p_2, [1:1:t]$. Moreover, $\kappa(0)$ equal to the linear automorphism $[x:y:z] \mapsto [x:y:-z]$. We can then define ν as $\nu(t) = \kappa(t) \circ \kappa(0)$.

The second case is when p_2 is infinitely near to p_1. We can then assume that $p_1 = [1:0:0]$ and that p_2 corresponds to the tangent direction $z = 0$. In this case, we choose $\kappa\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ given by

$$\kappa(t)\colon [x:y:z] \dashrightarrow [-xz+ty^2:yz:z^2].$$

For $t \neq 0$, $\kappa(t)$ is a quadratic birational involution with base-points p_1, p_2 and some point $p_3(t)$ infinitely near p_2. Moreover, $\kappa(0)$ equal to the linear automorphism $[x:y:z] \mapsto [-x:y:z]$. Again, choosing $\nu(t) = \kappa(t) \circ \kappa(0)$ works. $\square$

Proposition 4.3. *Suppose that $\gamma \in \mathrm{Bir}(\mathbb{P}^2)$ is a birational map of degree d and has two base-points p_1, p_2 of multiplicity m_1, m_2 with $m_1 + m_2 = d - k$, such that p_1 is a proper point of $\mathbb{P}^2$ and p_2 is either a proper point of $\mathbb{P}^2$ or in the first neighbourhood of p_1.*

Then, there exists a morphism $\rho\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ such that $\rho(0) = \gamma$ and $\rho(t)$ has degree $d + k$ for a general $t \neq 0$.

Proof. We use the morphism $\nu\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ given by Lemma 4.1 and define ρ as $\rho(t) = \gamma \circ \nu(t)^{-1}$. By construction, this is a morphism which satisfies $\rho(0) = \gamma$. Moreover, the degree of the map $\rho(t)$ for a general t is equal to $2d - m_1 - m_2 = d + k$. $\square$

Remark 4.4. If $p_1, \ldots, p_r$ are the base-points of γ of multiplicity $m_1, \ldots, m_r$, the degeneration provided by Proposition 4.3 gives a family of birational maps of degree $d+k$ with base-points of multiplicity $m_1+k, m_2+k, k, m_3, \ldots, m_r$. The point of multiplicity k created degenerates to a point collinear with the first two points, and the linear system becomes the union of the linear system of γ with k times the line through p_1 and p_2.

In order to be able to apply Proposition 4.3, we need to compute the multiplicities of birational maps and estimate the integer k which appears in the statement. This is done in the following lemma.

Lemma 4.5. *Let φ be a birational map of degree d. Then there exists two distinct points (proper or infinitely near) of multiplicity $m_1, m_2 \geq 0$, such that*

$$\begin{array}{rl} m_1 + m_2 = d - 1 & \text{if } d \in \{1, 2, \ldots, 6, 7, 9, 11\} \\ d - 2 \leq m_1 + m_2 \leq d - 1 & \text{if } d = 8 \\ d - 3 \leq m_1 + m_2 \leq d - 1 & \text{if } d = 10 \\ \frac{2d}{3} < m_1 + m_2 < d & \text{if } d \geq 12 \end{array}$$

Proof. If φ is of de Jonquières type, we can choose $m_1 = d - 1$ and $m_2 = 0$. If φ has an homaloidal type $(d; d-2, 2^{d-2}, 1^3)$, we can choose $m_1 = d - 2$ and $m_2 = 1$.

If $d \in \{1, 2, \ldots, 6, 7, 9, 11\}$, we find two base-points of multiplicity m_1, m_2 with $m_1 + m_2 = d - 1$ (see Example 2.9). For $d = 8, 10$, the result also follows from Example 2.9.

We can thus assume $d \geq 12$, and find, with Noether inequalities, two points of multiplicity m_1, m_2 with $m_1 + m_2 > \frac{2d}{3}$. If $m_1 + m_2 < d$, we are done, so we can assume that $m_1 + m_2 = d$ ($m_1 + m_2 > d$ is not possible by the Bézout theorem), i.e. $m_2 = d - m_1$. Either $m_1 > d/2$, or $m_1 = m_2 = d/2$. Moreover, Noether inequalities implies also that $m_3 > (d - m_1)/2 = m_2/2$ [A-C02, Lemma 8.2.6]. Hence, $m_1 + m_3 > (d + m_1)/2 \geq 3d/4 > 2d/3$, that is the assertion, unless $m_3 = m_2 = d - m_1$ too.

Let γ be the number of points, different from p_1, with multiplicity $d - m_1$. Either $m_1 > d/2$, or $m_1 = m_2 = \cdots = m_{\gamma+1} = d/2$.

In the latter case, applying a quadratic map centered at p_1, p_2, p_3, one finds a Cremona map of degree $d/2$ that must have a base-point q_4 of multiplicity m_4 with $d/2 > m_4 > (d/2)/3 = d/6$. This means that also φ has a point p_4 of multiplicity m_4. It follows that $d > m_1 + m_4 > 2d/3$ and the proof is concluded in this case.

In the former case, we claim that, if φ is not de Jonquières (in which case the assertion of the lemma is trivial, as we already observed), then φ has at least one further base-point $p_{\gamma+2}$ (cf. [Hud27, p. 75]). Suppose indeed that the number r of base-points is $\gamma+1$. Multiplying the first Noether equality in (1) by m_2 and subtracting the second Noether equality in (1), we find

$$\sum_{i=1}^{r} m_i(m_2 - m_i) = 3m_2(d-1) - (d^2 - 1) = (d-1)(2d - 3m_1 - 1)$$

that is

$$m_1(d - 2m_1) = (d-1)(2d - 3m_1 - 1)$$

which is impossible because $m_1 < d - 1$. So our claim is proved and there is at least another base-point $p_{\gamma+2}$ that we can use together with p_1. Note also that the assertion is trivial if $m_1 + 1 > 2d/3$, i.e. $m_1 > (2d-3)/3$. Hence, we may assume that $m_1 \leq (2d-3)/3$ and therefore $m_2 = d - m_1 \geq (d+3)/3$. Recalling that $m_2 = m_3 = d - m_1 < d/2$, it follows that $d > m_2 + m_3 > 2d/3$, as wanted. □

Corollary 4.6. *We have* $\mathrm{Bir}(\mathbb{P}^2)_d \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$ *for each*

$$d \in \{1, 2, \ldots, 6, 7, 9, 11\}$$

and $\mathrm{Bir}(\mathbb{P}^2)_8 \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{10}}$.

Proof. Let $\gamma \in \mathrm{Bir}(\mathbb{P}^2)$ be of degree $d \in \{1, 2, \ldots, 6, 7, 9, 11\}$. By Lemma 4.5 there exist two points p_1 and p_2 of respective multiplicity m_1 and m_2 with $m_1 + m_2 = d - 1$. If p_1, p_2 are proper points in $\mathbb{P}^2$, then $\gamma \in \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$ by Proposition 4.3.

The set of elements $\gamma \in \mathrm{Bir}(\mathbb{P}^2)_d$ such that p_1, p_2 are proper being dense in each irreducible component of $\mathrm{Bir}(\mathbb{P}^2)_d$ (Proposition 3.11), we obtain $\mathrm{Bir}(\mathbb{P}^2)_d \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$.

Similarly, if γ has degree $d = 8$, by Lemma 4.5 there exist p_1, p_2 with $6 = d - 2 \leq m_1 + m_2 \leq d - 1 = 7$, hence we conclude as above that γ is in either $\overline{\mathrm{Bir}(\mathbb{P}^2)_{10}}$ or $\overline{\mathrm{Bir}(\mathbb{P}^2)_9}$, the latter being included in $\overline{\mathrm{Bir}(\mathbb{P}^2)_{10}}$ by the first part of the proof. □

Corollary 4.7. *Let* $\gamma \in \mathrm{Bir}(\mathbb{P}^2)$ *be a birational map of degree* d*. There exists an integer* k *such that* $1 \leq k \leq \max\{1, \frac{d}{3}\}$ *and* $\gamma \in \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+k}}$.

Proof. The proof is similar as the one of Corollary 4.6. □

4.2. **Degeneration associated to five general base-points.** Let us give another degeneration process, similar to Lemma 4.1 and Proposition 4.3 but with more points. It will be useful to show that $\mathrm{Bir}(\mathbb{P}^2)_{10} \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{12}}$.

Lemma 4.8. *Let* $p_1, \ldots, p_5$ *be five distinct points of* $\mathbb{P}^2$*, such that no* 3 *of them are collinear.*

Then, there exists an open subset $U \subset \mathbb{A}^1$ *containing* 0 *and two morphisms* $\nu \colon U \to \mathrm{Bir}(\mathbb{P}^2)$ *and* $p_6 \colon U \to \mathbb{P}^2$*, such that the following hold:*

(1) *For* $t \neq 0$*, the map* $\nu(t)$ *has degree* 5 *and six base-points of multiplicity* 2*, being* $p_1, \ldots, p_5, p_6(t)$*, which are such that no* 3 *are collinear and which do not belong to the same conic.*
(2) *The map* $\nu(0)$ *is the identity and* $p_6(0)$ *belongs to to the conic passing through* $p_1, \ldots, p_5$.

Proof. Since no three of the points $p_1, \ldots, p_5$ are collinear, we can assume that $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$. We then denote by σ the standard quadratic transformation

$$\sigma\colon [x : y : z] \dashrightarrow [yz : xz : xy].$$

Note that σ is a local isomorphism at p_4, p_5 and that $p_1, p_2, p_3, \sigma(p_4), \sigma(p_5)$ are five points of $\mathbb{P}^2$ such that no 3 are collinear.

Applying Lemma 4.1, we obtain morphisms $\nu'\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ and $p_6'\colon \mathbb{A}^1 \to \mathbb{P}^2$ such that $\nu'(0)$ is the identity and $\nu'(t)$ is a quadratic map with base-points $\sigma(p_4), \sigma(p_5), p_6'(t)$. Moreover, $p_6'(t)$ is collinear with $\sigma(p_4)$ and $\sigma(p_5)$ if and only if $t = 0$. We can moreover choose that $p_6(0)$ does not belong to the triangle $xyz = 0$, conjugate ν' with an automorphism of $\mathbb{P}^2$ if needed.

We denote by $U' \subset \mathbb{A}^1$ the dense open subset such that $p_6'(t)$ is not collinear with two of the points p_1, p_2, p_3, $\sigma(p_4)$,$\sigma(p_5)$ and does not belong to the conic through these points. In particular, $\nu'(t)$ is a local isomorphism at p_1, p_2, p_3 for each $t \in U'$. We have then a morphism map $\psi\colon U' \to \mathrm{PGL}(3, \mathbf{k})$ (or equivalently an element of $\mathrm{PGL}(3, \mathbf{k}(t))$) such that $\psi(t)$ sends $\nu'(t)(p_i)$ onto p_i for $i = 1, 2, 3$. Since $\nu'(0)$ is the identity, we have $0 \in U'$ and can choose $\psi(0)$ to be the identity.

We then define a morphism $\nu\colon U' \to \mathrm{Bir}(\mathbb{P}^2)$ in the following way:

$$\nu(t) = \sigma\psi(t)\nu'(t)\sigma.$$

For $t = 0$, $\nu(t)$ is the identity, since $\psi(0)$ and $\nu'(0)$ are the identity. It remains to observe that for a general $t \in U'$ the linear system of $\nu(t)$ has the desired form.

For $t \neq 0$, the linear system of $\sigma\psi(t)$ consists of conics through $\nu'(t)(p_1)$, $\nu'(t)(p_2)$, $\nu'(t)(p_3)$. The linear system of $\sigma\psi(t)\nu'(t)$ consists then of quartics having multiplicity two at $\sigma(p_4), \sigma(p_5), p_6'(t)$ and multiplicity one at p_1, p_2, p_3. The linear system of ν has then multiplicity 5 and multiplicity 2 at $p_1, p_2, p_3, p_4, p_5, \sigma(p_6'(t))$. We define $U \subset U'$ to be the union of 0 with the points of U' such that $p_6(t) = \sigma(p_6'(t))$ is a proper point of $\mathbb{P}^2$ and obtain the result. The degeneration of $\nu(t)$ comes because $p_6(0)$ belongs to the conic through $p_1, \ldots, p_5$, which is the image by σ of the line through $\sigma(p_4)$ and $\sigma(p_5)$. □

Proposition 4.9. *Suppose that $\gamma \in \mathrm{Bir}(\mathbb{P}^2)$ is a birational map of degree d and has five proper base-points $p_1, \ldots, p_5$ of multiplicity $m_1, \ldots, m_5$, such that no 3 of them are collinear and such that $\sum_{i=1}^5 m_i = 2d - k$.*

Then, there exists a morphism $\rho\colon U \to \mathrm{Bir}(\mathbb{P}^2)$, where $U \subset \mathbb{A}^1$ is an open subset containing 0, such that $\rho(0) = \gamma$ and $\rho(t)$ has degree $d + 2k$ for a general $t \neq 0$.

Proof. We use the morphism $\nu\colon U \to \mathrm{Bir}(\mathbb{P}^2)$ given by Lemma 4.8 and define ρ as $\rho(t) = \gamma \circ \nu(t)^{-1}$. By construction, this is a morphism which satisfies $\rho(0) = \gamma$. Moreover, the degree of the map $\rho(t)$ for a general t is equal to $5d - 2m_1 - 2m_2 - 2m_3 - 2m_4 - 2m_5 = d + 2k$. □

Remark 4.10. If $p_1, \ldots, p_r$ are the base-points of γ of multiplicity $m_1, \ldots, m_r$, the degeneration provided by Proposition 4.9 gives a family of birational maps of degree $d+2k$ with base-points of multiplicity $m_1+k, m_2+k, m_3+k, m_4+k, m_5+k, k, m_6, \ldots, m_r$. The point of multiplicity k created degenerates to a point which belongs to the conic through the first five points, and the linear system becomes the union of the linear system of γ with k times the conic.

Corollary 4.11. *We have $\mathrm{Bir}(\mathbb{P}^2)_{10} \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{12}}$.*

Proof. Each irreducible component of $\mathrm{Bir}(\mathbb{P}^2)_{10}$ corresponds to a $\mathrm{Bir}_\Lambda^\circ$ where $\Lambda = (d; m_1, \ldots, m_k)$ is a proper homaloidal type (Proposition 3.11 and Notation 3.12). If there are two multiplicities m_i, m_j such that $m_i + m_j = 9$ or $m_i + m_j = 8$, then Proposition 4.3 shows that a general element of $\mathrm{Bir}_\Lambda^\circ$ belongs to $\overline{\mathrm{Bir}(\mathbb{P}^2)_{11} \cup \mathrm{Bir}(\mathbb{P}^2)_{12}} =$

$\overline{\mathrm{Bir}(\mathbb{P}^2)_{12}}$, where the last equality follows from the fact that $\mathrm{Bir}(\mathbb{P}^2)_{11} \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{12}}$ by Corollary 4.6. Looking at Example 2.9, one sees that this holds for each proper homaloidal type of degree 10, except for $\Lambda = (10; 5^3, 2^6)$. We then apply Proposition 4.9 to the first 5 multiplicities, and obtain that a general element of $\mathrm{Bir}_\Lambda^\circ$ is contained in $\overline{\mathrm{Bir}(\mathbb{P}^2)_{12}}$. □

4.3. **Degeneration associated to five base-points, three of them being collinear.** We finish this section with another degeneration process, which works with maps having three collinear points. It will be useful to show that some maps of type $(8; 4^3, 2^3, 1^3)$ belong to $\overline{\mathrm{Bir}(\mathbb{P}^2)_9}$ (Corollary 4.14), although this is not true for a general element of type $(8; 4^3, 2^3, 1^3)$ (Corollary 5.2), and the same for maps of type $(10; 5^3, 2^6)$.

Lemma 4.12. *Let $p_1, \dots, p_5$ be five distinct points of $\mathbb{P}^2$, such that p_1, p_2, p_3 are collinear but no other triple of points belongs to the same line.*

Then, there exists an open subset $U \subset \mathbb{A}^1$ containing 0 and two morphisms $\nu\colon U \to \mathrm{Bir}(\mathbb{P}^2)$ and $p_6\colon U \to \mathbb{P}^2$, such that the following hold:

(1) *For $t \neq 0$, the map $\nu(t)$ has degree 4 and six base-points, namely p_1, p_2, p_3 with multiplicity 1 and $p_4, p_5, p_6(t)$ with multiplicity 2, and $p_6(t)$ is not collinear with any other base-point.*
(2) *The map $\nu(0)$ is the identity and $p_6(0)$ belongs to to the line through p_4 and p_5.*

Proof. The points p_1, p_2, p_4, p_5 being in general position, we can assume, up to change of cooordinates, that

$$p_1 = [0 : 0 : 1],\ p_2 = [1 : 1 : 1],\ p_4 = [0 : 1 : 0],\ p_5 = [1 : 0 : 0].$$

This implies that $p_3 = [1 : 1 : a]$ for some $a \in \mathbf{k}^*$.

We consider the morphisms $\kappa, \rho, \tau\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ defined by

$$\begin{array}{rclcl} \kappa(t) & : & [x : y : z] & \dashrightarrow & [(ty - z)x : (tx - z)y : (tx - z)(ty - z)] \\ \rho(t) & : & [x : y : z] & \dashrightarrow & [x(yt + z) : yz : -z(yt + z)] \\ \tau(t) & : & [x : y : z] & \mapsto & [(a + t)y + z : (a - 1)y : ax - (a + t)y] \end{array}$$

(the map κ is the same as in Lemma 4.1). Observe that $\tau(t) \in \mathrm{Aut}(\mathbb{P}^2)$ for each t, that $\rho(0), \kappa(0) \in \mathrm{Aut}(\mathbb{P}^2)$ and that for a general t, $\kappa(t), \rho(t)$ are quadratic birational involutions of $\mathbb{P}^2$. Moreover, the base-points of $\rho(t)$ are p_4, p_5 and the point infinitely to p_5 that corresponds to the line $yt + z = 0$, that we will denote $p_6(t)$. The base-points of $\kappa(t)$ are $p_4, p_5, [1 : 1 : t]$.

We then define $\nu\colon \mathbb{A}^1 \dashrightarrow \mathrm{Bir}(\mathbb{P}^2)$ as $\nu(t) = \rho(0)\tau(0)^{-1}\kappa(0)\kappa(t)\tau(t)\rho(t)$. The linear system of $\nu(t)$ is given by comparing the linear system of $\kappa(t)\tau(t)$ with the one of $\rho(t)^{-1} = \rho(t)$. The linear system of $\kappa(t)\tau(t)$ consists of conics through

$$\tau(t)^{-1}(\{p_4, p_5, [1 : 1 : t]\}) = \{[a + t : a : -a(a + t)], p_1, [t + 1 : 1 : -t - 1]\}.$$

For $t \notin \{0, -1, a\}$, the three points are different from the three base-points of $\rho(t)$, so $\nu(t)$ has degree 4, multiplicity 2 at the three base-points of $\rho(t)$ and multiplicity 1 at

$$\rho(t)(\tau(t)^{-1})(\{p_4, p_5, [1 : 1 : t]\}) = \{p_2, p_1, p_3\}.$$

Choosing $U = \mathbb{A}^1 \setminus \{-1, a\}$, we obtain the result. □

Proposition 4.13. *Suppose that $\gamma \in \mathrm{Bir}(\mathbb{P}^2)$ is a birational map of degree d and has five proper base-points $p_1, \dots, p_5$ of multiplicity $m_1, \dots, m_5$, such that p_1, p_2, p_3 are collinear but no other triple of points belongs to the same line and such that $m_1 + m_2 + m_3 + 2m_4 + 2m_5 = 3d - k$.*

Then, there exists a morphism $\rho\colon U \to \mathrm{Bir}(\mathbb{P}^2)$, where $U \subset \mathbb{A}^1$ is an open subset containing 0, such that $\rho(0) = \gamma$ and $\rho(t)$ has degree $d + k$ for a general $t \neq 0$.

Proof. We use the morphism $\nu\colon U \to \mathrm{Bir}(\mathbb{P}^2)$ given by Lemma 4.12 and define ρ as $\rho(t) = \gamma\circ\nu(t)^{-1}$. By construction, this is a morphism which satisfies $\rho(0) = \gamma$. Moreover, the degree of the map $\rho(t)$ for a general t is equal to $4d - m_1 - m_2 - m_3 - 2m_4 - 2m_5 = d + k$. $\square$

Corollary 4.14. *If $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ is a map of type $(d; m_1, \ldots, m_r)$ with five proper base-points p_1, p_2, p_3, p_4, p_5 of multiplicity $m_1, \ldots, m_5$ respectively, such that p_1, p_2, p_3 are collinear but no other triple of points belongs to the same line. If $m_1 + m_2 + m_3 = d - 1$ and $m_4 + m_5 = d$ then $\varphi \in \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$.*

In particular, there exist some elements in $\overline{\mathrm{Bir}(\mathbb{P}^2)_9}$ of type $(8; 4^3, 2^3, 1^3)$ and elements in $\overline{\mathrm{Bir}(\mathbb{P}^2)_{11}}$ of type $(10; 5^3, 2^6)$.

Proof. The second part directly follows from Proposition 4.13. The second part follows by taking $(m_1, \ldots, m_5)$ to be respectively $(1, 2, 4, 4, 4)$ and $(2, 2, 5, 5, 5)$. $\square$

5. Restrictions on the degeneration in one degree less

Proposition 5.1. *Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)_d$ be a birational map of degree $d \geq 2$ with only proper base-points (but φ^{-1} can have infinitely near base-points), and assume that φ belongs to the closure of $\mathrm{Bir}(\mathbb{P}^2)_{d+1}$.*

Then, there exist a set Ω consisting of one, two, three or four base-points of φ, which are collinear and such that the sum of their multiplicities is equal to $d - 1$.

Proof. Suppose that φ belongs to the closure of $\mathrm{Bir}(\mathbb{P}^2)_{d+1}$, which is equivalent to the fact that φ^{-1} belongs to the closure of $\mathrm{Bir}(\mathbb{P}^2)_{d+1}$.

By Corollary 3.9, there exist elements $\hat{f}, \hat{g} \in \overline{\mathrm{Bir}^\circ_{d+1}}$ which are sent by π_{d+1} onto φ^{-1} and φ respectively. Denoting by $f = [f_0 : f_1 : f_2] \in \mathrm{Bir}^\circ_d$ and $g = [g_0 : g_1 : g_2] \in \mathrm{Bir}^\circ_d$ the elements corresponding to φ^{-1} and φ respectively, there exists thus homogeneous polynomials $\alpha, \beta \in \mathbf{k}[x_0, x_1, x_2]$ of degree 1, such that $\hat{f} = [\alpha f_0 : \alpha f_1 : \alpha f_2]$ and $\hat{g} = [\beta g_0 : \beta g_1 : \beta g_2]$ belong to $\overline{\mathrm{Bir}^\circ_{d+1}}$. By Proposition 3.21, there is a homaloidal type Λ such that $\hat{f} \in \mathrm{Bir}_\Lambda$.

Changing maybe β and writing $\Lambda^* = (d+1; n_1, \ldots, n_r)$, this yields the existence (see Definition 3.20) of homogeneous polynomials $\hat{p}_1, \ldots, \hat{p}_r$ of degree $n_1, \ldots, n_r$ respectively, each of them contracted by $\hat{f}$ onto points $q_1, \ldots, q_r \in \mathbb{P}^2$, being base-points of $\hat{g}$ of multiplicity at least $n_1, \ldots, n_r$, and such that $J(\hat{f}) = \prod_{i=1}^{r} \hat{p}_i$.

For each i, the fact that $\hat{p}_i$ is contracted by $\hat{f} = [\alpha f_0 : \alpha f_1 : \alpha f_2]$, and that $f = [f_0 : f_1 : f_2]$ is without common component imply that one of the following holds:

(1) $\hat{p}_i = \alpha$ and q_i is any point of $\mathbb{P}^2$;
(2) $\hat{p}_i = \alpha p_i$, where p_i is a polynomial contracted by f onto q_i;
(3) $\hat{p}_i = p_i$, where p_i is a polynomial contracted by f onto q_i.

Because φ has only proper base-points, if a polynomial p of degree k is contracted by f onto a point q, then q is a base-point of φ, and thus of g, of multiplicity k.

The polynomials p_i defined in (2), (3) above are thus irreductible factors of the Jacobian $J(f)$. Note that $\prod_{i=1}^{r} \hat{p}_i = J(\hat{f}) = J(f)\alpha^3$, so each polynomial contracted by f appears exactly once in this decomposition, except if this polynomial is α itself.

(a) We assume first that α is not a divisor of $J(f)$, which is the easiest case. We write $p_i = 1$ in the case where $\hat{p}_i = \alpha$, and obtain then $\prod_{i=1}^r p_i = J(f)$. For each i, we denote by m_i the degree of p_i, and obtain $m_i \in \{n_i, n_i - 1\}$, and obtain, via Noether equalities

$$\sum_{i=1}^{r} n_i = 3d + 6 = 3 + \sum_{i=1}^{r} m_i,$$

$$\sum_{i=1}^{r}(n_i)^2 = (d+1)^2 - 1 = (2d+1) + d^2 - 1 = (2d+1) + \sum_{i=1}^{r}(m_i)^2.$$

There are then exactly three values of i such that $n_i = m_i + 1$. Reordering such that the three indices are $1, 2, 3$, we obtain

$$2d+1 = \sum_{i=1}^{r}(n_i)^2 - \sum_{i=1}^{r}(m_i)^2 = (2m_1+1) + (2m_2+1) + (2m_3+1)$$

which yields $m_1 + m_2 + m_3 = d - 1$.

If $i \in \{1,2,3\}$ is such that $m_i > 0$, then $\hat{p}_i = \alpha p_i$ and p_i is contracted by f onto q_i (case (2) above). Moreover, q_i is a base-point of multiplicity m_i of g and of multiplicy $m_i + 1$ of $\hat{g}$. This implies that q_i belongs to the line given by $\beta = 0$. Choosing

$$\Omega = \{q_i \mid i \in \{1,2,3\} \text{ and } m_i > 0\},$$

we obtain the result.

(b) Assume now that α is a divisor of $J(f)$. Each other irreducible factor of $J(f)$ is then equal to exactly one p_i, and α appears four times in $J(f)\alpha^3 = J(\hat{f}) = \prod_{i=1}^{r} \hat{p}_i$.

If $\hat{p}_i$ is equal to α or to α^2 for some i, we will choose that $p_i = 1$. In all other cases, the polynomial p_i is defined as before. This implies that $J(f) = \alpha \prod_{i=1}^{r} p_i$. As before, we denote by m_i the degree of p_i and the Noether equalities yield $1 + \sum_{i=1}^{r} m_i = 3d - 3$ and $1 + \sum_{i=1}^{r}(m_i)^2 = d^2 - 1$.

We consider now the following possibilities, which describe which one of the $\hat{p}_i$ are multiple of α.

(i) Suppose that α^2 is equal to two different $\hat{p}_i$, that we can choose to be $\hat{p}_1$ and $\hat{p}_2$. We have then $n_1 = n_2 = 2$, $m_1 = m_2 = 0$ and $m_i = n_i$ for $i \geq 3$. Then $2d + 1 = \sum(n_i)^2 - (\sum(m_i)^2 + 1) = 3$, which is not possible since $d \geq 2$.

(ii) Suppose that $\hat{p}_1 = \alpha^2$ and that α divides two other $\hat{p}_i$, that we can assume to be $\hat{p}_2$ and $\hat{p}_3$. We have then $n_1 = 2$, $m_1 = 0$, $n_2 = m_2 + 1$, $n_3 = m_2 + 1$ and $n_i = m_i$ for $i \geq 3$. Then $2d+1 = \sum(n_i)^2 - (\sum(m_i)^2 + 1) = 1 + (2m_1+1) + (2m_2+1)$, which yields $m_1 + m_2 = d - 1$. We can conclude as before: if $m_i > 0$ with $i \in \{2,3\}$, then p_i is contracted onto q_i, which is a base-point of φ of multiplicity m_i.

(iii) The last case is when α^2 does not divide any of the $\hat{p}_i$. There are thus exactly four values of i such that α divides $\hat{p}_i$. We can choose that these are $1, 2, 3, 4$, and obtain $\hat{p}_i = \alpha p_i$ for $i = 1,2,3,4$, and $\hat{p}_i = p_i$ for $i > 4$. So $n_i = m_i + 1$ for $i \leq 4$ and $n_i = m_i$ for $i > 4$. In particular, we obtain $2d + 1 = \sum(n_i)^2 - (\sum(m_i)^2 + 1) = (2m_1+1)+(2m_2+1)+(2m_2+1)+(2m_3+1)-1$, which yields $m_1+m_2+m_3+m_4 = d-1$. $\square$

Corollary 5.2. *Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)_d$ be a birational map of degree $d \geq 2$ with only proper base-points (but φ^{-1} can have infinitely near base-points), such that no three are collinear.*

Then, the following conditions are equivalent:

(1) *The map φ belongs to the closure of $\mathrm{Bir}(\mathbb{P}^2)_{d+1}$.*
(2) *There exist a set Ω consisting of one or two base-points of φ such that the sum of their multiplicities is equal to $d - 1$.*

Proof. The implication (1) $\Rightarrow$ (2) is given by Proposition 5.1. The implication (2) $\Rightarrow$ (1) is given by Proposition 4.3. $\square$

Proposition 5.3. *Let $\Lambda = (d; m_1, m_2, \ldots, m_r)$ be a proper homaloidal type.*

The irreducible component $\pi_d(\mathrm{Bir}_\Lambda^\circ)$ of $\mathrm{Bir}(\mathbb{P}^2)_d$ lies in the closure of $\mathrm{Bir}(\mathbb{P}^2)_{d+1}$ if and only if there exists m_i and m_j such that $m_i + m_j = d - 1$ or $m_i = d - 1$.

Proof. The necessity of the condition on the multiplicities is given by Corollary 5.2.

Conversely, suppose that there exists m_i and m_j such that $m_i + m_j = d-1$ or $m_i = d-1$. By Corollary 5.2, a general element of $\pi_d(\mathrm{Bir}^\circ_\Lambda)$ is in the closure of $\mathrm{Bir}(\mathbb{P}^2)_{d+1}$. This gives the result. □

Remark 5.4. The product of three general quadratic transformations is a general map of type $(8; 4^3, 2^3, 1^3)$ that does not belong to $\overline{\mathrm{Bir}(\mathbb{P}^2)_9}$. However, there are some maps $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ of type $(8; 4^3, 2^3, 1^3)$ that belong to $\overline{\mathrm{Bir}(\mathbb{P}^2)_9}$ (Corollary 4.14). The same phenomenon occurs for $(10; 5^3, 2^6)$.

Proposition 5.3 implies the following:

Corollary 5.5. *One has that* $\mathrm{Bir}(\mathbb{P}^2)_d \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$ *if and only if* $d \in \{1, 2, \dots, 6, 7, 9, 11\}$.

Proof. The "if" part follows from Corollary 4.6. When $d = 8$, there are exactly two proper homaloidal types, namely $(8; 4^3, 2^3, 1^3)$ and $(8; 3^7)$, cf. Table 1, whose corresponding irreducible components of $\mathrm{Bir}(\mathbb{P}^2)_8$ are not contained in $\overline{\mathrm{Bir}(\mathbb{P}^2)_9}$ by Proposition 5.3. Similarly, when $d = 10$, there are exactly 7 proper homaloidal types, cf. Table 1, whose corresponding irreducible components of $\mathrm{Bir}(\mathbb{P}^2)_{10}$ are not contained in $\overline{\mathrm{Bir}(\mathbb{P}^2)_{11}}$.

It remains to see that $\mathrm{Bir}(\mathbb{P}^2)_d \not\subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$ for each $d \geq 12$. To do this, we use the proper homaloidal types $(3m; 3m-6, 6^{m-3}, 4^3, 3^2, 2, 1)$ for $m \geq 4$, $(3m+1; 3m-5, 6^{m-2}, 4, 3^3, 1^4)$ for $m \geq 4$ and $(3m+2; 3m-4, 6^{m-2}, 4^2, 3^2, 2^2, 1)$ for $m \geq 4$ (see Example 2.10). For each of these types of degree $d \in \{3m, 3m+1, 3m+2\}$, there are no two multiplicities m_i and m_j with $m_i + m_j = d-1$, hence Proposition 5.3 says that the corresponding irreducible components of $\mathrm{Bir}(\mathbb{P}^2)_d$ are not contained in $\overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$. □

We can now give the proof of Theorem 1:

Proof of Theorem 1. It follows from Corollary 5.5 that $\overline{\mathrm{Bir}(\mathbb{P}^2)_d} = \mathrm{Bir}(\mathbb{P}^2)_{\leq d}$ for $d \leq 8$, that $\overline{\mathrm{Bir}(\mathbb{P}^2)_d} \neq \mathrm{Bir}(\mathbb{P}^2)_{\leq d}$ for $d \in \{9, 11\}$ and $d \geq 13$, and that $\mathrm{Bir}(\mathbb{P}^2)_{d-1} \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_d}$ for $d \in \{10, 12\}$.

The inclusions $\mathrm{Bir}(\mathbb{P}^2)_8 \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{10}}$ and $\mathrm{Bir}(\mathbb{P}^2)_{10} \subset \overline{\mathrm{Bir}(\mathbb{P}^2)_{12}}$, given by Corollaries 4.6 and 4.11, conclude the proof that $\mathrm{Bir}(\mathbb{P}^2)_{\leq d} = \overline{\mathrm{Bir}(\mathbb{P}^2)_d}$ for each $d \in \{10, 12\}$. □

5.1. **Examples.** The following example shows that it is possible that $[f_0 h : f_1 h : f_2 h] \in \overline{\mathrm{Bir}^\circ_{d+m}}$ corresponds to a birational map $[f_0 : f_1 : f_2] \in \mathrm{Bir}^\circ_d$ that contracts the curve given by $h = 0$.

Example 5.6. Let $\tilde{\kappa} \colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ be given by

$$\tilde{\kappa}(t) \colon [x : y : z] \dashrightarrow [t(x^2 - y^2) - xz : -yz : (t(x+y) - z)(t(x-y) - z)].$$

For each $t \neq 0$, $\tilde{\kappa}(t)$ is a quadratic birational involution, whose three base-points are

$$[1 : -1 : 0], [1 : 1 : 0], [1 : 0 : t],$$

and $\tilde{\kappa}(0)$ is the automorphism

$$[x : y : z] \mapsto [-xz : -yz : z^2] = [-x : -y : z].$$

We now consider $\kappa \colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ be the morphism that is given by $\kappa(t) = \tilde{\kappa}(t) \circ \sigma$, where $\sigma \colon [x : y : z] \dashrightarrow [yz : xz : xy]$ is the standard quadratic involution of $\mathbb{P}^2$:

$$\kappa(t) \colon [x : y : z] \dashrightarrow [(tz^2(y^2 - x^2) - xy^2 z) : -x^2 yz : (tz(x+y) - xy)(tz(y-x) - xy)].$$

For $t \neq 0$, the linear system of the birational map $\kappa(t)$ consists of quartics having multiplicity 2 at $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, and having one tangent direction infinitely near to $[0 : 1 : 0]$ and the two tangent directions infinitely near to $[0 : 0 : 1]$.

For $t = 0$, we obtain explicitely an element

$$[-xy^2 z : -x^2 yz : x^2 y^2] \in \overline{\mathrm{Bir}^\circ_4} \setminus \mathrm{Bir}^\circ_4$$

which corresponds to the birational map of degree 2 given by

$$\kappa(0)\colon [x:y:z] \dashmapsto [-yz:-xz:xy].$$

The polynomial which multiplies this element of Bir_2° to get an element of Bir_4 is xy, and is here contracted by $\kappa(0)$.

The following example shows that one can obtain a general map of degree 2 (no infinitely near base-points) as a limit of special maps of degree 3 (having infinitely near base-points).

Example 5.7. Let $\sigma_1 \in \mathrm{Bir}(\mathbb{P}^2)$ be the quadratic birational involution given by

$$[x:y:z] \dashmapsto [(x-y)(x-z) : y(z-x), z(y-x)]$$

whose three base-points are

$$[0:0:1], [0:1:0], [1:1:1].$$

Let $\sigma_2\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ be the morphism given by

$$\sigma_2(t)\colon [x:y:z] \dashmapsto [y(tx+z(1-t^2)) : z(x-zt) : y(x-zt)]$$

For each t, the map $\sigma_2(t)$ is a birational quadratic involution whose three base-points are

$$[1:0:0], [0:1:0], [t:0:1],$$

and $\sigma_2(0)$ is the standard quadratic transformation.

In particular, the morphism $\sigma_2\sigma_1\colon \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^2)$ gives a degeneration of a family of cubic birational maps $\sigma_2\sigma_1(t)$ for $t \neq 0$ to a quadratic map $\sigma_2\sigma_1(0)$ having only proper base-points. Moreover, for $t \neq 0$ the fact that $[t:0:1]$, $[1:0:0]$ and $[0:0:1]$ are collinear implies that $\sigma_1\sigma_2(t)$ has one base-point infinitely near. In coordinates, we find

$$\sigma_2(t)\sigma_1\colon [x:y:z] \dashmapsto [(x-z)y(tx+z(t^2-t-1)) : (x-y)(x+z(t-1))z : (x-z)y(x+z(t-1))]$$

for $t=0$ we find an element

$$[-(x-z)yz : (x-y)(x-z)z : (x-z)^2y] \in \overline{\mathrm{Bir}_3^\circ} \setminus \mathrm{Bir}_3^\circ$$

which corresponds to the birational map

$$[x:y:z] \dashmapsto [-yz : (x-y)z : (x-z)y]$$

having base-points at $[1:0:0]$, $[0:1:0]$, $[0:0:1]$.

6. Halphen maps

6.1. **Preliminaries on a family of Halphen homaloidal types.** Let us recall the notation of §2.1: we consider the free $\mathbb{Z}$-module V of infinite countable rank, whose basis is $\{e_i\}_{i\in\mathbb{N}}$ and denote by W the group of automorphisms of V generated by σ_0 and by the permutations of the e_i fixing e_0 (see §2.1 for the definition of σ_0, which corresponds to the action of the standard quadratic transformation).

Lemma 6.1. (1) *The automorphism B of V that fixes e_i for $i \geq 10$ and acts on $\bigoplus_{i=0}^{9} \mathbb{Z}e_i$ via the matrix*

$$\begin{pmatrix} 17 & 0 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & -3 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & 0 & -2 & -3 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & 0 & -2 & -2 & -3 & -2 & -2 & -2 & -2 & -2 \\ -6 & 0 & -2 & -2 & -2 & -3 & -2 & -2 & -2 & -2 \\ -6 & 0 & -2 & -2 & -2 & -2 & -3 & -2 & -2 & -2 \\ -6 & 0 & -2 & -2 & -2 & -2 & -2 & -3 & -2 & -2 \\ -6 & 0 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -2 \\ -6 & 0 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -3 \end{pmatrix}$$

respectively to the basis $(e_0, \dots, e_9)$ belongs to the group W.

(2) *Denoting by $\nu \in W$ the transposition that exchanges e_1 and e_2, the matrix of $(\nu B)^{2a} \in W$ relative to $(e_0, \dots, e_9)$ is equal to*

$$\begin{pmatrix} 36a^2 & 12a^2-6a & 12a^2+6a & 12a^2 & 12a^2 & \dots & 12a^2 \\ -12a^2-6a & -4a^2 & -4a^2-4a & -4a^2-2a & -4a^2-2a & \dots & -4a^2-2a \\ -12a^2+6a & -4a^2+4a & -4a^2 & -4a^2+2a & -4a^2+2a & \dots & -4a^2+2a \\ -12a^2 & -4a^2+2a & -4a^2-2a & -4a^2 & -4a^2 & \dots & -4a^2 \\ -12a^2 & -4a^2+2a & -4a^2-2a & -4a^2 & -4a^2 & \dots & -4a^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ -12a^2 & -4a^2+2a & -4a^2-2a & -4a^2 & -4a^2 & \dots & -4a^2 \end{pmatrix} + I$$

for each integer $a \in \mathbb{Z}$, where $I \in \mathrm{GL}(10, \mathbb{Z})$ is the identity matrix.

Proof. Assertion (1) can be proven by hand, following the Hudson's test on the coefficients and applying then σ_0 and permutations. It can also be seen by observing that it is the action of a Bertini involution on the blow-up of 8 general base-points.

Assertion (2) is a straight-forward computation for $a = \pm 1$ and can be proved by induction on $|a|$ for the other integers. □

Remark 6.2. If we take nine points $p_1, \dots, p_9 \in \mathbb{P}^2$ given by the intersection of two general cubics, the blow-up $X \to \mathbb{P}^2$ of these points gives a Halphen surface, whose anti-canonical morphism yields an elliptic fibration.

Moreover, the Bertini involutions (see [BB00, §(1.3)]) associated to 8 of the 9 points lift to automorphisms of X having actions on $\mathrm{Pic}(X)$ which are given by the first matrix of Lemma 6.1, up to permutation. The second matrix, for $a = 1$ is then equal to the matrix of an automorphism $\tau \in X$. This implies that the matrix for $a \in \mathbb{Z}$ is the one given by τ^a.

See [Giz80] for more details on the possible automorphisms of the Halphen surfaces.

Corollary 6.3. *For each $a \geq 1$,*

$$\Lambda_a = (36a^2+1; 12a^2+6a, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2-6a) \qquad (5)$$

is a proper homaloidal type that satisfies $(\Lambda_a)^* = \Lambda_a$.

Proof. According to Proposition 2.4, Λ_a is proper if and only if it belongs to the orbit $W(e_0)$ of e_0 under the action of e_0.

It follows from Lemma 6.1 that $\nu B \in W$, and that

$$(\nu B)^{2a}(e_0) = (36a^2+1)e_0 - (12a^2+6a)e_1 - (12a^2-6a)e_2 - \sum_{i=3}^{9} 12a^2 e_i,$$

which corresponds to the homaloidal type Λ_a. Hence Λ_a is a proper homaloidal type for each $a \geq 1$.

Moreover, the homoloidal type $(\Lambda_a)^*$ is obtained by $(\nu B)^{-2a}(e_0)$ (Remark 2.5). Using again Lemma 6.1, we obtain the equality

$$(\nu B)^{-2a}(e_0) = (36a^2+1)e_0 - (12a^2-6a)e_1 - (12a^2+6a)e_2 - \sum_{i=3}^{9} 12a^2 e_i,$$

which implies that $(\Lambda_a)^* = \Lambda_a$. □

The sequence of proper homaloidal types of Example 2.10 suffices to show that for d large, there are elements in $\mathrm{Bir}(\mathbb{P}^2)_d \backslash \overline{\mathrm{Bir}(\mathbb{P}^2)_{d+1}}$. However, all such families belong in fact to $\overline{\mathrm{Bir}(\mathbb{P}^2)_{d+2}}$. The following family of examples will be sufficient to prove Theorem 2.

Proposition 6.4. *For each $a \geq 1$, there exists a birational map τ_a of degree $d = 36a^2+1$, which is of type*

$$(36a^2+1; 12a^2+6a, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2, 12a^2-6a),$$

and which contracts exactly 9 irreducible curves, 7 of degree $12a^2$, one of degree $12a^2+6a$ and one of degree $12a^2-6a$.

Moreover $\tau_a \in \mathrm{Bir}(\mathbb{P}^2)_d$ does not belong to $\overline{\mathrm{Bir}(\mathbb{P}^2)_{d+k}}$ if $1 \leq k \leq a$.

Proof. By Corollary 6.3, the type given above is a proper homaloidal type which is self-dual. Hence, by Proposition 2.4 there is a birational map τ_a having this type and having only proper base-points. We can moreover assume that $(\tau_a)^{-1}$ also has only proper base-points. This implies that τ_a contracts exactly 9 irreducible curves, 7 of degree $12a^2$, one of degree $12a^2+6a$ and one of degree $12a^2-6a$ (Lemma 3.19).

We write $d = 36a^2+1$ and $f = [f_0 : f_1 : f_2] \in \mathrm{Bir}_d^\circ$ the element sent on τ_a by π_d and suppose that $\hat{f} = [\alpha f_0 : \alpha f_1 : \alpha f_2] \in \mathrm{Bir}_{d+k}$ belongs to the closure of Bir_{d+k}°, for some homogeneous polynomial α of degree k. Hence, $\hat{f}$ belongs to Bir_Λ for some homaloidal type $(d+k; m_1, \ldots, m_r)$ (Proposition 3.21). There exist then polynomials $p_1, \ldots, p_r$ of degree $m_1, \ldots, m_r$ respectively, each of them contracted by $\hat{f}$ onto points $q_1, \ldots, q_r$, being base-points of $\hat{g} = [g_0 : g_1 : g_2]$ of multiplicity at least $m_1, \ldots, m_r$ and satisfying that $J(\hat{f}) = \prod_{i=1}^{r} p_i$. Moreover, $\pi_{d+k}(\hat{g})^{-1} = \pi_{d+k}(\hat{f})$.

Denote by $l_1, \ldots, l_9$ the irreducible polynomials contracted by $[f_0 : f_1 : f_2]$, of degree $n_1, \ldots, n_9$ respectively, with

$$n_1 = 12a^2+6a, n_2 = \cdots = n_8 = 12a^2, n_9 = 12a^2-6a.$$

We have then

$$\prod_{i=1}^{r} p_i = J(\hat{f}) = \alpha^3 J([f_0 : f_1 : f_2]) = \alpha^3 \prod_{i=1}^{9} l_i.$$

The polynomial α having degree $k \leq a < 12a^2 - 6a$, it is not a multiple of l_i for any i. This implies that $l_i l_j P$ is not contracted by $\hat{f}$ for any $1 \leq i, j \leq 9$ and any homogeneous polynomial $P \neq 0$. We can then reorder the p_i such that:

(1) for $i = 1, \ldots, 9$, l_i divides p_i and p_i divides $l_i\alpha$;
(2) for $i \geq 10$, p_i divides α.

Writing $m_i = n_i + \epsilon_i$ for $i = 1, \ldots, 9$ and $m_i = \epsilon_i$ for $i \geq 10$ we have then $0 \leq \epsilon_i \leq k$ for each i.

Applying Noether inequalities we obtain

$$\begin{array}{rcl} 3k & = & (3(d+k)-3)-(3d-3) \\ & = & \sum m_i - \sum n_i \\ & = & \sum \epsilon_i \\ (d+k)^2-1 & = & \sum (m_i)^2 \\ & = & \sum\limits_{i\le 9} (n_i+\epsilon_i)^2 + \sum\limits_{i\ge 10} (\epsilon_i)^2 \\ & = & d^2-1+\sum\limits_{i\le 9}(2n_i\epsilon_i)+\sum(\epsilon_i)^2 \\ & = & d^2-1+24a^2\sum\limits_{i\le 9}\epsilon_i + 12a(\epsilon_1-\epsilon_9)+\sum(\epsilon_i)^2 \end{array}$$

The difference of both sides of the equation yields then

$$\begin{array}{rcl} 0 & = & k^2+2dk-24a^2\sum\limits_{i\le 9}\epsilon_i - 12a(\epsilon_1-\epsilon_9)-\sum(\epsilon_i)^2 \\ & = & k^2+2dk-24a^2(3k-\sum\limits_{i\ge 10}\epsilon_i)-12a(\epsilon_1-\epsilon_9)-\sum(\epsilon_i)^2 \\ & = & k^2+2k(d-36a^2)+\sum\limits_{i\ge 10}\epsilon_i(24a^2-\epsilon_i)-12a(\epsilon_1-\epsilon_9)-\sum\limits_{i\le 9}(\epsilon_i)^2 \\ & \ge & k^2+2k+\sum\limits_{i\ge 10}\epsilon_i(24a^2-\epsilon_i)-12ak-9k^2 \\ & = & 2k(1-6a-4k)+\sum\limits_{i\ge 10}\epsilon_i(24a^2-\epsilon_i) \\ & \ge & 2k-20a^2+\sum\limits_{i\ge 10}\epsilon_i(24a^2-\epsilon_i). \end{array}$$

If $\epsilon_j > 0$ for some $j > 9$, we find

$$\sum_{i\ge 10}\epsilon_i(24a^2-\epsilon_i)\ge 24a^2-k > 20a^2-2k,$$

so the above inequality implies that $\epsilon_j = 0$ for all $j \ge 10$, which means that $r = 9$.

Note that $f = [f_0 : f_1 : f_2]$ contracts $l_1, \ldots, l_9$ onto $q_1, \ldots, q_9$, which are then the base-points of $\pi_d(f)^{-1} = (\tau_a)^{-1}$. This implies that $\hat{f}$ contracts $p_1, \ldots, p_9$ onto $q_1, \ldots, q_9$.

Moreover, the points $q_1, \ldots, q_9$ are base-points of $\hat{g} = [g_0 : g_1 : g_2]$ of multiplicity at least $m_1, \ldots, m_9$. As we can choose the 9 points in general position and since $(d+k; m_1, \ldots, m_9)$ is a proper homaloidal type, the linear system of curves of degree $d+k$ having multiplicity at least m_i at q_i has dimension 2 and corresponds to a birational map of degree $d+k$. This implies that the linear system $\sum \lambda_i g_i$ is irreducible, which leads to a contradiction. □

We can now finish the text with the proof of Theorem 2:

Proof of Theorem 2. The first part follows from Proposition 6.4, the second part follows from Corollary 4.7. □

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