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THE PATH DECOMPOSITION TECHNIQUE FOR SYSTEMS OF HYPERBOLIC CONSERVATION LAWS

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Abstract. We are concerned with the problem of the global (in time) existence of weak solutions to hyperbolic systems of conservation laws, in one spatial dimension. First, we provide a survey of the different facets of a technique that has been used in several papers in the last years: the path decomposition. Then, we report on two very recent results that have been achieved by means of suitable applications of this technique. The first one concerns a system of three equations arising in the dynamic modeling of phase transitions, the second one is the famous Euler system for nonisentropic fluid flow. In both cases, the results concern classes of initial data with possibly large total variation.

1. **Introduction.** We consider the following initial-value problem for a 1-d system of conservation laws

$$U_t + F(U)_x = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1}$$

$$U(x,0) = U_0(x), \qquad x \in \mathbb{R}, \tag{2}$$

where $U = {}^t(u_1, u_2, \ldots, u_n)$ is an n-tuple of conserved quantities taking values in a simply connected region $\Omega \subset \mathbb{R}^n$ and $F(U)$ is the *flux function*, which is a smooth map from Ω to $\mathbb{R}^n$. We assume that the system is *strictly hyperbolic*, which means that $F'(U)$ has real and distinct eigenvalues

$$\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_n(U).$$

The corresponding right eigenvectors $R_1(U), R_2(U), \ldots, R_n(U)$ are called *characteristic vector fields*. We further assume that each characteristic direction is

$$\begin{array}{ll} \text{either } \textit{genuinely nonlinear}: & \nabla_U \lambda_j(U) \neq 0, \\ \text{or } \textit{linearly degenerate}: & \nabla_U \lambda_j(U) \equiv 0. \end{array} \tag{3}$$

The initial-value problem 1-2 does not possess, in general, *smooth* solutions that are defined globally in time, even if the initial data are sufficiently smooth and

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small. Therefore, solutions will be meant in the weak sense: a bounded measurable function is said to be a *weak solution* if

$$\iint_{\mathbb{R}\times\mathbb{R}_+} [U\cdot\Phi_t + F(U)\cdot\Phi_x]\,dxdt + \int_{\mathbb{R}} U_0(x)\cdot\Phi(x,0)\,dx = 0,$$

for any smooth function $\Phi(x,t)$ with bounded support in $\mathbb{R}\times\mathbb{R}_+$.

Since the system of equations 1 is invariant with respect to the similarity transformation $U(x,t)\to U(\alpha x,\alpha t)$, for every $\alpha>0$, the local structure and the asymptotic properties of the solutions are described by *self-similar solutions*, which are solutions satisfying

$$U(\alpha x,\alpha t) = U(x,t).$$

Let $U^\pm\in\Omega$ be constant vectors; then, the *Riemann's initial data*

$$U_0(x) = \begin{cases} U^- & \text{if } x<0,\\ U^+ & \text{if } x>0,\end{cases} \tag{4}$$

satisfy $U_0(\alpha x)=U_0(x)$. The Cauchy problem 1-4 is the *Riemann initial-value problem*.

We recall the following classical result, due to Lax [17], that provides the fundamental existence theorem of self-similar solutions to the Riemann problem.

Theorem 1.1 ([17]). *Assume condition 3. If $|U^- - U^+|$ is sufficiently small, then there exists a unique admissible self-similar solution with small total variation to the Riemann problem 1-4, which consists of $(n+1)$ constant states $U^- = U_0, U_1, \ldots, U_n = U^+$ connected by rarefaction waves, contact discontinuities or shock waves.*

Suppose that U is a weak solution such that the limits $U(x_0\pm 0,t_0)$ exist and the jump $|U(x_0+0,t_0)-U(x_0-0,t_0)|$ is small. The Riemann solution provided by Theorem 1.1 with initial data $U^\pm = U(x_0\pm 0,t_0)$ is a good approximation of U in a neighborhood of (x_0,t_0) and, then, can be used to construct *approximate* solutions. More precisely, let h, k be positive numbers satisfying

$$\frac{h}{k} \ge \max_{\substack{1\le j\le n\\ U\in\Omega}} |\lambda_j(U)|.$$

Consider the points (mh,nk) as points of a grid, with $m,n\in\mathbb{Z}$, $n\ge 0$ and $m+n=$ even. For example, at $t=0$, the initial data are approximated by a function that is constant in every interval $mh<x<(m+2)h$, $m=0,\pm2,\ldots$ Theorem 1.1 provides us of a unique admissible solution in a neighborhood of $(2mh,0)$, for $0<t<k$, as long as the amplitude of $|U^- - U^+|$ is sufficiently small. Since two neighboring Riemann solutions are connected by a constant state, the set of Riemann solutions constitutes an explicit weak solution.

In order to construct approximate solution *beyond* $t=k$, one has to define a piecewise approximation at $t=k$. Godunov [15] adopted the average of the solution in the interval $(2m-1)h<x<(2m+1)h$ while Glimm [14] used the random sampling point $\big((2m-1+2\theta_1)h,k\big)$. In this way, one can repeat the above arguments and construct an approximate solution as long as each jump $U^- - U^+$ is kept sufficiently small. In particular, Glimm [14] obtained the following remarkable existence theorem (see also Lax [16]) for general initial data with *small* total variation.

Theorem 1.2 ([14]). *Assume condition 3. If the total variation of the initial data is sufficiently small, then there exists a global in time solution to the initial-value* *problem 1-2. Moreover, if the system admits a convex entropy function, such a weak solution satisfies the entropy condition.*

About the proof of Theorem 1.2, the first task consists of estimating the Riemann solution at (mh, nk) by those at $\big((m-1)h, (n-1)k\big)$ and $\big((m+1)h, (k-1)k\big)$ in the approximate solutions. Since system 1 is invariant with respect to similarity transformations, we can enlarge both time and space scales simultaneously and turn a local phenomenon into an asymptotic one: two sets of waves interact and then a set of outgoing waves is generated. The task is to estimate the amplitudes of the outgoing waves by those of incoming waves; this gives rise to the so-called *local interaction estimates*.

For approximate solutions with small spatial total variation, the amplitude of the outgoing wave of the j-th characteristic direction is a linear superposition of those of the two incoming waves of the same direction *plus* a quadratic term, see Lemma 2.1. The second task is to show that the sum of the quadratic terms is uniformly bounded with respect to h and to the initial data; this gives rise to the so-called *global interaction estimates*.

Glimm existence Theorem 1.2 has a stochastic nature in the sense that the limit function satisfies the equations for *almost* every choice of the random sequence $\{\theta_n\}$. Indeed, Liu [18] proved that the limit function satisfies the equations for *every* choice of an equidistributed sequence. Later, DiPerna [13], Bressan [9] and Risebro [23] proposed the method of the *wave-front tracking*, an alternative to the random sampling method, and obtained the same general existence result. We note that solutions obtained via wave-front tracking are more appropriate for studying the uniqueness of admissible weak solution, see Bressan [10, 11].

The requirement that the initial data have small total variation can be removed in some cases, one of them being the *isothermal gasdynamics* equations:

$$v_t - u_x = 0, \quad u_t + \Big(\frac{a^2}{v}\Big)_x = 0. \tag{5}$$

Nishida [21], by using the Glimm difference scheme, showed that initial data with merely bounded total variation give rise to global solutions.

Theorem 1.3 ([21]). *If the total variation of the initial data (v_0, u_0) is finite, then the initial-value problem for system 5 has a global solution.*

In this case, the local interaction estimates are provided by the famous Nishida lemma, see Lemma 2.2. Surprisingly, these estimates require *no* quadratic term. Thus, the global interaction estimates are not necessary and Nishida showed instead that the total amount of negative variation (the total amount of shock waves) is uniformly bounded.

Though Nishida lemma is strong enough to yield global solutions, more information on the approximate solutions is required to study decay properties, the local structure of the solutions and so on. Asakura [4] proved that Nishida solutions actually decay by introducing the notion of *approximate shock fronts* and the partition of elementary waves with large amplitudes. There, the crucial point is that the amplitudes of the approximate shock fronts decay with the rate of a geometric series as the characteristic directions change after the interaction with other waves. However, as long as Glimm difference scheme is concerned, tracing back the strengths is hard. This is due to the fact that the local interaction potential contains not only the *present* amount of wave interactions but also the *future* amount of wave interactions.

The study of the local interactions looks much easier in the approximate solutions constructed by the wave-front tracking scheme than in those obtained by the Glimm scheme. The reason is that, in the former case, the local interaction potential contains only the present amount of wave interactions. This is why Asakura [5] first introduced the *path decomposition* in the framework of a wave-front tracking algorithm to study the equations of the *isentropic gas dynamics* (also including the isothermal case)

$$v_t - u_x = 0, \quad u_t + \Big(\frac{a^2}{v^\gamma}\Big)_x = 0, \qquad (\gamma \geq 1). \tag{6}$$

The decay property of Nishida solutions obtained in the case $\gamma = 1$ still holds for the solutions of system 6 and is interpreted as the *pathwise* Nishida lemma. This method has been used in subsequent papers, see Asakura-Corli [7, 8]. We note that also Temple and Young [25] introduced a (different) notion of *path*; a short account of their work is found in Section 2.

After this general introduction, the paper goes on as follows. In the next Section 2 we provide some more details about the constructions briefly outlined in this Introduction, to the aim of motivating and introducing the *path decomposition method*. Links to similar ideas are also given in a sort of short survey. Some recent results obtained by the authors [7, 8] are then reported in Sections 3 and 4. In particular, in Section 3 we show with full details this technique when applied to a simple system of three equations arising in phase transition modeling; moreover, a slightly stronger result than that given in [7] is proved.

2. **A survey of the path decomposition method.** In this section we trace back the main ideas that led to the path decomposition method. Most results quoted here are classical; we refer the reader to the book of Dafermos [12] for a comprehensive treatment of the subject.

We begin with providing some more details about the Glimm scheme. Let $\theta = \{\theta_n\}$ be a sequence of random numbers in $(0,1)$; we may assume that θ is an equidistributed sequence in $(0,1)$, see Liu [18]. For $m, n \in \mathbb{Z}$, $n \geq 0$, $m + n =$ even, the mesh points in the Glimm scheme are defined by

$$A_{m+1,n} = \big((m + 2\theta_n)h, nk\big),$$

for $n \geq 0$; see Figure 1. The half plane $\mathbb{R} \times \mathbb{R}_+$ is divided into a countable number of diamond shaped domains $\Delta_{m,n}$, which are defined by the vertices

$$A_{m,n+1},\ A_{m-1,n},\ A_{m,n-1},\ A_{m+1,n}.$$

The domain $\Delta_{m,n}$ is called the *interaction diamond* centered at (mh, nk). A curve consisting of segments joining $A_{m,n}$ to $A_{m+1,n+1}$ and $A_{m,n}$ to $A_{m+1,n-1}$ is called an I-curve. We can partially order the I-curves: we say that $I > J$ if every point of J is either on I or it is contained between I and $t = 0$.

Next, we define Glimm approximations $U = U_{h,\theta}$ as follows. For $n = 0$ we denote

$$U_{h,\theta}(A_{m,0}) = U_0\big((m + \theta_0)h\big).$$

Then, assume that $U_{h,\theta}$ is defined at $(x,t) = A_{m\pm1,n}$ and that the Riemann problem:

$$\begin{aligned} V_t + F(V)_x &= 0 \quad (m-1)h \leq x \leq (m+1)h,\ nk < t < (n+1)k, \\ V(x, nk) &= \begin{cases} V(A_{m-1,n}) & (m-1)h \leq x < mh, \\ V(A_{m+1,n}) & mh < x \leq (m+1)h, \end{cases} \end{aligned}$$

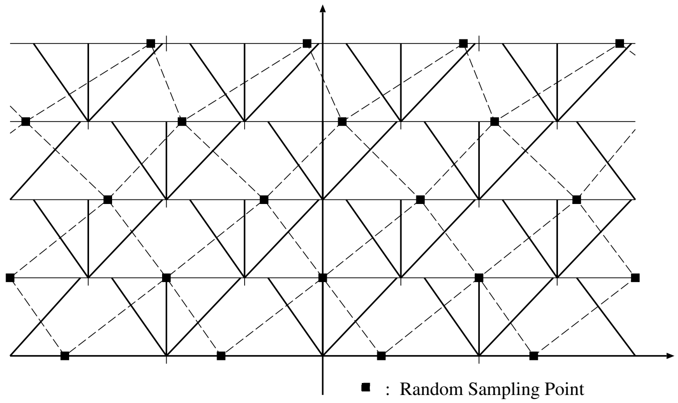

FIGURE 1. Glimm approximate solution.

is solved. Then we define $U_{h,\theta}(A_{m,n+1}) = V(A_{m,n+1})$. It is convenient to set

$$U_{h,\theta}(x,t) = V(x,t) \quad \text{for} \quad (m-1)h \le x < (m+1)h \text{ and } nk \le t < (n+1)k.$$

By using the above difference scheme, Glimm [14] proved Theorem 1.2.

Let us consider the approximate solution in an interaction diamond $D = \Delta_{m,n}$, see Figure 2. For $1 \le j \le n$ we denote by α_j the left incoming waves, by β_j the

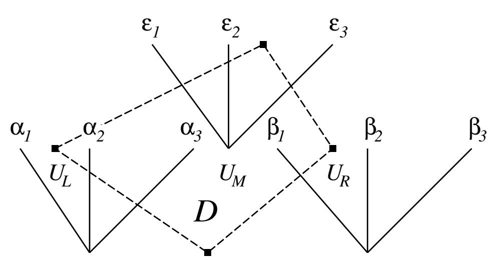

FIGURE 2. Wave interaction in the Glimm scheme.

right incoming waves and finally by ε_j the outgoing waves. Each wave is either a shock wave, a rarefaction wave or a contact discontinuity; all of them are called *elementary waves*. The waves α_j, β_k are said to be *approaching* if either $j > k$ or $j = k$ but then at least one of them is a shock wave. With a little abuse of notation, we also denote by α_j, β_j and ε_j the *strengths* of the waves. Then, we define the *local interaction potential*:

$$Q(D) = \sum_{\substack{\alpha_j,\beta_k \\ \text{approaching}}} |\alpha_j \beta_k|.$$

The following *local interaction estimates* are crucial in the Glimm scheme.

Lemma 2.1 (Local Interaction Estimates, [14]). *With reference to Figure 2 we have, for $i = 1, 2, \ldots, n$,*

$$\varepsilon_i = \alpha_i + \beta_i + O(1)Q(D).$$

We observe that the amplitude of an outgoing wave corresponding to the characteristic direction i is the sum of the amplitudes of the incoming waves of the same characteristic direction *plus* a quadratic term. Once that we have briefly recalled this background topic, we can focus on subsequent research.

In 1977, Liu [18] further developed the above issues and introduced the method of partition of elementary waves, in order that the strengths of the elementary waves at time $T > 0$ can be traced back to the initial data. Let $\varepsilon_i^{(m,n)}$ denote the strength of the i-th wave issuing from (mh, nk) and M a sufficiently large number. By partitioning $\varepsilon_i^{(m,n)}$ into smaller pieces, Liu constructed a one-to-one correspondence between waves at n, where$(p-1)M \leq n \leq pM$ and $p \in \mathbb{N}$, and those at $(p-1)M$, modulo the total amount of interactions and cancelations. This fundamental paper was the main source of ideas for Asakura [4].

Later on, in 1993, in order to study how the above quadratic terms Q were generated, Young [26] picked up a pure quadratic wave from the local interaction potential, modulo third order waves, and provided two nice tools for tracing back the speeds and strengths of the elementary waves: the *reordering* and the *interaction maps*. More precisely, let $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ denote a sequence of elementary waves such that γ_j and γ_{j+1} are separated by constant states. As long as the linear superposition of amplitudes in each interaction is concerned, the interaction is described by a *permutation* of the above sequence: this is called the *reordering*. When the quadratic part of the interaction has to be concerned, Young takes the k-th quadratic wave into the sequence of elementary waves by adding its amplitude to that of the nearest k-wave: this is the *interaction map*. Thus, in general, a reordering determines an interaction map.

By following the thread of some ideas of Young [26], Temple and Young [25] introduced in 1996 the notion of *path* for the system of compressible Euler equations. Let j_p denote a 1 or 3-wave and k_p a contact discontinuity which interacts with j_p. A sequence (j_p, k_p), for $p = 1, 2, \dots N$, such that j_{p+1} is the image of the interaction map induced by the (j_p, k_p) interaction, is called a path in [25]. Temple and Young also provided various calculi along the paths and in particular introduced the notion of *path integral*.

One of the main ideas leading to the path decomposition technique comes from the paper of Nishida [21]. In the system 5 of the isothermal gasdynamics, a pair of Riemann invariants is

$$w = u - \log p, \quad z = u + \log p.$$

Then, the amplitudes of the waves are defined by

$$\varepsilon_1 = w_R - w_L, \quad \varepsilon_2 = z_R - z_L.$$

In particular, according to this definition, $\varepsilon_j > 0$ if and only if ε_j is a rarefaction wave; then $\varepsilon_j < 0$ if and only if ε_j is a shock wave. For a real number ε we define its positive and negative part as $\varepsilon^+ = \max\{\varepsilon, 0\}$ and $\varepsilon^- = \max\{-\varepsilon, 0\}$, respectively. We have the following result.

Lemma 2.2 (Nishida Lemma, [21]). *Consider system 5. Then we have*

$$\sum_{j=1,2} \varepsilon_j^- \leq \sum_{j=1,2} (\alpha_j^- + \beta_j^-).$$

This lemma shows that the *sum* of the strengths of the shock waves at $t = nk$ is *decreasing*. However, for example, when a 1-shock and a 1-rarefaction wave interact then an outgoing 2-shock wave is generated, which is not quadratically estimated by the interacting waves. Hence, the strengths of the shock waves do not decrease in each direction.

In order to study the time decay of solutions to the system of isothermal gas dynamics, in 1993 Asakura [4] constructed a partition of elementary waves with large amplitudes. Then, a wave partitioned at time $T > 0$ can be traced back to the initial data. If a partitioned wave *does not change* its characteristic direction between 0 and T, then its amplitude never increases; if a partitioned wave *changes* characteristic direction, then the amplitude decreases with the rate of a geometric series as the characteristic direction changes after the interaction.

A short description of the *path decomposition method* is now provided; details of construction in the special case of a system of three equations will be given in Section 3. Consider a wave-front tracking approximate solution defined for $0 \le t < T$. Let $\mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_n$ be a sequence of interaction points in the plane xt, such that P_{m-1} and P_m are connected by a shock wave. According to [5, 7, 8], a *path* is a polygonal line joining the points $\mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_n$. A single shock front is composed of several segments of paths. The path decomposition consists in assigning a path's strength to all segments so that the strength of a single shock front is the summation of paths' strengths. *Primary paths* are generated at the initial line $t = 0$. At some interaction points we have quadratic waves which generate *secondary paths*. The strength of a single path never increases; if its characteristic direction changes, its strength decreases at a geometric rate.

3. **A model from phase transitions.**

We consider the following system of conservation laws arising in modeling phase transitions in fluids:

$$\begin{cases} v_t - u_x &= 0, \\ u_t + p(v,\lambda)_x &= 0, \\ \lambda_t &= 0, \end{cases} \tag{7}$$

for $t > 0$ and $x \in \mathbb{R}$. Here $v > 0$ stands for the specific volume, u the velocity and $\lambda \in [0, 1]$ the mass density fraction of vapor in the fluid. About the pressure p we assume that, for a C^1 function a defined on $[0, 1]$,

$$p(v,\lambda) = \frac{a^2(\lambda)}{v}, \qquad \text{with } a(\lambda) > 0,\ a'(\lambda) > 0. \tag{8}$$

Under assumption 8 on the pressure, system 7 is easily proved to be strictly hyperbolic for $(v, u, \lambda) \in \Omega = (0, +\infty)\times\mathbb{R}\times[0, 1]$ with characteristic speeds $e_1 = -a(\lambda)/v$, $e_2 = 0$, $e_3 = a(\lambda)/v$; the characteristic speed e_2 is linearly degenerate, the other ones are genuinely nonlinear.

We refer to [7] for more information on system 7. We just remark that if λ is constant then we recover system 5; on the other hand, system 7 can be read as the system of non-isentropic gas-dynamics in Lagrangian coordinates in case of smooth solutions, if we intend λ to represent the entropy; see Liu [19, 20]. Moreover, our main result, see Theorem 3.1, reminds of an analogous result due to Nishida and Smoller [22] for system 6.

The global existence of weak solutions to system 7 for initial data with large total variation was first proved by Amadori and Corli [1] by using a front-tracking scheme inspired by a paper of Amadori and Guerra [3]. The approach in [7] differs from that in [1] under several respects. In particular, Riemann coordinates as in [22] are exploited; as a consequence, the treatment of wave curves, Riemann problem and the study of wave interactions is mostly given by geometric considerations, following [22]. The final results are not easily comparable, since they refer to related but different quantities. However, as we stressed above, the main difference lies in the introduction of the path decomposition in the algorithm.

3.1. **The main result.** We denote by $\mathrm{TV}(f)$ the total variation of $f:\mathbb{R}\to\mathbb{R}$. If f is strictly positive we also use the *weighted total variation* of f,

$$\mathrm{WTV}(f) = 2\sup\sum_{j=1}^{n}\frac{|f(x_j)-f(x_{j-1})|}{f(x_j)+f(x_{j-1})}, \tag{9}$$

where the supremum is taken over all $n\geq 1$ and $(n+1)$-tuples of points x_j, $x_o < x_1 < \ldots < x_n$. The weighted total variation is close to the logarithmic variation, as the following estimate shows [1]:

$$\frac{\inf f}{\sup f}\mathrm{TV}\left(\log(f)\right) \leq \mathrm{WTV}(f) \leq \mathrm{TV}\left(\log(f)\right).$$

We denote the initial data for system 7 at $t=0$ by

$$\left(\bar v(x),\bar u(x),\bar\lambda(x)\right) \qquad \text{with} \qquad \bar v(x)\geq v_0>0 \ \text{ and } \ 0\leq\bar\lambda(x)\leq 1. \tag{10}$$

Moreover, we write $\bar a(x) = a(\bar\lambda(x))$ and

$$\bar a_* = \inf_{x\in\mathbb{R}}\bar a(x)\,, \quad \bar a^* = \sup_{x\in\mathbb{R}}\bar a(x), \quad [\bar a]_* = \frac{\bar a^*-\bar a_*}{\bar a^*+\bar a_*}.$$

Theorem 3.1 ([7]). *Consider system 7 under the assumption 8 and initial data 10. There exists a positive constant $c\in(0,1)$ such that if*

$$\frac{1}{2}\frac{\mathrm{WTV}(\bar a)}{1-[\bar a]_*}\,\mathrm{e}^{\frac{1}{2}\frac{\mathrm{WTV}(\bar a)}{1-[\bar a]_*}}\cdot\frac{2-c}{1-c} < \frac{1}{2} \tag{11}$$

and

$$\mathrm{WTV}(\bar a)\cdot\mathrm{TV}(\bar v,\bar u) \quad \textit{is sufficiently small,} \tag{12}$$

then the Cauchy problem 7-10 has a weak entropic solution defined for $t>0$.

We observe that condition 11 requires that $\mathrm{WTV}(\bar a)<1/2$, as in [1]. Throughout the paper [7], Riemann coordinates r and s for the first two equations in 7 with respect to $\bar a_*$ are exploited; they are $r = u-\bar a_*\log p$ and $s = u+\bar a_*\log p$. The strength of an i-wave joining (p_o,u_o,λ_o) to (p,u,λ) is defined by

$$\varepsilon_1 = r-r_o\,, \qquad \varepsilon_2 = 2\,\frac{a-a_o}{a+a_o}\,, \qquad \varepsilon_3 = s-s_o\,. \tag{13}$$

The total variation in 12 is computed by using the above Riemann coordinates and the max metric; more precisely,

$$|(v',u')-(v,u)| = \max\left\{|r(v',u')-r(v,u)|,\,|s(v',u')-s(v,u)|\right\}. \tag{14}$$

The proof of Theorem 3.1 is based on a wave-front tracking algorithm. First, the initial data $(\bar v,\bar u,\bar\lambda)$ are approximated by a sequence of piecewise constant functions $(\bar v^\nu,\bar u^\nu,\bar\lambda^\nu)$, $\nu\in\mathbb{N}$, with a finite number of jumps. Second, an accurate and a simplified Riemann solver are introduced as in the book by Bressan [11]. About the functionals, we denote by $S_i(t)$ ($R_i(t)$) the i-shock (resp., rarefaction) waves at time t and by CD the contact discontinuities in the approximate solution; by $\mathcal{A}(t)$ we denote the set of approaching waves. Then,

$$L(t) = \sum_{\substack{\delta_i \in S_i(t)\\ i=1,3}} |\delta_i|, \quad L^+(t) = \sum_{\substack{\delta_i \in R_i(t)\\ i=1,3}} |\delta_i|, \quad L_{\text{tot}}(t) = L(t) + L^+(t),$$

$$Q(t) = \sum_{\substack{\delta_i, \delta_i' \in S_i(t)\\ i=1,3}} |\delta_i \delta_i'|, \quad Q_2(t) = \sum_{\substack{(\delta_2, \delta_i) \in \mathcal{A}(t)\\ i=1,3}} |\delta_2 \delta_i|,$$

and finally $F(t) = L(t) + KQ(t) + K_2 Q_2(t)$. We also denote $L_2 = \sum_{\delta_2 \in CD} |\delta_2|$.

Now, we may comment on the quantities introduced in Theorem 3.1. The term c depends on the slopes of the shock curves in the region where the solution takes its values and is related to the damping of waves emitted in an interaction. More precisely, it depends on $\bar{M}$, $\bar{a}_*$ and $\bar{a}^*$, where $\bar{M}$ is a uniform bound to the strengths of the shocks in the approximated solutions. In [7] we showed that we can take

$$\bar{M} = 2|(\bar{v}_{+\infty}, \bar{u}_{+\infty}) - (\bar{v}_{-\infty}, \bar{u}_{-\infty})| + \frac{24}{1 - [\bar{a}]_*} \mathrm{TV}\,(\bar{v}, \bar{u}).$$

For $y < 0$ we denote by $g(y, a) = 2\bar{a}_* y + 2a \sinh y$ a (negative) function arising in the parametrization of shock curves. For any fixed a, the function g is invertible with respect to y and we denote $f(x, a) = 2a \sinh\left(g^{-1}(x, a)\right) - 2\bar{a}_* g^{-1}(x, a)$, which is again negative. Notice that we have

$$f_x(x, a) = \frac{a \cosh(y) - \bar{a}_*}{a \cosh(y) + \bar{a}_*}, \qquad \text{for } y = g^{-1}(x).$$

The function f_x is positive when $a \geq \bar{a}_*$. Then, the term c can be estimated by

$$\begin{aligned} &c(\bar{M}, \bar{a}_*, \bar{a}^*) \\ &\leq \max\left\{ \max_{\substack{\bar{a}_* \leq a \leq \bar{a}^*\\ -\bar{M} \leq \delta \leq 0}} f_x(\delta, a),\ \max_{\substack{\bar{a}_* \leq a \leq \bar{a}^*\\ -\bar{M} \leq \delta, \delta' \leq 0}} f_x\left(\delta + \delta' - \frac{a - \bar{a}_*}{2(a + \bar{a}_*)^2} \delta\delta', a\right) \right\}. \end{aligned} \tag{15}$$

Indeed, we can simplify a bit the previous estimate.

Proposition 1. *We have*

$$c(\bar{M}, \bar{a}_*, \bar{a}^*) \leq \max_{\bar{a}_* \leq a \leq \bar{a}^*} f_x\left(-2\bar{M} - \frac{a - \bar{a}_*}{2(a + \bar{a}_*)^2} \bar{M}^2, a\right). \tag{16}$$

Moreover, if $\bar{a}^* < 3\bar{a}_*$ we have

$$c \leq f_x\left(-2\bar{M} - \frac{\bar{a}^* - \bar{a}_*}{2(\bar{a}^* + \bar{a}_*)^2} \bar{M}^2, \bar{a}^*\right). \tag{17}$$

Proof. For simplicity we drop all $\bar{\cdot}$'s in M, a_* and a^*. We see that for δ and δ' negative we have

$$\delta + \delta' - \frac{a - a_*}{2(a + a_*)^2} \delta\delta' \leq \delta.$$

We proved in [7, Lemma 3.1] that $x \to f(x, a)$ is concave for every $a > 0$; then $x \to f_x(x, a)$ is decreasing for every $a > 0$. Therefore we have

$$f_x\left(\delta + \delta' - \frac{a - a_*}{2(a + a_*)^2} \delta\delta', a\right) \geq f_x(\delta, a).$$

As a consequence we deduce that

$$c(M, a_*, a^*) \leq \max_{\substack{a_* \leq a \leq a^* \\ -M \leq \delta, \delta' \leq 0}} f_x \left(\delta + \delta' - \frac{a - a_*}{2(a + a_*)^2} \delta\delta', a \right).$$

Moreover, for any $a \in [\bar{a}_*, \bar{a}^*]$ the negative function $[-\bar{M}, 0] \times [-\bar{M}, 0] \ni (\delta, \delta') \to \delta + \delta' - \frac{a-a_*}{2(a+a_*)^2}\delta\delta'$ reaches it minimum when $\delta = \delta' = -M$ and then we get 16.

To prove 17 we define

$$\Phi(a) = f_x \left(-2M - \frac{a - a_*}{2(a + a_*)^2} M^2, a \right).$$

We need to maximize Φ. We have

$$\Phi'(a) = -\frac{M^2}{2} \frac{3a_* - a}{2(a + a_*)^3} \cdot f_{xx} \left(-2M - \frac{a - a_*}{2(a + a_*)^2} M^2, a \right) + f_{xa} \left(-2M - \frac{a - a_*}{2(a + a_*)^2} M^2, a \right).$$

By differentiating the expression $g\left(g^{-1}(x, a), a\right) = x$ with respect to a we deduce

$$\partial_a g^{-1}(x, a) = -\frac{\sinh y}{a \cosh y + a_*}.$$

Moreover, we have

$$f_{xa}(x, a) = \frac{2a_*(a_* \cosh y + a)}{(a \cosh y + a_*)^3}, \quad y = g^{-1}(x, a).$$

Then

$$\Phi'(a) = \frac{a_*}{(a \cosh y + a_*)^3} \left[-\frac{M^2}{2} \frac{3a_* - a}{(a + a_*)^3} a \sinh y_* + 2(a_* \cosh y_* + a) \right],$$

where

$$y_* = -2M - \frac{a - a_*}{2(a + a_*)^2} M^2.$$

Since $f_{xx} < 0$, $f_{xa} > 0$, we deduce that if $a < 3a_*$ then $\Phi'(a) > 0$, whence 17. $\square$

We emphasize that c depends on $\mathrm{TV}(\bar{v}, \bar{u})$ through $\bar{M}$. Condition 11 concerns the size of 2-waves. If $\bar{\lambda}$ is constant then c only depends on $\bar{M}$ and 11-12 are trivially satisfied; in this way we recover the result of Nishida [21].

A precise threshold for the smallness of the quantity $\mathrm{WTV}(\bar{a}) \cdot \mathrm{TV}(\bar{v}, \bar{u})$ in 12 is that both inequalities below are satisfied (the related formulas in [7, Remark 11.9], namely (11.40) and (11.42), have a couple of typos):

$$\frac{2 - c}{2\bar{a}_*} e^{\frac{1}{2} \frac{L_2}{1 - [\bar{a}]_*}} \left\{ 1 + \frac{1 - c}{8c} \left(\frac{2 - c}{2\bar{a}_*} L_{\mathrm{tot}}(0) + \frac{1}{1 - [\bar{a}]_*} \right) \right\} L_2 L_{\mathrm{tot}}(0) < \frac{1}{2} \frac{(1 - c)^2}{2 - c} - \frac{1}{2} \frac{\mathrm{WTV}(\bar{a})}{1 - [\bar{a}]_*} e^{\frac{1}{2} \frac{\mathrm{WTV}(\bar{a})}{1 - [\bar{a}]_*}}, \tag{18}$$

$$L_2 L_{\mathrm{tot}}(0) \leq 2\bar{a}_* \frac{2(1 - [\bar{a}]_*)(1 - c) - (2 - c)L_2}{(2 - c)^2(1 - [\bar{a}]_*)}. \tag{19}$$

They are both satisfied if $L_2 L_{\mathrm{tot}}(0)$ is small. These conditions are expressed in terms of the *approximate initial data*; now, we write these conditions in terms of the *real initial data* and show that they can be reduced to a single condition. To this aim, we introduce the following notation

$$\begin{aligned} V &= \mathrm{TV}(\bar v, \bar u), & W &= \mathrm{WTV}(\bar a), \\ \mathcal{V} &= \frac{2(2-c)V}{\bar a_*}, & \mathcal{W} &= \frac{1}{2}\frac{W}{1-[\bar a]_*}. \end{aligned} \tag{20}$$

Proposition 2. *Conditions 11 and 12 in Theorem 3.1 are implied by the single condition*

$$2\mathcal{V}\mathcal{W}\left\{1+\frac{1-c}{8c(1-[\bar a]_*)}(1+\mathcal{V})\right\}\mathrm{e}^{\mathcal{W}}+\mathcal{W}\mathrm{e}^{\mathcal{W}}<\frac{1}{2}\frac{(1-c)^2}{2-c}. \tag{21}$$

Proof. First, notice that by 13_2, 9 and [7, (6.1)] it follows that

$$L_2 \le \mathrm{WTV}(\bar a).$$

Second, consider a Riemann problem with initial data (v_L, u_L, λ_L), (v_R, u_R, λ_R) and solved by waves ε_i, $i = 1, 2, 3$; by [7, (4.4)] it follows that

$$|\varepsilon_1| + |\varepsilon_3| \le \frac{2}{1-\max\{k_L, k_R\}}\left(|r_R - r_L| + |s_R - s_L|\right).$$

Here, $k_L = \frac{a_L - \bar a_*}{a_L + \bar a_*}$, $k_R = \frac{a_R - \bar a_*}{a_R + \bar a_*}$. Since $k_L, k_R \le [\bar a]_*$, it follows that $|\varepsilon_1| + |\varepsilon_3| \le \frac{2}{1-[\bar a]_*}\left(|r_R - r_L| + |s_R - s_L|\right)$. By the inequality $a + b \le 2\max\{a, b\}$, which is valid for $a \ge 0$, $b \ge 0$, we deduce that

$$L_{\mathrm{tot}}(0) \le \frac{4}{1-[\bar a^\nu]_*}\mathrm{TV}(\bar v^\nu, \bar u^\nu) \le \frac{4}{1-[\bar a]_*}\mathrm{TV}(\bar v, \bar u),$$

by the choice of the approximating sequence, see [7, §6]. Using notation 20 we see that conditions 18 and 19 are satisfied if

$$\begin{aligned} \frac{2(2-c)VW}{\bar a_*(1-[\bar a]_*)}\mathrm{e}^{\frac{1}{2}\frac{W}{1-[\bar a]_*}}\left\{1+\frac{1-c}{8c(1-[\bar a]_*)}\left(1+\frac{2(2-c)}{\bar a_*}V\right)\right\} \\ < \frac{1}{2}\frac{(1-c)^2}{2-c} - \frac{1}{2}\frac{W}{1-[\bar a]_*}\mathrm{e}^{\frac{1}{2}\frac{W}{1-[\bar a]_*}}, \\ \frac{2}{1-[\bar a]_*}VW \le \bar a_*\frac{2(1-[\bar a]_*)(1-c)-(2-c)W}{(2-c)^2(1-[\bar a]_*)}. \end{aligned}$$

By using the quantities $\mathcal{V}$ and $\mathcal{W}$ introduced above we can simplify the above conditions:

$$\begin{aligned} 2\mathcal{V}\mathcal{W}\left\{1+\frac{1-c}{8c(1-[\bar a]_*)}(1+\mathcal{V})\right\}\mathrm{e}^{\mathcal{W}}+\mathcal{W}\mathrm{e}^{\mathcal{W}}<\frac{1}{2}\frac{(1-c)^2}{2-c}, \\ \mathcal{V}\mathcal{W}+\mathcal{W} \le \frac{1-c}{2-c}. \end{aligned}$$

The second condition is clearly implied by the first one. Then, we see that 18-19 reduce to a *single* condition, namely 21. We observe that condition 21 also contains condition 11 as a necessary condition; then, 21 is the only condition required in order that Theorem 3.1 holds. □

We remark that, even if the product $\mathcal{V}\mathcal{W}$ is small, condition 21 imposes a bound on $\mathrm{WTV}(\bar a)$. This is not surprising: an analogous bound for the same quantity was required in [1, (2.8)], see also the slightly different statement in [2, Theorem 3.1].

Moreover, notice that while the conditions in [1, 2] only related the *(weighted) total variations*, condition 21 also involves the L^∞ bounds of the initial data of the solution by means of the quantities c and $[\bar{a}]_*$. This means that some compatibility conditions between the (weighted) total variation and the oscillation of the initial data must be satisfied; these compatibility conditions are expressed as bounds from below of the variation. For instance, by [7, (6.4)] we now that $\mathrm{WTV}(\bar{a}) \geq 2[\bar{a}]_*$; on the other hand, since $|(\bar{v}_{+\infty}, \bar{u}_{+\infty}) - (\bar{v}_{-\infty}, \bar{u}_{-\infty})| \leq \mathrm{TV}(\bar{v}, \bar{u})$, we deduce that

$$\bar{M} \leq 2\left(1 + \frac{12}{1-[\bar{a}]_*}\right)\mathrm{TV}\left(\bar{v}, \bar{u}\right),$$

which gives a bound to $\mathrm{TV}\left(\bar{v}, \bar{u}\right)$.

3.2. **The path decomposition technique.** First, we need some results on wave interactions. We first deal with interactions of the 2-wave. Below, we write R, S and C for rarefactions, shocks and contact discontinuities, respectively; an index denotes the family.

Proposition 3. *Assume that a 1- or a 3-wave δ_i interacts with a 2-wave δ_2 of side states λ_L, λ_R. Then, the outgoing waves are as follows:*

(i) C_2R_1, $a_L < a_R$ *and* R_3C_2, $a_L > a_R$ $\;\to\;$ $R_1C_2R_3$*;*
(ii) C_2R_1, $a_L > a_R$ $\;\to\;$ $R_1C_2S_3$ *and* R_3C_2, $a_L < a_R$ $\;\to\;$ $S_1C_2R_3$*;*
(iii) C_2S_1, $a_L < a_R$ *and* S_3C_2, $a_L > a_R$ $\;\to\;$ $S_1C_2S_3$*;*
(iv) C_2S_1, $a_L > a_R$ $\;\to\;$ $S_1C_2R_3$ *and* S_3C_2, $a_L < a_R$ $\;\to\;$ $R_1C_2S_3$.

The strengths ε_i of the transmitted wave, ε_j of the reflected wave and ε_2 satisfy

$$|\varepsilon_i - \delta_i| \leq |\varepsilon_j|, \qquad \varepsilon_2 = \delta_2.$$

At last, the following quadratic interaction estimates hold for $i, j = 1, 3$, $i \neq j$:

$$|\varepsilon_j| \leq \tfrac{1}{2}\,|\delta_2\delta_i|, \qquad |\varepsilon_1| + |\varepsilon_3| \leq |\delta_i| + |\delta_2\delta_i|; \qquad \textit{in all cases but (ii),}$$

$$|\varepsilon_j| \leq \tfrac{1}{2}\,\tfrac{1}{1-[a]_*}\,|\delta_2\delta_i|, \quad |\varepsilon_1| + |\varepsilon_3| \leq |\delta_i| + \tfrac{1}{1-[a]_*}\,|\delta_2\delta_i|, \qquad \textit{in case (ii).}$$

Now, we focus on interactions of waves of families 1 and 3; we denote by λ_o the constant value of λ in the interaction and

$$k_o \doteq \frac{a_o - a_*}{a_o + a_*}, \qquad l_o \doteq \frac{1}{2(a_o + a_*)}.$$

Proposition 4. *Assume that an i-wave W_i and a j-wave W_j, $i, j \in \{1, 3\}$ interact at time $t > 0$. Let δ_i, δ_j (or δ_i, δ_i' if they are of the same type) be their strengths, ε_k be the strengths of the outgoing waves, $k = 1, 3$, U_L, U_R the outer states in the interaction.*

Then, there exists a constant $0 \leq c_{LR} < 1$, depending on U_L and U_R and uniformly on λ_o, such that the following estimates hold.

(1) $W_3W_1 \to W_1W_3$*:* $\varepsilon_i = \delta_i$ *for* $i = 1, 3$.
(2) $S_iR_i' \to S_iS_j$*:* $|\varepsilon_j| \leq c_{LR}\zeta_i$ *for* $\zeta_i \doteq |\delta_i| - |\varepsilon_i| > 0$.
(3) $S_iR_i' \to R_iS_j$*:* $|\varepsilon_j| \leq c_{LR}|\delta_i|$.
(4) $S_iS_i' \to S_iR_j$*:* $|\varepsilon_j| \leq \min\{c_{LR}|\delta_i|, c_{LR}|\delta_i'|, l_o|\delta_i\delta_i'|\}$, $|\varepsilon_i| \leq |\delta_i| + |\delta_i'| + k_ol_o|\delta_i\delta_i'|$.

The constants c_{LR} are uniformly estimated if there exists $M > 0$ such that

$$\textit{the size of any interacting shock wave is less than } M. \tag{22}$$

Then there exists $c = c(M) \in (0, 1)$ bounding from above all c_{LR}'s and such that

$$|\varepsilon_j| \leq \begin{cases} c \cdot \zeta_i & \text{in case } \textit{(2)}, \\ c \cdot |\delta_i| & \text{in case } \textit{(3)}, \\ \min\{c|\delta_i|, c|\delta_i'|, l_o|\delta_i\delta_i'|\} & \text{in case } \textit{(4)}. \end{cases} \tag{23}$$

We resume below what we stated on *shocks* produced by physical interactions.

Proposition 5. *Consider an interaction between physical waves, under the notation of Propositions 3,4 and assume 22. Then, there exist $h' \geq 0$ and $h'' \geq 0$, possibly changing from line to line, such that:*

$$
\begin{array}{llll}
2\text{-}int\text{'}s: & (ii) & & |\varepsilon_j| = h''|\delta_2\delta_i|, \quad h'' \leq \dfrac{1}{2(1-[a]_*)}, \\
& (iii) & |\varepsilon_i| \leq |\delta_i| & |\varepsilon_j| = h''|\delta_2\delta_i|, \quad h'' \leq 1/2, \\
& (iv) & |\varepsilon_i| = |\delta_i| + h''|\delta_2\delta_i| & \qquad\qquad\qquad\quad h'' \leq 1/2, \\
13\text{-}int\text{'}s: & (2) & |\varepsilon_i| = |\delta_i| - \zeta_i & |\varepsilon_j| = h'\zeta_i, \qquad\;\; h' \leq c, \\
& (3) & & |\varepsilon_j| = h'|\delta_i|, \qquad h' \leq c, \\
& (4) & |\varepsilon_i| = |\delta_i| + |\delta_i'| + h''|\delta_i\delta_i'| & \qquad\qquad\qquad\quad h'' \leq k_o l_o\,.
\end{array}
$$

We finally introduce in detail the technique of *decomposition by paths.* Consider an approximate solution defined for $0 \leq t < T$ and a sequence P_0, P_1, …, P_n of interaction points in the plane xt, with $\mathrm{P}_m = (x_m, t_m)$ and $0 \leq t_0 < t_1 < \ldots < t_n$, such that P_{m-1} and P_m are connected by a shock wave, for $m = 1, \ldots, n$. A *path* $\Gamma: \ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$ is a polygonal line joining the points P_0, P_1, …, P_n. At any interaction of waves of families 1 or 3, the strength of an outgoing shock is decomposed as in Proposition 5 into a linear and a quadratic part, which are the *strengths* of the segment defined by that shock; a 2-wave is never decomposed. A path is *primary* and denoted Γ^P if it starts at $t = 0$ and its segments have strengths obtained either from linear parts of decompositions or from shocks transmitted through an interaction with a 2-wave. *Secondary* paths are denoted by Γ^S and are associated to the quadratic parts of the decomposition. Paths are constructed according to the inductive process below and a *generation order* (order, for brevity) is assigned.

• $t = 0$. Any shock generated at $t = 0$ gives raise to a primary path, which is composed by a single segment. If δ_i is a shock from $\mathrm{P}_0 \in \{t = 0\}$ and interacting at $\mathrm{P}_1 = (x_1, t_1)$, then we have a primary path $\Gamma^P: \ \mathrm{P}_0 \to \mathrm{P}_1$ of order 1 and strength $|\delta_i|$ composed by the segment $\mathrm{P}_0\mathrm{P}_1$. We also assign order 1 to all rarefaction waves generated at $t = 0$.

• $t = t_1$. Let P_1 be the first interaction point and assume that the accurate Riemann solver is used in solving the interaction; we refer to Figure 3.

If no shock is involved in the interaction, then three cases are possible: in cases *(i)* and *(1)*, two interacting rarefactions, no path is generated; in case *(ii)* we define a secondary path $\Gamma^S: \ \mathrm{P}_1 \to \mathrm{P}_2'$ of order 2 and strength $h''|\delta_2| \cdot |\delta_i|$.

Assume that a shock δ_i from $\mathrm{P}_0 \in \{t = 0\}$ interacts at P_1 with a wave δ. Let ε_i, ε_j be the emitted waves and P_2, P_2' the next interaction points for the waves ε_i, ε_j, respectively; we have the following cases.

(iii) We decompose Γ^P into Γ_1^P and Γ_2^P; Γ_1^P has order 1 and strength $|\varepsilon_i|$, Γ_2^P has order 1 and strength $|\delta_i| - |\varepsilon_i|$. The path Γ_1^P is extended to P_2, Γ_2^P stops at P_1. A secondary path $\Gamma'^S: \ \mathrm{P}_1 \to \mathrm{P}_2'$ is generated, of order 2 and strength $h''|\delta_2| \cdot |\delta_i|$.

(iv) The path Γ^P is extended to P_2; order and strength of $\mathrm{P}_0\mathrm{P}_1$ are unchanged, the order of $\mathrm{P}_1\mathrm{P}_2$ is 1, the strength $|\delta_i|$. A secondary path $\Gamma'^S: \ \mathrm{P}_1 \to \mathrm{P}_2$ is generated with order 2 and strength $h''|\delta_2| \cdot |\delta_i|$.

(1) If δ_i is a shock, then Γ^P is extended to P_2, with order and strength unchanged.

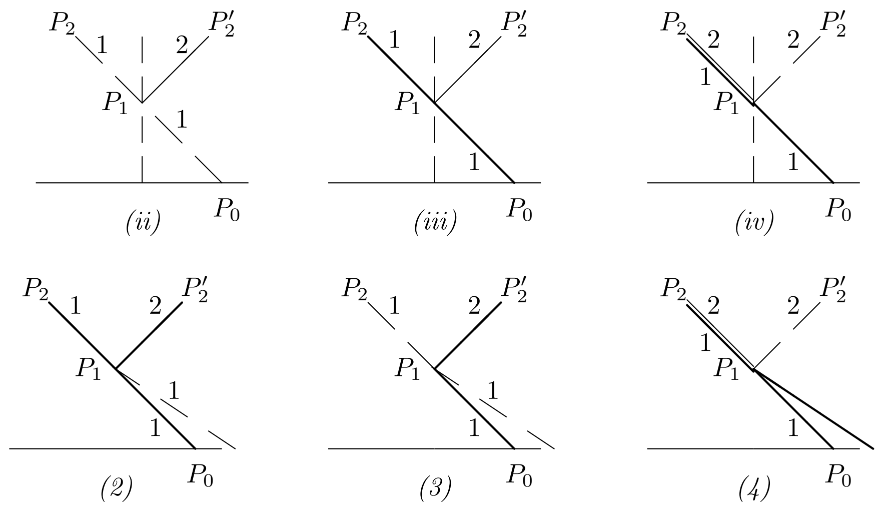

FIGURE 3. Interactions of a 1-wave with a 2-wave (above) and with another 1-wave (below). Primary paths are depicted with thick lines, secondary paths with thin lines, rarefactions and contact discontinuities with broken lines. Numbers refer to generation orders.

(2) Two paths Γ_1^P, Γ_2^P from P_0 are defined. Γ_1^P has order 1 and strength $|\varepsilon_i| = |\delta_i| - \zeta_i$ in the segment $\mathrm{P}_0\mathrm{P}_1$; it is extended to P_2 where the order of $\mathrm{P}_1\mathrm{P}_2$ is 1, the strength $|\varepsilon_i|$. Γ_2^P has order 1 and strength ζ_i in the segment $\mathrm{P}_0\mathrm{P}_1$; it is extended to P_2' where the order of $\mathrm{P}_1\mathrm{P}_2'$ is 2, the strength $|\varepsilon_j| = h'\zeta_i$.

(3) The path Γ^P is extended to P_2'; order and strength of $\mathrm{P}_0\mathrm{P}_1$ are unchanged, the order of $\mathrm{P}_1\mathrm{P}_2'$ is 2, the strength $|\varepsilon_j| = h'|\delta_i|$.

(4) The path Γ^P is extended to P_2. Generation order and strength of $\mathrm{P}_0\mathrm{P}_1$ are unchanged; the order of $\mathrm{P}_1\mathrm{P}_2$ is 1, the strength $|\delta_i|$. A secondary path Γ^S : $\mathrm{P}_1 \to \mathrm{P}_2$ is generated with order 2 and strength $h''|\delta_i\delta_i'|$.

If the wave δ_i interacts with a physical wave δ but $|\delta_i\delta| < \rho$, then the simplified Riemann solver is used. Cases *(i)*, *(ii)* and *(1)*, two interacting rarefactions, do not generate paths. In cases *(iii)* and *(iv)* a path is extended with the same order; the same happens in case *(1)* if a shock interacts. When a shock δ_i interacts with rarefaction δ_i' of the same family, then the path is extended with strength $|\delta_i| - \delta_i'$ if $|\delta_i| - \delta_i' > 0$, otherwise it stops. If a shock δ_i meets another shock δ_i' of the same family then the path is extended with the same strength.

• *Definition of orders.* Let $\mathbf{\Gamma} = \{\Gamma_\ell\}$ be the collection of paths in the approximate solution up to the interaction time t_n. The segment associated to a shock γ belongs to N paths which can be ordered by putting first the p primary paths for increasing order, then the $s = N - p$ secondary paths, again for increasing order; by dropping the indexes P and S and denoting k_m the order of a segment, we write

$$\Gamma_1, \ \ldots \ \Gamma_p, \ \Gamma_{p+1}, \ \ldots \ \Gamma_N, \qquad \text{for} \qquad k_1 \le \ldots \le k_p, \ k_{p+1} \le \ldots \le k_N. \tag{24}$$

The order of the shock wave γ is defined by $k_\gamma = \min_{1 \le l \le N}\{k_l\}$.

We also define the generation order of rarefactions. A rarefaction of size σ contains $N = [\sigma/\eta] + 1$ fronts and each has strength $\sigma/N < \eta$. If a rarefaction wave of order k interacts with a 2-wave, then the possibly transmitted (reflected) rarefaction has order k ($k+1$, respectively). By Proposition 3 it follows

$$|\varepsilon_i - \delta_i| \leq |\varepsilon_j| \leq \frac{1}{2}\frac{1}{1-[a]_*}|\delta_2||\delta_i| \leq |\delta_i|. \tag{25}$$

Then, the outgoing i-rarefaction wave is decomposed into at most two rarefaction fronts of order k so that $\varepsilon_i = \varepsilon_i^{(1)} + \varepsilon_i^{(2)}$, where

$$\varepsilon_i = \begin{cases} \varepsilon_i^{(1)} & \text{if } \delta_i < \varepsilon_i \\ \delta_i + \varepsilon_i^{(2)} & \text{if } \varepsilon_i \leq \delta_i \leq 2\varepsilon_i\,. \end{cases} \tag{26}$$

When an i-rarefaction front interacts with a j-wave, with $i,j = 1,3$, $i \neq j$, strengths and orders do not change. In case *(3)* the outgoing rarefaction keeps the same order of the interacting one, in case *(4)* it is assigned order $\max\{k,k'\}+1$, where k and k' are the orders of the colliding shocks.

The generation order of nonphysical waves is defined as follows. If a wave of order k interacts with a 2-wave, then the order of the outgoing nonphysical wave is $k+1$. If two waves of the same family 1 or 3 interact and a nonphysical wave is generated, then it is assigned order $\max\{k,k'\}+1$, where k, k' are the orders of interacting shocks.

• $t = t_n$. Consider a path $\Gamma:\ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$ to $t = t_n$ and δ_i a shock connecting P_{n-1} to P_n. Let the segment $\mathrm{P}_{n-1}\mathrm{P}_n$ be contained in paths Γ_l of strengths $|\delta_l|$, $1 \leq l \leq N$, so that, as in (24),

$$|\delta_i| = \sum_{l=1}^{p} |\delta_l| + \sum_{l=p+1}^{N} |\delta_l|, \tag{27}$$

where we omitted the dependence on i in the strengths $|\delta_l|$. Assume that δ_i interacts at P_n with a physical wave δ and $|\delta_i\delta| \geq \rho$. Let ε_i, ε_j be the emitted waves ($i,j \in \{1,3\}$) and P_{n+1}, P'_{n+1} the next interaction points for ε_i, ε_j, respectively. Only the following cases deserve consideration.

(iii) There exist $1 \leq r \leq N$ and $\bar\delta_r$ with $0 \leq |\bar\delta_r| < |\delta_r|$ such that $|\delta_i| - |\varepsilon_i| = |\bar\delta_r| + \sum_{l=r+1}^{N}|\delta_l|$. We split Γ_r into $\Gamma_r^{(1)}$ and $\Gamma_r^{(2)}$ so that the orders of the paths are not changed while the absolute values $\alpha_r^{(1)}$, $\alpha_r^{(2)}$ of their strengths are decomposed according to $\alpha_r^{(2)} : \alpha_r^{(1)} = |\bar\delta_r| : \big(|\delta_r| - |\bar\delta_r|\big)$. We extend every Γ_l for $1 \leq l \leq r-1$ and $\Gamma_r^{(1)}$ up to P_{n+1} with orders and strengths unchanged. The paths $\Gamma_r^{(2)}$ and Γ_l for $r+1 \leq l \leq N$ stop. A secondary path $\Gamma'^S:\ \mathrm{P}_n \to \mathrm{P}'_{n+1}$ is generated with order $k_{\delta_i}+1$ and strength $h''|\delta_2|\cdot|\delta_i|$.

(iv) All paths Γ_l, $1 \leq l \leq N$, are extended to P_{n+1} with equal orders and strengths. A secondary path $\Gamma''^S:\ \mathrm{P}_n \to \mathrm{P}_{n+1}$ is generated with order $k_{\delta_i}+1$ and strength $h''|\delta_2|\cdot|\delta_i|$.

(2) This case is analogous to case *(iii)*.

(3) The paths Γ_l, $1 \leq l \leq N$, are extended to P'_{n+1} leaving unchanged the orders and strengths of their segments until P_n; the order of $\mathrm{P}_n\mathrm{P}'_{n+1}$ is $k_{\delta_i}+1$, the strength $|\varepsilon_j| = h'|\delta_i|$.

(4) Let the shock $\delta'_i : \mathrm{P}'_{n-1}\mathrm{P}_n$ be contained in the paths Γ'_l of strengths $|\delta'_l|$, $1 \leq l \leq N'$. All paths Γ_l, $1 \leq l \leq N$, and Γ'_l, $1 \leq l \leq N'$, are extended to P_{n+1} with orders and strengths unchanged. A secondary path $\Gamma''^S:\ \mathrm{P}_n \to \mathrm{P}_{n+1}$ is generated with order $\max\{k_{\delta_i}, k_{\delta'_i}\}+1$ and strength $h''|\delta_i\delta'_i|$.

The case when δ_i interacts with a physical wave δ and $|\delta_i\delta| < \rho$ is dealt as in the previous step. At last, in the interaction of a physical wave with a nonphysical wave a path is extended with the same order.

This concludes the definitions of paths. In this way, a collection of primary paths $\mathbf{\Gamma}^P = \{\Gamma_\ell^P\}$ and secondary paths $\mathbf{\Gamma}^S = \{\Gamma_\ell^S\}$ is defined up to the next interaction time t_{n+1} and hence as long as the approximate solution exists; we denote $\mathbf{\Gamma} = \mathbf{\Gamma}^P \cup \mathbf{\Gamma}^S$.

Without entering in the details of the proof of Theorem 3.1, we now state some immediate consequences of the above construction. The first result shows how the generation order is related to the decreasing of the strength along a path.

Lemma 3.2. *Consider any approximate solution valued in a bounded domain and any path* $\Gamma :\ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$*; assume 22. Let* k_m *be the order and* α_m *the strength of the segment* $\mathrm{P}_{m-1}\mathrm{P}_m$ *in* Γ*. Then*

$$\begin{array}{lcl} k_{m+1} = k_m & \Rightarrow & \alpha_{m+1} = \alpha_m \\ k_{m+1} = k_m + 1 & \Rightarrow & \alpha_{m+1} \leq c \cdot \alpha_m \,. \end{array} \tag{28}$$

By the above lemma we deduce the following important result on the decreasing of the strength along a path.

Lemma 3.3. *Assume 22. For any approximate solution valued in a bounded domain we have:*

(i) $\alpha_\Gamma(t) \leq c^{k_\Gamma(t) - k_\Gamma(t')}\alpha_\Gamma(t')$*, for any* $\Gamma \in \mathbf{\Gamma}$ *and* $0 \leq t' \leq t$*;*
(ii) $L(t) = \sum_{\Gamma\in\mathbf{\Gamma}} \alpha_\Gamma(t)$*.*

For $\Gamma \in \mathbf{\Gamma}$ denote by $t_{0,\Gamma}$ the time at which Γ is generated. Then, by Lemma 3.3 we have

$$\alpha_\Gamma(t) \leq c^{k_\Gamma(t) - k_\Gamma(t_{0,\Gamma})}\alpha_\Gamma(t_{0,\Gamma}) \,. \tag{29}$$

We denote the total amount of the strengths at time t of all primary paths whose order at time t is k (resp., larger than k) by

$$L_k^P(t) = \sum_{\substack{\Gamma\in\mathbf{\Gamma}^P \\ k_\Gamma(t)=k}} \alpha_\Gamma(t) \qquad V_k^P(t) = \sum_{l\geq k} L_l^P(t) \,.$$

By 29 we finally deduce the decreasing of the collections of primary paths $L_k^P(t)$ and $V_k^P(t)$.

Proposition 6. *Assume 22. For every approximate solution valued in a bounded domain we have, for* $k \geq 1$*,*

$$\begin{aligned} L_k^P(t) &\leq c^{k-1}L(0) \,, \\ V_k^P(t) &\leq L(0)\sum_{l\geq k} c^{l-1} = \frac{c^{k-1}}{1-c}L(0) \,. \end{aligned}$$

We refer to [7] for further properties of the path decomposition and, in particular, for how it is exploited to control the wave interactions.

4. **The system of nonisentropic gasdynamics.** In this section we briefly show another important application of the method of path decomposition. We are concerned with the non-isentropic system of gasdynamics; in Lagrangian coordinates

it is written as

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(\mathcal{E} + \frac{1}{2}u^2\right)_t + (pu)_x = 0. \end{cases} \tag{30}$$

Here, v and u are as in Section 3; the pressure p and the internal energy $\mathcal{E}$ are linked together with the temperature Θ and the entropy S by the second law of thermodynamics $d\mathcal{E} = \Theta dS - pdv$. We assume that the gas is ideal and polytropic; this means that $pv = R\Theta$ and $\mathcal{E} = C_v\Theta$, respectively, where R and C_v are positive constants. We denote by $\gamma = 1 + R/C_v > 1$ the adiabatic gas constant.

We provide system 30 with the initial data

$$(v, u, S)|_{t=0} = \left(\bar{v}(x), \bar{u}(x), \bar{S}(x)\right), \qquad \text{with} \qquad \bar{v}(x) \geq v_0 > 0. \tag{31}$$

In 1977 Liu [20] proved the following fundamental result.

Theorem 4.1 ([20]). *Assume $1 < \gamma \leq \frac{5}{3}$. If $(\gamma-1)\mathrm{TV}(\bar{v}, \bar{u}, \bar{S})$ is sufficiently small, then there exists a global solution with bounded total variation to 30-31.*

In the degenerate case $\gamma = 1$, the two first equations in 30 decouple from the third one; then, the entropy is expressed in terms of v and we recover the result by Nishida [21]. Another proof of Theorem 4.1, which still relied on the Glimm scheme, was given in [24].

Here follows the main result on the Cauchy problem for system 30, which, on the contrary, is based on the wave-front tracking algorithm.

Theorem 4.2 ([8]). *Under the same assumptions of Theorem 4.1, the wave-front tracking scheme is stable and provides a global solution with bounded total variation to 30–31.*

We briefly show here the main features of the proof. First, as in Theorem 3.1 and [20], we introduce Riemann coordinates to analyze the wave curves. The study of the interactions is particularly heavy, as the related abridged and condensed part in [20] showed, where eleven patterns of interactions had to be taken into consideration. A fully detailed analysis was performed by Asakura [6], where also some new and refined interaction estimates, which are exploited in the proof of Theorem 4.2, were provided.

Second, the path decomposition technique is applied not only to shocks fronts but also to entropy fronts (contact discontinuities); that was not the case for the simpler system 7. Moreover, the notion of secondary rarefaction (analogous to that of secondary path) needs to be introduced, even if we never need to deal with rarefaction paths.

Apart from these difficulties, results analogous to those provided in Lemmas 3.2 and 3.3 still hold, where now $L(t)$ includes not only shocks but also the entropy waves. In particular, the analog of Lemma 3.3 could be interpreted as the *pathwise version* of Nishida's lemma [21]. Moreover, we show that the total amount of secondary waves (which include secondary rarefactions) is bounded from above by $(\gamma - 1)$ times the *interaction potential*; this leads to a further understanding of the assumptions made in Theorem 4.1.

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