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On the modeling of learning dynamics in large living systems

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Abstract

This paper deals with the modeling of learning dynamics in a large system of interacting entities. The mathematical approach is based on the kinetic theory on active particles. Their microscopic state is modeled by a scalar variable called activity, which is assumed to be heterogeneously distributed among the particles. Nonlinear interactions lead to collective phenomena of learning. The structure allows the derivation of specific models and of numerical simulations related to real systems.

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1. Introduction.

A new approach to collective learning and hiding processes has recently been proposed in [1] based on suitable developments of the kinetic theory of active particles [2]. The leading idea proposed in [1] is that individuals learn by interactions, whose frequency depends on a suitable metric distance, while learning and hiding are related to the state of the interacting individuals, who may either approach their state (*consensus*) or increase such a distance (*competition*).

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The kinetic theory of active particles was specifically developed to model living, hence complex, systems constituted by several multi-agents interacting by linear or nonlinear rules. The modeling of such systems was approached by different theoretical tools such as statistical mechanics [3] and kinetic theory [4]. An already vast bibliography is reported in these books. Additional titles can be found in [5] and in the survey [2].

The hallmarks of the kinetic theory of active particles are the following:

- the overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*;
- the state of each functional subsystem is defined by a suitable, time dependent, probability distribution over the activity variable;
- interactions are modeled by games, more precisely stochastic games [6], where the state of the interacting particles and the output of the interactions are known in probability;
- the evolution of the aforesaid probability distribution is obtained by a balance of particles within elementary volumes of the space of the microscopic states, where the dynamics of the inflow and the outflow of particles is related to the interactions at the microscopic scale.

This approach has been applied in a variety of different fields such as spread of epidemics [7], social systems [8], opinion formation [9], micro-scale Darwinian evolution and selection [10]. Mathematical tools can be developed to include space dynamics, for instance in vehicular traffic [11], crowd dynamics [12], and many others. Space can be represented by continuous variables or correspond to networks representation [13,14].

The aim of this paper is to present a detailed analysis of a collective learning dynamic for a large system of interacting individuals. The learning process is characterized by nonlinear interactions among individuals with a different level of knowledge. The contents of the paper is presented through four more sections. Namely, the reference mathematical framework is reported in Section 2, while the qualitative analysis of the initial value problem is presented in Section 3. Section 4 is devoted to the derivation of a specific two populations learning dynamic model and to the related numerical simulations. In the first part of the section we discuss the modeling methods both in the continuum and in the discrete case. We specify the form of the encounter rate and of the nonlinearly additive interactions characterizing the model. We remark that in our model the interactions are

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nonlinearly additive: in fact, the result of the interaction between a pair of active particles is nonlinearly dependent not only on the states of the two particles, but also on the states of all the particles in a defined interaction domain.

In the second part of the section we specialize the treatment to the case of a two populations learning dynamic, described by two functional subsystems characterized by different levels of knowledge. Individuals of the less evolved subsystem increase their own knowledge by taking advantage of the interactions with the individuals of the other subsystem. We conclude the section with a few numerical simulations which visualize the predictive ability of the model. Finally, some concluding remarks and perspectives are reported in the last section.

2. A reference mathematical structure.

In this section, for sake of completeness, following [1] and [2] we review the essential aspects and definitions of the reference mathematical framework, which will be necessary in the following of this paper.

Let us consider a large system of many interacting entities, called *active particles* grouped into n different *functional subsystems*. Each subsystem consists of entities which collectively express the same strategy through a scalar variable $u \in D_u \subseteq \mathbb{R}^+$, called *activity*, with a possibly different meaning in each functional subsystem. In the modeling approach under study space and velocity variables have no relevant physical meaning. As a consequence, such variables do not play any role in the interactions.
We assume that there are only interactions which modify the microscopic state of the particles. *Interactions* involve three kinds of particles: *candidate*, *test* and *field*. The interaction rule is as follows: *candidate* particles can acquire, in probability, the state of the *test* particles, after an interaction with *field* particles; *test* particles instead, lose their state after interactions.

In the following we assume that the activity of individuals is heterogeneously distributed in each functional subsystem, with the overall state described by the probability distributions:

$$f_i = f_i(t,u) : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+, \qquad i \in \{1,...,n\}\,, \tag{1}$$

which correspond to the *test* particles, representative of each subsystem. Moreover, the f_is in (1) are divided by the number of particles in each subsystem, which is assumed to be constant. As a consequence, the f_is are probability densities. There follows that, under suitable integrability assumptions, $f_i(t,u)du$ denotes, for all $t \geq 0$, the probability of finding an active particle whose state is in the elementary volume $[u, u+du]$ at time t. Moreover for $u^p f_i \in L_1(\mathbb{R}^+)$, $p \in \mathbb{N}$, higher order moments corresponding to the macroscopic description are to be computed as:

$$\mathbb{E}_i^p[f_i](t) = \int_{\mathbb{R}^+} u^p f_i(t,u)du, \tag{2}$$

where

$$p = 0 \quad \Rightarrow \quad \mathbb{E}_i^0[f_i](t) = \int_{\mathbb{R}^+} f_i(t,u)du = 1, \quad \forall\, t \geq 0, \quad i \in \{1,...,n\}\,. \tag{3}$$

The time evolution of f_i is obtained as a balance equation between the inlet-outlet flows in the elementary volume $[u, u+du]$ of the space of the microscopic states. Such equation can be derived once the following quantities are specified through a detailed modeling of the microscopic interactions:

- $\eta_{ij} = \eta_{ij}[\mathbf{f}](u_*, u^*)$ is the encounter rate between the *candidate* (or *test*) active particle with state u_* of the i-th functional subsystem and the *field* active particle with state u^* of the j-th functional subsystem.
- $\mathscr{B}_{ij} = \mathscr{B}_{ij}[\mathbf{f}](u_* \to u|u^*, u_*)$ is the probability density that a *candidate* particle with state u_* of the i-th functional subsystem ends up into the state u of the *test* particle in the same subsystem, after interacting with the *field* particle, with state u^*, of the j-th functional subsystem.

Both quantities η_{ij} and $\mathscr{B}_{ij}$ can be conditioned by the probability distributions $\mathbf{f} = \{f_1, ..., f_n\}$ of the interacting pairs. As a consequence, due to the influence of collective distributions, the resulting interactions are nonlinearly additive.

Remark 1. $\mathscr{B}_{ij}$ satisfies for all $i, j \in \{1, ..., n\}$, the following condition:

$$\int_{\mathbb{R}^+} \mathscr{B}_{ij}[\mathbf{f}](u_* \to u\;|u^*, u_*)du = 1, \qquad \forall u_*, u* \in \mathbb{R}^+, \quad \forall \mathbf{f}. \tag{4}$$

The balance of particles in the elementary volume of the microscopic states

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yields:

$$\partial_t f_i(t,u) = J_i\,[\mathbf{f}]\,(t,u) =$$
$$= \sum_{j=1}^{n} \int_{\mathbb{R}^+\times\mathbb{R}^+} \eta_{ij}\,[\mathbf{f}]\,(u_*,u^*)\mathscr{B}_{ij}\,[\mathbf{f}]\,(u_* \to u \mid u^*, u_*) f_i(t,u_*) f_j(t,u^*) du_* du^*$$
(5)
$$- f_i(t,u) \sum_{j=1}^{n} \int_{\mathbb{R}^+} \eta_{ij}\,[\mathbf{f}]\,(u,u^*) f_j(t,u^*) du^*.$$

The above equation (5) has been derived under the assumption that the variable u is continuous over $\mathbb{R}^+$. Condition (1) therefore implies that the densities f_i decay rapidly to zero at infinity. On the other hand, it is also useful in several applications to consider a discrete set for the values of the activity variable. This is indeed the case of the learning dynamics, where it is useful to consider a discrete variable $u \in [0,1]$ describing the different levels of knowledge. In particular, $u = 0$ and $u = 1$ correspond to the lowest and to the highest levels of knowledge, respectively.

Accordingly, let us introduce the set $I_u = \{u_1 = 0, ..., u_r, ..., u_m = 1\}$, such that discrete probability densities follow:

$$f_{ir} = f_i(t, u = u_r) : [0,T] \to \mathbb{R}^+, \qquad i \in \{1,...,n\}, \quad r = 1,...,m \tag{6}$$

such that

$$\sum_{r=1}^{m} f_{ir}(t) = 1, \qquad \forall t \geq 0, \quad i \in \{1,...,n\}. \tag{7}$$

The interaction terms are defined as follows:

- $\eta_{ij}^{hk} = \eta_{ij}\,[\mathbf{f}]\,(u_h, u_k)$ is the encounter rate between the i-h candidate (or test) particle, with state u_h and the j-th field particle, with state u_k;
- $\mathscr{B}_{ij}^{hk}(r) = \mathscr{B}_{ij}\,[\mathbf{f}]\,(u_h \to u_r \mid u_h, u_k)$ is the probability that an i-th candidate particle, with state u_h ends up into the state u_r of the test particle of the same functional subsystem, after interacting with the j-th field particle with state u^k.

Remark 2. $\mathscr{B}_{ij}^{hk}(r)$ satisfies for all $i,j \in \{1,...,n\}$ and for all $h,k \in \{1,...,m\}$ the following conditions:

$$\eta_{ij}^{hk} \geq 0, \quad \mathscr{B}_{ij}^{hk}(r) \geq 0, \quad \sum_{r=1}^{m} \mathscr{B}_{ij}\,[\mathbf{f}]\,(u_h \to u_r \mid u_h, u_k) = 1, \qquad \forall\, \mathbf{f}. \tag{8}$$

The evolution in the discrete case then follows as:

$$\frac{d}{dt} f_{ir}(t) = Q_{ir}\left[\mathbf{f}\right](t) =$$
$$= \sum_{j=1}^{n} \sum_{h,k=1}^{m} \eta_{ij}\left[\mathbf{f}\right](u_h, u_k) \mathscr{B}_{ij}\left[\mathbf{f}\right](u_h \to u_r \mid u_h, u_k) f_{ih}(t) f_{jk}(t)$$
$$- f_{ir}(t) \sum_{j=1}^{n} \sum_{k=1}^{m} \eta_{ij}\left[\mathbf{f}\right](u_h, u_k) f_{jk}(t)\,, \quad i = 1, \ldots, n;\ r = 1, \ldots, m\,, \tag{9}$$

where $\mathbf{f}$ denotes the set of all f_{ir} components of the probability density.

3. Qualitative analysis.

In view of the derivation of a specific two populations learning dynamic model (see Section 4), in this section the initial value (I.V.) problem in the discrete case is considered. It is shown that the solution of such I.V. problem exists and is a positive, regular function of time, of class $C^1([0,T])$. We point out that global existence and uniqueness of the solution for the I.V. problems belonging to the same class was reported in [15] for the continuous case.

In order to obtain the time evolution of the distribution functions $f_{ir}(t)$, $i \in \{1, ..., n\}$, we consider the I.V. problem for equation (9):

$$\begin{cases} \frac{d}{dt} f_{ir}(t) = Q_{ir}\left[\mathbf{f}\right](t)\,, \\ f_{ir}(0) = f_i(0, u_r)\,, \end{cases} \quad i = 1, \ldots, n;\ r = 1, \ldots, m\,. \tag{10}$$

We introduce the space:

$$X = \left\{ f_i : [0,T] \to \mathbb{R}\ ,\ \ f_i \in C^1\left([0,T]\right), i = 1, ..., n;\ T > 0 \right\} \tag{11}$$

equipped with the norm:

$$\| f_i(t) \|_X = \sum_{r=1}^{m} |f_{ir}(t)|\,. \tag{12}$$

Moreover, we introduce the space $\mathbf{X} = X^n$ equipped with the norm:

$$\| \mathbf{f}(t) \|_{\mathbf{X}} = \sum_{i=1}^{n} \| f_i(t) \|_X\ , \tag{13}$$

and set:

$$\mathbf{X}_+ = \{ \mathbf{f} \in \mathbf{X} \mid f_i \geq 0,\ i = 1, ..., n \}\,. \tag{14}$$

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The following theorem states a result of local existence and uniqueness for the solution of the I.V. problem (10).

Theorem 1. *Consider the I.V. problem (10) with* $\mathbf{f}_0 = \{f_1(0,u), ..., f_n(0,u)\} \in \mathbf{X}_+$. *Assume that (8) holds, together with the following hypotheses:*

- *the encounter rate* η_{ij}^{hk} *satisfies the following condition:*

$$\sum_{r=1}^{m} \eta_{ij}[\mathbf{f}](u_h, u_k) \leq C,$$

  $\forall\, i,j = 1,...,n$, $\forall\, h,k \in \{1,...,m\}$ *and* $\forall\, \mathbf{f} \in \mathbf{X}$ *with* C *positive constant;*
- $\forall\ \mathbf{f}, \mathbf{g} \in \mathbf{X}$ *the probability* $\mathscr{B}_{ij}^{hk}$ *and the encounter rate* η_{ij}^{hk} *are Lipschitz continuous in* X, *that is,* $\forall\, h,k \in \{1,...,m\}$ *it results*

$$\sum_{i,j=1}^{n}\sum_{r=1}^{m} |\mathscr{B}_{ij}[\mathbf{f}](u_h \to u_r \mid u_h, u_k) - \mathscr{B}_{ij}[\mathbf{g}](u_h \to u_r \mid u_h, u_k)| \leq L_1 \|\mathbf{f} - \mathbf{g}\|_{\mathbf{X}},$$

$$\sum_{i,j=1}^{n}\sum_{r=1}^{m} |\eta_{ij}[\mathbf{f}](u_h, u_k) - \eta_{ij}[\mathbf{g}](u_h, u_k)| \leq L_2 \|\mathbf{f} - \mathbf{g}\|_{\mathbf{X}},$$

  with L_1, L_2 *a positive constants.*

Then, there exist $T > 0$ *and a unique solution* $\mathbf{f}(t)$ *in* $\mathbf{X}$ *for the I.V. problem (10) on the time interval* $[0, T]$. *Moreover,* $\mathbf{f}(t) \in \mathbf{X}_+$, $t \in [0, T]$.

Proof. We start observing that if we formally sum the equations $(10)_1$ on $r = 1, \ldots, m$ and use the condition (8), we get:

$$\frac{d}{dt}\sum_{r=1}^{m} f_{ir}(t) = 0\,, \tag{15}$$

that implies:

$$||\mathbf{f}(t)||_{\mathbf{X}} = ||\mathbf{f}(0)||_{\mathbf{X}}\,, \qquad \text{for any } t \geq 0. \tag{16}$$

Therefore the solution of (10), if it exists, remains bounded in $\mathbf{X}$ for any time $t \geq 0$. The latter observation assures that the operator $Q_i[\mathbf{f}](t)$ in the right hand side of (10) is a closed map in $\mathbf{X}$. Let us now prove that $Q_i[\mathbf{f}](t)$ is Lipschitz continuous in $\mathbf{X}$, i.e. given $\|\mathbf{f}\|_{\mathbf{X}}$ and $\|\mathbf{g}\|_{\mathbf{X}} \leq M$, it follows that:

$$\|Q_i[\mathbf{f}](t) - Q_i[\mathbf{g}](t)\|_{\mathbf{X}} \leq L\,\|\mathbf{f} - \mathbf{g}\|_{\mathbf{X}} \tag{17}$$

with L a positive constant depending on M. Indeed, when (9) is used together with (12) and (13), for the right hand side of (17) we can write:

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{r=1}^{m}\Bigg|\Bigg[\sum_{j=1}^{n}\sum_{h,k=1}^{m}\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\ \mathscr{B}_{ij}\left[\mathbf{f}\right](u_h\to u_r\mid u_h,u_k)\,f_{ih}(t)f_{jk}(t)\\
&-f_{ir}(t)\sum_{j=1}^{n}\sum_{k=1}^{m}\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\,f_{jk}(t)\Bigg]\\
&-\Bigg[\sum_{j=1}^{n}\sum_{h,k=1}^{m}\eta_{ij}\left[\mathbf{g}\right](u_h,u_k)\ \mathscr{B}_{ij}\left[\mathbf{g}\right](u_h\to u_r\mid u_h,u_k)\,g_{ih}(t)g_{jk}(t)\\
&-g_{ir}(t)\sum_{j=1}^{n}\sum_{k=1}^{m}\eta_{ij}\left[\mathbf{g}\right](u_h,u_k)\,g_{jk}(t)\Bigg]\Bigg|\le\\
&\sum_{i=1}^{n}\sum_{r=1}^{m}\Bigg\{\sum_{j=1}^{n}\sum_{h,k=1}^{m}\Big|\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\ \mathscr{B}_{ij}\left[\mathbf{f}\right](u_h\to u_r\mid u_h,u_k)\left[f_{ih}(t)f_{jk}(t)-g_{ih}(t)g_{jk}(t)\right]\\
&+g_{ih}(t)g_{jk}(t)\left[\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\mathscr{B}_{ij}\left[\mathbf{f}\right](u_h\to u_r\mid u_h,u_k)\right.\\
&\left.-\ \eta_{ij}\left[\mathbf{g}\right](u_h,u_k)\mathscr{B}_{ij}\left[\mathbf{g}\right](u_h\to u_r\mid u_h,u_k)\right]\Big|\\
&+\sum_{j=1}^{n}\sum_{k=1}^{m}\Big|\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\left[f_{ir}(t)f_{jk}(t)-g_{ir}(t)g_{jk}(t)\right]\\
&+g_{ir}(t)g_{jk}(t)\left[\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)-\eta_{ij}\left[\mathbf{g}\right](u_h,u_k)\right]\Big|\Bigg\}\le\\
&\sum_{i=1}^{n}\sum_{r=1}^{m}\Bigg[\sum_{j=1}^{n}\sum_{h,k=1}^{m}\Big|\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\mathscr{B}_{ij}\left[\mathbf{f}\right](u_h\to u_r\mid u_h,u_k)\,f_{ih}(t)\left[f_{jk}(t)-g_{jk}(t)\right]\Big|\\
&+\sum_{j=1}^{n}\sum_{h,k=1}^{m}\Big|\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\mathscr{B}_{ij}\left[\mathbf{f}\right](u_h\to u_r\mid u_h,u_k)\,g_{jk}(t)\left[f_{ih}(t)-g_{ih}(t)\right]\Big|\\
&+\sum_{j=1}^{n}\sum_{h,k=1}^{m}\Big|g_{ih}(t)g_{jk}(t)\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\left[\mathscr{B}_{ij}\left[\mathbf{f}\right](u_h\to u_r\mid u_h,u_k)\right.\\
&\left.-\mathscr{B}_{ij}\left[\mathbf{g}\right](u_h\to u_r\mid u_h,u_k)\right]\Big|\\
&+\sum_{j=1}^{n}\sum_{h,k=1}^{m}\Big|g_{ih}(t)g_{jk}(t)\mathscr{B}_{ij}\left[\mathbf{g}\right](u_h\to u_r\mid u_h,u_k)\left[\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\right.\\
&\left.-\eta_{ij}\left[\mathbf{g}\right](u_h,u_k)\right]\Big|\\
&+\sum_{j=1}^{n}\sum_{k=1}^{m}\Big|\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)f_{ir}(t)\left[f_{jk}(t)-g_{jk}(t)\right]\Big|\\
&+\sum_{j=1}^{n}\sum_{k=1}^{m}\Big|\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)g_{jk}(t)\left[f_{ir}(t)-g_{ir}(t)\right]\Big|\\
&+\sum_{j=1}^{n}\sum_{k=1}^{m}\Big|g_{ir}(t)g_{jk}(t)\left[\eta_{ij}\left[\mathbf{f}\right](u_h,u_k)-\eta_{ij}\left[\mathbf{g}\right](u_h,u_k)\right]\Big|\Bigg]\le\\
&2n^2m^2CM\left\|\mathbf{f}-\mathbf{g}\right\|_{\mathbf{X}}+m^2M^2CL_1\left\|\mathbf{f}-\mathbf{g}\right\|_{\mathbf{X}}+m^2M^2L_2\left\|\mathbf{f}-\mathbf{g}\right\|_{\mathbf{X}}\\
&+2n^2mCM\left\|\mathbf{f}-\mathbf{g}\right\|_{\mathbf{X}}+M^2mL_2\left\|\mathbf{f}-\mathbf{g}\right\|_{\mathbf{X}}\ \le\ L\left\|\mathbf{f}-\mathbf{g}\right\|_{\mathbf{X}},
\end{aligned}
$$

that proves (17). Then, the existence of a unique solution $\mathbf{f}(t)$ in $\mathbf{X}$, local in time, to (10) follows. Non negativity of such a solution is easily obtained

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observing that the components $f_{ir}(t)$ of the solution satisfy the condition:

$$f_{ir}(t) \geq 0 \qquad \forall i = 1, ..., n \quad \text{e} \quad \forall r = 1, ..., m, \tag{18}$$

when $\mathbf{f}(0) \in \mathbf{X}_+$. We set:

$$R^i(f,f)(t) = \sum_{j=1}^{n} \sum_{h,k=1}^{m} \eta_{ij}\left[\mathbf{f}\right](u_h,u_k) \cdot \mathscr{B}_{ij}\left[\mathbf{f}\right](u_h \to u_r \mid u_h,u_k)\, f_{ih}(t) f_{jk}(t),$$

$$S^i(f)(t) = \sum_{j=1}^{n} \sum_{k=1}^{m} \eta_{ij}\left[\mathbf{f}\right](u_h,u_k)\,.$$

Equation (9) can be rewritten as

$$\frac{d}{dt} f_{ir}(t) + f_{ir}(t) S^i(f)(t) = R^i(f,f)(t). \tag{19}$$

Now we call

$$\lambda_i(t) = \int_0^t S^i(f)(t')dt'. \tag{20}$$

If $f_{ir}(t)$ is solution of (19), it then follows

$$\frac{d}{dt}\left(\exp\left(\lambda_i(t)\right) f_{ir}(t)\right) = \exp\left(\lambda_i(t)\right) R^i(f,f)(t) \tag{21}$$

which implies

$$f_{ir}(t) = \exp\left(-\lambda_i(t)\right) f_{ir}(0) + \int_0^t \left[\exp\left(\lambda_i(t') - \lambda_i(t)\right) R^i(f,f)(t')\right] dt'. \square \tag{22}$$

The relation (22) allows us to conclude that, given $\mathbf{f}(0) \in \mathbf{X}_+$ and the positivity of the integral function, the function $f_{ir}(t)$ satisfies the condition of non-negativity (18) in its domain of existence. Moreover, when (18) is used together with (16), we obtain that the solution to (10) is uniformly bounded on any compact time interval $[0,T]$, $T > 0$. This latter observation leads immediately to the following result of global existence of the solution in $\mathbf{X}_+$.

Theorem 2. *Consider the I.V. problem (10), under the assumptions of the Theorem 1. Then, the solution* $\mathbf{f}(t)$ *exists for any finite time* $t \geq 0$.

4. Modeling and simulations.

4.1. *Modeling methods.*

In order to model a collective learning dynamic, following [1], it is necessary to introduce a distance between the probability densities. We first consider the continuous case, where condition (2) allows to use the $L_1(\mathbb{R}^+)$ space. The distance will therefore be given by the norm of the difference of the two distributions:

$$d\left(f_i, f_j\right)[\mathbf{f}](t) = \int_{\mathbb{R}^+} |f_i(t,u) - f_j(t,u)|\, du, \qquad i,\, j \in \{1, ..., n\}\,. \tag{23}$$

Similarly, in the discrete case we get:

$$d\left(f_i, f_j\right)[\mathbf{f}](t) = \sum_{r=1}^{m} |f_{ir}(t) - f_{jr}(t)|, \qquad i,\, j \in \{1, ..., n\}\,. \tag{24}$$

Let us now turn our attention to the modeling of the encounter rate. We assume that such rate decays with the distance between the interacting active particles. Accordingly, we write:

$$\eta_{ij} = \eta_{ij}^0\, e^{-\alpha_{ij}(1 + |u_* - u^*|)(1 + d\left(f_i, f_j\right)[\mathbf{f}](t))} \tag{25}$$

where η_{ij}^0 and α_{ij} are positive constants characterizing the specific system. These quantities are defined respectively as *basic interaction rate* and *decay rate*. Analogous considerations for the discrete case yield:

$$\eta_{ij}^{hk} = \eta_{ij}^0\, e^{-\alpha_{ij}(1 + |u_h - u_k|)(1 + d\left(f_i, f_j\right)[\mathbf{f}](t))}. \tag{26}$$

In order to model the terms $\mathscr{B}_{ij}$, we assume that the result of the interaction is influenced not only by the states of the interacting particles but also, at macroscopic level, by the collective action of all the other particles.

In the following we limit our analysis only to the low order moments of the action of this particles. In particular, we consider the first order moment:

$$\mathbb{E}_i^1[f_i](t) = \int_{\mathbb{R}^+} u f_i(t,u) du. \tag{27}$$

When (27) is taken into account, the formal expression of the transition probability density takes the form:

$$\mathscr{B}_{ij} = \mathscr{B}_{ij}(u_* \to u \mid u_*, u^*, \mathbb{E}_i^1[f_i](t), \mathbb{E}_j^1[f_j](t)), \qquad \forall u^* \in D_u{}^*. \tag{28}$$

Analogous considerations in the discrete case, yield:

$$\mathscr{B}_{ij}^{hk}(r) = \mathscr{B}_{ij}(u_h \to u_r \mid u_h, u_k, \mathbb{E}_i^1[f_i](t), \mathbb{E}_j^1[f_j](t)), \tag{29}$$

with

$$\mathbb{E}_i^1[f_i](t) = \sum_{r=1}^{m} u_r f_{ir}(t). \tag{30}$$

When the transition probability densities (28) and (29) are used in (5) and (9) respectively, the detailed expression for the class of evolution equations characterizing the model is obtained. In the continuous case the result is:

$$\begin{aligned}
&\partial_t f_i(t,u) = J_i\left[\mathbf{f}\right](t,u) = \\
&= \sum_{j=1}^{n} \eta_{ij}^0 \int_{\mathbb{R}^+ \times \mathbb{R}^+} e^{-\alpha_{ij}(1+|u_* - u^*|)(1 + d\left(f_i, f_j\right)\left[\mathbf{f}\right](t))} \\
&\times \mathscr{B}_{ij}(u_* \to u \mid \; u_h, u_k, \mathbb{E}_i^1[f_i](t), \mathbb{E}_j^1[f_j](t)) f_i(t,u_*) f_j(t,u^*) du_* du^* \\
&- f_i(t,u) \sum_{j=1}^{n} \eta_{ij}^0 \int_{\mathbb{R}^+} e^{-\alpha_{ij}(1+|u - u^*|)(1 + d\left(f_i, f_j\right)\left[\mathbf{f}\right](t))} f_j(t,u^*) du^*.
\end{aligned} \tag{31}$$

Similarly, in the discrete case we get:

$$\begin{aligned}
&\frac{d}{dt} f_{ir}(t) = Q_i\left[\mathbf{f}\right](t) = \\
&= \sum_{j=1}^{n} \sum_{h,k=1}^{m} \eta_{ij}^0 e^{-\alpha_{ij}(1+|u_h - u_k|)(1 + d\left(f_i, f_j\right)\left[\mathbf{f}\right](t))} \\
&\times \mathscr{B}_{ij}^{hk}(u_h \to u_r \mid \; u_h, u_k, \mathbb{E}_i^1[f_i](t), \mathbb{E}_j^1[f_j](t)) f_{ih}(t) f_{jk}(t) \\
&- f_{ir}(t) \sum_{j=1}^{n} \sum_{k=1}^{m} \eta_{ij}^0 e^{-\alpha_{ij}(1+|u_r - u_k|)(1 + d\left(f_i, f_j\right)\left[\mathbf{f}\right](t))} f_{jk}(t).
\end{aligned} \tag{32}$$

Remark 3. In the modeling approach developed above, the interactions are nonlinearly additive. Indeed, the outcome of the interactions is not simply given by the sum of the actions of the field particles on the candidate (or test) particles. The result is in fact nonlinearly dependent not only on the states of the interacting particles, but also on the states of all the particles in the interaction domain.

4.2. *Two populations dynamics.*

Let us now turn our attention to the discrete case, leaving the continuum case as a future application. Namely, here we consider two populations learning dynamic model, characterized by two functional subsystems, the first one collecting more evolved individuals autonomously progressing toward an higher knowledge level, the second one made by less evolved agents. These latter try to increase their own knowledge by taking advantage of the interactions with individuals from the first subsystem. Then, it is clear that the *activity* associated to each active particle has different meaning according to which different subsystem the active particle belongs to. For particles of the first subsystem, the activity represents the ability to increase and develop the knowledge through the interactions with particles of the same subsystem. In the second subsystem, instead, the activity is intended as the ability of learning through the interactions with particles of the more evolved subsystem.

The probability densities are defined as:

$$f_{1r} = f_1(t, u = u_r) : [0, T] \to \mathbb{R}^+, \qquad r = 1, ..., m, \tag{33}$$

for the evolved subsystem, and

$$f_{2r} = f_2(t, u = u_r) : [0, T] \to \mathbb{R}^+, \qquad r = 1, ..., m, \tag{34}$$

for the second subsystem.

In the case under study, the encounter rate is modeled as:

$$\begin{aligned} \eta_{11}^{hk} &= \eta_{11}^0 \; e^{-\alpha_{11}(1 + |u_h - u_k|)} \\ \eta_{22}^{hk} &= \eta_{22}^0 \; e^{-\alpha_{22}(1 + |u_h - u_k|)} \\ \eta_{12}^{hk} &= \eta_{12}^0 \; e^{-\alpha_{12}(1 + |u_h - u_k|)(1 + d\,(f_1, f_2)\,(t))} \\ \eta_{21}^{hk} &= \eta_{21}^0 \; e^{-\alpha_{21}(1 + |u_h - u_k|)(1 + d\,(f_1, f_2)\,(t))} \end{aligned} \tag{35}$$

where η_{ij}^0 and α_{ij}, $i, j \in \{1, 2\}$, are positive constants and $h, k \in \{1, ..., m\}$. Moreover, we assume the following hypotheses:

(1) The knowledge level of the active particles in the first subsystem is always higher than that of the particles in the second subsystem. As a consequence, when the second subsystem interacts with the first one there will never be a regression as a possible output.
(2) The output of the interactions could depend on the activity distance of the two interacting particles in addition to the presence of an effect caused by the influence of the mean value of each population.

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In the corrisponding evolution equations (32), the transition probability densities $\mathscr{B}_{ij}^{hk}$ consist of a set of $4m$ matrices with dimension $m \times m$. These matrices will be called *tables of the game* and define the rules to be followed by the pairs in each encounter. They are stochastic tables due to the fact that the output is a discrete random variable. In the case under analysis, the collective learning dynamics can be formalized in the following tables of game.

Table of game for $\mathscr{B}_{11}^{hk}(r)$ and $\mathscr{B}_{22}^{hk}(r)$.

In order to create the table of game for $\mathscr{B}_{11}^{hk}(r)$, we consider the term:

$$B_1^h\left(\mathbb{E}_1^1[f_1]\right) = \varepsilon_1 \left|\mathbb{E}_1^1[f_1] - u_h\right| \tag{36}$$

which is proportional to the distance between the activity of the interacting particle h and the mean value $\mathbb{E}_1^1[f_1]$, with $h = 1, ..., m$ and $0 < \varepsilon_1 \leq 1$.

Then, omitting the dependence on $\mathbb{E}_1^1[f_1]$ of $B_1^h\left(\mathbb{E}_1^1[f_1]\right)$ and considering $0 < \delta_1 \leq 1$, interactions for $i = j = 1$ are modeled as follows:

$$u_h < u_k \begin{cases} u_h \geq \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{11}^{hk}(r = h-1) & = 0 \\ \mathscr{B}_{11}^{hk}(r = h) & = B_1^h \\ \mathscr{B}_{11}^{hk}(r = h+1) & = 1 - B_1^h \\ \mathscr{B}_{11}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \\ u_h < \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{11}^{hk}(r = h-1) & = 0 \\ \mathscr{B}_{11}^{hk}(r = h) & = 1 - \delta_1 \left|u_k - u_h\right| - B_1^h \\ \mathscr{B}_{11}^{hk}(r = h+1) & = \delta_1 \left|u_k - u_h\right| + B_1^h \\ \mathscr{B}_{11}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \end{cases}$$

$$u_h \geq u_k \begin{cases} u_h \geq \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{11}^{hk}(r = h-1) & = B_1^h \\ \mathscr{B}_{11}^{hk}(r = h) & = 1 - B_1^h \\ \mathscr{B}_{11}^{hk}(r = h+1) & = 0 \\ \mathscr{B}_{11}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \\ u_h < \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{11}^{hk}(r = h-1) & = 1 - \delta_1 \left|u_k - u_h\right| - B_1^h \\ \mathscr{B}_{11}^{hk}(r = h) & = \delta_1 \left|u_k - u_h\right| + B_1^h \\ \mathscr{B}_{11}^{hk}(r = h+1) & = 0 \\ \mathscr{B}_{11}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \end{cases}$$

This table is described in terms of the effect due to the distance between u_k and u_h plus that due to the distance with respect to the mean value. The mean value of the population has a dragging effect on the activity of the particle h. This means that, for example, when $u_h \geq \mathbb{E}_1^1[f_1]$ the effect of the mean value, represented by B_1^h, tries to oppose the growth of the particle h or tries to take it back one level. Moreover, a phenomenon of regression appears when the particle h interacts with a particle k with a lower level of knowledge.

The table of game for $\mathscr{B}_{22}^{hk}(r)$ is constructed in a similar way to the table for $\mathscr{B}_{11}^{hk}(r)$, so we do not write it. Because it refers to the second population, let's consider in this case the constant $0 < \delta_2 \leq 1$ instead of δ_1 and the term

$$B_2^h\left(\mathbb{E}_2^1[f_2]\right) = \varepsilon_2\left|\mathbb{E}_2^1[f_2] - u_h\right| \tag{39}$$

with $h = 1, ..., m$ and $0 < \varepsilon_2 \leq 1$, instead of $B_1^h\left(\mathbb{E}_1^1[f_1]\right)$.

Table of game for $\mathscr{B}_{12}^{hk}(r)$ and $\mathscr{B}_{21}^{hk}(r)$.

To create the table of game for $\mathscr{B}_{12}^{hk}(r)$, we note that, due to assumption (1), the only possibility is $u_h > u_k \ \forall h, k = 1, ..., m$ (with no dependence on the position of u_h with respect to the mean value $\mathbb{E}_1^1[f_1]$). Such interactions will never modify the state u_h of the particle h belonging to the first subsystem, so that the table of game in this case will be described trivially as follows.

$$i = 1,\, j = 2 \begin{cases} \mathscr{B}_{12}^{hk}(r = h-1) & = 0 \\ \mathscr{B}_{12}^{hk}(r = h) & = 1 \\ \mathscr{B}_{12}^{hk}(r = h+1) & = 0 \\ \mathscr{B}_{12}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases}$$

The table for $\mathscr{B}_{21}^{hk}(r)$ models the transition probability densities when a particle of the second subsystem interacts with a particle of the first subsystem. Then, still for hypothesis (1), here there is only the possibility $u_h < u_k \ \forall h, k = 1, ..., m$. We therefore have the following cases for $i = 2$ and $j = 1$ in which we consider both the positions of u_h and u_k with respect

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to their own mean value.

$$u_h \geq \mathbb{E}_2^1[f_2] \begin{cases} h \neq m \begin{cases} u_k \geq \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{21}^{hk}(r = h-1) & = 0 \\ \mathscr{B}_{21}^{hk}(r = h) & = 0 \\ \mathscr{B}_{21}^{hk}(r = h+1) & = 1 \\ \mathscr{B}_{21}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \\ u_k < \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{21}^{hk}(r = h-1) & = 0 \\ \mathscr{B}_{21}^{hk}(r = h) & = B_2^h \\ \mathscr{B}_{21}^{hk}(r = h+1) & = 1 - B_2^h \\ \mathscr{B}_{21}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \end{cases} \\ h = m \begin{cases} \mathscr{B}_{21}^{mk}(r = m-1) & = 0 \\ \mathscr{B}_{21}^{mk}(r = m) & = 1 \\ \mathscr{B}_{21}^{mk}(r = m+1) & = 0 \\ \mathscr{B}_{21}^{mk}(r \neq m-1, m, m+1) & = 0 \end{cases} \end{cases}$$

$$u_h < \mathbb{E}_2^1[f_2] \begin{cases} u_k \geq \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{21}^{hk}(r = h-1) & = 0 \\ \mathscr{B}_{21}^{hk}(r = h) & = 1 - B_2^h \\ \mathscr{B}_{21}^{hk}(r = h+1) & = B_2^h \\ \mathscr{B}_{21}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \\ u_k < \mathbb{E}_1^1[f_1] \begin{cases} \mathscr{B}_{21}^{hk}(r = h-1) & = 0 \\ \mathscr{B}_{21}^{hk}(r = h) & = 1 - B_2^h \cdot B_1^h \\ \mathscr{B}_{21}^{hk}(r = h+1) & = B_2^h \cdot B_1^h \\ \mathscr{B}_{21}^{hk}(r \neq h-1, h, h+1) & = 0 \end{cases} \end{cases}$$

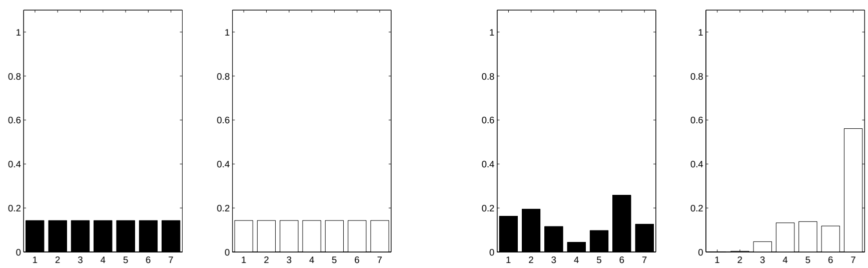
If $u_h \geq \mathbb{E}_2^1[f_2]$, when the particle h interacts with a particle k (belonging to the first subsystem) which is above the mean value $\mathbb{E}_1^1[f_1]$, the activity of h certainly improves its level of one. Otherwise, if the activity of k is under the mean value of the evolved subsystem, the particle h tends to improve its level of activity but it is opposed by the effect of the mean value $\mathbb{E}_2^1[f_2]$. In the case $h = m$, the table of game shows no transitions due to the conservative character of the subsystem.

When $u_h < \mathbb{E}_2^1[f_2]$, the particle h is encouraged to improve its level by the driving effects of the two mean values, according to the position of the activity of k.

4.3. *Simulations.*

In this subsection we show the results of numerical simulations. We set some specific parameters which will not be the subject of the sensitivity analysis: $\varepsilon_1 = \varepsilon_2 = \delta_1 = \delta_2 = \frac{1}{2}$, $\eta^0_{11} = \frac{3}{4}$, $\eta^0_{22} = \frac{3}{4}$, $\eta^0_{12} = \frac{1}{2}$, $\eta^0_{21} = \frac{1}{2}$, $\alpha_{11} = \frac{1}{2}$, $\alpha_{22} = \frac{1}{2}$, $\alpha_{12} = 1$, $\alpha_{21} = 1$. Modifying these parameters, the qualitative behavior of the simulations does not change. We fix $m = 7$, i.e. for each subsystem we consider seven different levels for the variable u which represent the activity of the interacting particles. Then, fixed an initial point $t = 0$ we observe numerically the time evolution of the two probability distribution $f_1(t, u_1, ..., u_7)$ and $f_2(t, u_1, ..., u_7)$ until their asymptotic configuration.

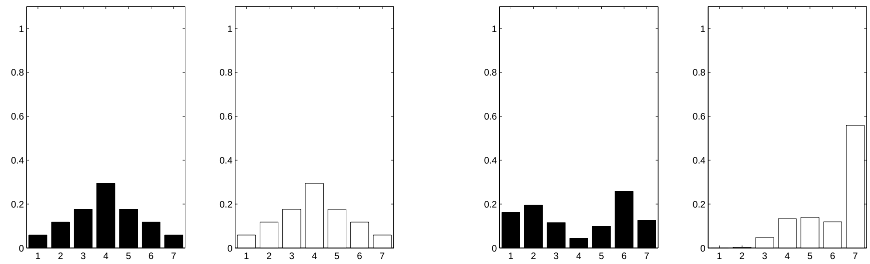
The aim of our simulations is to test the influence of different initial configurations, which correspond to different distributions of the active particles on the levels of knowledge, on the learning trends exhibited by the two subsystems. In the following figures, we represent the first subsystem (f_1) in black and the second (f_2) in white. The results reported in Figures 1 and 2 show that the final configurations of two subsystems are not modified when the initial configuration is changed from uniform to normal distribution.



(a) Initial configuration. (b) Asymptotic configuration.

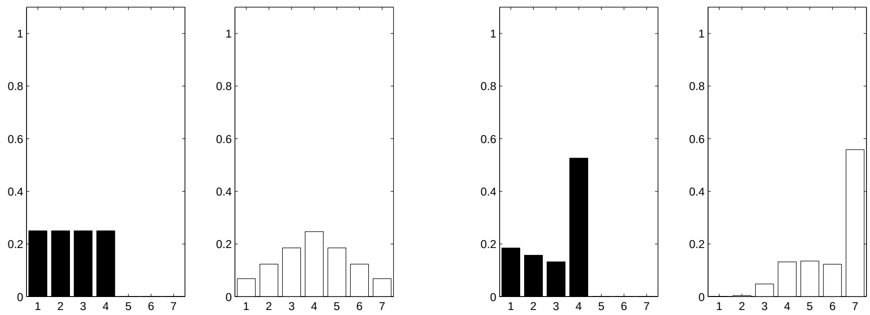
Figure 1. Uniform initial distribution for both f_1 and f_2.

As we see from the following Figures 3-8, the second subsystem always shows the same qualitative behavior, as in Figures 1 and 2, reaching the same final configuration. The outcome of the numerical simulations clearly indicate the different "learning strategy" characterizing the individuals of the two subsystems. Indeed, individuals belonging to the first group cannot take any advantage from the interactions with individuals of the second one. They can increase their level of knowledge only through interactions



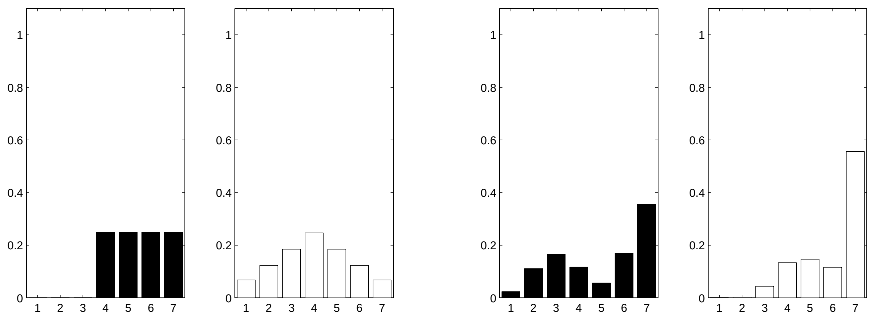
(a) Initial configuration. (b) Asymptotic configuration.

Figure 2. Normal initial distribution for both f_1 and f_2.



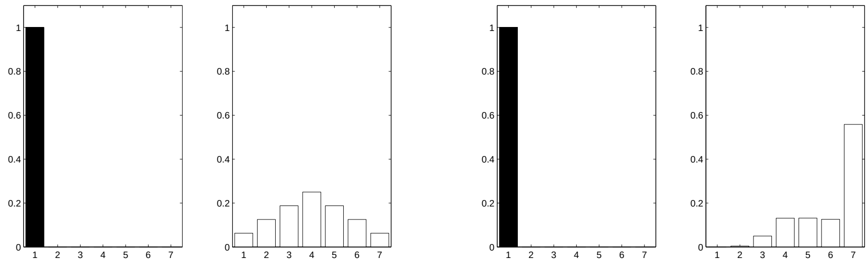
(a) Initial configuration. (b) Asymptotic configuration.

Figure 3. Initial distribution only on the first four levels for f_1 and normal for f_2.



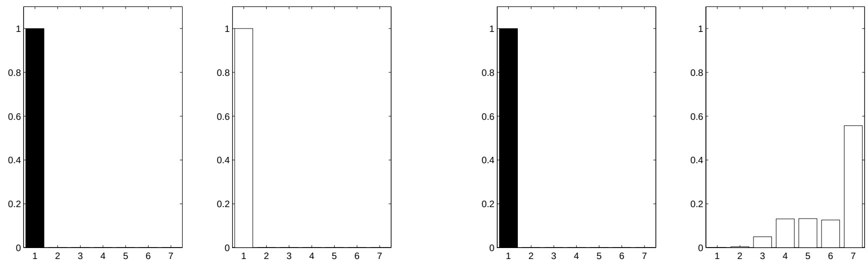
(a) Initial configuration. (b) Asymptotic configuration.

Figure 4. Initial distribution only on the last four levels for f_1 and normal for f_2.



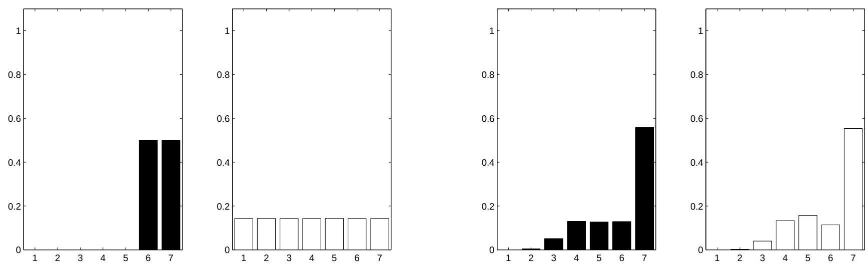
(a) Initial configuration. (b) Asymptotic configuration.

Figure 5. Initial distribution on the first level for f_1 and normal for f_2.



(a) Initial configuration. (b) Asymptotic configuration.

Figure 6. Initial distribution on the first level for f_1 and f_2.



(a) Initial configuration. (b) Asymptotic configuration.

Figure 7. Initial distribution on the last two levels for f_1 and uniform for f_2.

with individuals belonging to the same group. Therefore the asymptotic configuration of the first subsystem only depends on its initial distribution of knowledge. In other words, the first subsystem evolves only through internal interactions, while the interactions with the second subsystem have

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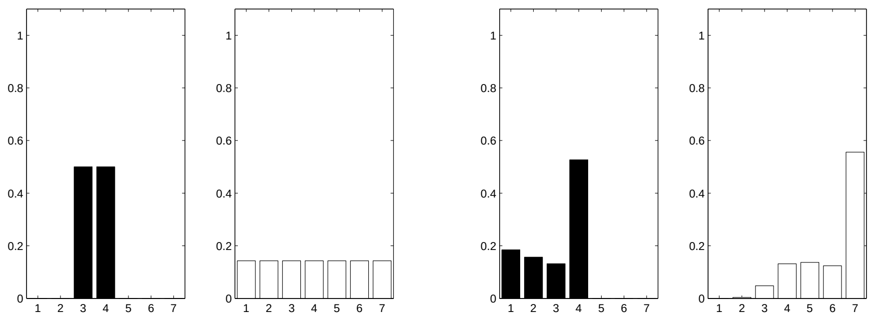

(a) Initial configuration. (b) Asymptotic configuration.

Figure 8. Initial distribution on two levels for f_1 and uniform for f_2.

no effects. In this case we obtain several final configurations that depend on the different initial conditions.

Individuals of the second subsystem instead, always tend to increase their level of knowledge, since they can develop a strategy of learning through the interactions with the more evolved individuals of the first subsystem. Looking at the above result we can conclude that they always take advantage of such interactions, reaching the same final configuration of knowledge, independently from the initial conditions of both the subsystems.

5. Perspectives.

The model of learning treated in this paper is based on the hallmarks of the kinetic theory of active particles described in Section 1. In particular, the *learning ability* of the active particles induces modifications of the *activity* distributions by nonlinear interactions. Moreover, the modeling involves different representation scales. Active particles are the minimal entities of the system; they therefore define the *microscopic scale*. On the other hand, interactions involving active particles produce collective behaviors which are observables at the *macroscopic scale*.

Focusing on future developments, the modeling approach can be generalized by a representation in networks, namely, active particles occupy various nodes and communicate over the network. Its topology [13,14] would in this case influence the frequency of the interactions and therefore the dynamics of the system.

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