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Triad's consonance and dissonance: a detailed analysis of compactness models

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Abstract Theories of consonance and dissonance based on the "compactness" approach include the two sub-categories of periodicity and harmonicity. In a previous work, we discussed the related consonance and dissonance indicators for dyads; as they are given by discontinuous functions of the dyad frequency ratio, we proposed a method to extend them to the continuum, based on the auditory discrimination limen. Here, we generalize the compactness indicators to describe the consonance and dissonance for triads and discuss their extension to the continuum. We compare our model predictions with perceptual data from a recent psychoacoustic test by means of a Chi-square analysis. The result is that compactness indicators provide a quite effective, but not fully satisfactory, description of consonance, and dissonance for triads.

1 Introduction

Explaining the auditory perceptions of consonance and dissonance (C and D) in music[1] is a fascinating issue that challenged many generations of scientists over centuries. Even nowadays, it is a subject open to scientific discussion, the actual functioning of our hearing system being not fully understood. The aim of the present paper is to extend from dyads (intervals) to triads (triadic chords) our previous work [1], contributing to the scientific discussion about C and D from the point of view of physics and its methodologies.[2]

A brief historical review is useful to understand the status of the art. With the beginning of modern science in the 17th century, the problem of justifying on a physical basis the perceptions of C and D started to be formulated in a quantitative way and with an increasing amount of mathematical sophistication. The coincidence theory proposed by Galilei [4], likely building on arguments by Benedetti and previous scholars [5], is in fact based on dyad's waveform periodicity [1]. These ideas triggered contributions and debates from both musicians and scientists deep into the 18th century [5], including, e.g. Euler [6] and Riccati [7]. In relation to the discoveries about higher harmonics pioneered by Mersenne et al., arguments about the role of the fundamental bass in music were formulated by Rameau [8] and Tartini [9]; Estève [10] related dyad's consonance to the largest presence of common harmonics, as did Pizzati [11], who further discussed the issue in relation to the fundamental bass [12]. The main criticism against these periodicity and harmonicity theories was the fact that the associated C and D indicators are discontinuous functions of the frequency ratios characterizing dyads and chords. The first pioneering tentatives to obtain an experimentally-based continuous C and D function where carried out by Foderà [13] in the first half of the 19th century. A different approach to the field appeared in the fall of the 19th century, when Helmholtz [14] suggested C and D to be related to the absence of the roughness sensation due to beats; the associated C and D indicator being naturally continuous, this approach dominated until the fall of the 20th century and was further refined by Plomp and Levelt [15] and Hutkinson and Knopoff [16, 17]. More recently, the drawbacks of the roughness approach [18] stimulated a re-evaluation of the periodicity and harmonicity approaches, as done, e.g. by Tenney [19] and Gill and Purves [20].

Summarizing, the scientific literature developed two main categories of explanations for C and D, that we denote here for short by "roughness" and "compactness", the latter including the two sub-categories of periodicity and harmonicity. For an extensive review of recent models, see for instance Ref. [21]. As already mentioned, the compactness and roughness approaches have traditionally been considered to be alternative and somewhat competing. However, none of their representative models emerged as a fully satisfactory explanation of the perceptual data about C and D.

Focusing on dyads, in Ref. [1], we: (1) considered various historical and recent indicators, establishing whether (or not) they have some physical foundation, related to the periodicity or harmonicity approaches; (2) showed that periodicity and harmonicity

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[1] C and D are here defined according to their literal meaning, that is whether two sounds are perceived to blend well or not. In Latin, the prefixes *con* and *dis* stand for unity and division, respectively.

[2] For other recent approaches to music based on physical concepts, see, e.g. Refs. [2, 3].

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indicators are essentially equivalent; (3) proposed a method to extend compactness models to the continuum; and (4) carried out a complete analysis to assess whether a single refined model among the two categories of compactness and roughness, or rather a combination of them, provides a satisfactory explanation of the perceptual data. We found that combined models, with similar weight attributed to compactness and roughness, are highly successful when confronted to the perceptual observations, and perform significantly better than the constituent models: This nontrivial result demonstrates that compactness and roughness are both fundamental ingredients of an effective explanation of C and D [1].

The goal of the present work is to extend compactness models from dyads to triads. We first find a generalization to triads for each compactness model studied in Ref. [1]. This requires to study mathematically the fundamental bass, that is the virtual frequency with period equal to that of the wave function obtained by superposing the triad's constituent wave functions. This virtual frequency is nevertheless "real" and extremely important from the psychoacoustic point of view. The predictions of the compactness models have then to be confronted with the perceptual data. In particular, we exploit the results of a psychoacoustic test performed in Bowling et al. [22], where just intonation was used to generate triadic sounds within an octave. In order to numerically estimate the performance of each compactness model, we carry out a Chi-square analysis. The C and D indicators associated to compactness models being intrinsically discontinuous, we develop a procedure to extend them to the continuum in the case of triads. In particular, following the approach of Ref. [1], we exploit a Gaussian smoothing with standard deviation equal to the discrimination limen of the hearing system. This allows also to estimate C and D for other temperaments, like for instance the equal temperament.

The paper is organized as follows. In Sect. 2, we comment on the methodology adopted by Bowling et al. [22] to carry out their test and discuss their results. Section 3 is devoted to the mathematical study of the fundamental bass. In Sect. 4, we consider compactness models, of the periodicity and harmonicity type, showing that they are essentially equivalent. In Sect. 5, we compare compactness model predictions with perceptual data by means of a Chi-square analysis. In Sect. 6, the compactness indicators are extended to the continuum, and comments on the equal temperament are provided. Our comments and conclusions are discussed in Sect. 7. App. A is devoted to the Euler's indicator [6].

2 Previous tests

There are not many psychoacoustic tests available for triads. An interesting one has been performed by Rasmussen et al. [18] in 2014, with the goal of comparing the C and D rankings of dyads and triads. Overall, the latter were not found to be more dissonant than dyads. The authors thus suggested that the hypothesis according to which consonance decreases with the amount of interaction between present harmonics might not hold for chords. In other words, the roughness approach cannot account for the observed results: This supports the relevance of the compactness approach. Unfortunately, Ref. [18] considers only a small set of triads, so it is not suitable for the sake of our analysis.

A relevant test for the sake of our discussion has been carried out by Bowling et al. [22] in 2018. They tested all 12 dyads, 66 triads, and 220 tetrads that can be formed using the intervals specified by the chromatic scale over one octave, with just intonation ratios. Individual tones were created using a synthesized piano. Participants were instructed about the test, defining consonance as the musical pleasantness or attractiveness of a sound, and the opposite for dissonance. On each trial of the experiment, the participants heard a single chord and provided a rating of consonance/dissonance using a four-point scale: "quite consonant"; "moderately consonant"; "moderately dissonant"; and "quite dissonant".

The results of Ref. [22] for the 66 triads, normalized to a scale from 0 to 1 in order of increasing consonance, are reproduced in Table 1. The first column displays the number assigned to each chord, according to an organization criterion linked to the pitch class concept. The pitch class of a frequency f_i takes values in the interval $[0, 12)$ and is given by $p_{c_i} = [9 + 12 \log_2(f_i/440\,\text{Hz})]\mod 12$. Introducing the three frequencies $f_1 < f_2 < f_3$, a triad can be characterized by a pitch class set, $(p_{c_1}, p_{c_2}, p_{c_3})$, as shown in the second column[3] of Table 1. In this way, for instance, all C's and G's have pitch class 0 and 7, respectively. For an easier understanding of the type of chord, we display in the the third column the notes associated to the pitch class set.

In the fourth column of Table 1, we include, when possible, the name of the triad according to the common denomination used in music theory. From the music theory point of view, the notation should describe the constituents intervals. Indeed, triads are denoted in reference to the type of interval formed by the dyad f_1 and f_2, and the one formed by the dyad f_1 and f_3, as shown in the fifth column. For instance, if such intervals are a M3 and a P5, the chord is called a major chord in the rest position, and it is denoted by maj (r); if such intervals are instead a m3 and a P5, the chord is called a minor chord in the rest position, and it is denoted by min (r). Diminished and augmented chords at rest are characterized, respectively, by m3 and TT and by M3 and m6. In addition, chords can be in one among three possible positions: rest (r), first inversion (1), and second inversion (2). A relevant category of triads is the so-called power chords, where f_3 is fixed to be the octave of f_1, that is $f_3/f_1 = 2$; the intermediate frequency f_2 thus fully characterizes power chords. We denote them by "pc X", where X stands for the interval formed by f_2 and f_1, for instance X=P5, P4, M3.

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[3] Bowling et al. [22] adopted this notation, even though its use in a context dealing with just intervals is a bit misleading as, strictly speaking, such notation only applies to equal temperament (see, e.g. [23] and references therein); for this reason, we will not use it in the following sections.

Table 1 Triad ratings according to Bowling et al. [22], rescaled to the range [0, 1]

Num	Pitch class	Notes	Name	Intervals	$(f_2/f_1, f_3/f_1)$	Mean	σ	n, m, ℓ
1	(0,1,2)	$C - C\sharp - D$		m2, M2	(16/15, 9/8)	0.022	0.084	120,128,135
2	(0,1,3)	$C - C\sharp - D\sharp$		m2, m3	(16/15, 6/5)	0.222	0.253	15,16,18
3	(0,1,4)	$C - C\sharp - E$		m2, M3	(16/15, 5/4)	0.167	0.191	60,64,75
4	(0,1,5)	$C - C\sharp - F.$		m2, P4	(16/15, 4/3)	0.367	0.220	15,16,20
5	(0,1,6)	$C - C\sharp - F\sharp$		m2, TT	(16/15, 7/5)	0.267	0.203	15,16,21
6	(0,1,7)	$C - C\sharp - G$		m2, P5	(16/15, 3/2)	0.133	0.167	30,32,45
7	(0,1,8)	$C - C\sharp - G\sharp$		m2, m6	(16/15, 8/5)	0.156	0.169	15,16,24
8	(0,1,9)	$C - C\sharp - A$		m2, M6	(16/15, 5/3)	0.089	0.150	15,16,25
9	(0,1,10)	$C - C\sharp - A\sharp$		m2, m7	(16/15, 9/5)	0.056	0.126	15,16,27
10	(0,1,11)	$C - C\sharp - B$		m2, M7	(16/15, 15/8)	0.067	0.161	120,128,225
11	(0,1,12)	$C - C\sharp - C$		m2, P8	(16/15, 2/1).	0.156	0.190	15,16,30
12	(0,2,3)	$C - D - D\sharp$		M2, m3	(9/8, 6/5)	0.200	0.241	40,45,48
13	(0,2,4)	$C - D - E$		M2, M3	(9/8, 5/4)	0.533	0.271	8,9,10
14	(0,2,5)	$C - D - F$		M2, P4	(9/8, 4/3)	0.567	0.265	24,27,32
15	(0,2,6)	$C - D - F\sharp$		M2, TT	(9/8, 7/5)	0.456	0.205	40,45,56
16	(0,2,7)	$C - D - G$	sus 2	M2, P5	(9/8, 3/2)	0.644	0.230	8,9,12
17	(0,2,8)	$C - D - G\sharp$		M2, m6	(9/8, 8/5)	0.256	0.226	40,45,64
18	(0,2,9)	$C - D - A$		M2, M6	(9/8, 5/3)	0.300	0.268	24,27,40
19	(0,2,10)	$C - D - A\sharp$		M2, m7	(9/8, 9/5)	0.267	0.203	40,45,72
20	(0,2,11)	$C - D - B$		M2, M7	(9/8, 15/8)	0.344	0.270	8,9,15
21	(0,2,12)	$C - D - C$		M2, P8	(9/8, 2/1)	0.400	0.238	8,9,16
22	(0,3,4)	$C - D\sharp - E$		m3, M3	(6/5, 5/4)	0.067	0.136	20,24,25
23	(0,3,5)	$C - D\sharp - F$		m3, P4	(6/5, 4/3)	0.467	0.241	15,18,20
24	(0,3,6)	$C - D\sharp - F\sharp$	dim (r)	m3, TT	(6/5, 7/5)	0.456	0.239	5,6,7
25	(0,3,7)	$C - D\sharp - G$	min (r)	m3, P5	(6/5, 3/2)	0.800	0.225	10,12,15
26	(0,3,8)	$C - D\sharp - G\sharp$	maj (1)	m3, m6	(6/5, 8/5)	0.878	0.205	5,6,8
27	(0,3,9)	$C - D\sharp - A$		m3, M6	(6/5, 5/3)	0.411	0.226	15,18,25
28	(0,3,10)	$C - D\sharp - A\sharp$	dim (1)	m3, m7	(6/5, 9/5)	0.511	0.243	5,6,9
29	(0,3,11)	$C - D\sharp - B$		m3, M7	(6/5, 15/8)	0.178	0.190	40,48,75
30	(0,3,12)	$C - D\sharp - C$	pc m3	m3, P8	(6/5, 2/1)	0.600	0.308	5,6,10
31	(0,4,5)	$C - E - F$		M3, P4	(5/4, 4/3)	0.378	0.259	12,15,16
32	(0,4,6)	$C - E - F\sharp$		M3, TT	(5/4, 7/5)	0.300	0.182	20,25,28
33	(0,4,7)	$C - E - G$	maj (r)	M3, P5	(5/4, 3/2)	0.933	0.136	4,5,6
34	(0,4,8)	$C - E - G\sharp$	aug (r)	M3, m6	(5/4, 8/5)	0.311	0.213	20,25,32
35	(0,4,9)	$C - E - A$	min (1)	M3, M6	(5/4, 5/3)	0.678	0.270	12,15,20
36	(0,4,10)	$C - E - A\sharp$		M3, m7	(5/4, 9/5)	0.478	0.258	20,25,36
37	(0,4,11)	$C - E - B$		M3, M7	(5/4, 15/8)	0.533	0.207	8,10,15
38	(0,4,12)	$C - E - C$	pc M3	M3, P8	(5/4, 2/1)	0.844	0.190	4,5,8
39	(0,5,6)	$C - F - F\sharp$		P4, TT	(4/3, 7/5)	0.122	0.223	15,20,21
40	(0,5,7)	$C - F - G$	sus 4	P4, P5	(4/3, 3/2)	0.711	0.273	6,8,9
41	(0,5,8)	$C - F - G\sharp$	min (2)	P4, m6	(4/3, 8/5)	0.600	0.221	15,20,24
42	(0,5,9)	$C - F - A$	maj (2)	P4, M6	(4/3, 5/3)	0.878	0.255	3,4,5
43	(0,5,10)	$C - F - A\sharp$		P4, m7	(4/3, 9/5)	0.456	0.223	15,20,27
44	(0,5,11)	$C - F - B$		P4, M7	(4/3, 15/8)	0.433	0.217	24,32,45
45	(0,5,12)	$C - F - C$	pc P4	P4, P8	(4/3, 2/1)	0.889	0.160	3,4,6
46	(0,6,7)	$C - F\sharp - G$		TT, P5	(7/5, 3/2)	0.222	0.202	10,14,15
47	(0,6,8)	$C - F\sharp - G\sharp$		TT, m6	(7/5, 8/5)	0.544	0.223	5,7,8
48	(0,6,9)	$C - F\sharp - A$	dim (2)	TT, M6	(7/5, 5/3)	0.411	0.209	15,21,25
49	(0,6,10)	$C - F\sharp - A\sharp$		TT, m7	(7/5, 9/5)	0.311	0.276	5,7,9
50	(0,6,11)	$C - F\sharp - B$		TT, M7	(7/5, 15/8)	0.300	0.237	40,56,75
51	(0,6,12)	$C - F\sharp - C$		TT, P8	(7/5, 2/1)	0.456	0.283	5,7,10
52	(0,7,8)	$C - G - G\sharp$		P5, m6	(3/2, 8/5)	0.278	0.177	10,15,16
53	(0,7,9)	$C - G - A$		P5, M6	(3/2, 5/3)	0.500	0.273	6,9,10

Table 1 continued

Num	Pitch class	Notes	Name	Intervals	$(f_2/f_1,\ f_3/f_1)$	Mean	σ	n, m, ℓ
54	(0,7,10)	$C - G - A\sharp$		P5, m7	(3/2, 9/5)	0.389	0.304	10,15,18
55	(0,7,11)	$C - G - B$		P5, M7	(3/2, 15/8)	0.422	0.276	8,12,15
56	(0,7,12)	$C - G - C$	pc P5	P5, P8	(3/2, 2/1)	0.944	0.126	2,3,4
57	(0,8,9)	$C - G\sharp - A$		m6, M6	(8/5, 5/3)	0.067	0.136	15,24,25
58	(0,8,10)	$C - G\sharp - A\sharp$		m6, m7	(8/5, 9/5)	0.322	0.239	5,8,9
59	(0,8,11)	$C - G\sharp - B$		m6, M7	(8/5, 15/8)	0.211	0.185	40,64,75
60	(0,8,12)	$C - G\sharp - C$	pc m6	m6, P8	(8/5, 2/1)	0.733	0.238	5,8,10
61	(0,9,10)	$C - A - A\sharp$		M6, m7	(5/3, 9/5)	0.078	0.143	15,25,27
62	(0,9,11)	$C - A - B$		M6, M7	(5/3, 15/8)	0.178	0.169	24,40,45
63	(0,9,12)	$C - A - C$	pc M6	M6, P8	(5/3, 2/1)	0.678	0.297	3,5,6
64	(0,10,11)	$C - A\sharp - B$		m7, M7	(9/5, 15/8)	0.067	0.184	40,72,75
65	(0,10,12)	$C - A\sharp - C$		m7, P8	(9/5, 2/1)	0.267	0.221	5,9,10
66	(0,11,12)	$C - B - C$		M7, P8	(15/8, 2/1)	0.067	0.136	8,15,16

The last column shows the values of (n, m, ℓ), according to Eq. (4)
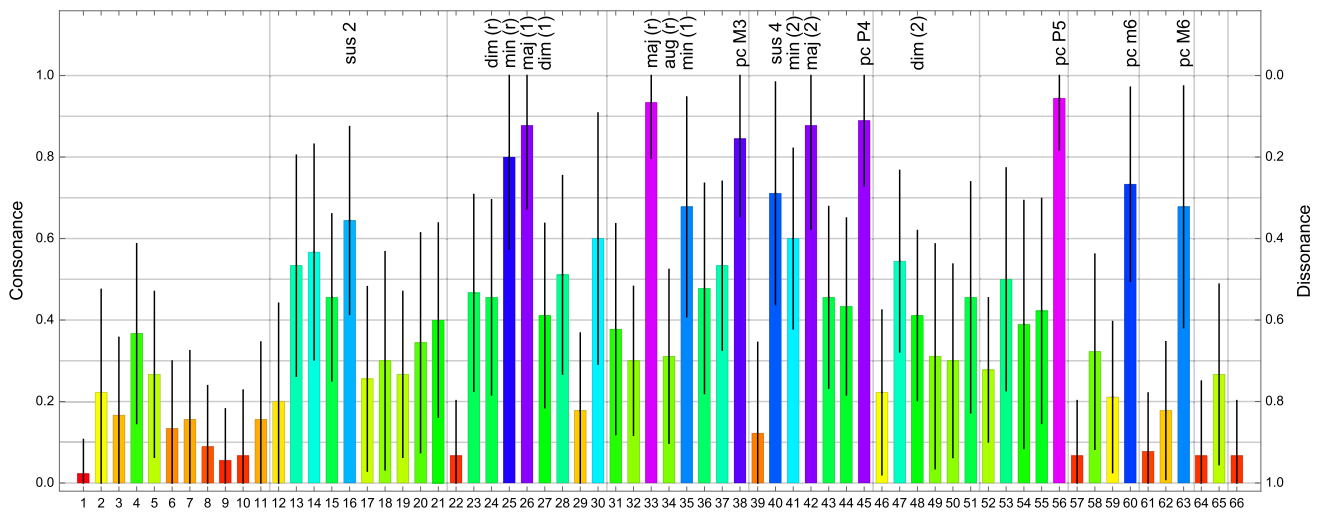
Fig. 1 Bowling et al. [22] results, rescaled to the C and D range [0, 1]

From the physical point of view, it is useful to characterize a triad by the pair of frequency ratios (f_2/f_1, f_3/f_1), written in simple form (that is without common factors between numerator and denominator), as shown in the sixth column of Table 1; the third ratio f_3/f_2 can thus be easily determined. The mean value and the standard deviation σ of the score assigned to each triad [22] (upon rescaling the results to the range [0, 1]) are reported in the seventh and eight columns. For an easy visual comparison, these normalized results are reproduced in Fig. 1.

The two best performing triads are pc P5 and maj (r), with a mean consonance value larger than 0.9. This is not surprising: Only the consonant P5 interval is involved, together with the M3 for the latter. The third best performing triad is pc P4; the fourth best performing are maj (1) and maj (2), with the same score. Then, in decreasing order of consonance, we find: pc M3, min (r), pc m6, sus 4, pc M6, min (1), sus 2, min (2), triad n.30, etc. As for the most dissonant triads, in order of increasing consonance, we find: triad n.1 (or cluster chord); triad n.9; and triads n.10, 22, 57, 65, and 66, all with the same score.

The results of the test by Bowling et al. [22] are globally reasonable, but there are some critical points. It is generally agreed by musicians that maj (2) is more consonant than maj (r); the opposite result obtained by the participants to the test might be due to a cultural bias, maj (r) being very popular in Western music. The same cultural bias might apply to min (r), which also gets a high score from the test; musicians would for instance typically consider sus 4 and pc m6 to be more consonant than min (r).

3 The fundamental bass

For dyads, compactness models and their generalization to continuum were studied in detail in Ref. [1], stressing the relevance of the fundamental frequency f_0 of the sound wave obtained by summing the two sound waves having (fundamental) frequencies f_1

and f_2. This frequency is important for both periodicity and harmonicity models. It is referred to in many ways, including *missing fundamental* or *virtual pitch*.

It is indeed a virtual frequency, in the sense, that it is not actually present in the spectrum of the dyad; nevertheless, it plays an important role in psychoacoustics, as the hearing system is able to reconstruct it (see, e.g. [24] and references therein). It is by now accepted that the brain processes the information present in a set of tones to calculate the associated fundamental frequency f_0; the precise way in which it does so is still a matter of debate, but the processing seems to be based on an autocorrelation involving the timing of neural impulses in the auditory nerve [25]. In music theory, the relevance of this fundamental frequency f_0 is also largely recognized: It corresponds to the frequency of the fundamental bass which, as well known, plays a crucial role in the characterization of harmonic structures [8, 9]. In the following, we will adopt the music theory denomination.

Since our aim is to generalize compactness models to triads, which are composed by three tones with frequencies $f_1 < f_2 < f_3$, we first calculate the fundamental bass of such triad. By definition, this corresponds the fundamental frequency of the sound wave obtained by summing the three sound waves having (fundamental) frequencies f_1, f_2, and f_3. We fix f_1 to some frequency and define $f_2/f_1 = M_{12}/N_{12}$ and $f_3/f_1 = M_{13}/N_{13}$, where M_{12}, N_{12} and M_{13}, N_{13} are integers. For $ij = 12, 13$ (but not for $ij = 23$), we introduce the integers

$$m_{ij} = \frac{M_{ij}}{\mathrm{GCD}[N_{ij}, M_{ij}]}, \quad n_{ij} = \frac{N_{ij}}{\mathrm{GCD}[N_{ij}, M_{ij}]}, \quad \text{for } ij = 12, 13. \tag{1}$$

With this notation, $f_2/f_1 = m_{12}/n_{12}$ and $f_3/f_1 = m_{13}/n_{13}$ are written in the simplest form, namely with numerator and denominator having no prime factors in common.

As for the ratio f_3/f_2, since it can be written as the product of f_3/f_1 and f_1/f_2, one would obtain $f_3/f_2 = m_{13}n_{12}/(n_{13}m_{12})$, which in general is not in its simplest form. Nevertheless, it can be written in its simplest form by introducing the simple ratio $m_{23}/n_{23} = f_3/f_2$, where

$$m_{23} = \frac{m_{13}\, n_{12}}{\mathrm{GCD}[n_{13}\, m_{12}, m_{13}\, n_{12}]}, \quad n_{23} = \frac{n_{13}\, m_{12}}{\mathrm{GCD}[n_{13}\, m_{12}, m_{13}\, n_{12}]}. \tag{2}$$

The fundamental bass of the dyad with frequency ratio f_j/f_i, with $ij = 12, 13$ and 23, has frequency given by:

$$f_0^{(12)} = \frac{1}{n_{12}}\, f_1 = \frac{1}{m_{12}}\, f_2, \quad f_0^{(13)} = \frac{1}{n_{13}}\, f_1 = \frac{1}{m_{13}}\, f_3, \quad f_0^{(23)} = \frac{1}{n_{23}}\, f_1 = \frac{1}{m_{23}}\, f_3. \tag{3}$$

The fundamental bass of these three fundamental basses can be calculated as follows. Defining

$$n = \frac{n_{12}\, n_{13}}{\mathrm{GCD}[n_{12}, n_{13}]}, \quad m = n\frac{m_{12}}{n_{12}} = \frac{m_{12}\, n_{13}}{\mathrm{GCD}[n_{12}, n_{13}]}, \quad \ell = n\frac{m_{13}}{n_{13}} = \frac{n_{12}\, m_{13}}{\mathrm{GCD}[n_{12}, n_{13}]}, \tag{4}$$

the fundamental bass of the three (fundamental) frequencies constituting the triad is

$$f_0 = \frac{1}{n}\, f_1 = \frac{1}{m}\, f_2 = \frac{1}{\ell}\, f_3. \tag{5}$$

As already stressed, it is a virtual sound, not present in the spectrum of the triad, but it is of remarkable importance for the hearing system and for music theory. So, let us focus on the harmonic spectrum of a tone with (fundamental) frequency f_0 and denote it by $\{n_0 f_0\}$ with $n_0 = 1, 2, 3, \ldots$. Clearly, the constituent (fundamental) tones of the triad discussed previously correspond to the harmonic numbers $n_0 = (n, m, \ell)$ of the harmonic spectrum of f_0. This offers another useful notation to denote a chord; in the last column of Table 1, we show the values of (n, m, ℓ) for each of the 66 triads that can be built within the octave, using a just intonation scale.

The highness of the fundamental bass has been itself interpreted as a C and D indicator [11, 12]: The more (n, m, l) are small, the more the three frequencies belong to the low harmonics of the fundamental bass, and the more there is consonance.

In particular, the knowledge of n for each of the 66 triads of Table 1 allows for an easy calculation of f_0, according to Eq. (5). In Fig. 2, we show a graphical representation of the magnitude of f_0, taking f_1 fixed, say for instance to C_4; a larger ellipse is associated to a higher value of f_0.

From Table 1 and Fig. 2, we can see that, for triads within the octave, the highest value of the fundamental bass occurs for pc P5, and it is one octave below f_1; indeed, in this case $n = 2$, so that $f_0 = f_1/2 = C_3$. The second highest value of the fundamental bass happens when $n = 3$ and corresponds to $f_0 = f_1/3 = F_2$: This is the case for pc P4, pc M6, and maj (2). The third highest fundamental bass is lower than f_1 by two octaves: $n = 4$ for both maj (r) and cp M3, so that $f_0 = f_1/4 = C_2$. This is in agreement with musicians common experience that maj (2) is more consonant than maj (r)—although the results of Bowling et al. [22] test showed the opposite. The fourth highest fundamental bass is $f_0 = f_1/5$, which is obtained by dim (r), maj (1), and dim (1). The other triads have a lower f_0. For instance, sus 4 has $n = 6$, while min (r) has $n = 10$. The cluster chord n.1 and the triad n.10 are the chords with the lowest fundamental bass, having $n = 120$.

This shows that the height of the fundamental bass is indeed an efficient C and D indicator. It has, however, some weak points: dim (r) has higher fundamental bass than min (r), but the common experience by musicians is that dim (r) is more dissonant than min (r)—in agreement with Bowling et al. [22] test. It is reasonable to expect that C and D indicators including the effect of roughness should be better suited to explain this feature.

Fig. 2 Visual representation of the highness of the fundamental bass f_0, for the 66 just intonation triads within one octave, taking f_1 fixed (say to C_4). A larger ellipse and colder colors are associated to triads with increasingly higher values of f_0
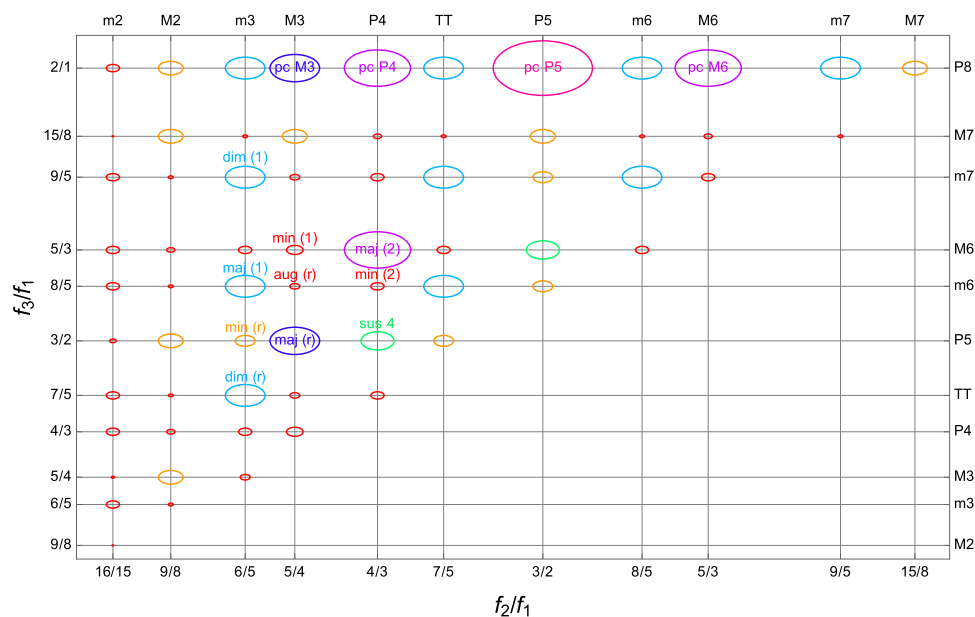

Table 2 Examples of triads beyond the octave having $f_0 = f_1$

Notes	Intervals	$(f_2/f_1,\ f_3/f_1)$	n, m, l
$C_4 - C_5 - G_5$	P8,cP5	(2/1, 3/1)	1,2,3
$C_4 - C_5 - C_6$	P8,cP8	(2/1, 4/1)	1,2,4
$C_4 - G_5 - C_6$	cP5,cP8	(3/1, 4/1)	1,3,4

It is worth to consider which triads have the fundamental bass as high as possibile, that is $f_0 = f_1$; this requires $n = 1$, while m and ℓ are not constrained. We call them super harmonic chords, as f_2 and f_3 belong to the harmonic series of f_1. Necessarily, f_3 is in the second octave, so they are compound chords, which have not been tested by Bowling et al. [22]. Table 2 displays three such super harmonic chords, those with (n, m, ℓ) equal to $(1, 2, 3)$, $(1, 2, 4)$, and $(1, 3, 4)$.

Notice also that, in general, the ratio f_3/f_2, even in its simple form, might involve larger integers in the numerator and denominator with respect to those characterizing f_2/f_1 and f_3/f_1 in the just intonation scale. This is for instance the case for the triad n.1, for which $f_3/f_2 = (3^4 \times 5)/2^7$. In general, this happens for triads having a low fundamental bass. On the contrary, triads with a high fundamental bass typically display small integers in the numerator and denominator of f_3/f_2. This for instance the case for maj (2), which has $f_3/f_2 = 5/4$.

4 Compactness models

There are two categories of compactness models, those based on periodicity and those on harmonicity. In Ref. [1], we showed explicitly for dyads that they are associated to indicators that are numerically much similar. Indeed, the criterion of having the period of the dyadic sound wave as short as possible and the criterion of having the highest possible number of common harmonics in the constituent dyad tones are practically equivalent. We expect that the same holds in the case of triads.

4.1 Periodicity indicators

We start by analysing periodicity models for triads. According to the periodicity approach, the more the period of the triadic sound wave is short with respect to the period of its component tones, the more the triad is consonant. So, more consonant chords should be those with the highest value of f_0 with respect to the chord's frequencies.

Comparing the fundamental bass f_0 with the lowest/middle/highest sound of the chord gives the following three periodicity consonance indicators

$$I_1^P = \frac{f_0}{f_1} = \frac{1}{n}, \quad I_2^P = \frac{f_0}{f_2} = \frac{1}{m}, \quad I_3^P = \frac{f_0}{f_3} = \frac{1}{\ell}. \tag{6}$$

These indicators span a smaller range than the interval $[0, 1]$ for the 66 triads of Table 1; in the following, we will discuss how to normalize them. Notice that I_1^P corresponds to using just the fundamental bass as an indicator, since it compares f_0 with the

lowest tone of the triad f_1, taken to be fixed; the dependence on the values of f_2 and f_3 is then fully encoded in the value of f_0. The indicators I_2^P and I_3^P have instead a stronger dependence on the fact that the middle and highest tone of the triad should be close to f_0.

The indicator I_3^P is particularly interesting. It compares the period of the fundamental bass with the period of the highest frequency of the triad: The more ℓ is small, the more the triad is compact from the periodicity point of view. This indicator is somewhat related to an extension to triads of Galilei's arguments [4] about dyads coincidence theory.[4]

Comparing the fundamental bass frequency f_0 with some mean value of the triad frequencies might be an interesting option to find more sophisticated periodicity indicators. We define the arithmetic, geometric, and harmonic means

$$f_A = \frac{f_1 + f_2 + f_3}{3}, \quad f_G = (f_1 f_2 f_3)^{1/3}, \quad f_H = \frac{3}{\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}}, \tag{7}$$

so that

$$I_A^P = \frac{f_0}{f_A} = \frac{3}{n + m + \ell}, \quad I_G^P = \frac{f_0}{f_G} = \frac{1}{(n\,m\,\ell)^{1/3}}, \quad I_H^P = \frac{f_0}{f_H} = \frac{m\,\ell + n\,\ell + n\,m}{3\,n\,m\,\ell}. \tag{8}$$

These indicators are expected to give predictions in between I_1^P and I_3^P, and they correspond to the following combinations of the three indicators above,

$$I_A^P = \frac{3}{\frac{1}{I_1^P} + \frac{1}{I_2^P} + \frac{1}{I_3^P}}, \quad I_G^P = (I_1^P I_2^P I_3^P)^{1/3}, \quad I_H^P = \frac{1}{3}(I_1^P + I_2^P + I_3^P). \tag{9}$$

The indicator I_H^P can be seen as an extension to triads of the dyadic consonance indicator suggested, on the basis of very different arguments, in Ref. [26].

More in general, any non-decreasing monotonic function of the three indicators I_i^P ($i = 1, 2, 3$) can be used to define a consonance indicator somewhat related to periodicity. For instance, even the function $\ln I_3^P = -\ln \ell$ can be viewed as a consonance indicator related to periodicity.

Equivalently, any non-decreasing monotonic function of the three numbers n, m, ℓ can be used to define a dissonance indicator somewhat related to periodicity. For instance, the product nm was suggested by G.B. Benedetti as an estimator for dyads rankings [5]. Of course, not all such guessed indicators turn out to explain perceptual results in a satisfactory way. This is for instance the case for the indicator proposed by L. Euler [6] and discussed in app. A for historical completeness.

4.2 Harmonicity indicators

According to the harmonicity approach, more harmonics the tones of a chord have in common, more consonance is achieved. We showed in Ref. [1] that harmonicity models for dyads are essentially equivalent to periodicity ones. In order to generalize to triads the harmonicity models defined for dyads, we first introduce a few definitions.

The first (triple) coincidence of the harmonic sounds of a triad happens at

$$n_1^{c_1} f_1 = n_2^{c_1} f_2 = n_3^{c_1} f_3, \tag{10}$$

where $n_i^{c_1}$, $i = 1, 2, 3$, are integer numbers, with $n_3^{c_1} \geq 1$ (the equality is included in order to include possible coincidences with the fundamental of f_3). Hence, we have that the ratios can be expressed as

$$\frac{f_2}{f_1} = \frac{m_{12}}{n_{12}} = \frac{n_1^{c_1}}{n_2^{c_1}}, \quad \frac{f_3}{f_1} = \frac{m_{13}}{n_{13}} = \frac{n_1^{c_1}}{n_3^{c_1}}, \quad \frac{f_3}{f_2} = \frac{m_{23}}{n_{23}} = \frac{n_2^{c_1}}{n_3^{c_1}}, \tag{11}$$

where the ratios of the type m_{ij}/n_{ij} are in simple form, while those of the form $n_i^{c_1}/n_j^{c_1}$ are in general not in simple form.

We define the integers α_{12}, α_{13} and α_{23} to be precisely those integers responsible for the non-simple form of $n_i^{c_1}/n_j^{c_1}$, that is

$$n_1^{c_1} = \alpha_{12} m_{12} = \alpha_{13} m_{13}, \quad n_2^{c_1} = \alpha_{12} n_{12} = \alpha_{23} m_{23}, \quad n_3^{c_1} = \alpha_{13} n_{13} = \alpha_{23} n_{23}. \tag{12}$$

Hence, it turns out that

$$\alpha_{12} = \frac{m_{13}}{\text{GCD}[m_{12}, m_{13}]}, \quad \alpha_{13} = \frac{m_{12}}{\text{GCD}[m_{12}, m_{13}]}, \quad \alpha_{23} = \alpha_{12} \frac{n_{12}}{m_{23}}, \tag{13}$$

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[4] This indicator corresponds to the fraction of "concordant pulses", discussed by G. Galilei in Ref. [4]. During the time interval (equal to the period of the fundamental bass) in which the lowest string (with frequency f_1) gives n vibrations, the highest string (with frequency f_3) gives ℓ vibrations. So, numbering the vibrations of the sharper string during this time interval, it happens that just $1/\ell$ of them agree to strike simultaneously with the lowest string (i.e. $\ell - 1$ single pulses intervene between each pair of concordant pulses). We thus name after Galilei's the coincidence indicator $1/\ell$, even though he never suggested this identification explicitly.

so that the integers $n_i^{c_1}$ are given by

$$n_1^{c_1} = \frac{m_{13}m_{12}}{\mathrm{GCD}[m_{12}, m_{13}]}, \quad n_2^{c_1} = \frac{m_{13}n_{12}}{\mathrm{GCD}[m_{12}, m_{13}]}, \quad n_3^{c_1} = \frac{m_{12}n_{13}}{\mathrm{GCD}[m_{12}, m_{13}]}. \tag{14}$$

Clearly, the smaller are the integers $n_i^{c_1}$, the larger is the number of coincidences in the lower harmonics of the triad spectrum (which are typically those with the highest amplitudes), and the larger is consonance. So, any non-decreasing function of the ratios $1/n_i^{c_1}$ can be used as a consonance indicator somewhat related to harmonicity, which turns out to be numerically similar to the periodicity indicators discussed previously.

A valuable argument to define a harmonicity indicator is to count the overlaps among the harmonic spectrum of the fundamental bass and each component of the triad. As already mentioned, the first coincidence is found at the frequency $n_1^{c_1} f_1$; this corresponds to the harmonic number

$$n_0^{c_1} = n\, n_1^{c_1} = n\, \frac{m_{13}m_{12}}{\mathrm{GCD}[m_{12}, m_{13}]} \tag{15}$$

in the harmonic series expansion of f_0. The common harmonics of f_0 and f_i, with $i = 1, 2, 3$, up to the first coincidence, are given by $n_i^{c_1}$. Hence, a simple harmonicity indicator corresponds to the ratio between the number of harmonics that f_0 shares with any harmonic of f_i, divided by the number of all harmonics of f_0, up to the first coincidence. We thus obtain

$$I_S^H = \frac{n_1^{c_1} + n_2^{c_1} + n_3^{c_1}}{n_0^{c_1}} = \frac{m_{13}m_{12} + m_{13}n_{12} + m_{12}n_{13}}{n\, m_{13}m_{12}}, \tag{16}$$

where twofold common harmonics are counted twice, and the first coincidence is counted three times.

If one wants to count just once the twofold and threefold coincidences, as suggested by Gill and Purves in Ref. [20], one can proceed as follows.[5] Up to the first coincidence, the harmonics in common between f_i and f_j are given by α_{ij}. Hence, double counting of harmonics is avoided by considering the ratio

$$I_{GP}^H = \frac{(n_1^{c_1} + n_2^{c_1} + n_3^{c_1}) - (\alpha_{12} + \alpha_{13} + \alpha_{23}) + 1}{n_0^{c_1}}. \tag{17}$$

Notice that for super harmonic chords $I_{GP}^H = 1$, while this is not the case for I_S^H, which is equal to 11/6, 7/4, and 19/12 for triads with $(f_2/f_1, f_3/f_1) = (2/1, 3/1)$, $(2/1, 4/1)$ and $(3/1, 4/1)$, respectively. We now turn to discuss a proper normalization of the compactness indicators.

5 Comparison with perceptual data

In order to compare effectively Bowling et al. [22] results for triads within one octave with the predictions of the periodicity and harmonicity indicators previously introduced, we have to normalize the latter so that they all predict maximal consonance (that is 1) for pc P5. So, we introduce normalized consonance indicators as

$$\tilde{I}_X^C = \frac{I_X^C}{I_X^C|_{pcP5}}, X = 1, 2, 3, A, G, H, S, GP, \tag{18}$$

where $C = P, H$ stands for compactness, while P and H stand for the sub-categories of periodicity and harmonicity.

In Fig. 3, Bowling et al. [22] results (red) are compared with the predictions of all the periodicity and harmonicity indicators previously introduced. In general, the predictions of compactness models agree with perceptual data within one standard deviation. The error bars are, however, quite large. Notice also that periodicity and harmonicity models have quite similar predictions (the same happens for dyads [1]): This shows that the compactness of the waveform period and the compactness of the harmonic spectrum are practically equivalent requirements. Inside each sub-category, one can recognize that there are small differences between the models. For instance, the Galileo's inspired indicator $\tilde{I}_3^P$ is slightly larger than $\tilde{I}_1^P$; the indicators $\tilde{I}_{A,G,H}^P$ instead display negligible differences among them; the harmonicity indicator $\tilde{I}_{GP}^H$ is slightly larger than $\tilde{I}_3^P$.

From Fig. 3, we can see that the most consonant triads according to the compactness models, do get a high score in the perceptual test. However, a high test score is also associated to some triads that should be not so consonant according to the models: This is the case in particular for maj (r), maj (1), and for all minor chords, for which the test scores are much above the model expectations. As mentioned, this might be due to cultural familiarity, whose effect is possibly enhanced by the choice of the piano timbre in the test by Bowling et al. [22]. It would be interesting to perform a perceptual test with tones generated with a "neutral" timbre, as done in our previous work [1], to check whether such difference persists; it would also be interesting to include triads beyond one octave. Notice also that not only $\tilde{I}_1^P$, but all compactness models predict dim (r) to be more consonant than min (r); this unsatisfactory prediction should disappear by including the effect of roughness.

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[5] Actually, Gill and Purves did not provide a formula in Ref. [20]. It is possible that they just did the calculation numerically; anyway, our formula agrees with the values of their indicator.
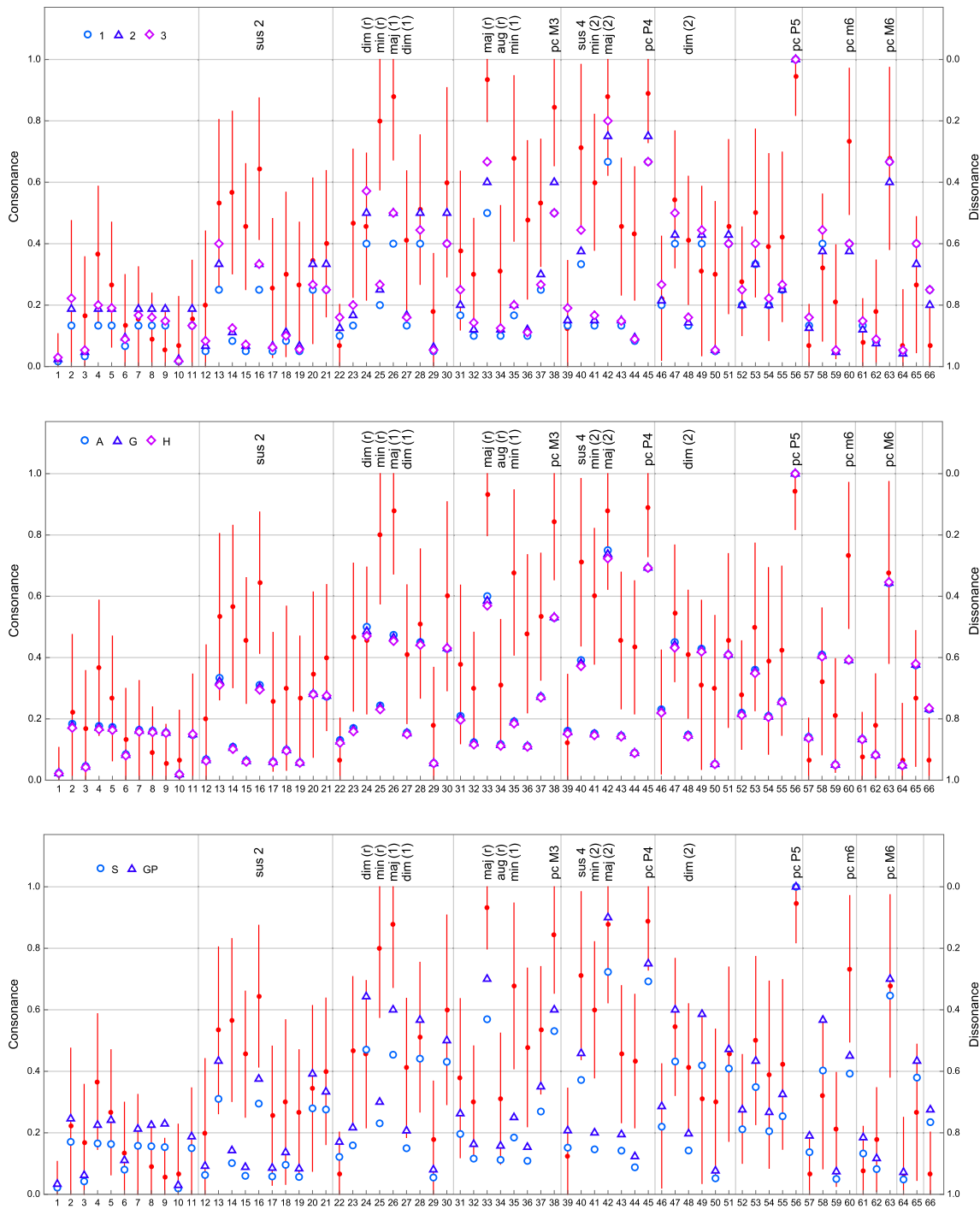
Fig. 3 Bowling et al. [22] results (red) compared with the predictions of $\tilde{I}^P_{1,2,3}$, $\tilde{I}^P_{A,G,H}$, and $\tilde{I}^H_{S,GP}$, in the top, middle, and bottom panels, respectively

Using the results at our disposal now, we perform a reduced Chi-square analysis to compare periodicity and harmonicity models in a more quantitative way, for triads within one octave:

$$\tilde{\chi}^2 = \frac{1}{66} \sum_{n=1,\ldots,66} \frac{(\tilde{I}^C_X(n) - m(n))^2}{\sigma(n)^2},\tag{19}$$

where $m(n)$ and $\sigma(n)$ are Bowling et al. [22] means and standard deviations, displayed in Table 1. The results of the calculations are summarized in Table 3: For all models, a reduced Chi-square around unity is achieved, which means that compactness models reproduce experimental data in a satisfactory way. The differences in the reduced Chi-square among the compactness models are

Table 3 Reduced Chi-square for indicated periodicity and harmonicity models, calculated by using the 66 triads within one octave according to Bowling et al. [22]

	$\tilde{I}_1^P$	$\tilde{I}_2^P$	$\tilde{I}_3^P$	$\tilde{I}_A^P$	$\tilde{I}_G^P$	$\tilde{I}_H^P$	$\tilde{I}_S^H$	$\tilde{I}_{GP}^H$
$\tilde{\chi}^2$	1.26	0.97	0.96	1.03	1.05	1.08	1.08	0.81

not very significant: This is also related to the fact that the standard deviations of the perceptual test of Ref. [22] are quite large. We can see that the best periodicity model is $\tilde{I}_3^P$, the Galilei's inspired model. The smallest reduced Chi-square among all compactness models is, however, obtained by $\tilde{I}_{GP}^H$, as was the case also for dyads [1]: This allows to conclude that the extension from dyads to triads provides consistent results.

6 Extension to continuum

The extension to continuum is quite a delicate task and was addressed in many different ways, from adopting arbitrary mathematical simplifications [26, 27] to exploiting signal processing techniques [28]. In [1], we proposed an analytical procedure for the extension to the continuum, based on the hearing system auditory property called discrimination limen (DL) [29]. Here, we extend this procedure to triads, by considering the plane with horizontal axis x and vertical axis y, to be identified, respectively, with f_2/f_1 and f_3/f_1.

Suppose that $M_{12,13}$ and $N_{12,13}$ take all integer values from 1 up to 30, for instance. To study triads inside the octave, we select (avoiding double counting) the k different points of the type $(m_{12}/n_{12}, m_{13}/n_{13})$, such that $1 < m_{12}/n_{12} < 2$ and $m_{12}/n_{12} < m_{13}/n_{13} < 2$, and we denote them by (x_i, y_i), with $i = 1, .., k$. The associated normalized consonance indicator, $\tilde{I}_X^C(x_i, y_i)$, is thus defined in the plane (x, y) only for the k discrete values (x_i, y_i).

Our aim is to extend the indicator to any value of the (x, y) domain such that $1 < x < 2$ and $x < y$, thus rendering it more "physical". We recall that, as derived by Zwicker et al. [29], the DL turns out to be about 1/30 of the critical bandwidth (CB) [30], which is frequency dependent. In particular, the ear has a DL of about 3 Hz at the frequency of middle C, that is $f_{DL}(C_4) = 3$ Hz, where $C_4 = 261.63$ Hz; the DL increases up to 6 Hz two octaves above [29].

We propose to simulate the effect of the DL by smoothing the consonance peaks with a Gaussian [6] characterized by a standard deviation which reflects the magnitude of the DL at the frequencies characteristics of the triads to be considered, $\sigma \equiv f_{DL}(\bar{f})/f_1$, where $\bar{f}$ is some mean value between f_2 and f_3. Indeed, a variation of $f_{DL}(\bar{f})$ in the value of f_2 and f_3 is associated to a variation of σ along the x and y axes, respectively[7]:

$$\frac{f_2 \pm f_{DL}(\bar{f})}{f_1} = x \pm \sigma, \quad \frac{f_3 \pm f_{DL}(\bar{f})}{f_1} = y \pm \sigma. \tag{20}$$

Fixing for definiteness $f_1 = C_4$, we have that σ spans the range $f_{DL}(\bar{f})/f_1 \approx (3-6)/261.63 = (0.0115 - 0.0229)$ when $\bar{f}$ goes from C_4 up to C_6. In the following, we take for definiteness $\sigma = 0.02$, which corresponds to a DL of 5.25 Hz. The distance between the point $P = (x, y)$ and the point $P_i = (x_i, y_i)$ (with $i = 1, ..., k$) is given by $d(P, P_i) = ((x - x_i)^2 + (y - y_i)^2)^{1/2}$. We can define the continuum consonance indicator $C_X^C(x, y)$ as the maximum consonance value resulting from smoothing with a Gaussian the consonance peaks placed at positions (x_i, y_i), that is

$$C_X^C(x, y) = \text{Max}_i \, \tilde{I}_X^C(x_i, y_i) \, e^{-\frac{(x-x_i)^2+(y-y_i)^2}{2\sigma^2}}, \quad \text{where } i \text{ is such that } d(P, P_i) < 2\sigma. \tag{21}$$

The constraint on the distance between P and P_i allows to speed up the numerical calculation: One could indeed include all k points P_i in the calculation of the maximum, but this is not necessary as only points within a circle with radius of a couple of standard deviations significantly contribute. The result is a continuous surface function, with smoothed peaks such that, within (beyond) about 3 (6) Hz from a peak, the consonance function does not (does) change significantly.[8]

Figure 4 shows the extension to continuum, taking $\sigma = 0.02$, for the compactness models C_1^P, C_3^P, and C_{GP}^H. It turns out that C_1^P is symmetric with respect to the diagonal axis from the top left to the bottom right of the figure; while the other two display more consonance in the region below such axis. Indeed, notice that with C_3^P and C_{GP}^H, the major third and many other relevant triads get a higher score with respect to C_1^P.

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[6] Our choice of a Gaussian is reasonable but, of course, it is not the unique possibility.

[7] In principle, one might consider an elliptical Gaussian with different standard deviations along x and y axes: $\sigma_i = f_{DL}(f_i)/f_1$, with $i = 2, 3$. In practice, this is not necessary as here we focus on triads within one octave.

[8] For dyads, we performed a psychoacoustic test to check that this is indeed the case: For example, focussing on the peak of the P5, within 3 Hz from the peak, we found, according to our judgement, that the quality of the perceived consonance is not much affected despite the appearance of some roughness, while beyond about 6 Hz, we assessed that the perception turns into dissonance [1].
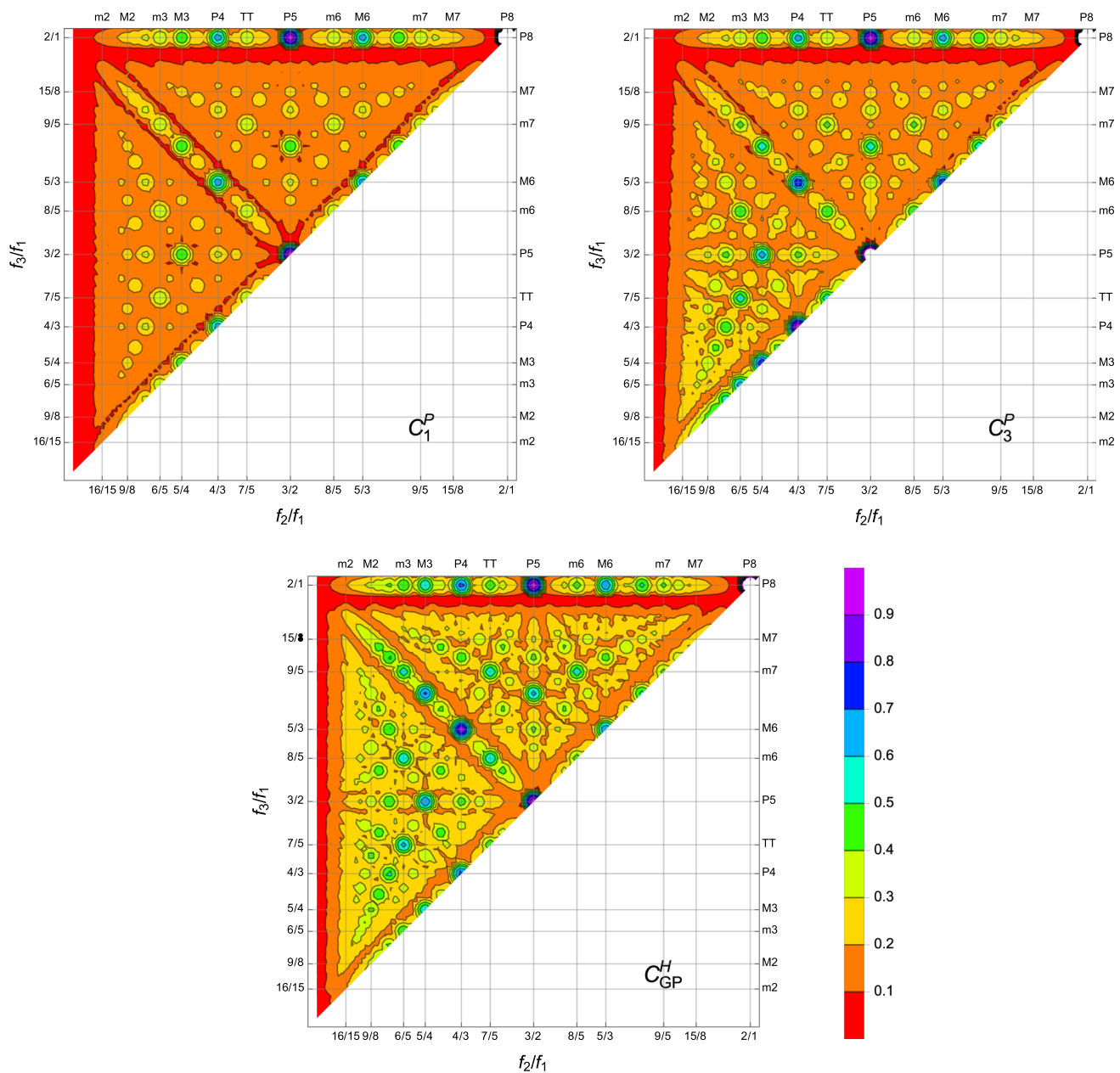
Fig. 4 Top left, top right, and bottom: continuum consonance indicators C_1^P, C_3^P, and C_{GP}^H

This can be better seen by zooming on the region around major and minor triads, as done in Fig. 5. We can clearly see the high consonance peaks associated to the major triads. Notice, however, that also dim (r) has a quite high consonance peak, even at the same level as maj (r) for C_{GP}^H. On the contrary, notice the low consonance value of min (r) and the fact that there is no peak associated with it: min (r) actually is in the tail of the peak of a more consonant close triad, which is associated to frequency ratios (7/6, 3/2), that is a septimal minor third with a fifth. It is reasonable to expect that these not fully satisfactory features would disappear by including the effect of roughness.

These plots are also useful to comment on the major and minor triads of the equal temperament, whose location is shown in Fig. 5 by using dark dots. As well known, equal temperament triads have a lower consonance with respect to just intonation ones: Indeed, being not too far from the peak of the just maj (r), the equally tempered maj (r) gets a quite high consonance value. On the contrary, it would be unreasonable to find the tempered maj (r) down in the dissonance valley around the just maj (r). Similar considerations apply for the other major and minor triads. Notice that min (r) in the equal temperament gets closer to the peak of the triad with the septimal minor third.
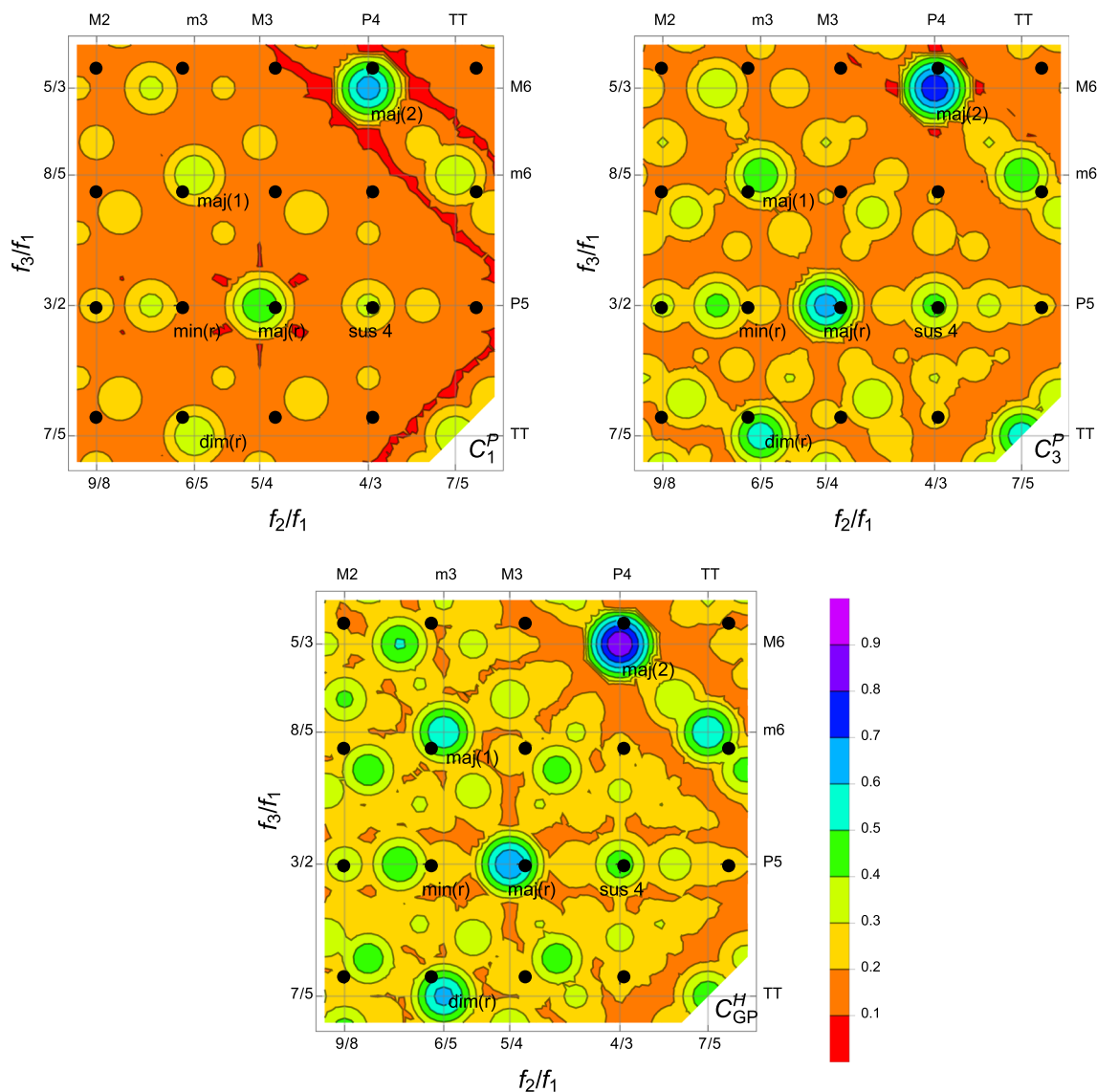
Fig. 5 Top left, top right, and bottom: zoom on the region around major and minor triads for the continuum consonance indicators C_1^P, C_3^P, and C_{GP}^H

As these considerations depend on the value of σ adopted in the procedure of the extension to continuum, it would be interesting to further explore the effect of varying σ on the various models. We leave this for future work.

7 Overview and conclusions

In this work, we studied in detail the viability of compactness models in explaining C and D perceptual data for triads. Compactness models can be divided in the two sub-categories of periodicity and harmonicity, according to the fact that one chooses to relate consonance to a compact period of the waveform signal or rather to a compact structure of harmonics. In both cases, the role of the fundamental bass turns out to be outstanding in order to identify consonance indicators.

Compactness models for dyads were studied extensively in Ref. [1], where it was shown that the two compactness requirements mentioned above (i.e. periodicity and harmonicity) are practically equivalent. Here, we showed that this turns out to be the case also for triads, see Fig. 3.

We found that compactness models are quite satisfactory in reproducing perceptual data. In particular, here, we calculated the reduced Chi-square of our compactness indicators by using the perceptual data of Bowling et al. [22], obtained by testing all 66 triads that can be formed using the chromatic just scale within one octave. We found that, for all the compactness indicators proposed,

the reduced Chi-square is of order one, see Table 3. In particular, the best performing model turned out to be the Gill and Purves harmonicity indicator [20], followed by the periodicity indicator inspired to Galilei's arguments [4].

One of the most serious criticisms formulated against this category of C and D models is the fact that they are naturally associated to indicators which are discontinuous [5]. This represents a problem, especially in order to assess the predictions of these models for the various temperaments proposed in the history of music. In Ref. [1], a procedure to obtain continuum indicators for dyads was suggested, based on smoothing the just intonation consonance peaks by means of a Gaussian with a standard deviation σ taken to be equal to the discrimination limen. Here, we extended this procedure to compactness models for triads, obtaining continuum consonance indicators describing a surface in the plane f_2/f_1 and f_3/f_1, see Fig. 4. This allows for an effective comparison of the just scale predictions with those of various temperaments. In particular, we discussed here the case of the equal temperament, see Fig. 5.

Our general result is that compactness indicators provide a quite effective, but not fully satisfactory, description of C and D for triads. It would be interesting to consider also roughness models for triads and explore the effect of combining them with compactness models, as done for dyads in [1]. The extension to tetrads would be also worth to explore. This is postponed to future work, together with a more refined study of various temperaments and the effect of varying the parameter σ. From the point of view of the perceptual data, it would be worth to carry out a test with sounds not having the piano timbre (as done in [22]) and to consider triads and tetrads beyond one octave. The above findings and considerations show that justifying on a physical basis the auditory perception of C and D for triads is a subject open to further research improvements.

Data Availability Statement The authors declare that data supporting the findings of this study are available within the article.

A Euler's indicator

A mathematical proposal, not based on physical grounds, was proposed by Euler [6]. Despite its name, the Euler's *gradus suavitatis* [6], d_E, is a dissonance indicator, as it increases with the dissonance. We discussed it for dyads in the appendix of [1] and discuss it now for triads.

For triads associated to frequency ratios $f_2/f_1 = f_2/f_0 \times f_0/f_1 = m/n$ and $f_3/f_1 = f_3/f_0 \times f_0/f_1 = \ell/n$, the frequencies have proportions $f_1 : f_2 : f_3 = n : m : \ell$. As n, m, ℓ might have prime factors in common (e.g. for maj (r), they are 4:5:6), one has to consider the "exponent", namely the least common multiple of the product $nm\ell$: $E_E = \mathrm{LCM}(nm\ell)$. Focusing on its factorization, $E_E = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$, where p_1, p_2,... are the prime factors, and e_i are integers and nonzero, the Euler's dissonance indicator is defined as

$$d_E = 1 + \sum_{i=1,...,k} e_i (p_i - 1). \tag{22}$$

For the frequency ratios of the 66 triads inside one octave, d_E ranges from 5 to 20; one can define a normalized consonance indicator C_E, with values in the range [0, 1], as $C_E = 1 - (d_E - 5)/15$. The associated reduced Chi-square is 1.43: It is even not so bad (but worse than the compactness models studied previously), because the results of the test by Bowling et al. [22] have large error bars.

References

1. I. Masina, G. Lo Presti, D. Stanzia, Dyad's consonance and dissonance: combining the compactness and roughness approaches. Eur. Phys. J. Plus **137**, 1254 (2022)
2. M. Blasone, A physicist's view on chopin's études. Eur. Phys. J. Spec. Top. **226**, 2715–2728 (2017)
3. H. Din, J. Berezovsky, Critical behavior and the Kibble–Zurek mechanism in a musical phase transition. PLoS ONE **18**(1), e0280227 (2023)
4. G. Galilei, Discorsi e dimostrazioni matematiche intorno a due nuove scienze, Leida (1638)
5. P. Barbieri, Galileo's coincidence theory of consonances, from Nicomachus to Sauveur. Recercare **13**, 201–232 (2001)
6. L. Euler, Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae, Petropoli (1739)
7. G. Riccati, Saggio sopra le leggi del contrappunto, Castelfranco, Giulio Trento (1762)
8. J. Rameau, Génération harmonique, Paris (1737)

9. G. Tartini, Trattato di musica secondo la vera scienza dell'armonia, Padova (1754)

10. P. Estève, Nouvelle découverte du principe de l'harmonie, Paris (1752)

11. G. Pizzati, La scienza de' suoni e dell'armonia, Venezia (1782)

12. P. Barbieri, Quarrels on harmonic theories in the venetian enlightenment, LIM Ed. (2020)

13. P. Barbieri, La nascita delle teorie 'continue' della consonanza. La ignorata curva di Draghetti e Foderà, poi di Helmholtz (1771-). Acta Musicol. 74 (1), 55–75 (2002)

14. H. von Helmholtz, On the Sensation of Tone (1863)

15. R. Plomp, W.J. Levelt, Tonal consonance and critical bandwidth. J. Acoust. Soc. Am. **38**(4), 548–560 (1965)

16. W. Hutchinson, L. Knopoff, The acoustic component of western consonance. Interface **7**(1), 1–29 (1978)

17. W. Hutchinson, L. Knopoff, The significance of the acoustic component of consonance in western triads. J. Musicol. Res. **3**(1–2), 5–22 (1979)

18. M. Rasmussen, S. Santurette, E. MacDonald, *Consonance perception of complex-tone dyads and chords, Proceedings of Forum Acusticum* (2014)

19. J. Tenney, *A History of Consonance and Dissonance* (Excelsior Music Publishing Company, New York, 1988)

20. K. Gill, D. Purves, A biological rationale for musical scales. PLoS One **4**(12), e8144 (2009)

21. P. Harrison, M. Pearce, Simultaneous consonance in music perception and composition. Psychol. Rev. **127**(2), 216–244 (2020)

22. D. Bowling, D. Purves, K. Gill, Vocal similarity predicts the relative attraction of musical chords. PNAS **115**(1), 216–221 (2018)

23. P. Harrison, M. Pearce, *An energy-based generative sequence model for testing sensory theories of Western harmony*, in *Proceedings of the 19th International Society for Music Information Retrieval Conference, Paris, France* (2018)

24. D. Schwartz, D. Purves, Pitch is determined by naturally occurring periodic sounds. Hearing Res. **23**(18), 31–46 (2004). (**194**)

25. P. Cariani, B. Delgutte, Neural correlates of the pitch of complex tones. I. Pitch and pitch salience. J. Neurophysiol. **76**(3), 1698–1716 (1996)

26. A. Frova, Fisica nella musica, Zanichelli Ed. (1999)

27. F. Stolzenburg, Harmony perception by periodicity detection. J. Math. Music **9**(3), 215–238 (2015)

28. A. Milne, S. Holland, Empirically testing tonnetz, voice-leading, and spectral models of perceived triadic distance. J. Math. Music **10**(1), 59–85 (2016)

29. E. Zwicker, G. Flottorp, S.S. Stevens, Critical band width in loudness summation. J. Acoust. Soc. Am. **29**, 548 (1957)

30. H. Fletcher, Auditory patterns. Rev. Mod. Phys. **12**, 47–65 (1940)