Journal of Magnetism and Magnetic Materials 581 (2023) 170957

Contents lists available at ScienceDirect

Journal of Magnetism and Magnetic Materials

journal homepage: www.elsevier.com/locate/jmmm

Research article

Poiseuille–Couette flow of a hybrid nanofluid in a vertical channel: Mixed magneto-convection

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ARTICLE INFO

Keywords:
Hybrid nanofluid
Exact solutions
MHD Couette flow
Mixed convection
Reverse flow

ABSTRACT

The study of mutual interaction between flow and external magnetic field, as well as the influence of temperature on the motion, is crucial for new classes of materials involved in nanotechnologies. This paper considers a very common situation where a hybrid nanofluid fills a vertical plane channel with a moving wall. Since the nanofluid is Boussinesquian the flow is induced by the buoyancy and Lorentz forces together with a constant pressure gradient. This problem has many industrial applications so that it is of relevant interest. Using a steady and laminar flow, an exact solution for the ODEs which govern the motion has been found. This is the first time an analytical solution is developed for the problem here considered. Analytical expressions for velocity profile and magnetic field are exhibited graphically. Effect of parameters on the flow characteristics has been discussed also in the case of some real hybrid nanofluids (H_2O with Al_2O_3 and Cu, H_2O with Ag and MgO, $C_2H_6O_2$ with TiO_2 and Fe_3O_4). We also find that the presence of two different types of particles determines an increase in the velocity of the nanofluid in accordance with experimental studies. As usual the presence of the external magnetic field causes a decrease in the velocity. Finally, the reverse flow phenomenon is discussed.

1. Introduction

Nanofluids are colloidal mixtures of nanoparticles (1–100 nm) and a base liquid which describe the new class of nanotechnology-based heat transfer fluids with enhanced thermal properties with respect to usual fluids. The base fluid may be water, oil, glycol or polymeric solution.

The name "nanofluid" given to this continuum is due to Choi [1] in 1995 and since that date there have been many studies on nanofluids ([2–9], see [10–12] for extensive bibliography). The thermal conductivity properties of this medium suggest to study the motion of nanofluids in the case of heat transfer. Many are the possible uses of nanofluids as, for example, transportation, electronics and nuclear systems, heat exchangers, biomedical applications, lubrications and cleaning.

As mentioned above there is a substantial number of numerical and experimental studies on nanofluids convective heat transfer while the theoretical investigations are limited [13,14]. The present paper is in the last area and concerns the study of the influence of an external uniform magnetic field on the mixed convection in the fully developed flow of a hybrid nanofluid filling a vertical channel under the Oberbeck–Boussinesq approximation. Flow and heat transfer for a nanofluid flowing inside a channel are very important in cooling applications. In our study we treat the Poiseuille–Couette flow. There are some recent papers dealing with the Couette motion for nanofluids, but the induced magnetic field is not taken into account and the physical problem is different (see [15,16] and the papers cited there). In our research the starting point is the model of Buongiorno [2] which we extend to the case of two types of particles dispersed in the base fluid. Indeed, in a hybrid nanofluid two types of particles (typically alumina, copper, magnetite, titania, ...) are suspended in the base fluid giving rise to a homogeneous and good dispersion. The purpose of combining two different particles in a fluid is to enhance its thermal conductivity ([17–22] for a review).

Convection flow of an electrically conducting fluid in a channel under the effect of a transverse magnetic field has a relevant technical significance because of its many industrial applications such as geothermal reservoirs, cooling of nuclear reactors, electric transmission cables, thermal insulation and petroleum reservoirs. To this regard we notice that recent studies have shown relevant enhancement in the thermal conductivity of magnetic nanofluids when a magnetic field orthogonal to the flow is applied [23,24]. Many papers study the effect of the magnetic field on the flow of a nanofluid or hybrid nanofluid in several

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https://doi.org/10.1016/j.jmmm.2023.170957
Received 19 December 2022; Received in revised form 9 May 2023; Accepted 19 June 2023
Available online 28 June 2023

Nomenclature

A_i	dimensionless nanoparameters defined by $(13)_2$, $i = 1,2$
c	nanofluid specific heat
c_{p_i}	nanoparticle specific heats
C	constant such that $p^* = -Cx_1 + p_0$
d	channel width
D_{B_i}	Brownian diffusion coefficients
D_{T_i}	thermophoresis diffusion coefficients
$\mathbf{E}$	electric field
$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$	canonical base of $\mathbb{R}^3$
$\mathbf{g} = -g\mathbf{e}_1$	gravity acceleration
$\mathbf{H}$	total magnetic field
$h(y)$	dimensionless function describing the induced magnetic field defined by $(6)_3$
$H_0\mathbf{e}_2$	external uniform magnetic field ($H_0 > 0$)
$H_1(x_2)$	induced magnetic field component in the x_1-direction
k	nanofluid thermal conductivity
Le_i	Lewis numbers defined in $(9)_1$
M^2	Hartmann number defined in $(6)_7$
N_{BT_i}	ratio of Brownian and thermophoretic diffusivities defined in $(9)_2$
Nu	Nusselt number
p	pressure
$p^* = p + \mu_e \frac{H_1^2}{2} + \rho_R g x_1$	modified pressure: difference between the hydromagnetic pressure and the hydrostatic pressure
p_0	arbitrary constant
P	dimensional parameter defined by $(6)_8$
P^*	dimensional parameter defined by $(13)_3$
S	vertical channel
$T = T(x_2)$	temperature
T_R	reference temperature
T_{Π_1}, T_{Π_2}	uniform temperatures at the walls ($T_{\Pi_2} > T_{\Pi_1}$)
$\mathbf{v}$	velocity field
$V(y)$	dimensionless function describing the velocity defined by $(6)_2$
$V_0\,\mathbf{e}_1$	velocity of Π_2
$v_1(x_2)$	velocity component in the x_1-direction
y	dimensionless transverse coordinate defined by $(6)_1$

Greek symbols

α_{Φ_i}	composition coefficients of volume expansion
α_T	thermal coefficient of volume expansion
η_e	electrical permittivity $\left(\eta_e = \frac{1}{\mu_e \sigma_e}\right)$
$\vartheta(y)$	dimensionless temperature defined by $(6)_4$
λ_T, λ_{Φ_i}	buoyancy coefficients defined by $(6)_{9,10,11}$
λ	dimensional parameter defined by $(15)_1$
μ	nanofluid dynamical viscosity ($\mu > 0$)
μ_{bf}	base fluid dynamical viscosity ($\mu_{bf} > 0$)
μ_e	magnetic permeability of free space
Π_i	channel walls
ρ	nanofluid mass density
ρ_{p_i}	nanoparticle mass densities
ρ_{bf}	base fluid mass density
ρ_R	nanofluid mass density at the temperature T_R
σ_e	electrical conductivity
Φ_i	nanoparticle volume fractions
Φ_{R_i}	reference nanoparticle volume fractions
Φ_{0i}	nanoparticle volume fractions at Π_1
$\tau_{1,2}$	skin friction at the walls
$\varphi_i(y)$	dimensionless nanoparticle volume fraction defined by $(6)_{5,6}$

geometries and using different models [14,25–27]. Some other good works on heat transport, mass transfer and influence of the magnetic field on the flow for different types of fluids have been reported in the literature [28–35].

In our problem the nanofluid fills a vertical channel under the action of a uniform magnetic field applied normal to the direction of velocity. The infinite rigid walls are maintained at a fixed distance and a different constant temperatures. The horizontal difference of temperature determines heat transfer. Moreover we take into account the induced magnetic field h and give its expression, unlike most of the papers concerning MHD flows. Neglecting h does not allow to highlight the existence of a constant electric field associated with the magnetic field and the influence that h has in hindering the reverse flow phenomenon.

The paper is organized as follows.

In Section 2 we formulate the Poiseuille–Couette problem for a hybrid nanofluid. The obtained ODEs problem is solved in Section 3: the analytical solution is obtained in the general case and in some particular cases. We also highlight the main differences with the non-hybrid nanofluid and also analyze the Newtonian fluid.

Section 4 discusses the trend of solution when the physical parameters vary and outlining the principal differences with respect to Newtonian fluids. Effect of parameters on the flow characteristics has been discussed also in the case of some real hybrid nanofluids (H_2O with Al_2O_3 and Cu, H_2O with Ag and MgO, $C_2H_6O_2$ with TiO_2 and Fe_3O_4). We find that the presence of two different types of particles determines an increase of the nanofluid velocity. As usual, the presence of the external magnetic field causes a decrease in the velocity. Moreover, we treat the reverse flow phenomenon in some illustrative cases.

Section 5 concludes the work.

2. Mathematical formulation

We consider an electrically conducting hybrid nanofluid filling the region S between two infinite rigid, non-electrically conducting vertical plates Π_1, Π_2 separated by a distance d (see Fig. 1).

We assume the regions outside the plane to be a vacuum (free space).

The coordinate axes are fixed in order to have

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2,\ x_2 \in (0, d)\},$$
$$\Pi_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2,\ x_2 = 0\},$$
$$\Pi_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_3) \in \mathbb{R}^2,\ x_2 = d\} \quad (1)$$

and x_1-axis is vertical upward.

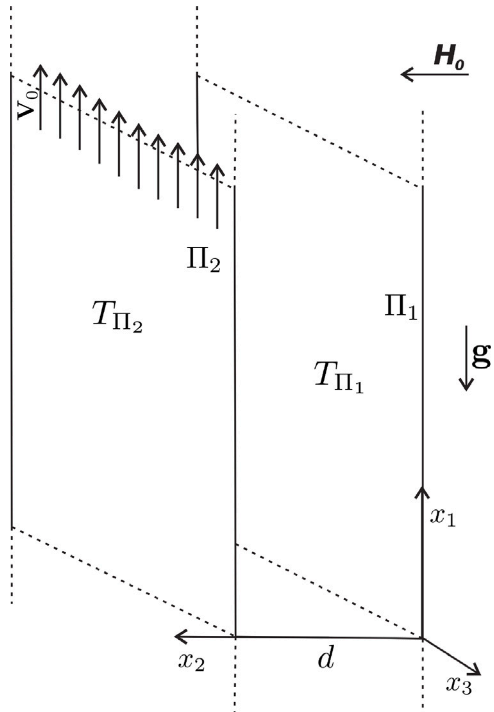

Fig. 1. Physical configuration and coordinate system.

The plane Π_1 is fixed while Π_2 moves along x_1-direction with uniform velocity $V_0\,\mathbf{e}_1$.

We study the steady mixed convection in the fully developed flow of a hybrid nanofluid in which two types of nanoparticles are dispersed in the basic fluid.

We distinguish the physical quantities of the two types of nanoparticles by the subscripts 1, 2.

We assume that an external uniform magnetic field $\mathbf{H}_0 = H_0\mathbf{e}_2$ normal to planes $\Pi_{1,2}$ ($H_0 > 0$) is applied.

For such a nanofluid we adopt the Buongiorno two-phase model [2]. This model has been modified in order to take into consideration the two types of particles dispersed in the base fluid: the buoyancy force has been modified in the momentum equation and we have two nanoparticle volume fractions in the energy equations. Therefore, under the Oberbeck–Boussinesq approximation and in the absence of free electric charges, the governing equations can be written as follows

$$
\begin{aligned}
&\rho_R \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p^* + \mu \triangle \mathbf{v} + \mu_e(\nabla\times\mathbf{H})\times\mathbf{H} \\
&\qquad - \rho_R[\alpha_T(T-T_R) - \alpha_{\Phi_1}(\Phi_1-\Phi_{R_1}) - \alpha_{\Phi_2}(\Phi_2-\Phi_{R_2})]\mathbf{g},\\
&\nabla\cdot\mathbf{v} = 0, \qquad \nabla\cdot\mathbf{H} = 0,\\
&\eta_e \triangle \mathbf{H} + \nabla\times(\mathbf{v}\times\mathbf{H}) = \mathbf{0},\\
&\rho_R c \nabla T\cdot\mathbf{v} = k\triangle T + \rho_{p_1}c_{p_1}\left(D_{B_1}\nabla T\cdot\nabla\Phi_1 + D_{T_1}\frac{|\nabla T|^2}{T_R}\right)\\
&\qquad + \rho_{p_2}c_{p_2}\left(D_{B_2}\nabla T\cdot\nabla\Phi_2 + D_{T_2}\frac{|\nabla T|^2}{T_R}\right),\\
&\nabla\Phi_1\cdot\mathbf{v} = \nabla\cdot\left(D_{B_1}\nabla\Phi_1 + D_{T_1}\frac{\nabla T}{T_R}\right),\\
&\nabla\Phi_2\cdot\mathbf{v} = \nabla\cdot\left(D_{B_2}\nabla\Phi_2 + D_{T_2}\frac{\nabla T}{T_R}\right) \qquad \text{in } \mathcal{S}.
\end{aligned}
\tag{2}
$$

In (2), all material parameters are positive constants and μ_e is equal to the magnetic permeability of free space.

We have supposed slightly compressibility in order to apply the Boussinesq approximation. In our study we have assumed that there exists a configuration in which the temperature T and the nanoparticle volume fractions Φ_1, Φ_2 take the constant values T_R and Φ_{R_1}, Φ_{R_2}, respectively. Moreover, we have assumed that the change of temperature in the nanofluid is small comparing to the reference temperature T_R so that T is replaced by T_R in the denominators of $(2)_{5,6,7}$ (see [2]).

The nanofluid density ρ depends on T, Φ_1, Φ_2 [2,36] in the following way

$$\rho = \Phi_2\rho_{p_2} + (1-\Phi_2)[\Phi_1\rho_{p_1} + (1-\Phi_1)\rho_{bf}].$$

We put $\rho_R = \rho(T_R, \Phi_{R_1}, \Phi_{R_2})$ and

$$
\begin{aligned}
&\alpha_T = -\frac{1}{\rho_R}\frac{\partial\rho}{\partial T}(T_R, \Phi_{R_1}, \Phi_{R_2}),\\
&\alpha_{\Phi_1} = \frac{1}{\rho_R}\frac{\partial\rho}{\partial\Phi_1}(T_R, \Phi_{R_1}, \Phi_{R_2}), \quad \alpha_{\Phi_2} = \frac{1}{\rho_R}\frac{\partial\rho}{\partial\Phi_2}(T_R, \Phi_{R_1}, \Phi_{R_2}).
\end{aligned}
$$

We remark that from physical considerations the thermal coefficient of volume expansion α_T and the composition coefficients of volume expansion $\alpha_{\Phi 1}, \alpha_{\Phi 2}$ are positive. Moreover, we can assume $\rho \simeq \rho_{bf}$ because $\Phi_1, \Phi_2 \ll 1$ [37]. By means of these latter assumptions on Φ_1, Φ_2, the viscosity μ is such that $\mu \simeq \mu_{bf}$ [2].

The coefficients D_{B_1}, D_{B_2} and D_{T_1}, D_{T_2} take into account Brownian diffusion and thermophoresis. Finally, we neglect viscous and ohmic dissipative terms in $(2)_5$.

Since we study the Poiseuille–Couette flow, we search $\mathbf{v}, \mathbf{H}, T, \Phi_1, \Phi_2$ in the following form:

$$
\begin{aligned}
&\mathbf{v} = v_1(x_2)\mathbf{e}_1, \quad \mathbf{H} = H_1(x_2)\mathbf{e}_1 + H_0\mathbf{e}_2, \quad T = T(x_2),\\
&\Phi_1 = \Phi_1(x_2), \quad \Phi_2 = \Phi_2(x_2),
\end{aligned}
\tag{3}
$$

so that $\mathbf{v}, \mathbf{H}$ are divergence free.

By virtue of (2) and $(3)_1$, we easily deduce that the modified pressure p^* is

$$p^* = p^*(x_1) \equiv p + \mu_e\frac{H_1^2}{2} + \rho_R g\, x_1 = -Cx_1 + p_0,$$

where C, p_0 are some constants so that the Eqs. (2) become

$$
\begin{aligned}
&\mu v_1'' + \mu_e H_0 H_1' + \rho_R[\alpha_T(T-T_R) - \alpha_{\Phi_1}(\Phi_1-\Phi_{R_1}) - \alpha_{\Phi_2}(\Phi_2-\Phi_{R_2})]g = -C,\\
&\eta_e H_1'' + H_0 v_1' = 0,\\
&kT'' + \rho_{p_1}c_{p_1}\left(D_{B_1}T'\Phi_1' + \frac{D_{T_1}}{T_R}T'^2\right) + \rho_{p_2}c_{p_2}\left(D_{B_2}T'\Phi_2' + \frac{D_{T_2}}{T_R}T'^2\right) = 0,\\
&D_{B_1}\Phi_1' + \frac{D_{T_1}}{T_R}T' = 0,\\
&D_{B_2}\Phi_2' + \frac{D_{T_2}}{T_R}T' = 0 \qquad \text{in } (0, d).
\end{aligned}
\tag{4}
$$

Eqs. $(4)_4$, $(4)_5$ are obtained taking into account that the walls are impermeable and so the diffusion mass fluxes for the nanoparticles vanish at the walls.

As far as boundary conditions are concerned, we note that $(4)_{4,5}$ hold also at $x_2 = 0, x_2 = d$ and we assign

$$
\begin{aligned}
&v_1(0) = 0, \quad v_1(d) = V_0, \quad H_1(0) = H_1(d) = 0, \quad T(0) = T_{\Pi_1}, \quad T(d) = T_{\Pi_2},\\
&\Phi_1(0) = \Phi_{0_1}, \quad \Phi_2(0) = \Phi_{0_2}.
\end{aligned}
\tag{5}
$$

$(5)_{1,2}$ are the usual no-slip boundary conditions, $(5)_{3,4}$ express the continuity of the tangential component of the magnetic field, $(5)_{7,8}$ are the nanoparticle volume fractions at the fixed wall which are necessary in order to have the uniqueness of the solution.

We assume

$$T_{\Pi_2} > T_{\Pi_1}, \quad T_R = \frac{T_{\Pi_1} + T_{\Pi_2}}{2}, \quad \Phi_{R_1} > \Phi_{0_1}, \quad \Phi_{R_2} > \Phi_{0_2}.$$

We introduce the following dimensionless quantities:

$$
\begin{aligned}
&y = \frac{x_2}{d}, \quad V(y) = \frac{v_1(dy)}{V_0}, \quad h(y) = \frac{H_1(dy)}{V_0\sqrt{\sigma_e\mu}}, \quad \vartheta(y) = \frac{T(dy) - T_R}{T_{\Pi_2} - T_{\Pi_1}},\\
&\varphi_1(y) = \frac{\Phi_1(dy) - \Phi_{R_1}}{\Phi_{R_1} - \Phi_{0_1}}, \quad \varphi_2(y) = \frac{\Phi_2(dy) - \Phi_{R_2}}{\Phi_{R_2} - \Phi_{0_2}},
\end{aligned}
$$

$$M^2 = \frac{\sigma_e}{\mu}\mu_e^2 H_0^2 d^2, \quad P = \frac{Cd^2}{2\mu V_0}, \quad \lambda_T = \frac{\rho_R \alpha_T g(T_{\Pi_2} - T_{\Pi_1})d^2}{\mu V_0},$$
$$\lambda_{\Phi_1} = \frac{\rho_R \alpha_{\Phi_1} g(\Phi_{R_1} - \Phi_{0_1})d^2}{\mu V_0}, \quad \lambda_{\Phi_2} = \frac{\rho_R \alpha_{\Phi_2} g(\Phi_{R_2} - \Phi_{0_2})d^2}{\mu V_0}. \tag{6}$$

Then, system (4) and boundary conditions (5) take the form:

$$\begin{aligned}
&V'' + Mh' + \lambda_T \vartheta - \lambda_{\Phi_1}\varphi_1 - \lambda_{\Phi_2}\varphi_2 + 2P = 0,\\
&h'' + MV' = 0,\\
&\vartheta'' + \frac{1}{Le_1}\left(\vartheta'\varphi_1' + \frac{1}{N_{BT_1}}\vartheta'^2\right) + \frac{1}{Le_2}\left(\vartheta'\varphi_2' + \frac{1}{N_{BT_2}}\vartheta'^2\right) = 0,\\
&\varphi_1' + \frac{1}{N_{BT_1}}\vartheta' = 0,\\
&\varphi_2' + \frac{1}{N_{BT_2}}\vartheta' = 0 \qquad \text{in } (0,1),
\end{aligned} \tag{7}$$

$$\begin{aligned}
&V(0) = 0, \quad V(1) = 1, \quad h(0) = h(1) = 0, \quad \vartheta(0) = -\frac{1}{2}, \quad \vartheta(1) = \frac{1}{2},\\
&\varphi_1(0) = -1, \quad \varphi_2(0) = -1.
\end{aligned} \tag{8}$$

In (7), we have also introduced the Lewis numbers Le_i and the ratio of Brownian and thermophoretic diffusivities N_{BT_i}:

$$Le_i = \frac{k}{\rho_{p_i} c_{p_i} D_{B_i}(\Phi_{R_i} - \Phi_{0_i})}, \quad N_{BT_i} = \frac{D_{B_i} T_R(\Phi_{R_i} - \Phi_{0_i})}{D_{T_i}(T_{\Pi_2} - T_{\Pi_1})}, \quad i = 1,2. \tag{9}$$

Integration of Eqs. $(7)_{3,4,5}$ together with $(8)_{5,6,7,8}$ furnishes

$$\vartheta(y) = y - \frac{1}{2}, \quad \varphi_i(y) = -\frac{1}{N_{BT_i}} y - 1, \quad i = 1,2 \quad \text{in } [0,1]. \tag{10}$$

Taking into account these results, we solve problem $(7)_{1,2}$, $(8)_{1,2,3,4}$.

3. Explicit solutions

We proceed by considering some particular cases.

3.1. Case 1

Let us assume that the hybrid nanofluid is incompressible and the external magnetic field is absent.

Then, the equation for V is

$$V'' + 2P = 0$$

from which we get

$$V(y) = -Py^2 + (1 + P)y. \tag{11}$$

3.2. Case 2

In this case we consider an hybrid nanofluid which is Boussinesquian in the absence of external magnetic field.

The velocity satisfies the following equation

$$V'' + \lambda_T \vartheta - \lambda_{\Phi_1}\varphi_1 - \lambda_{\Phi_2}\varphi_2 + 2P = 0,$$

which by virtue of (10) takes the form:

$$V'' = -\lambda\left(y - \frac{1}{2}\right) - 2P^*, \tag{12}$$

where

$$\lambda = \lambda_T + \frac{\lambda_{\Phi_1}}{N_{BT_1}} + \frac{\lambda_{\Phi_2}}{N_{BT_2}}, \quad A_i = 1 + \frac{1}{2N_{BT_i}}, \quad i = 1,2$$
$$P^* = P + A_1\frac{\lambda_{\Phi_1}}{2} + A_2\frac{\lambda_{\Phi_2}}{2}. \tag{13}$$

Therefore we find

$$V(y) = -\frac{y}{12}[2\lambda y^2 - 3(\lambda - 4P^*)y + \lambda - 12(P^* + 1)]. \tag{14}$$

3.3. Case 3

If the hybrid nanofluid is incompressible and a uniform external magnetic field is impressed, then V and h satisfy system

$$\begin{aligned}
&V'' + Mh' = -2P,\\
&h'' + MV' = 0
\end{aligned} \tag{15}$$

together with boundary conditions $(8)_{1,2,3,4}$.

In this physical situation V and h are given by:

$$\begin{aligned}
V(y) &= \frac{(M + 2P)(e^{-My} - 1)}{2M(e^{-M} - 1)} + \frac{(M - 2P)(e^{My} - 1)}{2M(e^{M} - 1)}\\
h(y) &= \frac{(M + 2P)(e^{-My} - 1)}{2M(e^{-M} - 1)} - \frac{(M - 2P)(e^{My} - 1)}{2M(e^{M} - 1)} - \frac{2P}{M}y.
\end{aligned} \tag{16}$$

3.4. Case 4 - General case

Finally, we consider the case when the hybrid nanofluid is Boussinesquian and a uniform external magnetic field is impressed.

The two unknown fields V and h satisfy system

$$\begin{aligned}
&V'' + Mh' = -\lambda(y - \frac{1}{2}) - 2P^*,\\
&h'' + MV' = 0
\end{aligned} \tag{17}$$

with the boundary conditions $(8)_{1,2,3,4}$. The solution is

$$\begin{aligned}
V(y) &= \frac{[M(M + 2P^*) - \lambda](e^{-My} - 1)}{2M^2(e^{-M} - 1)} + \frac{[M(M - 2P^*) - \lambda](e^{My} - 1)}{2M^2(e^{M} - 1)} + \frac{\lambda}{M^2}y\\
h(y) &= \frac{[M(M + 2P^*) - \lambda](e^{-My} - 1)}{2M^2(e^{-M} - 1)} - \frac{[M(M - 2P^*) - \lambda](e^{My} - 1)}{2M^2(e^{M} - 1)} +\\
&\quad - \frac{\lambda}{2M}y^2 + \frac{\lambda - 4P^*}{2M}y.
\end{aligned} \tag{18}$$

Remark 1. If the nanofluid is not hybrid, we have only one type of particles. Hence we get similar profiles assuming that in Eqs. (2) the terms with subscript 2 (i.e. $\alpha_{\Phi_2}, D_{B_2}, \Phi_2, D_{T_2}$) vanish.

Remark 2. We notice that if the channel is filled with a Newtonian fluid, the non-Boussinesquian cases (**Case 1** and **Case 3**) have the same solutions (11), (16). In the Boussinesquian cases (**Case 2** and **Case 4**), the solutions are given by (14), (18), respectively, with $\lambda \equiv \lambda_T$ and $P^* \equiv P$.

Remark 3. In **Cases 3** and **4** it is interesting to compute the electric field $\mathbf{E}$ associated to the magnetic field. Actually, from the Maxwell equation:

$$\mathbf{E} = \frac{1}{\sigma_e}\nabla \times \mathbf{H} + \mu_e \mathbf{H} \times \mathbf{v}$$

and from Eq. $(3)_{1,2}$, $(4)_2$, we deduce that the electric field is constant and orthogonal to $\mathbf{v}$ *and* $\mathbf{H}$:

$$\mathbf{E} = -\frac{\mu_e H_0 V_0}{M} h'(0)\mathbf{e}_3 \equiv \mathbf{E}_0.$$

If one neglects the induced magnetic field, then one cannot highlight the existence of the constant electric field $\mathbf{E}_0$.

Outside the channel, where vacuum is, we get

$$\mathbf{E} = \mathbf{E}_0, \quad \mathbf{H} = \mathbf{H}_0$$

thanks to the usual transmission conditions for the electromagnetic field across the vertical walls.

Remark 4. From the practical point of view, the following physical quantities are of relevant interest: the *skin frictions* $\tau_{\Pi_1}, \tau_{\Pi_2}$, the *Nusselt numbers* Nu_{Π_1}, Nu_{Π_2} and the *Sherwood numbers* $\mathrm{Sh}^{(i)}_{\Pi_1}$, $\mathrm{Sh}^{(i)}_{\Pi_2}$, i = 1,2 given by

$$\tau_{\Pi_1} = \frac{\mu V_0}{d}V'(0)\mathbf{e}_1, \quad \tau_{\Pi_2} = \frac{\mu V_0}{d}V'(1)\mathbf{e}_1,$$

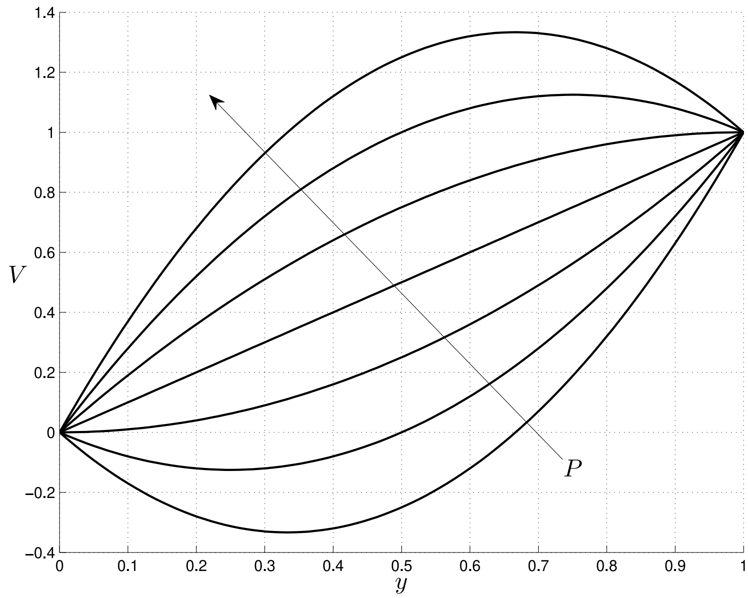

Fig. 2. Velocity of **Case 1** for $P = -3, -2, -1, 0, 1, 2, 3$.

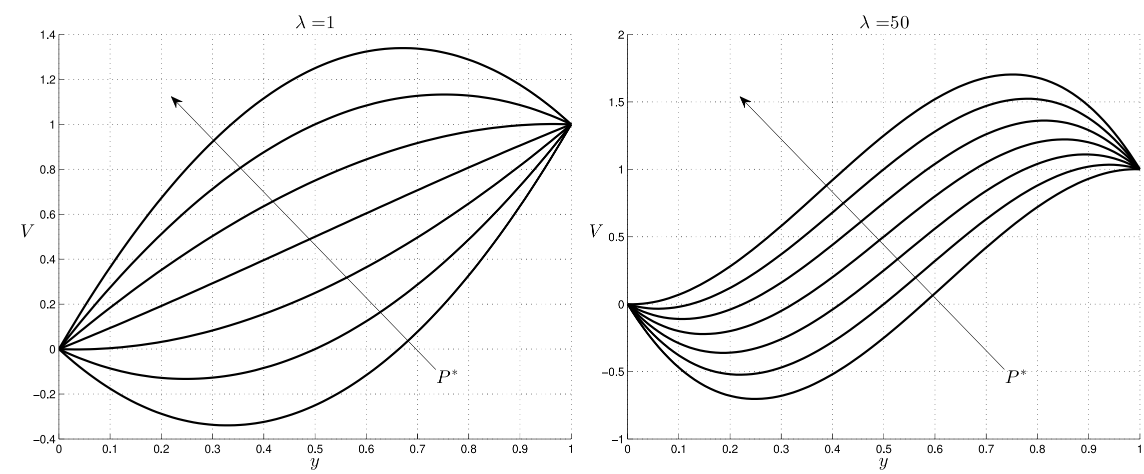

Fig. 3. Velocity of **Case 2** with $P^* = -3, -2, -1, 0, 1, 2, 3$.

$$Nu_{\Pi_1} = \frac{d}{T_2 - T_1} T'(0) = Nu_{\Pi_2} = \frac{d}{T_2 - T_1} T'(d) = 1,$$

$$Sh^{(i)}_{\Pi_1} = \frac{d}{\Phi_{R_i} - \Phi_{0_i}} \Phi'_i(0) = Sh^{(i)}_{\Pi_2} = \frac{d}{\Phi_{R_i} - \Phi_{0_i}} \Phi'_i(d) = -\frac{1}{N_{BT_i}}, \quad i = 1, 2.$$

We notice that the Nusselt numbers do not depend on the type of the nanofluid unlike the Sherwood numbers (see Table 1).

4. Results and discussion

In this paper, the mixed magneto-convection of a hybrid nanofluid has been considered when the Poiseuille–Couette flow occurs in a vertical channel.

The detailed discussion and graphical representation of the explicit solutions previously derived are reported in this section.

We begin our analysis by illustrating the trend of solutions with toy parameters, and then we will take care of several real hybrid nanofluids.

Fig. 2 shows the trend of velocity in **Case 1** for several values of P.

In this case the nanofluid is incompressible and the external magnetic field is absent, so that the velocity behaves as in the Newtonian case.

If the nanofluid is Boussinesquian, then the parameter λ summarizes thermal and nanoparticles properties in **Case 2**. In Fig. 3 we can see the plot of velocity when P^* varies for two fixed values of λ.

The trend of velocity when P^* is fixed and λ varies is showed in Fig. 4.

As λ increases, the velocity is not always monotonous: its minimum and maximum increase in modulus.

We now deal with the situation in which a magnetic field is impressed. We first look to the solution in **Case 3** where the fluid is assumed to be incompressible.

Since the velocity trend is very similar to the first case when M is fixed, we now display the motion when M changes and P is fixed (see Fig. 5).

We see that as M increases the velocity can change its concavity. This can be proved by differentiate twice the analytical expression of V given in (16).

The induced magnetic field h is displayed in Fig. 6.

The function h increases with M until the Hartmann number reaches a critical value (see Fig. 7). This trend has already been noticed in similar situations [38,39].

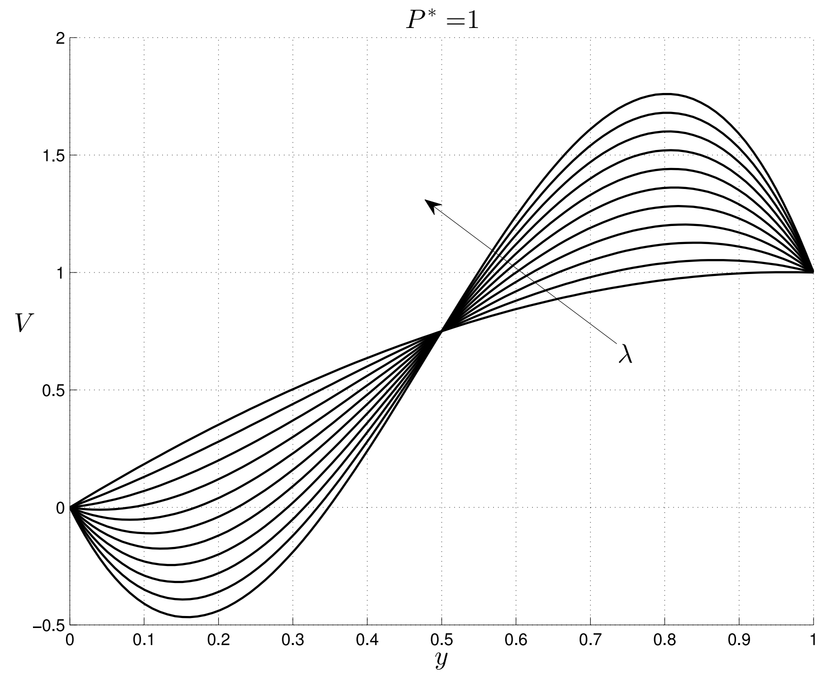

Fig. 4. Velocity of **Case 2** with $\lambda = 1, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$.

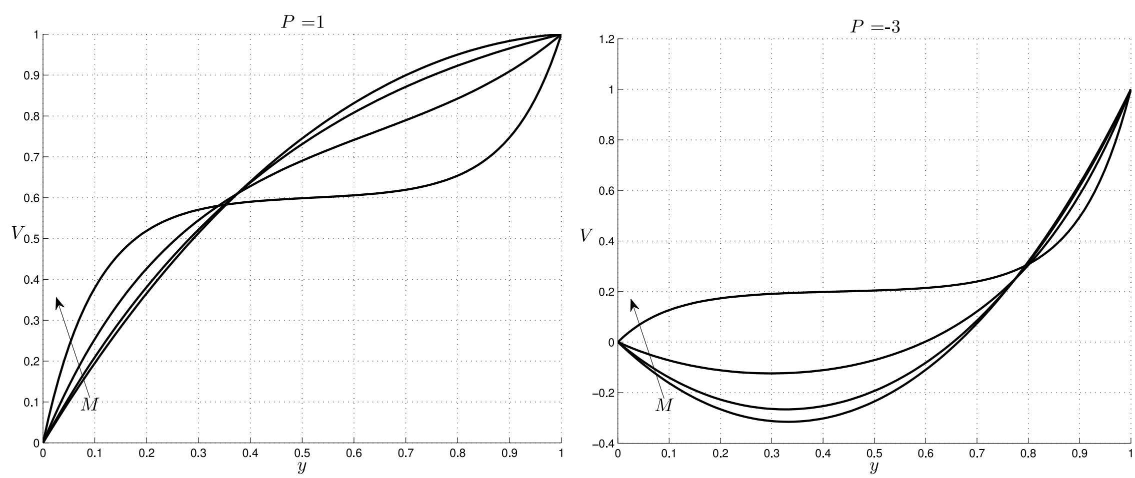

Fig. 5. Velocity in **Case 3** with $M = 1, 2, 4, 10$.

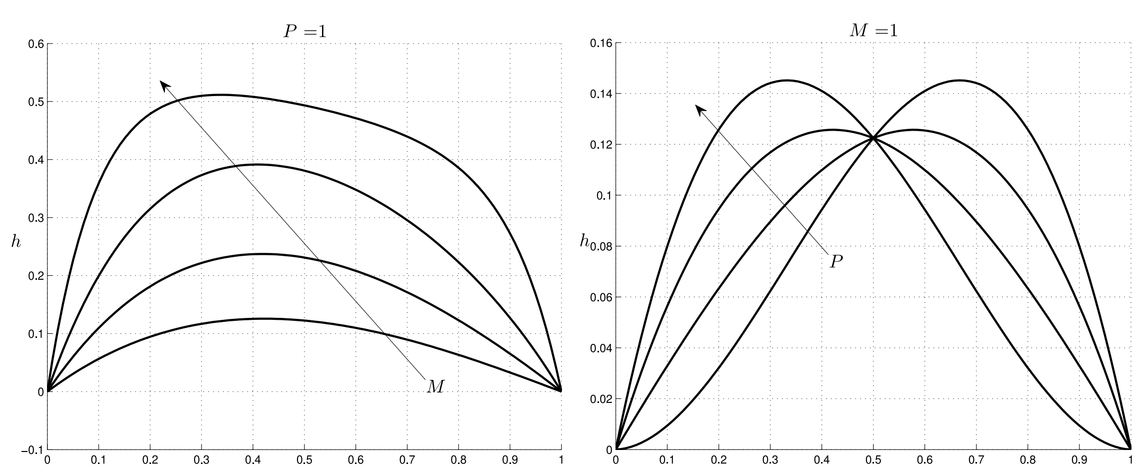

Fig. 6. Trend of the magnetic field in **Case 3**. Left $M = 1, 2, 4, 10$, right $P = -3, -1, 1, 3$.

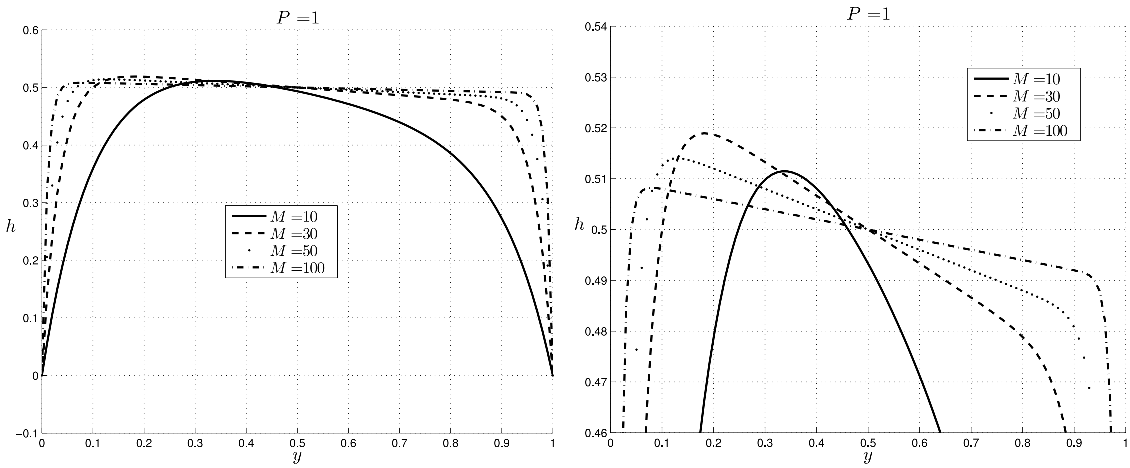

Fig. 7. Trend of the magnetic field in **Case 3** when M is big (in the right plot there is a zoom of h).

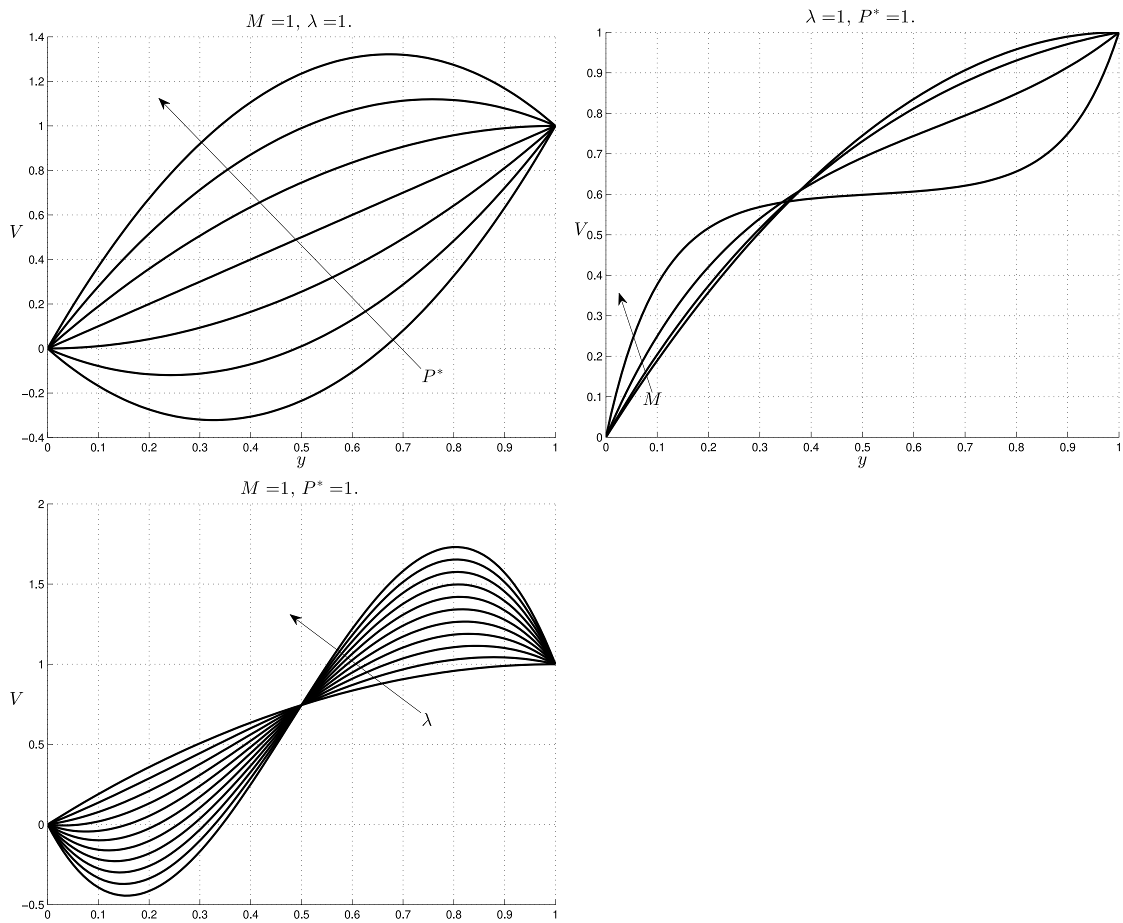

Fig. 8. Velocity in **Case 4** ($P^* = -3, -2, -1, 0, 1, 2, 3,\ M = 1, 2, 4, 10,\ \lambda = 1, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$).

The Figs. 8 and 9 show the behavior of the motion in the general case (**Case 4**).

It is interesting to notice that h increases with λ.

4.1. Flow behavior of real hybrid nanofluids

In this section we look to the solution of flow for real hybrid nanofluids, whose physical parameters are give in Table 1 ($C_2H_6O_2$ ethylene glycol, Al_2O_3 alumina, Cu copper, Ag silver, MgO magnesium oxide, TiO_2 titanium dioxide, Fe_3O_4 magnetite).

The behavior of velocity in **Case 2** for two real hybrid nanofluids with water base fluid is showed in Fig. 10. One can see that the trend is very similar and the two curves almost overlap as P increases.

It is also interesting to show the difference between hybrid and not hybrid nanofluids (see Fig. 11). We notice that the velocity increases with the dispersion of two different types of particles according to experimental studies.

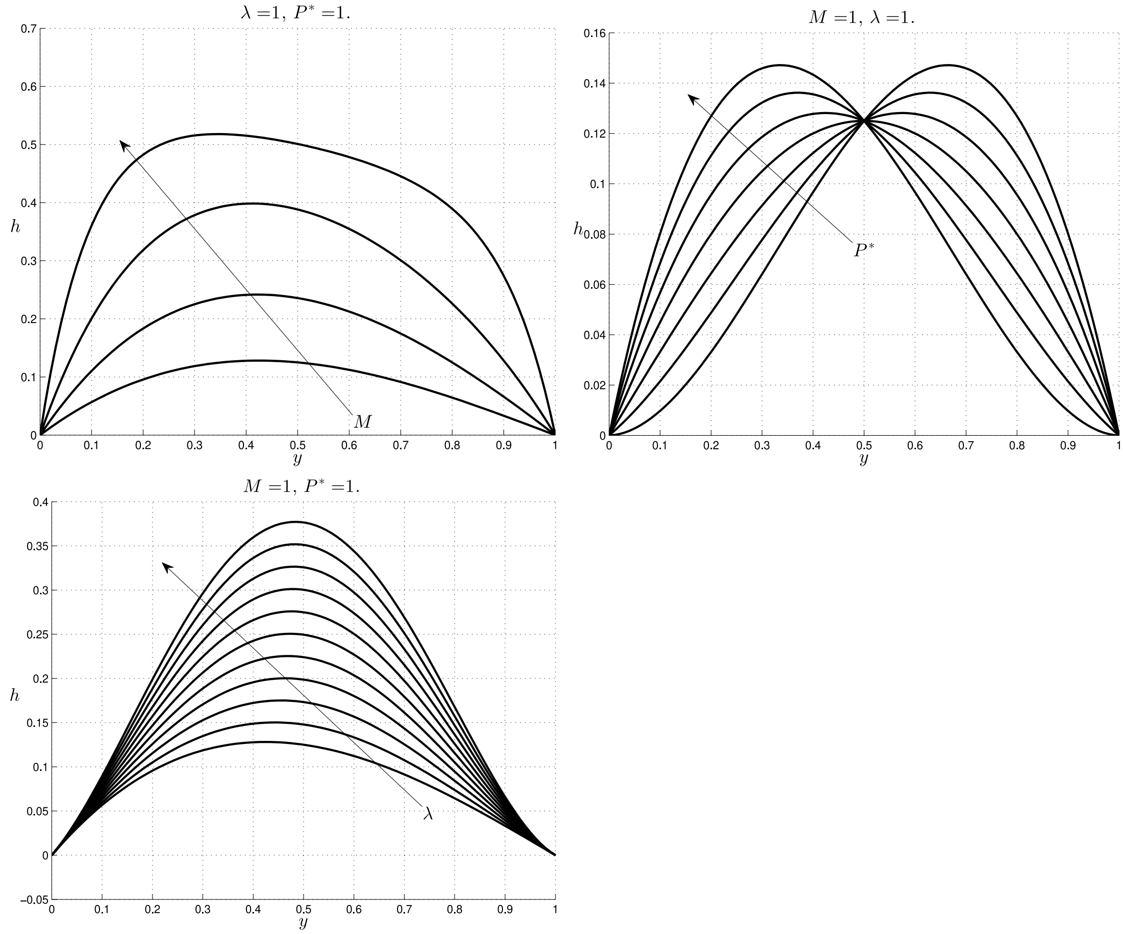

Fig. 9. Magnetic field in **Case 4** ($M = 1, 2, 4, 10,\ P^* = -3, -2, -1, 0, 1, 2, 3,\ \lambda = 1, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$).

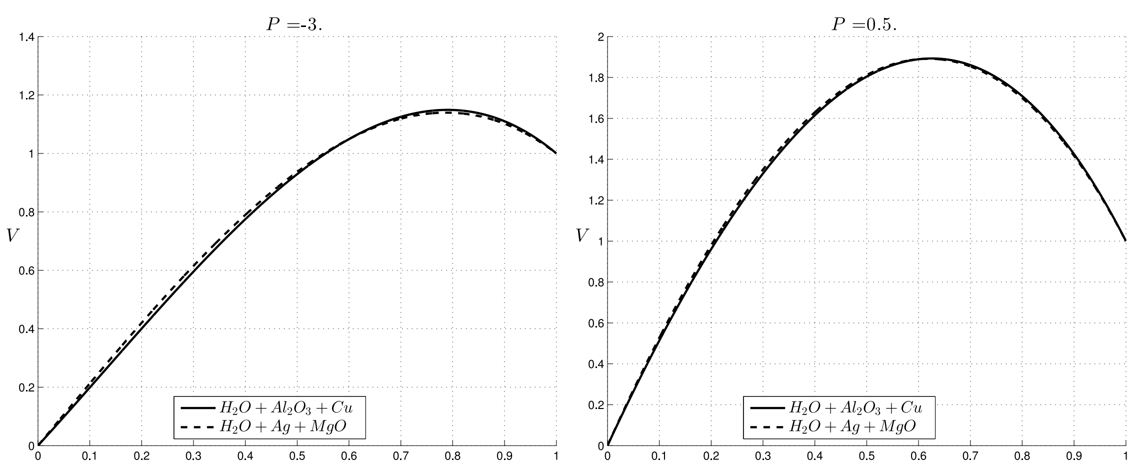

Fig. 10. Velocity in **Case 2** of two real hybrid nanofluids.

In order to compare the hybrid alumina–copper–water nanofluid with the hybrid titanium dioxide–ethylene–glycol nanofluid we provide Fig. 12.

In this case, one has to vary the value of the pressure drop C because the parameter P differs in the two cases as it depends on the viscosity value μ.

The influence of magnetic field is shown in Fig. 13.

4.2. The reverse flow phenomenon

Looking to the previous velocity plots, one can notice that for some values of the parameters it can happen that the velocity changes sign: this is a well-known behavior called *reverse flow*.

The reverse flow phenomenon occurs when the fluid flowing in the channel changes direction of its velocity. In the most cases it is

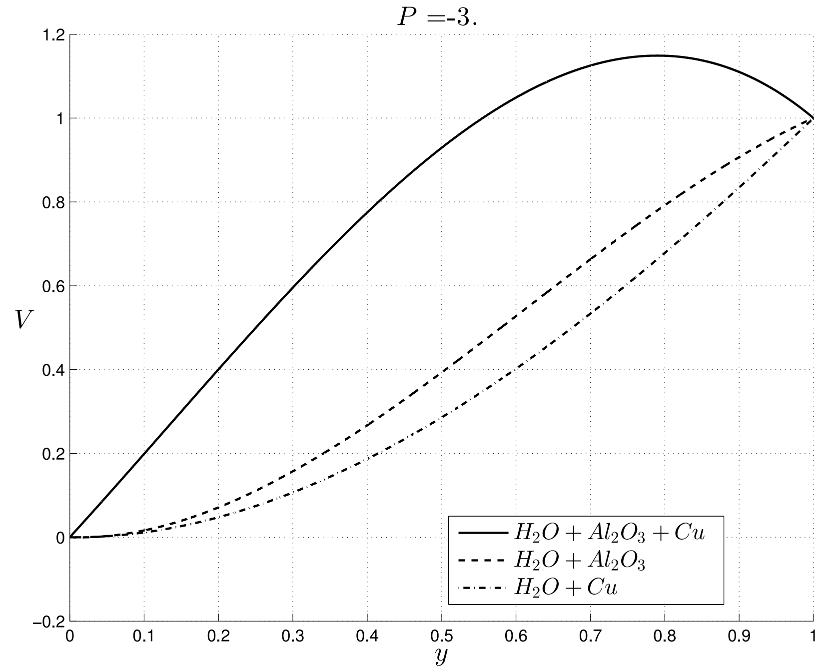

Fig. 11. Velocity in **Case 2** of hybrid and not hybrid nanofluids.

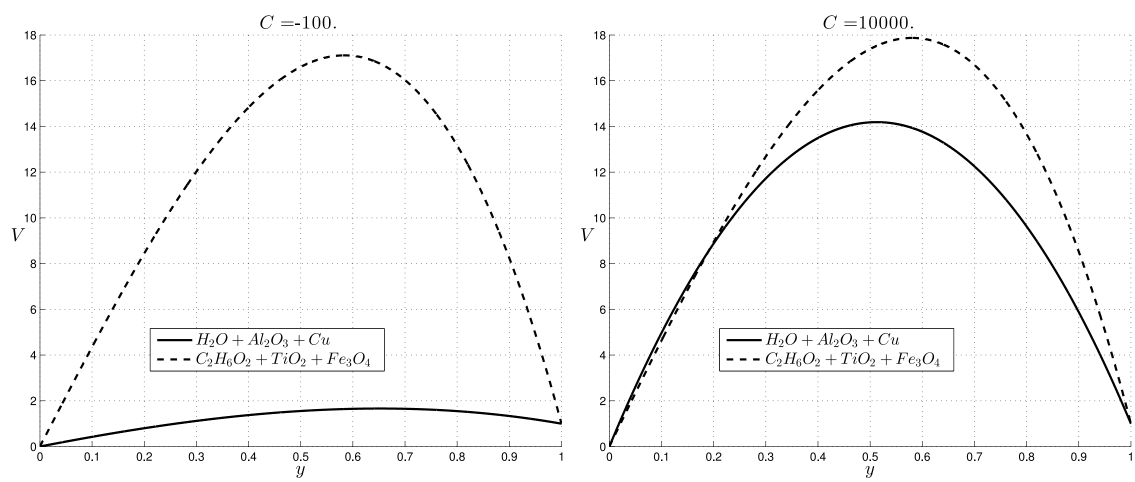

Fig. 12. Velocity in **Case 2** of two real hybrid nanofluids with different base fluid. The parameter C is expressed in kg m^{-2} s^{-2}.

Table 1
Thermo-physical properties of motion, base fluids and nanoparticles.

Motion properties			
g (m s^{-2})	9.8		
$\frac{d^2}{V_0}$ (s^{-2})	10^{-5}		
$T_{\Pi_2} - T_{\Pi_1}$ (K)	10		
$\Phi_{R_i} - \Phi_{0_i}$	0.005		
Base fluids properties	H_2O	$C_2H_6O_2$	
μ (Pa s)	10^{-3}	1.61×10^{-2}	
α_T (K^{-1})	0.2×10^{-3}	0.57×10^{-3}	
ρ_R (kg m^{-3})	10^3	1114	
Nanoparticles properties	$Al_2O_3 + Cu$	$Ag + MgO$	$TiO_2 + Fe_3O_4$
α_{Φ_1}	3	9	2.82
α_{Φ_2}	7	3	3.65
N_{BT_1}	0.2	0.64	8.17×10^{-4}
N_{BT_2}	2	4.33	8.82×10^{-4}

related to the sign of pressure jump C. As it is known, the Poiseuille–Couette flow is the simplest example of motion where this phenomenon can occur. Unlike what happens for the plane Poiseuille motion, this phenomenon can also occur even if the fluid is not Boussinesquian. Obviously, in order to have the occurrence of reverse flow, the velocity of the wall, the temperatures of the walls and the type of nanofluid must satisfy appropriate conditions. Therefore there are many material parameters that contribute to the occurrence of the phenomenon. For this reason, we only study a few illustrative cases.

4.2.1. Case 1 - the nanofluid is incompressible and the external magnetic field is absent.

As it is immediately to verify, the reverse flow phenomenon occurs if

$$P < -1$$

(C and V_0 have discordant sign) and the separation point is $\overline{y} = \dfrac{1+P}{P}$. This behavior is underlined in Fig. 2.

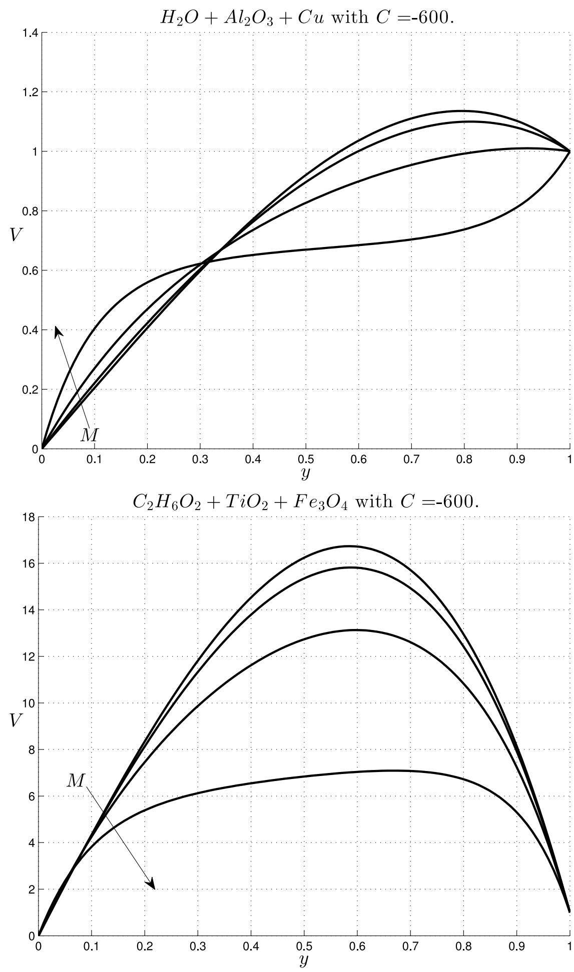

Fig. 13. Velocity in **Case 4** where C is expressed in kg m^{-2} s^{-2} ($M = 1, 2, 4, 10$).

As P decreases, the separation point $\overline{y}$ gets closer to the movable wall. Moreover the minimum of velocity decreases with P.

4.2.2. Case 2 - the nanofluid is boussinesquian and the external magnetic field is absent.

Let us begin with the case of nanofluid which is not hybrid.

In this case the velocity given by (14) depends explicitly on two parameters: λ and

$$P^* = P + \left(1 + \frac{1}{2N_{BT}}\right)\frac{\lambda_\Phi}{2}.$$

We recall that P depends on C and so we have to distinguish between natural ($C = 0$) and mixed ($C \neq 0$) convection.

We limit ourselves to natural convection, providing only a few numerical examples for the mixed convection.

In the first case $C = 0$ and so $P = 0$. Moreover λ_T, λ_Φ, P^* have the sign of V_0.

Subcase $\mathbf{2}_{N+}$: $V_0 > 0$ (**g** and $\mathbf{V}_0$ have opposite direction) .

By studying $V(y)$ given by (14), we deduce (see Fig. 14) that reverse flow occurs if

$$\lambda_T > 2\left(3 + \frac{1}{N_{BT}}\right)\lambda_\Phi + 12. \tag{19}$$

If the fluid is Newtonian we find $\lambda_T > 12$, as it is easily to see.

Subcase $\mathbf{2}_{N-}$: $V_0 < 0$ (**g** and $\mathbf{V}_0$ have the same direction).

The parameters λ_T, λ_Φ, P^* are negative. There are two possibilities depending on whether $P^* \leq -2$ or $-2 < P^* < 0$. More precisely we deduce:

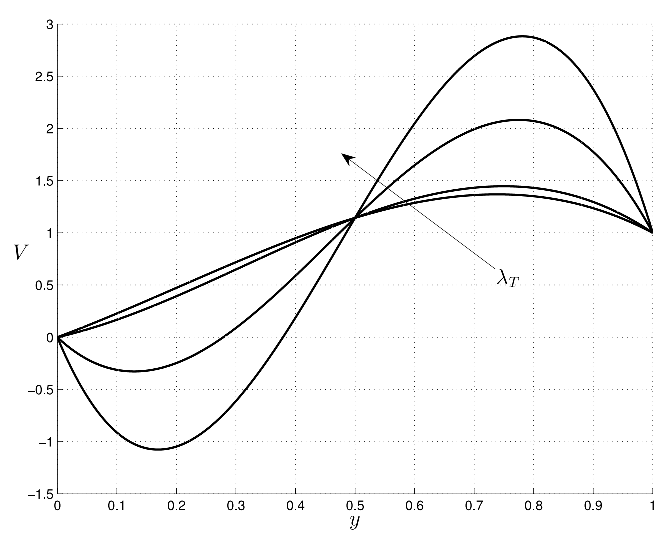

Fig. 14. Occurrence of reverse flow in the natural convection of a not hybrid nanofluid ($\lambda_T = 10, 20, 100, 200$). The velocity is given by (14).

(a) if

$$\lambda_\Phi \leq -8\frac{N_{BT}}{1 + 2N_{BT}},$$

then the reverse flow occurs, for every value of $\lambda_T < 0$;

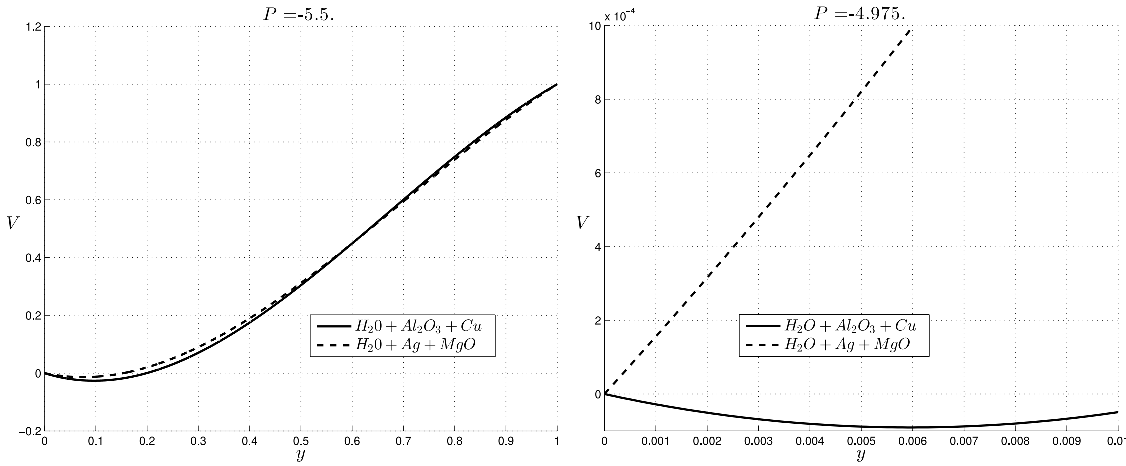

Fig. 15. Reverse flow in **Case 2** of two real hybrid nanofluids.

(b) if

$$\lambda_{\Phi} > -8\frac{N_{BT}}{1+2N_{BT}}$$

then the reverse flow occurs provided

$$\lambda_T < -2\left[\left(3+\frac{2}{N_{BT}}\right)\lambda_{\Phi} + 12\sqrt{4+\left(1+\frac{1}{2N_{BT}}\right)\lambda_{\Phi}} + 24\right] \tag{20}$$

or

$$-2\left[\left(3+\frac{2}{N_{BT}}\right)\lambda_{\Phi} - 12\sqrt{4+\left(1+\frac{1}{2N_{BT}}\right)\lambda_{\Phi}} + 24\right] < \lambda_T < 0. \tag{21}$$

In the mixed convection (second subcase) $C \neq 0$ and so $P \neq 0$.

Subcase $\mathbf{2}_{M++}$: $V_0 > 0$ (**g** and $\mathbf{V}_0$ have opposite direction), $C > 0$.

All the material parameters are positive. The reverse phenomenon occurs if

$$\lambda_T > 2\left(3+\frac{1}{N_{BT}}\right)\lambda_{\Phi} + 12(1+P). \tag{22}$$

Then to complete the analysis one should study all other cases concerning the sings of V_0 and C.

The occurrence of reverse flow for hybrid fluids is more subtle because many parameters are involved. We just show some pictures in order to underline that a region of reverse flow can appear.

Figs. 15 and 16 show the phenomenon for real hybrid nanofluids.

Finally, in the general case (**Case 4**) we have that the presence of the magnetic field hinders the phenomenon of reverse flow (Fig. 17). This property of the magnetic field is known in the literature (see, for example, [38,39]) and is due to the Lorentz forces. As underlined in the literature, neglecting the induced magnetic field leads to affirm the opposite effect. Therefore the approximation of neglecting h, which is often found in literature, appears physically unreasonable.

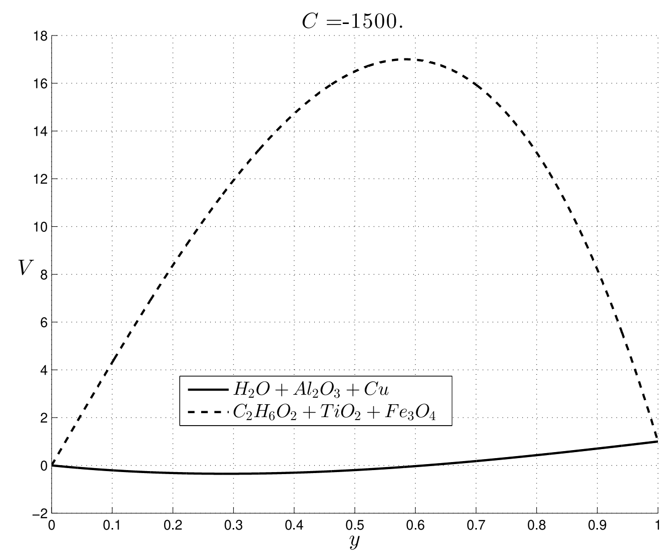

Fig. 16. Velocity in **Case 2** of two real hybrid nanofluids with different base fluid. The parameter C is expressed in kg m^{-2} s^{-2}.

5. Conclusions

In this article we have studied the influence of magnetic field and temperature on the Poiseuille–Couette steady flow of a hybrid nanofluid. The fluid is assumed to be Boussinesquian and the steady motion occurs in a vertical channel under the action of a uniform magnetic field applied normal to the direction of velocity. We have analytically solved the problem. Some special cases were also considered: incompressible fluid, absence of magnetic field, not hybrid nanofluid.

The following conclusions are drawn.

- This is the first time an analytical solution is developed for the problem considered.
- If the hybrid nanofluid is incompressible and the external magnetic field is absent then the velocity behaves as in Newtonian case.
- If the hybrid nanofluid is Boussinesquian and the external magnetic field is absent then as the parameter λ increases the velocity is not monotonous and its minimum and maximum increase in modulus.
- If the hybrid nanofluid is incompressible and a uniform external magnetic field is impressed, then as Hartmann number M increases the velocity can change its concavity. As far as the induced magnetic field h is concerned, it increases with M until M reaches a critical value.
- In the general case (**Case 4**) the function h increases with λ.
- Some real hybrid nanofluids have been considered.
- The presence of two different types of particles determines an increase in the velocity of the nanofluid.
- The presence of the external magnetic field causes a decrease in the velocity.
- In correspondence of particular values of the material parameter the reverse flow phenomenon occurs. The presence of the magnetic field hinders this phenomenon.

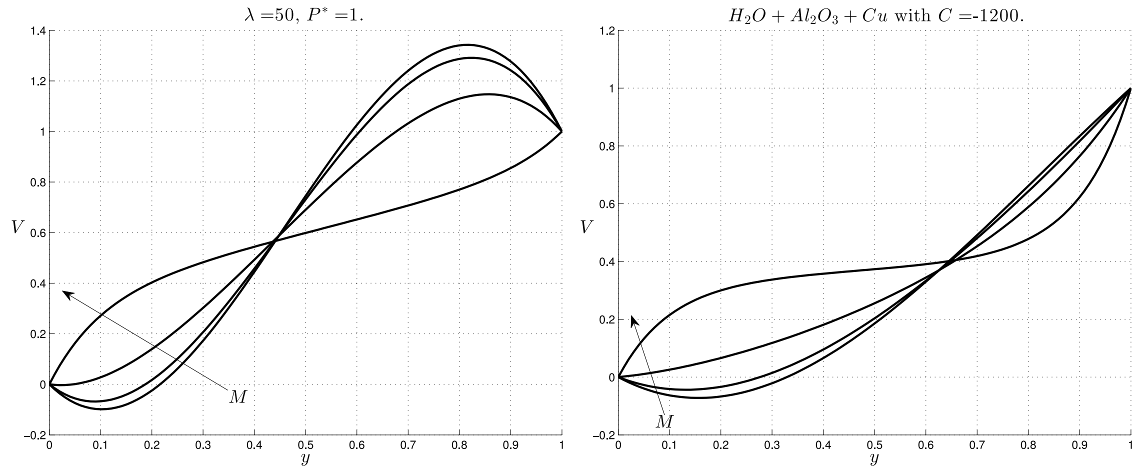

Fig. 17. Reverse flow when the magnetic field is impressed (**Case 4**). Left: toy model. Right: real hybrid nanofluid (the parameter C is expressed in kg m^{-2} s^{-2}).

- The results were compared with the Newtonian case.
- Several pictures were provided in order to understand the trend of motion and the occurrence of reverse flow.

An important extension of this work is to consider the case in which the governing equations (in particular the energy equations) are not linearized in order to understand if the phenomenon of dual solutions appears also for nanofluids [40]. Another possible future development is to consider similar three-dimensional motion situations, such as inside a cylindrical duct.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors thank the anonymous reviewers for their comments and suggestions.

This work was partially supported by National Group of Mathematical Physics (GNFM-INdAM).

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