



On the Cauchy problem for p -evolution equations with variable coefficients: A necessary condition for Gevrey well-posedness [☆]

Alexandre Arias Junior ^{a,b}, Alessia Ascanelli ^c, Marco Cappiello ^{a,*}

^a *Dipartimento di Matematica “G. Peano”, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy*

^b *Department of Computing and Mathematics, Universidade de São Paulo, Ribeirão Preto, Brazil*

^c *Dipartimento di Matematica ed Informatica, Università di Ferrara, Via Machiavelli 30, 44121 Ferrara, Italy*

Received 15 September 2023; revised 12 March 2024; accepted 29 May 2024

Abstract

In this paper we consider a class of p -evolution equations of arbitrary order with variable coefficients depending on time and space variables (t, x) . We prove necessary conditions on the decay rates of the coefficients for the well-posedness of the related Cauchy problem in Gevrey spaces.

© 2024 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

MSC: 35G10; 35S05; 35B65; 46F05

Keywords: p -Evolution equations; Gevrey classes; Well-posedness; Necessary conditions

[☆] The first author was supported by São Paulo Research Foundation (FAPESP), Brazil, grant 2022/01712-3. The second and third author were supported by the INdAM-GNAMPA project CUP E53C23001670001.

* Corresponding author.

E-mail addresses: alexandre.ariasjunior@usp.br (A. Arias Junior), alessia.ascanelli@unife.it (A. Ascanelli), marco.cappiello@unito.it (M. Cappiello).

<https://doi.org/10.1016/j.jde.2024.05.051>

0022-0396/© 2024 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The main concern in this paper is the Gevrey well-posedness of the Cauchy problem

$$\begin{cases} Pu(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, x) = \phi(x), & x \in \mathbb{R} \end{cases} \tag{1.1}$$

where, for a fixed $p \in \mathbb{N}$, $p \geq 2$, P is a non-kowalewskian linear evolution operator of the form

$$P = D_t + a_p(t)D_x^p + \sum_{j=1}^p a_{p-j}(t, x)D_x^{p-j}, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad D = -i\partial. \tag{1.2}$$

We assume that $a_p \in C([0, T]; \mathbb{R})$, $a_p(t) \neq 0 \forall t \in [0, T]$ and $a_{p-j} \in C([0, T]; \mathcal{B}^\infty(\mathbb{R}))$, $j = 1, \dots, p$, where $\mathcal{B}^\infty(\mathbb{R})$ stands for the space of complex-valued functions bounded on \mathbb{R} together with all their derivatives. The operator P is known in the literature as a *p-evolution operator with real characteristics* (cf. [15]). To point out some outstanding particular cases, we recover a Schrödinger-type operator when $p = 2$ and linearized KdV-type equations when $p = 3$, cf. [1]. Also for higher values of p the linearizations of several dispersive evolution equations can be written using an operator of the form (1.2), see e.g. [16] and the references therein. The Cauchy problem (1.1) for (1.2) has been intensively investigated and well understood in $H^\infty(\mathbb{R}) = \bigcap_{m \in \mathbb{N}} H^m(\mathbb{R})$, see [4,5,8,11–13] and in the Schwartz spaces $\mathcal{S}(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, see [6].

In fact, our assumption $a_p(t) \in \mathbb{R}$ agrees with the necessary condition for well-posedness in $H^\infty(\mathbb{R})$ given by Theorem 3 on page 31 of [15]: $\text{Im } a_p(t) \leq 0 \forall t \in [0, T]$. Under this condition, if the coefficients a_j of the lower order terms are all real-valued, then the problem (1.1) is L^2 well-posed; well-posedness of (1.1) may fail if $\text{Im } a_j \neq 0$ for some $1 \leq j \leq p - 1$.

A necessary condition on the coefficient $a_{p-1}(t, x)$ of the subprincipal part of P for well-posedness in $H^\infty(\mathbb{R})$ has been given in [11] when $p = 2$ and in [5] for general $p \geq 3$; it reads as follows:

if (1.1) is well-posed in $H^\infty(\mathbb{R})$, then there exist $M, N > 0$ such that

$$\min_{0 \leq \tau \leq t \leq T} \int_{-q}^q \text{Im } a_{p-1}(t, x + pa_p(\tau)\theta) d\theta \leq M \log(1 + q) + N, \quad \forall q > 0, \forall x \in \mathbb{R}. \tag{1.3}$$

We recall that dealing with one space dimension, for $p = 2$ condition (1.3) is also sufficient for H^∞ well-posedness and with $M = 0$ it is necessary and sufficient for L^2 well-posedness, cf. [12].

The condition (1.3) clearly implies that if a_{p-1} does not depend on x , the problem (1.1) is not H^∞ well-posed. On the other hand, x -dependent coefficients with bounded primitive are allowed. For instance, the operator $D_t + D_x^2 + i \cos x D_x$ fulfills (1.3) with $M = 0$ and the associated Cauchy problem is L^2 well-posed, cf. [15]. Nevertheless, if we test condition (1.3) on the model operator

$$D_t + D_x^p + i \langle x \rangle^{-\sigma} D_x^{p-1}, \quad \langle x \rangle := \sqrt{1 + x^2}, \tag{1.4}$$

we immediately realize that for $\sigma \in (0, 1)$ H^∞ well-posedness does not hold. In other words, if a_{p-1} is vanishing for $|x| \rightarrow \infty$, then its decay must be fast enough to hope for H^∞ well posedness.

In the study of sufficient conditions for the H^∞ well-posedness of (1.1) it is customary to impose some pointwise decay conditions as $|x| \rightarrow \infty$ on the coefficients $a_{p-j}(t, x)$, $1 \leq j \leq p - 1$, see [8,13] for $p = 2$ and [4] for general $p \geq 3$. These decay conditions are of the form

$$|\operatorname{Im} a_{p-j}(t, x)| \leq C_T \langle x \rangle^{-\frac{p-j}{p-1}}, \quad 1 \leq j \leq p - 1,$$

matched with suitable decay conditions on the x -derivatives of a_j .

Coming now to well-posedness in Gevrey type spaces, the known results are restricted to the case $p = 2$ for necessary conditions, see [9,10], and to the cases $p = 2$ and $p = 3$ for sufficient conditions, see [13] and [2] respectively. Concerning sufficient conditions, as in the H^∞ case, they are expressed as decay conditions for $|x| \rightarrow \infty$ on the coefficients of the lower order terms. As far as we know, there are no results concerning necessary conditions for Gevrey well-posedness for $p \geq 3$. The above mentioned results are settled in the following class of Gevrey functions:

$$\mathcal{H}_\theta^\infty(\mathbb{R}) := \bigcup_{\rho > 0} H_{\rho;\theta}^0(\mathbb{R}), \quad H_{\rho;\theta}^0(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : e^{\rho\langle \xi \rangle^{\frac{1}{\theta}}} \widehat{u}(\xi) \in L^2(\mathbb{R})\}$$

endowed with the norm

$$\|u\|_{H_{\rho;\theta}^0(\mathbb{R})} := \|e^{\rho\langle \cdot \rangle^{\frac{1}{\theta}}} \widehat{u}(\cdot)\|_{L^2(\mathbb{R})},$$

where $\rho > 0$ and \widehat{u} denotes the Fourier transform of u . The space $\mathcal{H}_\theta^\infty(\mathbb{R})$ is related to the class of Gevrey functions in the following sense:

$$G_0^\theta(\mathbb{R}) \subset \mathcal{H}_\theta^\infty(\mathbb{R}) \subset G^\theta(\mathbb{R}),$$

where $G^\theta(\mathbb{R})$ denotes the space of all smooth functions f such that for some $C > 0$

$$\sup_{\alpha \in \mathbb{N}_0} \sup_{x \in \mathbb{R}} |\partial_x^\alpha f(x)| C^{-\alpha} \alpha!^{-\theta} < +\infty,$$

and $G_0^\theta(\mathbb{R})$ is the space of all compactly supported functions contained in $G^\theta(\mathbb{R})$.

The definition of well-posedness in $\mathcal{H}_\theta^\infty(\mathbb{R})$ for the Cauchy problem (1.1) reads as follows:

Definition 1. We say that the Cauchy problem (1.1) is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$ if for any given $\rho_0 > 0$ there exist $\rho > 0$ and $C := C(\rho, T) > 0$ such that for all $\phi \in H_{\rho_0;\theta}^0(\mathbb{R})$ there exists a unique solution $u \in C^1([0, T]; H_{\rho;\theta}^0(\mathbb{R}))$ and the following energy inequality holds

$$\|u(t, \cdot)\|_{H_{\rho;\theta}^0(\mathbb{R})} \leq C \|\phi\|_{H_{\rho_0;\theta}^0(\mathbb{R})}, \quad \forall t \in [0, T].$$

To explain the available results in the literature in a simple way it is convenient to use the model operator (1.4) for $p = 2$ and $p = 3$ with $\sigma \in (0, 1)$. For results concerning the more general operator (1.2) we refer to [13, Theorem 1.1] in the case $p = 2$ and to [2, Theorem 1] in the case $p = 3$.

- By [10] and [13], for the operator $D_t + D_x^2 + i\langle x \rangle^{-\sigma} D_x$ we have:

$$\sigma \in (0, 1) \text{ and } 1 - \sigma \leq \frac{1}{\theta} \iff \text{well-posedness in } \mathcal{H}_\theta^\infty(\mathbb{R}), \theta > 1.$$

In the limit case $1 - \sigma = \frac{1}{\theta}$ we have local in time well-posedness, whereas when $1 - \sigma < \frac{1}{\theta}$ we get it on the whole interval $[0, T]$. The number $1/(1 - \sigma)$ works so as a threshold for the Gevrey indices θ for which well-posedness results can be found. The case $1 - \sigma > \frac{1}{\theta}$ has been also investigated in [7], where under additional exponential decay conditions on the datum g a solution in suitable Gevrey classes has been obtained.

- By [2], for the operator $D_t + D_x^3 + i\langle x \rangle^{-\sigma} D_x^2$ we have:

$$\sigma \in \left(\frac{1}{2}, 1\right) \text{ and } 2(1 - \sigma) \leq \frac{1}{\theta} \implies \text{well-posedness in } \mathcal{H}_\theta^\infty(\mathbb{R}), \theta > 1.$$

Again, in the limit case $2(1 - \sigma) = \frac{1}{\theta}$ we have local in time well-posedness, whereas when $2(1 - \sigma) < \frac{1}{\theta}$ we get it on the whole interval $[0, T]$.

Thus, in the case $p = 3$ the following two questions arise:

- Q1)** What happens when the decay rate σ is less than or equal to $\frac{1}{2}$?
- Q2)** What happens when $2(1 - \sigma) > \frac{1}{\theta}$?

The main goal of this manuscript is to answer the two above questions, and at the same time to generalize both the questions and the answers to the more general operator (1.2). Namely, we give a necessary condition on the Gevrey index θ and on the decay rates of the imaginary parts of the coefficients of (1.2) for the well-posedness of the Cauchy problem (1.1) in $\mathcal{H}_\theta^\infty(\mathbb{R})$, $\theta > 1$. Our main theorem reads as follows.

Theorem 1. *Let $\theta > 1$. Let P be an operator of the form (1.2) with $a_p \in C([0, T]; \mathbb{R})$ and $a_p(t) \neq 0 \forall t \in [0, T]$, and assume that the coefficients a_{p-j} satisfy the following conditions:*

- (i) *there exist $R, A > 0$ and $\sigma_{p-j} \in [0, 1]$, $j = 1, \dots, p - 1$, such that*

$$\text{Im } a_{p-j}(t, x) \geq A \langle x \rangle^{-\sigma_{p-j}}, \quad x > R \text{ (or } x < -R), t \in [0, T], j = 1, \dots, p - 1;$$

- (ii) *there exists $C > 0$ such that for every $\beta \in \mathbb{N}$:*

$$|\partial_x^\beta a_{p-j}(t, x)| \leq C^{\beta+1} \beta! \langle x \rangle^{-\beta}, \quad x \in \mathbb{R}, t \in [0, T], \quad j = 1, \dots, p.$$

If the Cauchy problem (1.1) is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$, then

$$\Xi := \max_{j=1, \dots, p-1} \{(p-1)(1-\sigma_{p-j}) - j + 1\} \leq \frac{1}{\theta}. \tag{1.5}$$

Remark 1. Notice that in the case $p = 2$, the condition i) in Theorem 1 is equivalent in one space dimension to the *slow decay condition* assumed in [10].

Remark 2. Let us notice that since $(p-1)(1-\sigma_{p-1}) > 0$ we always have $\Xi > 0$. We also point out the following inequalities

$$(p-1)(1-\sigma_{p-j}) - j + 1 \geq 1 \iff \sigma_{p-j} \leq \frac{p-1-j}{p-1},$$

$$(p-1)(1-\sigma_{p-j}) - j + 1 \leq 0 \iff \sigma_{p-j} \geq \frac{p-j}{p-1}.$$

Therefore, as a consequence of (1.5), when $\theta > 1$ we conclude the following:

- If $\sigma_{p-j} \leq \frac{p-1-j}{p-1}$ for some $j = 1, \dots, p-1$, the Cauchy problem is not well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$;
- If $\sigma_{p-j} \geq \frac{p-j}{p-1}$ for some $j = 1, \dots, p-1$, then the power σ_{p-j} has no effect on the $\mathcal{H}_\theta^\infty$ well-posedness;
- If $\sigma_{p-j} \in \left(\frac{p-1-j}{p-1}, \frac{p-j}{p-1}\right)$ for some $j = 1, \dots, p-1$, then the power σ_{p-j} imposes the restriction

$$(p-1)(1-\sigma_{p-j}) - j + 1 \leq \frac{1}{\theta}$$

for the indices θ where $\mathcal{H}_\theta^\infty$ well-posedness can be found.

In particular, we conclude that a first order coefficient of the type $a_1(t, x) = i\langle x \rangle^{-\sigma_1}$, $\sigma_1 \in (0, 1)$ cannot affect the $\mathcal{H}_\theta^\infty$ well-posedness of (1.1) for all $\theta > 1$, whereas $a_{p-j}(t, x) = i\langle x \rangle^{-\sigma_{p-j}}$, $j = 1, \dots, p-2$, can compromise well-posedness provided that σ_{p-j} is close enough to zero.

Remark 3. We also notice that if at least one of the coefficients $a_{p-j}(t, x)$ decays less rapidly than any negative power of $\langle x \rangle$, e.g. if $a_{p-j}(t, x) = i(\log(\sqrt{2+x^2}))^{-1}$ for some j , then a_{p-j} fulfills the assumption (ii) for any $\sigma_{p-j} > 0$. So, for any fixed $\theta > 1$ we can find $\sigma_{p-j} \in (0, 1)$ so that $(p-1)(1-\sigma_{p-j}) - j + 1 > \theta^{-1}$. Hence, there is no well-posedness in $\mathcal{H}_\theta^\infty(\mathbb{R})$, $\theta > 1$ in this case. This opens the new question of finding a suitable functional setting where the Cauchy problem is well-posed in this situation.

Considering the model operator $D_t + D_x^3 + i\langle x \rangle^{-\sigma} D_x^2$, $\sigma \in (0, 1)$, from Theorem 1 and the above remark we obtain the answer to the two questions posed before.

A1) There is no well-posedness in $\mathcal{H}_\theta^\infty(\mathbb{R})$ of the Cauchy problem for $D_t + D_x^3 + i\langle x \rangle^{-\sigma} D_x^2$ when $\sigma \leq \frac{1}{2}$.

A2) If $\sigma \in (1/2, 1)$, the Cauchy problem associated with $D_t + D_x^3 + i\langle x \rangle^{-\sigma} D_x^2$ is not well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$ when $2(1 - \sigma) > \frac{1}{\theta}$. Thus, $2(1 - \sigma)$ is the threshold for the Gevrey indices θ for which well-posedness results in $\mathcal{H}_\theta^\infty(\mathbb{R})$ can be obtained.

Remark 4. By the known results for $p = 2, 3$ and by Theorem 1, we conjecture that we can obtain well-posedness for (1.1) in $\mathcal{H}_\theta^\infty(\mathbb{R})$ for all $\theta \in (1, \Xi^{-1})$ for a generic p . Notice that the latter interval becomes more and more narrow as p increases. We intend to treat this problem in a future paper.

The paper is organized as follows. In Section 2 we recall some classical definitions and preliminary results that will be needed for the subsequent parts of the work. Section 3 is devoted to define the principal tools for the proof of Theorem 1 and to explain the strategy we will follow to prove it. Section 4 contains several technical estimates needed to obtain our main result. Finally, in Section 5, we present the proof of Theorem 1.

2. Preliminaries

In this section we fix some notation and recall the basic definitions and results we will employ in the subsequent sections. Throughout the paper we shall denote respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the scalar product and the norm in $L^2(\mathbb{R})$. For functions depending on (t, x) appearing in the next sections, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ will always denote the scalar product and the norm in $L^2(\mathbb{R}_x)$ for a fixed $t \in [0, T]$.

Given $m \in \mathbb{R}$, we denote by $S_{0,0}^m(\mathbb{R})$ the space of all functions $p \in C^\infty(\mathbb{R}^2)$ such that for any $\alpha, \beta \in \mathbb{N}_0$ the following estimate holds

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^m$$

for a positive constant $C_{\alpha,\beta}$. The topology of the space $S_{0,0}^m(\mathbb{R})$ is induced by the following family of seminorms

$$|p|_\ell^{(m)} := \max_{\alpha \leq \ell, \beta \leq \ell} \sup_{x, \xi \in \mathbb{R}} |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \langle \xi \rangle^{-m}, \quad p \in S_{0,0}^m(\mathbb{R}), \ell \in \mathbb{N}_0.$$

As usual we associate to every symbol $p \in S_{0,0}^m(\mathbb{R})$ the continuous operator on the Schwartz space of rapidly decreasing functions $p(x, D) : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$, known as pseudodifferential operator, given by

$$p(x, D)u(x) = \int e^{i\xi x} p(x, \xi) \widehat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}),$$

where $d\xi := \frac{d\xi}{2\pi}$. Sometimes we will write $p(x, D) = \text{op}(p(x, \xi))$. The next Theorem 2 gives the action of operators coming from symbols $S_{0,0}^m(\mathbb{R})$ in the standard Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, defined by

$$H^s(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) : \langle \xi \rangle^s \widehat{u}(\xi) \in L^2(\mathbb{R})\}, \quad \|u\|_{H^s(\mathbb{R})} := \|\langle \xi \rangle^s u\|.$$

We recall that, when s is a positive integer we can replace $\|u\|_{H^s(\mathbb{R})}$ by the equivalent norm

$$\|u\|_s := \sum_{j=0}^s \|D_x^j u\|.$$

Theorem 2. [Calderón-Vaillancourt] Let $p \in S_{0,0}^m(\mathbb{R})$. Then for any real number $s \in \mathbb{R}$ there exist $\ell := \ell(s, m) \in \mathbb{N}_0$ and $C := C_{s,m} > 0$ such that

$$\|p(\cdot, D)u\|_{H^s(\mathbb{R})} \leq C |p|_{\ell}^{(m)} \|u\|_{H^{s+m}(\mathbb{R})}, \quad \forall u \in H^{s+m}(\mathbb{R}).$$

Besides, when $m = s = 0$ we can replace $|p|_{\ell}^{(m)}$ by

$$\max_{\alpha, \beta \leq 2} \sup_{x, \xi \in \mathbb{R}} |\partial_{\xi}^{\alpha} \partial_x^{\beta} p(x, \xi)|.$$

For a proof of Theorem 2 we address the reader to Theorem 1.6 on page 224 of [14]. Now we consider the algebra properties of $S_{0,0}^m(\mathbb{R})$ with respect to the composition of operators. Let $p_j \in S_{0,0}^{m_j}(\mathbb{R})$, $j = 1, 2$, and define

$$\begin{aligned} q(x, \xi) &= Os - \iint e^{-iy\eta} p_1(x, \xi + \eta) p_2(x + y, \xi) dy d\eta \\ &= \lim_{\varepsilon \rightarrow 0} \iint e^{-iy\eta} p_1(x, \xi + \eta) p_2(x + y, \xi) e^{-\varepsilon^2 y^2} e^{-\varepsilon^2 \eta^2} dy d\eta. \end{aligned} \tag{2.1}$$

Then we have the following theorem (for a proof see Lemma 2.4 on page 69 and Theorem 1.4 on page 223 of [14]).

Theorem 3. Let $p_j \in S_{0,0}^{m_j}(\mathbb{R})$, $j = 1, 2$, and consider q defined by (2.1). Then $q \in S_{0,0}^{m_1+m_2}(\mathbb{R})$ and $q(x, D) = p_1(x, D)p_2(x, D)$. Moreover, the symbol q has the following asymptotic expansion

$$q(x, \xi) = \sum_{\alpha < N} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p_1(x, \xi) D_x^{\alpha} p_2(x, \xi) + r_N(x, \xi),$$

where

$$r_N(x, \xi) = N \int_0^1 \frac{(1-\theta)^{N-1}}{N!} Os - \iint e^{-iy\eta} \partial_{\xi}^N p_1(x, \xi + \theta\eta) D_x^N p_2(x + y, \xi) dy d\eta d\theta,$$

and the seminorms of r_N may be estimated in the following way: for any $\ell_0 \in \mathbb{N}_0$ there exists $\ell_1 := \ell_1(\ell_0) \in \mathbb{N}_0$ such that

$$|r_N|_{\ell_0}^{(m_1+m_2)} \leq C_{\ell_0} |\partial_{\xi}^N p_1|_{\ell_1}^{(m_1)} |\partial_x^N p_2|_{\ell_1}^{(m_2)}.$$

The last theorem that we recall is a version of the so-called sharp Gårding inequality which takes into account the polynomial behavior at infinity on $|x|$ of the symbol. To state this result it is convenient to introduce the SG symbol classes. Given $m_1, m_2 \in \mathbb{R}$, we say that a smooth

function $p \in C^\infty(\mathbb{R}^2)$ belongs to the class $SG^{m_1, m_2}(\mathbb{R}^2)$ if for every $\alpha, \beta \in \mathbb{N}_0$ there exists a positive constant $C_{\alpha, \beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m_1 - \alpha} \langle x \rangle^{m_2 - \beta}.$$

Let $p \in SG^{m_1, m_2}(\mathbb{R}^2)$, then we consider the following special type of Friedrichs' symmetrization

$$p_{F, L}(x, \xi) = Os - \iint e^{-iy\eta} p_F(\xi + \eta, x + y, \xi) dy d\eta, \tag{2.2}$$

where

$$p_F(\xi, x', \xi') = \int F(x', \xi, \zeta) p(x', \zeta) F(x', \xi', \zeta) d\zeta, \quad x', \xi, \xi' \in \mathbb{R},$$

$$F(x', \xi, \zeta) = q(\langle x' \rangle^{\frac{1}{2}} \langle \xi \rangle^{-\frac{1}{2}} (\xi - \zeta)) \langle \xi \rangle^{-\frac{1}{4}} \langle x' \rangle^{\frac{1}{4}}, \quad x', \xi, \zeta \in \mathbb{R},$$

for some even cutoff function $q \in C_0^\infty(\mathbb{R})$ such that $\int q^2 = 1$ (cf. Definition 11 of [3]). Then we have the following result (see Theorem 4 and Proposition 6 of [3]).

Theorem 4. *Let $p \in SG^{m_1, m_2}(\mathbb{R})$ and let moreover $p_{F, L}$ be the symbol defined by (2.2). Then $p_{F, L} \in SG^{m_1, m_2}(\mathbb{R})$ and $p - p_{F, L} \in SG^{m_1 - 1, m_2 - 1}(\mathbb{R})$. Moreover, if $p(x, \xi) \geq 0$ then $\langle p_{F, L}(x, D)u, u \rangle \geq 0$ for all $u \in \mathcal{S}(\mathbb{R})$.*

3. Idea of the proof

To prove our result we follow an argument inspired by [10]. We shall prove that if the Cauchy problem (1.1) is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$, then the denial of (1.5) leads to a contradiction. With this idea in mind let us start by defining the main ingredients to get the desired contradiction. Consider a Gevrey cutoff function $h \in G_0^{\theta_h}(\mathbb{R})$ for some $\theta_h > 1$ close to 1 such that

$$h(x) = \begin{cases} 1, & |x| \leq \frac{1}{2}, \\ 0, & |x| \geq 1. \end{cases}$$

For a sequence $\{\nu_k\}$ of positive real numbers such that $\nu_k \rightarrow \infty$ as $k \rightarrow \infty$, we define the following sequence of symbols

$$w_k(x, \xi) = h\left(\frac{x - 4\nu_k^{p-1}}{\nu_k^{p-1}}\right) h\left(\frac{\xi - \nu_k}{\frac{1}{4}\nu_k}\right). \tag{3.1}$$

Note that $w_k(x, \xi)$ is a symbol localized around the bicharacteristic curve of ξ^p passing through the point $(0, \nu_k)$ at some fixed time t (in this case $t = 4/p$). Indeed, the bicharacteristic curve of ξ^p (usually called *Hamilton flow* generated by the operator $D_t + D_x^p$) passing through a point $(x_0, \xi_0) \in \mathbb{R}^2$, is the solution of

$$\begin{cases} x'(t) = p\xi(t)^{p-1}, & x(0) = x_0, \\ \xi'(t) = 0, & \xi(0) = \xi_0, \end{cases} \tag{3.2}$$

that is $(x(t), \xi(t)) = (x_0 + pt\xi_0^{p-1}, \xi_0)$.

Remark 5. On the support of $w_k(\cdot, \cdot)$ we have that ξ is comparable with v_k and x is comparable with v_k^{p-1} . Indeed, if $\xi \in \text{supp } w_k(x, \cdot)$ then

$$|\xi - v_k| \leq \frac{1}{4}v_k \iff \frac{3v_k}{4} \leq \xi \leq \frac{5v_k}{4},$$

for all $k \in \mathbb{N}_0$. Similarly, if $x \in \text{supp } w_k(\cdot, \xi)$ then

$$|x - 4v_k^{p-1}| \leq v_k^{p-1} \iff 3v_k^{p-1} \leq x \leq 5v_k^{p-1},$$

for all $k \in \mathbb{N}_0$.

Consider now $\phi \in G^\theta(\mathbb{R})$ such that

$$\widehat{\phi}(\xi) = e^{-2\rho_0(\xi)^{\frac{1}{\theta}}},$$

for some $\rho_0 > 0$ and define $\phi_k(x) = \phi(x - 4v_k^{p-1})$ for all $k \in \mathbb{N}_0$. From the assumed $\mathcal{H}_\theta^\infty$ well-posedness, let $u_k \in C^1([0, T]; H_{\rho, \theta}^0(\mathbb{R}))$ be the solution of (1.1) with initial datum $\phi_k \in H_{\rho_0, \theta}^0(\mathbb{R})$. Then for $\lambda \in (0, 1)$ and $\theta_1 > \theta_h$ (still close to 1) to be chosen later, we define

$$N_k = \lfloor v_k^{\frac{\lambda}{\theta_1}} \rfloor \tag{3.3}$$

and then we introduce the energy

$$E_k(t) = \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha! \beta!)^{\theta_1}} \|w_k^{(\alpha\beta)}(x, D)u_k(t, x)\| = \sum_{\alpha \leq N_k, \beta \leq N_k} E_{k, \alpha, \beta}(t), \tag{3.4}$$

where

$$w_k^{(\alpha\beta)}(x, \xi) = h^{(\alpha)}\left(\frac{x - 4v_k^{p-1}}{v_k^{p-1}}\right) h^{(\beta)}\left(\frac{\xi - v_k}{\frac{1}{4}v_k}\right).$$

The next lemma, whose proof follows from Remark 5 and a simple computation, gives estimates for the norms of w_k .

Lemma 1. Let $\alpha, \beta, \gamma, \delta, \mu, \ell \in \mathbb{N}_0$. Then $w_k^{(\alpha\beta)} \in S_{0,0}^0(\mathbb{R}^2)$ and there exists $C := C(\theta_h)$ such that

$$|\xi^\mu \partial_x^\delta \partial_\xi^\gamma w_k^{(\alpha\beta)}(x, \xi)|_\ell^{(0)} \leq C^{\alpha+\beta+\gamma+\delta+\mu+\ell+1} (\alpha! \beta! \gamma! \delta! \ell!)^{\theta_h} v_k^{\mu-\gamma} v_k^{-\delta(p-1)}.$$

Assuming that the Cauchy problem (1.1) is $\mathcal{H}_\theta^\infty$ well-posed, by the energy estimate for the solution (see Definition 1) we simply obtain a uniform upper bound for the energies $E_k(t)$ with respect to both $k \in \mathbb{N}_0$ and $t \in [0, T]$. Indeed, the Calderón-Vaillancourt theorem implies (for the same constant C of Lemma 1) that

$$E_{k,\alpha,\beta}(t) \leq C^{\alpha+\beta+1} (\alpha!\beta!)^{\theta_h-\theta_1} \|u_k(t)\| \leq C^{\alpha+\beta+1} (\alpha!\beta!)^{\theta_h-\theta_1} \|u_k(t)\|_{H_{\rho,\theta}^0(\mathbb{R})}$$

and then from the $\mathcal{H}_\theta^\infty$ well-posedness

$$\begin{aligned} E_{k,\alpha,\beta}(t) &\leq C_{T,\rho_0} C^{\alpha+\beta} (\alpha!\beta!)^{\theta_h-\theta_1} \|\phi_k\|_{H_{\rho_0,\theta}^0(\mathbb{R})} \\ &= C_{T,\rho_0} C^{\alpha+\beta} (\alpha!\beta!)^{\theta_h-\theta_1} \|\phi\|_{H_{\rho_0,\theta}^0(\mathbb{R})} \\ &= C_{T,\rho_0,\phi} C^{\alpha+\beta} (\alpha!\beta!)^{\theta_h-\theta_1}. \end{aligned} \tag{3.5}$$

Recalling that $\theta_1 > \theta_h$, we conclude that

$$\begin{aligned} E_k(t) &\leq C_{T,\rho_0,\phi} \sum_{\alpha \leq N_k, \beta \leq N_k} C^{\alpha+\beta} (\alpha!\beta!)^{\theta_h-\theta_1} \leq C_{T,\rho_0,\phi} \sum_{\alpha,\beta \geq 0} C^{\alpha+\beta} (\alpha!\beta!)^{\theta_h-\theta_1} \\ &= C_1 C_{T,\rho_0,\phi}, \quad \forall t \in [0, T], k \in \mathbb{N}_0. \end{aligned} \tag{3.6}$$

The idea to get a contradiction is to use the energy method to obtain an estimate from below for $E_k(t)$, $t \in [0, T]$, of the type

$$E_k(t) \geq f(v_k),$$

where $f(v_k) \rightarrow +\infty$ if (1.5) does not hold. Obtaining such estimate is the most involving part of the proof, so we shall dedicate the next section to deal with this problem.

4. Estimates from below for $E_k(t)$

We denote

$$v_k^{(\alpha\beta)}(t, x) = w_k^{(\alpha\beta)}(x, D)u_k(t, x). \tag{4.1}$$

Then we have

$$P v_k^{(\alpha\beta)} = w_k^{(\alpha\beta)} \underbrace{P u_k}_{=0} + [P, w_k^{(\alpha\beta)}]u_k = [P, w_k^{(\alpha\beta)}]u_k =: f_k^{(\alpha\beta)},$$

and therefore

$$\begin{aligned}
 \|v_k^{(\alpha\beta)}\| \|\partial_t v_k^{(\alpha\beta)}\| &= \frac{1}{2} \partial_t \{\|v_k^{(\alpha\beta)}\|^2\} \\
 &= \operatorname{Re} \langle \partial_t v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \\
 &= \operatorname{Re} \langle i f_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle - \underbrace{\operatorname{Re} \langle i a_p(t) D_x^p v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle}_{=0} - \sum_{j=1}^p \operatorname{Re} \langle i a_{p-j}(t, x) D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \\
 &\geq -\|f_k^{(\alpha\beta)}\| \|v_k^{(\alpha\beta)}\| - \sum_{j=1}^{p-1} \operatorname{Re} \langle i a_{p-j}(t, x) D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle - C_{a_0} \|v_k^{(\alpha\beta)}\|^2.
 \end{aligned}
 \tag{4.2}$$

The terms $\operatorname{Re} \langle i a_{p-j}(t, x) D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle$ ($j = 1, \dots, p - 1$) will provide the contradiction, whereas $\|f_k^{(\alpha\beta)}\|$ will be negligible in some sense. Being more precise, we shall obtain estimates from above for $\|f_k^{(\alpha\beta)}\|$ and absorb them into an estimate from below for the terms $\operatorname{Re} \langle i a_{p-j}(t, x) D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle$. In the next two subsections we shall discuss these estimates.

4.1. Estimate from below for $\operatorname{Re} \langle i a_{p-j}(t, x) D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle$

For $j = 1, \dots, p - 1$ we write

$$\begin{aligned}
 &-\operatorname{Re} \langle i a_{p-j} D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \\
 &= \operatorname{Re} \langle \operatorname{Im} a_{p-j} D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle - \operatorname{Re} \langle i \operatorname{Re} a_{p-j} D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \\
 &= \operatorname{Re} \langle \operatorname{Im} a_{p-j} D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle + \frac{1}{2} \sum_{s=0}^{p-j-1} \binom{p-j}{s} \langle i D_x^{p-j-s} \operatorname{Re} a_{p-j} D_x^s v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle.
 \end{aligned}$$

Then we consider the following cutoff functions

$$\chi_k(\xi) = h\left(\frac{\xi - v_k}{\frac{3}{4}v_k}\right), \quad \psi_k(x) = h\left(\frac{x - 4v_k^{p-1}}{3v_k^{p-1}}\right).
 \tag{4.3}$$

Note that on the support of $\psi_k(x)\chi_k(\xi)$ we have the following

$$\begin{aligned}
 |\xi - v_k| \leq \frac{3}{4}v_k &\iff \frac{v_k}{4} \leq \xi \leq \frac{7v_k}{4}, \\
 |x - 4v_k^{p-1}| \leq 3v_k^{p-1} &\iff v_k^{p-1} \leq x \leq 7v_k^{p-1},
 \end{aligned}$$

for all $k \in \mathbb{N}_0$. Therefore,

$$(x, \xi) \in \operatorname{supp} \psi_k(x)\chi_k(\xi) \implies \xi^{p-j} \geq \frac{v_k^{p-j}}{4^{p-j}}, \quad \langle x \rangle^{-\sigma_{p-j}} \geq 7^{-\sigma_{p-j}} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}}.
 \tag{4.4}$$

Denoting

$$c_{p-j} = A \frac{7^{-\sigma_{p-j}}}{4^{p-j}}, \tag{4.5}$$

where A is the constant appearing in condition i) of Theorem 1, we decompose the symbol of $\text{Im } a_{p-j}(t, x) D_x^{p-j}$ as follows

$$\begin{aligned} & \text{Im } a_{p-j}(t, x) \xi^{p-j} \\ &= c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} + \left(\text{Im } a_{p-j}(t, x) \xi^{p-j} - c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} \right) \\ &= c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} + \left(\text{Im } a_{p-j}(t, x) \xi^{p-j} - c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} \right) \psi_k(x) \chi_k(\xi) \\ & \quad + \left(\text{Im } a_{p-j}(t, x) \xi^{p-j} - c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} \right) (1 - \psi_k(x) \chi_k(\xi)). \end{aligned}$$

Using assumption (i) we get for k sufficiently large

$$\text{Im } a_{p-j}(t, x) \xi^{p-j} \geq c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} + I_{p-j,k}(x, \xi) + J_{p-j,k}(t, x, \xi), \tag{4.6}$$

where

$$I_{p-j,k}(x, \xi) = \left(A \langle x \rangle^{-\sigma_{p-j}} \xi^{p-j} - c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} \right) \psi_k(x) \chi_k(\xi), \tag{4.7}$$

$$J_{p-j,k}(t, x, \xi) = \left(\text{Im } a_{p-j}(t, x) \xi^{p-j} - c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} \right) (1 - \psi_k(x) \chi_k(\xi)). \tag{4.8}$$

Let us immediately notice that by (4.4), (4.7) and the choice (4.5) of c_{p-j} we have

$$I_{p-j,k}(x, \xi) \geq 0, \quad \forall (x, \xi) \in \mathbb{R}^2. \tag{4.9}$$

Hence

$$\begin{aligned} & \text{Re} \langle \text{Im } a_{p-j}(t, x) D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \geq \text{Re} \langle c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \tag{4.10} \\ & \quad + \text{Re} \langle I_{p-j,k}(x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle + \text{Re} \langle J_{p-j,k}(t, x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \\ &= c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} \|v_k^{(\alpha\beta)}\|^2 \\ & \quad + \text{Re} \langle I_{p-j,k}(x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle + \text{Re} \langle J_{p-j,k}(t, x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \\ & \geq c_{p-j} 2^{-\sigma_{p-j}/2} v_k^{(p-1)(1-\sigma_{p-j})-j+1} \|v_k^{(\alpha\beta)}\|^2 \\ & \quad + \text{Re} \langle I_{p-j,k}(x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle + \text{Re} \langle J_{p-j,k}(t, x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle. \end{aligned}$$

From now on, we shall denote by C a positive constant independent of k, α, β, N_k but possibly depending on ϕ, ρ_0, T and on the coefficients a_{p-j} .

Lemma 2. *If (1.1) is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$, then for any $M \in \mathbb{N}$ the following estimate holds:*

$$\|D_x^r v_k^{(\alpha\beta)}\| \leq C v_k^r \|v_k^{(\alpha\beta)}\| + C^{\alpha+\beta+M+1} \{\alpha! \beta!\}^{\theta_h} M!^{2\theta_h-1} v_k^{r-M}$$

for some $C > 0$ independent of k .

Proof. We split the symbol of D_x^r as

$$\xi^r = \xi^r \chi_k(\xi) + \xi^r \{1 - \chi_k(\xi)\}.$$

Using the properties of the support of χ_k (ξ comparable with v_k) and the Calderón-Vaillancourt theorem we obtain

$$\|D_x^r \chi_k(D) v_k^{(\alpha\beta)}\| \leq C v_k^r \|v_k^{(\alpha\beta)}\|. \tag{4.11}$$

Since the supports of $w_k^{(\alpha\beta)}(x, \xi)$ and of $1 - \chi_k(\xi)$ are disjoint, recalling (4.1) we get

$$\text{op}(\xi^r \{1 - \chi_k(\xi)\}) w_k^{(\alpha\beta)}(x, D) = r_{k,M}^{(\alpha\beta)}(x, D), \tag{4.12}$$

where, defining $q_k(\xi) = \xi^r \{1 - \chi_k(\xi)\}$,

$$r_{k,M}^{(\alpha\beta)}(x, \xi) = M \int_0^1 \frac{(1-\theta)^{M-1}}{M!} O_S - \iint e^{-iy\eta} \partial_\xi^M q_k(\xi + \theta\eta) D_x^M w_k^{(\alpha\beta)}(x+y, \xi) dy d\eta d\theta.$$

We write $\partial_\xi^M q_k$ in the following way:

$$\begin{aligned} & \partial_\xi^M q_k(\xi) \\ &= \begin{cases} \{1 - \chi_k(\xi)\} \frac{r!}{(r-M)!} \xi^{r-M} - \sum_{\substack{M_1+M_2=M \\ M_1 \geq 1}} \frac{M!}{M_1! M_2!} \partial_\xi^{M_1} \chi_k(\xi) \frac{r!}{(r-M_2)!} \xi^{r-M_2}, & M \leq r, \\ - \sum_{\substack{M_1+M_2=M \\ M_2 \leq r}} \frac{M!}{M_1! M_2!} \partial_\xi^{M_1} \chi_k(\xi) \frac{r!}{(r-M_2)!} \xi^{r-M_2}, & M > r. \end{cases} \end{aligned} \tag{4.13}$$

If $M > r$ we see that $\text{supp } \partial_\xi^M q_k \subset \text{supp } \chi_k$, so we can estimate the seminorms of $r_{k,M}^{\alpha\beta}$ as follows: for every $\ell_0 \in \mathbb{N}_0$ there exists $\ell_1 := \ell_1(\ell_0)$ such that

$$|r_{k,M}^{(\alpha\beta)}|_{\ell_0}^{(0)} \leq C(\ell_0) \frac{M}{M!} |\partial_\xi^M q_k|_{\ell_1}^{(0)} |\partial_x^M w_k^{(\alpha\beta)}|_{\ell_1}^{(0)}.$$

From Lemma 1 we get

$$|\partial_x^M w_k^{(\alpha\beta)}|_{\ell_1}^{(0)} \leq C^{\ell_1+\alpha+\beta+M+1} \ell_1!^{2\theta_h} \{\alpha! \beta! M!\}^{\theta_h} v_k^{-M(p-1)}.$$

On the other hand (using the support of χ_k) we have

$$|\partial_{\xi}^M q_k(\xi)|_{\ell_1}^{(0)} \leq C^{\ell_1+M+1} \ell_1!^{\theta_h} M!^{\theta_h} v_k^{r-M}.$$

Therefore

$$|r_{k,M}^{(\alpha\beta)}|_{\ell_0}^{(0)} \leq C^{\alpha+\beta+M+1} \{\alpha!\beta!\}^{\theta_h} M!^{2\theta_h-1} v_k^{r-M} v_k^{-M(p-1)}.$$

When $M \leq r$, from (4.13) we have $\partial_{\xi}^M q_k(\xi) = \{1 - \chi_k(\xi)\}r!(r - M)!^{-1}\xi^{r-M} + b_k(\xi)$ where $\text{supp } b_k \subset \text{supp } \chi_k$. Since the part of $r_{k,M}^{(\alpha\beta)}$ involving b_k can be estimated as before, we only explain how to treat the part regarding $\{1 - \chi_k(\xi)\}r!(r - M)!^{-1}\xi^{r-M}$ which will be denoted by $a_{k,M}^{(\alpha\beta)}$ in the following. Writing

$$(\xi + \theta\eta)^{r-M} = \sum_{r_1+r_2=r-M} \frac{(r - M)!}{r_1!r_2!} \xi^{r_1} (\theta\eta)^{r_2},$$

integration by parts gives

$$a_{k,M}^{(\alpha\beta)}(x, \xi) = \frac{r!}{(r - M)!} \sum_{r_1+r_2=r-M} \frac{(r - M)!}{r_1!r_2!} M \int_0^1 \frac{(1 - \theta)^{M-1}}{M!} \theta^{r_2} Os - \iint e^{-iy\eta} \{1 - \chi_k\}(\xi + \theta\eta) D_x^{M+r_2} w_k^{(\alpha\beta)}(x + y, \xi) \xi^{r_1} dy d\eta d\theta.$$

In this way, for any $\ell_0 \in \mathbb{N}_0$ there exists $\ell_1 := \ell_1(\ell_0)$ such that

$$\begin{aligned} |a_{k,M}^{(\alpha\beta)}(x, \xi)|_{\ell_0}^{(0)} &\leq \frac{r!M}{M!} \sum_{r_1+r_2=r-M} \frac{1}{r_1!r_2!} |\{1 - \chi_k\}(\xi)|_{\ell_1}^{(0)} |\xi^{r_1} D_x^{M+r_2} w_k^{(\alpha\beta)}(x, \xi)|_{\ell_1}^{(0)} \\ &\leq \frac{r!M}{M!} \sum_{r_1+r_2=r-M} \frac{1}{r_1!r_2!} C_{\ell_0}^{\alpha+\beta+r+1} \{\alpha!\beta!M!r_2!\}^{\theta_h} v_k^{-(p-1)(M+r_2)} v_k^{r_1} \\ &\leq C_{\ell_1,r}^{\alpha+\beta+M+1} \{\alpha!\beta!\}^{\theta_h} M!^{2\theta_h-1} v_k^{r-M} v_k^{-(p-1)M}. \end{aligned}$$

Thus, we also have

$$|r_{k,M}^{(\alpha\beta)}|_{\ell_0}^{(0)} \leq C^{\alpha+\beta+M+1} \{\alpha!\beta!\}^{\theta_h} M!^{2\theta_h-1} v_k^{r-pM}, \tag{4.14}$$

when $M \leq r$. Therefore, from (4.1), (4.12), the Calderón-Vaillancourt Theorem, (4.14) and the energy estimate for u_k (coming from the assumed $\mathcal{H}_{\theta}^{\infty}(\mathbb{R})$ well-posedness) we get

$$\|D_x^r (1 - \chi_k(D)) v_k^{(\alpha\beta)}\| \leq C^{\alpha+\beta+M+1} \{\alpha!\beta!\}^{\theta_h} M!^{2\theta_h-1} v_k^{r-pM}. \tag{4.15}$$

From (4.11) and (4.15) we conclude the desired estimate. \square

4.1.1. Estimate of $\text{Re} \langle I_{p-j,k}(x, D)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle$

The symbol $I_{p-j,k}$ defined by (4.7) belongs to $SG^{p-j, -\sigma_{p-j}}(\mathbb{R}^2)$ and its seminorms are uniformly bounded with respect to k . Indeed, since x is comparable with v_k^{p-1} and ξ is comparable with v_k on the support of $\psi_k(x)\chi_k(\xi)$ we obtain

$$\begin{aligned} & |\partial_\xi^\gamma \partial_x^\delta I_{p-j,k}(x, \xi)| \\ & \leq \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \delta_1+\delta_2=\delta}} \frac{\gamma! \delta!}{\gamma_1! \gamma_2! \delta_1! \delta_2!} |\partial_\xi^{\gamma_1} \partial_x^{\delta_1} \{A(x)^{-\sigma_{p-j}} \xi^{p-j} - c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-1}\}| \\ & \quad \times |\partial_x^{\delta_2} \psi_k(x) \partial_\xi^{\gamma_2} \chi_k(\xi)| \\ & \leq \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \delta_1+\delta_2=\delta}} \frac{\gamma! \delta!}{\gamma_1! \gamma_2! \delta_1! \delta_2!} C^{\gamma_1+\delta_1+1} \gamma_1! \delta_1! \langle \xi \rangle^{p-j-\gamma_1} \langle x \rangle^{-\sigma_{p-j}-\delta_1} C^{\gamma_2+\delta_2+1} \gamma_2! \delta_2! \langle \theta_h \rangle^{\theta_h} v_k^{-\gamma_2-\delta_2(p-1)} \\ & \leq C^{\gamma+\delta+1} \{\gamma! \delta!\}^{\theta_h} \langle \xi \rangle^{p-j-\gamma} \langle x \rangle^{-\sigma_{p-j}-\delta}. \end{aligned}$$

Thanks to (4.9), we can apply Theorem 4 to $I_{p-j,k}(x, D)$ to get the following decomposition

$$I_{p-j,k}(x, D) = p_{p-j,k}(x, D) + r_{p-j,k}(x, D),$$

where $p_{p-j,k}(x, D)$ is a positive operator and $r_{p-j,k} \in SG^{p-j-1, -\sigma_{p-j-1}}(\mathbb{R}^2)$. In addition, since the seminorms of $I_{p-j,k}$ are uniformly bounded with respect to k , the same holds for the seminorms of $r_{p-j,k}$. In this way we conclude that

$$\begin{aligned} \text{Re} \langle I_{p-j,k}(x, D)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle & \geq \text{Re} \langle r_{p-j,k}(x, D)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \\ & = \text{Re} \underbrace{\langle r_{p-j,k}(x, D)\langle x \rangle^{\sigma_{p-j+1}} \langle x \rangle^{-\sigma_{p-j-1}} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle}_{\text{order } p-j-1} \\ & \geq -C \|\langle x \rangle^{-\sigma_{p-j-1}} v_k^{(\alpha\beta)}\|_{H^{p-j-1}(\mathbb{R})} \|v_k^{(\alpha\beta)}\| \\ & \geq -C \|v_k^{(\alpha\beta)}\| \sum_{r=0}^{p-j-1} \|D_x^r \{\langle x \rangle^{-1-\sigma_{p-j}} v_k^{(\alpha\beta)}\}\|. \end{aligned}$$

To handle with the terms $\|D_x^r \{\langle x \rangle^{-\sigma_{p-j-1}} v_k^{(\alpha\beta)}\}\|$ we first write

$$D_x^r \{\langle x \rangle^{-\sigma_{p-j-1}} v_k^{(\alpha\beta)}\} = \sum_{r'=0}^r \binom{r}{r'} D_x^{r'} \langle x \rangle^{-\sigma_{p-j-1}} D_x^{r-r'} v_k^{(\alpha\beta)}.$$

On the support of $D_x^{r-r'} v_k^{(\alpha\beta)}$ we have that x is comparable with v_k^{p-1} , so

$$\|D_x^r \{\langle x \rangle^{-\sigma_{p-j-1}} v_k^{(\alpha\beta)}\}\| \leq C^{r+1} v_k^{-(p-1)(\sigma_{p-j+1})} \sum_{r'=0}^r \frac{r!}{(r-r')!} \|D_x^{r-r'} v_k^{(\alpha\beta)}\|.$$

Applying Lemma 2 for $M = N_k$ we obtain

$$\|D_x^r \{ \langle x \rangle^{-\sigma_{p-j-1}} v_k^{(\alpha\beta)} \} \| \leq C v_k^{-(p-1)(\sigma_{p-j+1})+r} \|v_k^{(\alpha\beta)}\| + C^{\alpha+\beta+N_k+1} \{ \alpha! \beta! \}^{\theta_h} N_k!^{2\theta_h-1} v_k^{-(p-1)(\sigma_{p-j+1})} v_k^{r-N_k}.$$

Hence,

$$\begin{aligned} \operatorname{Re} \langle I_{p-j,k}(x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle &\geq -C v_k^{-(p-1)\sigma_{p-j-j}} \|v_k^{(\alpha\beta)}\|^2 \\ &\quad - C^{\alpha+\beta+N_k+1} \{ \alpha! \beta! \}^{\theta_h} N_k!^{2\theta_h-1} v_k^{-(p-1)\sigma_{p-j-j-N_k}} \|v_k^{(\alpha\beta)}\|. \end{aligned} \tag{4.16}$$

4.1.2. Estimate of $\operatorname{Re} \langle J_{p-j,k}(t, x, D) v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle$

Since the supports of $1 - \psi_k(x) \chi_k(\xi)$ and of $w_k^{(\alpha\beta)}(x, \xi)$ are disjoint, we may write

$$J_{p-j,k}(t, x, D) w_k^{(\alpha\beta)}(x, D) = R_{p-j,k}^{(\alpha\beta)}(t, x, D),$$

where

$$\begin{aligned} R_{p-j,k}^{(\alpha\beta)}(t, x, \xi) &= N_k \int_0^1 \frac{(1-\theta)^{N_k-1}}{N_k!} \mathcal{O}_S \\ &\quad - \iint e^{-iy\eta} \partial_\xi^{N_k} J_{p-j,k}(t, x, \xi + \theta\eta) D_x^{N_k} w_k^{(\alpha\beta)}(x + y, \xi) dy d\eta d\theta. \end{aligned}$$

The seminorms of $R_k^{(\alpha\beta)}$ can be estimated in the following way: for every $\ell_0 \in \mathbb{N}_0$ there exists $\ell_1 = \ell_1(\ell_0)$ such that

$$|R_{p-j,k}^{(\alpha\beta)}|_{\ell_0}^{(0)} \leq C(\ell_0) \frac{N_k}{N_k!} |\partial_\xi^{N_k} J_{p-j,k}|_{\ell_1}^{(0)} |\partial_x^{N_k} w_k^{(\alpha\beta)}|_{\ell_1}^{(0)}.$$

From Lemma 1 we get

$$|\partial_x^{N_k} w_k^{(\alpha\beta)}|_{\ell_1}^{(0)} \leq C^{\ell_1+\alpha+\beta+N_k+1} \ell_1!^{2\theta_h} \{ \alpha! \beta! N_k! \}^{\theta_h} v_k^{-N_k(p-1)}.$$

On the other hand, with similar computations as in (4.13) and since there is no harm into assuming $N_k \geq p$ (because $v_k \rightarrow +\infty$ and N_k is defined by (3.3)),

$$\begin{aligned} &\partial_\xi^{N_k} J_{p-j,k}(t, x, \xi) \\ &= - \sum_{\substack{N_1+N_2=N_k \\ N_1 \leq p-j}} \frac{N_k!}{N_1! N_2!} \partial_\xi^{N_1} \{ \operatorname{Im} a_{p-j}(t, x) \xi^{p-j} - c_{p-j} \langle v_k^{p-1} \rangle^{-\sigma_{p-j}} v_k^{p-j} \} \psi_k(x) \partial_\xi^{N_2} \chi_k(\xi) \end{aligned}$$

hence

$$|\partial_\xi^{N_k} J_{p-j,k}(t, x, \xi)|_{\ell_1}^{(0)} \leq C^{\ell_1+N_k+1} \ell_1!^{2\theta_h} N_k!^{\theta_h} v_k^{p-j-N_k}.$$

Therefore

$$|R_{p-j,k}^{(\alpha\beta)}|_{\ell_0}^{(0)} \leq C^{\alpha+\beta+N_k+1} \{\alpha!\beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-j-N_k} v_k^{-N_k(p-1)},$$

which allows us to conclude

$$\|J_{p-j,k}(t, x, D)v_k^{(\alpha\beta)}\| \leq C^{\alpha+\beta+N_k+1} \{\alpha!\beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-j-N_k} v_k^{-N_k(p-1)}.$$

So,

$$\operatorname{Re} \langle J_{p-j,k}(t, x, D)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)} \rangle \geq -C^{\alpha+\beta+N_k+1} \{\alpha!\beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-j-N_k} v_k^{-N_k(p-1)} \|v_k^{(\alpha\beta)}\|. \tag{4.17}$$

4.1.3. Estimate of $\sum_{s=0}^{p-j-1} \binom{p-j}{s} (i D_x^{p-j-s} \operatorname{Re} a_{p-j} D_x^s v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)})$

From assumption (ii), the support of $D_x^s v_k^{(\alpha\beta)}$ and Lemma 2 we get, for $s \leq p - j - 1$,

$$\begin{aligned} & \|D_x^{p-j-s} \operatorname{Re} a_{p-j} D_x^s v_k^{(\alpha\beta)}\| \\ & \leq C v_k^{-(p-j-s)(p-1)} \|D_x^s v_k^{(\alpha\beta)}\| \\ & \leq C v_k^{-(p-j-s)(p-1)} \{C v_k^s \|v_k^{(\alpha\beta)}\| + C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{s-N_k}\} \\ & \leq C v_k^{-j} \|v_k^{(\alpha\beta)}\| + C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{-j-N_k}. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{1}{2} \sum_{s=0}^{p-j-1} \binom{p-j}{s} (i D_x^{p-j-s} \operatorname{Re} a_{p-j} D_x^s v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}) \\ & \geq -C v_k^{-j} \|v_k^{(\alpha\beta)}\|^2 - C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{-j-N_k} \|v_k^{(\alpha\beta)}\|. \end{aligned} \tag{4.18}$$

From (4.10), (4.16), (4.17) and (4.18) using the fact that $v_k \rightarrow +\infty$ we obtain the following lemma.

Lemma 3. *If (1.1) is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$, then for all k sufficiently large the estimate*

$$\begin{aligned} - \sum_{j=1}^{p-1} \operatorname{Re} \langle i a_{p-j}(t, x) D_x^{p-j} v_k^{(\alpha\beta)}, v_k^{\alpha\beta} \rangle & \geq c_1 v_k^\Xi \|v_k^{(\alpha\beta)}\|^2 \\ & - C^{\alpha+\beta+N_k+1} \{\alpha!\beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{-1-N_k} \|v_k^{(\alpha\beta)}\|, \end{aligned}$$

holds for some $c_1 > 0$ independent of k, α, β and N_k , where Ξ is defined by (1.5).

4.2. Estimate from below of $f_k^{(\alpha\beta)}$

We start recalling that

$$f_k^{(\alpha\beta)} = [P, w_k^{(\alpha\beta)}]u_k = [D_t + a_p(t)D_x^p, w_k^{(\alpha\beta)}]u_k + \sum_{j=1}^p [a_{p-j}(t, x)D_x^{p-j}, w_k^{(\alpha\beta)}]u_k.$$

In the sequel we shall explain how to estimate the above commutators.

4.2.1. Estimate of $[D_t + D_x^p, w_k^{(\alpha\beta)}]u_k$

Since $w_k^{(\alpha\beta)}$ is independent of t we have

$$[D_t + a_p(t)D_x^p, w_k^{(\alpha\beta)}] = a_p(t) \sum_{\gamma=1}^p \frac{1}{\gamma!} \frac{p!}{(p-\gamma)!} D_x^\gamma w_k^{(\alpha\beta)} D_x^{p-\gamma}.$$

Now we use the formula

$$f(x)D^\lambda g(x) = \sum_{j=0}^\lambda (-1)^j \binom{\lambda}{j} D^{\lambda-j} (g(x)D^j f(x)) \tag{4.19}$$

for smooth functions f, g , and the following elementary identities:

$$\sum_{\gamma=1}^p \sum_{j=0}^{p-\gamma} c_{\gamma,j} a_{\gamma+j} = \sum_{\ell=1}^p \left\{ \sum_{s=1}^\ell c_{s,\ell-s} \right\} a_\ell, \quad \frac{(-1)^{\ell+1}}{\ell!} = \sum_{s=1}^\ell \frac{(-1)^{\ell-s}}{s!(\ell-s)!},$$

to deduce that

$$\begin{aligned} & [D_t + a_p(t)D_x^p, w_k^{(\alpha\beta)}] \\ &= a_p(t) \sum_{\gamma=1}^p \sum_{j=0}^{p-\gamma} \frac{(-1)^j p!}{\gamma! j! (p-\gamma-j)!} \left(\frac{1}{i v_k^{p-1}} \right)^{\gamma+j} D_x^{p-\gamma-j} \circ w_k^{((\alpha+\gamma+j)\beta)}(x, D) \\ &= a_p(t) \sum_{\ell=1}^p \left\{ \sum_{s=1}^\ell \frac{(-1)^{\ell-s}}{s!(\ell-s)!} \right\} \frac{p!}{(p-\ell)!} \left(\frac{1}{i v_k^{p-1}} \right)^\ell D_x^{p-\ell} \circ w_k^{((\alpha+\ell)\beta)}(x, D) \\ &= a_p(t) \sum_{\ell=1}^p (-1)^{\ell+1} \binom{p}{\ell} \left(\frac{1}{i v_k^{p-1}} \right)^\ell D_x^{p-\ell} \circ w_k^{((\alpha+\ell)\beta)}(x, D). \end{aligned}$$

Thus, from Lemma 2 and using the fact that $a_p(t)$ is continuous on $[0, T]$ we get

$$\begin{aligned}
 & \| [D_t + a_p(t)D_x^p, w_k^{(\alpha\beta)}]u_k \| \\
 & \leq C \sum_{\ell=1}^p \binom{p}{\ell} \left(\frac{1}{v_k^{p-1}} \right)^\ell \| D_x^{p-\ell} v_k^{((\alpha+\ell)\beta)} \| \\
 & \leq C \sum_{\ell=1}^p \frac{1}{v_k^{p(\ell-1)}} \| v_k^{((\alpha+\ell)\beta)} \| + C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-1-N_k}. \tag{4.20}
 \end{aligned}$$

4.2.2. Estimate of $[ia_{p-j}(t, x)D_x^{p-j}, w_k^{(\alpha\beta)}]u_k$

We first observe that

$$\begin{aligned}
 [ia_{p-j}(t, x)D_x^{p-j}, w_k^{(\alpha\beta)}] &= ia_{p-j}(t, x) \sum_{\gamma=1}^{p-j} \binom{p-j}{\gamma} D_x^\gamma w_k^{(\alpha\beta)}(x, D) D_x^{p-j-\gamma} \\
 &\quad - i \sum_{\gamma=1}^{N_k-1} \frac{1}{\gamma!} D_x^\gamma a_{p-j}(t, x) \partial_\xi^\gamma w_k^{(\alpha\beta)}(x, D) D_x^{p-j} + r_{p-j,k}^{(\alpha\beta)}(t, x, D), \tag{4.21}
 \end{aligned}$$

where

$$\begin{aligned}
 r_{p-j,k}^{(\alpha\beta)}(t, x, \xi) &= -iN_k \int_0^1 \frac{(1-\theta)^{N_k-1}}{N_k!} \\
 &\quad \times Os - \iint e^{-iy\eta} \partial_\xi^{N_k} w_k^{(\alpha\beta)}(x, \xi + \theta\eta) D_x^{N_k} a_{p-j}(t, x + y) \xi^{p-j} dy \bar{d}\eta d\theta.
 \end{aligned}$$

To estimate $r_{p-j,k}^{(\alpha\beta)}$ we need to use the support properties of $w_k^{(\alpha\beta)}$, then we write

$$\xi^{p-j} = (\xi + \theta\eta - \theta\eta)^{p-j} = \sum_{\ell=0}^{p-j} \binom{p-j}{\ell} (\xi + \theta\eta)^\ell (-\theta\eta)^{p-j-\ell}.$$

Now, integration by parts gives

$$\begin{aligned}
 & Os - \iint e^{-iy\eta} \partial_\xi^{N_k} w_k^{(\alpha\beta)}(x, \xi + \theta\eta) D_x^{N_k} a_{p-j}(t, x + y) \xi^{p-j} dy \bar{d}\eta \\
 &= \sum_{\ell=0}^{p-j} \binom{p-j}{\ell} \theta^{p-j-\ell} \\
 &\times Os - \iint D_y^{p-j-\ell} e^{-iy\eta} \partial_\xi^{N_k} w_k^{(\alpha\beta)}(x, \xi + \theta\eta) (\xi + \theta\eta)^\ell D_x^{N_k} a_{p-j}(t, x + y) dy \bar{d}\eta \\
 &= \sum_{\ell=0}^{p-j} \binom{p-j}{\ell} (-\theta)^{p-j-\ell}
 \end{aligned}$$

$$\times O_s - \iint e^{-iy\eta} \partial_{\xi}^{N_k} w_k^{(\alpha\beta)}(x, \xi + \theta\eta) (\xi + \theta\eta)^{\ell} D_x^{N_k+p-j-\ell} a_{p-j}(t, x + y) dy d\eta.$$

Hence, using assumption (ii), we estimate the seminorms of $r_k^{(\alpha\beta)}$ in the following way

$$\begin{aligned} |r_{p-j,k}^{(\alpha\beta)}|_{\ell_0}^0 &\leq \frac{C^{N_k+p-j}}{N_k!} \sum_{\ell=0}^{p-j} |\xi^{\ell} \partial_{\xi}^{N_k} w_k^{(\alpha\beta)}|_{\ell_1}^{(0)} |D_x^{N_k+p-j-\ell} a_{p-j}(t, x)|_{\ell_1}^{(0)} \\ &\leq C^{\alpha+\beta+N_k+1} \{\alpha! \beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-j-N_k}, \end{aligned}$$

which allows to conclude (using the assumed $\mathcal{H}_{\theta}^{\infty}$ well-posedness) that

$$\|r_{p-j,k}^{(\alpha\beta)}(\cdot, D)u_k\| \leq C^{\alpha+\beta+N_k+1} \{\alpha! \beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-j-N_k}. \tag{4.22}$$

Now we consider the second term in (4.21), and by (4.19) we write

$$\begin{aligned} &\sum_{\gamma=1}^{N_k-1} \frac{1}{\gamma!} D_x^{\gamma} a_{p-j}(t, x) \partial_{\xi}^{\gamma} w_k^{(\alpha\beta)}(x, D) D_x^{p-j} \\ &= \sum_{\gamma=1}^{N_k-1} \sum_{s=0}^{p-j} \frac{1}{\gamma!} \binom{p-j}{s} \left(\frac{4}{v_k}\right)^{\gamma} \left(\frac{i}{v_k^{p-1}}\right)^s D_x^{\gamma} a_{p-j}(t, x) D_x^{p-j-s} \circ w_k^{((\alpha+s)(\beta+\gamma))}(x, D). \end{aligned}$$

Using the support of $D_x^{p-j-s} v_k^{((\alpha+s)(\beta+\gamma))}$ and the assumption (ii) we get

$$\|D_x^{\gamma} a_{p-j}(t, x) D_x^{p-j-s} v_k^{((\alpha+s)(\beta+\gamma))}\| \leq C^{\gamma+1} \gamma! v_k^{-\gamma(p-1)} \|D_x^{p-j-s} v_k^{((\alpha+s)(\beta+\gamma))}\|.$$

Now, applying Lemma 2 with $M = N_k - \gamma$ we obtain

$$\begin{aligned} &\|D_x^{\gamma} a_{p-j}(t, x) D_x^{p-j-s} v_k^{((\alpha+s)(\beta+\gamma))}\| \leq C^{\gamma+1} \gamma! v_k^{-\gamma(p-1)} v_k^{p-j-s} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ &+ C^{\gamma+1} \gamma! v_k^{-\gamma(p-1)} C^{\alpha+s+\beta+N_k} \{\alpha! \beta! s! \gamma!\}^{\theta_h} (N_k - \gamma)!^{2\theta_h-1} v_k^{p-j-s-(N_k-\gamma)}. \end{aligned}$$

Hence

$$\begin{aligned} &\left\| \sum_{\gamma=1}^{N_k-1} \frac{1}{\gamma!} D_x^{\gamma} a_{p-j}(t, x) \partial_{\xi}^{\gamma} w_k^{(\alpha\beta)}(x, D) D_x^{p-j} u_k \right\| \leq \tag{4.23} \\ &\leq C \sum_{s=0}^{p-j} \sum_{\gamma=1}^{N_k-1} C^{\gamma} v_k^{-(p-j)(s+\gamma-1)-j(s+\gamma)} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ &+ C^{\alpha+\beta+N_k+1} \{\alpha! \beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{1-j-N_k}. \end{aligned}$$

Finally, for the first term in (4.21), we follow the same steps of Subsection 4.2.1 to get

$$\begin{aligned} & \sum_{\gamma=1}^{p-j} \binom{p-j}{\gamma} D_x^\gamma w_k^{(\alpha\beta)}(x, D) D_x^{p-j-\gamma} \\ &= \sum_{\ell=1}^{p-j} (-1)^{\ell+1} \binom{p-j}{\ell} \frac{1}{(i v_k^{p-1})^\ell} D_x^{p-j-\ell} \circ w_k^{((\alpha+\ell)\beta)}(x, D), \end{aligned}$$

and therefore

$$\begin{aligned} & \left\| i a_{p-j}(t, x) \sum_{\gamma=1}^{p-j} \binom{p-j}{\gamma} D_x^\gamma w_k^{(\alpha\beta)}(x, D) D_x^{p-j-\gamma} u_k \right\| \leq \tag{4.24} \\ & \leq C \sum_{\ell=1}^{p-j} \frac{1}{v_k^{j+p(\ell-1)}} \|v_k^{((\alpha+\ell)\beta)}\| + C^{\alpha+\beta+N_k+1} \{\alpha! \beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{1-j-N_k}. \end{aligned}$$

Gathering (4.22), (4.23) and (4.24) we obtain

$$\begin{aligned} & \sum_{j=1}^p \|[i a_{p-j}(t, x) D_x^{p-j}, w_j^{(\alpha\beta)}] u_k\| \leq \tag{4.25} \\ & \leq C \sum_{j=1}^p \sum_{s=0}^{p-j} \sum_{\gamma=1}^{N_k-1} C^\gamma v_k^{-(p-j)(s+\gamma-1)-j(s+\gamma)} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ & + C \sum_{j=1}^p \sum_{\ell=1}^{p-j} \frac{1}{v_k^{j+p(\ell-1)}} \|v_k^{((\alpha+\ell)\beta)}\| + C^{\alpha+\beta+N_k+1} \{\alpha! \beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-1-N_k} \\ & \leq C \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} C^\gamma v_k^{-(p-1)(s+\gamma-1)-(s+\gamma)} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ & + C \sum_{\ell=1}^{p-1} \frac{1}{v_k^{1+p(\ell-1)}} \|v_k^{((\alpha+\ell)\beta)}\| + C^{\alpha+\beta+N_k+1} \{\alpha! \beta!\}^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-1-N_k}. \end{aligned}$$

Now, combining (4.20) and (4.25) we can summarize the computations of Subsection 4.2 in the following lemma.

Lemma 4. *If (1.1) is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$, then*

$$\begin{aligned} \|f_k^{(\alpha\beta)}\| & \leq C \sum_{\ell=1}^p \frac{1}{v_k^{p(\ell-1)}} \|v_k^{((\alpha+\ell)\beta)}\| + C \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} C^\gamma v_k^{-(p-1)(s+\gamma-1)-s-\gamma} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ & + C^{\alpha+\beta+N_k+1} (\alpha! \beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-1-N_k} \end{aligned}$$

for some constant $C > 0$ independent from k, α, β and N_k .

4.3. A lower bound estimate for $\partial_t E_k(t)$

Lemmas 3, 4 and (4.2) give the following

$$\begin{aligned} \partial_t \|v_k^{(\alpha\beta)}\| &\geq c_1 v_k^\Xi \|v_k^{(\alpha\beta)}\| - C_{a_0} \|v_k^{(\alpha\beta)}\| - C \sum_{\ell=1}^p \frac{1}{v_k^{p(\ell-1)}} \|v_k^{((\alpha+\ell)\beta)}\| \\ &\quad - C \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} C^\gamma v_k^{-(p-1)(s+\gamma-1)-s-\gamma} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ &\quad - C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-1-N_k}. \end{aligned}$$

Therefore

$$\begin{aligned} \partial_t E_k(t) &= \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \partial_t \|v_k^{(\alpha\beta)}(t, \cdot)\| \tag{4.26} \\ &\geq \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \{c_1 v_k^\Xi - C_{a_0}\} \|v_k^{(\alpha\beta)}\| \\ &\quad - C \sum_{\ell=1}^p \frac{1}{v_k^{p(\ell-1)}} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\| \\ &\quad - C \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} C^\gamma v_k^{-(p-1)(s+\gamma-1)-s-\gamma} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ &\quad - \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} C^{\alpha+\beta+N_k+1} (\alpha!\beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-1-N_k}. \end{aligned}$$

In the sequel we shall discuss how to treat all the terms appearing in the above summation. For the first one, by (3.4) and (4.1) we simply have

$$\sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \{c_1 v_k^\Xi - C_{a_0}\} \|v_k^{(\alpha\beta)}\| = \{c_1 v_k^\Xi - C_{a_0}\} E_k(t). \tag{4.27}$$

For the second one we proceed as follows

$$\begin{aligned} \sum_{\ell=1}^p \frac{C}{v_k^{p(\ell-1)}} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha!\beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\| &= \sum_{\ell=1}^p \frac{C}{v_k^{p(\ell-1)}} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{(\alpha+\ell)!^{\theta_1}}{\alpha!^{\theta_1}} E_{k, \alpha+\ell, \beta} \\ &\leq \sum_{\ell=1}^p \frac{C N_k^{\ell\theta_1}}{v_k^{p(\ell-1)}} \left\{ E_k + \sum_{\alpha=N_k-\ell+1}^{N_k} \sum_{\beta \leq N_k} E_{k, \alpha+\ell, \beta} \right\}. \end{aligned}$$

Recalling (3.5) we get the upper bound

$$E_{k,\alpha+\ell,\beta} \leq C^{\alpha+\beta+\ell+1} \{(\alpha + \ell)! \beta!\}^{\theta_h - \theta_1},$$

so, since $\alpha + \ell \geq N_k$, we obtain

$$\sum_{\ell=1}^p \frac{C}{v_k^{p(\ell-1)}} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha! \beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\| \leq \sum_{\ell=1}^p \frac{C N_k^{\ell \theta_1}}{v_k^{p(\ell-1)}} \left\{ E_k + C^{N_k+1} N_k!^{\theta_h - \theta_1} \right\}.$$

Now we use the definition of $N_k := \lfloor v_k^{\frac{\lambda}{\theta_1}} \rfloor$ and the inequality $N_k^{N_k} \leq e^{N_k} N_k!$ to conclude that

$$\begin{aligned} \sum_{\ell=1}^p \frac{C}{v_k^{p(\ell-1)}} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha! \beta!)^{\theta_1}} \|v_k^{((\alpha+\ell)\beta)}\| & \tag{4.28} \\ & \leq C \sum_{\ell=1}^p \frac{v_k^{\ell \lambda}}{v_k^{p(\ell-1)}} \left\{ E_k + C^{N_k+1} e^{N_k(\theta_h - \theta_1)} v_k^{\frac{\lambda(\theta_h - \theta_1)N_k}{\theta_1}} \right\} \\ & \leq C \sum_{\ell=1}^p \frac{v_k^{\ell \lambda}}{v_k^{p(\ell-1)}} E_k + C^{N_k+1} v_k^{C - cN_k}, \end{aligned}$$

where, from now on, $c > 0$ shall denote a constant independent from k .

For the third term in (4.26) we recall the following inequality

$$\frac{(\beta + \gamma)!}{\beta!} \leq (\beta + \gamma)^\gamma \leq (r N_k)^\gamma \leq r^\gamma (v_k^{\frac{\lambda}{\theta_1}})^\gamma, \quad \text{provided that } \beta + \gamma \leq r N_k, r \in \mathbb{N},$$

thus, since $\lambda \in (0, 1)$, for k sufficiently large so that $C v_k^{\lambda-1} < 1$ we have

$$\begin{aligned} & \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} C^\gamma v_k^{-(p-1)(s+\gamma-1)-s-\gamma} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha! \beta!)^{\theta_1}} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \\ & = \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} \left\{ \sum_{\substack{\alpha \leq N_k-s \\ \beta \leq N_k-\gamma}} + \sum_{\substack{\alpha \leq N_k, \beta \leq N_k \\ \alpha+s > N_k \text{ or } \beta+\gamma > N_k}} \right\} C^\gamma v_k^{-(p-1)(s+\gamma-1)-s-\gamma} \left(\frac{(\alpha + s)! (\beta + \gamma)!}{\alpha! \beta!} \right)^{\theta_1} \\ & \quad \times E_{k,\alpha+s,\beta+\gamma} \\ & \leq \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} \sum_{\alpha \leq N_k, \beta \leq N_k} \left\{ \sum_{\substack{\alpha \leq N_k-s \\ \beta \leq N_k-\gamma}} + \sum_{\substack{\alpha \leq N_k, \beta \leq N_k \\ \alpha+s > N_k \text{ or } \beta+\gamma > N_k}} \right\} (C v_k^{\lambda-1})^{\gamma+s} v_k^{-(p-1)(s+\gamma-1)} E_{k,\alpha+s,\beta+\gamma} \\ & \leq \left[E_k + \sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} \sum_{\substack{\alpha \leq N_k, \beta \leq N_k \\ \alpha+s > N_k \text{ or } \beta+\gamma > N_k}} C^{\alpha+s+\beta+\gamma+1} \{(\alpha + s)! (\beta + \gamma)!\}^{\theta_h - \theta_1} \right] \end{aligned}$$

$$\leq E_k + C^{N_k+1} N_k!^{\theta_h - \theta_1}.$$

From the definition of $N_k := \lfloor v_k^{\frac{\lambda}{\theta_1}} \rfloor$ we obtain

$$\sum_{s=0}^{p-1} \sum_{\gamma=1}^{N_k-1} C^\gamma v_k^{-(p-1)(s+\gamma-1)-s-\gamma} \sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha! \beta!)^{\theta_1}} \|v_k^{((\alpha+s)(\beta+\gamma))}\| \leq E_k + C^{N_k+1} v_k^{C-cN_k} \tag{4.29}$$

for all k large enough, where C, c are positive constants independent of k .

Finally, for the last term in (4.26), using the definition of $N_k := \lfloor v_k^{\frac{\lambda}{\theta_1}} \rfloor$ and recalling that $\theta_1 > \theta_h$ we easily conclude that

$$\sum_{\alpha \leq N_k, \beta \leq N_k} \frac{1}{(\alpha! \beta!)^{\theta_1}} C^{\alpha+\beta+N_k+1} (\alpha! \beta!)^{\theta_h} N_k!^{2\theta_h-1} v_k^{p-1-N_k} \leq C^{N_k+1} v_k^{C-cN_k}. \tag{4.30}$$

From (4.27), (4.28), (4.29) and (4.30) we obtain the following proposition.

Proposition 1. *If (1.1) is well-posed in $\mathcal{H}_\theta^\infty(\mathbb{R})$, then for all $t \in [0, T]$ and k sufficiently large the inequality*

$$\partial_t E_k(t) \geq \left[c_1 v_k^\Xi - C \sum_{\ell=1}^p \frac{v_k^{\ell\lambda}}{v_k^{p(\ell-1)}} \right] E_k(t) - C^{N_k+1} v_k^{C-cN_k}$$

holds for some $C, c > 0$ independent from k and N_k .

5. The proof of Theorem 1

We are now ready to prove our main result.

Proof of Theorem 1. Denote by

$$A_k := c_1 v_k^\Xi - C \sum_{\ell=1}^p \frac{v_k^{\ell\lambda}}{v_k^{p(\ell-1)}}, \quad R_k = C^{N_k+1} v_k^{C-cN_k}. \tag{5.1}$$

By Proposition 1 we have $\partial_t E_k(t) \geq A_k E_k(t) - R_k$.

Choosing the parameter $\lambda < \min\{\Xi, 1\}$, then we have $(\ell - 1)(\lambda - p) + \lambda \leq \lambda < \Xi$, and so $\ell\lambda - p(\ell - 1) < \Xi$, and this means that the leading term in A_k is the first one; so for k sufficiently large we obtain

$$A_k \geq \frac{c_1}{2} v_k^\Xi.$$

In the next estimates we shall always consider k sufficiently large. Applying Gronwall’s inequality we obtain

$$E_k(t) \geq e^{A_k t} \left(E_k(0) - R_k \int_0^t e^{-A_k \tau} d\tau \right) \geq e^{t \frac{c_1}{2} v_k^{\frac{\lambda}{\theta_1}}} (E_k(0) - t R_k), \quad t \in (0, T]. \quad (5.2)$$

Now we need to estimate properly the term R_k from above and $E_k(0)$ from below. Recalling that $R_k = C v_k^{C-cN_k}$ and $N_k := \lfloor v_k^{\frac{\lambda}{\theta_1}} \rfloor$ we may estimate R_k from above in the following way

$$R_k \leq C e^{-c v_k^{\frac{\lambda}{\theta_1}}}. \quad (5.3)$$

The estimate for $E_k(0)$ is more involved. We have

$$\begin{aligned} E_k(0) &\geq \|w_k(x, D)\phi_k\|_{L^2(\mathbb{R}_x)} = \left\| h \left(\frac{x - 4v_k^{p-1}}{v_k^{p-1}} \right) h \left(\frac{D - v_k}{\frac{1}{4}v_k} \right) \phi_k \right\|_{L^2(\mathbb{R}_x)} \\ &= \left\| \mathcal{F} \left[h \left(\frac{x - 4v_k^{p-1}}{v_k^{p-1}} \right) \right] (\xi) * h \left(\frac{\xi - v_k}{\frac{1}{4}v_k} \right) \widehat{\phi}_k(\xi) \right\|_{L^2(\mathbb{R}_\xi)} \\ &= v_k^{p-1} \left\| e^{-4i v_k^{p-1} \xi} \widehat{h}(v_k^{p-1} \xi) * h \left(\frac{\xi - v_k}{\frac{1}{4}v_k} \right) e^{-4i v_k^{p-1} \xi} \widehat{\phi}_k(\xi) \right\|_{L^2(\mathbb{R}_\xi)} \end{aligned}$$

hence

$$E_k^2(0) \geq v_k^{2(p-1)} \int_{\mathbb{R}_\xi} \left| \int_{\mathbb{R}_\eta} \widehat{h}(v_k^{p-1}(\xi - \eta)) h \left(\frac{\eta - v_k}{\frac{1}{4}v_k} \right) \widehat{\phi}_k(\eta) d\eta \right|^2 d\xi.$$

We choose the Gevrey cutoff function h in such a way that $\widehat{h}(0) > 0$ and $\widehat{h}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Then we get an estimate from below for $E_k^2(0)$ by restricting the integration domain. Indeed, we set

$$\begin{aligned} G_{1,k} &= \left[v_k - \frac{v_k}{8}, v_k - \frac{v_k}{8} + v_k^{-p} \right] \cup \left[v_k + \frac{v_k}{8} - v_k^{-p}, v_k + \frac{v_k}{8} \right], \\ G_{2,k} &= \left[v_k - \frac{v_k}{8} - v_k^{-p}, v_k - \frac{v_k}{8} + v_k^{-p} \right] \cup \left[v_k + \frac{v_k}{8} - v_k^{-p}, v_k + \frac{v_k}{8} + v_k^{-p} \right]. \end{aligned}$$

Then $|\eta - v_k| \leq 8^{-1} v_k$ for all $\eta \in G_{1,k}$ and $v_k^{p-1} |\xi - \eta| \leq 2v_k^{-1}$ for all $\eta \in G_{1,k}$ and all $\xi \in G_{2,k}$. Now, if $(\xi, \eta) \in G_{2,k} \times G_{1,k}$, then $v_k^{p-1}(\xi - \eta)$ is close to zero at least for k large enough, and so by the choice $\widehat{h}(0) > 0$ and $\widehat{h}(\xi) \geq 0$ there exists a positive constant C such that for k large enough we have $\widehat{h}(v_k^{p-1}(\xi - \eta)) > C$. Moreover, if $\eta \in G_{1,k}$ then $h \left(\frac{\eta - v_k}{\frac{1}{4}v_k} \right) = 1$, and finally using the fact that $\widehat{\phi}_k(\eta) = e^{-2\rho_0 \langle \eta \rangle^{\frac{1}{\theta}}}$, but η is comparable with v_k on $G_{1,k}$, we can write $\widehat{\phi}_k(\eta) \geq e^{-c_{\rho_0} v_k^{\frac{1}{\theta}}}$ for a positive constant c_{ρ_0} depending on ρ_0 . Summing up, by restricting the domain of integration we have

$$\begin{aligned}
 E_k^2(0) &\geq C v_k^{2(p-1)} e^{-2c_{\rho_0} v_k^{\frac{1}{\theta}}} \int_{G_{2,k}} \left| \int_{G_{1,k}} d\eta \right|^2 d\xi \\
 &= C v_k^{-(p+2)} e^{-2c_{\rho_0} v_k^{\frac{1}{\theta}}}.
 \end{aligned}$$

In this way we get the following estimate

$$E_k(0) \geq C v_k^{-(p+2)/2} e^{-c_{\rho_0} v_k^{\frac{1}{\theta}}}. \tag{5.4}$$

From (5.2), (5.3) and (5.4) we conclude that for all $t \in (0, T]$

$$E_k(t) \geq C e^{t \frac{c_1}{2} v_k^{\tilde{\varepsilon}}} \left[v_k^{-(p+2)/2} e^{-c_{\rho_0} v_k^{\frac{1}{\theta}}} - t e^{-c v_k^{\frac{\lambda}{\theta_1}}} \right], \tag{5.5}$$

provided that $\lambda < \min\{\Xi, 1\}$ and k being sufficiently large.

Suppose now that (1.1) is $\mathcal{H}_\theta^\infty(\mathbb{R})$ well-posed and assume by contradiction that $\frac{1}{\theta} < \Xi$. Then we take $\lambda = \frac{1}{\theta} + \tilde{\varepsilon}$ with $\tilde{\varepsilon} > 0$ small so that $\lambda < \Xi$ and $\lambda < 1$. After that, we choose θ_1 very close to 1 to get $\frac{\lambda}{\theta_1} > \frac{1}{\theta}$. In this way (5.5) implies for k sufficiently large

$$E_k(t) \geq C_2 e^{\tilde{c}_0 v_k^{\tilde{\varepsilon}}} e^{-\tilde{c}_{\rho_0} v_k^{\frac{1}{\theta}}}.$$

Thus, for k sufficiently large

$$E_k(t) \geq C_3 e^{\frac{\tilde{c}_0}{2} v_k^{\tilde{\varepsilon}}} \rightarrow \infty \quad \text{for } k \rightarrow \infty.$$

The latter inequality provides a contradiction, since the assumed $\mathcal{H}_\theta^\infty$ well-posedness implies that the energy $E_k(t)$ is uniformly bounded from above, see (3.6). \square

Data availability

No data was used for the research described in the article.

References

- [1] A. Arias Junior, A. Ascanelli, M. Capiello, KdV-type equations in projective Gevrey spaces, *J. Math. Pures Appl.* 178 (2023) 110–140.
- [2] A. Arias Junior, A. Ascanelli, M. Capiello, Gevrey well posedness for 3-evolution equations with variable coefficients, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* XXV (1) (2024) 1–31.
- [3] A. Arias Junior, M. Capiello, On the sharp Gårding inequality for operators with polynomially bounded and Gevrey regular symbols, *Mathematics* 8 (2020).
- [4] A. Ascanelli, C. Boiti, L. Zanghirati, Well-posedness of the Cauchy problem for p-evolution equations, *J. Differ. Equ.* 253 (10) (2012) 2765–2795.
- [5] A. Ascanelli, C. Boiti, L. Zanghirati, A necessary condition for H^∞ well-posedness of p -evolution equations, *Adv. Differ. Equ.* 21 (2016) 1165–1196.

- [6] A. Ascanelli, M. Capiello, Weighted energy estimates for p-evolution equations in SG classes, *J. Evol. Equ.* 15 (3) (2015) 583–607.
- [7] A. Ascanelli, M. Cicognani, M. Reissig, The interplay between decay of the data and regularity of the solution in Schrödinger equations, *Ann. Mat. Pura Appl.* 199 (4) (2020) 1649–1671.
- [8] M. Cicognani, M. Reissig, Well-posedness for degenerate Schrödinger equations, *Evol. Equ. Control Theory* 3 (1) (2014) 15–33.
- [9] M. Cicognani, M. Reissig, Necessity of Gevrey-type Levi conditions for degenerate Schrödinger equations, *J. Abstr. Differ. Equ. Appl.* 5 (1) (2014) 52–70.
- [10] M. Dreher, Necessary conditions for the well-posedness of Schrödinger type equations in Gevrey spaces, *Bull. Sci. Math.* 127 (6) (2003) 485–503.
- [11] W. Ichinose, Some remarks on the Cauchy problem for Schrödinger type equations, *Osaka J. Math.* 21 (3) (1984) 565–581.
- [12] W. Ichinose, Sufficient condition on H^∞ well-posedness for Schrödinger type equations, *Commun. Partial Differ. Equ.* 9 (1) (1984) 33–48.
- [13] K. Kajitani, A. Baba, The Cauchy problem for Schrödinger type equations, *Bull. Sci. Math.* 119 (5) (1995) 459–473.
- [14] H. Kumano-Go, *Pseudo-Differential Operators*, The MIT Press, Cambridge, London, 1982.
- [15] S. Mizohata, *On the Cauchy Problem*, vol. 3, Academic Press, 2014.
- [16] A.D. Polyanin, V.F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, CRC Press, Boca Raton, FL, 2012.