Graphical Recognition of Antiderivatives: Analysis of Different Strategies Reflecting Level of Expertise Using Eye-Tracker Tool

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Abstract: The aim of this research is to identify the difficulties of the students with Calculus tasks. This study is a qualitative analysis carried out using the eye-tracker tool. The data collected allowed us to study the differences and analogies between the experts and novices in their cognitive processes. The way of reading a text can provide a lot of information about cognitive and resolution processes. Through the eye tracker instrument, it is possible to observe the ability of the subject to switch between different registers of representation. Calculus tasks concern the concept of derivative and antiderivative; in particular, students were asked to recognise the graph of an antiderivative function. Finally, this allowed us to put forward some suggestions that, in our opinion, could improve the Didactics of Mathematics at the level of the first years of academic studies, in the delicate period that accompanies the student in the transition from secondary school to university.

1 INTRODUCTION

Eye-tracking, recently, was used in the research of Mathematics Education, but the use of this technique was present in other fields, such as Psychology, Neuroscience or Linguistics or about cognitive processes' creativity (Schindler & Lilienthal, 2020). The eye-tracker tool collects data from eye movements and obtains important information about cognitive processes. Some studies (e.g. Notaro et al., 2019), indicate that in small, involuntary eye movements lies the key to decoding whether a person has learned what they are observing. In particular, on research in maths education it is possible to analyse, through the eye movements, the solving process in a mathematical task.

The transition from secondary to tertiary education is characterised by a number of difficulties for students of mathematics and for students of Science faculties. The difficulties encountered are divided into epistemological and cognitive, sociological and cultural and didactical. Students, starting a new cycle of studies, encounter new concepts and new methodologies that are not in continuity with the previous studies. For example, one epistemological and cognitive difficulty is a lack of a deep understanding of a mathematical concept. In fact, high school students generally apply algorithms and procedures but rarely master mathematical objects. Moreover, high school students have difficulties in taking notes and reading a text (De Guzmán et al., 1998). This is related to our research because the way a student reads a problem gives important information about cognitive processes.

One of the main difficulties in the transition between secondary and tertiary school is the different use of the mathematical object. During high school, the students usually use the mathematical objects in the calculation procedures, but at the university a greater grade of abstraction is required to them. In particular, university requires students to be able to handle the theoretical aspects, whereas during secondary school the focus was usually less on these aspects (Gueudet, 2008). Moreover, the ability to switch from one register of representation to another and between frames is required (Duval, 2006).

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In this study we analysed the processes involved of the students in Calculus tasks about the concept of integrals. The participants have been divided into two groups: experts and novices, this group is composed of students of Science faculty who have attended the Calculus course, while the experts are PhD students and graduate students (Andrà et al., 2009). The research of Andrà et al. (2009) showed a comparative analysis between the gaze of the experts and novices. In particular, the attention was focused on the way mathematical representations are studied. In addition, one purpose of our research is to understand, through the comparison of the two groups, which elements capture the attention of experts and novices the longest, highlighting the differences. In the literature, it has been repeatedly observed that there are quantitative and qualitative differences between novices and experts in their approach to reading a mathematical text. Peters (2010) used the Eye-tracker to analyse how mathematical constructs are perceived by students and showed that an experienced mathematician is able to identify and process relevant information more quickly than a novice. (Chumachemko et al, 2015), through an example of scanning a Cartesian coordinate system to locate a target point on it, found a greater ability of experts to use additional essential information and discard unnecessary data than novices.

Eye-tracker is a tool that allows you to study what a subject observes and the time spent on objects, tracing the trace of the person's gaze. This tool has been used to analyse cognitive processes, for example, Geometry (Simon et al., 2021), Algebra (Obersteiner & Tumpek, 2016) and interpretation of motion graphs (Ferrara & Nemirovsky, 2005). Recent research has used the eye-tracker tool to analyse the cognitive processes in high school (Spagnolo et al., 2021, 2024). The study of Just and Carpenter (1976) showed that the subject's gaze gives important information about their cognitive and learning processes, this hypothesis has been called Eye-Mind Hypothesis. Moreover, thanks to the eye-tracker instrument it is possible to analyse the ability to switch between registers (Scheiter & van Gog, 2009; Andrà et al., 2009; Spagnolo et al., 2024).

2 PURPOSE OF THE RESEARCH

In this paper, we propose to study the way in which a student, enrolled in a course of a scientific faculty, faces an analytical problem of integral type. Our intention is to try first of all to determine which tools learned during the basic mathematics courses of the first academic year he/she is able to use in order to tackle the task. The acquisition of these tools is one of the fundamental objectives of a course in Calculus (Gueudet, 2008), whatever the scientific path undertaken by the student, because of the many applications they have in the most varied fields of pure and/or applied pure sciences. A direct or indirect verification of the degree of acquisition of these competences by a student should provide a test-bed for understanding whether the didactic objectives have been optimally realised, or whether to possibly try to find alternative approaches or methodologies to improve didactics at university level. However, in solving a concrete mathematical problem, the mere acquisition of tools is not enough, without a wise and appropriate use of these tools. The latter requires an elaboration of the concepts learned, together with a comparison with the mathematical objects that constitute the problem being tackled, to determine a measure of the degree of reliability of a possible answer, followed finally by a possible verification of the veracity of the cognitive pathway used. In other words, it is fundamental to know the strategy used in facing and solving the proposed problem. At the end of a university mathematics course, therefore, the expectation is to provide not only the theoretical and practical tools of the fundamental objects dealt with in the course, but also that set of hidden and difficult-to-teach rules that regulate how these tools coexist, how they interface and how their combined action determines the cognitive process underway: what is usually called the way of reasoning. The analysis of the strategy adopted in tackling a mathematical problem should therefore also provide useful didactic suggestions for pursuing this aim more and more effectively.

One of the main problems in dealing with a mathematical problem consists in a skillful ability to operate between the different semiotic registers (textual, graphic, algebraic, functional, pictorial, etc.) with which the question is formulated (Duval, 2006). This is particularly true in a multiple-choice question, both in terms of acquisition of the information contained in the request of the problem, and in terms of comparison with the information contained in the alternative answers, and in terms of verification with the theoretical knowledge possessed by the candidate. The two questions we propose in our survey concern the ability to be able to read the differential and integral properties of the graph of a function and to be able to transform this information passing from one representation to the other in both directions (Ferrari, 2017). The advantage of this approach is twofold: on the one hand, the eye-tracker tool, by its very nature, is particularly suited to the study of the visual aspects of a mathematical question; on the other hand, a graph contains an enormous amount of information that is spatially dispersed over a vast area and has many iconic aspects. They therefore make it easier to deduce the candidate's cognitive process underlying his or her solution strategy. Graphs can be read as a whole and not be the result of unpacking (as is the case with formulae). If a student does not know how to read a graph, he/she can only focus on one part of it and lose useful information in another part. A graph should be read by decoding the relevant information, finding relationships at syntax level, discerning its global meaning, and positioning it correctly with respect to the phenomenon it is modelling.

On the basis of the above, our research questions are as follows:

(RQ1) What information is acquired from observing a graph to deduce the graph of its antiderivative? How is this information supplemented by the information content of a possible formula and how is this information used when compared to alternative graphs?

(RQ2) Are there significant differences between experts and novices in solving strategies based on the graphical properties of antiderivative functions?

(RQ3) Is it possible to draw suggestions on Didactics of Mathematics in a basic university course?

3 PRESENTATION OF THE TASK

The task concerns the ability of the candidate to deduce the possible graph of a particular antiderivative of an assigned function (Fig.1). The question presents five distinct *areas of interest*, one or the input stimulus, containing the combination of three different semiotic registers (text, formula and graph) and four for the alternatives, containing only one semiotic register (graph).

The stimulus, consisting of the combination of text, graph and formula, contains two specific cognitive difficulties in two of the registers used. Firstly, the assigned function is discontinuous, and secondly, essential information is contained in the RHS of the formula expressing the antiderivative sought. This question has a twofold interpretation: the first is purely analytical, in which, using the regularity properties of the antiderivative functions with respect

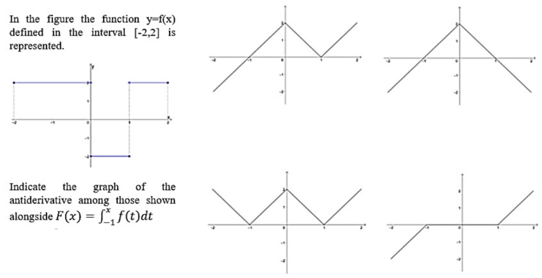

Figure 1: Identifying the antiderivative function.

to those of the integrand, one can deduce graphical properties of the antiderivative functions. On closer inspection, however, the question is in fact solvable by means of simple evaluations (*mental calculation* operations) of *areas* of geometric shapes such as triangles and rectangles. The underlying cognitive processes are therefore varied, and it is interesting to understand, by means of the eye-tracker analysis, which strategy is mainly adopted by the students when faced with a question that seems to require more abstract reasoning skills than the others. Again, the clear division into spatially distinct areas of interest will allow various levels of analysis of the acquired data.

Stimulus: the stimulus of the question is divided into three different registers: the *textual register* in the question prompt, the *graphical register* of the function *f(x)* and the *algebraic/analytical register* of the integral function. The eye-tracker analysis therefore considers the attention the candidate pays to the three registers, how he/she intersects the information he/she obtains from them and how he/she uses this information when comparing the alternatives.

The input function has a piecewise graph *with two points of discontinuity* and therefore *derivable almost everywhere*. The integral function, expressed by a *formula*, has as its lower extreme *x=-1*.

We can summarise the characteristics of the various aspects of the question by means of Fig. 2.

Visual input clues	**Analytical deduction**	**Graphical characteristics of F(x)**	**Register**
(p1) two distinct points of discon-tinuity of the first kind at *x=0* and at *x=1*	***(dp1)*** Fundamental theorem of cal-culus	***(sp1)*** singularity in the graph of *F(x)* at *x=0* and at *x=1*	graph (local)
(p2) geometric integrability from *x=-1*	***(dp2)*** Geometric interpretation of the definite integral	***(sp2)*** sign and zeros of the graph of *F(x)*	graph (global)
(p3) *f(x)* constant at times	***(dp3)*** Meaning of antiderivative in intervals	***(sp3)*** angular coefficient of straight lines in the graph of *F(x)*	graph (global)
(p4) integral function formula	***(dp4)*** Torricelli-Barrow theorem	***(sp4)*** explicit expression of *F(x)*	algebraic/ analytical

Figure 2: Analytical characteristics of the stimulus.

Alternatives: The four alternatives present graphs that are *derivable almost everywhere* with *singularity of non-derivability* and have been generated in such a way that only the correct graph satisfies all the properties ***(sp1)***, (***sp2), (sp3),*** while the other three present at least one graphical feature that makes them invalid. Property ***(sp4)*** is closely related to the presence of a formula in the stimulus. In detail:

(a) (top left graph) is the correct answer. The graph has all four of the above characteristics. It should also be noted that this graph is the only one that is not symmetrical with respect to the origin;
(b) (top right graph) *does* not have a singularity predicted by ***(sp1)*** and neither the sign nor the angular coefficient of the segment $1< x<2$ is correct for ***(sp2)*** and (***sp3)*** respectively;
(c) (bottom left graph) there is an incorrect singularity at $x=-1$ for ***(sp1)*** and neither the sign nor the angular coefficient of the segment is correct for $-2< x<-1$ for ***(sp2)*** and (***sp3)*** respectively;
(d) (bottom right graph) *does* not have a singularity at $x=0$ and, moreover, in the interval $(-1,1)$ the integral function is identically zero, thus negating the properties ***(sp1)***, (***sp2)*** and (***sp3)***.

It should be noted that although no specific distractor elements are present here, as all the graphical elements present can be considered important factors in solving the question itself, the particular *appearance of* the function can be considered a distractor: ***(PX)*** Piecewise function.

Also in the case of alternatives, as with the stimulus graph, we can consider a distractor element: ***(px)*** functions defined at intervals.

4 METHODOLOGY

Our analysis is based on the study of some typical elements used in the study of visual behaviour using the eye-tracker tool. A *fixation* is a point in the visual field where the eyes remain for a relatively long period of time, commonly in the order of tenths of a second. The most relevant characteristic parameters for fixations are their temporal duration and their number. A *saccade,* on the other hand, is a rapid spatial transition between two consecutive fixations and its characteristic parameters are their length (metric) and their frequency. A *scan path* is the study in time of repeated cycles or trajectories identified by a sequence of successive fixations. It allows, for example, to understand if the visual interest is focused on the comparison between two objects or two areas of interest in the question, or on some global aspects of a part of it. An *area of interest* is a tool to select regions of a displayed stimulus and to extract specific measures for those regions. Finally, there is a range of techniques to visualise the data recorded by an eye tracker. *Heat maps* and *gaze plots* are two methods that allow important aspects of visual behaviour to be communicated clearly and effectively. A heat map is a graphical representation that illustrates how visual attention is distributed over the various areas of interest in the question: warmer colours indicate areas where fixations have been longer lasting or more frequent. The main function of the gaze plot, on the other hand, is to reveal information about the temporal sequence of gaze or where and when the candidate looks at an object. The time spent looking, most expressed as the duration of fixation, is shown by the diameter of the fixation circles. The longer the gaze, the larger the circle.

The instrument we used to conduct the test is Tobii Pro Nano ®, a screen-based eye-tracker that acquires data obtained from the pupillary and corneal reflection with dark and bright pupillary illumination modes. The output data thus obtained, are subsequently analysed by an external processing unit capable of performing gaze calculations, using dedicated software capable of collecting eye-tracking data, observing and qualitatively analysing both individual recordings and aggregated data for comparative analysis, such as that used in this work. The latter was possible because the software used (Tobii Pro Lab ®) was able to provide us with useful and powerful analysis tools, such as video recording of eye movements, segmentation of the data with times of interest, calculation of areas of interest, and creation of heatmaps and gaze plots.

The input image was fed through the software and displayed on the screen of a monitor to which the eye-tracker camera was connected and on which the test was performed. The candidate initially faced the question with no time limit while the eye-tracker detected and recorded his eye movements. After indicating the correct answer among the four options, the candidate was subjected to a voice-recorded interview in which he tried to explain the reasons that led him to choose that solution. In addition, during the interview with the subjects, they were shown the task again and the audio recording was synchronised with this second visual analysis carried out by the eye-tracker, in order both to compare as correctly as possible the characteristics of the eye pattern detected by the instrument during the test administration phase with the cognitive motivations that generated it, and to analyse the argumentative abilities of the candidates.

The study was conducted on a novice sample consisting of 13 students enrolled in the first or second year of a science faculty (6 engineering, 3 physics, 2 chemistry, 1 mathematics, and 1 statistics) with an age range of 19 to 24 years and a sample of 7 experts consisting of mathematics undergraduates (3), doctoral students (2), and high school mathematics teachers (2).

The analysis can be done by studying the characteristics of eye patterns by studying their distribution on several levels (Andrà et al., 2009):

- *micro-level*: analysis of a specific part of a graph or formula;
- *meso-level*: analysis within each area of interest;
- *macro-level*: analysis between the different areas of interest in the question.

The eye-tracker tool allows us to *translate* our research questions in terms of the fixations, saccades and scan paths used by various candidates in the visual analysis of a mathematics question and reformulate them as follows:

- What are the visual elements that attract the most attention in students when they look at the stimulus graph?
- How much visual attention are students paid to a possible formula?
- What are the visual elements that attract the most attention when students look at the graphs of alternatives? How do they relate to those observed in the stimulus?
- What different eye trajectories are used by students when tackling a maths question?

5 QUALITATIVE DATA ANALYSIS

5.1 Experts: General Considerations

The analysis of the videos of the eye movements recorded by the eye-tracker regarding presents some general aspects (Fig. 3). The expert candidate presents a more ordered gaze plot, the fixations appear to be concentrated mainly on the graph of the input and on that of the correct answer, with an eye-tracker based mainly on the comparison between the two graphs.

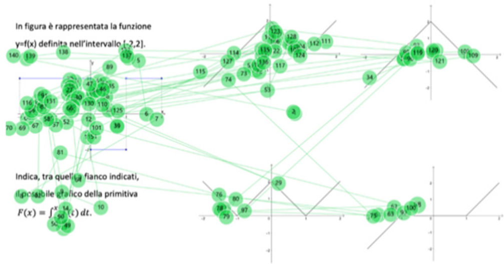

Figure 3: Heat map of an expert candidate.

There is generally a first exploratory phase of the input graph, in which the saccades involved are generally short and frequent and distributed over the whole area of interest. The presence of an analytical formula in the text acts as a partial catalyst of the expert candidate's visual attention, with some fixations on the lower extreme of the integral and some saccades with the graph of the function. The algebraic expression of the defining interval is also of partial visual interest, at least in the first phase. In the next phase, the observation of the input graph is often interspersed with sparse long saccades with some peculiar points of the alternative graphs: this is the comparison phase, in which the candidate transforms the graphical information obtained from the exploratory analysis into as much graphical information that characterises the graphs in the answers. The general behaviour of an expert candidate is to switch from a *global* type of analysis on the input to a *local* type of analysis in the answers. Despite this, the choice of the solving strategy is not the same for all expert candidates, but there are different types of approaches. Based on the data we collected, we were able to interpret and select two types of expert candidates, which we named with the acronyms "EA1" (Expert-Antiderivative 1) and "EA2", which we will now briefly discuss.

Candidate *EA1*

The first type of expert candidate, after an exploratory phase by means of short saccades on the input graph and some brief fixation on the expression of the integral, focuses its visual interest on the alternatives (Fig. 4).

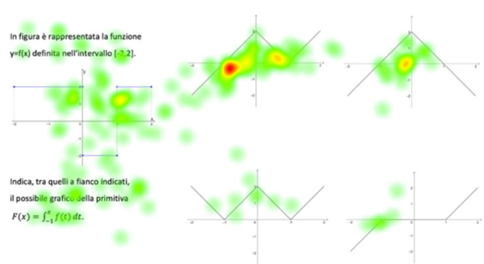

Figure 4: Heat map of candidate EA1.

In particular, his attention is initially devoted to zero at *x=-1*: the candidate immediately verifies on the graph the nulling of the integral function at that point. The candidate immediately verifies on the graph the nulling of the integral function at that point. He then transforms an analytical information contained in the expression of the integral function, namely that *F(-1)=0*, into a graphical information about the alternative answers. A candidate explicitly states: “The first thing that struck me in the integral is the lower extremity and that therefore the integral function should cancel at *-1*”. The process of comparison thus begins with a visual path that goes from the input to the alternatives, using a cognitive instrument based on a change of semantic register between the analytical and the graphical one. Once it is realised that this property is not selective as it does not eliminate any alternative response, the ocular trajectory changes abruptly. Numerous rapid saccades between the course of the alternatives’ graph around *x=-1* alternated to long saccades with the corresponding points of the graph in the input indicate that the cognitive path is now centered in the research of a characteristic of the alternatives’ graph under examination to be compared with the graphical properties of the input. In this phase, therefore, the comparison goes from the alternatives to the input, that is in the *opposite* direction to what happened previously. This is confirmed by the candidate’s own explanation: “Then I calculated the derivative of the graphs on the right and checked that it was compatible with the function on the left”. From this point of view, there is a reversal of the cognitive process: the question on the search for the antiderivative is reduced to a question on the search for the derivative. The visual analysis, therefore, of the coordinates of the singularities in the alternatives, by means of fixations on some of them, indicates a *mental calculation* for the determination of the angular coefficient of the segments present in the alternative graphs. The process of eliminating the alternatives is soon resolved: alternatives (c) and (d) are immediately eliminated, and the verification process remains isolated to only alternatives (a) and (b), which differ only in the interval *(1,2]. The* presence of the singularity at *x=1*, on which the candidate’s visual attention is concentrated at this stage through long fixations, is the decisive verification element for the resolution of the question.

Candidate *EA2*

The second type of expert candidate presents a heat map that is qualitatively different from the one examined previously (Fig. 5). Although the visual attention is distributed on the whole of the input graph, a greater number of fixations are present than in the first candidate, while the formula visual attention is captured by the lower extreme of the integral. The input graph is therefore examined with more attention in several of its aspects, such as the point *x=-1*, the share of the segments, the discontinuity points and the jump in them. This phase of analysis is interspersed with some long saccades with the graphs of the alternatives, starting from the observation around all four zeros in *x=-1* and continuing in a rather methodical way by gradually moving such saccades from the left to the right of the alternative graphs.

As the observation of the input graph proceeds in time, however, the saccades progressively decrease and soon the ocular trajectory begins to exclude the alternatives from the visual analysis. The first to disappear is (c), followed shortly after by alternative (d). This process continues until alternative (b) is also excluded from the visual path. From this point on, the saccades between the only alternative left, (a), and the input graph become sparser and are gradually replaced by saccades that follow the course of the graph (a) and by short fixations in some of its points.

Figure 5: Heat map of candidate EA2.

The qualitative analysis of the ocular behaviour of this type of candidate is centred on the fact that the discriminating element for the resolution of the question is the *area* subtended to the graph of the input function. In the discussion that followed the test, in fact, an EA2 candidate states: “I noticed that the function *f(x)* was defined in strokes through horizontal segments and the underlying area was easily calculable” and, to confirm this, the ocular trajectory seems to almost *fill* the area of the rectangle *[-2,0]×[0,2]. In* this case the exploratory phase of the initial input graph contains in itself also a *calculation* phase, in which the candidate mentally *measures* the areas and uses them for the selection of the correct candidate among the alternative ones. In this sense, candidate EA2 prefers a *geometric* approach to a purely analytical one, having to calculate areas of known geometric figures and exploiting the fact that calculating definite integrals is equivalent to calculating areas *with sign* subtended by curves. The cognitive path was therefore the following: one starts from the analytical expression of the antiderivative; one carries out the mental calculation of areas in a geometric way on the input graph and one translates this result on a graphic information of the antiderivative. Following this scheme, the mental process was gradually *exclusive*: starting from the point $x=-1$ we can immediately exclude the alternative (c) because it presents positive values of the area for $x<-1$. Immediately afterwards, alternative (d) can be excluded because, for $x>-1$ the graph presents null values of the area. As far as the remaining two alternatives (a) and (b) are concerned, it is necessary to analyse the graph up to values immediately following the second zero $x=1$, to exclude alternative (b) as it presents negative area values. The remaining phase of the candidate's visual analysis is devoted to the final *verification* phase, that the properties of (a) were compatible with the correct answer.

We report, as usual, a summary table (Fig. 6) of the main characteristics detected by the eye-tracker for the two types of expert candidates.

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	Fixations and saccades	**Prevailing comparison**	**Interest in stimulus**	**Interest in alternatives**
EA1	• short fixations on the input graph; • brief fixation on the formula • long fixations on the graph (a) • numerous long saccades between alternative graphs and the graph in the input • numerous and short saccades in (a) and (b)	• graphical input - (a) • (a) - (b)	• height of the various sections • full	• trend of (a) and its zeros
EA2	• long fixations on the input graph; • brief fixation on the formula • brief fixations on graphs (a) and (b); • numerous short saccades on the input graph	• Input chart zones • graphical input - (a)	• area subtended by the graph of $f(x)$ • point $x=-1$ • discontinuity in origin	• zeros of the function (a) • zeros of the function (b) • some non-specific points in (a)

Figure 6: Eye movement characteristics of the various types of expert candidates.

5.2 Novices: General Considerations

The novice candidate presents a rather chaotic eye path and the gaze plot in Fig. 7 is a clear example. There are long fixations distributed over most of the areas of interest of the five graphs, connected by saccades that densely fill the work sheet.

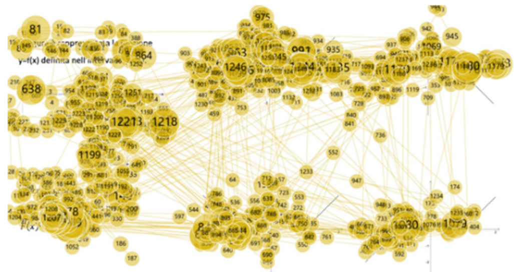

Figure 7: Typical gaze plot of the novice candidate.

For the novice candidate, moreover, the integral function formula acts to all intents and purposes as a catalyst for visual attention. A good part of the time spent observing the question is devoted to it, with long and repeated fixations on its area of interest. A formula, to all intents and purposes, appears as the most complicated stimulus for a novice to interpret, as the rules for reading a formula require a more complex semiotic register. It should also be added that for a novice, the presence of a formula in a question is considered as an implicit request to perform *explicit* (algebraic or analytical) *calculations*. In the case of the question, such a methodological approach requires knowledge of the analytical expression of the integrand function, which is absent. Such a solving method would therefore require several changes of semiotic register on the part of the candidate, who would first have to pass from the graphical register of the form of the integrand function to the analytical register, obtaining the analytical formula of the function itself (from graph to formula); the next step would be to carry out a mental calculation of the definite integral in order to obtain the explicit expression of the integral function $F(x)$ (from formula to formula) and finally to translate this analytical expression in the graphical register of the alternatives in order to determine the correct answer (from formula to graph). Moreover, the complete impossibility of the candidate using pen and paper during the test to carry out any kind of calculation made this type of approach impracticable to all intents and purposes. The presence of a formula, therefore, seems to "force" the novice candidate to select a tortuous path, even in the presence of much easier ways, which are not even considered. This generates dismay and perplexity in the candidate, who frantically searches for alternative clues using several disordered saccades on both the input graph and the four alternatives graph.

Given this complex background, for the antiderivative question we were able to identify two types of novice candidates from the data we collected with the instrument, named “NA1” and “NA2”.

Candidate *NA1*

As already mentioned above and as can be clearly seen in Fig. 8, the element that attracts the most visual attention for this type of candidate is the formula expressing the integral function of the request. However, the ocular trajectory describes a dynamic that presents peculiar and interesting cues.

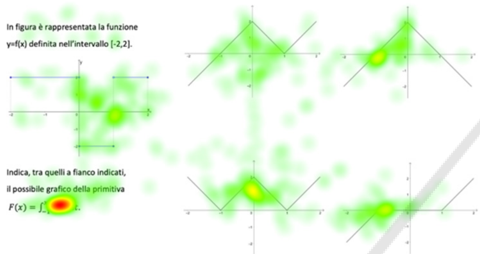

Figure 8: Heat map of candidate NA1.

In a first phase, the input graph is analysed by means of an eye trajectory characterised by some short saccades that analyse its global behaviour in its whole domain, with some short fixations on the discontinuity points. After that, the visual attention is completely captured by the formula in the request to which long fixations are devoted. From this point onwards, ocular behaviour is essentially divided into two distinct processes. On the one hand, saccades begin to occur between the formula and some points on the segments where *f(x)* is defined, while on the other hand, alternative graphs are observed with some short saccades starting from the point *x=-1* and following the trend of the represented functions. These two processes are interspersed, by means of some saccades, with different fixations on some points considered irrelevant for the resolution of the question, either belonging to the area of interest of the graph in the stimulus or, even, on *empty* points of the screen, typically included in the area in between the alternative graphs. A candidate in the following interview states: “When I saw the question, I understood that I had to calculate an integral”. The mere presence of a *formula* in the request is a sign of the need to *calculate* something: for this type of candidate the formula does not represent an alternative register to the expression “antiderivative of *f(x)* that cancels in the point *x=-1*” but represents an implicit request for *formality* that cannot be manifested except through *calculations.* From this point of view, the graphical register with which the integrand function is expressed is only an *indirect means to* be able to carry out such calculations formally and explicitly and not a *direct* means to reach resolution. Since fixations in points without *distracting* elements can be used by the brain to carry out *abstract* calculations, i.e., unrelated to what is being sensorially experienced, we were able to provide an interpretation of the candidate’s ocular behaviour. In fact, in the absence of pen and paper, the candidate is forced to mentally perform two types of calculation: first, he must find the analytical expression of *f(x)*, then he must explicitly calculate the integral. Fixations in irrelevant points of the input graph would therefore be symptomatic of the first type of calculation, those in empty points of the second. The further verification with the four alternatives is finally carried out by evaluating the trend of the segments of the answers on the basis of the calculations previously carried out. The re-transformation phase between the analytical form just calculated and the trend of the graph, necessary for the comparison of the information acquired with the four alternatives, turned out to be simpler and the candidate used several saccades to follow the trend of the various segments in the answers, with some fixations on the singularities and on the zeros, probable consequence of a final verification of the choice made. It is interesting, finally, to observe that a few candidates of this type gave as an answer option (d), and that no other category of candidates gave such an answer. The observation that the areas of the two rectangles defined in the intervals *(-1,0)* and *(0,1) are* equal in modulus but opposite in sign when calculated as integrals led to the belief that the area subtended in the interval *(-1,1)* is null because it is the *average* of two objects that are equal but opposite in sign. This seems to imply that in the purely analytical process undertaken by the candidate there is also a geometrical aspect involved, but that the two types of information are not well separated. This obviously has consequences for the distinction between the local concept of an integral function (defined point by point) and the global result of an integral defined on an interval.

Candidate *NA2*

The second type of novice candidate uses two distinct phases of visual analysis of the question, very different from each other (Fig. 9). The first phase, the exploratory phase of the stimulus, has strong similarities with the initial phase of the first type of candidate. The graphs of the alternatives are practically ignored, while the fixations on the stimulus graph are generally of short duration except for the point of discontinuity in the origin. The others are distributed mainly on the point *x=-1* and on the discontinuity jumps (identified by means of very rapid saccades).

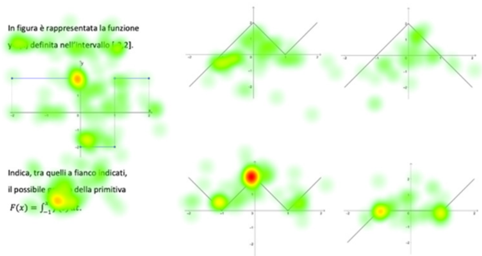

Figure 9: Heat map of candidate NA2.

We also point out that this type of candidate also dwells his gaze on the textual part of the request, even if the catalysing elements are always linked to the analytical/algebraic register contained in the symbols “f(x)” for the integrand function and “[-2,2]” of the definition interval. The visual attention is, however, mainly captured by the formula expressing the integral function, to which long fixations are devoted. Some quick saccades between the formula and the above-mentioned points of the stimulus graph finally define the ocular trajectory of this first phase. In the second phase, on the other hand, the main protagonists are the alternatives graphs, while the visual analysis of the stimulus is strongly reduced. On them there are fixations of a duration comparable to those used for the analysis of the stimulus, with some longer fixations concentrated mainly on the zeros and the singularities. Particular attention is paid to the maximum point of the graph (c) to which the longest fixations are dedicated. The saccades at this stage are rapid and sequential when analysing a single graph, while they become longer and oscillating when moving between alternative graphs. The preferred eye-paths in this sense are between alternatives (a) and (b) and between alternatives (c) and (d): of these pairs of graphs the corresponding points on the graphs and considered relevant by the candidate (zeros and singularities) are compared two by two.

One of the key elements in the NA2 candidate’s approach is the psychological aspect of the particular shape of the graph in the stimulus. One of the candidates says: “As soon as I saw the function, I knew immediately that I would have difficulty”. Asking what aspect of this function created such discomfort, the candidate replied: “I don’t know, this function had a strange form”. The adjective “strange” is linked to the fact that the function is discontinuous and defined at intervals: although analytically more complex, the function of the derivative question, continuous and smooth in all points of its domain, appears closer to the idea of the graph of a function for a novice, thus making him more comfortable in facing the question and also more self-confident, because he believes he is dealing with objects he knows better. This mental mechanism is instantaneously triggered once the visual stimulus has been received, and therefore *before* (“As soon as I saw the function”) any possible consideration about the graph of the function and its properties, with respect to the actual request, self-assigning to it a high level of difficulty a *priori* (“I knew immediately that I would have difficulties”). Any *subsequent* choice is therefore influenced by the psychological situation triggered by this process, considerably lowering the level of *awareness of* the path one intends to follow to solve the question. The candidate subsequently states: “Anyway, I think that in order to solve this exercise I should have calculated the integral, but I did not manage to do so”. The cognitive path, therefore, had been decided, but the ocular trajectory shows that the attempt is soon abandoned because it is not correctly guided as to where and what to observe of the graph of the function to acquire the appropriate tools to calculate the integral (first phase). Having discarded this solution, moreover the only one considered viable (“I should have calculated the integral”), the candidate abandons the idea of looking for the answer in the stimulus but starts to look for it directly in the alternatives. The failure to acquire information considered useful for the choice of the correct alternative triggers a process of visual comparison between the points considered representative in the alternatives in search of some usable clue. It is evident that in this phase it is the eye that determines the cognitive pathway and not vice versa.

In Fig. 10, we have listed the summary characteristics of novice candidates for the anti-derivative question.

	Fixations and saccades	Prevailing scan path	Interest in stimulus	Interest in alternatives
NA1	• short fixations on the stimulus graph • long fixations on the formula • brief fixations on some irrelevant points of the stimulus and on some 'blank' points between alternatives • numerous short diffuse saccades on both the stimulus graph and those of the four alternatives	• formula - stimulus chart • stimulus diagram - alternatives (a), (b) and (c) • (a) - (b)	• formula • points of discontinuity	• singularities • zeros
NA2	• short fixations and quick saccades on the stimulus graph • long fixations on the formula • brief fixation on alternatives • short saccades mainly on and between the graphs of alternatives	• formula - stimulus chart • (a) - (b) • (c) - (d)	• formula • discontinuity in origin	• singularities • maximum in (c) • zeros

Figure 10: Eye movement characteristics of the various types of novice candidates.

6 COMPARATIVE ANALYSES

The information collected allowed us to carry out comparative analyses between the different types of candidates and the two different questions, especially in terms of the solving strategies actually used and deduced through the analysis of the data collected by the eye-tracker.

	Stimulus data acquisition	Data use	Alternative data acquisition	Main strategies used	Main answers
EA1	*(p1), (p3)*	*(dp3)*	*(sp3)*	*(sp3) (dp3)* → *(p3)*	*(a)*
EA2	*(p1), (p2)*	*(dp2)*	*(sp2)*	*(p2) (dp2)* → *(dp2)*	*(a)*
NA1	*(p1), (p4), (PX)*	*(dp4)*	*(sp1), (sp4), (px)*	*(p4) (dp4)* → *(sp4)*	*(a), (d)*
NA2	*(p1), (p4), (PX)*	*(?)*	*(sp1), (sp2), (px)*	*[(p4), (PX), (?)]* → *[(sp1), (sp2)]*	*(b), (c)*

Figure 11: Strategies used.

To this end, the discussion carried out previously can be summarised by means of Fig. 11. Using the classification and the relative symbology described in Fig. 2, we have reported on the rows the type of candidate, while in the columns we have reported *what kind of* data and *where* they are acquired on the stimulus ("Acquisition of stimulus data"), *how* this information is used in the underlying cognitive process ("Use of data"), *what* kind of information and *where* it is acquired in the alternative graphs ("Acquisition of alternative data"), the main *strategy* (or *strategies*) used by the candidate to try to solve the question and finally the answers given by the various candidates belonging to that particular typology. As regards the strategy column, the symbology we have used is as follows:

[(hypothesis 1), (hypothesis 2), ...] (cognitive process)→***[(conclusion 1), (conclusion 2), ...].***

On a couple of occasions, a question mark ***(?)*** was inserted in the column relating to the use of the data: these were the cases in which a definite analytical deduction could not be established and the visual comparison between the stimulus and the alternatives was the result of unconscious cognitive processes, by instinct or pure chance. In both the hypotheses and the conclusions, any distracting elements that influenced the candidate's choice were reported.

7 CONCLUSIONS

From these considerations, we can draw some general conclusions and some guidelines for teaching at university level. As already widely discussed, from the point of view of the analysis (by means of the eye-tracker instrument), we can affirm that it is possible to highlight different patterns in eye movements between an expert candidate and a novice one, when facing a question on the search of the antiderivative. Moreover, this seems to signal a difference in basic cognitive approach, which could be an indicator of how the concept of *function* and its *graph, as* well as the concept of derivative and integral, is perceived by a student. In fact, the analysis of the attempts of strategies implemented to (attempt to) solve the task seems to provide useful indications on both *cognitive* and *methodological* difficulties encountered by students enrolled in the first year of a university faculty of science when they are faced with problems of an integral-differential nature and, more generally, in the interpretation or even in the construction of theoretical scientific models using these mathematical tools. In our work we have tried to *identify* the cognitive pathways involved in the candidates' approach to the proposed questions through specific eye patterns detected and to *encode them,* to try to obtain an analysis as *abstract as* possible of the solving strategies used. However, the different ocular behaviors are also indicators of *how* these mental processes are implemented and of failed attempts to derive useful information for solving the problem. A cognitive pathway at the basis of the resolution of questions such as those proposed is an extremely complex process, based on the concomitant sensory (visual) processing of many stimuli of different nature from each other. How far and to what extent the data obtained by the eye-tracker are effective indicators of such a complex system and provide a *scale of* measurement is yet to be defined by further future studies of an exquisitely quantitative nature, based on a large number of data and interpreted by means of statistical tools. Since our study is qualitative and administered to a small number of experts and novices, a fundamental weight is given to the candidate's personal *experience* which, in some way, is closely intertwined with his or her objective analytical *skills* required to solve the proposed questions. However, from the analysis carried out on the strategies used and the way of arriving at them, we have tried to identify some characteristics for two classes of candidates.

7.1 Ability to Formulate Hypotheses → Thesis Development and Verification

The cognitive path to explore a mathematical question is almost always based on a continuous alternation of processes of hypothesis formulation, thesis elaboration and verification. This process often proceeds in small steps, carried out in a sequential manner: at each success in the verification of a thesis based on a previously formulated hypothesis one proceeds with the next hypothesis, while at each failure one tries to modify or completely change the previous hypothesis. From the point of view of ocular behavior, it is very difficult to distinguish the three phases, because each individual fixation may correspond to different underlying mental processes. Nevertheless, the general analysis of ocular patterns, supported by the candidate's subsequent interview, allows us to obtain useful information. We can affirm that the quantity of the elements collected and used in the construction of the resolution strategy for the novice candidate is greater than for the expert candidate. In general, the phase of gathering information is the one, which corresponds to the formulation of hypotheses. The choice of the decisive elements for the resolution of the question corresponds to the capacity to formulate theses (analytical deduction) and to verify them (conclusions). This means that the overabundance of elements gathered by the novice inhibits or reduces the capacity to formulate valid hypotheses with a consequent increase in the difficulty of formulating theses and their possible verification. On the other hand, the acquisition of data takes place in a very different way and, consequently, the capacity to formulate hypotheses. The presence of the formula is a very important element because it represents a diversification of the semiotic register contained in the question and its role will be described later. It is already known (e.g. Andrà et al., 2009) that students pay more cognitive attention to formulas: this can be explained by the fact that a formula often condenses the information it contains more *compactly* than other registers, but also more *cryptically.* In our analysis, the presence of a formula is a catalyst for the visual (and cognitive) attention of the novice, while it does not play a fundamental role for an expert. In our opinion, however, this aspect is also linked to a greater *need for formal justification* (Brousseau, 1988) on the part of the novice, according to which the solution to a given mathematical problem is only correct if calculations and operations (or more generally formal procedures) are performed. Moreover, this need is translated in this case into the felt necessity of explicitly evaluating an integral, i.e. a calculation that is in general *more difficult* than that of the calculation of a derivative, because the same formal algorithms are not available. This difficulty has repercussions in the failure to elaborate a thesis or in the difficulty of verifying it, blocking the cognitive process since the phase of acceptance or rejection of the hypothesis formulated is missing.A last consideration on what has been said concerns the more or less accentuated use of reasoning by *exclusion of* alternatives. The use of this strategy is favoured when one is not able to formulate totally selective hypotheses on the stimulus or, in any case, when it is easier to do so on the alternatives: this process, therefore, implies a *shift of* visual attention from the graph of the stimulus to that of the alternatives for the phase of formulation of the hypotheses and vice versa for the phase of verification of the thesis. Our analysis shows that the exclusion process is more marked in experts than in novices. This seems to confirm that the process of formulating hypotheses is more difficult for novices due to their reduced ability to invert the deductive process and thus be able to "swap" the role of the hypotheses with that of the thesis.

7.2 Ability to Change the Direction of the Cognitive Process

Closely related to what has just been said is the candidate's ability to change the direction of the cognitive process, i.e. the ability to interchange the area of interest in which information is collected (and hypotheses are formulated) with that in which such information is compared (verification of the thesis). This entails not only the counter-nominal reasoning described above, in which the negation of a previously elaborated thesis is formulated as a hypothesis, but also a *transformation of* this thesis into a different one, more easily verifiable. An example is the case of the type of expert candidate who transforms the question about finding the antiderivative of the stimulus among the alternatives into a question about finding the derivative of the alternatives in the stimulus. The preferred path for a novice candidate is in general *unidirectional* and goes from the stimulus to the alternatives, both for the question about finding the derivative and the antiderivative; on the contrary, the expert candidate is more ductile in changing roles between the two areas of interest (Fig. 12).

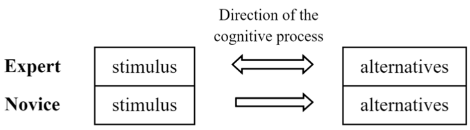

Figure 12: Direction of cognitive process in experts and novices.

Moreover, we recall that the analysis of expert candidates is global type in the stimulus and local type in the alternatives. The ability to change direction of the cognitive process is therefore linked to the ability to compare and verify local and global properties of functions and, consequently, to the ability to handle the concept of an *integral function.* From this point of view, the novice candidate appears much more attached to the punctual concepts. In particular, the choice of a point x_0 in the integral function seems to lead the novice to believe that he has to compare the *local* value at x_0 in the alternative with the *local* value in x_0 in the stimulus, regardless of the global behaviour of the integrand function in the interval $[-1, x_0]$. A significant example of this is the already mentioned choice of answer (d) by some novice students who, at point x_0=1 they assign to the whole interval *[-1,1]* the null value to the antiderivative, since $\int_{-1}^{1} f(t)dt = 0$.

7.3 Ability to Interpret Different Semantic Fields

Another characteristic that seems to distinguish an expert from a novice who tries to solve a mathematical problem is the ability to handle the different semantic fields with which the mathematical objects in the problem are described. We observe that, as it has been constructed, the semantic field expressed in the stimulus graph is different from the semantic field of a graph of alternatives, because the information expressed by it is different. In solving this question, therefore, the candidate must have a dual competence: knowing how to *read* and *translate* two different semiotic registers and knowing how to *interpret* and *transform* the two semantic fields of the stimulus graphs and the alternatives.

From our analysis, we can conclude that an expert candidate, regardless of the specific strategy used to solve the task, has a greater ability both in using several semiotic registers simultaneously and in interpreting the different semantic fields of the graphic register. A significant example is the interpretation of the graph of the antiderivative either as a function expressing an area (*geometric* interpretation), or as a function whose derivative (almost everywhere) provides the integrand function (*analytic* interpretation). For a novice candidate there does not seem to be a clear distinction between the *geometrical* and *analytical* aspects and, probably, this is one of the main problems of the reduced ability, compared to the expert colleague, to change the direction of the cognitive process.

7.4 Consequences for the Didactics of Mathematics at University Level

The discussion we have presented in this paper provides, in our opinion, some suggestions for improving the teaching of mathematics at university level. In general, we can state that students need to learn *how to* look at a mathematical task and how to correctly *direct* their gaze in order to obtain the relevant information to address the problem at hand. Mason (2008) argues that learning is an education in *awareness* that is closely related to *attention* and *observation*. Teaching therefore means directing a student's attention to becoming aware of what they are not yet aware of and one of the roles of the teacher is to push students to become familiar with the different semiotic registers of mathematical objects and the different semantic fields for each register. In the specific case of a basic analysis course, however, we believe that there are some unavoidable objectives, such as the knowledge of the concepts of derivative and integral and their applications. This can then be achieved by constructing appropriate examples and exercises that highlight how a problem should be read: for example, how a certain graph can have different properties and provide different information depending on whether it is the graph of a function describing a certain model, or of its derivative or, again, of one of its antiderivatives. Furthermore, we believe it is appropriate to help develop the students' ability to formulate hypotheses and elaborate theses, because this process is fundamental in any scientific reasoning, whether theoretical or applied. This could be remedied by administering small tests, similar to the ones we propose in this paper, specially designed to encourage students to use logical deductions and to highlight the difference between the concepts of "integral" and "integral function". Finally, we would like to draw attention to the didactic problem of introducing the integral-differential concepts in a course of Calculus. What usually happens is that first the concept of derivative of a function is defined, then the computational algorithms to determine it are illustrated, and finally all its properties are discussed, by means of the well-known theorems, useful above all for the construction of the graph of a function. Only at this point is the concept of the antiderivative introduced, but the greater difficulty of integral calculus and the lack of general algorithms relegates the role of teaching to the teaching of the various calculation techniques, thus creating, in fact, a distancing from its antiderivative functional meaning and its graphic link with the integrand function. Students may thus be given the idea that the study and information of antiderivatives is in reality only connected with the difficulty of the explicit calculation of an integral. Our suggestion is to try to bring forward in a *parallel* way the concept of derivative and antiderivative from the beginning, at least on the elementary functions, presenting the link between the graphs of functions and those of derivatives and antiderivatives, and only afterwards to present the differential applications linked to the complete study of a function and the techniques of integral calculation.

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