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An optimal lower bound in fractional spectral geometry for planar sets with topological constraints

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Abstract

We prove a lower bound on the first eigenvalue of the fractional Dirichlet–Laplacian of order s on planar open sets, in terms of their inradius and topology. The result is optimal, in many respects. In particular, we recover a classical result proved independently by Croke, Osserman, and Taylor, in the limit as s goes to 1. The limit as s goes to $1/2$ is carefully analyzed, as well.

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Contents

1 | INTRODUCTION

1.1 | Goal of the paper

In this paper, we pursue our investigation on geometric estimates for the following sharp fractional Poincaré constant

$$\lambda_1^s(\Omega) := \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{[u]^2_{W^{s,2}(\mathbb{R}^2)}}{\|u\|^2_{L^2(\Omega)}}, \tag{1.1}$$

on planar open sets $\Omega \subseteq \mathbb{R}^2$. Here the parameter $0 < s < 1$ represents a fractional order of differentiation and the quantity $[\,\cdot\,]_{W^{s,2}(\mathbb{R}^2)}$ is given by

$$[u]_{W^{s,2}(\mathbb{R}^2)} = \left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy\right)^{\frac{1}{2}}, \qquad \text{for every } u \in C_0^\infty(\mathbb{R}^2).$$

All functions in $C_0^\infty(\Omega)$ are considered as elements of $C_0^\infty(\mathbb{R}^2)$, by extending them to be zero outside Ω. The infimum in (1.1) can be equivalently performed on the space $\widetilde{W}_0^{s,2}(\Omega)$. The latter is defined as the closure of $C_0^\infty(\Omega)$ in the fractional Sobolev–Slobodeckiĭ space

$$W^{s,2}(\mathbb{R}^2) = \left\{ u \in L^2(\mathbb{R}^2) \,:\, [u]_{W^{s,2}(\mathbb{R}^2)} < +\infty \right\},$$

endowed with its natural norm. Whenever the infimum (1.1) becomes a minimum on this larger space $\widetilde{W}_0^{s,2}(\Omega)$, the quantity $\lambda_1^s(\Omega)$ will be called *first eigenvalue of the fractional Dirichlet–Laplacian of order s on* Ω.

The constant $\lambda_1^s(\Omega)$ can be seen as a fractional counterpart of

$$\lambda_1(\Omega) := \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}},$$

which coincides with the bottom of the spectrum of the more familiar Dirichlet–Laplacian on Ω. The link between λ_1^s and λ_1 can be made more precise by recalling that

$$\lim_{s\nearrow 1}(1-s)\,[u]^2_{W^{s,2}(\mathbb{R}^2)} = C\,\|\nabla u\|^2_{L^2(\Omega)}, \qquad \text{for every } u \in C_0^\infty(\Omega),$$

for some universal constant $C > 0$, see [8] or [14, chapter 3].

The present paper is a continuation of our previous work [6], to which we refer for more background material. In particular, we still focus on getting lower bounds on $\lambda_1^s(\Omega)$, in terms of the *inradius* of Ω, which is defined by

$$r_\Omega := \sup\left\{ r > 0 \;:\; \exists\, x_0 \in \Omega \text{ such that } B_r(x_0) \subseteq \Omega \right\},$$

where $B_r(x_0)$ is the open disk of center x_0 and radius r.

In [6, Theorem 1.1], extending a classical result of Makai [21] and Hayman [18] valid for λ_1 (see also [2, 3] and [4]), we showed that we have

$$\lambda_1^s(\Omega) \geqslant C_s \left(\frac{1}{r_\Omega}\right)^{2\,s},$$

for every *simply connected* open set $\Omega \subseteq \mathbb{R}^2$ with finite inradius and for every $1/2 < s < 1$. Here the constant C_s depends on s only and it has the following asymptotic behaviors[†]

$$C_s \sim (2\,s-1) \;\text{ for } s \searrow \frac{1}{2} \qquad \text{and} \qquad C_s \sim \frac{1}{1-s} \;\text{ for } s \nearrow 1.$$

Moreover, we showed by means of a counterexample, that for $0 < s \leqslant 1/2$ such a lower bound is not possible (see [6, Theorem 1.3]).

In the present paper, we considerably extend this result, by considering open connected planar sets having *nontrivial topology*. More precisely, we will work with the following class of sets:

Definition. Let us indicate by $(\mathbb{R}^2)^*$ the *one-point compactification* of $\mathbb{R}^2$, that is, the compact space obtained by adding to $\mathbb{R}^2$ the point at infinity. We say that an open connected set $\Omega \subseteq \mathbb{R}^2$ is *multiply connected of order k* if its complement in $(\mathbb{R}^2)^*$ has k connected components. When $k = 1$, we will simply say that Ω is *simply connected*.

We thus seek for an estimate of the type

$$\lambda_1^s(\Omega) \geqslant C_{s,k} \left(\frac{1}{r_\Omega}\right)^{2\,s},$$

for open multiply connected sets of order k in the plane. In light of the simply connected case recalled above, we can directly restrict our analysis to the case $1/2 < s < 1$ only.

[†] Here, the writing "$f \sim g$ for $x \to x_0$" has to be intended in the following sense

$$0 < \liminf_{x\to x_0} \frac{f(x)}{g(x)} \leqslant \limsup_{x\to x_0} \frac{f(x)}{g(x)} < +\infty.$$

1.2 | The Croke–Osserman–Taylor inequality

For the classical case of λ_1, the first lower bound of this type is due to Osserman. Notably, [23, Theorem, p. 546] shows that

$$\lambda_1(\Omega) \geqslant \min\left\{\frac{1}{4}, \frac{1}{k^2}\right\}\left(\frac{1}{r_\Omega}\right)^2,$$

for every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order k. The proof by Osserman is based on a refinement of the so-called *Cheeger's inequality*, in conjunction with Bonnesen-type inequalities.

It turns out that the estimate by Osserman does not display the sharp dependence on the topology of the sets, that is, the term $1/k^2$ is suboptimal, as k diverges to ∞. Indeed, the result by Osserman has been improved by Taylor in [30, Theorem 2], showing that

$$\lambda_1(\Omega) \geqslant \frac{C}{k}\left(\frac{1}{r_\Omega}\right)^2,$$

for some constant $C > 0$ that is not made explicit in [30]. The dependence on k is now optimal, for k going to ∞. The proof by Taylor is quite sophisticated and completely different from Osserman's one: it is based on estimating the first eigenvalue of the Laplacian with mixed boundary conditions (i.e., Dirichlet and Neumann) of a set, in terms of the *capacity* of the "Dirichlet region". Such an estimate is achieved by means of heat kernel estimates. This method is connected with Taylor's work [31] on the *scattering length* of a positive potential, which acts as a perturbation of the Laplacian (see also [28] for a generalization to the case of the fractional Laplacian). We will come back in a moment on Taylor's proof, as our main result will be based on the same arguments.

An improvement of Taylor's estimate has been given by Croke, who gives the explicit lower bound

$$\lambda_1(\Omega) \geqslant \frac{1}{2\,k}\left(\frac{1}{r_\Omega}\right)^2,$$

for $k \geqslant 2$ (see [12, Theorem]). The proof by Croke is more elementary and based on refining Osserman's argument.

Finally, for completeness we mention [17, Theorem 3] by Graversen and Rao, which proves the following lower bound

$$\lambda_1(\Omega) \geqslant \frac{C_k}{r_\Omega^2}, \qquad \text{where} \qquad C_k = \begin{cases} 1/4, & \text{if } k = 1, \\ \dfrac{A}{k\,\log k}, & \text{if } k \geqslant 2, \end{cases}$$

for some $A > 0$ (see [17, Theorem 3]). Their result is slightly worse when compared with the ones by Croke and Taylor. We notice that the proof in [17] uses techniques from the theory of Brownian motion, which are quite close to the ideas by Taylor.

1.3 | Main results

Our goal is to generalize the Croke–Osserman–Taylor result to the setting of fractional Sobolev spaces. We also want to discuss the optimality of the estimate we obtain, with respect to the parameters k and s.

Theorem 1.1 (Main Theorem). *Let $1/2 < s < 1$, there exists a constant $\vartheta_s > 0$ such that for every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order $k \in \mathbb{N} \setminus \{0\}$, we have*

$$\lambda_1^s(\Omega) \geqslant \frac{\vartheta_s}{k^s}\left(\frac{1}{r_\Omega}\right)^{2\,s}. \tag{1.2}$$

Moreover, the constant ϑ_s has the following asymptotic behaviors

$$\vartheta_s \sim (2\,s-1) \text{ for } s \searrow \frac{1}{2} \qquad \text{and} \qquad \vartheta_s \sim \frac{1}{1-s} \text{ for } s \nearrow 1.$$

The next result shows that the estimate (1.2) is sharp, *apart from the evaluation of the absolute constant*[†].

Theorem 1.2 (Optimality). *The following facts hold.*

(1) *For every $\Omega \subseteq \mathbb{R}^2$ open set, we have*

$$\limsup_{s\nearrow 1}(1-s)\,\lambda_1^s(\Omega) \leqslant \frac{1}{2}\,\lambda_1(\Omega).$$

Thus, the estimate (1.2) is sharp in its dependence on $s \nearrow 1$. In particular, by taking the limit as s goes to 1 in (1.2), we get the classical Croke–Osserman–Taylor inequality, possibly with a worse constant.

(2) *Let $1/2 < s < 1$, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}\setminus\{0\}} \subseteq \mathbb{R}^2$ of open sets such that Ω_k is multiply connected of order k*

$$r_{\Omega_k} \leqslant C \qquad \text{and} \qquad \limsup_{k\to\infty} k^s\,\lambda_1^s(\Omega_k) < +\infty.$$

Thus, the estimate (1.2) is sharp in its dependence on $k \to \infty$.

(3) *For every $k \in \mathbb{N} \setminus \{0\}$, there exists $\Theta_k \subseteq \mathbb{R}^2$ an open multiply connected set of order k, such that*

$$r_{\Theta_k} < +\infty \qquad \text{and} \qquad \limsup_{s\searrow \frac{1}{2}} \frac{\lambda_1^s(\Theta_k)}{2\,s-1} < +\infty.$$

Thus, the estimate (1.2) is sharp in its dependence on $s \searrow 1/2$.

1.4 | Comments on the proofs

As anticipated above, the statement of Theorem 1.1 contains our previous result [6, Theorem 1.1] as a particular case. Indeed, the latter was concerned with simply connected sets, that is, with

[†] This is a quotation from Taylor's paper, see [30, p. 452].

the case $k = 1$. However, the proof given here is completely different: the elegant and elementary argument used in [6], taken from [18], crucially exploited the simple connectedness and would not work here. Actually, a much more sophisticated argument is needed now. We also point out that it seems extremely complicated to adapt the proof by Osserman (and Croke), because a genuine Cheeger's inequality is still missing in the fractional case.

The general strategy for proving Theorem 1.1 will be the same as in [30]. However, even if we closely follow Taylor's ideas, some important modifications are needed and new technical difficulties arise. In addition, we tried to simplify and/or expand some of the arguments contained in [30]. We now expose the overall strategy of the proof and highlight the main changes needed to cope with the fractional case.

(1) At first, we tile the whole plane $\mathbb{R}^2$ by a family of squares $\{Q_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$. By observing that for every $u \in C_0^\infty(\Omega)$, we have

$$[u]^2_{W^{s,2}(\mathbb{R}^2)} \geqslant \sum_{(i,j)\in\mathbb{Z}^2} \iint_{Q_{i,j}\times Q_{i,j}} \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy,$$

we can reduce the problem to proving a "regional" fractional Poincaré inequality on squares such that $Q_{i,j} \cap \Omega \neq \emptyset$. Of course, the main difficulty lies in getting such an inequality with an explicit constant, which only depends on the geometry (i.e., on r_Ω) and topology (i.e., on k) of the open set Ω.

(2) This type of Poincaré inequality is possible only if $u \in C_0^\infty(\Omega)$ vanishes on "sufficiently large portions" of $Q_{i,j}$, for every square $Q_{i,j}$ intersecting Ω. Here "largeness" has to be intended in the sense of *fractional Sobolev capacity*. Thus, the first important step of this strategy is to prove a *Maz'ya-type Poincaré inequality* on a square, for functions vanishing on a compact subset Σ of positive fractional capacity (see Proposition 4.3). The constant in such an inequality can be estimated from below in terms of the capacity of the "Dirichlet region" Σ.

(3) The second step consists in converting the previous *analytic* estimate into a *geometric* one. In other words, we have to bound from below the fractional capacity of the "Dirichlet region" Σ in terms of some of its geometric features. This can be done by using orthogonal projections, which enable a dimensional reduction argument. In the two-dimensional setting, this permits to estimate the fractional capacity of Σ in terms of the length of its orthogonal projection on a line. Such an estimate is possible as soon as *points have positive fractional capacity in dimension 1.* This happens precisely if and only if $s > 1/2$.

(4) The previous two points clarify that, in order to conclude the proof, we need to know that in each square $Q_{i,j}$ intersecting Ω, there is a "Dirichlet region" $\Sigma_{i,j}$ having at least an orthogonal projection "large enough," that is, with a length bounded from below in terms of r_Ω and k, in a uniform way.

Here we crucially rely on a topological argument by Taylor, that we have called "Taylor's fatness lemma" (see Lemma 2.1). In a nutshell, it asserts that any multiply connected planar set Ω with finite inradius has a "locally uniformly fat" complement. This means that, if we choose the size of $Q_{i,j}$ sufficiently large (in terms of r_Ω and k), then this square must contain a portion of $\mathbb{R}^2 \setminus \Omega$ that has an orthogonal projection with

$$\text{length} \simeq \sqrt{k}\, r_\Omega,$$

in a universal fashion, that is, no matter the location of the square.

Differently from Taylor's paper, we work here with a *variational* definition of (fractional) capacity (see, for example, [1, 25, 26, 32]), which appears more natural and well-adapted to the problem. This permits to prove the Poincaré inequality at point (2) above in an elementary way, by avoiding both the heat kernel estimate and the reference to an eigenvalue with mixed boundary conditions used in [30]. Both points would have been problematic (or at least complicated) in the fractional setting. Also, we point out that our proof of the Maz'ya-type Poincaré inequality is genuinely *nonlinear* in nature.

As for point (3): with respect to [30], we expand the explanations and try to make the geometric estimates as much quantitative as possible. There is in addition a technical difficulty linked to the fractional case: in the classical case treated by Taylor, one essential ingredient of the dimensional reduction argument is the following simple algebraic fact

$$|\partial_\omega u|^2 \leqslant |\nabla u|^2, \qquad \text{for every } \omega \in \mathbb{S}^{N-1}.$$

In the fractional case, there is no direct analogue of this simple formula. Nevertheless, it is possible to give a sort of fractional counterpart of this property (see Proposition 3.3), but the proof is by far less straightforward: in order to prove it, we find it useful to resort to some *real interpolation techniques* (see also [9, appendix B]). These permit to "localize the nonlocality," in a sense. We think this part to be interesting in itself.

The "fatness lemma" of point (4) would be just a topological fact and could be directly recycled in the fractional case. However, in [30] this is not explicitly stated in the form that can be found below. Here as well, we tried to add some details and precisions. We believe that the final outcome should be useful to have a better understanding of Taylor's proof.

Finally, in all the estimates presented above, a great effort is needed in order to obtain the correct asymptotic behavior of $s \mapsto \vartheta_s$ claimed in Theorem 1.1. In particular, getting the sharp asymptotic behavior for $s \searrow 1/2$ requires a very careful analysis. Accordingly, proving that the set Θ_k in Theorem 1.2 provides the sharp decay rate at 0 needs quite refined (though elementary) estimates. We point out that this part is new already for the simply connected case, previously considered in [6].

1.5 | Plan of the paper

All the needed notations are settled in Section 2. Here, we also state and prove Taylor's fatness lemma. Section 3 contains some technical facts on fractional Sobolev spaces that are useful for our main result, though hard to trace back in the literature. The uninterested reader may skip this part on a first reading. We then introduce the relevant notion of fractional capacity in Section 4 and prove the main building blocks for obtaining the fractional Croke–Osserman–Taylor inequality. Sections 5 and 6 contain the proofs of Theorems 1.1 and 1.2, respectively. Finally, the paper is concluded by two appendices (Appendices A and B).

2 | PRELIMINARIES

2.1 | Notation

For every $\alpha \in \mathbb{R}$, we denote its *integer part* by

$$\lfloor \alpha \rfloor = \max \left\{ n \in \mathbb{Z} \,:\, \alpha \geqslant n \right\}.$$

We recall that

$$\alpha - 1 \leqslant \lfloor \alpha \rfloor \leqslant \alpha, \qquad \text{for every } \alpha \in \mathbb{R}. \tag{2.1}$$

For $r > 0$ and $x_0 \in \mathbb{R}^N$, we will indicate

$$B_r(x_0) = \left\{ x \in \mathbb{R}^N \ : \ |x - x_0| < r \right\},$$

and

$$Q_r(x_0) = \prod_{i=1}^{N} (x_0^i - r, x_0^i + r), \qquad \text{where } x_0 = (x_0^1, \dots, x_0^N).$$

When the center x_0 coincides with the origin, we will simply write B_r and Q_r, respectively. We will indicate by ω_N the N-dimensional Lebesgue measure of B_1.

For completeness, we also recall the following classical definition from point-set topology.

Definition. Let $K \subseteq \mathbb{R}^N$, we say that K is a *continuum* if it is a nonempty compact and connected set.

For every $\omega \in \mathbb{S}^{N-1}$, we will indicate by

$$\langle \omega \rangle^\perp = \left\{ x \in \mathbb{R}^N \ : \ \langle x, \omega \rangle = 0 \right\},$$

the orthogonal space to ω. We will also set

$$\begin{array}{rccc} \Pi_\omega & : & \mathbb{R}^N & \to & \langle \omega \rangle^\perp \\ & & x & \mapsto & x - \langle x, \omega \rangle \, \omega, \end{array} \tag{2.2}$$

that is, this is the orthogonal projection on $\langle \omega \rangle^\perp$. In particular, for $N = 2$, if we indicate by $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ the normal vectors of the canonical basis, we get that

$$\Pi_{\mathbf{e}_1}(x_1, x_2) = (0, x_2), \qquad \Pi_{\mathbf{e}_2}(x_1, x_2) = (x_1, 0), \qquad \text{for every } (x_1, x_2) \in \mathbb{R}^2.$$

For $m \in \mathbb{N} \setminus \{0\}$, we will indicate by $\mathcal{H}^m$ the m-dimensional Hausdorff measure.

Finally, for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a bounded measurable set $E \subseteq \mathbb{R}^N$ with positive measure, we set

$$\mathrm{av}(u; E) := \fint_E u \, dx = \frac{1}{|E|} \int_E u \, dx,$$

the integral average of u over E.

2.2 | Fatness of the complement of a multiply connected set

As explained in the introduction, the following geometric fact will be a crucial ingredient of our main result.

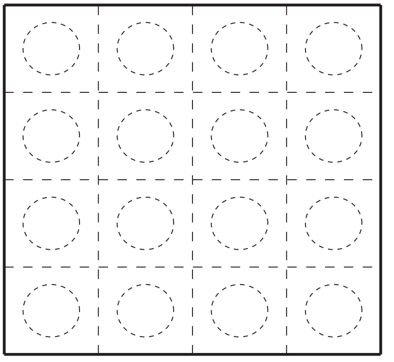

FIGURE 1 The construction of disks and squares in the proof of Lemma 2.1, for the cases $k = 1$, $k = 2$ or $k = 3$ (i.e., $\delta = 2$). Each disk contains at least a point belonging to $\mathbb{R}^2 \setminus \Omega$. The *reliable* squares are those for which such a point can be "connected" to the boundary of the "cell" containing it, with a continuum lying outside of Ω.

Lemma 2.1 (Taylor's fatness lemma). *Let $k \in \mathbb{N} \setminus \{0\}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k, with finite inradius. Let Q be an open square with side length $10\,(\lfloor\sqrt{k}\rfloor + 1)\, r_\Omega$, whose sides are parallel to the coordinate axes. Then there exists a compact set $\Sigma \subseteq \overline{Q} \setminus \Omega$ such that*

$$\max\left\{\mathcal{H}^1(\Pi_{\mathbf{e}_1}(\Sigma)),\, \mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma))\right\} \geqslant \frac{\sqrt{k}}{4}\, r_\Omega. \tag{2.3}$$

Proof. Let us set $\delta = \lfloor\sqrt{k}\rfloor + 1$, for notational simplicity. By dilating and translating, there is no loss of generality in assuming $r_\Omega = 1$ and

$$Q = Q_{5\,\delta}(0) = (-5\,\delta, 5\,\delta) \times (-5\,\delta, 5\,\delta).$$

We can suppose that $Q \cap \Omega \neq \emptyset$, otherwise the proof is trivial: it would be sufficient to take $\Sigma = \overline{Q}$ to get the desired conclusion.

We then fix the following set of $4\,\delta^2$ centers

$$P_{j,m} = \left(-5\,\delta + \frac{5}{2} + 5\,j,\ 5\,\delta - \frac{5}{2} - 5\,m\right), \qquad \text{for } j, m = 0, \dots, 2\,\delta - 1,$$

and take accordingly the two family of squares and disks (Figures 1 and 2), given by

$$B_{\frac{3}{2}}(P_{j,m}) \subseteq Q_{\frac{5}{2}}(P_{j,m}), \qquad \text{for } j, m = 0, \dots, 2\,\delta - 1.$$

We observe that by construction we have

$$\operatorname{dist}\Big(B_{\frac{3}{2}}(P_{j,m}), \partial Q_{\frac{5}{2}}(P_{j,m})\Big) = 1, \qquad \text{for every } j, m = 0, \dots, 2\,\delta - 1. \tag{2.4}$$

As $r_\Omega = 1$, our open set Ω can not entirely contain an open disk with radius larger than 1. Thus, we have that each disk $B_{3/2}(P_{j,m})$ must intersect the complement $\mathbb{R}^2 \setminus \Omega$. Let us select a point $X_{j,m} \in B_{3/2}(P_{j,m}) \setminus \Omega$. We will say that a square $Q_{5/2}(P_{j,m})$ is:

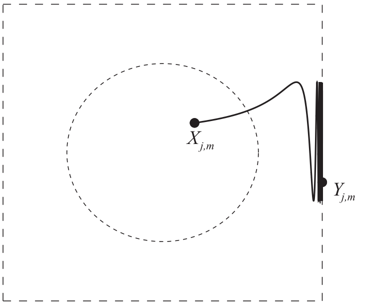

FIGURE 2 A zooming on a reliable square $Q_{5/2}(P_{j,m})$. The bold line corresponds to a continuum that connects the point $X_{j,m}$ to the boundary of the "cell," lying outside of Ω.

- *unreliable* if for every continuum $K \subseteq \overline{Q_{\frac{5}{2}}(P_{j,m})} \setminus \Omega$ such that $X_{j,m} \in K$, we have

$$K \cap \partial Q_{\frac{5}{2}}(P_{j,m}) = \emptyset;$$

- *reliable* if there exists a continuum $K_{j,m} \subseteq \overline{Q_{\frac{5}{2}}(P_{j,m})} \setminus \Omega$ such that $X_{j,m} \in K_{j,m}$ and

$$K_{j,m} \cap \partial Q_{\frac{5}{2}}(P_{j,m}) \neq \emptyset.$$

We observe that every unreliable square must contain at least a connected component of $(\mathbb{R}^2)^* \setminus \Omega$. Thus, by definition of multiply connected set of order k, the unreliable squares can be at most k. Thus, if we set

$$\mathcal{N} = \left\{(j,m) \,:\, Q_{\frac{5}{2}}(P_{j,m}) \text{ is reliable}\right\},$$

we get[†]

$$\#\mathcal{N} \geqslant 4\,\delta^2 - k = 4\left(\lfloor\sqrt{k}\rfloor + 1\right)^2 - k \geqslant 3\left(\lfloor\sqrt{k}\rfloor + 1\right)^2 = 3\,\delta^2.$$

That is, our square Q contains at least $3\,\delta^2$ reliable squares. We want to work with these squares and their continua $K_{j,m}$ defined above. By construction, we have

$$K_{j,m} \subseteq \overline{Q} \setminus \Omega.$$

We are ready to construct the compact set Σ of the statement: this is given by[‡]

$$\Sigma = \bigcup_{(j,m)\in\mathcal{N}} K_{j,m}.$$

[†] We denote by # the cardinality of a discrete set.

[‡] We notice that this union is not necessarily a disjoint one.

We need to show that its projections along the coordinate axes satisfy (2.3). At this aim, we first observe that $K_{j,m}$ is a connected set, containing both the point $X_{j,m} \in B_{3/2}(P_{j,m})$ and a point $Y_{j,m} \in \partial Q_{5/2}(P_{j,m})$. By recalling (2.4), we have that

$$|X_{j,m} - Y_{j,m}| \geqslant 1.$$

Moreover, we have that at least one of the two quantities

$$|\Pi_{\mathbf{e}_1}(X_{j,m}) - \Pi_{\mathbf{e}_1}(Y_{j,m})| \qquad \text{or} \qquad |\Pi_{\mathbf{e}_2}(X_{j,m}) - \Pi_{\mathbf{e}_2}(Y_{j,m})|,$$

is larger than or equal to 1 (recall that all the squares involved have sides parallel to the coordinate axes). By using this fact, together with the fact that both projections

$$\Pi_{\mathbf{e}_i}\big(K_{j,m}\big), \qquad \text{for } i = 1, 2,$$

coincide with a segment containing both $\Pi_{\mathbf{e}_i}(X_{j,m})$ and $\Pi_{\mathbf{e}_i}(Y_{j,m})$, we can finally assure that at least one of the two projections of $K_{j,m}$ has a length larger than or equal to 1. To conclude, we need to take care of the possible overlaps in these projections. Let us denote by $J_1, J_2 \in \mathbb{N}$ the following numbers

$$J_i = \#\Big\{K_{j,m} \,:\, \mathcal{H}^1(\Pi_{\mathbf{e}_i}(K_{j,m})) \geqslant 1\Big\}, \qquad \text{for } i = 1, 2.$$

According to the previous discussion, we have

$$J_1 + J_2 \geqslant 3\,\delta^2 \qquad \text{and thus in particular} \qquad \max\{J_1, J_2\} \geqslant \delta^2.$$

Without loss of generality, we can suppose that $J_2 \geqslant J_1$. This implies that there are at least δ^2 "good" projections, that is, projections with length at least 1, on the first coordinate axis. We need to estimate the number of such projections, modulo possible overlaps: observe that for every fixed $m \in \{0, \dots, 2\,\delta - 1\}$, the array of squares

$$\overline{Q_{m,0}(P_{m,0})}, \dots, \overline{Q_{m,2\,\delta-1}(P_{m,2\,\delta-1})},$$

all have the same projection on the first coordinate axis. Thus, the number of distinct projections is at least

$$\frac{\delta^2}{2\,\delta} = \frac{\delta}{2}.$$

As a technical and annoying fact, we record that this could fail to be a natural number. However, if we set

$$\Lambda_k = \begin{cases} 1, & \text{for } k \in \{1, 2, 3\}, \\[2mm] \dfrac{\lfloor \sqrt{k} \rfloor}{2}, & \text{for } k \geqslant 4 \text{ such that } \lfloor \sqrt{k} \rfloor \text{ is even}, \\[2mm] \dfrac{\lfloor \sqrt{k} \rfloor - 1}{2}, & \text{for } k \geqslant 4 \text{ such that } \lfloor \sqrt{k} \rfloor \text{ is odd}, \end{cases}$$

we have

$$\frac{\delta}{2} \geqslant \Lambda_k.$$

Thus, we have at least Λ_k distinct projections on the first coordinate axis, each having length at least 1. This in turn yields

$$\mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma)) \geqslant \Lambda_k.$$

Finally, by observing that $\Lambda_k \geqslant \sqrt{k}/4$, we get the claimed conclusion. □

2.3 | Functional spaces

We need some definitions from the theory of fractional Sobolev spaces. We refer the reader to [13, 14] for a brief introduction to these spaces, as well as for further references.

Let $0 < s < 1$ and $1 < p < \infty$, for a measurable set $E \subseteq \mathbb{R}^N$ we recall the definition of Sobolev–Slobodeckiĭ seminorm

$$[u]_{W^{s,p}(E)} := \left(\iint_{E\times E} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\, dx\, dy \right)^{\frac{1}{p}}, \qquad \text{for } u \in L^1_{\rm loc}(E).$$

Accordingly, we consider

$$W^{s,p}(E) = \left\{ u \in L^p(E) \,:\, [u]_{W^{s,p}(E)} < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(E)} = \|u\|_{L^p(E)} + [u]_{W^{s,p}(E)}, \qquad \text{for every } u \in W^{s,p}(E).$$

Occasionally, we will need these definitions for $p = \infty$. For $0 < s < 1$, we set

$$W^{s,\infty}(E) = \left\{ u \in L^\infty(E) \,:\, [u]_{W^{s,\infty}(E)} < +\infty \right\},$$

where

$$[u]_{W^{s,\infty}(E)} := \sup_{x,y\in E, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^s}.$$

When $E \subseteq \mathbb{R}^N$ is an open set, we will also consider the classical Sobolev space

$$W^{1,p}(E) = \left\{ u \in L^p(E) \,:\, [u]_{W^{1,p}(E)} < +\infty \right\},$$

where we used the symbol

$$[u]_{W^{1,p}(E)} := \|\nabla u\|_{L^p(E)}, \qquad \text{for every } u \in W^{1,p}(E).$$

The space $W^{1,p}(E)$ will be endowed with the norm

$$\|u\|_{W^{1,p}(E)} = \|u\|_{L^p(E)} + [u]_{W^{1,p}(E)}, \qquad \text{for every } u \in W^{1,p}(E).$$

In the case $p = \infty$, the definition of this space does not need any further precision. Finally, for $0 < s \leqslant 1$ and $1 < p \leqslant \infty$, the symbol $\widetilde{W}_0^{s,p}(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ in the space $W^{s,p}(\mathbb{R}^N)$. By $W^{s,p}_{\mathrm{loc}}(\mathbb{R}^N)$, we mean the collection of functions that are in $W^{s,p}(B_R)$, for every $R > 0$.

3 | SOME FACTS FROM THE THEORY OF FRACTIONAL SOBOLEV SPACES

Unless otherwise stated, all the results of this section are valid in every dimension $N \geqslant 1$.

We start with the following generalization of [6, Lemma 2.2]. The main focus is on the precise form of the estimates.

Proposition 3.1. *Let $r > 0$ and $x_0 \in \mathbb{R}^N$, there exists a linear extension operator*

$$\mathcal{E}_r : L^1(B_r(x_0)) \to L^1_{\mathrm{loc}}(\mathbb{R}^N),$$

with the following property:

for $0 < s \leqslant 1$ and $1 < p \leqslant \infty$ it maps $W^{s,p}(B_r(x_0))$ to $W^{s,p}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, for every $u \in W^{s,p}(B_r(x_0))$ and every $R > r$ we have[†]

$$[\mathcal{E}_r[u]]_{W^{s,p}(B_R(x_0))} \leqslant 4^{\frac{1}{p}} \left(\frac{R}{r}\right)^{\frac{4N}{p}} [u]_{W^{s,p}(B_r(x_0))}, \tag{3.1}$$

and

$$\left\|\mathcal{E}_r[u]\right\|_{L^p(B_R(x_0))} \leqslant 2^{\frac{1}{p}} \left(\frac{R}{r}\right)^{\frac{2N}{p}} \|u\|_{L^p(B_r(x_0))}. \tag{3.2}$$

Proof. We first prove the result at scale 1, that is, when $r = 1$. Then we will show how to get the general result, by an easy scaling argument.

Case $r = 1$. For $0 < s < 1$ and $p = 2$, this is exactly [6, Lemma 2.2]. We also observe that the very same proof applies to the case $1 < p \leqslant \infty$, thus we omit the straightforward modifications.

We now come to the case $s = 1$ and $1 < p < \infty$. We take $u \in W^{1,p}(B_1(x_0))$, thus by [14, Proposition 3.1] we have $u \in W^{s,p}(B_1(x_0))$ for every $0 < s < 1$, as well. From the previous step, we know that

$$(1-s)^{\frac{1}{p}} [\mathcal{E}_1[u]]_{W^{s,p}(B_R(x_0))} \leqslant 4^{\frac{1}{p}} R^{\frac{4N}{p}} (1-s)^{\frac{1}{p}} [u]_{W^{s,p}(B_1(x_0))}.$$

By using [8, Theorem 2], we get the desired result by taking the limit as s goes to 1, that is

$$[\mathcal{E}_1[u]]_{W^{1,p}(B_R(x_0))} \leqslant 4^{\frac{1}{p}} R^{\frac{4N}{p}} [u]_{W^{1,p}(B_1(x_0))}.$$

[†] In the case $p = \infty$, we use the convention $1/\infty = 0$.

Finally, the case $p=\infty$ can be obtained from the last formula in display, by taking the limit as p goes to ∞.

Case $r\neq 1$. At first, we need a notation. For every $\tau>0$, we indicate by

$$\mathcal{T}_\tau(x)=\tau\,(x-x_0)+x_0,\qquad \text{for every } x\in\mathbb{R}^N.$$

Then the operator $\mathcal{E}_r$ can be simply defined as

$$\mathcal{E}_r[u]:=(\mathcal{E}_1[u\circ\mathcal{T}_r])\circ\mathcal{T}_{\frac{1}{r}}.$$

In other words, given a function $u\in L^1(B_r(x_0))$, we first scale it to a function defined on $B_1(x_0)$, then extend it with $\mathcal{E}_1$ and finally scale back this extension. Observe that for $x\in B_r(x_0)$, we have

$$\mathcal{E}_r[u](x)=\mathcal{E}_1[u\circ\mathcal{T}_r]\Big(\frac{x-x_0}{r}+x_0\Big)=u\Big(\mathcal{T}_r\Big(\frac{x-x_0}{r}+x_0\Big)\Big)=u(x).$$

By using the scaling properties of the norms involved, it is easy to see that this operator has the desired properties. □

By combining Proposition 3.1 with Lemma A.1 in Appendix A, we can get a universal linear extension operator for any $K\subseteq\mathbb{R}^N$ open bounded convex set. The control on the relevant constants is quite precise and useful for our scopes. In what follows, for every $x_0\in K$, we introduce the following geometric quantities

$$d_K(x_0)=\min_{x\in\partial K}|x-x_0|,\qquad D_K(x_0)=\max_{x\in\partial K}|x-x_0|.$$

Corollary 3.2. *Let $K\subseteq\mathbb{R}^N$ be an open bounded convex set and $x_0\in K$, there exists a linear extension operator*

$$\mathcal{E}_K:L^1(K)\to L^1_{\rm loc}(\mathbb{R}^N),$$

with the following property:

for $0<s\leqslant 1$ and $1<p\leqslant\infty$ it maps $W^{s,p}(K)$ to $W^{s,p}_{\rm loc}(\mathbb{R}^N)$. Moreover, for every $u\in W^{s,p}(K)$ and every $R>1$ we have

$$[\mathcal{E}_K(u)]_{W^{s,p}(K_R(x_0))}\leqslant\left(4\cdot 6^{3\,N+s\,p}\right)^{\frac{1}{p}}R^{\frac{4\,N}{p}}\left(\frac{D_K(x_0)}{d_K(x_0)}\right)^{\frac{6\,N}{p}+2\,s}[u]_{W^{s,p}(K)},\tag{3.3}$$

and

$$\|\mathcal{E}_K(u)\|_{L^p(K_R(x_0))}\leqslant\left(2\cdot 6^{N}\right)^{\frac{1}{p}}R^{\frac{2\,N}{p}}\left(\frac{D_K(x_0)}{d_K(x_0)}\right)^{\frac{2\,N}{p}}\|u\|_{L^p(K)},\tag{3.4}$$

where

$$K_R(x_0):=R\,(K-x_0)+x_0=\Big\{R\,(x-x_0)+x_0\ :\ x\in K\Big\}.$$

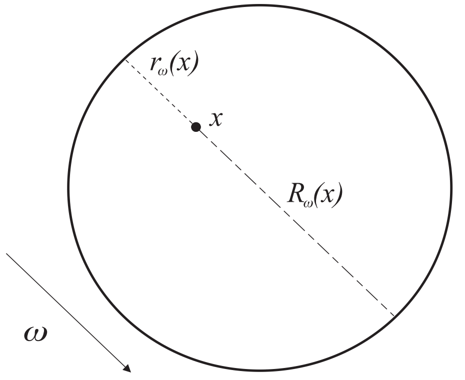

FIGURE 3 The two quantities $R_\omega(x)$ and $r_\omega(x)$.

Proof. The operator $\mathcal{E}_K$ is constructed as follows: by indicating with $\Phi_{K,x_0} : \mathbb{R}^N \to \mathbb{R}^N$ the bi-Lipschitz homeomorphism of Lemma A.1, for every $u \in L^1_{\mathrm{loc}}(K)$, we define

$$\mathcal{E}_K[u] := \left(\mathcal{E}_1[u \circ \Phi_{K,x_0}^{-1}]\right) \circ \Phi_{K,x_0},$$

where $\mathcal{E}_1$ is the operator of Proposition 3.1. In other words, we transplant u to the unit ball centered at x_0, then we extend this function to the whole $\mathbb{R}^N$ by means of $\mathcal{E}_1$ and finally compose the resulting function with Φ_{K,x_0}.

By construction, it is clear that $\mathcal{E}_K$ is linear and such that

$$\mathcal{E}_K[u](x) = u(x), \qquad \text{for } x \in K.$$

The continuity estimates (3.3) and (3.4) can now be proved from the corresponding estimates for $\mathcal{E}_1$, by using the properties of Φ_{K,x_0} and Φ_{K,x_0}^{-1}, encoded by Lemma A.1. We leave the details to the reader. □

In what follows, given a ball $B_r(x_0) \subseteq \mathbb{R}^N$, a point $x \in B_r(x_0)$ and a direction $\omega \in \mathbb{S}^{N-1}$, we set

$$R_\omega(x) = \sup\left\{\varrho \in \mathbb{R} \ : \ x + \varrho\,\omega \in B_r(x_0)\right\},$$

and

$$r_\omega(x) = \inf\left\{\varrho \in \mathbb{R} \ : \ x + \varrho\,\omega \in B_r(x_0)\right\},$$

see Figure 3. The following result is interesting in itself.

Proposition 3.3 (Directional fractional derivatives). *Let $0 < s < 1$ and $r > 0$, for every $u \in C^1(\overline{B_r(x_0)})$ and every $\omega \in \mathbb{S}^{N-1}$, we have*

$$\int_{B_r(x_0)} \left(\int_{r_\omega(x)}^{R_\omega(x)} \frac{|u(x) - u(x+\varrho\,\omega)|^2}{|\varrho|^{1+2s}}\,d\varrho\right) dx \leqslant \mathcal{A}\,[u]^2_{W^{s,2}(B_r(x_0))}, \tag{3.5}$$

for some $\mathcal{A} = \mathcal{A}(N) > 0$.

Proof. Without loss of generality, we can assume that x_0 coincides with the origin. We use Proposition 3.1 and estimate (3.1) with $R = 4\,r$, so to get

$$\begin{aligned}\iint_{B_r\times B_r}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\,s}}\,dx\,dy &\geqslant \frac{1}{C}\iint_{B_{4r}\times B_{4r}}\frac{|\mathcal{E}_r[u](x)-\mathcal{E}_r[u](y)|^2}{|x-y|^{N+2\,s}}\,dx\,dy\\ &\geqslant \frac{1}{C}\int_{B_r}\left(\int_{B_{2r}(x)}\frac{|\mathcal{E}_r[u](x)-\mathcal{E}_r[u](y)|^2}{|x-y|^{N+2\,s}}\,dy\right)dx \qquad (3.6)\\ &= \frac{1}{C}\iint_{B_r\times B_{2r}}\frac{|\mathcal{E}_r[u](x)-\mathcal{E}_r[u](x+h)|^2}{|h|^{N+2\,s}}\,dx\,dh,\end{aligned}$$

where C only depends on the dimension N. In the last identity, we used the change of variable $y = x + h$.

From now on, we will write $\widetilde{u}$ in place of $\mathcal{E}_r[u]$, for notational simplicity. We then introduce the following *K-functional*

$$\mathcal{K}(t,u) = \min_{v\in W^{1,2}(B_r)}\Big[\|u-v\|_{L^2(B_r)} + t\,[v]_{W^{1,2}(B_r)}\Big], \qquad \text{for } t\in[0,2r]. \tag{3.7}$$

We claim that the following two estimates hold: there exist two constants $A_1, A_2 > 0$ depending on the dimension N only, such that

$$\int_0^{2r}\left(\frac{\mathcal{K}(t,u)}{t^s}\right)^2\frac{dt}{t} \leqslant A_1\iint_{B_r\times B_{2r}}\frac{|\widetilde{u}(x)-\widetilde{u}(x+h)|^2}{|h|^{N+2\,s}}\,dx\,dh, \tag{3.8}$$

and

$$\int_{B_r}\left(\int_{-2r}^{2r}\frac{|\widetilde{u}(x)-\widetilde{u}(x+\varrho\,\omega)|^2}{|\varrho|^{1+2\,s}}\,d\varrho\right)dx \leqslant A_2\int_0^{2r}\left(\frac{\mathcal{K}(t,u)}{t^s}\right)^2\frac{dt}{t}, \qquad \text{for every } \omega\in\mathbb{S}^{N-1}. \tag{3.9}$$

Observe that by joining (3.6), (3.8) and (3.9), we would get

$$\iint_{B_r\times B_r}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\,s}}\,dx\,dy \geqslant \frac{1}{C\cdot A_1\cdot A_2}\int_{B_r}\left(\int_{-2r}^{2r}\frac{|\widetilde{u}(x)-\widetilde{u}(x+\varrho\,\omega)|^2}{|\varrho|^{1+2\,s}}\,d\varrho\right)dx,$$

and thus the desired conclusion (3.5) would follow, once observed that $R_\omega(x) \leqslant 2\,r$ and $r_\omega(x) \geqslant -2\,r$, together with the fact that $\widetilde{u} = u$ on B_r. Thus, we are left with establishing the validity of both (3.8) and (3.9).

To prove (3.8), we proceed exactly as in the proof of [11, Proposition 4.5], up to some necessary modifications. At first, it is useful to define

$$U(h) = \left(\int_{B_r}|\widetilde{u}(x+h)-\widetilde{u}(x)|^2\,dx\right)^{\frac{1}{2}}, \qquad h\in B_{2r}.$$

Thus, by definition, the right-hand side of (3.8) can be rewritten as

$$\iint_{B_r\times B_{2r}}\frac{|\widetilde{u}(x)-\widetilde{u}(x+h)|^2}{|h|^{N+2\,s}}\,dx\,dh = \int_{B_{2r}}\frac{U(h)^2}{|h|^{N+2\,s}}\,dh.$$

We also define

$$\overline{U}(\varrho) = \fint_{\partial B_\varrho} U \, d\mathcal{H}^{N-1}, \qquad \text{for } 0 < \varrho \leqslant 2r.$$

By Jensen's inequality, we obtain

$$\begin{aligned}\int_0^{2r} \overline{U}^2 \, \frac{d\varrho}{\varrho^{1+2s}} &\leqslant \frac{1}{N\,\omega_N} \int_0^{2r} \left(\int_{\partial B_\varrho} U^2 \, d\mathcal{H}^{N-1} \right) \frac{d\varrho}{\varrho^{N+2s}} \\ &= \frac{1}{N\,\omega_N} \int_{B_{2r}} \frac{U(h)^2}{|h|^{N+2s}} \, dh = \frac{1}{N\,\omega_N} \iint_{B_r \times B_{2r}} \frac{|\widetilde{u}(x) - \widetilde{u}(x+h)|^2}{|h|^{N+2s}} \, dx \, dh.\end{aligned} \tag{3.10}$$

We now take the compactly supported Lipschitz function

$$\psi(x) = \frac{N+1}{\omega_N} \, (1 - |x|)_+,$$

where $(\,\cdot\,)_+$ stands for the positive part. Observe that ψ has unit L^1 norm, by construction. We then define the rescaled function

$$\psi_t(x) = \frac{1}{t^N} \, \psi\left(\frac{x}{t}\right), \qquad \text{for } 0 < t \leqslant 2r,$$

which is supported on $\overline{B_t}$. By observing that $\psi_t * \widetilde{u} \in W^{1,2}(B_r)$, from the definition of $\mathcal{K}(t,u)$ we have

$$\mathcal{K}(t,u) \leqslant \|u - \psi_t * \widetilde{u}\|_{L^2(B_r)} + t \, [\psi_t * \widetilde{u}]_{W^{1,2}(B_r)}.$$

We estimate the two norms in the right-hand side separately: for the first one, by Minkowski's inequality and Fubini's theorem, we obtain

$$\begin{aligned}\|u - \psi_t * \widetilde{u}\|_{L^2(B_r)} &= \left\| \int_{B_t} [\widetilde{u}(\cdot) - \widetilde{u}(\cdot - y)] \, \psi_t(y) \, dy \right\|_{L^2(B_r)} \\ &\leqslant \int_{B_t} \left(\int_{B_r} |\widetilde{u}(x) - \widetilde{u}(x-y)|^2 \, dx \right)^{\frac{1}{2}} \psi_t(y) \, dy \\ &= \int_{B_t} U(-y) \, \psi_t(y) \, dy \leqslant \frac{N+1}{\omega_N \, t^N} \int_{B_t} U(-y) \, dy \\ &= \frac{N\,(N+1)}{t^N} \int_0^t \overline{U} \, \varrho^{N-1} \, d\varrho \leqslant \frac{N\,(N+1)}{t} \int_0^t \overline{U} \, d\varrho.\end{aligned}$$

In the first identity, we used that $\widetilde{u} = u$ in B_r, in the last inequality we used that $\varrho^{N-1} \leqslant t^{N-1}$. For the second norm, we first observe that the Divergence Theorem gives

$$\int_{B_t} \nabla \psi_t(y) \, dy = 0,$$

thus we can write

$$\nabla\psi_t * \widetilde{u} = (\nabla\psi_t) * \widetilde{u} = \int_{B_t} \nabla\psi_t(y)\,[\widetilde{u}(x-y)-\widetilde{u}(x)]\,dy.$$

Thus, again Minkowski's inequality yields

$$\begin{aligned}[\psi_t * \widetilde{u}]_{W^{1,2}(B_r)} &= \left\| \int_{B_t} \nabla\psi_t(y)\,[\widetilde{u}(\cdot-y)-\widetilde{u}(\cdot)]\,dy \right\|_{L^2(B_r)} \\ &\leqslant \int_{B_t} \left(\int_{B_r} |\widetilde{u}(x-y)-\widetilde{u}(x)|^2\,dx \right)^{\frac{1}{2}} |\nabla\psi_t(y)|\,dy \\ &\leqslant \frac{N+1}{\omega_N\,t^{N+1}} \int_{B_t} U(-y)\,dy \leqslant \frac{N\,(N+1)}{t^2} \int_0^t \overline{U}\,d\varrho.\end{aligned}$$

In conclusion, we have obtained

$$\mathcal{K}(t,u) \leqslant \frac{2\,N\,(N+1)}{t} \int_0^t \overline{U}\,d\varrho, \qquad \text{for every } 0<t\leqslant 2\,r. \tag{3.11}$$

By raising to the power 2, dividing by $t^{2\,s+1}$ and integrating, the previous estimate yields

$$\int_0^{2r} \left(\frac{\mathcal{K}(t,u)}{t^s}\right)^2 \frac{dt}{t} \leqslant (2\,N\,(N+1))^2 \int_0^{2r} \left(\frac{1}{t}\int_0^t \overline{U}\,d\varrho\right)^2 \frac{dt}{t^{1+2\,s}}.$$

If we now use the one-dimensional Hardy inequality (see [29, Teorema 1]) for the function $t \mapsto \int_0^t \overline{U}\,d\varrho$, we get

$$\begin{aligned}\int_0^{2r} \left(\frac{\mathcal{K}(t,u)}{t^s}\right)^2 \frac{dt}{t} &\leqslant \left(\frac{2\,N\,(N+1)}{s+1}\right)^2 \int_0^{2\,r} \overline{U}^2\,\frac{dt}{t^{1+2\,s}} \\ &\leqslant \frac{4\,N\,(N+1)^2}{\omega_N} \iint_{B_r\times B_{2r}} \frac{|\widetilde{u}(x)-\widetilde{u}(x+h)|^2}{|h|^{N+2\,s}}\,dx\,dh,\end{aligned}$$

where we used (3.10) in the second inequality. This proves (3.8), as desired.

The proof of (3.9) is similar to that of [9, Proposition B.1], but some technical modifications are needed, here as well. We take $0<|\varrho|\leqslant 2\,r$, by definition of the $K-$functional there exists $v_\varrho \in W^{1,2}(B_r)$ such that

$$\|u-v_\varrho\|_{L^2(B_r)} + |\varrho|\,\|\nabla v_\varrho\|_{L^2(B_r)} = \mathcal{K}(|\varrho|,u). \tag{3.12}$$

For notational simplicity, we simply write v in place of v_ϱ. We also denote by $\widetilde{v}$ the extension of v given by $\mathcal{E}_r[v]$. For $\omega \in \mathbb{S}^{N-1}$ and $|\varrho|\leqslant 2\,r$, we get†

† In the second inequality, we use that for every $\omega\in\mathbb{S}^{N-1}$ and every $|\varrho|\leqslant 2r$, we have

$$\left(\int_{B_r} |\widetilde{v}(x+\varrho\,\omega)-\widetilde{v}(x)|^2\,dx\right)^{\frac{1}{2}} \leqslant |\varrho| \left(\int_{B_{3r}} |\partial_\omega \widetilde{v}|^2\,dx\right)^{\frac{1}{2}}.$$

$$\left(\int_{B_r} |\widetilde{u}(x+\varrho\,\omega)-\widetilde{u}(x)|^2\,dx\right)^{\frac{1}{2}} \leqslant \left(\int_{B_r} |\widetilde{u}(x+\varrho\,\omega)-\widetilde{v}(x+\varrho\,\omega)-\widetilde{u}(x)+\widetilde{v}(x)|^2\,dx\right)^{\frac{1}{2}}$$
$$+\left(\int_{B_r} |\widetilde{v}(x+\varrho\,\omega)-\widetilde{v}(x)|^2\,dx\right)^{\frac{1}{2}}$$
$$\leqslant 2\,\|\widetilde{u}-\widetilde{v}\|_{L^2(B_{3r})} + |\varrho|\,\|\partial_\omega \widetilde{v}\|_{L^2(B_{3r})}$$
$$\leqslant 2\left(\|\widetilde{u}-\widetilde{v}\|_{L^2(B_{3r})} + |\varrho|\,\|\nabla \widetilde{v}\|_{L^2(B_{3r})}\right).$$

In the last estimate, we used the pointwise inequality $|\partial_\omega \widetilde{v}| \leqslant |\nabla \widetilde{v}|$ and the trivial estimate $|\varrho| \leqslant 2\,|\varrho|$. We can now use the properties of our extension operator $\mathcal{E}_r$, in order to replace the norms over B_{3r} with those on B_r. By Proposition 3.1, we have

$$\|\widetilde{u}-\widetilde{v}\|_{L^2(B_{3r})} = \|\mathcal{E}_r[u]-\mathcal{E}_r[v]\|_{L^2(B_{3r})} = \|\mathcal{E}_r[u-v]\|_{L^2(B_{3r})} \leqslant \sqrt{2}\cdot 3^N\,\|u-v\|_{L^2(B_r)},$$

and also

$$\|\nabla \widetilde{v}\|_{L^2(B_{3r})} = [\mathcal{E}_r[v]]_{W^{1,2}(B_{3r})} \leqslant 2\cdot 9^N\,[v]_{W^{1,2}(B_r)}.$$

This leads to

$$\left(\int_{B_r} |\tilde{u}(x+\varrho\,\omega)-\tilde{u}(x)|^2\,dx\right)^{\frac{1}{2}} \leqslant C\,\left(\|u-v\|_{L^2(B_r)} + |\varrho|\,[v]_{W^{1,2}(B_r)}\right).$$

By combining this estimate with (3.12), we then obtain for $0<|\varrho|\leqslant 2r$

$$\int_{B_r} \frac{|\tilde{u}(x+\varrho\,\omega)-\tilde{u}(x)|^2}{|\varrho|^{1+2\,s}}\,dx \leqslant C^2\,|\varrho|^{-1-2\,s}\,\mathcal{K}(|\varrho|,u)^2.$$

If we now integrate with respect to ϱ we get (3.9), as desired. The proof is now over. □

As a straightforward consequence of Proposition 3.3, we also get the following result (see also [5, Lemma A.4]).

Corollary 3.4. *Let $0<s<1$, for every $u\in C_0^\infty(\mathbb{R}^N)$ and every $\omega\in\mathbb{S}^{N-1}$, we have*

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}} \frac{|u(x)-u(x+\varrho\,\omega)|^2}{|\varrho|^{1+2\,s}}\,d\varrho\right)dx \leqslant \mathcal{A}\,[u]^2_{W^{s,2}(\mathbb{R}^N)}, \tag{3.13}$$

for the same constant $\mathcal{A}=\mathcal{A}(N)>0$ appearing in (3.5).

The next result can be found in [22] and [24, Corollary 1]. In the latter, the estimate is slightly worse in its dependence on s, while in the former the result is not explicitly stated, but it must be extrapolated from the proof of [22, Corollary 1, p. 524]. For these reasons, we prefer to provide a full proof, which in any case is different from those of the aforementioned references. As before, we state and prove it for smooth functions, for simplicity: it can then be extended to $W^{s,2}$ functions, by using standard density results (see, for example, [20, Theorem 6.70]).

Lemma 3.5 (Fractional Poincaré–Wirtinger inequality). *Let* $0 < s < 1$, *for every* $u \in C^1(\overline{B_r(x_0)})$ *we have*

$$\Big\|u - \mathrm{av}(u; B_r(x_0))\Big\|^2_{L^2(B_r(x_0))} \leqslant \mathcal{M}\,(1-s)\,r^{2\,s}\,[u]^2_{W^{s,2}(B_r(x_0))},$$

for some $\mathcal{M} = \mathcal{M}(N) > 0$.

Proof. We can suppose that $x_0 = 0$, without loss of generality. We use real interpolation techniques, as in the previous result. By combining (3.6) and (3.8), we have

$$[u]^2_{W^{s,2}(B_r)} \geq \frac{1}{C}\int_0^{2\,r}\left(\frac{\mathcal{K}(t,u)}{t^s}\right)^2 \frac{dt}{t}, \tag{3.14}$$

where C depends on the dimension N only and $\mathcal{K}(t,u)$ is still defined by (3.7). We now take $0 < t \leqslant 2\,r$ and $v \in W^{1,2}(B_r)$, by the triangle inequality we get

$$\begin{aligned} t\,\|u - \mathrm{av}(u; B_r)\|_{L^2(B_r)} &\leqslant t\,\|u - v\|_{L^2(B_r)} + t\,\|v - \mathrm{av}(v; B_r)\|_{L^2(B_r)} + t\,\|\mathrm{av}(v; B_r) - \mathrm{av}(u; B_r)\|_{L^2(B_r)} \\ &\leqslant 2\,r\left(\|u - v\|_{L^2(B_r)} + \|\mathrm{av}(v; B_r) - \mathrm{av}(u; B_r)\|_{L^2(B_r)}\right) \\ &\quad + t\,\|v - \mathrm{av}(v; B_r)\|_{L^2(B_r)}. \end{aligned}$$

By using Jensen's inequality, we have

$$\|\mathrm{av}(v; B_r) - \mathrm{av}(u; B_r)\|_{L^2(B_r)} \leqslant \|u - v\|_{L^2(B_r)},$$

while by using the classical Poincaré–Wirtinger inequality we have

$$\|v - \mathrm{av}(v; B_r)\|_{L^2(B_r)} \leqslant \frac{r}{\mu}\,[v]_{W^{1,2}(B_r)},$$

for some $\mu = \mu(N) > 0$. By keeping all these estimates together, we obtain

$$\begin{aligned} t\,\|u - \mathrm{av}(u; B_r)\|_{L^2(B_r)} &\leqslant 4\,r\,\|u - v\|_{L^2(B_r)} + \frac{t\,r}{\mu}\,[v]_{W^{1,2}(B_r)} \\ &\leqslant C\,r\left(\|u - v\|_{L^2(B_r)} + t\,[v]_{W^{1,2}(B_r)}\right), \end{aligned}$$

where $C = \max\{4, 1/\mu\}$ depends on N only. If we now take the infimum over $v \in W^{1,2}(B_r)$, we get

$$t\,\|u - \mathrm{av}(u; B_r)\|_{L^2(B_r)} \leqslant C\,r\,\mathcal{K}(t,u), \qquad \text{for } 0 < t \leqslant 2\,r.$$

By raising to the power 2, dividing by t^{2s+1} and integrating over $(0, 2r)$, this yields

$$\|u - \mathrm{av}(u; B_r)\|^2_{L^2(B_r)} \frac{(2r)^{2-2s}}{2(1-s)} \leqslant C^2 r^2 \int_0^{2r} \left(\frac{\mathcal{K}(t,u)}{t^s}\right)^2 \frac{dt}{t}.$$

By using this estimate in (3.14), we finally get the desired conclusion. □

We conclude this section with a particular case of the well-known fractional Morrey-type embedding in the space of continuous functions (see, for example, [19, Corollary 7.9.4]). For our scopes, we need a precise "quantitative" behavior of the relevant constant, as s goes to 1 or $1/2$. Here we take $N = 1$.

Theorem 3.6 (Fractional Morrey–Sobolev inequality). *For every $1/2 < s < 1$, there exists a constant $\mathfrak{m}_s > 0$ depending on s only, such that*

$$\mathfrak{m}_s\, [u]^2_{W^{s-\frac{1}{2},\infty}(\mathbb{R})} \leqslant [u]^2_{W^{s,2}(\mathbb{R})}, \qquad \text{for every } u \in C_0^\infty(\mathbb{R}). \tag{3.15}$$

In particular, if $a < b$ we have

$$\mathfrak{m}_s\, \|u\|^2_{L^\infty((a,b))} \leqslant (b-a)^{2s-1}\, [u]^2_{W^{s,2}(\mathbb{R})}, \qquad \text{for every } u \in C_0^\infty((a,b)). \tag{3.16}$$

Moreover, the constant $\mathfrak{m}_s$ has the following asymptotic behaviors

$$\mathfrak{m}_s \sim 2s - 1, \quad \text{as } s \searrow 1/2, \qquad \text{and} \qquad \mathfrak{m}_s \sim \frac{1}{1-s}, \quad \text{as } s \nearrow 1.$$

Proof. We first observe that (3.16) is an easy consequence of (3.15). Indeed, for every $u \in C_0^\infty((a,b))$ and every $x \in (a,b)$, by (3.15) we would get

$$|u(x)|^2 = |u(x) - u(a)|^2 \leqslant \frac{1}{\mathfrak{m}_s} (x-a)^{2s-1}\, [u]^2_{W^{s,2}(\mathbb{R})} \leqslant \frac{1}{\mathfrak{m}_s} (b-a)^{2s-1}\, [u]^2_{W^{s,2}(\mathbb{R})},$$

as desired.

To establish (3.15), let us take $\varphi \in C_0^\infty(\mathbb{R})$. We indicate by $\mathcal{F}[\varphi]$ its *Fourier transform*, defined by

$$\mathcal{F}[\varphi](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t)\, e^{-i t \xi}\, dt, \qquad \text{for } \xi \in \mathbb{R}.$$

From the inversion formula (see [19, chapter VII, section 1]), we can write

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}[\varphi](\xi)\, e^{i t \xi}\, d\xi, \qquad \text{for } t \in \mathbb{R}.$$

Thus, for every $t, \tau \in \mathbb{R}$ we get

$$\begin{aligned} |\varphi(t) - \varphi(\tau)| &\leqslant \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left|\mathcal{F}[\varphi](\xi)\right| |e^{i t \xi} - e^{i \tau \xi}|\, d\xi \\ &\leqslant \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\xi|^{2s} \left|\mathcal{F}[\varphi](\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{|e^{i t \xi} - e^{i \tau \xi}|^2}{|\xi|^{2s}}\, d\xi\right)^{\frac{1}{2}}. \end{aligned} \tag{3.17}$$

We now recall that by [19, chapter VII, section 9], we have

$$\int_{\mathbb{R}} |\xi|^{2s} \left|\mathcal{F}[\varphi](\xi)\right|^2 d\xi = 2\pi\, A_s\, [\varphi]^2_{W^{s,2}(\mathbb{R})},$$

with the constant A_s given by

$$A_s = \left(\int_{\mathbb{R}} \frac{|e^{i\,t} - 1|^2}{|t|^{1+2\,s}}\, dt\right)^{-1},$$

which satisfies

$$A_s \sim 1 - s, \quad \text{for } s \nearrow 1 \qquad \text{and} \qquad A_s \sim s \quad \text{for } s \searrow 0.$$

From (3.17), we obtain

$$|\varphi(t) - \varphi(\tau)| \leqslant \sqrt{A_s} \left(\int_{\mathbb{R}} \frac{|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}|^2}{|\xi|^{2\,s}}\, d\xi\right)^{\frac{1}{2}} [\varphi]_{W^{s,2}(\mathbb{R})}. \tag{3.18}$$

To conclude, we are only left with handling the integral on the right-hand side. For every $\alpha > 0$, we split this integral as follows

$$\int_{\mathbb{R}} \frac{|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}|^2}{|\xi|^{2\,s}}\, d\xi = \int_{\{|\xi|\leqslant\alpha\}} \frac{|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}|^2}{|\xi|^{2\,s}}\, d\xi + \int_{\{|\xi|>\alpha\}} \frac{|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}|^2}{|\xi|^{2\,s}}\, d\xi.$$

To estimate the low frequencies, we use the 1-Lipschitz character of $\vartheta \mapsto e^{i\,\vartheta}$ to infer that

$$|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}| \leqslant |t - \tau|\, |\xi|.$$

The high frequencies are dealt with by using that

$$|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}| \leqslant |e^{i\,t\,\xi}| + |e^{i\,\tau\,\xi}| = 2.$$

These lead to

$$\begin{aligned}\int_{\mathbb{R}} \frac{|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}|^2}{|\xi|^{2\,s}}\, d\xi &\leqslant 2\,|t-\tau|^2 \int_0^{\alpha} \xi^{2-2\,s}\, d\xi + 8 \int_{\alpha}^{+\infty} \xi^{-2\,s}\, d\xi \\ &= \frac{2}{3-2\,s}\, |t-\tau|^2\, \alpha^{3-2\,s} + \frac{8}{2\,s-1}\, \frac{1}{\alpha^{2\,s-1}},\end{aligned}$$

which is valid for every $\alpha > 0$. We can now optimize this estimate with respect to α: indeed, the quantity on the right-hand side is minimal for† $\alpha = \alpha_0 = 2/|t - \tau|$. With such a choice, we get

$$\int_{\mathbb{R}} \frac{|e^{i\,t\,\xi} - e^{i\,\tau\,\xi}|^2}{|\xi|^{2\,s}}\, d\xi \leqslant 4^{2-s}\, \frac{2}{(3-2\,s)\,(2\,s-1)}\, |t-\tau|^{2\,s-1}.$$

† We can obviously suppose that $t \neq \tau$, otherwise there is nothing to prove.

By inserting this estimate in (3.18), we finally get (3.15) with

$$\mathfrak{m}_s = \frac{(3-2\,s)\,(2\,s-1)}{2\cdot 4^{2-s}\,A_s},$$

which has the claimed asymptotic behavior. ☐

Remark 3.7. We point out the reference [27], which keeps track of the dependence on s in the one-dimensional fractional Morrey estimate, as this parameter goes to the borderline situation $s = 1/2$ (see [27, Corollary 26]). However, the asymptotic behavior detected in this reference is suboptimal. Moreover, the asymptotic behavior as s goes to 1 is not taken into account. For these reasons, the estimates of [27] are not suitable for our needing.

4 | BASICS OF FRACTIONAL CAPACITY

We start with the definition of fractional capacity.

Definition. Let $\Sigma \subseteq \mathbb{R}^N$ be a compact set and let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\Sigma \Subset \Omega$. For $0 < s < 1$, we define the *fractional capacity of* Σ of order s relative to Ω as the quantity

$$\widetilde{\mathrm{cap}}_s(\Sigma;\Omega) = \inf_{u\in C_0^\infty(\Omega)} \left\{ [u]^2_{W^{s,2}(\mathbb{R}^N)} \;:\; u \geqslant 1_\Sigma \right\}.$$

Here 1_Σ denotes the characteristic function of Σ.

Remark 4.1. By standard approximation arguments based on convolutions, it is easy to see that in the definition of $\widetilde{\mathrm{cap}}_s(\Sigma;\Omega)$ we can replace $C_0^\infty(\Omega)$ with Lipschitz functions having compact support in Ω. We leave the details to the reader.

As a straightforward consequence of both the definition and the Morrey-type inequality, we have an explicit lower bound for the fractional capacity of a point. As simple as it is, this will play a crucial role in our main result.

Lemma 4.2 (One-dimensional capacity of a point). *Let* $1/2 < s < 1$ *and* $x_0 \in (a,b)$. *Then*

$$\widetilde{\mathrm{cap}}_s(\{x_0\};(a,b)) \geqslant (b-a)^{1-2\,s}\,\mathfrak{m}_s,$$

where $\mathfrak{m}_s$ *is the same constant as in Theorem 3.6.*

Proof. Let us take $u \in C_0^\infty((a,b))$ such that $u(x_0) \geqslant 1$. Hence, from (3.16), we get

$$1 \leqslant |u(x_0)|^2 \leqslant \frac{(b-a)^{2\,s-1}}{\mathfrak{m}_s}\,[u]^2_{W^{s,2}(\mathbb{R})}.$$

The thesis follows by taking the infimum over the admissible functions u. ☐

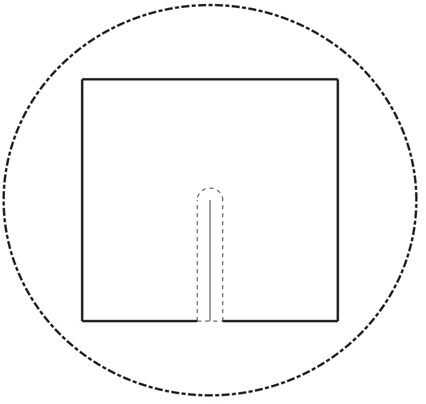

FIGURE 4 The geometric configuration of Proposition 4.3: we have a smooth function defined on the square, which vanishes on the dashed neighborhood of the vertical line (i.e., the set Σ). The relative fractional capacity of Σ is computed with respect to the surrounding disk.

4.1 | A Maz'ya-type Poincaré inequality

We will need the following fractional Poincaré inequality for functions on a cube, which vanish in a neighborhood of a set with positive fractional capacity (see Figure 4). This is analogous to the result of [30, Theorem A], but we will follow the approach of [22, chapter 14], which is more suitable for our framework. In particular, we will not explicitly relate this result to eigenvalues with mixed boundary conditions, differently from [30].

Proposition 4.3. *Let $0 < s < 1$ and let $\Sigma \subseteq \overline{Q_r(x_0)} \subseteq \mathbb{R}^N$ be a compact set. For every $R > \sqrt{N}\,r$, there exists a constant $\phi(N, R/r) > 0$ such that the following Poincaré inequality holds*

$$[u]^2_{W^{s,2}(Q_r(x_0))} \geqslant \left[\frac{s}{r^N}\,\phi\left(N, \frac{R}{r}\right)\right] \widetilde{\mathrm{cap}}_s(\Sigma; B_R(x_0))\, \|u\|^2_{L^2(Q_r(x_0))}, \tag{4.1}$$

for every $u \in C^\infty(\overline{Q_r(x_0)})$ with $\mathrm{dist}(\mathrm{supp}(u), \Sigma) > 0$. Moreover, we have

$$\lim_{t \to +\infty} \phi(N, t) = \lim_{t \searrow \sqrt{N}} \phi(N, t) = 0.$$

Proof. The proof is lengthy, though elementary. Without loss of generality, we can assume $x_0 = 0$. Let $u \in C^\infty(\overline{Q_r})$ be as in the statement, we can additionally assume that

$$\fint_{Q_r} |u|^2\, dx = 1, \tag{4.2}$$

still without loss of generality. We now use the extension operator $\mathcal{E}_K$ of Corollary 3.2, with the choices

$$K = Q_r \quad \text{and} \quad x_0 = 0, \qquad \text{so that } \frac{D_K(x_0)}{d_K(x_0)} = \sqrt{N}.$$

In order not to overburden the presentation, we will use the symbol $\widetilde{u}$ in place of $\mathcal{E}_K[u]$. By the properties of our extension operator, we get in particular that $\widetilde{u}$ is locally Lipschitz continuous and from (3.3) with $p = 2$ we also have

$$[\widetilde{u}]_{W^{s,2}(B_R)} \leqslant [\widetilde{u}]_{W^{s,2}(Q_R)} \leqslant C_N \left(\frac{R}{r}\right)^{2N} [u]_{W^{s,2}(Q_r)}. \tag{4.3}$$

Without loss of generality, we can further suppose that

$$\mathrm{av}(\widetilde{u}; B_R) \geqslant 0. \tag{4.4}$$

We take a Lipschitz cut-off function η such that

$$0 \leqslant \eta \leqslant 1, \qquad \eta \equiv 1 \text{ in } \overline{B_{\sqrt{N}\,r}}, \qquad \eta \equiv 0 \text{ in } \mathbb{R}^N \setminus B_{\frac{R+\sqrt{N}\,r}{2}}, \qquad |\nabla \eta| \leqslant \frac{2}{R - \sqrt{N}\,r}$$

and we define $\psi = (1 - \widetilde{u})\,\eta$. By recalling Remark 4.1, we have that ψ is an admissible trial function for the variational problem defining $\widetilde{\mathrm{cap}}_s(\Sigma; B_R)$. By using this fact and some algebraic manipulations, we get

$$\begin{aligned} \sqrt{\widetilde{\mathrm{cap}}_s(\Sigma; B_R)} &\leqslant [\psi]_{W^{s,2}(\mathbb{R}^N)} \\ &= \left([\psi]^2_{W^{s,2}(B_R)} + 2 \int_{B_R} |\psi(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}}\right) dx\right)^{\frac{1}{2}} \\ &\leqslant [\psi]_{W^{s,2}(B_R)} + \sqrt{2}\left(\int_{B_R} |\psi(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}}\right) dx\right)^{\frac{1}{2}}. \end{aligned} \tag{4.5}$$

In turn, by using the definition of ψ and Minkowki's inequality, we have

$$\begin{aligned} [\psi]_{W^{s,2}(B_R)} &\leqslant \left(\int_{B_R} |\eta(x)|^2 \left(\int_{B_R} \frac{|\widetilde{u}(x) - \widetilde{u}(y)|^2}{|x-y|^{N+2s}}\, dy\right) dx\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{B_R} |1 - \widetilde{u}(y)|^2 \left(\int_{B_R} \frac{|\eta(x) - \eta(y)|^2}{|x-y|^{N+2s}}\, dx\right) dy\right)^{\frac{1}{2}} \\ &\leqslant [\widetilde{u}]_{W^{s,2}(B_R)} + \|1 - \widetilde{u}\|_{L^2(B_R)} \sqrt{\frac{C}{s\,(1-s)}}\, \|\nabla \eta\|^s_{L^\infty(B_R)}\, \|\eta\|^{1-s}_{L^\infty(B_R)}, \end{aligned}$$

for some $C = C(N) > 0$. In the last inequality, we used that for every Lipschitz function φ with compact support, we have

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{N+2s}}\, dy \leqslant \frac{C}{s\,(1-s)}\, \|\nabla \varphi\|^{2s}_{L^\infty(\mathbb{R}^N)}\, \|\varphi\|^{2(1-s)}_{L^\infty(\mathbb{R}^N)},$$

see [7, Lemma 2.6]. If we now use (4.3) to bound the seminorm of $\widetilde{u}$ and the properties of η, from (4.5) we get

$$\begin{aligned}\sqrt{\widetilde{\mathrm{cap}}_s(\Sigma;B_R)} \leqslant C_N\left(\frac{R}{r}\right)^{2N}[u]_{W^{s,2}(Q_r)} + \frac{2}{(R-\sqrt{N}\,r)^s}\sqrt{\frac{C}{s\,(1-s)}}\,\|1-\widetilde{u}\|_{L^2(B_R)} \\ + \sqrt{2}\left(\int_{B_R}|\psi(x)|^2\left(\int_{\mathbb{R}^N\setminus B_R}\frac{dy}{|x-y|^{N+2\,s}}\right)dx\right)^{\frac{1}{2}}.\end{aligned} \tag{4.6}$$

To handle the last term, we recall that ψ identically vanishes outside $B_{(R+\sqrt{N}\,r)/2}$. Thus, we actually have

$$\begin{aligned}\int_{B_R}|\psi(x)|^2\left(\int_{\mathbb{R}^N\setminus B_R}\frac{dy}{|x-y|^{N+2\,s}}\right) &= \int_{B_{\frac{R+\sqrt{N}\,r}{2}}}|\eta(x)|^2\,|1-\widetilde{u}(x)|^2\left(\int_{\mathbb{R}^N\setminus B_R}\frac{dy}{|x-y|^{N+2\,s}}\right)dx \\ &\leqslant \int_{B_{\frac{R+\sqrt{N}\,r}{2}}}|1-\widetilde{u}(x)|^2\left(\int_{\mathbb{R}^N\setminus B_R}\frac{dy}{|x-y|^{N+2\,s}}\right)dx.\end{aligned}$$

We now observe that, for every $x \in B_{(R+\sqrt{N}\,r)/2}$ and $y \notin B_R$ we have

$$|x-y| \geqslant |y|-|x| \geqslant |y| - \frac{R+\sqrt{N}\,r}{2} \geqslant |y| - \frac{R+\sqrt{N}\,r}{2\,R}\,|y| = \left(\frac{R-\sqrt{N}\,r}{2\,R}\right)|y|.$$

Thus, for every $x \in B_{(R+\sqrt{N}\,r)/2}$, we get

$$\int_{\mathbb{R}^N\setminus B_R}\frac{dy}{|x-y|^{N+2\,s}}\,dy \leqslant \frac{N\,\omega_N}{2\,s}\left(\frac{2\,R}{R-\sqrt{N}\,r}\right)^{N+2\,s}\frac{1}{R^{2\,s}}.$$

By collecting the previous estimates, we obtain from (4.6)

$$\begin{aligned}\sqrt{\widetilde{\mathrm{cap}}_s(\Sigma;B_R)} \leqslant C_N\left(\frac{R}{r}\right)^{2N}[u]_{W^{s,2}(Q_r)} + \frac{2}{(R-\sqrt{N}\,r)^s}\sqrt{\frac{C}{s\,(1-s)}}\,\|1-\widetilde{u}\|_{L^2(B_R)} \\ + \sqrt{\frac{N\,\omega_N}{s}}\left(\frac{2\,R}{R-\sqrt{N}\,r}\right)^{\frac{N}{2}+s}\frac{1}{R^s}\,\|1-\widetilde{u}\|_{L^2(B_{\frac{R+\sqrt{N}\,r}{2}})}.\end{aligned}$$

We need to estimate the L^2 norm of $1-\widetilde{u}$. For this, we use the triangle inequality

$$\|1-\widetilde{u}\|_{L^2(B_{\frac{R+\sqrt{N}\,r}{2}})} \leqslant \|1-\widetilde{u}\|_{L^2(B_R)} \leqslant \|1-\mathrm{av}(\widetilde{u};B_R)\|_{L^2(B_R)} + \|\mathrm{av}(\widetilde{u};B_R)-\widetilde{u}\|_{L^2(B_R)} := \mathcal{I}_1+\mathcal{I}_2,$$

so that

$$\sqrt{\widetilde{\mathrm{cap}}_s(\Sigma;B_R)} \leqslant C_N \left(\frac{R}{r}\right)^{2N} [u]_{W^{s,2}(Q_r)} + \frac{2}{(R-\sqrt{N}\,r)^s}\sqrt{\frac{2\,C}{s\,(1-s)}}\,(\mathcal{I}_1+\mathcal{I}_2) \\ + \sqrt{\frac{N\,\omega_N}{s}}\left(\frac{2\,R}{R-\sqrt{N}\,r}\right)^{\frac{N}{2}+s}\frac{1}{R^s}\,(\mathcal{I}_1+\mathcal{I}_2). \tag{4.7}$$

In turn, the term $\mathcal{I}_1$ can be bounded by $\mathcal{I}_2$. Indeed, by observing that the integrand of $\mathcal{I}_1$ is constant and using the normalization (4.2), we get

$$\mathcal{I}_1 = \sqrt{|B_R|}\,|1-\mathrm{av}(\widetilde{u};B_R)| = \sqrt{\frac{|B_R|}{|Q_r|}}\,\Big|\|u\|_{L^2(Q_r)} - \|\mathrm{av}(\widetilde{u};B_R)\|_{L^2(Q_r)}\Big| \\ \leqslant \sqrt{\frac{|B_R|}{|Q_r|}}\,\big\|u-\mathrm{av}(\widetilde{u};B_R)\big\|_{L^2(Q_r)} \leqslant \sqrt{\frac{|B_R|}{|Q_r|}}\,\mathcal{I}_2.$$

Observe that we also used the condition (4.4) in the second identity. As for the integral $\mathcal{I}_2$, by Lemma 3.5 we directly get

$$\mathcal{I}_2 \leqslant \sqrt{\mathcal{M}\,(1-s)}\,R^s\,[\widetilde{u}]_{W^{s,2}(B_R)}.$$

Then the last term can be estimated by (4.3), again. By inserting these estimates in (4.7) we eventually conclude the proof. □

4.2 | A geometric lower bound in the plane

In dimension $N=2$ and for $s>1/2$, by exploiting the fact that points on the line have positive relative fractional capacity (see Lemma 4.2), it is possible to give a geometric lower bound for the term

$$\widetilde{\mathrm{cap}}_s(\Sigma;B_R(x_0)),$$

appearing in (4.1). We will follow the idea of [22, chapter 3, section 1.2, Proposition 1], which is quite close to that used by Taylor, even if the latter worked with a different notion of *capacity* coming from Potential Theory. The proof will also crucially exploits the result on "directional" fractional derivatives (Proposition 3.3 and Corollary 3.4). We still use the symbol Π_ω defined in (2.2).

Proposition 4.4. *Let $N=2$, $1/2<s<1$ and let $\Sigma \Subset B_r(x_0)$ be a compact set. For every direction $\omega\in\mathbb{S}^1$, it holds that*

$$\widetilde{\mathrm{cap}}_s(\Sigma;B_r(x_0)) \geq \frac{\mathfrak{m}_s}{\mathcal{A}}\,(2r)^{1-2\,s}\mathcal{H}^1(\Pi_\omega(\Sigma)).$$

Here $\mathcal{A}$ is the same constant as in Proposition 3.3 and $\mathfrak{m}_s$ is the same constant as in Theorem 3.6.

Proof. We observe that we can assume $\mathcal{H}^1(\Pi_\omega(\Sigma)) > 0$, otherwise there is nothing to prove. We may suppose as always that $x_0 = 0$, without loss of generality.

We start by noticing that every $x \in \mathbb{R}^2$ can be written as

$$x = x' + t\,\omega, \qquad \text{with } x' \in \Pi_\omega(\mathbb{R}^2) \text{ and } t \in \mathbb{R}.$$

We also set

$$R_\omega(x') = \sup\left\{\varrho \in \mathbb{R} \,:\, x' + \varrho\,\omega \in B_r\right\} \qquad \text{and} \qquad r_\omega(x') = \inf\left\{\varrho \in \mathbb{R} \,:\, x' + \varrho\,\omega \in B_r\right\}.$$

We take $u \in C_0^\infty(B_r)$ such that $u \geqslant 1_\Sigma$. By using Corollary 3.4 and Fubini's theorem, we can infer

$$\begin{aligned} [u]^2_{W^{s,2}(\mathbb{R}^2)} &\geq \frac{1}{\mathcal{A}} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \frac{|u(x) - u(x+\varrho\,\omega)|^2}{|\varrho|^{1+2\,s}}\, d\varrho \right) dx \\ &= \frac{1}{\mathcal{A}} \int_{\Pi_\omega(\mathbb{R}^2)} \left(\int\!\!\int_{\mathbb{R}\times\mathbb{R}} \frac{|u(x'+t\,\omega) - u(x'+t\,\omega+\varrho\,\omega)|^2}{|\varrho|^{1+2\,s}}\, dt\, d\varrho \right) d\mathcal{H}^1(x') \\ &\geq \frac{1}{\mathcal{A}} \int_{\Pi_\omega(\Sigma)} [u(x' + \cdot\,\omega)]^2_{W^{s,2}(\mathbb{R})}\, d\mathcal{H}^1(x'). \end{aligned} \tag{4.8}$$

Recalling that $u \geqslant 1$ on Σ, it follows that for every $x' \in \Pi_\omega(\Sigma)$ there exists t_0 such that $u(x' + t_0\,\omega) \geqslant 1$. Hence, by using the trial function

$$\psi_{x'} = u(x' + \cdot\,\omega) \in C_0^\infty((r_\omega(x'), R_\omega(x'))),$$

we have

$$[u(x' + \cdot\,\omega)]^2_{W^{s,2}(\mathbb{R})} = [\psi_{x'}]^2_{W^{s,2}(\mathbb{R})} \geqslant \widetilde{\mathrm{cap}}_s\big(\{t_0\}; (r_\omega(x'), R_\omega(x'))\big), \qquad \text{for } x' \in \Pi_\omega(\Sigma),$$

by the very definition of capacity. In turn, by applying Lemma 4.2 in the right-hand side above, we get

$$[u(x' + \cdot\,\omega)]^2_{W^{s,2}(\mathbb{R})} \geq \mathfrak{m}_s\left(R_\omega(x') - r_\omega(x')\right)^{1-2\,s} \geq \mathfrak{m}_s(2r)^{1-2\,s}.$$

To get a lower bound for the last term, we set $\ell = \mathrm{dist}(\Sigma, \partial B_r) > 0$. Then in particular we have

$$R_\omega(x') - r_\omega(x') \geqslant \sqrt{r^2 - (r-\ell)^2} \geqslant \sqrt{r\,\ell}, \qquad \text{for every } x' \in \Pi_\omega(\Sigma).$$

By spending this information in (4.8), we can obtain

$$[u]^2_{W^{s,2}(B_r)} \geq \frac{\mathfrak{m}_s}{\mathcal{A}}\,(2r)^{1-2\,s}\,\mathcal{H}^1(\Pi_\omega(\Sigma)).$$

The thesis follows by taking the infimum over the admissible trial functions u. □

5 | PROOF OF THEOREM 1.1

Without loss of generality, we can assume $r_\Omega = 1$. As in the proof of Lemma 2.1, we consider the natural number $\delta = \lfloor \sqrt{k} \rfloor + 1$ and take the family of squares $\{Q_{i,j}\}_{(i,j)\in\mathbb{Z}^2} \subseteq \mathbb{R}^2$ given by

$$Q_{i,j} := Q_{5\,\delta}(10\,\delta\, i, 10\,\delta\, j), \qquad \text{for } (i,j)\in\mathbb{Z}^2.$$

We observe that they form a tiling of the whole plane, more precisely they are pairwise disjoint and the union of their closures covers the whole $\mathbb{R}^2$. We also introduce the set of indexes

$$\mathbb{Z}^2_\Omega = \left\{(i,j)\in\mathbb{Z}^2 \ :\ Q_{i,j}\cap\Omega\neq\emptyset\right\},$$

and for every $(i,j)\in\mathbb{Z}^2_\Omega$, we indicate by $\Sigma_{i,j}\subseteq\overline{Q_{i,j}}\setminus\Omega$ the compact set provided by Lemma 2.1. By using the tiling properties of these squares, for a function $u\in C^\infty_0(\Omega)$ we have

$$\begin{aligned}[u]^2_{W^{s,2}(\mathbb{R}^2)} &= \sum_{(i,j)\in\mathbb{Z}^2}\int\!\!\int_{Q_{i,j}\times\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy\\ &\geq \sum_{(i,j)\in\mathbb{Z}^2}\int\!\!\int_{Q_{i,j}\times Q_{i,j}}\frac{|u(x)-u(y)|^2}{|x-y|^{2+2\,s}}\,dx\,dy = \sum_{(i,j)\in\mathbb{Z}^2_\Omega}[u]^2_{W^{s,2}(Q_{i,j})}.\end{aligned}$$

For every $(i,j)\in\mathbb{Z}^2_\Omega$, we can use the fractional Poincaré inequality of Proposition 4.3, with the choices

$$r = 5\,\delta \qquad \text{and} \qquad R = 2\,r = 10\,\delta.$$

By setting for brevity $\mathcal{B}_{i,j} := B_{10\,\delta}(10\,\delta\, i, 10\,\delta\, j)$, this leads to

$$[u]^2_{W^{s,2}(Q_{i,j})} \geqslant \left[\frac{1}{50\,\delta^2}\,\phi(2,2)\right]\widetilde{\mathrm{cap}}_s\big(\Sigma_{i,j};\mathcal{B}_{i,j}\big)\,\|u\|^2_{L^2(Q_{ij})}, \qquad \text{for every } (i,j)\in\mathbb{Z}^2_\Omega,$$

where we also used that $s > 1/2$. We now have to estimate from below the relative fractional capacity of each compact set $\Sigma_{i,j}$. By combining Lemma 2.1 and Proposition 4.4, we have

$$\begin{aligned}\widetilde{\mathrm{cap}}_s(\Sigma_{i,j};\mathcal{B}_{i,j}) &\geq (20)^{1-2\,s}\,\frac{\mathfrak{m}_s}{\mathcal{A}}\,\delta^{1-2\,s}\,\max\left\{\mathcal{H}^1(\Pi_{\mathbf{e}_1}(\Sigma_{i,j})),\,\mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma_{i,j}))\right\}\\ &\geq (20)^{1-2\,s}\,\frac{\mathfrak{m}_s}{4\,\mathcal{A}}\,\delta^{1-2\,s}\sqrt{k}.\end{aligned}$$

By collecting the estimates above, we obtain

$$\begin{aligned}[u]^2_{W^{s,2}(\mathbb{R}^2)} &\geq (20)^{1-2\,s}\,\frac{\mathfrak{m}_s\,\phi(2,2)}{200\,\mathcal{A}}\,\sqrt{k}\,\delta^{-1-2\,s}\sum_{(i,j)\in\mathbb{Z}^2_\Omega}\|u\|^2_{L^2(Q_{ij})}\\ &= (20)^{1-2\,s}\,\frac{\mathfrak{m}_s\,\phi(2,2)}{200\,\mathcal{A}}\,\sqrt{k}\,\delta^{-1-2\,s}\,\|u\|^2_{L^2(\Omega)},\end{aligned} \tag{5.1}$$

where the last identity follows by the tiling property of the family $\{\mathcal{Q}_{ij}\}_{i,j}$. By recalling the definition of δ and using (2.1), we get

$$\sqrt{k}\,\delta^{-1-2\,s} \geqslant \sqrt{k}\left(\sqrt{k}+1\right)^{-1-2\,s} \geqslant \frac{1}{2^{1+2\,s}}\,\frac{1}{k^s}.$$

By the arbitrariness of $u \in C_0^\infty(\Omega)$, from (5.1) we get the claimed lower bound on $\lambda_1^s(\Omega)$, with

$$\vartheta_s = \frac{(20)^{1-2\,s}}{2^{1+2\,s}}\,\frac{\mathfrak{m}_s\,\phi(2,2)}{200\,\mathcal{A}}.$$

Finally, the claimed asymptotic behavior of ϑ_s simply follows from its definition and the properties of $\mathfrak{m}_s$, encoded in Theorem 3.6.

6 | PROOF OF THEOREM 1.2

6.1 | Proof of point (1)

This is a straightforward consequence of the *Bourgain–Brezis–Mironescu formula*. Indeed, for every $\Omega \subseteq \mathbb{R}^2$ open set, let $u \in C_0^\infty(\Omega)\setminus\{0\}$. Then by [14, Corollary 3.20] we have

$$\lim_{s\nearrow 1}(1-s)\,[u]^2_{W^{s,2}(\mathbb{R}^2)} = \frac{\pi}{2}\int_\Omega |\nabla u|^2\,dx.$$

This implies that

$$\limsup_{s\nearrow 1}(1-s)\,\lambda_1^s(\Omega) \leqslant \lim_{s\nearrow 1}\frac{(1-s)\,[u]^2_{W^{s,2}(\mathbb{R}^N)}}{\|u\|^2_{L^2(\Omega)}} = \frac{\pi}{2}\,\frac{\int_\Omega |\nabla u|^2\,dx}{\|u\|^2_{L^2(\Omega)}}.$$

By taking the infimum over $C_0^\infty(\Omega)\setminus\{0\}$, we get

$$\limsup_{s\nearrow 1}(1-s)\,\lambda_1^s(\Omega) \leqslant \frac{\pi}{2}\,\lambda_1(\Omega),$$

as claimed. Thus, by multiplying both sides of (1.2) by the factor $(1-s)$, using the previous property and the asymptotic behavior of ϑ_s, we get back the classical Croke–Osserman–Taylor estimate, in the limit as s goes to 1.

6.2 | Proof of point (2)

We need at first the following

Lemma 6.1. *Let $0 < s < 1$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set. Then for every $\{x_0,\dots,x_m\} \subseteq \Omega$, we have*

$$\lambda_1^s(\Omega\setminus\{x_0,\dots,x_m\}) = \lambda_1^s(\Omega).$$

Proof. We may suppose that the points $\{x_0, \dots, x_m\}$ are distinct. We first observe that

$$\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\}) \geqslant \lambda_1^s(\Omega),$$

as $\Omega \setminus \{x_0, \dots, x_m\} \subseteq \Omega$. To prove the converse inequality, we set

$$\varepsilon_0 = \frac{1}{4} \min_{i,j\in\{0,m\}} \left\{ |x_i - x_j| \ : \ i \neq j \right\}.$$

Then we take a cut-off function $\eta \in C_0^\infty(B_1)$ such that

$$\eta \equiv 1 \text{ in } B_{\frac{1}{2}}, \qquad 0 \leqslant \eta \leqslant 1, \qquad |\nabla \eta| \leqslant C,$$

and define for every $0 < \varepsilon < \varepsilon_0$

$$\Psi_\varepsilon(x) = \sum_{i=0}^m \eta\Big(\frac{x - x_i}{\varepsilon}\Big).$$

We now take $u \in C_0^\infty(\Omega) \setminus \{0\}$ and observe that $u\,(1 - \Psi_\varepsilon)$ is a feasible trial function for the variational problem that defines $\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\})$. Thus, by using Minkowski's inequality, we get for every $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned}
\sqrt{\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\})} &\leqslant \frac{[u\,(1-\Psi_\varepsilon)]_{W^{s,2}(\mathbb{R}^2)}}{\|u\,(1-\Psi_\varepsilon)\|_{L^2(\Omega)}} \\
&\leqslant \frac{[u]_{W^{s,2}(\mathbb{R}^2)}\,\|1-\Psi_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)}\,[\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)}}{\|u\,(1-\Psi_\varepsilon)\|_{L^2(\Omega)}} \qquad (6.1)\\
&= \frac{[u]_{W^{s,2}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)}\,[\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)}}{\|u\,(1-\Psi_\varepsilon)\|_{L^2(\Omega)}}.
\end{aligned}$$

By applying the Dominated Convergence Theorem, we easily get that

$$\lim_{\varepsilon\to 0} \|u\,(1-\Psi_\varepsilon)\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}.$$

As for the second term in the numerator, we observe that by Minkowski's inequality again, we have

$$[\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)} = \left[\sum_{i=0}^m \eta\Big(\frac{\cdot - x_i}{\varepsilon}\Big)\right]_{W^{s,2}(\mathbb{R}^2)} \leqslant (m+1)\,\varepsilon^{1-s}\,[\eta]_{W^{s,2}(\mathbb{R}^2)}.$$

We also used the scaling properties of the fractional seminorm. This in turn implies that

$$\lim_{\varepsilon\to 0} [\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)} = 0.$$

Thus, by taking the limit as ε goes to 0 in (6.1), we end up with

$$\sqrt{\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\})} \leqslant \frac{[u]_{W^{s,2}(\mathbb{R}^2)}}{\|u\|_{L^2(\Omega)}}.$$

By arbitrariness of u, we get the desired conclusion. □

Remark 6.2. The previous result is a particular case of the following general fact: removing sets with zero fractional capacity does not alter the relevant fractional Sobolev space. Consequently, fractional Poincaré constants are insensitive to removal of these sets. We refer, for example, to [1, Proposition 2.6 and Corollary 2.7] for this general result.

The sequence $\{\Omega_k\}_{k\in\mathbb{N}\setminus\{0,1\}}$ is then constructed as follows: for every $k \in \mathbb{N} \setminus \{0, 1\}$, we set

$$n_k = \lfloor \sqrt{k-1} \rfloor \qquad \text{and} \qquad m_k = (k-1) - n_k^2.$$

Then, we take the set

$$\mathrm{Shell}_k = ([0, n_k] \times [0, n_k]) \setminus \bigcup_{i,j=0}^{n_k-1} \left\{ \left(i + \frac{1}{2}, j + \frac{1}{2}\right) \right\}, \text{ for } k \geqslant 2,$$

which consists of a square with n_k^2 equally spaced points removed. More precisely, we remove the centers of the squares

$$[i, i+1] \times [j, j+1], \qquad \text{for } i, j = 0, \dots, n_k - 1.$$

We also introduce the set

$$\mathrm{Slug}_k = ([0, m_k] \times [-1, 0]) \setminus \bigcup_{i=0}^{m_k-1} \left\{ \left(i + \frac{1}{2}, -\frac{1}{2}\right) \right\},$$

which consists of an horizontal strip of width 1 and length m_k, from which we removed the centers of the squares

$$[i, i+1] \times [-1, 0], \qquad \text{for } i = 0, \dots, m_k - 1.$$

Finally, we define the open bounded set

$$\Omega_k = \mathrm{int}\big(\mathrm{Shell}_k \cup \mathrm{Slug}_k\big), \qquad \text{for every } k \geqslant 2,$$

that is, the interior points of the union of Shell_k and Slug_k (see Figure 5). By construction, we have that Ω_k is multiply connected of order k. Moreover, we have

$$r_{\Omega_k} \leqslant \frac{\sqrt{2}}{2}, \qquad \text{for every } k \geqslant 2,$$

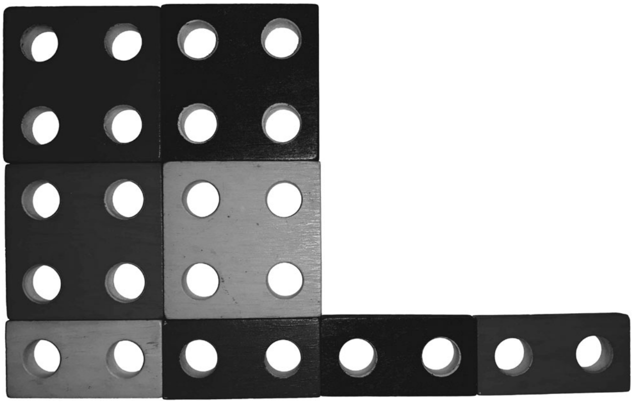

FIGURE 5 The set Ω_k of Theorem 1.2, point (2), for $k = 25$.

and

$$\Omega_k \supseteq \mathrm{int}(\mathrm{Shell}_k) = ((0, n_k) \times (0, n_k)) \setminus \bigcup_{i,j=0}^{n_k-1} \left\{ \left(i + \frac{1}{2}, j + \frac{1}{2} \right) \right\}.$$

By using the monotonicity of λ_1^s with respect to set inclusion and Lemma 6.1 for $\mathrm{int}(\mathrm{Shell}_k)$, we can then infer

$$\lambda_1^s(\Omega_k) \leqslant \lambda_1^s((0, n_k) \times (0, n_k)) = n_k^{-2s}\, \lambda_1^s((0, 1) \times (0, 1)).$$

By recalling the definition of n_k, this finally gives the desired result.

6.3 | Proof of point (3)

We divide the proof in various steps, for ease of presentation.

Step 1: Construction of the set. We define

$$\Sigma = \bigcup_{i \in \mathbb{Z}} \Sigma^{(i)}, \qquad \text{where } \Sigma^{(i)} := \left\{ (x_1, i) \in \mathbb{R}^2 \ : \ |x_1| \geqslant 1 \right\},$$

and then consider the *infinite complement comb*

$$\Theta := \mathbb{R}^2 \setminus \Sigma,$$

as in [6, section 5]. The set Θ_k of the statement is then constructed by simply removing $k - 1$ distinct points from Θ, that is, we set

$$\Theta_k = \Theta \setminus \{(0, i) \ : \ i = 1, \dots, k-1\},$$

see Figure 6. By construction, we have that Θ_k is multiply connected of order k, with finite

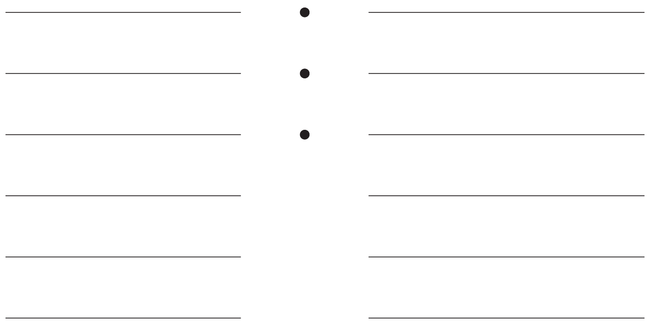

FIGURE 6 The set Θ_k for $k = 4$ of Theorem 1.2, point (3): it is has been obtained by removing the black dots from Θ.

inradius. Thus, by Theorem 1.1 we have $\lambda_1^s(\Theta_k) > 0$, for every $s > 1/2$. We claim that

$$\limsup_{s \searrow \frac{1}{2}} \frac{\lambda_1^s(\Theta_k)}{2\,s-1} < +\infty. \tag{6.2}$$

Step 2: One-dimensional reduction. Here we need the following result.

Lemma 6.3. *Let* $0 < s < 1$ *and let* $A \subseteq \mathbb{R}$ *be an open set. Then we have*

$$\lambda_1^s(A \times \mathbb{R}) \leqslant \alpha_s\,\lambda_1^s(A), \qquad \text{where } \alpha_s = \int_{\mathbb{R}} \frac{dt}{(1+t^2)^{\frac{2+2\,s}{2}}}. \tag{6.3}$$

Proof. We proceed as in the proof of [15, Lemma 2.4]. For every $x \in \mathbb{R}^2$, we will use the notation $x = (x_1, x_2)$. We take $u \in C_0^\infty(A) \setminus \{0\}$ and $\varphi \in C_0^\infty(\mathbb{R}) \setminus \{0\}$. We first observe that by Fubini's theorem, for the function $v(x_1, x_2) = u(x_1)\,\varphi(x_2)$ we have

$$\|u\,\varphi\|_{L^2(A\times\mathbb{R})} = \|u\|_{L^2(A)}\,\|\varphi\|_{L^2(\mathbb{R})}.$$

We then estimate the fractional seminorm of $v = u\,\varphi$. By Minkowksi's inequality, we have

$$\begin{aligned}[u\,\varphi]_{W^{s,2}(\mathbb{R}^2)} \leqslant &\left(\int\!\!\int_{\mathbb{R}^2\times\mathbb{R}^2} |u(x_1)|^2\,\frac{|\varphi(x_2)-\varphi(y_2)|^2}{|x-y|^{2+2\,s}}\,dx\,dy\right)^{1/2} \\ &+ \left(\int\!\!\int_{\mathbb{R}^2\times\mathbb{R}^2} |\varphi(y_2)|^2\,\frac{|u(x_1)-u(y_1)|^2}{|x-y|^{2+2\,s}}\,dx\,dy\right)^{1/2}.\end{aligned}$$

By using Fubini's theorem, we have

$$\begin{aligned}&\int\!\!\int_{\mathbb{R}^2\times\mathbb{R}^2} |u(x_1)|^2\,\frac{|\varphi(x_2)-\varphi(y_2)|^2}{|x-y|^{2+2\,s}}\,dx\,dy \\ &= \int_{\mathbb{R}} |u(x_1)|^2 \left(\int\!\!\int_{\mathbb{R}\times\mathbb{R}} |\varphi(x_2)-\varphi(y_2)|^2 \left(\int_{\mathbb{R}} \frac{dy_1}{((x_1-y_1)^2+(x_2-y_2)^2)^{\frac{2+2\,s}{2}}}\right) dx_2\,dy_2\right) dx_1.\end{aligned}$$

By using a change of variable, we get

$$\int_{\mathbb{R}} \frac{dx_2}{\big((x_1-y_1)^2+(x_2-y_2)^2\big)^{\frac{2+2s}{2}}} = \frac{\alpha_s}{|x_2-y_2|^{1+2s}}.$$

Thus, we obtain

$$\int\int_{\mathbb{R}^2\times\mathbb{R}^2} |u(x_1)|^2 \frac{|\varphi(x_2)-\varphi(y_2)|^2}{|x-y|^{2+2s}}\,dx\,dy = \alpha_s\,\|u\|^2_{L^2(A)}\,[\varphi]^2_{W^{s,2}(\mathbb{R})}.$$

With a similar computation, we also get

$$\int\int_{\mathbb{R}^2\times\mathbb{R}^2} |\varphi(y_2)|^2 \frac{|u(x_1)-u(y_1)|^2}{|x-y|^{2+2s}}\,dx\,dy = \alpha_s\,\|\varphi\|^2_{L^2(\mathbb{R})}\,[u]^2_{W^{s,2}(\mathbb{R})}.$$

Thus, from the variational definition of $\lambda_1^s(A\times\mathbb{R})$, we get

$$\begin{aligned}\sqrt{\lambda_1^s(A\times\mathbb{R})} &\leqslant \frac{[u\,\varphi]_{W^{s,2}(\mathbb{R}^2)}}{\|u\,\varphi\|_{L^2(A\times\mathbb{R})}} \leqslant \sqrt{\alpha_s}\,\frac{\|u\|_{L^2(A)}\,[\varphi]_{W^{s,2}(\mathbb{R})}+\|\varphi\|_{L^2(\mathbb{R})}\,[u]_{W^{s,2}(\mathbb{R})}}{\|u\|_{L^2(A)}\,\|\varphi\|_{L^2(\mathbb{R})}}\\ &= \sqrt{\alpha_s}\left(\frac{[\varphi]_{W^{s,2}(\mathbb{R})}}{\|\varphi\|_{L^2(\mathbb{R})}}+\frac{[u]_{W^{s,2}(\mathbb{R})}}{\|u\|_{L^2(A)}}\right).\end{aligned}$$

By taking the infimum over u and φ, recalling that $\lambda_1^s(\mathbb{R})=0$, we get the desired conclusion □

In particular, from the previous result with $A=\mathbb{R}\setminus\mathbb{Z}$, we get that

$$\lambda_1^s(\Theta_k)\leqslant\lambda_1^s(\mathbb{R}\times(\mathbb{R}\setminus\mathbb{Z}))\leqslant\alpha_s\,\lambda_1^s(\mathbb{R}\setminus\mathbb{Z}).$$

In the first inequality, we used that

$$\mathbb{R}\times(\mathbb{R}\setminus\mathbb{Z})\subseteq\Theta_k.$$

From its definition (6.3), it is easy to see that α_s various continuously with respect to $s\in(0,1)$. Thus, in order to prove (6.2), it will be sufficient to establish that

$$\limsup_{s\searrow\frac{1}{2}} \frac{\lambda_1^s(\mathbb{R}\setminus\mathbb{Z})}{2\,s-1}<+\infty. \tag{6.4}$$

Step 3: Choice of the trial functions. To prove (6.4), we will need to carefully construct a suitable family of s-depending trial functions, which provides an upper bound on $\lambda_1^s(\mathbb{R}\setminus\mathbb{Z})$ with the correct asymptotic behavior. For every

$$n\in\mathbb{N}\setminus\{0\},\qquad s>1/2\qquad\text{and}\qquad 0<\varepsilon<\frac{1}{10},$$

we consider the trial function $u_n\,\varphi_{n,s,\varepsilon}$, where:

- $u_n \in C_0^\infty((-n,n))$ has the form

$$u_n(x) = u\Big(\frac{x}{n}\Big),$$

  for some $u \in C_0^\infty((-1,1))$ such that $\|u\|_{L^2((-1,1))} = 1$;
- the *multiple funnel-type* cut-off function $\varphi_{n,s,\varepsilon} \in W^{s,2}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ has the form

$$\varphi_{n,s,\varepsilon} = 1 - \sum_{j=-n}^{n} \zeta_s\Big(\frac{x-j}{\varepsilon}\Big),$$

  where ζ_s is the function given by

$$\zeta_s(x) = \big(1 - |x|^{2\,s-1}\big)_+.$$

Thanks to [7, Lemma 2.7], we see that

$$u_n\,\varphi_{n,s,\varepsilon} \in \widetilde{W}^{s,2}_0((-n,n)) \subseteq \widetilde{W}^{s,2}_0(\mathbb{R}\setminus\mathbb{Z}).$$

Thus, it is a feasible trial function. By using again Minkowski's inequality, this yields

$$\sqrt{\lambda_1^s(\mathbb{R}\setminus\mathbb{Z})} \leqslant \frac{[u_n]_{W^{s,2}(\mathbb{R})} + \|u_n\|_{L^\infty((-n,n))}\,[\varphi_{n,\varepsilon,s}]_{W^{s,2}(\mathbb{R})}}{\|u_n\,\varphi_{n,\varepsilon,s}\|_{L^2((-n,n))}}.$$

Step 4: Estimate of the quotient. Let us start by handling the terms at the numerator. We consider at first the terms containing u_n, which are simpler. By recalling the definition of u_n, we have

$$[u_n]_{W^{s,2}(\mathbb{R})} = n^{\frac{1}{2}-s}\,[u]_{W^{s,2}(\mathbb{R})}.$$

The last term can be estimated by using the interpolation inequality [10, Corollary 2.2], which gives

$$[u]_{W^{s,2}(\mathbb{R})} \leqslant \sqrt{\frac{C}{s\,(1-s)}}\,\|u\|^{1-s}_{L^2((-1,1))}\,\|u'\|^{s}_{L^2((-1,1))},$$

for some $C > 0$ independent of s. This guarantees that we have

$$[u_n]_{W^{s,2}(\mathbb{R})} \leqslant n^{\frac{1}{2}-s}\sqrt{\frac{C}{s\,(1-s)}}\,\|u'\|^{s}_{L^2((-1,1))}. \tag{6.5}$$

The term with the L^∞ norm is easy to handle, as we simply have

$$\|u_n\|_{L^\infty((-n,n))} = \|u\|_{L^\infty((-1,1))}. \tag{6.6}$$

The term containing the cut-off is the most delicate one. To estimate it, we observe that

$$\sum_{j=-n}^{n} \zeta_s\left(\frac{x-j}{\varepsilon}\right) = \max_{j=-n,\dots,n} \zeta_s\left(\frac{x-j}{\varepsilon}\right),$$

thanks to the fact that all the functions involved in the sum have disjoint support. We can then use the submodularity of the Sobolev–Slobodeckiĭ seminorm (see [16, Theorem 3.2 and Remark 3.3]) and obtain

$$\begin{aligned}
[\varphi_{n,\varepsilon,s}]_{W^{s,2}(\mathbb{R})} &= \left[\sum_{j=-n}^{n} \zeta_s\left(\frac{\cdot-j}{\varepsilon}\right)\right]_{W^{s,2}(\mathbb{R})} \\
&= \left[\max_{j=-n,\dots,n} \zeta_s\left(\frac{\cdot-j}{\varepsilon}\right)\right]_{W^{s,2}(\mathbb{R})} \\
&\leqslant \left(\sum_{j=-n}^{n} \left[\zeta_s\left(\frac{\cdot-j}{\varepsilon}\right)\right]^2_{W^{s,2}(\mathbb{R})}\right)^{\frac{1}{2}} = \sqrt{2\,n+1}\,\varepsilon^{\frac{1}{2}-s}\,[\zeta_s]_{W^{s,2}(\mathbb{R})}.
\end{aligned}$$

To conclude, the key fact is a very precise asymptotic estimate of the last term, as s goes to $1/2$. This is contained in Lemma B.1 in Appendix B, which permits to infer

$$[\varphi_{n,\varepsilon,s}]_{W^{s,2}(\mathbb{R})} \leqslant C\,\sqrt{2\,n+1}\,\varepsilon^{\frac{1}{2}-s}\,\sqrt{2\,s-1}, \qquad \text{for } \frac{1}{2} < s < \frac{3}{4}, \tag{6.7}$$

with $C > 0$ not depending on s.

We now pass to examine the denominator. In this case, we have

$$\|u_n\,\varphi_{n,\varepsilon,s}\|_{L^2((-n,n))} = n^{\frac{1}{2}} \left(\int_{-1}^{1} |u(y)|^2 \left(1 - \sum_{j=-n}^{n} \zeta_s\left(\frac{n\,y-j}{\varepsilon}\right)\right)^2 dy\right)^{\frac{1}{2}} \geqslant n^{\frac{1}{2}}\,\|u\|_{L^2(A_\varepsilon)}, \tag{6.8}$$

where

$$A_\varepsilon = (-1,1) \setminus \bigcup_{j=-n}^{n} \left(\frac{j-\varepsilon}{n}, \frac{j+\varepsilon}{n}\right).$$

Step 5: Conclusion. By collecting the estimates (6.5), (6.6), (6.7), and (6.8), we obtain

$$\begin{aligned}
\sqrt{\frac{\lambda_1^s(\mathbb{R}\setminus\mathbb{Z})}{2\,s-1}} &\leqslant \frac{n^{\frac{1}{2}-s}\sqrt{\dfrac{C}{s\,(1-s)}}\,\|u'\|^s_{L^2((-1,1))} + C\,\|u\|_{L^\infty((-1,1))}\,\sqrt{2\,n+1}\,\varepsilon^{\frac{1}{2}-s}\,\sqrt{2\,s-1}}{n^{\frac{1}{2}}\,\sqrt{2\,s-1}\,\|u\|_{L^2(A_\varepsilon)}} \\
&\leqslant \sqrt{\frac{C}{s\,(1-s)}}\,\frac{\|u'\|^s_{L^2((-1,1))}}{\|u\|_{L^2(A_\varepsilon)}}\,\frac{n^{-s}}{\sqrt{2\,s-1}} + C\,\frac{\|u\|_{L^\infty((-1,1))}}{\|u\|_{L^2(A_\varepsilon)}}\,\sqrt{3}\,\varepsilon^{\frac{1}{2}-s}.
\end{aligned}$$

It is now important to make a good choice of the parameters n and ε: we take them to be

$$\varepsilon = \left(\frac{1}{10}\right)^{\frac{1}{2s-1}} \qquad \text{and} \qquad n = \left(\left\lfloor \frac{1}{2s-1} \right\rfloor + 1\right)^2.$$

Observe that with these choices, we have

$$\lim_{s \searrow \frac{1}{2}} \varepsilon = 0 \qquad \text{and} \qquad \varepsilon^{\frac{1}{2}-s} = \sqrt{10},$$

and

$$\lim_{s \searrow \frac{1}{2}} \frac{n^{-s}}{\sqrt{2s-1}} \leqslant \lim_{s \searrow \frac{1}{2}} (2s-1)^{2s-\frac{1}{2}} = 0,$$

where we also used (2.1). Moreover, by using the Dominated Convergence Theorem, we also have

$$\lim_{s \searrow \frac{1}{2}} \|u\|_{L^2(A_\varepsilon)} = \|u\|_{L^2((-1,1))} = 1.$$

These facts finally enable us to conclude that

$$\limsup_{s \searrow \frac{1}{2}} \sqrt{\frac{\lambda_1^s(\mathbb{R} \setminus \mathbb{Z})}{2s-1}} \leqslant \sqrt{30}\, C\, \|u\|_{L^\infty((-1,1))} < +\infty.$$

The proof is now over.

APPENDIX A: A BI-LIPSCHITZ HOMEOMORPHISM

In what follows, for every open bounded set $K \subseteq \mathbb{R}^N$ and every $x_0 \in K$, we define

$$d_K(x_0) = \min_{x \in \partial K} |x - x_0|, \qquad D_K(x_0) = \max_{x \in \partial K} |x - x_0|.$$

Lemma A.1. *Let $K \subseteq \mathbb{R}^N$ be an open bounded convex set and $x_0 \in K$. There exists a bi-Lipschitz homeomorphism $\Phi_{K,x_0} : \mathbb{R}^N \to \mathbb{R}^N$ with the following properties:*

- *$\Phi_{K,x_0}(x_0) = x_0$ and $\Phi_{K,x_0}(r\,(K - x_0) + x_0) = B_r(x_0)$, for every $r > 0$;*
- *Φ_{K,x_0} is L_K-Lipschitz with*

$$L_K = \frac{2}{d_K(x_0)};$$

- *Φ_{K,x_0}^{-1} is M_K-Lipschitz with*

$$M_K = D_K(x_0)\left(2 + \frac{D_K(x_0)}{d_K(x_0)}\right).$$

Moreover, we have

$$\left(\frac{1}{M_K}\right)^N \leqslant |\det \nabla \Phi_{K,x_0}(x)| \leqslant (L_K)^N, \qquad \textit{for a. e. } x \in \mathbb{R}^N. \tag{A.1}$$

and

$$\left(\frac{1}{L_K}\right)^N \leqslant |\det \nabla \Phi_{K,x_0}^{-1}(x)| \leqslant (M_K)^N, \qquad \textit{for a. e. } x \in \mathbb{R}^N. \tag{A.2}$$

Proof. For notational simplicity, we omit to indicate the subscript x_0 everywhere. We recall at first the definition of the *Minkowski functional of* K centered at x_0, that is,

$$j_K(x) = \inf \left\{ \lambda > 0 \ : \ x \in \lambda\,(K - x_0) + x_0 \right\}.$$

We recall that this is a Lipschitz function, which verifies the following homogeneity property

$$j_K(t\,(x - x_0) + x_0) = t\, j_K(x), \qquad \text{for every } x \in \mathbb{R}^N,\ t > 0. \tag{A.3}$$

We also observe that by construction it holds

$$j_K(x) < r \quad \text{if and only if} \quad x \in r\,(K - x_0) + x_0,$$

and that

$$j_K(x) = r \quad \text{if and only if} \quad x \in r\,(\partial K - x_0) + x_0.$$

Moreover, j_K satisfies

$$|j_K(x) - j_K(y)| \leqslant \frac{1}{d_K(x_0)}\, |x - y|, \qquad \text{for every } x, y \in \mathbb{R}^N. \tag{A.4}$$

Last, but not least, we have the following lower bound

$$j_K(x) = |x - x_0|\, j_K\left(\frac{x - x_0}{|x - x_0|} + x_0\right) \geqslant \frac{|x - x_0|}{D_K(x_0)}, \qquad \text{for every } x \in \mathbb{R}^N. \tag{A.5}$$

Then we define Φ_K as follows

$$\Phi_K(x_0) = x_0, \qquad \Phi_K(x) = \frac{x - x_0}{|x - x_0|}\, j_K(x) + x_0, \qquad \text{if } x \in \mathbb{R}^N \setminus \{x_0\}.$$

Thanks to the properties of the Minkowski functional, we have that Φ_K is injective. To verify that Φ_K is bijective, let us take $y \in \mathbb{R}^N \setminus \{x_0\}$. We then define

$$\overline{x} = \frac{|y - x_0|}{j_K(y)}\,(y - x_0) + x_0, \tag{A.6}$$

we claim that $\Phi_K(\overline{x}) = y$. Indeed, by construction we have

$$\Phi_K(\overline{x}) = \frac{\overline{x} - x_0}{|\overline{x} - x_0|}\, j_K(\overline{x}) + x_0 = \frac{y - x_0}{|y - x_0|}\, j_K\left(\frac{|y - x_0|}{j_K(y)}\,(y - x_0) + x_0\right) + x_0.$$

From property (A.3), we get

$$\Phi_K(\overline{x}) = \frac{y - x_0}{|y - x_0|}\, j_K\left(\frac{|y - x_0|}{j_K(y)}\,(y - x_0) + x_0\right) + x_0 = \frac{y - x_0}{|y - x_0|}\, j_K(y)\, \frac{|y - x_0|}{j_K(y)} + x_0 = y,$$

as desired. This shows that Φ_K is bijective and from (A.6) we get

$$\Phi_K^{-1}(y) = \frac{|y - x_0|}{j_K(y)}\,(y - x_0) + x_0, \qquad \text{for } y \in \mathbb{R}^N \setminus \{x_0\}.$$

Thanks to the properties of the Minkowski functional, it is easily seen that

$$\Phi_K(r\,(K - x_0) + x_0) = B_r(x_0), \qquad \text{for every } r > 0.$$

We now claim that both Φ_K and its inverse are Lipschitz continuous. We start with Φ_K: we take $x, y \in \mathbb{R}^N \setminus \{x_0\}$ and, without loss of generality, we can suppose that $|y - x_0| \leqslant |x - x_0|$. By the triangle inequality, we get

$$\begin{aligned} |\Phi_K(x) - \Phi_K(y)| &\leqslant j_K(y) \left| \frac{x - x_0}{|x - x_0|} - \frac{y - x_0}{|y - x_0|} \right| + |j_K(x) - j_K(y)| \\ &\leqslant j_K(y)\, \frac{|x - y|}{\sqrt{|x - x_0|\,|y - x_0|}} + |j_K(x) - j_K(y)|, \end{aligned} \tag{A.7}$$

where we used that

$$\begin{aligned} \left| \frac{x - x_0}{|x - x_0|} - \frac{y - x_0}{|y - x_0|} \right|^2 &= 2 - 2\, \frac{\langle x - x_0, y - x_0 \rangle}{|x - x_0|\,|y - x_0|} \\ &\leqslant \frac{|x - x_0|^2 + |y - x_0|^2}{|x - x_0|\,|y - x_0|} - 2\, \frac{\langle x - x_0, y - x_0 \rangle}{|x - x_0|\,|y - x_0|} = \frac{|x - y|^2}{|x - x_0|\,|y - x_0|}, \end{aligned}$$

thanks to Young's inequality. By using (A.4), the fact that $j_K(x_0) = 0$ and the assumption $|y - x_0| \leqslant |x - x_0|$, we get from (A.7)

$$|\Phi_K(x) - \Phi_K(y)| \leqslant \frac{1}{d_K(x_0)} \left[|y - x_0|\, \frac{|x - y|}{\sqrt{|x - x_0|\,|y - x_0|}} + |x - y| \right] \leqslant \frac{2}{d_K(x_0)}\, |x - y|.$$

This proves the claimed Lipschitz regularity of Φ_K.

We now turn our attention to the inverse function Φ_K^{-1}. We proceed in a similar way: we take $x, y \in \mathbb{R}^N \setminus \{x_0\}$ and we can suppose that $|y - x_0| \leqslant |x - x_0|$. Then by the triangle inequality

$$|\Phi_K^{-1}(x)-\Phi_K^{-1}(y)| \leqslant \frac{1}{j_K(x)}\Big||x-x_0|\,(x-x_0)-|y-x_0|\,(y-x_0)\Big| + |y-x_0|^2\left|\frac{1}{j_K(x)}-\frac{1}{j_K(y)}\right|.$$

By using (A.5) and observing that

$$\Big||x-x_0|\,(x-x_0)-|y-x_0|\,(y-x_0)\Big| \leqslant (|x-x_0|+|y-x_0|)\,|x-y \leqslant 2\,|x-x_0|\,|x-y|,$$

we get that

$$\begin{aligned}|\Phi_K^{-1}(x)-\Phi_K^{-1}(y)| &\leqslant \frac{2\,|x-x_0|}{j_K(x)}\,|x-y| + \frac{|y-x_0|^2}{j_K(x)\,j_K(y)}\,|j_K(x)-j_K(y)| \\ &\leqslant 2\,D_K(x_0)\,|x-y| + D_K(x_0)^2\,\frac{|y-x_0|^2}{|x-x_0|\,|y-x_0|}\,\frac{|x-y|}{d_K(x_0)} \\ &\leqslant 2\,D_K(x_0)\,|x-y| + D_K(x_0)^2\,\frac{|x-y|}{d_K(x_0)} \\ &= D_K(x_0)\left(2+\frac{D_K(x_0)}{d_K(x_0)}\right)|x-y|.\end{aligned}$$

This gives the desired Lipschitz estimate for Φ_K^{-1}, as well.

Finally, the two-sided estimates (A.1) and (A.2) are a standard consequence of the Lipschitz estimates on Φ_K and Φ_K^{-1}, in conjunction with the Area Formula for Lipschitz functions and Rademacher's theorem. □

APPENDIX B: A SPECIAL CUT-OFF FUNCTION

Lemma B.1. *Let $1/2 < s < 1$ and let*

$$\zeta_s(x) = \left(1-|x|^{2s-1}\right)_+, \qquad \textit{for } x \in \mathbb{R}.$$

Then we have

$$[\zeta_s]_{W^{s,2}(\mathbb{R})} \leqslant C\,\frac{\sqrt{2s-1}}{\sqrt{1-s}}, \tag{B.1}$$

with a constant $C > 0$ independent of $s \in (1/2, 1)$.

Proof. We decompose the seminorm as follows

$$\begin{aligned}[\zeta_s]^2_{W^{s,2}(\mathbb{R})} &= \iint_{(-1,1)\times(-1,1)} \frac{\Big||x|^{2s-1}-|y|^{2s-1}\Big|^2}{|x-y|^{1+2s}}\,dx\,dy \\ &+ \frac{1}{s}\int_{-1}^{1}\frac{\Big|1-|x|^{2s-1}\Big|^2}{(1-x)^{2s}}\,dx + \frac{1}{s}\int_{-1}^{1}\frac{\Big|1-|x|^{2s-1}\Big|^2}{(1+x)^{2s}}\,dx = \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned} \tag{B.2}$$

To prove (B.1), we will prove that

$$\mathcal{I}_i \leqslant C\,\frac{2\,s-1}{1-s}, \qquad \text{for } i=1,2,3. \tag{B.3}$$

For the first term $\mathcal{I}_1$, we observe that by using the symmetry of the set and of the integrand, we have

$$\mathcal{I}_1 \leqslant 4 \iint_{(0,1)\times(0,1)} \frac{\left||x|^{2\,s-1}-|y|^{2\,s-1}\right|^2}{|x-y|^{1+2\,s}}\,dx\,dy.$$

By using [7, Remark 4.2, formula (4.3)] with the choice $\beta = 2\,s-1$ there, we can estimate the last double integral as follows

$$\mathcal{I}_1 \leqslant 4 \iint_{(0,1)\times(0,1)} \frac{\left||x|^{2\,s-1}-|y|^{2\,s-1}\right|^2}{|x-y|^{1+2\,s}}\,dx\,dy \leqslant \left(\int_0^1 \frac{|1-\tau^{2\,s-1}|^2}{|1-\tau|^{1+2\,s}}\left(1+\tau^{1-2\,s}\right)d\tau\right)\frac{4}{2\,s-1}.$$

We then write

$$\begin{aligned}\int_0^1 \frac{|1-\tau^{2\,s-1}|^2}{|1-\tau|^{1+2\,s}}\left(1+\tau^{1-2\,s}\right)d\tau &= \int_0^{\frac12} \frac{|1-\tau^{2\,s-1}|^2}{|1-\tau|^{1+2\,s}}\left(1+\tau^{1-2\,s}\right)d\tau \\ &\quad + \int_{\frac12}^1 \frac{|1-\tau^{2\,s-1}|^2}{|1-\tau|^{1+2\,s}}\left(1+\tau^{1-2\,s}\right)d\tau \\ &\leqslant C\int_0^{\frac12} |1-\tau^{2\,s-1}|^2\left(1+\tau^{1-2\,s}\right)d\tau \\ &\quad + C\int_{\frac12}^1 \frac{|1-\tau^{2\,s-1}|^2}{|1-\tau|^{1+2\,s}}\,d\tau =: \mathcal{I}_{1,1}+\mathcal{I}_{1,2}.\end{aligned}$$

The constant $C>0$ can be taken independent of s. We start by estimating $\mathcal{I}_{1,2}$, which is simpler: we use the following pointwise inequality

$$a^\alpha - b^\alpha \leqslant \alpha\,b^{\alpha-1}\,(a-b), \qquad \text{for } 0<b\leqslant a,\ 0<\alpha<1,$$

which just follows from concavity of the map $\tau \mapsto \tau^\alpha$. This gives

$$\mathcal{I}_{1,2} \leqslant C\,4^{1-s}\,(2\,s-1)^2 \int_{\frac12}^1 (1-\tau)^{1-2\,s}\,d\tau = \frac{C}{2\,(1-s)}\,(2\,s-1)^2,$$

as desired. We now come to $\mathcal{I}_{1,1}$, which is the most subtle. We have to distinguish two cases: $1/2<s<3/4$ and $3/4\leqslant s<1$. In the first case, we set for simplicity

$$f_\tau(s) = \tau^{2\,s-1}, \qquad \text{for } \tau>0,\ s>\frac12.$$

Then we have

$$\left|f_\tau(s) - f_\tau\left(\frac{1}{2}\right)\right| = \left|\int_{\frac{1}{2}}^{s} f_\tau'(t)\,dt\right|,$$

that is for $0 < \tau \leqslant 1/2$

$$|1-\tau^{2s-1}| = 2\,|\log\tau|\left|\int_{\frac{1}{2}}^{s} \tau^{2t-1}\,dt\right| \leqslant 2\,(-\log\tau)\left(s-\frac{1}{2}\right) = (-\log\tau)\,(2s-1).$$

Thus, we get for $1/2 < s < 3/4$

$$\mathcal{I}_{1,1} \leqslant C\,(2s-1)^2 \int_0^{\frac{1}{2}} (-\log\tau)^2\,(1+\tau^{1-2s})\,d\tau \leqslant 2\,C\,(2s-1)^2 \int_0^{\frac{1}{2}} (-\log\tau)^2\,\frac{d\tau}{\sqrt{\tau}}. \tag{B.4}$$

This gives the desired estimate for $1/2 < s < 3/4$, as the last integral is finite and independent of s. On the other hand, for $3/4 \leqslant s < 1$, we can simply estimate

$$\mathcal{I}_{1,1} \leqslant C \int_0^{\frac{1}{2}} (1+\tau^{1-2s})\,d\tau \leqslant 2\,C \int_0^{\frac{1}{2}} \tau^{1-2s}\,d\tau = \frac{C}{1-s}\left(\frac{1}{2}\right)^{2-2s} \leqslant \frac{1}{1-s}. \tag{B.5}$$

In particular, we get from (B.4) and (B.5)

$$\mathcal{I}_{1,1} \leqslant C\,\frac{(2s-1)^2}{1-s}, \qquad \text{for } \frac{1}{2} < s < 1,$$

possibly for a different $C > 0$, still independent of s. By collecting the estimates for $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$, we thus get (B.1) for $\mathcal{I}_1$.

We now consider $\mathcal{I}_2$ and $\mathcal{I}_3$. We only estimate the first one, as the estimate for the second one is similar. For $s > 1/2$, we have

$$\begin{aligned}
\frac{1}{s}\int_{-1}^{1} \frac{\left|1-|x|^{2s-1}\right|^2}{(1-x)^{2s}}\,dx &\leqslant 2\int_0^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}}\,dx + 2\int_{-1}^{0} \frac{(1-|x|^{2s-1})^2}{(1-x)^{2s}}\,dx \\
&\leqslant 2\int_0^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}}\,dx + 2\int_{-1}^{0} (1-|x|^{2s-1})^2\,dx \\
&= 2\int_{\frac{1}{2}}^{1} \frac{(1-x^{2s-1})^2}{(1-x)^{2s}}\,dx + 2\int_0^{\frac{1}{2}} \frac{(1-x^{2s-1})^2}{(1-x)^{2s}}\,dx \\
&\quad + 2\int_0^1 (1-x^{2s-1})^2\,dx \\
&\leqslant 2\int_{\frac{1}{2}}^{1} \frac{(1-x^{2s-1})^2}{(1-x)^{2s}}\,dx + 2\cdot 4^s \int_0^{\frac{1}{2}} (1-x^{2s-1})^2\,dx
\end{aligned}$$

$$+ 2 \int_0^1 (1-x^{2s-1})^2\,dx$$

$$\leqslant 2 \int_{\frac{1}{2}}^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}}\,dx + 2\,(4^s+1) \int_0^1 (1-x^{2s-1})\,dx.$$

By computing the last integral, this gives in particular

$$\frac{1}{s} \int_{-1}^1 \frac{\left|1-|x|^{2s-1}\right|^2}{(1-x)^{2s}}\,dx \leqslant 2 \int_{\frac{1}{2}}^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}}\,dx + (4^s+1)\,\frac{2s-1}{s}.$$

At this point, the integral in the right-hand side can be estimated as we did for $\mathcal{I}_{1,2}$ above. By proceeding as before, we get (B.3) for $\mathcal{I}_2$ (and thus for $\mathcal{I}_3$), as well. □

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