



Hardy–Littlewood–Sobolev inequalities for a class of non-symmetric and non-doubling hypoelliptic semigroups

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Abstract

In his seminal 1934 paper on Brownian motion and the theory of gases Kolmogorov introduced a second order evolution equation which displays some challenging features. In the opening of his 1967 hypoellipticity paper Hörmander discussed a general class of degenerate Ornstein–Uhlenbeck operators that includes Kolmogorov’s as a special case. In this note we combine semigroup theory with a nonlocal calculus for these hypoelliptic operators to establish new inequalities of Hardy–Littlewood–Sobolev type in the situation when the drift matrix has nonnegative trace. Our work has been influenced by ideas of E. Stein and Varopoulos in the framework of symmetric semigroups. One of our objectives is to show that such ideas can be pushed to successfully handle the present degenerate non-symmetric setting.

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1 Introduction

Sobolev inequalities occupy a central position in analysis, geometry and physics. Typically, in such a priori estimates one is able to control a certain L^q norm of a derivative of a function in terms of a L^p norm of derivatives of higher order. One distinctive aspect of these inequalities is that there is gain in the exponent of integrability, i.e., $q > p$. For instance, the prototypical Sobolev inequality in \mathbb{R}^N states that for any $1 \leq p < N$, there exists a constant $S_{N,p}$ such that for any function f in the Schwartz class \mathcal{S} , one has

$$(\star) \quad \|f\|_q \leq S_{N,p} \|\nabla f\|_p \iff \frac{1}{p} - \frac{1}{q} = \frac{1}{N}.$$

In such framework, (\star) is referred to as the embedding theorem $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. The relation between the exponents p and q in (\star) is the well-known Hardy–Littlewood–Sobolev condition, and the if and only if character is connected with the interplay between the differential operator ∇ and the homogeneous structure (Euclidean dilations) of the ambient space.

In this paper we are concerned with a scale of global inequalities such as the one above for the following class of second-order partial differential equations in \mathbb{R}^{N+1} ,

$$\mathcal{K}u = \mathcal{A}u - \partial_t u \stackrel{\text{def}}{=} \text{tr}(Q\nabla^2 u) + \langle BX, \nabla u \rangle - \partial_t u = 0, \quad (1.1)$$

where the $N \times N$ matrices Q and B have real, constant coefficients, and $Q = Q^* \geq 0$. We assume throughout that $N \geq 2$, and we indicate with X the generic point in \mathbb{R}^N , with (X, t) the one in \mathbb{R}^{N+1} . Such class has been intensely studied during the past thirty years by several schools worldwide. Presently, many mathematicians are actively involved in developing a De Giorgi–Nash–Moser theory for such equations, i.e., a theory which includes the treatment of rough variable coefficients. It is well-known that at the very foundations of such theory are Sobolev and Poincaré inequalities. No global sharp results of this kind are presently available for the time-independent part $\mathcal{A} = \text{tr}(Q\nabla^2 u) + \langle BX, \nabla u \rangle$ of the operator in (1.1), and to the best of our knowledge our Sobolev embedding theorems are the first in this respect.

It is worth noting here that when $Q = I_N$ and $B = O_N$, then (1.1) becomes the standard heat operator $\Delta - \partial_t$ in \mathbb{R}^{N+1} , and we are back into the well-developed framework of (\star) . Our interest is exclusively in the degenerate case when $Q \geq 0$ and $B \neq O_N$. Then, the evolution of equations such as (1.1) is driven by semigroups

$P_t = e^{-t\mathcal{A}}$ which, in general, are non-symmetric and non-doubling. Furthermore, there is no global homogeneous structure associated with them, and they lack an obvious notion of “gradient”. For instance, a tool like the P.A. Meyer *carré du champ* $\Gamma(f) = \frac{1}{2}[\mathcal{A}(f^2) - 2f\mathcal{A}f]$ is not directly effective here since $\Gamma(f) = \langle Q\nabla f, \nabla f \rangle$. This misses all directions of non-ellipticity in the degenerate case, and also does not provide control on the drift.

The class (1.1) first appeared in the 1967 work of Hörmander [35], in which he proved his celebrated hypoellipticity theorem asserting that if smooth vector fields Y_0, Y_1, \dots, Y_m in \mathbb{R}^{N+1} verify the finite rank condition on the Lie algebra, then the operator $\sum_{i=1}^m Y_i^2 + Y_0$ is hypoelliptic. To motivate this result, in the opening of his paper he discussed (1.1) and showed that \mathcal{K} is hypoelliptic if and only if $\text{Ker } Q$ does not contain any non-trivial subspace which is invariant for B^* . This condition can be equivalently expressed in terms of the strict positivity, hence invertibility, of the covariance matrix

$$K(t) = \frac{1}{t} \int_0^t e^{sB} Q e^{sB^*} ds \tag{1.2}$$

for every $t > 0$. We note that, in the degenerate case when Q fails to be elliptic, this property becomes void at $t = 0$, since $K(0) = Q$. Also, it is easy to see that $K(t) > 0$ for every $t > 0$ if and only if $K(t_0) > 0$ for one $t_0 > 0$. Under the hypoellipticity assumption Hörmander constructed a fundamental solution $p(X, Y, t) > 0$ for (1.1), and proved that, given $f \in \mathcal{S}$, the Cauchy problem $\mathcal{K}u = 0, u(X, 0) = f(X)$ admits a unique bounded solution given by $P_t f(X) = \int_{\mathbb{R}^N} p(X, Y, t) f(Y) dY$. This defines a non-symmetric semigroup $\{P_t\}_{t>0}$ which is strongly continuous in $L^p, 1 \leq p < \infty$, satisfies $P_t 1 = 1$, but which, however, is not contractive in general.

Our primary interest in this paper is on the subclass of (1.1) which, besides Hörmander’s hypoellipticity condition $K(t) > 0$, also satisfy the assumption

$$\text{tr } B \geq 0. \tag{1.3}$$

This serves to guarantee that the semigroup $\{P_t\}_{t>0}$ be contractive in L^p for $1 \leq p < \infty$, a fact that plays a pervasive role in our work. Although to an unfamiliar reader (1.3) might seem restrictive, we emphasise that this subclass is sufficiently wide to include many examples of considerable physical relevance for which, prior to the present work, the development of sharp global results has resisted the efforts of many groups of mathematicians. For instance, a prototypical example to keep in mind is the operator

$$\mathcal{K}_0 u = \Delta_v u + \langle v, \nabla_x u \rangle - \partial_t u,$$

introduced by Kolmogorov in his seminal 1934 note [38] on Brownian motion and the theory of gases. Here, we have let $N = 2n$, and $X = (v, x)$, with $v, x \in \mathbb{R}^n$. Such \mathcal{K}_0 fails to be parabolic since it is missing the diffusive term $\Delta_x u$, but it is easily seen to satisfy Hörmander’s finite rank condition for the hypoellipticity. Equivalently, one can verify that $K(t) = \begin{pmatrix} I_n & t/2 I_n \\ t/2 I_n & t^2/3 I_n \end{pmatrix} > 0$ for every $t > 0$. Remarkably, Kolmogorov himself had already produced the following explicit fundamental solution

$$p_0(X, Y, t) = \frac{c_n}{t^{2n}} \exp \left\{ -\frac{1}{t} (|v-w|^2 + \frac{3}{t} (v-w, y-x-tv) + \frac{3}{t^2} |x-y+tv|^2) \right\},$$

where $Y = (w, y)$. Since such function is smooth off the diagonal, it follows that he had proved that \mathcal{H}_0 is hypoelliptic more than thirty years before [35]. We note that the hypothesis (1.3) trivially includes Kolmogorov's operator \mathcal{H}_0 since for the latter we have $\text{tr } B = 0$, but also encompasses several different examples of interest in mathematics and physics. For a short list the reader can see the items in bold in the table in Fig. 1 in Sect. 3. For the items in black (see [12,25,52,64]) we have $\text{tr } B < 0$, thus they are not covered by our results. Such subclass of (1.1) will be analysed in a future study.

To provide the reader with some perspective we mention that during the last three decades there has been considerable progress in the study of the equations (1.1). The existing approaches are essentially of two types: (a) far reaching adaptations of direct methods from partial differential equations combined with Lie group theory and analysis in spaces with homogeneous and non-homogeneous structures; or (b) powerful combinations of ideas from probability and semigroup theory. For the existing literature covering either (a) or (b), the reader should see [1,3,8,10,11,16,19,22,26,28,32,39,41,42,46–48,50,51,53–57], but such list is by no means exhaustive. One should also consult the survey papers [9,40], and the books [17,18,34,45,69]. Despite such large body of works, some aspects presently remain elusive, such as: (i) a systematic development of a sharp intrinsic Hardy–Littlewood–Sobolev theory; (ii) the analysis of local and nonlocal isoperimetric inequalities. The aim of the present paper is to take a first step in the program (i). In our sequel paper [30] we address (ii).

Our approach combines semigroup theory with the nonlocal calculus for (1.1) recently developed in [29], and it has been influenced by the ideas of Stein in [60] and Varopoulos in [67] in the setting of positive symmetric semigroups. In fact, one of the objectives of the present paper is to show that their powerful ideas can be pushed to successfully handle the degenerate non-symmetric setting of (1.1).

A discussion of the main results and techniques seems in order at this point. Section 2 is devoted to collecting the known background results on the semigroup $\{P_t\}_{t>0}$. We introduce the intertwined non-symmetric pseudo-distance $m_t(X, Y)$, and the time-dependent pseudo-balls $B_t(X, r)$. The volume function $V(t) = \text{Vol}_N(B_t(X, \sqrt{t}))$ is defined in (2.5). The relevance of such function is demonstrated by its place in Hörmander's probability transition density (2.6). We also recall for completeness an important result from [41] stating that as $t \rightarrow 0^+$ the small-time behaviour of $V(t)$ is governed by a suitable infinitesimal homogeneous structure. Using such information one can show that there exists $D_0 \geq N \geq 2$ such that $V(t) \cong t^{D_0/2}$ as $t \rightarrow 0^+$. We call the number D_0 the intrinsic dimension of the semigroup at zero.

As it became evident from the work [67] (see also [65,66,68]), in Varopoulos' semigroup approach to the Hardy–Littlewood theory the evolution is driven by the large time behaviour of the semigroup. It should thus come as no surprise that the functional inequalities in this paper hinge on the behaviour of the volume function

$V(t)$ as $t \rightarrow \infty$. Section 3 is dedicated to the analysis of this aspect. The first key result is Proposition 3.1 in which we show that, under the hypothesis (1.3), the function $V(t)$ must blow-up at least linearly as $t \rightarrow \infty$ (note that for the Ornstein–Uhlenbeck operator $\Delta_X - \langle X, \nabla_X \rangle - \partial_t$, for which $\text{tr}(B) < 0$, one has instead $V(t) \rightarrow c_N > 0$ as $t \rightarrow \infty$). Furthermore, if the drift matrix B has at least one eigenvalue with strictly positive real part, then $V(t)$ blows up exponentially and is not doubling. In other words, in such situation the drift induces a negative “curvature” in the ambient space \mathbb{R}^N . In Definition 3.4 we introduce the key notion of *intrinsic dimension at infinity* of the semigroup, and we indicate such number with D_∞ . We note that the above mentioned minimal linear growth of $V(t)$ at infinity, provides the basic information that $D_\infty \geq 2$. The reader should see the table in Fig. 1 where the quantities D_0 and D_∞ are compared for several differential operators of interest in mathematics and physics. The second result of the section is Proposition 3.6 which establishes the $L^p - L^\infty$ ultracontractivity of the semigroup $\{P_t\}_{t>0}$ for $1 \leq p < \infty$. As the reader can surmise from the seminal work [67, Theorem 1] in the symmetric case, such property plays a central role in our work as well.

In Sect. 4 we introduce the relevant Sobolev spaces. One of the difficulties in the analysis of (1.1), already hinted at above, is that a “gradient” is not readily available. This problem is circumvented using the nonlocal operator $(-\mathcal{A})^{1/2}$ as a gradient since it intrinsically contains the appropriate fractional order of differentiation along the drift, which is instead missing in the above mentioned carré du champ. By means of Balakrishnan’s formula (4.1), we can precisely identify the nonlocal operators $(-\mathcal{A})^s$ by means of the semigroup $\{P_t\}_{t>0}$. This allows to introduce spaces of Sobolev type as follows. Given $0 < s < 1$ and $1 \leq p < \infty$, we define the Banach space

$$\mathcal{L}^{2s,p} = \overline{\mathcal{F}}^{\|\cdot\|_{\mathcal{L}^{2s,p}}},$$

where for a function in Schwartz class \mathcal{S} we have denoted by $\|f\|_{\mathcal{L}^{2s,p}} \stackrel{\text{def}}{=} \|f\|_{L^p} + \|(-\mathcal{A})^s f\|_{L^p}$. We stress that, when $\mathcal{A} = \Delta$, $s = 1/2$ and $1 < p < \infty$, the classical Calderón–Zygmund theory guarantees that the space $\mathcal{L}^{1,p}$ coincides with the standard Sobolev space $W^{1,p} = \{f \in L^p \mid \nabla f \in L^p\}$.

In Sect. 5, under the hypothesis (1.3), we establish a Littlewood–Paley estimate that has been so far missing in the analysis of the class (1.1). To achieve this we have combined a far reaching idea of Stein in [60] with the kernel associated with the Poisson semigroup $\mathcal{P}_z = e^{z(-\mathcal{A})^{1/2}}$ in [29]. Combining such tools with the powerful abstract Hopf–Dunford–Schwartz ergodic theorem in [20] we obtain the main weak $-L^1$ estimate in Theorem 5.5.

In Sect. 6 we introduce, for any $0 < \alpha < D_\infty$, the Riesz potential operators \mathcal{I}_α . Our central result is Theorem 6.3 that shows that for any $0 < \alpha < 2$ and $f \in \mathcal{S}$, one has

$$f = \mathcal{I}_\alpha \circ (-\mathcal{A})^{\alpha/2} f = (-\mathcal{A})^{\alpha/2} \circ \mathcal{I}_\alpha f. \tag{1.4}$$

This proves that $\mathcal{I}_\alpha = (-\mathcal{A})^{-\alpha/2}$. Again, the hypothesis (1.3) is essential. The reader should pay attention here to the fact noted above that, under such assumption, we have $D_\infty \geq 2$, and thus (1.4) covers the whole range $0 < \alpha < 2$. We note that, once again, the semigroup $\mathcal{P}_z = e^{z(-\mathcal{A})^{1/2}}$, $z > 0$, is in the background here.

In Sect. 7 we establish our main Hardy–Littlewood–Sobolev embedding, Theorem 7.4. Suppose that there exist $D, \gamma_D > 0$ such that

$$V(t) \geq \gamma_D t^{D/2}, \quad \forall t > 0. \quad (1.5)$$

Then, for every $0 < \alpha < D$ the operator \mathcal{J}_α maps L^1 into $L^{\frac{D}{D-\alpha}, \infty}$. If instead $1 < p < D/\alpha$, then \mathcal{J}_α maps L^p to L^q , with $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{D}$. Combining this result with (1.4) we finally obtain the Sobolev embedding Theorem 7.5. We mention that in the “negative curvature” situation when $D_\infty = \infty$, see in this respect the operator of Kolmogorov with friction in ex.6⁺ in Fig. 1, given any $1 \leq p < \infty$ we are free to chose $D > \max\{D_0, 2sp\}$ such that (1.5) hold. For such D we thus obtain $\mathcal{L}^{2s,p} \hookrightarrow L^{pD/(D-2sp)}$. The reader should note that (1.5) implies that $2 \leq D_0 \leq D \leq D_\infty$, and thus Theorems 7.4 and 7.5 do not cover the possibility $D_0 > D_\infty$. In the degenerate setting this case can occur, see the Ex. 4 of the Kramers’ operator in Fig. 1. When $D_0 > D_\infty$ the estimate (1.5) must be replaced by (7.9) below and, under such hypothesis, we obtain appropriate versions of the above described results, see Theorems 7.6 and 7.7.

In closing, we compare our results with the available literature. We first recall the deep developments, which began with the works of E. Stein and collaborators, for the class of Hörmander operators $\sum_i X_i^* X_i$ and $\sum_i X_i^* X_i - \partial_t$, where X_i are smooth vector fields in \mathbb{R}^N satisfying Hörmander’s finite rank condition. These operators, which are seemingly close relatives of (1.1), have a well-understood underlying geometry which stands in stark contrast with the one discussed in the present paper. In fact, there is a well-defined metric structure attached to the vector fields given by the Carnot–Carathéodory distance. In this respect, three groundbreaking results in the field were the Rothschild–Stein’s lifting theorem in [59] and its consequences, the Nagel–Stein–Wainger’s local doubling properties for the metric balls in [49], and Jerison’s local Poincaré inequality in [37]. Because of the drift, the class of operators we study has infinitely many pseudometrics driving the evolution, there is in general no doubling condition for the volume function, and there is no natural “gradient” associated.

Let us mention the few Sobolev-type estimates related to the class of degenerate operators (1.1) present in the literature we are aware of. In [16,53] the authors prove some interesting local results for nonnegative solutions to equations modelled on (1.1). They use tools from potential theory and representation formulas. The restriction to solutions, however, does not allow to obtain a priori information for arbitrary functions. For kinetic Fokker–Planck equations (where in particular we have $X = (v, x)$, with v indicating velocity and x position), we mention the recent papers [3,32]. In the former the authors prove a local gain of integrability for nonnegative sub-solutions via a non-trivial adaptation of the so-called velocity averaging method. In the latter the authors obtain a Poincaré inequality in a weighted L^2 space by means of an ad-hoc variational space. Our results differ from either one of these works since our Sobolev spaces $\mathcal{L}^{2s,p}$ are defined with the aid of the nonlocal operators $(-\mathcal{A})^s$. Similarly to the classical potential estimate $|f(X)| \leq c_N I_1(|\nabla f|)(X)$, our formula (1.4), combined with Theorem 7.5, provides the sharp a priori control of the L^q norm of a function, in

terms of the appropriate fractional order of differentiation. Both, along the directions of ellipticity, and of the drift.

We also mention [10], in which the author obtained L^2 a priori estimates for the above discussed homogeneous Kolmogorov's operator \mathcal{K}_0 , and the work [11], where the authors prove some Calderón-Zygmund type estimates (both in L^p and weak- L^1) for the operator \mathcal{A} . The interesting analysis in [11] combines local singular integral estimates with suitable coverings that exploit the homogeneous structure discovered in [41] (see also Sect. 2.4 below). Our approach, based on the semigroup $\mathcal{P}_z = e^{z\sqrt{-\mathcal{A}}}$, is different and allows to obtain results of a global nature, both in space and time.

1.1 Notation

The notation $\text{tr } A$ indicates the trace of a matrix A , A^* is the transpose of A , and $\nabla^2 u$ denotes the Hessian matrix of a function u . All the function spaces in this paper are based on \mathbb{R}^N , thus we will routinely avoid reference to the ambient space throughout this work. For instance, the Schwartz space of rapidly decreasing functions in \mathbb{R}^N will be denoted by \mathcal{S} , and for $1 \leq p \leq \infty$ we let $L^p = L^p(\mathbb{R}^N)$. The norm in L^p will be denoted by $\|\cdot\|_p$, instead of $\|\cdot\|_{L^p}$. We will indicate with L_0^∞ the Banach space of the $f \in C(\mathbb{R}^N)$ such that $\lim_{|X| \rightarrow \infty} |f(X)| = 0$ with the norm $\|\cdot\|_\infty$. The reader should keep in mind the following simple facts: (1) $P_t : L_0^\infty \rightarrow L_0^\infty$ for every $t > 0$; (2) \mathcal{S} is dense in L_0^∞ . The notation $|E|$ will indicate the N -dimensional Lebesgue measure of a set E . If $T : L^p \rightarrow L^q$ is a bounded linear map, we will indicate with $\|T\|_{p \rightarrow q}$ its operator norm. If $q = p$, the spectrum of T on L^p will be denoted by $\sigma_p(T)$, the resolvent set by $\rho_p(T)$, the resolvent operator by $R(\lambda, T) = (\lambda I - T)^{-1}$. For $x > 0$ we will indicate with $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ Euler's gamma function. For any $N \in \mathbb{N}$ we will use the standard notation $\sigma_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$, $\omega_N = \frac{\sigma_{N-1}}{N}$, respectively for the $(N-1)$ -dimensional measure of the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$, and N -dimensional measure of the unit ball. We adopt the convention that $a/\infty = 0$ for any $a \in \mathbb{R}$.

2 Preliminaries

In this section we collect, mostly without proofs, various properties of the semigroup associated with (1.1) which will be used throughout the rest of the paper. One should see [29, Section 2], where some of the results in this section are discussed in detail. We also refer to the same paper for all proofs which are omitted in this section.

2.1 One-parameter intertwined pseudo-distances

Given matrices Q and B as in (1.1) we introduce a one-parameter family of intertwined pseudo-distances which plays a key role in the analysis of the relevant operators \mathcal{K} . For $X, Y \in \mathbb{R}^N$ we define

$$m_t(X, Y) = \sqrt{K(t)^{-1}(Y - e^{tB}X), Y - e^{tB}X)}, \quad t > 0. \quad (2.1)$$

It is obvious that, when $B \neq O_N$, we have $m_t(X, Y) \neq m_t(Y, X)$ for every $t > 0$. Given $X \in \mathbb{R}^N$ and $r > 0$, we consider the set

$$B_t(X, r) = \{Y \in \mathbb{R}^N \mid m_t(X, Y) < r\},$$

and call it the time-varying pseudo-ball. We will need the following simple result.

Lemma 2.1 *Let $g \in L^1(0, \infty)$. Then, for every $X \in \mathbb{R}^N$ and $t > 0$ one has*

$$\int_{\mathbb{R}^N} g(m_t(X, Y)) dY = \sigma_{N-1}(\det K(t))^{1/2} \int_0^\infty g(r) r^{N-1} dr, \quad (2.2)$$

and

$$\int_{\mathbb{R}^N} g(m_t(X, Y)) dX = \sigma_{N-1} e^{-t \operatorname{tr} B} (\det K(t))^{1/2} \int_0^\infty g(r) r^{N-1} dr. \quad (2.3)$$

In particular, we have

$$\operatorname{Vol}_N(B_t(X, r)) = \omega_N r^N (\det K(t))^{1/2}. \quad (2.4)$$

Proof Formula (2.2) easily follows from (2.1) by the change of variable $Z = K(t)^{-1/2}(Y - e^{tB}X)$. The latter gives

$$\begin{aligned} \int_{\mathbb{R}^N} g(m_t(X, Y)) dY &= (\det K(t))^{1/2} \int_{\mathbb{R}^N} g(|Z|) dZ \\ &= \sigma_{N-1} (\det K(t))^{1/2} \int_0^\infty g(r) r^{N-1} dr. \end{aligned}$$

The proof of (2.3) is similar and we leave it to the reader. To obtain (2.4) it suffices to apply (2.2) with $g = \mathbf{1}_{(0,r)}$. \square

We stress that the quantity in the right-hand side of (2.4) is independent of $X \in \mathbb{R}^N$, a reflection of the underlying Lie group structure induced by the matrix B , see Remark 2.3. As a consequence, we will hereafter drop the dependence in such variable and indicate $\operatorname{Vol}_N(B_t(X, r)) = V_t(r)$. When $r = \sqrt{t}$, we will simply write $V(t)$, instead of $V_t(\sqrt{t})$, i.e.,

$$V(t) = \operatorname{Vol}_N(B_t(X, \sqrt{t})) = \omega_N (\det(tK(t)))^{1/2}. \quad (2.5)$$

2.2 The Cauchy problem

We next recall the theorem in the opening of [35] which constitutes the starting point of the present work. We warn the unfamiliar reader that our presentation of the fundamental solution (2.6) of (1.1) differs from that in [35]. This is done to emphasise the role of the one-parameter intertwined pseudo-distances (2.1) and of the corresponding volume function $V(t)$ defined by (2.5). In (2.6) below we have let $c_N = (4\pi)^{-N/2} \omega_N$.

Theorem 2.2 (Hörmander) *Given Q and B as in (1.1), for every $t > 0$ consider the covariance matrix (1.2). Then, the operator \mathcal{K} is hypoelliptic if and only if $\det K(t) > 0$ for every $t > 0$. In such case, given $f \in \mathcal{S}$, the unique solution to the Cauchy problem $\mathcal{K}u = 0$ in \mathbb{R}_+^{N+1} , $u(X, 0) = f$, is given by $P_t f(X) = \int_{\mathbb{R}^N} p(X, Y, t) f(Y) dY$, where*

$$p(X, Y, t) = \frac{c_N}{V(t)} \exp\left(-\frac{m_t(X, Y)^2}{4t}\right). \quad (2.6)$$

For a small list of differential operators of interest in mathematics and physics that are encompassed by Theorem 2.2 the reader should see the table in Fig. 1 at the end of this section.

Remark 2.3 We mention that it was noted in [41] that the class (1.1) is invariant with respect to the following non commutative group law $(X, s) \circ (Y, t) = (Y + e^{-tB} X, s + t)$. Endowed with the latter, the space $(\mathbb{R}^{N+1}, \circ)$ becomes a non-Abelian Lie group. This aspect is reflected in the expression (2.6), as well as in the invariance with respect to \circ of the volume of the intertwined pseudoballs, see (2.4) in Lemma 2.1. Except for this, such Lie group structure will play no role in our work.

2.3 Semigroup aspects

In the following lemmas we collect the main (well-known) properties of the semigroup $\{P_t\}_{t>0}$ defined by Theorem 2.2.

Lemma 2.4 *For any $t > 0$ we have:*

- $\mathcal{A}(\mathcal{S}) \subset \mathcal{S}$ and $P_t(\mathcal{S}) \subset \mathcal{S}$;
- For any $f \in \mathcal{S}$ and $X \in \mathbb{R}^N$ one has $\frac{\partial}{\partial t} P_t f(X) = \mathcal{A} P_t f(X)$;
- For every $f \in \mathcal{S}$, $X \in \mathbb{R}^N$ and $t > 0$, the commutation property is true $\mathcal{A} P_t f(X) = P_t \mathcal{A} f(X)$.

Lemma 2.5 *The following properties hold:*

- For every $X \in \mathbb{R}^N$ and $t > 0$ we have $P_t 1(X) = \int_{\mathbb{R}^N} p(X, Y, t) dY = 1$;
- $P_t : L^\infty \rightarrow L^\infty$ with $\|P_t\|_{L^\infty \rightarrow L^\infty} \leq 1$;
- For every $Y \in \mathbb{R}^N$ and $t > 0$ one has $\int_{\mathbb{R}^N} p(X, Y, t) dX = e^{-t \operatorname{tr} B}$.
- Let $1 \leq p < \infty$, then $P_t : L^p \rightarrow L^p$ with $\|P_t\|_{L^p \rightarrow L^p} \leq e^{-\frac{t \operatorname{tr} B}{p}}$. If $\operatorname{tr} B \geq 0$, P_t is a contraction on L^p for every $t > 0$;
- [Chapman–Kolmogorov equation] for every $X, Y \in \mathbb{R}^N$ and $t > 0$ one has $p(X, Y, s+t) = \int_{\mathbb{R}^N} p(X, Z, s) p(Z, Y, t) dZ$. Equivalently, one has $P_{t+s} = P_t \circ P_s$ for every $s, t > 0$.

Lemma 2.6 *Let $1 \leq p \leq \infty$. Given any $f \in \mathcal{S}$ for any $t \in [0, 1]$ we have*

$$\|P_t f - f\|_p \leq \|\mathcal{A} f\|_p \max\{1, e^{-\frac{\operatorname{tr} B}{p}}\} t.$$

Corollary 2.7 *Let $1 \leq p < \infty$. For every $f \in L^p$, we have $\|P_t f - f\|_p \rightarrow 0$ as $t \rightarrow 0^+$. Consequently, $\{P_t\}_{t>0}$ is a strongly continuous semigroup on L^p . The same is true when $p = \infty$, if we replace L^∞ by the space L^∞_0 .*

Remark 2.8 The reader should keep in mind that from this point on when we consider $\{P_t\}_{t>0}$ as a strongly continuous semigroup in L^p , we always intend to use L_0^∞ when $p = \infty$.

Denote by (\mathcal{A}_p, D_p) the infinitesimal generator of the semigroup $\{P_t\}_{t>0}$ on L^p with domain

$$D_p = \left\{ f \in L^p \mid \mathcal{A}_p f \stackrel{\text{def}}{=} \lim_{t \rightarrow 0^+} \frac{P_t f - f}{t} \text{ exists in } L^p \right\}.$$

One knows that (\mathcal{A}_p, D_p) is closed and densely defined (see [21, Theorem 1.4]).

Corollary 2.9 *We have $\mathcal{S} \subset D_p$. Furthermore, $\mathcal{A}_p f = \mathcal{A} f$ for any $f \in \mathcal{S}$, and \mathcal{S} is a core for (\mathcal{A}_p, D_p) .*

Remark 2.10 From now on for a given $p \in [1, \infty]$ with a slight abuse of notation we write $\mathcal{A} : D_p \rightarrow L^p$ instead of \mathcal{A}_p . In so doing, we must keep in mind that \mathcal{A} actually indicates the closed operator \mathcal{A}_p that, thanks to Corollary 2.9, coincides with the differential operator \mathcal{A} on \mathcal{S} . Using this identification we will henceforth say that (\mathcal{A}, D_p) is the infinitesimal generator of the semigroup $\{P_t\}_{t>0}$ on L^p .

Lemma 2.11 *Assume that (1.3) be in force, and let $1 \leq p \leq \infty$. Then:*

- (1) *For any $\lambda \in \mathbb{C}$ such that $\Re \lambda > 0$, we have $\lambda \in \rho_p(\mathcal{A})$;*
- (2) *If $\lambda \in \mathbb{C}$ such that $\Re \lambda > 0$, then $R(\lambda, \mathcal{A})$ exists and for any $f \in L^p$ it is given by the formula $R(\lambda, \mathcal{A})f = \int_0^\infty e^{-\lambda t} P_t f dt$;*
- (3) *For any $\Re \lambda > 0$ we have $\|R(\lambda, \mathcal{A})\|_{p \rightarrow p} \leq \frac{1}{\Re \lambda}$.*

2.4 Small-time behaviour of the volume function

The small-time behaviour of the function $V(t)$ was studied in the paper [41], where it was shown that the class of operators (1.1) possesses an infinitesimal osculating structure. For completeness of presentation we recall it in this subsection. We begin with the following known result, see [35,41,45,46].

Proposition 2.12 *The following are equivalent:*

- (i) *condition $K(t) > 0$ for every $t > 0$;*
- (ii) *$\text{Ker } Q$ does not contain any non-trivial subspace which is invariant for B^* ;*
- (iii) *$\text{Rank}[Q, BQ, \dots, B^{N-1}Q] = N$. (Kalman rank condition)*
- (iv) *the vector fields defined by $Y_0 u = \langle BX, \nabla u \rangle$, $Y_i u = \sum_{j=1}^N a_{ij} \partial_{x_j} u$, $i = 1, \dots, N$, where $A = [a_{ij}] = Q^{1/2}$, satisfy the finite rank condition*

$$\text{Rank Lie}[Y_0, Y_1, \dots, Y_N](X) = N, \quad \forall X \in \mathbb{R}^N.$$

(v) in a suitable basis of \mathbb{R}^N the matrices Q and B assume the following form

$$Q = \begin{pmatrix} Q_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \star & \star & \cdots & \star & \star \\ B_1 & \star & \cdots & \star & \star \\ 0 & B_2 & \star & \star & \star \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & B_r & \star \end{pmatrix},$$

where Q_0 is a $p_0 \times p_0$ non-singular matrix, and B_j is a $p_j \times p_{j-1}$ matrix having rank p_j , $j = 1, \dots, r$, with $p_0 \geq p_1 \geq \dots \geq p_r \geq 1$, and $p_0 + p_1 + \dots + p_r = N$. The \star blocks in the canonical form of B can be arbitrary matrices.

Let us now suppose that in a given basis of \mathbb{R}^N the matrices Q and B are given as in (v) of Proposition 2.12. Recall that Q_0 is a $p_0 \times p_0$ positive matrix. We form a new matrix \bar{B} by replacing all the elements with a \star in B with a zero matrix of the same dimensions, i.e.,

$$\bar{B} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & B_r & 0 \end{pmatrix}.$$

We recall that B_j is a $p_j \times p_{j-1}$ matrix having rank p_j . If we denote by $X = (x^{(p_0)}, x^{(p_1)}, \dots, x^{(p_r)})$ the generic point of $\mathbb{R}^N = \mathbb{R}^{p_0} \times \mathbb{R}^{p_1} \times \dots \times \mathbb{R}^{p_r}$, then the differential operator associated to the matrices Q and \bar{B} is given by

$$\bar{\mathcal{K}} = \text{tr} \left(Q_0 D_{x^{(p_0)}}^2 \right) + \sum_{j=1}^r \left\langle B_j x^{(p_{j-1})}, \nabla_{x^{(p_j)}} \right\rangle - \partial_t. \tag{2.7}$$

The fact that the blocks B_j have maximal rank allows to easily check the condition (iv) in Proposition 2.12, therefore also $\bar{\mathcal{K}}$ verifies the Hörmander’s condition (i) in Proposition 2.12, with a matrix $\bar{K}(t)$ defined as in (1.2) with \bar{B} in place of B . Furthermore, $\bar{\mathcal{K}}$ is left-invariant with respect to the group law \circ in Remark 2.3, in which B has been replaced by \bar{B} . We remark that $\text{tr} \bar{B} = 0$, and that \bar{B} is nilpotent, therefore $e^{s\bar{B}}$ is in fact a finite sum. One important aspect of the operator $\bar{\mathcal{K}}$ is that, unlike \mathcal{K} , it possesses a homogeneous structure: it is invariant of degree 2 with respect to the group of anisotropic dilations $\delta_\lambda : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$ defined by

$$\delta_\lambda(X, t) = \left(\lambda x^{(p_0)}, \lambda^3 x^{(p_1)}, \dots, \lambda^{2r+1} x^{(p_r)}, \lambda^2 t \right). \tag{2.8}$$

We mention that it was proved in [41, Proposition 2.2] that a necessary and sufficient condition for the existence of a family of non-isotropic dilations δ_λ associated with the operator \mathcal{K} in (1.1) is that B in (v) takes precisely the special form \bar{B} . The *homogeneous dimension* of $(\mathbb{R}^{N+1}, \circ, \delta_\lambda)$ is given by

$$D_0 + 2 = p_0 + 3p_1 + \cdots + (2r + 1)p_r + 2. \quad (2.9)$$

Returning to the general discussion, we consider the one-parameter group of anisotropic dilations $D_\lambda : \mathbb{R}^N \rightarrow \mathbb{R}^N$ associated with the matrix \bar{B}

$$D_\lambda(X) = \left(\lambda x^{(p_0)}, \lambda^3 x^{(p_1)}, \dots, \lambda^{2r+1} x^{(p_r)} \right). \quad (2.10)$$

The fact that δ_λ are group automorphisms with respect to \circ is a consequence of the following commutation property valid for any $\lambda > 0$ and $\tau \in \mathbb{R}$,

$$e^{-\lambda^2 \tau \bar{B}} = D_\lambda e^{-\tau \bar{B}} D_{\lambda^{-1}},$$

(see [41, eq. (2.20)] and also [39]). From this, and the fact that $\text{tr } \bar{B} = 0$, one can see that the positive definite matrix $\bar{K}(t)$, defined in (1.2) with \bar{B} instead of B , satisfies

$$\det(t \bar{K}(t)) = t^{D_0} \det(\bar{K}(1)).$$

Denoting with $\bar{V}(t)$ the volume of the pseudoballs $\bar{B}_t(X, \sqrt{t})$ associated with \bar{K} , we thus conclude that we must have for every $t > 0$,

$$\bar{V}(t) = c_N \det(\bar{K}(1))^{1/2} t^{D_0/2} = \gamma t^{D_0/2}. \quad (2.11)$$

The result in [41, eq. (3.14) and Remark 3.1] gives us the following asymptotic.

Proposition 2.13 *One has $\lim_{t \rightarrow 0^+} \frac{V(t)}{\bar{V}(t)} = 1$.*

Proposition 2.13 and (2.11) motivate the following.

Definition 2.14 We call the number D_0 in (2.9) the *intrinsic dimension at zero* of the Hörmander semigroup $\{P_t\}_{t>0}$. Note that it follows from (2.9) that it must be $D_0 \geq N \geq 2$.

3 Large time behaviour of the volume function and ultracontractivity

The analysis of the semigroup $\{P_t\}_{t>0}$ revolves on the large time behaviour of the volume function $V(t)$. In this section we analyse this behaviour under the assumption (1.3). Our main result, Proposition 3.1, plays a pervasive role in the rest of the paper since: (1) it shows that $V(t)$ grows at infinity at least linearly; and, (2) it says that when at least one of the eigenvalues of the drift matrix B has a strictly positive real part, then $V(t)$ must blow up exponentially. In what follows we will make use of the

equivalence (i) \iff (ii) in Proposition 2.12. The notation $\sigma(B)$ indicates the spectrum of B .

Proposition 3.1 *Suppose that (1.3) hold. Then:*

- (i) *there exists a constant $c_1 > 0$ such that $V(t) \geq c_1 t$ for all $t \geq 1$;*
- (ii) *moreover, if $\max\{\Re(\lambda) \mid \lambda \in \sigma(B)\} = L_0 > 0$, there exists a constant c_0 such that $V(t) \geq c_0 e^{L_0 t}$ for all $t \geq 1$.*

Proof As it will be evident from the proof, we first establish (ii) and then (i). Up to a change of variables in \mathbb{R}^N , we can assume that B^* is in the following block-diagonal real Jordan canonical form (see, e.g., [36, Theorem 3.4.5])

$$B^* = \begin{pmatrix} J_{n_1}(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & J_{n_q}(\lambda_q) & & & \\ & & & C_{m_1}(a_1, b_1) & & \\ & & & & \ddots & \\ & & & & & C_{m_p}(a_p, b_p) \end{pmatrix},$$

where $\sigma(B) = \sigma(B^*) = \{\lambda_1, \dots, \lambda_q, a_1 \pm ib_1, \dots, a_p \pm ib_p\}$ with $\lambda_k, a_\ell, b_\ell \in \mathbb{R}$ ($b_\ell \neq 0$), $n_1 + \dots + n_q + 2m_1 + \dots + 2m_p = N$ with $n_k, m_\ell \in \mathbb{N}$, and the $n_k \times n_k$ matrix $J_{n_k}(\lambda_k)$ and the $2m_\ell \times 2m_\ell$ matrix $C_{m_\ell}(a_\ell, b_\ell)$ are respectively in the form

$$J_{n_k}(\lambda_k) = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ 0 & \lambda_k & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_k & 1 \\ 0 & 0 & \dots & 0 & \lambda_k \end{pmatrix}, \quad C_{m_\ell}(a_\ell, b_\ell) = \begin{pmatrix} a_\ell - b_\ell & 1 & 0 & 0 & \dots & \dots & 0 \\ b_\ell & a_\ell & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & a_\ell & -b_\ell & 1 & 0 & \dots & 0 \\ 0 & 0 & b_\ell & a_\ell & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 & a_\ell & -b_\ell \\ 0 & 0 & \dots & \dots & 0 & 0 & b_\ell & a_\ell \end{pmatrix}.$$

Since $\text{tr } B = \sum_{k=1}^q n_k \lambda_k + 2 \sum_{\ell=1}^p m_\ell a_\ell \geq 0$, we have two cases:

- either $L_0 = \max\{\lambda_k, a_\ell\} > 0$
- or $\lambda_k = 0 = a_\ell \quad \forall k \in \{1, \dots, q\}, \ell \in \{1, \dots, p\}$.

Suppose $L_0 > 0$. We are going to show that, for some $C_0 > 0$, we have

$$\det(tK(t)) \geq C_0 e^{2L_0 t} \quad \text{for all } t \geq 1. \tag{3.1}$$

To do this, it is enough to show that

$$\lambda_M(t) \geq C e^{2L_0 t} \quad \text{for all } t \geq 1, \tag{3.2}$$

where $\lambda_M(t)$ is the largest eigenvalue of $tK(t)$. In fact, since $t \mapsto tK(t)$ is monotone increasing in the sense of matrices, for $t \geq 1$ all the eigenvalues of $tK(t)$ are larger than the minimum eigenvalue of $K(1)$, which is strictly positive by Hörmander condition: this tells us that (3.2) implies (3.1). To prove (3.2), we notice that at least one of the following two possibilities occurs:

- (a) there is $k_0 \in \{1, \dots, q\}$ such that $\lambda_{k_0} = L_0$;
- (b) there is $\ell_0 \in \{1, \dots, p\}$ such that $a_{\ell_0} = L_0$.

Suppose case (a) occurs. It is not restrictive to assume $k_0 = 1$. Then, $v_0 = (1, 0, \dots, 0) \in \mathbb{R}^N$ is an eigenvector for B^* with relative eigenvalue L_0 . Thus $e^{sB^*}v_0 = e^{L_0s}v_0$, for all $s \in \mathbb{R}$. From (ii) in Proposition 2.12 we know that $v_0 \notin \text{Ker } Q$, i.e. $\langle Qv_0, v_0 \rangle > 0$. Therefore, we have

$$\begin{aligned} \lambda_M(t) &\geq \langle tK(t)v_0, v_0 \rangle = \int_0^t \langle Qe^{sB^*}v_0, e^{sB^*}v_0 \rangle ds \\ &= \langle Qv_0, v_0 \rangle \int_0^t e^{2L_0s} ds = \frac{e^{2L_0t} - 1}{2L_0} \langle Qv_0, v_0 \rangle, \end{aligned}$$

which proves (3.2) in case (a). Suppose now case (b) occurs. As before, let us assume also $\ell_0 = 1$ (that is $a_1 = L_0$). If $e_1, e_2 \in \mathbb{R}^{2m_1}$ are the vectors $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, we denote the correspondent vectors in \mathbb{R}^N by $v_1 = (0, \dots, 0, e_1, 0, \dots, 0)$, $v_2 = (0, \dots, 0, e_2, 0, \dots, 0)$. With these notations, we have that $\text{span}\{v_1, v_2\}$ is an invariant subspace for B^* . From (ii) in Proposition 2.12 we know that $\text{span}\{v_1, v_2\}$ is not contained in $\text{Ker } Q$. Moreover, denoting by J the symplectic matrix restricted to $\text{span}\{v_1, v_2\}$ such that $Jv_1 = v_2$ and $Jv_2 = -v_1$, we have

$$e^{sB^*}v = e^{L_0s} (\cos(b_1s)v + \sin(b_1s)Jv) \quad \text{for all } s \in \mathbb{R} \text{ and for any } v \in \text{span}\{v_1, v_2\}. \tag{3.3}$$

Hence, for $v \in \text{span}\{v_1, v_2\}$ with unit norm, we have

$$\begin{aligned} \lambda_M(t) &\geq \langle tK(t)v, v \rangle = \int_0^t \langle Qe^{sB^*}v, e^{sB^*}v \rangle ds \\ &= \int_0^t e^{2L_0s} (\cos^2(b_1s) \langle Qv, v \rangle + \sin^2(b_1s) \langle QJv, Jv \rangle + \sin(2b_1s) \langle Qv, Jv \rangle) ds \\ &= \frac{e^{2L_0t}}{4L_0} \left(\langle Q(\cos(b_1t)v + \sin(b_1t)Jv), (\cos(b_1t)v + \sin(b_1t)Jv) \rangle + \frac{1}{L_0^2 + b_1^2} \langle Qv_t, v_t \rangle \right) \\ &\quad - \frac{1}{4L_0} (\langle Qv, v \rangle + \langle QJv, Jv \rangle) - \frac{L_0}{4(L_0^2 + b_1^2)} (\langle Qv, v \rangle - \langle QJv, Jv \rangle) \\ &\quad + \frac{b_1}{2(L_0^2 + b_1^2)} \langle Qv, Jv \rangle, \end{aligned}$$

where we have denoted $v_t = (L_0 \cos(b_1t) + b_1 \sin(b_1t))v - (b_1 \cos(b_1t) - L_0 \sin(b_1t))Jv$. The fact that v and Jv cannot belong both to $\text{Ker } Q$ implies that, for any $t > 0$, also v_t and $\cos(b_1t)v + \sin(b_1t)Jv$ cannot be in $\text{Ker } Q$ at the same

time. From this, using also $Q \geq 0$ and $b_1 \neq 0$, we can deduce the existence of $\bar{c} > 0$ such that

$$\langle Q(\cos(b_1t)v + \sin(b_1t)Jv), (\cos(b_1t)v + \sin(b_1t)Jv) \rangle + \frac{1}{L_0^2 + b_1^2} \langle Qv_t, v_t \rangle \geq \bar{c},$$

from which we infer

$$\lambda_M(t) \geq \frac{\bar{c}}{8L_0} e^{2L_0t} \quad \text{for large } t.$$

This proves (3.2) and concludes case (b). This establishes (ii) in the statement of the proposition.

We next turn to proving (i). Suppose that

$$\lambda_k = 0 = a_\ell \quad \forall k \in \{1, \dots, q\}, \ell \in \{1, \dots, p\}.$$

Since $N \geq 2$, at least one of the following possibilities must occur:

- (1) there are two linearly independent eigenvectors of B^* with eigenvalue 0;
- (2) there exists $k_0 \in \{1, \dots, q\}$ such that $n_{k_0} \geq 2$;
- (3) there exists $\ell_0 \in \{1, \dots, p\}$ such that $m_{\ell_0} \geq 1$.

In each of these three cases we are going to show that, for some $C_1 > 0$, we have

$$\det(tK(t)) \geq C_1 t^2 \quad \text{for all } t \geq 1. \tag{3.4}$$

The case (1) is the easiest since we have two linearly independent eigenvectors $v_{0,1}, v_{0,2}$ such that $e^{sB^*} v_{0,j} = v_{0,j}$ for all $s \in \mathbb{R}$ and $j \in \{1, 2\}$. Furthermore, for any $v \in \text{span}\{v_{0,1}, v_{0,2}\}$, $\langle Qv, v \rangle > 0$ since the eigenvectors cannot belong to $\text{Ker } Q$ by (ii). Hence,

$$\langle tK(t)v, v \rangle = \int_0^t \langle Qe^{sB^*} v, e^{sB^*} v \rangle ds = t \langle Qv, v \rangle$$

for all $t > 0$ and $v \in \text{span}\{v_{0,1}, v_{0,2}\}$. Then, the symmetric matrix $tK(t)$ has at least two eigenvalues growing as t . By monotonicity the other eigenvalues are bounded below by a positive constant for $t \geq 1$. This yields (3.4).

Suppose case (2) occurs. Again, it is not restrictive to assume $k_0 = 1$. Then, we have $n_1 \geq 2$ and we know that

$$e^{sJ_{n_1}(0)} = \begin{pmatrix} 1 & s & \frac{s^2}{2} & \dots & \frac{s^{n_1-1}}{(n_1-1)!} \\ 0 & 1 & s & \dots & \frac{s^{n_1-2}}{(n_1-2)!} \\ 0 & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & s \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

From this we infer that $e^{sB^*} e_{n_1} = \frac{s^{n_1-1}}{(n_1-1)!} e_1 + o(s^{n_1-1})$ as $s \rightarrow +\infty$, where we have denoted by e_j the canonical basis of \mathbb{R}^N . We recall that, being e_1 an eigenvector for B^* , we have $\langle Qe_1, e_1 \rangle > 0$. Therefore, we obtain

$$\langle Qe^{sB^*} e_{n_1}, e^{sB^*} e_{n_1} \rangle = \langle Qe_1, e_1 \rangle \frac{s^{2n_1-2}}{(n_1-1)!^2} + o(s^{2n_1-2})$$

as $s \rightarrow +\infty$. In particular, for $s \geq s_0$, we deduce

$$\langle Qe^{sB^*} e_{n_1}, e^{sB^*} e_{n_1} \rangle \geq cs^{2n_1-2},$$

for some positive constant c . This implies

$$\langle tK(t)e_{n_1}, e_{n_1} \rangle \geq \int_{s_0}^t \langle Qe^{sB^*} e_{n_1}, e^{sB^*} e_{n_1} \rangle ds \geq c \int_{s_0}^t s^{2n_1-2} ds \geq \tilde{c}t^{2n_1-1}$$

for large t . This tells us that $\lambda_M(t) \geq \tilde{c}t^{2n_1-1}$, from which (3.4) follows since $n_1 \geq 2$. We are left with case (3). As before, assume $\ell_0 = 1$ and denote by $v_1 = (0, \dots, 0, e_1, 0, \dots, 0)$, $v_2 = (0, \dots, 0, e_2, 0, \dots, 0)$ the vectors in \mathbb{R}^N which correspond to the canonical vectors $e_1, e_2 \in \mathbb{R}^{2m_1}$. We have that $\text{span}\{v_1, v_2\}$ is an invariant subspace for B^* . From (ii) in Proposition 2.12 we know that $\text{span}\{v_1, v_2\}$ is not contained in $\text{Ker } Q$. Thus, at least one of $\langle Qv_1, v_1 \rangle$ and $\langle Qv_2, v_2 \rangle$ is strictly positive. With the same notations as in (3.3), for $v \in \text{span}\{v_1, v_2\}$ we find

$$e^{sB^*} v = \cos(b_1s)v + \sin(b_1s)Jv.$$

Hence, for all $t > 0$ we have

$$\begin{aligned} \langle tK(t)v, v \rangle &= \int_0^t \langle Qe^{sB^*} v, e^{sB^*} v \rangle ds \\ &= \langle Qv, v \rangle \int_0^t \cos^2(b_1s) ds + \langle QJv, Jv \rangle \int_0^t \sin^2(b_1s) ds \\ &\quad + 2 \langle Qv, Jv \rangle \int_0^t \cos(b_1s) \sin(b_1s) ds \\ &= \left(\frac{t}{2} + \frac{\sin(2b_1t)}{4b_1} \right) \langle Qv, v \rangle + \left(\frac{t}{2} - \frac{\sin(2b_1t)}{4b_1} \right) \langle QJv, Jv \rangle \\ &\quad + \frac{1 - \cos(2b_1t)}{2b_1} \langle Qv, Jv \rangle. \end{aligned}$$

Exploiting the fact that $\langle Qv, v \rangle + \langle QJv, Jv \rangle = \langle Qv_1, v_1 \rangle + \langle Qv_2, v_2 \rangle > 0$, we can make the quantity $\langle tK(t)v, v \rangle \geq ct$ for large t and for any unit vector $v \in \text{span}\{v_1, v_2\}$. This shows the validity of (3.4), thus completing the proof. \square

Remark 3.2 We mention explicitly that, while (i) of Proposition 3.1 fails when $\text{tr } B < 0$ (see e.g. the Ex. 2 in Fig. 1 below), part (ii) instead continues to be true under such assumption. We refer the reader to [13, Section 2] for the study of the behaviour of $V(t)$ for large t regardless on the sign assumption of $\text{tr } B$.

Proposition 3.1 has the following basic consequence.

Corollary 3.3 *Suppose that (1.3) hold. Then, $\lim_{t \rightarrow \infty} V(t) = \infty$.*

Proof Recalling that $t \rightarrow tK(t)$ is monotone increasing in the sense of matrices, we have that $t \rightarrow V(t)$ is also a monotone function. Then, the conclusion immediately follows from Proposition 3.1. □

3.1 Intrinsic dimension at infinity

In dealing with the general class (1.1) the first question that comes to mind is: what number occupies the role of the dimension N in the analysis of the semigroup $\{P_t\}_{t>0}$? This question is central since, as one can see in Fig. 1, the behaviour for large times of the volume function $V(t) = \text{Vol}_N(B_t(X, \sqrt{t}))$ can be quite diverse, depending on the structure of the matrix B , and in fact non-doubling in general. The next definition introduces a notion which allows to successfully handle this matter.

Definition 3.4 Consider the set $\Sigma_\infty \stackrel{\text{def}}{=} \{\alpha > 0 \mid \int_1^\infty \frac{t^{\alpha/2-1}}{V(t)} dt < \infty\}$. We call the number $D_\infty = \sup \Sigma_\infty$ the *intrinsic dimension at infinity* of the semigroup $\{P_t\}_{t>0}$.

When $\Sigma_\infty = \emptyset$ we set $D_\infty = 0$. If $\Sigma_\infty \neq \emptyset$ we clearly have $0 < D_\infty \leq \infty$.

Remark 3.5 Some comments are in order:

- (1) when $\mathcal{A} = \Delta$, the standard Laplacian in \mathbb{R}^N , then $V(t) = \omega_N t^{N/2}$. In such case, $\alpha \in \Sigma_\infty$ if and only if $0 < \alpha < N$, and thus $D_\infty = N$;
- (2) if the operator \mathcal{K} in (1.1) admits a homogeneous structure (as $\bar{\mathcal{K}}$ in Sect. 2.4), we have $V(t) = \gamma_N t^{D_0/2}$, where D_0 is the intrinsic dimension at zero of the semigroup. In such case, we have $D_\infty = D_0$;
- (3) more in general, if there exist constants $T, \gamma, D > 0$ such that $V(t) \geq \gamma_D t^{D/2}$ for every $t \geq T$, then we must have $(0, D) \subset \Sigma_\infty$, and therefore $D_\infty \geq D$; from this observation and (i) in Proposition 3.1 we infer $D_\infty \geq 2$;
- (4) the reader should note that the assumption $\Sigma_\infty \neq \emptyset$ hides a condition on the matrix B in (1.1). For instance, when $Q = I_N$ and $B = -I_N$, then $\mathcal{K}u = \Delta u - \langle X, \nabla u \rangle - \partial_t u$ is the Ornstein–Uhlenbeck operator, see Ex.2 in Fig. 1 below. In such case we have $V(t) = c_N (1 - e^{-2t})^N$, and therefore in particular $\lim_{t \rightarrow \infty} V(t) = c_N > 0$. It follows that $\Sigma_\infty = \emptyset$ and thus $D_\infty = 0$. The same happens with the Smoluchowski–Kramers’ operator in Ex.5 below. In both cases the theory developed in this paper does not apply (we will return to this aspect in a future study);
- (5) it can happen that $D_\infty < D_0$, see Ex.4;
- (6) finally, one can have $D_\infty = \infty$, see Ex.6⁺.

Ex.	\mathcal{K}	$V(t)$	$\text{tr}(B)$	N	D_0	D_∞
(1)	$\Delta - \partial_t$ Heat	$\omega_N t^{\frac{N}{2}}$	0	N	N	N
(2)	$\Delta - \langle X, \nabla \rangle - \partial_t$ Ornstein-Uhlenbeck	$\omega_N 2^{-\frac{N}{2}} (1 - e^{-2t})^{\frac{N}{2}}$	$-N$	N	N	0
(3)	$\Delta_v + \langle v, \nabla_x \rangle - \partial_t$ Kolmogorov	$\omega_{2n} 12^{-\frac{n}{2}} t^{2n}$	0	$2n$	$4n$	$4n$
(4)	$\partial_{vv} - x\partial_v + v\partial_x - \partial_t$ Kramers	$\pi \left(\frac{t^2}{4} + \frac{1}{8} (\cos(2t) - 1) \right)^{\frac{1}{2}}$	0	2	4	2
(5)	$\partial_{vv} - 2(v+x)\partial_v + v\partial_x - \partial_t$ Smoluchowski-Kramers	$\frac{\pi}{4\sqrt{2}} (e^{-4t} + 1 - 2e^{-2t}(2 - \cos(2t)))^{\frac{1}{2}}$	-2	2	4	0
(6 ⁺)	$\Delta_v + \langle v, \nabla_v \rangle + \langle v, \nabla_x \rangle - \partial_t$ Kolmogorov with friction	$\omega_{2n} (2e^t - \frac{t}{2} - 1 + \frac{t}{2} e^{2t} - e^{2t})^n$	n	$2n$	$4n$	∞
(6 ⁻)	$\Delta_v - \langle v, \nabla_v \rangle + \langle v, \nabla_x \rangle - \partial_t$ degenerate Ornstein-Uhlenbeck	$\omega_{2n} (2e^{-t} + \frac{t}{2} - 1 - \frac{t}{2} e^{-2t} - e^{-2t})^n$	$-n$	$2n$	$4n$	$2n$

Fig. 1 Some significant examples

In the above table we illustrate the different behaviours of the volume function $V(t)$ on a significant sample of operators. The items in red refer to situations in which the drift matrix satisfies $\text{tr}(B) \geq 0$. This is the situation covered by this paper.

3.2 Ultracontractivity

We next establish a crucial geometric property of the Hörmander semigroup that plays a pervasive role in the remainder of our work. The reader should note that we do not assume (1.3) in Proposition 3.6. As a consequence, such result alone does not imply a decay of the semigroup. In this respect, see Corollary 3.7.

Proposition 3.6 ($L^p \rightarrow L^\infty$ Ultracontractivity) *Let $1 \leq p < \infty$ and $f \in L^p$. For every $X \in \mathbb{R}^N$ and $t > 0$ we have*

$$|P_t f(X)| \leq \frac{c_{N,p}}{V(t)^{1/p}} \|f\|_p,$$

for a certain constant $c_{N,p} > 0$.

Proof Applying Hölder’s inequality to $P_t f(X) = \int_{\mathbb{R}^N} p(X, Y, t) f(Y) dY$, we find

$$|P_t f(X)| \leq \|f\|_p \left(\int_{\mathbb{R}^N} p(X, Y, t)^{p'} dY \right)^{\frac{1}{p'}}$$

with $1/p + 1/p' = 1$. Using (2.1), (2.6) and the change of variable $Z = (tK(t))^{-1/2}(Y - e^{tB}X)$, it is now easy to recognise that for any $1 \leq r < \infty$, one has

$$\begin{aligned} \left(\int_{\mathbb{R}^N} p(X, Y, t)^r dY \right)^{\frac{1}{r}} &= \frac{c_N}{V(t)} \left(\int_{\mathbb{R}^N} \exp\left(-\frac{r m_t(X, Y)^2}{4t}\right) dY \right)^{\frac{1}{r}} \\ &= \frac{c_N}{V(t)} (\det tK(t))^{\frac{1}{2r}} \left(\int_{\mathbb{R}^N} \exp\left(-\frac{r|Z|^2}{4}\right) dZ \right)^{\frac{1}{r}}. \end{aligned}$$

In view of (2.4) we infer that there exists a universal constant $c_{N,r} > 0$ such that

$$\left(\int_{\mathbb{R}^N} p(X, Y, t)^r dY \right)^{\frac{1}{r}} = \frac{c_{N,r}}{V(t)^{1-\frac{1}{r}}}. \tag{3.5}$$

The desired conclusion now follows taking $r = p'$ in (3.5). □

For later use, we also record the following formula, dual to (3.5), which easily follows by (2.3)

$$\left(\int_{\mathbb{R}^N} p(X, Y, t)^r dX \right)^{\frac{1}{r}} = \frac{c_{N,r} e^{-t\frac{rB}{r}}}{V(t)^{1-\frac{1}{r}}}.$$

Corollary 3.7 *Assume (1.3) and let $1 \leq p < \infty$. For every $f \in L^p$ and $X \in \mathbb{R}^N$, we have*

$$\lim_{t \rightarrow \infty} |P_t f(X)| = 0.$$

Proof By Proposition 3.6 we have for every $X \in \mathbb{R}^N$ and $t > 0$

$$|P_t f(X)| \leq \frac{c_N}{V(t)^{1/p}} \|f\|_p.$$

Combining this estimate with Corollary 3.3 we find

$$\lim_{t \rightarrow \infty} |P_t f(X)| \leq c_N \|f\|_p \lim_{t \rightarrow \infty} \frac{1}{V(t)^{1/p}} = 0.$$

□

4 Sobolev spaces

In the recent work [29] we developed a fractional calculus for the operators \mathcal{K} in (1.1) and solved the so-called *extension problem*. This is a generalisation of the famous work by Caffarelli and Silvestre for the fractional Laplacian $(-\Delta)^s$, see [14]. As a by-product of our work, we obtained a nonlocal calculus for the “time-independent” part of the operators \mathcal{K} , namely the second order partial differential operator

$$\mathcal{A}u = \text{tr}(Q\nabla^2 u) + \langle BX, \nabla u \rangle.$$

It is worth mentioning here that boundary value problems for these elliptic-parabolic operators were studied by Fichera in his pioneering works [23,24].

Since the nonlocal operators $(-\mathcal{A})^s$ play a central role in the present work we now recall their definition from [29, Definition 3.1]. Hereafter, when considering the

action of the operators \mathcal{A} or $(-\mathcal{A})^s$ on a given L^p , the reader should keep in mind Remark 2.10.

Definition 4.1 Let $0 < s < 1$. For any $f \in \mathcal{S}$ we define the nonlocal operator $(-\mathcal{A})^s$ by the following pointwise formula

$$(-\mathcal{A})^s f(X) = -\frac{s}{\Gamma(1-s)} \int_0^\infty t^{-(1+s)} [P_t f(X) - f(X)] dt, \quad X \in \mathbb{R}^N. \quad (4.1)$$

We mention that it was shown in [29] that the right-hand side of (4.1) is a convergent integral (in the sense of Bochner) in L^∞ , and also in L^p for any $p \in [1, \infty]$ when (1.3) holds. We note that, when $\mathcal{A} = \Delta$, it is easy to see that formula (4.1) allows to recover M. Riesz’ definition in [58] of the fractional powers of the Laplacian

$$(-\Delta)^s f(X) = \frac{s2^{2s-1}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)} \int_{\mathbb{R}^N} \frac{2u(X) - u(X+Y) - u(X-Y)}{|Y|^{N+2s}} dY.$$

Definition (4.1) comes from Balakrishnan’s seminal work [5]. The nonlocal operators (4.1) enjoy the following semigroup property (see [5]; [31, Section 2]).

Proposition 4.2 Let $s, s' \in (0, 1)$ and suppose that $s + s' \in (0, 1]$. Then, for every $f \in \mathcal{S}$ we have

$$(-\mathcal{A})^{s+s'} f = (-\mathcal{A})^s \circ (\mathcal{A})^{s'} f.$$

For any given $1 \leq p < \infty$, and any $0 < s < 1$, we denote by

$$D_{p,s} = \{f \in L^p \mid (-\mathcal{A})^s f \in L^p\},$$

the domain of $(-\mathcal{A})^s$ in L^p . The operator $(-\mathcal{A})^s$ can be extended to a closed operator on its domain, see [5, Lemma 2.1]. Therefore, endowed with the graph norm

$$\|f\|_{D_{p,s}} \stackrel{def}{=} \|f\|_p + \|(-\mathcal{A})^s f\|_p,$$

$D_{p,s}$ becomes a Banach space. The next lemma shows that, when (1.3) holds, then $\mathcal{S} \subset D_{p,s}$.

Lemma 4.3 Assume (1.3), and let $0 < s < 1$. Given $1 \leq p \leq \infty$, one has

$$(-\mathcal{A})^s(\mathcal{S}) \subset L^p.$$

Proof In view of (4.1) we have

$$\begin{aligned} \|(-\mathcal{A})^s f\|_p &\leq \frac{s}{\Gamma(1-s)} \int_0^\infty t^{-(1+s)} \|P_t f - f\|_p dt \\ &= \frac{s}{\Gamma(1-s)} \left\{ \int_0^1 t^{-(1+s)} \|P_t f - f\|_p dt + \int_1^\infty t^{-(1+s)} \|P_t f - f\|_p dt \right\}. \end{aligned}$$

Thanks to Lemma 2.6 we now have for some universal constant $C > 0$,

$$\int_0^1 t^{-(1+s)} \|P_t f - f\|_p dt \leq C \|\mathcal{A} f\|_p \int_0^1 \frac{dt}{t^s} < \infty.$$

On the other hand, by (iv) in Lemma 2.5 we know that, under the hypothesis (1.3), P_t is a contraction in L^p . We thus obtain

$$\int_1^\infty t^{-(1+s)} \|P_t f - f\|_p dt \leq 2 \|f\|_p \int_1^\infty \frac{dt}{t^{1+s}} < \infty.$$

This proves the desired conclusion. □

We now use the nonlocal operators $(-\mathcal{A})^s$ to introduce the functional spaces naturally attached to the operator \mathcal{A} . These spaces involve a fractional order of differentiation that is intrinsically calibrated both on the directions of ellipticity of the second order part of (1.1), as well as on the drift.

Definition 4.4 (*Sobolev spaces*) Assume (1.3), and let $1 \leq p < \infty$ and $0 < s < 1$. We define the Sobolev space as $\mathcal{L}^{2s,p} = \overline{\mathcal{F}}^{\|\cdot\|_{D_{p,s}}}$.

Remark 4.5 Some comments are in order:

(i) When $Q = I_N$ and $B = O_N$, and thus $\mathcal{A} = \Delta$, the space $\mathcal{L}^{2s,p}$ coincides, for $1 < p < \infty$ and $s = 1/2$, with the classical Sobolev space $W^{1,p} = \{f \in L^p \mid \nabla f \in L^p\}$, endowed with the usual norm $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|\nabla f\|_{L^p}$. In other words, one has $\mathcal{L}^{1,p} = W^{1,p}$, for $1 < p < \infty$. This follows from the well-known fact that $W^{1,p} = \overline{\mathcal{F}}^{\|\cdot\|_{W^{1,p}}}$ (Friedrich’s mollifiers, see [27]), combined with the L^p continuity of the singular integrals (Riesz transforms) in the range $1 < p < \infty$, see [61, Ch. 3]. This implies the double inequality

$$A_p \|\Delta^{1/2} f\|_p \leq \|\nabla f\|_p \leq B_p \|\Delta^{1/2} f\|_p, \quad f \in \mathcal{S}.$$

- (ii) We mention that such inequality, and therefore the identity $\mathcal{L}^{1,p} = W^{1,p}$, continue to be valid on any complete Riemannian manifold with Ricci lower bound $\text{Ric} \geq -\kappa$, where $\kappa \geq 0$. This was proved by Bakry in [4]. A generalisation to the larger class of sub-Riemannian manifolds with transverse symmetries was subsequently obtained in [6].
- (iii) As a final comment we note that, when $p = 2$, and again $\mathcal{A} = \Delta$, then the space $\mathcal{L}^{2s,p}$ coincides with the classical Sobolev space of fractional order H^{2s} , see e.g. [44] or [2].

By Lemma 4.3 and Definition 4.4 we know that $\mathcal{L}^{2s,p} \subseteq D_{p,s}$. The next proposition says that also the reverse inclusion holds true (we refer to [30, Proposition 2.13] for the proof).

Proposition 4.6 Assume (1.3). Let $0 < s < 1$ and $p \geq 1$. We have

$$D_{p,s} = \mathcal{L}^{2s,p}.$$

We close this section by recalling the result from [29] that will be needed in the next one. Given $0 < s < 1$, let $a = 1 - 2s$. The extension problem for $(-\mathcal{A})^s$ consists in the following degenerate Dirichlet problem in the variables $(X, z) \in \mathbb{R}_+^{N+1}$, where $X \in \mathbb{R}^N$ and $z > 0$:

$$\begin{cases} \mathcal{A}_a U \stackrel{\text{def}}{=} \mathcal{A}U + \partial_{zz}U + \frac{a}{z}\partial_z U = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ U(X, 0) = f(X) & X \in \mathbb{R}^N, \end{cases} \tag{4.2}$$

where $f \in \mathcal{S}$. We note that, since $s \in (0, 1)$, the relation $a = 1 - 2s$ gives $a \in (-1, 1)$, and that, in particular, $a = 0$ when $s = 1/2$. For the following Poisson kernel for the problem (4.2), and for the subsequent Theorem 4.7, one should see [29, Def. 5.1 and Theor. 5.5],

$$\mathcal{P}^{(a)}(X, Y, z) = \frac{1}{2^{1-a}\Gamma(\frac{1-a}{2})} \int_0^\infty \frac{z^{1-a}}{t^{\frac{3-a}{2}}} e^{-\frac{z^2}{4t}} p(X, Y, t) dt, \quad X, Y \in \mathbb{R}^N, z > 0. \tag{4.3}$$

The next result generalises the famous one by Caffarelli and Silvestre in [14] for the nonlocal operator $(-\Delta)^s$.

Theorem 4.7 *The function $U(X, z) = \int_{\mathbb{R}^N} \mathcal{P}^{(a)}(X, Y, z) f(Y) dY$, belongs to $C^\infty(\mathbb{R}^N \times (0, \infty))$ and solves the extension problem (4.2). By this we mean that $\mathcal{A}_a U = 0$ in \mathbb{R}_+^{N+1} , and we have in L^∞*

$$\lim_{z \rightarrow 0^+} U(\cdot; z) = f. \tag{4.4}$$

Moreover, we also have in L^∞

$$-\frac{2^{-a}\Gamma(\frac{1-a}{2})}{\Gamma(\frac{1+a}{2})} \lim_{z \rightarrow 0^+} z^a \partial_z U(\cdot, z) = (-\mathcal{A})^s f. \tag{4.5}$$

If furthermore one has $\text{tr } B \geq 0$, then the convergence in (4.4), (4.5) is also in L^p for any $1 \leq p \leq \infty$.

5 The key Littlewood–Paley estimate

In the Hardy–Littlewood theory the weak L^1 continuity of the maximal function occupies a central position. It is natural to expect that such result play a similar role for the operators in the general class (1.1), but because of the intertwining of the X and t variables it is not obvious how to select a “good” maximal function. At first it seems natural to consider $\mathcal{M}f(X) = \sup_{t>0} |P_t f(X)|$. But when $B \neq O_N$ in (1.1), because of the lack of symmetry of the semigroup P_t , estimating $\mathcal{M}f$ presents an obstruction connected with the mapping properties of the Littlewood–Paley function that controls it. We have been able to circumvent this difficulty by combining a far-reaching idea of

Stein in [60] with our work in [29]. In this respect, the case $s = 1/2$ of Theorem 4.7 provides the main technical tool to bypass the above mentioned difficulties connected with P_t . It will lead us to Theorem 5.5, which is the main result of this section.

Since in what follows we are primarily interested in the nonlocal operator $(-\mathcal{A})^{1/2}$ (the case $a = 0$ in Theorem 4.7), we will focus our attention on the corresponding Poisson kernel, which for ease of notation we henceforth denote by $\mathcal{P}(X, Y, z) \stackrel{\text{def}}{=} \mathcal{P}^{(0)}(X, Y, z)$. In such case, formula (4.3) reads

$$\mathcal{P}(X, Y, z) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{z}{t^{3/2}} e^{-\frac{z^2}{4t}} p(X, Y, t) dt, \quad X, Y \in \mathbb{R}^N, \quad z > 0. \tag{5.1}$$

Definition 5.1 We define the *Poisson semigroup* as follows

$$\mathcal{P}_z f(X) = \int_{\mathbb{R}^N} \mathcal{P}(X, Y, z) f(Y) dY, \quad f \in \mathcal{S}.$$

Using (5.1) and exchanging the order of integration in the above definition, we obtain the following useful representation of the semigroup \mathcal{P}_z in terms of the Hörmander semigroup P_t

$$\mathcal{P}_z f(X) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{z}{t^{3/2}} e^{-\frac{z^2}{4t}} P_t f(X) dt. \tag{5.2}$$

This is of course an instance of Bochner’s subordination, see [7]. We note in passing that, when the operator $\mathcal{A} = \Delta$, from (5.2) we recover the classical Poisson kernel for the half-space \mathbb{R}_+^{N+1} , see [61, (15), p.61],

$$\mathcal{P}(X, Y, z) = \frac{\Gamma(\frac{N+1}{2})}{\pi^{\frac{N+1}{2}}} \frac{z}{(z^2 + |X - Y|^2)^{\frac{N+1}{2}}}.$$

Some basic facts that we need about $\{\mathcal{P}_z\}_{z>0}$ are contained in the next result.

Lemma 5.2 *The following properties hold:*

- (i) For every $X \in \mathbb{R}^N$ and $z > 0$ we have $\mathcal{P}_z 1(X) = 1$;
- (ii) $\mathcal{P}_z : L^\infty \rightarrow L^\infty$ with $\|\mathcal{P}_z\|_{\infty \rightarrow \infty} \leq 1$;
- (iii) let $1 \leq p < \infty$. If (1.3) holds, then $\mathcal{P}_z : L^p \rightarrow L^p$ with $\|\mathcal{P}_z\|_{p \rightarrow p} \leq 1$;
- (iv) let $f \in \mathcal{S}$. Then, $\lim_{z \rightarrow 0^+} \frac{\mathcal{P}_z f(X) - f(X)}{z} = (-\mathcal{A})^{1/2} f(X)$;
- (v) The function $U(X, z) = \mathcal{P}_z f(X)$ belongs to $C^\infty(\mathbb{R}_+^{N+1})$ and it satisfies the partial differential equation $\partial_{zz} U + \mathcal{A}U = 0$. Moreover, $\lim_{z \rightarrow 0^+} U(\cdot, z) = f$ in L^∞ and in L^p for every $1 \leq p < \infty$, when (1.3) holds.

Proof The proof of (i) follows by taking $a = 0$ in [29, Proposition 5.2]. (ii) is a direct consequence of (i). To establish (iii) we use (5.2), that gives

$$\|\mathcal{P}_z f\|_p \leq \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{z}{t^{3/2}} e^{-\frac{z^2}{4t}} \|P_t f\|_p dt \leq \frac{\|f\|_p}{\sqrt{4\pi}} \int_0^\infty \frac{z}{t^{3/2}} e^{-\frac{z^2}{4t}} dt = \|f\|_p,$$

where in second inequality we have used (iv) in Lemma 2.5, and in the last equality the fact that

$$\frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{z}{t^{3/2}} e^{-\frac{z^2}{4t}} dt = 1.$$

The properties (iv) and (v) follow from the case $a = 0$ of Theorem 4.7. □

Remark 5.3 We note explicitly that (iv) in Lemma 5.2 says, in particular, that the infinitesimal generator of \mathcal{P}_z is the nonlocal operator $(-\mathcal{A})^{1/2}$, i.e., $\mathcal{P}_z = e^{z\sqrt{-\mathcal{A}}}$. In the case when $\mathcal{A} = \Delta$ one should see the seminal work [62], where an extensive use of the Poisson semigroup was made in connection with smoothness properties of functions.

Given a reasonable function f (for instance, $f \in \mathcal{S}$) we now introduce its *Poisson radial maximal function* as follows

$$\mathcal{M}^* f(X) \stackrel{\text{def}}{=} \sup_{z>0} |\mathcal{P}_z f(X)|, \quad X \in \mathbb{R}^N. \tag{5.3}$$

Lemma 5.4 *There exists a universal constant $A > 0$ such that*

$$\mathcal{M}^* f(X) \leq A \sup_{t>0} \left| \frac{1}{t} \int_0^t P_s f(X) ds \right|. \tag{5.4}$$

Proof Adapting an idea in [60, p. 49], we can write (5.2) as

$$\mathcal{P}_z f(X) = \int_0^\infty g(z, t) \frac{d}{dt} (tF(t)) dt, \tag{5.5}$$

where $g(z, t) = \frac{zt^{-3/2}}{\sqrt{4\pi}} e^{-\frac{z^2}{4t}}$, and we have let $F(t) = \frac{1}{t} \int_0^t P_s f(X) ds$. Notice that by (ii) in Lemma 2.5, we can bound $|F(t)| \leq \|f\|_\infty$. Also observe that $tg(z, t) \rightarrow 0$ as $t \rightarrow \infty$, and that $t \rightarrow t \left| \frac{\partial g}{\partial t}(z, t) \right| \in L^1(0, \infty)$. We can thus integrate by parts in (5.5), obtaining

$$|\mathcal{P}_z f(X)| = \left| \int_0^\infty t \frac{\partial g}{\partial t}(z, t) F(t) dt \right| \leq A(z) \sup_{t>0} \left| \frac{1}{t} \int_0^t P_s f(X) ds \right|,$$

with

$$A(z) = \int_0^\infty t \left| \frac{\partial g}{\partial t}(z, t) \right| dt < \infty.$$

To complete the proof it suffices to observe that $A(z) \leq A = 7/2$ for every $z > 0$. This follows from the fact that $t \frac{\partial g}{\partial t}(z, t) = (\frac{z^2}{t} - \frac{3}{2})g(z, t)$, and that $\int_0^\infty g(z, t) dt = 1$, and $\int_0^\infty \frac{z^2}{t} g(z, t) dt = 2$. □

The next is the main result in this section. It provides the key maximal theorem for the class (1.1). As far as we know, such tool has so far been missing in the existing literature.

Theorem 5.5 *Assume (1.3). Then, the Poisson maximal function (5.3) enjoys the following properties: (a) there exists a universal constant $A > 0$ such that, given $f \in L^1$, for every $\lambda > 0$ one has*

$$|\{X \in \mathbb{R}^N \mid \mathcal{M}^* f(X) > \lambda\}| \leq \frac{2A}{\lambda} \|f\|_{L^1};$$

(b) let $1 < p \leq \infty$, then there exists a universal constant $A_p > 0$ such that for any $f \in L^p$ one has

$$\|\mathcal{M}^* f\|_{L^p} \leq A_p \|f\|_{L^p}.$$

Proof (a) In view of (iv) in Lemma 2.5, we know that $\{P_t\}_{t>0}$ is contractive in L^1 and in L^∞ . Furthermore, by Corollary 2.7 it is a strongly continuous semigroup in L^p , for every $1 < p < \infty$. We can thus apply the powerful Hopf–Dunford–Schwartz ergodic theorem, see [20, Lemma 6, p. 153], and infer that, if $f \in L^1$, then for every $\lambda > 0$ one has

$$|\{X \in \mathbb{R}^N \mid f^*(X) > \lambda\}| \leq \frac{2}{\lambda} \int_{\{X \in \mathbb{R}^N \mid |f(X)| > \lambda/2\}} |f(X)| dX \leq \frac{2}{\lambda} \|f\|_1, \quad (5.6)$$

where we have let

$$f^*(X) \stackrel{\text{def}}{=} \sup_{t>0} \left| \frac{1}{t} \int_0^t P_s f(X) ds \right|.$$

On the other hand, (5.4) in Lemma 5.4 gives

$$|\{X \in \mathbb{R}^N \mid \mathcal{M}^* f(X) > \lambda\}| \leq |\{X \in \mathbb{R}^N \mid f^*(X) > \lambda/A\}| \leq \frac{2A}{\lambda} \|f\|_1,$$

where in the second inequality we have used (5.6).

(b) We observe that from (ii) in Lemma 5.2 we trivially have

$$\mathcal{M}^* : L^\infty \longrightarrow L^\infty, \quad \text{with} \quad \|\mathcal{M}^*\|_{L^\infty \rightarrow L^\infty} \leq 1.$$

By (a) and the theorem of real interpolation of Marcinkiewicz (see [61, Chap. 1, Theor. 5]), we conclude that (b) is true for some $A_p > 0$. \square

6 The fractional integration operator \mathcal{I}_α

In the classical theory of Hardy–Littlewood–Sobolev the M. Riesz’ operator of fractional integration plays a pivotal role. We recall, see [58] and also [61, Chap. 5], that

given a number $0 < \alpha < N$, the latter is defined by the formula

$$I_\alpha f(X) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^N} \frac{f(Y)}{|X - Y|^{N-\alpha}} dY. \tag{6.1}$$

The essential feature of such operator is that it provides the inverse of the fractional powers of the Laplacian, in the sense that for any $f \in \mathcal{S}$ one has $f = I_\alpha \circ (-\Delta)^{\alpha/2} f$. Its role in the Hardy–Littlewood theory is perhaps best highlighted by the following interpolating inequality which goes back to [61, Chapter 5], see also [33]. Suppose $1 \leq p < n/\alpha$ and that $f \in L^p$. Then, one has for any $\varepsilon > 0$,

$$|I_\alpha f(x)| \leq C(n, \alpha, p) \left(Mf(x)\varepsilon^\alpha + \|f\|_p \varepsilon^{-\left(\frac{n}{p}-\alpha\right)} \right). \tag{6.2}$$

The usefulness of the inequality (6.2) is multi-faceted. One the one hand, when $p > 1$, combined with the strong L^p continuity of the maximal operator, it shows that $I_\alpha : L^p \rightarrow L^q$, provided that $1/p - 1/q = \alpha/n$. On the other hand, (6.2) allows to immediately establish the geometric weak end-point result $W^{1,1} \hookrightarrow L^{\frac{n}{n-1},\infty}$. This implies, in turn, the isoperimetric inequality $P(E) \geq C_n |E|^{\frac{n-1}{n}}$ and, equivalently, the strong geometric Sobolev embedding, $BV \hookrightarrow L^{\frac{n}{n-1}}$, where $P(E)$ denotes De Giorgi’s perimeter and BV the subspace of L^1 of functions with bounded variation (for these aspects we refer to [15], where these ideas were developed in the general framework of Carnot–Carthéodory spaces).

In this section, we use the Poisson semigroup $\{\mathcal{P}_z\}_{z>0}$ in Definition 5.1 to introduce, in our setting, the counterpart of the potential operators (6.1), see Lemma 6.2. Theorem 6.3 is the first main result of the section. It shows that the operator \mathcal{I}_{2s} inverts the nonlocal operator $(-\mathcal{A})^s$. In the next definition the reader needs to keep in mind the number D_∞ in Definition 3.4.

Definition 6.1 Let $0 < \alpha < D_\infty$. Given $f \in \mathcal{S}$, we define the *Riesz potential* of order α as follows

$$\mathcal{I}_\alpha f(X) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} P_t f(X) dt.$$

Let us observe that for every $X \in \mathbb{R}^N$ the integral in Definition 6.1 converges absolutely. To see this we write

$$\int_0^\infty t^{\alpha/2-1} P_t f(X) dt = \int_0^1 t^{\alpha/2-1} P_t f(X) dt + \int_1^\infty t^{\alpha/2-1} P_t f(X) dt.$$

The integral on $[0, 1]$ is absolutely convergent for any $\alpha > 0$ since, using (ii) in Lemma 2.5, we can bound $|P_t f(X)| \leq \|P_t f\|_\infty \leq \|f\|_\infty$. For the integral on $[1, \infty)$ we use the ultracontractivity of P_t in Proposition 3.6, which gives for any $X \in \mathbb{R}^N$ and $t > 0$,

$$\int_1^\infty t^{\alpha/2-1} |P_t f(X)| dt \leq c_N \|f\|_1 \int_1^\infty \frac{t^{\alpha/2-1}}{V(t)} < \infty,$$

since $0 < \alpha < D_\infty$. In the next lemma, using Bochner’s subordination, we recall a useful alternative expression of the potential operators \mathcal{I}_α based on the Poisson semigroup $\{\mathcal{P}_z\}_{z>0}$.

Lemma 6.2 *Let $0 < \alpha < D_\infty$. For any $f \in \mathcal{S}$ one has*

$$\mathcal{I}_\alpha f(X) = \frac{1}{\Gamma(\alpha)} \int_0^\infty z^{\alpha-1} \mathcal{P}_z f(X) dz.$$

Proof We have from (5.2)

$$\begin{aligned} \int_0^\infty z^{\alpha-1} \mathcal{P}_z f(X) dz &= \frac{1}{\sqrt{4\pi}} \int_0^\infty z^{\alpha-1} \int_0^\infty \frac{z}{t^{3/2}} e^{-\frac{z^2}{4t}} P_t f(X) dt dz \\ &= \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{1}{t^{3/2}} \left(\int_0^\infty z^{\alpha+1} e^{-\frac{z^2}{4t}} \frac{dz}{z} \right) P_t f(X) dt \\ &= \frac{2^{\alpha-1} \Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi}} \int_0^\infty t^{\alpha/2-1} P_t f(X) dt = \frac{2^{\alpha-1} \Gamma(\frac{\alpha+1}{2}) \Gamma(\alpha/2)}{\sqrt{\pi}} \mathcal{I}_\alpha f(X) \\ &= \Gamma(\alpha) \mathcal{I}_\alpha f(X), \end{aligned}$$

where in the last equality we have used, with $x = \alpha/2$, the well-known duplication formula for the gamma function $2^{2x-1} \Gamma(x) \Gamma(x + 1/2) = \sqrt{\pi} \Gamma(2x)$, see e.g. [43, formula (1.2.3)]. □

The next basic result plays a central role for the remainder of this paper. It shows that the integral operator \mathcal{I}_α is the inverse of the nonlocal operator $(-\mathcal{A})^{\alpha/2}$.

Theorem 6.3 *Suppose that (1.3) hold, and let $0 < s < 1$. Then, for any $f \in \mathcal{S}$ we have*

$$f = \mathcal{I}_{2s} \circ (-\mathcal{A})^s f = (-\mathcal{A})^s \circ \mathcal{I}_{2s} f.$$

Proof We only prove the first equality, the second is established similarly. It will be useful in what follows to adopt the following alternative expression, see [5], of the nonlocal operator (4.1)

$$\begin{aligned} (-\mathcal{A})^s f(X) &= \frac{\sin(\pi s)}{\pi} \int_0^\infty \lambda^{s-1} R(\lambda, \mathcal{A}) (-\mathcal{A}) f(X) d\lambda \\ &= \frac{\sin(\pi s)}{\pi} \int_0^\infty \lambda^{s-1} (I - \lambda R(\lambda, \mathcal{A})) f(X) d\lambda, \end{aligned} \tag{6.3}$$

where we have denoted by $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ the resolvent of \mathcal{A} in L_0^∞ (we are now identifying \mathcal{A} with \mathcal{A}_∞ , the infinitesimal generator of $\{P_t\}_{t>0}$ in L_0^∞ , see Remarks 2.8, 2.10 and Lemma 2.11). We remark that either one of the integrals

in the right-hand side of (6.3) converge in L^∞ . For instance, in the first integral there is no issue near $\lambda = 0$ since $s > 0$, whereas (3) in Lemma 2.11 gives $\lambda^{s-1} \|R(\lambda, \mathcal{A})(-\mathcal{A})f\|_\infty \leq \lambda^{s-2} \|\mathcal{A}f\|_\infty$, which is convergent near ∞ . Keeping in mind that by (2) in Lemma 2.11 we have $R(\lambda, \mathcal{A})f = \int_0^\infty e^{-\lambda t} P_t f dt$, we can alternatively express (6.3) as follows

$$(-\mathcal{A})^s f(X) = \frac{\sin(\pi s)}{\pi} \int_0^\infty \int_0^\infty \lambda^{s-1} e^{-\lambda \tau} P_\tau (-\mathcal{A}) f(X) d\lambda d\tau. \tag{6.4}$$

If we now combine Definition 6.1 with (6.4), we find

$$\begin{aligned} \mathcal{I}_{2s} ((-\mathcal{A})^s f)(X) &= \frac{\sin(\pi s)}{\pi \Gamma(s)} \int_0^\infty \left(\int_0^\infty \lambda^s e^{-\lambda \tau} \left(\int_0^\infty t^s P_{t+\tau} (-\mathcal{A}) f(X) \frac{dt}{t} \right) \frac{d\lambda}{\lambda} \right) d\tau \\ &= \frac{\sin(\pi s)}{\pi \Gamma(s)} \int_0^\infty \int_0^\infty u^s P_{\tau(1+u)} (-\mathcal{A}) f(X) \left(\int_0^\infty \lambda^s e^{-\lambda \tau} \tau^s \frac{d\lambda}{\lambda} \right) \frac{du}{u} d\tau \\ &= \frac{\sin(\pi s)}{\pi} \int_0^\infty u^s \int_0^\infty P_{\tau(1+u)} (-\mathcal{A}) f(X) d\tau \frac{du}{u} \\ &= -\frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{u^{s-1}}{1+u} du \int_0^\infty P_\rho \mathcal{A} f(X) d\rho, \end{aligned}$$

where in the innermost integral we have made the change of variables $\rho = \tau(1 + u)$. We notice that one can justify the above chain of equalities by a standard application of Fubini and Tonelli theorems once we recognize that, for large t , the ultracontractivity and the fact that $D_\infty \geq 2 > 2s$ ensure the right summability properties. We now make the key observation that (b) and (c) in Lemma 2.4 and the assumption (1.3) imply, in view of Corollary 3.7,

$$\int_0^\infty \mathcal{A} P_\rho f(X) d\rho = \int_0^\infty \frac{d}{d\rho} P_\rho f(X) d\rho = -f(X).$$

In order to reach the desired conclusion we are only left with observing, see e.g. [63, 3.123 on p.105], that $\int_0^\infty \frac{u^{s-1}}{1+u} du = \Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)}$. □

7 An intrinsic embedding theorem of Sobolev type

In this section we prove our main embedding of Sobolev type, Theorem 7.5. Our strategy follows the classical approach to the subject. We first establish the key Hardy–Littlewood–Sobolev type result, Theorem 7.4. With such tool in hands, we are easily able to obtain the Sobolev embedding, Theorem 7.5. We note that these results do not tell the whole story since, as noted in Remark 7.2, their main assumption (7.1) implies necessarily that $D_0 \leq D_\infty$. But we have seen in Ex.4 in Fig. 1 that there exist operators of interest in physics for which we have instead $D_0 > D_\infty$. These cases are handled by Theorems 7.6 and 7.7. Since we will need to have in place all the results from the previous sections, hereafter we assume without further mention that the assumption (1.3) be in force. Our first result shows a basic property of the Poisson semigroup.

Lemma 7.1 (Ultracontractivity of $\mathcal{P}_z = e^{z\sqrt{-\Delta}}$) *Suppose that there exist numbers $D, \gamma_D > 0$ such that for every $t > 0$ one has*

$$V(t) \geq \gamma_D t^{D/2}. \tag{7.1}$$

If $1 \leq p < \infty$ one has for $f \in L^p, X \in \mathbb{R}^N$ and any $z > 0$,

$$|\mathcal{P}_z f(X)| \leq \frac{C_1}{z^{D/p}} \|f\|_p,$$

where $C_1 = C_1(N, D, p) > 0$.

Proof From (5.2), Proposition 3.6 and (7.1) we find

$$\begin{aligned} |\mathcal{P}_z f(X)| &\leq \frac{c_{N,p}}{\sqrt{4\pi}} \|f\|_p \int_0^\infty \frac{z}{t^{1/2} V(t)^{1/p}} e^{-\frac{z^2}{4t}} \frac{dt}{t} \\ &\leq C \|f\|_p \int_0^\infty \frac{z}{t^{\frac{1}{2}(\frac{D}{p}+1)}} e^{-\frac{z^2}{4t}} \frac{dt}{t} = C_1 \|f\|_p z^{-D/p}, \end{aligned}$$

where $C_1 = C_1(N, D, \gamma_D, p) > 0$. □

Remark 7.2 Keeping Definitions 2.14 and 3.4 in mind, the reader should note that the assumption (7.1) implies necessarily that $D_0 \leq D \leq D_\infty$. Thus, the case $D_0 > D_\infty$ is left out, but it will be addressed in Theorems 7.6 and 7.7.

The next proposition contains an essential interpolation estimate which generalises to the degenerate non-symmetric setting of (1.1) the one in [67], see also [68]. Such tool represents the semigroup replacement of the Stein-Hedberg estimate (6.2).

Proposition 7.3 *Assume (7.1), and let $0 < \alpha < D$. Given $1 \leq p < D/\alpha$ there exist a constant $C_2 = C_2(N, D, \alpha, \gamma_D, p) > 0$, such that for every $f \in \mathcal{S}$ and $\varepsilon > 0$*

$$|\mathcal{I}_\alpha f(X)| \leq \frac{1}{\Gamma(\alpha + 1)} \mathcal{M}^* f(X) \varepsilon^\alpha + C_2 \|f\|_p \varepsilon^{\alpha - \frac{D}{p}}, \tag{7.2}$$

where \mathcal{M}^* is as in (5.3).

Proof We begin by noting that we know from (3) in Remark 3.5 that $D_\infty \geq D$, and thus \mathcal{I}_α is well defined for all $0 < \alpha < D$. Now, for a given $f \in \mathcal{S}$ using Lemma 6.2 we write for every $\varepsilon > 0$

$$|\mathcal{I}_\alpha f(X)| \leq \frac{1}{\Gamma(\alpha)} \int_0^\varepsilon z^{\alpha-1} |\mathcal{P}_z f(X)| dz + \frac{1}{\Gamma(\alpha)} \int_\varepsilon^\infty z^{\alpha-1} |\mathcal{P}_z f(X)| dz. \tag{7.3}$$

The first term is easily controlled by the estimate

$$\frac{1}{\Gamma(\alpha)} \int_0^\varepsilon z^{\alpha-1} |\mathcal{P}_z f(X)| dz \leq \frac{1}{\Gamma(\alpha + 1)} \mathcal{M}^* f(X) \varepsilon^\alpha. \tag{7.4}$$

Let now $1 \leq p < D/\alpha$. To control the second term we use Lemma 7.1 to obtain

$$\frac{1}{\Gamma(\alpha)} \int_{\varepsilon}^{\infty} z^{\alpha-1} |\mathcal{P}_z f(X)| dz \leq \frac{C_1}{\Gamma(\alpha)} \|f\|_p \int_{\varepsilon}^{\infty} z^{\alpha-D/p-1} dz = C_2 \|f\|_p \varepsilon^{\alpha-\frac{D}{p}},$$

where $C_2 = C_2(N, D, \alpha, \gamma_D, p) > 0$. Combining this estimate with (7.4) and (7.3), we conclude that (7.2) holds. \square

With Proposition 7.3 in hands, we can now establish the first main result of this section.

Theorem 7.4 (of Hardy–Littlewood–Sobolev type) *Assume that (7.1) be valid. Then, we have: (i) for every $0 < \alpha < D$ the operator \mathcal{I}_{α} maps L^1 into $L^{\frac{D}{D-\alpha}, \infty}$. Furthermore, there exists $S_1 = S_1(N, D, \alpha, \gamma_D) > 0$ such that for any $f \in L^1$ one has*

$$\sup_{\lambda > 0} \lambda |\{X \in \mathbb{R}^N \mid |\mathcal{I}_{\alpha} f(X)| > \lambda\}|^{\frac{D-\alpha}{D}} \leq S_1 \|f\|_1; \tag{7.5}$$

(ii) for every $1 < p < D/\alpha$ the operator \mathcal{I}_{α} maps L^p to L^q , with $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{D}$. Moreover, there exists $S_p = S_p(N, D, \alpha, \gamma_D, p) > 0$ such that one has for any $f \in L^p$

$$\|\mathcal{I}_{\alpha} f\|_q \leq S_p \|f\|_p. \tag{7.6}$$

Proof (i) Suppose $f \in L^1$, with $\|f\|_1 \neq 0$ (otherwise, there is nothing to prove). The estimate (7.2) reads in this case

$$|\mathcal{I}_{\alpha} f(X)| \leq \frac{1}{\Gamma(\alpha + 1)} \mathcal{M}^* f(X) \varepsilon^{\alpha} + C_2 \|f\|_1 \varepsilon^{\alpha-D}. \tag{7.7}$$

Given $\lambda > 0$ we choose $\varepsilon > 0$ such that $C_2 \|f\|_1 \varepsilon^{\alpha-D} = \lambda$. With such choice we see from (7.7) that

$$\begin{aligned} |\{X \in \mathbb{R}^N \mid |\mathcal{I}_{\alpha} f(X)| > 2\lambda\}| &\leq |\{X \in \mathbb{R}^N \mid \frac{1}{\Gamma(\alpha + 1)} \mathcal{M}^* f(X) \varepsilon^{\alpha} > \lambda\}| \\ &\leq \frac{2A\varepsilon^{\alpha}}{\lambda\Gamma(\alpha + 1)} \|f\|_1, \end{aligned}$$

where in the last inequality we have used (a) in Theorem 5.5. Keeping in mind that from our choice of ε we have $\varepsilon^{\alpha} = \frac{(C_2 \|f\|_1)^{\frac{D-\alpha}{D-\alpha}}}{\lambda^{\frac{D-\alpha}{D-\alpha}}}$, we conclude that (7.5) holds.

To prove (ii), we suppose now that $1 < p < D/\alpha$. Minimising with respect to ε in (7.2) we easily find for some constant $C_3 = C_3(N, D, \alpha, \gamma_D, p) > 0$

$$|\mathcal{I}_{\alpha} f(X)| \leq C_3 \mathcal{M}^* f(X)^{1-\frac{\alpha p}{D}} \|f\|_p^{\frac{\alpha p}{D}}. \tag{7.8}$$

The desired conclusion (7.6) now follows from (7.8) and from (b) in Theorem 5.5. \square

Theorem 7.4 is the keystone on which the second main result of this section leans. Before stating it, we emphasise that in view of (iii) in Remark 3.5 we know that $D_\infty \geq 2$. Therefore, if $0 < s < 1$ then $2s < 2 \leq D_\infty$.

Theorem 7.5 (of Sobolev type) *Suppose that (7.1) hold. Let $0 < s < 1$. Given $1 \leq p < D/2s$ let $q > p$ be such that $\frac{1}{p} - \frac{1}{q} = \frac{2s}{D}$.*

(a) *If $p > 1$ we have $\mathcal{L}^{2s,p} \hookrightarrow L^{\frac{pD}{D-2sp}}$. More precisely, there exists a constant $S_{p,s} > 0$, depending on N, D, s, γ_D, p , such that for any $f \in \mathcal{S}$ one has*

$$\|f\|_q \leq S_{p,s} \|(-\mathcal{A})^s f\|_p.$$

(b) *When $p = 1$ we have $\mathcal{L}^{2s,1} \hookrightarrow L^{\frac{D}{D-2s}, \infty}$. More precisely, there exists a constant $S_{1,s} > 0$, depending on N, D, s, γ_D , such that for any $f \in \mathcal{S}$ one has*

$$\sup_{\lambda > 0} \lambda |\{X \in \mathbb{R}^N \mid |f(X)| > \lambda\}|^{1/q} \leq S_{1,s} \|(-\mathcal{A})^s f\|_1.$$

Proof We observe that (3) in Remark 3.5 guarantees that $D \leq D_\infty$, and therefore \mathcal{S}_{2s} is well-defined. At this point, the proof is easily obtained by combining Theorem 6.3, which allows to write for every $X \in \mathbb{R}^N$

$$|f(X)| = |\mathcal{S}_{2s}(-\mathcal{A})^s f(X)|,$$

with Theorem 7.4. We leave the routine details to the interested reader. □

From Remark 7.2 we know that Theorem 7.4 does not cover situations, such as the Kramers’ operator in Ex.4 in Fig. 1, in which $D_0 > D_\infty$. When this happens we have the following substitute result. In the sequel, when we write $L^{q_1} + L^{q_2}$ we mean the Banach space of functions f which can be written as $f = f_1 + f_2$ with $f_1 \in L^{q_1}$ and $f_2 \in L^{q_2}$, endowed with the norm

$$\|f\|_{L^{q_1} + L^{q_2}} = \inf_{f=f_1+f_2, f_1 \in L^{q_1}, f_2 \in L^{q_2}} \|f_1\|_{L^{q_1}} + \|f_2\|_{L^{q_2}}.$$

Theorem 7.6 *Suppose there exist $\gamma > 0$ such that for every $t > 0$ one has*

$$V(t) \geq \gamma \min\{t^{D_0/2}, t^{D_\infty/2}\}. \tag{7.9}$$

Then, we have: (i) for every $0 < \alpha < D_\infty < D_0$ the operator \mathcal{S}_α maps L^1 into $L^{\frac{D_0}{D_0-\alpha}, \infty} + L^{\frac{D_\infty}{D_\infty-\alpha}, \infty}$. Furthermore, there exists $S_1 = S_1(N, D_0, D_\infty, \alpha, \gamma) > 0$ such that for any $f \in L^1$ one has

$$\min\{\sup_{\lambda > 0} \lambda |\{X \mid |\mathcal{S}_\alpha f(X)| > \lambda\}|^{\frac{D_0-\alpha}{D_0}}, \sup_{\lambda > 0} \lambda |\{X \mid |\mathcal{S}_\alpha f(X)| > \lambda\}|^{\frac{D_\infty-\alpha}{D_\infty}}\} \leq S_1 \|f\|_{L^1}; \tag{7.10}$$

(ii) for every $1 < p < D_\infty/\alpha < D_0/\alpha$ the operator \mathcal{I}_α maps L^p to $L^{q_0} + L^{q_\infty}$, with $\frac{1}{p} - \frac{1}{q_0} = \frac{\alpha}{D_0}$ and $\frac{1}{p} - \frac{1}{q_\infty} = \frac{\alpha}{D_\infty}$. Moreover, there exists $S_p = S_p(N, D_0, D_\infty, \alpha, \gamma, p) > 0$ such that one has for any $f \in L^p(\mathbb{R}^N)$

$$\|\mathcal{I}_\alpha f\|_{L^{q_0}+L^{q_\infty}} \leq S_p \|f\|_{L^p}. \tag{7.11}$$

Proof It suffices to prove the statements for $f \in \mathcal{S}$. Let $1 \leq p < +\infty$ and $X \in \mathbb{R}^N$. From the ultracontractive estimate in Propositions 3.6 and (7.9), we obtain from (5.2),

$$\begin{aligned} |\mathcal{P}_z f(X)| &\leq \frac{c_{N,p}}{\sqrt{4\pi}} \|f\|_p \int_0^\infty \frac{z}{t^{1/2} V(t)^{1/p}} e^{-\frac{z^2}{4t}} \frac{dt}{t} \\ &\leq \frac{c_{N,p}}{\gamma^{\frac{1}{p}} \sqrt{4\pi}} \|f\|_p \int_0^\infty \frac{z}{t^{1/2} \min\{t^{D_0/2p}, t^{D_\infty/2p}\}} e^{-\frac{z^2}{4t}} \frac{dt}{t} \\ &= C(N, p, \gamma) \|f\|_p \int_0^\infty \sqrt{u} e^{-u} \max\left\{\left(\frac{4u}{z^2}\right)^{D_0/2p}, \left(\frac{4u}{z^2}\right)^{D_\infty/2p}\right\} \frac{du}{u} \\ &\leq C(N, p, \gamma) \|f\|_p \max\{z^{-D_0/p}, z^{-D_\infty/p}\} \\ &\quad \int_0^\infty \sqrt{u} e^{-u} \max\{(4u)^{D_0/2p}, (4u)^{D_\infty/2p}\} \frac{du}{u}. \end{aligned}$$

For any $X \in \mathbb{R}^N$ and $z > 0$ we have thus proved

$$|\mathcal{P}_z f(X)| \leq \bar{C} \|f\|_p \max\{z^{-D_0/p}, z^{-D_\infty/p}\}, \tag{7.12}$$

where $\bar{C} = C(N, p, \gamma, D_0, D_\infty) > 0$. Next, let $0 < \alpha < D_\infty < D_0$ and $1 \leq p < D_\infty/\alpha < D_0/\alpha$. As in (7.3) and (7.4), for any $X \in \mathbb{R}^N$ we have

$$|\mathcal{I}_\alpha f(X)| \leq \frac{1}{\Gamma(\alpha + 1)} \mathcal{M}^* f(X) \varepsilon^\alpha + \frac{1}{\Gamma(\alpha)} \int_\varepsilon^\infty z^{\alpha-1} |\mathcal{P}_z f(X)| dz. \tag{7.13}$$

To bound the second integral we use (7.12) to find

$$\int_\varepsilon^\infty z^{\alpha-1} |\mathcal{P}_z f(X)| dz \leq \bar{C} \|f\|_p g(\varepsilon),$$

where

$$g(\varepsilon) = \int_\varepsilon^\infty h(z) dz = \int_\varepsilon^\infty z^{\alpha-1} \max\{z^{-D_0/p}, z^{-D_\infty/p}\} dz.$$

To see that $g(\varepsilon) < \infty$ for all $\varepsilon > 0$, it suffices to look at $g(1)$:

$$g(1) = \int_1^\infty z^{\alpha-1} \max\{z^{-D_0/p}, z^{-D_\infty/p}\} dz = \int_1^\infty z^{\alpha-1-D_\infty/p} dz < \infty$$

since we have assumed $p < \frac{D_\infty}{\alpha}$. Therefore, $g(\varepsilon)$ is well-defined, $g \in C^1(0, \infty)$, and

$$g'(\varepsilon) = -\varepsilon^{\alpha-1} \max\{\varepsilon^{-D_0/p}, \varepsilon^{-D_\infty/p}\} < 0,$$

which shows that g is strictly decreasing. We also know that $\lim_{\varepsilon \rightarrow +\infty} g(\varepsilon) = 0$. Furthermore, if $0 < \varepsilon < 1$, then

$$\begin{aligned} g(\varepsilon) &= \int_\varepsilon^1 h(z) dz + \int_1^\infty h(z) dz = \int_\varepsilon^1 z^{\alpha-1-\frac{D_0}{p}} dz + g(1) \\ &= \frac{\varepsilon^{-\left(\frac{D_0}{p}-\alpha\right)}}{\frac{D_0}{p}-\alpha} - \frac{1}{\frac{D_0}{p}-\alpha} + g(1). \end{aligned}$$

We infer that $\lim_{\varepsilon \rightarrow 0^+} g(\varepsilon) = +\infty$. Thus $g : (0, \infty) \rightarrow (0, \infty)$ is invertible.

Going back to (7.13) we conclude

$$\begin{aligned} |\mathcal{J}_\alpha f(X)| &\leq \frac{1}{\Gamma(\alpha+1)} \\ \mathcal{M}^* f(X) \varepsilon^\alpha + \frac{\bar{C}}{\Gamma(\alpha)} \|f\|_p g(\varepsilon) &=: G(\varepsilon) \quad \forall \varepsilon > 0. \end{aligned} \tag{7.14}$$

To prove (ii) we look for the minimum of G which is attained at some ε such that

$$\min\{\varepsilon^{D_0/p}, \varepsilon^{D_\infty/p}\} = \bar{C} \frac{\|f\|_p}{\mathcal{M}^* f(X)} =: A_f(X).$$

In other words

$$\varepsilon_{min} = \max\{A_f(X)^{p/D_0}, A_f(X)^{p/D_\infty}\}.$$

Going back to (7.14) we conclude

$$\begin{aligned} |\mathcal{J}_\alpha f(X)| &\leq \frac{1}{\Gamma(\alpha+1)} \mathcal{M}^* f(X) \\ \max \left\{ \left(\frac{\bar{C} \|f\|_p}{\mathcal{M}^* f(X)} \right)^{\frac{\alpha p}{D_0}}, \left(\frac{\bar{C} \|f\|_p}{\mathcal{M}^* f(X)} \right)^{\frac{\alpha p}{D_\infty}} \right\} &+ \frac{\bar{C}}{\Gamma(\alpha)} \|f\|_p g(\varepsilon_{min}). \end{aligned}$$

In the case $0 < A_f(X) < 1$, then we have

$$\max \left\{ \left(\frac{\bar{C} \|f\|_p}{\mathcal{M}^* f(X)} \right)^{\alpha p/D_0}, \left(\frac{\bar{C} \|f\|_p}{\mathcal{M}^* f(X)} \right)^{\alpha p/D_\infty} \right\} = \left(\frac{\bar{C} \|f\|_p}{\mathcal{M}^* f(X)} \right)^{\alpha p/D_0}, \quad \text{and}$$

$$g(\varepsilon_{min}) = \frac{1}{\frac{D_\infty}{p} - \alpha} - \frac{1}{\frac{D_0}{p} - \alpha} + \frac{1}{\frac{D_0}{p} - \alpha} \left(\frac{\bar{C} \|f\|_p}{\mathcal{M}^* f(X)} \right)^{-\frac{p}{D_0} (\frac{D_0}{p} - \alpha)}$$

$$\leq \frac{1}{\frac{D_\infty}{p} - \alpha} \left(\frac{\bar{C} \|f\|_p}{\mathcal{M}^* f(X)} \right)^{-\frac{p}{D_0} (\frac{D_0}{p} - \alpha)}.$$

We conclude that, if $A_f(X) < 1$, then

$$|\mathcal{I}_\alpha f(X)| \leq C_1 \mathcal{M}^* f(X)^{1 - \frac{\alpha p}{D_0}} \|f\|_p^{\frac{\alpha p}{D_0}}.$$

If instead $A_f(X) \geq 1$, we can show in a similar way that

$$|\mathcal{I}_\alpha f(X)| \leq C_2 \mathcal{M}^* f(X)^{1 - \frac{\alpha p}{D_\infty}} \|f\|_p^{\frac{\alpha p}{D_\infty}}.$$

If we write

$$\mathcal{I}_\alpha f = \mathcal{I}_\alpha f \cdot \chi_{\{A_f < 1\}} + \mathcal{I}_\alpha f \cdot \chi_{\{A_f \geq 1\}},$$

we deduce from (b) in Theorem 5.5 that $\mathcal{I}_\alpha f \chi_{\{A_f < 1\}} \in L^{q_0}$ and $\mathcal{I}_\alpha f \chi_{\{A_f \geq 1\}} \in L^{q_\infty}$ with the relative bounds

$$\|\mathcal{I}_\alpha f \chi_{\{A_f < 1\}}\|_{q_0} \leq c' \|f\|_p, \quad \|\mathcal{I}_\alpha f \chi_{\{A_f \geq 1\}}\|_{q_\infty} \leq c'' \|f\|_p.$$

This proves (7.11).

Let us turn to the proof of (i). Let $p = 1$, $0 < \alpha < D_\infty < D_0$, and suppose $\|f\|_1 \neq 0$. Recalling (7.14) and the invertibility of g , for every positive λ we can pick $\varepsilon > 0$ such that $\frac{\bar{C}}{\Gamma(\alpha)} \|f\|_1 g(\varepsilon) = \lambda$. From (a) in Theorem 5.5, we then get

$$\begin{aligned} |\{X \in \mathbb{R}^N \mid |\mathcal{I}_\alpha f(X)| > 2\lambda\}| &\leq \left| \left\{ X \in \mathbb{R}^N \mid \frac{1}{\Gamma(\alpha + 1)} \mathcal{M}^* f(X) \varepsilon^\alpha > \lambda \right\} \right| \\ &\leq \frac{2A\varepsilon^\alpha}{\lambda\Gamma(\alpha + 1)} \|f\|_1. \end{aligned} \tag{7.15}$$

We can compute explicitly the inverse of g and find an expression for ε . In fact, if y belongs to the interval $(0, (D_\infty - \alpha)^{-1})$ we have $g^{-1}(y) = ((D_\infty - \alpha)y)^{\frac{1}{\alpha - D_\infty}}$, otherwise we have $g^{-1}(y) = \left(1 - \frac{D_0 - \alpha}{D_\infty - \alpha} + (D_0 - \alpha)y\right)^{\frac{1}{\alpha - D_0}}$. Hence, if $\frac{\lambda\Gamma(\alpha)}{\bar{C}\|f\|_1} < (D_\infty - \alpha)^{-1}$, we deduce from (7.15) that

$$\begin{aligned} |\{X \in \mathbb{R}^N \mid |\mathcal{I}_\alpha f(X)| > 2\lambda\}| &\leq \frac{2A}{\lambda\Gamma(\alpha + 1)} \|f\|_1 \left(\frac{\lambda(D_\infty - \alpha)\Gamma(\alpha)}{\bar{C}\|f\|_1} \right)^{\frac{\alpha}{\alpha - D_\infty}} \\ &= C_m \left(\frac{\|f\|_1}{\lambda} \right)^{\frac{D_\infty}{D_\infty - \alpha}}. \end{aligned}$$

On the other hand, if $\frac{\lambda\Gamma(\alpha)}{\bar{C}\|f\|_1} \geq (D_\infty - \alpha)^{-1}$, we have

$$\begin{aligned} |\{X \in \mathbb{R}^N \mid |\mathcal{I}_\alpha f(X)| > 2\lambda\}| &\leq \frac{2A}{\lambda\Gamma(\alpha + 1)} \|f\|_1 \left(1 - \frac{D_0 - \alpha}{D_\infty - \alpha} + \frac{\lambda(D_0 - \alpha)\Gamma(\alpha)}{\bar{C}\|f\|_1} \right)^{\frac{\alpha}{\alpha - D_0}} \\ &= \frac{2A\bar{C}^{\frac{\alpha}{D_0 - \alpha}}}{\Gamma(\alpha + 1)((D_0 - \alpha)\Gamma(\alpha))^{\frac{\alpha}{D_0 - \alpha}}} \left(\frac{\|f\|_1}{\lambda} \right)^{\frac{D_0}{D_0 - \alpha}} \left(1 - \frac{\bar{C}\|f\|_1}{\lambda\Gamma(\alpha)} \left(\frac{1}{D_\infty - \alpha} - \frac{1}{D_0 - \alpha} \right) \right)^{-\frac{\alpha}{D_0 - \alpha}} \\ &\leq \frac{2A\bar{C}^{\frac{\alpha}{D_0 - \alpha}}}{\Gamma(\alpha + 1)((D_0 - \alpha)\Gamma(\alpha))^{\frac{\alpha}{D_0 - \alpha}}} (D_\infty - \alpha)^{-\frac{\alpha}{D_0 - \alpha}} \left(\frac{\|f\|_1}{\lambda} \right)^{\frac{D_0}{D_0 - \alpha}} = C_M \left(\frac{\|f\|_1}{\lambda} \right)^{\frac{D_0}{D_0 - \alpha}}. \end{aligned}$$

In any case, for any $\lambda > 0$, we get

$$\min \left\{ \lambda |\{X \mid |\mathcal{I}_\alpha f(X)| > 2\lambda\}|^{\frac{D_0 - \alpha}{D_0}}, \lambda |\{X \mid |\mathcal{I}_\alpha f(X)| > 2\lambda\}|^{\frac{D_\infty - \alpha}{D_\infty}} \right\} \leq S \|f\|_{L^1},$$

where $S_1 = S_1(D_0, D_\infty, \alpha, A, \bar{C}) > 0$. This proves (7.10). □

Using Theorem 7.6, we obtain the following substitute result for Theorem 7.5. We leave it to the interested reader to fill the necessary details.

Theorem 7.7 *Suppose that (7.9) hold. Let $0 < s < 1$. Given $1 \leq p < D_\infty/2s < D_0/2s$, let $q_\infty > q_0 > p$ be such that $\frac{1}{p} - \frac{1}{q_\infty} = \frac{2s}{D_\infty}$, $\frac{1}{p} - \frac{1}{q_0} = \frac{2s}{D_0}$.*

- (a) *If $p > 1$ we have $\mathcal{L}^{2s,p} \hookrightarrow L^{\frac{pD_\infty}{D_\infty - 2sp}} + L^{\frac{pD_0}{D_0 - 2sp}}$. More precisely, there exists a constant $S_{p,s} > 0$, depending on $N, D_\infty, D_0, s, \gamma, p$, such that for any $f \in \mathcal{S}$ one has*

$$\|f\|_{L^{q_0} + L^{q_\infty}} \leq S_{p,s} \|(-\mathcal{A})^s f\|_p.$$

- (b) *If instead $p = 1$, we have $\mathcal{L}^{2s,1} \hookrightarrow L^{\frac{D_0}{D_0 - 2s}, \infty} + L^{\frac{D_\infty}{D_\infty - 2s}, \infty}$. More precisely, there exists a constant $S_{1,s} > 0$, depending on $N, D_\infty, D_0, s, \gamma$, such that for any $f \in \mathcal{S}$ one has*

$$\min \left\{ \sup_{\lambda > 0} \lambda |\{X \mid |f(X)| > \lambda\}|^{\frac{1}{q_0}}, \sup_{\lambda > 0} \lambda |\{X \mid |f(X)| > \lambda\}|^{\frac{1}{q_\infty}} \right\} \leq S_{1,s} \|(-\mathcal{A})^s f\|_1.$$

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