A Separable Solution for the Oscillatory Structure of Plasma in Accretion Disks

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Abstract. - In this paper we provide a new analysis of the system of partial differential equations describing the radial and vertical equilibria of the plasma in accretion disks. In particular, we show that the partial differential system can be separated once a definite. oscillatory (or hyperbolic) form for the radial dependence of the relevant physical quantities is assumed. The system is thus reduced to an ordinary differential system in the vertical dimensionless coordinate. The resulting equations can be integrated analytically in the limit of small magnetic pressure. We complete our analysis with a direct numerical integration of the more general case. The main result is that a ring-like density profile (i.e., radial oscillations in the mass density) can appear even in the limit of small magnetic pressure.

Introduction. – The morphology of an accretion disk around an astrophysical compact object represents one of the most important open questions of stellar physics [1]. In fact, while in absence of a significant magnetic field of the central object the disk configuration is properly described by the fluid dynamics approach, the situation becomes puzzling when we deal with a strongly magnetized source, namely a pulsar, accreting from a less dense companion. As recently shown by B. Coppi, see [2]- [5], the plasma nature of the disk implies a significant coupling between the vertical and the radial equilibrium, as a consequence of the relevant Lorentz force acting inside the structure. The existence of such a coupling suggests a deep modification of the original point of view at the base of our understanding of the stellar accretion phenomenon; for a sample of the basic literature in the field, see [6]- [9]. In fact, the standard approach to the description of a thin disk relies on the idea that the vertical equilibrium can be averaged out when the viscoresistive MHD is applied to the plasma. Such a model seems to satisfactorily reproduce the *coarse-grain* phenomenology, but at the price of introdocing an anomalous resistivity of the disk plasma (unjustified by direct estimations), see Ref. [10,11].

The analyses in Refs. [2, 3] demostrate that the details

of the disk equilibria are relevant in establishing an oscillatory local structure inside the disk. In particular, in Ref. [3] it is shown that, for a disk having a sufficiently strong magnetic pressure, (i. e. a small enough ratio of the thermostatic pressure to the magnetic one), the mass density perturbations, due to the internal currents, are able to induce a ring-like profile. This ideal MHD result constitutes an opposite point of view with respect to the idea of a diffusive magnetic field within the disk, as discussed in Ref. [1]. The striking interest in the details of the local disk morphology consists in the possibility that jets of matter and radiation are emitted by virtue of the strong magnetic field and the axial symmetry, see for instance Ref. [12].

Here, we provide a novel analysis of the fundamental partial differential system derived in Refs. [2,3] for the radial and vertical equilibria in the disk plasma. Our study is based on a separable solution, able to reduce the coupled partial differential scheme to a simple ordinary differential system in the vertical dimensionless coordinate. The separation is realized by a suitable trigonometric expression for the radial dependence; the remaining unknowns are four functions, associated to the vertical dependence of the magnetic flux function, the mass density corrections and eventually to functions related to the different behavior of the thermostatic pressure term.

Indeed it is just the presence of two different radial behaviors of the pressure, the main new feature of the present analysis. We see that the mass density perturbations induced in the plasma are relevant even when the ratio of the thermostatic to the magnetic pressure is high (differently from the analysis in [2]. This correction to the mass distribution has an oscillating character, so that we see the formation of the ring-like profile even in the parameter region where the relevance of the magnetic field is not crucial on the background level. This result suggests that the presence of rings in the disk structure morphology is a very general feature of the accretion disk structure for magnetized stars.

The ordinary differential system we derive is analytically integrated for small values of the ratio of the magnetic pressure over the thermostatic one; the solution that is obtained is a good tool to fix the boundary conditions of the numerical analysis for the general case. In fact, the analvtic solution remains valid for general values of the free parameters, as far as we restrict ourselves sufficiently close to the equatorial plane, where the boundary conditions for the numerical analysis can be given. The main implication of this link between the analytical and numerical analyses is that we get a direct relation between the ratio of the magnetic to thermostatic pressure to the one between the perturbation wavenumber and the fundamental wavenumber of the plasma structure. Then our solution cannot explore the very extreme value of the parameters, where the magnetic pressure completely dominates the equilibrium configuration. The paper is organized as follows. In Sec. I, we describe the basic features of the disk. In Sec. II, we write the equations governing the radial and vertical equilibrium of the disk. In Sec. III we reduce the fundamental partial differential system to a system of ordinary differential equations, and we solve it analytically in the limit of small magnetic pressure, discussing the appearance of an oscillatory structure. In Sec. IV we show the results of the numerical integration, and finally in Sec. V we draw our conclusions.

Basic Features of the Disk. – The magnetic field, characterizing the central object, takes the form

$$\vec{B} = -\frac{1}{r}\partial_z\psi\vec{e}_r + \frac{I}{r}\vec{e}_\phi + \frac{1}{r}\partial_r\psi\vec{e}_z\,,\qquad(1)$$

with $\psi = \psi(r, z^2)$ and $I = I(\psi, z)$.

The matter flux associated with the disk morphology is:

$$\epsilon \vec{v} = -\frac{1}{r} \partial_z \Theta \vec{e}_r + \epsilon \omega(r, z^2) r \vec{e}_\phi + \frac{1}{r} \partial_r \Theta \vec{e}_z , \qquad (2)$$

where ϵ denotes the matter density and $\Theta(r, z)$ is an odd function of z, to deal with a non zero accretion rate, i.e.

$$\dot{M}_d = -2\pi r \int_{-z_0}^{z_0} \epsilon v_r dz = 4\pi \Theta(r, z_0) \equiv 2\pi I > 0, \quad (3)$$

 $z_0(r) \ll r$ being the half-width of the thin disk.

The similarity of the magnetic field and matter flux structure, is due to their common divergenceless nature. Since in the present analysis we are concentrating our attention on the formation of the ring profile within the disk, in what follows, we neglect the presence of the functions I and Θ , which are relevant for the characterization of the azimuthal equilibrium. In fact, as discussed in [2,3], the origin of the oscillatory structure comes out by the coupling of the vertical and the radial equilibria, when the internal currents rising in the plasma are taken into account.

We now develop a local model of the equilibrium, as settled down around a radius value $r = r_0$, in order to investigate analytically the effects induced on the disk profile by the electromagnetic reaction of the plasma. To this end we split the energy density and the pressure contributions as $\epsilon = \bar{\epsilon}(r_0, z^2) + \hat{\epsilon}$ and $p = \bar{p}(r_0, z^2) + \hat{p}$, respectively. The same way, we express the magnetic surface function in the form $\psi = \psi_0(r_0) + \psi_1(r_0, r - r_0, z^2)$, with $\psi_1 \ll \psi_0$. The quantities $\hat{\epsilon}$, \hat{p} and ψ_1 describe the change of the fundamental plasma functions due to the currents that emerge within the disk embedded into the external magnetic field of the central object. In general these corrections are small in amplitude but with a very short scale of variation. Thus, we are led to address the "drift ordering" for the behavior of the gradient amplitude, i.e. the first order gradients of the perturbations are of zero-order, while the second order ones dominate.

As ensured by the corotation theorem [13], the angular frequency of the disk rotation has to be expressed via the magnetic flux function as $\omega(\psi)$. As a consequence, in the present splitted scheme, we can take the decomposition $\omega = \omega_K + \omega'_0 \psi_1$, where ω_K is the Keplerian term and $\omega'_0 \equiv d\omega_0 d\psi_0 = \text{const.}$ This form for ω holds locally, as far as $(r-r_0)$ remains a sufficiently small quantity, so that the dominant deviation from the Keplerian contribution is due to ψ_1 .

Accordingly to the drift ordering, the profile of the toroidal currents rising in the disk, takes the expression $J_{\phi} \simeq -(c/4\pi r_0) \times (\partial_r^2 \psi_1 + \partial_z^2 \psi_1).$

Vertical and Radial Equilibrium. – We now fix the equations governing the vertical and the radial equilibrium of the disk, by separating the basic fluid component from the presence of the electromagnetic reaction. Such a splitting of the MHD equations for the vertical force balance gives

$$D(z^{2}) \equiv \frac{\bar{\epsilon}}{\epsilon_{0}(r_{0})} = e^{-\frac{z^{2}}{H_{0}^{2}}}, \ H_{0}^{2} \equiv \frac{4K_{B}\bar{T}}{m_{i}\omega_{K}^{2}}, \qquad (4)$$

$$\partial_z \hat{p} + \omega_K^2 z \hat{\epsilon} - \frac{1}{4\pi r_0^2} \left(\partial_z^2 \psi_1 + \partial_r^2 \psi_1 \right) \partial_z \psi_1 = 0, \qquad (5)$$

where $\epsilon_0(r_0) \equiv \epsilon(r_0, 0)$ and m_i is the ion mass. The behavior of the function $D(z^2)$ accounts for the pure thermostatic equilibrium holding in the disk when the vertical gravity (i. e. the Keplerian rotation) is large enough to

provide a confined thin configuration, while the temperature T admits the representation

$$2K_BT \equiv m_i \frac{p}{\epsilon} = m_i \frac{\bar{p} + \hat{p}}{\bar{\epsilon} + \hat{\epsilon}} \equiv 2K_B(\bar{T} + \hat{T}).$$
(6)

The radial equations underlying the equilibrium of the rotating layers of the disk, can be decomposed into the dominant character of the Keplerian angular velocity plus an equation describing the behavior of the deviation $\delta\omega$:

$$2\omega_{K}r_{0}(\bar{\epsilon}+\hat{\epsilon})\omega_{0}'\psi_{1} + \frac{1}{4\pi r_{0}^{2}}\left(\partial_{z}^{2}\psi_{1}+\partial_{r}^{2}\psi_{1}\right)\partial_{r}\psi_{1} = \\ = \partial_{r}\left[\hat{p}+\frac{1}{8\pi r_{0}^{2}}\left(\partial_{r}\psi_{1}\right)^{2}\right] + \frac{1}{4\pi r_{0}^{2}}\partial_{r}\psi_{1}\partial_{z}^{2}\psi_{1}.$$
(7)

We neglected, in the radial and vertical equilibria, the presence of the poloidal current associated with the azimuthal component of the magnetic field.

We define the dimensionless functions Y, \hat{D} and \hat{P} , in place of ψ_1 , $\hat{\epsilon}$ and \hat{p} , i. e.

$$Y \equiv \frac{k_0 \psi_1}{\partial_{r_0} \psi_0}, \, \hat{D} \equiv \frac{\beta \hat{\epsilon}}{\epsilon_0}, \, \hat{P} \equiv \beta \frac{\hat{p}}{p_0}, \tag{8}$$

where $p_0 \equiv 2K_B \hat{T} \epsilon_0/m_i$ and $\beta \equiv 8\pi p_0/B_{0z}^2 = 1/(3\epsilon_z^2) \equiv k_0^2 H_0^2/3$. We introduced the fundamental wavenumber k_0 of the radial equilibrium, defined as $k_0 \equiv 3\omega_K^2/v_A^2$, with $v_A^2 \equiv 4\pi\epsilon_0/B_{z0}^2$, recalling that $B_{z0} = \partial_{r_0}\psi_0/r_0$. It is then natural to deal with the dimensionless radial variable $x \equiv k_0(r-r_0)$, while assuming that the fundamental length in the vertical direction is $\Delta \equiv \sqrt{\epsilon_z}H_0$, leading to introduce $u \equiv z/\Delta.1$

By this definitions, the vertical and radial equilibrium equations can be restated respectively as

$$\partial_{u^2}\hat{P} + \epsilon_z\hat{D} + 2\left(\partial_{x^2}^2Y + \epsilon_z\partial_{u^2}^2Y\right)\partial_{u^2}Y = 0, \qquad (9)$$

$$\left(D + \frac{1}{\beta}\hat{D}\right)Y + \partial_{x^2}^2 Y + \epsilon_z \partial_{u^2}^2 Y + \frac{1}{2}\partial_x \hat{P} + \left(\partial_{x^2}^2 Y + \epsilon_z \partial_{u^2}^2 Y\right)\partial_x Y = 0.$$
 (10)

Once D and \hat{D} are assigned, the equations above provide a coupled system for \hat{P} and Y, allowing to fix the disk configuration due to the toroidal currents.

Reduction of the Fundamental System. – The analysis of the partial differential system derived above has been performed in Ref. [2] in the limit of small values of ϵ_z and an approximated solution was found as an expansion in such a parameter. Instead in Ref. [3], the study has been extended to the case $\epsilon_z > 1$, by requiring that the function Y satisfied the basic eigenstate equation

$$\partial_{x^2}^2 Y + \epsilon_z \partial_{u^2}^2 Y = -DY.$$
(11)

Here we show that the two partial differential equations (9) and (10) can be treated separating the radial and vertical dependence, thus reducing them to an ordinary differential

system. In fact, we easily get such a reduction by the following positions

$$Y = F(u^2)\sin(\alpha x) \tag{12}$$

$$\hat{P} = L(u^2)\cos(\alpha x) + M(u^2)\sin^2(\alpha x) \tag{13}$$

$$\hat{D} = d(u^2)\cos(\alpha x). \tag{14}$$

and by imposing the vanishing of the coefficients of each type of trigonometrical terms. A simple calculation shows that the vertical equilibrium (9) yields the two equations

$$\frac{dL}{du^2} + \epsilon_z d = 0 \tag{15}$$

$$\frac{dM}{du^2} + 2\left(-\alpha^2 F + \epsilon_z \frac{d^2 F}{du^2}\right)\frac{dF}{du^2} = 0, \qquad (16)$$

while the radial equation (10) gives

$$D(u^2)F - \alpha^2 F + \epsilon_z \frac{d^2 F}{du^2} - \frac{\alpha}{2}L = 0$$
(17)

$$\frac{1}{\beta}dF + \alpha M + \alpha F\left(-\alpha^2 F + \epsilon_z \frac{d^2 F}{du^2}\right) = 0, \qquad (18)$$

These two pairs of equations form an ordinary differential system in the variable u of four coupled second order equation in the four unknowns $F(u^2)$, $L(u^2)$, $M(u^2)$ and $d(u^2)$ respectively. The quantities ϵ_z and α (we recall that $\beta = 1/3\epsilon_z^2$ play the role of free parameters of the problem. In particular ϵ_z measures the relevance of the electromagnetic interaction in the establishment of the equilibrium configuration of the disk plasma. The greater ϵ_z is, the stronger the internal currents deform the background distribution of matter and magnetic field. The parameter α fixes the amplitude of the radial wavenumber (with respect to the fundamental one k_0) associated to the perturbations. The greater is α , the smaller is the wavelenght of the radial plasma structures. The same way, also ϵ_z can be regarded as the parameter which gives the scale of the vertcal confinament, according to the relation, introduced above, $\Delta = \sqrt{\epsilon_z} H_0$. Such a relation, together with the definition $\Delta = \sqrt{H_0/k_0}$, allows to express the function $D(u^2)$ from $D(z^2)$ introduced in (4) as $D(u^2) = \exp\{-\epsilon_z u^2\}$.

Analytical Solution for Small ϵ_z Values. Let us study the system of configuration equations in the limit of small values of the parameter ϵ_z , when we can use the expansion $D(u^2) = 1 - \epsilon_z u^2$. In this way, we are led to search a solution to the four equations above, in the form

$$F = A \exp\left(-\frac{u^2}{2}\right); \quad L = lF; \tag{19}$$

$$d = kF;$$
 $M = C(u^2)F^2.$ (20)

In other words we assume that the function Y is confined around the equatorial plane and that the other functions can be expressed through $F(u^2)$, in agreement to the structure of the four equations. Substituting expressions (19) into the equations (17), we get the algebraic relations

$$\alpha = \frac{1}{2}\sqrt{3(1-\epsilon_z)}; \quad l = \frac{2}{3}\alpha; \tag{21}$$

$$k = \frac{\alpha}{3\epsilon_z};$$
 $C(u^2) = \alpha^2 - \epsilon_z u^2.$ (22)

Thus, for small values of ϵ_z , we are able to provide an analytic solution describing the detailed features of the disk plasma. We see that the wavenumber of the perturbations is not very different, in this limit, from k_0 , while the function d is much greater than F and, as we shall see below, this is an important peculiar feature of this solution.

We also remark that, since this solution relies on the expansion $e^{-\epsilon_z u^2} \simeq 1 - \epsilon_z u^2$, its range of validity is actually broader than the $\epsilon_z \ll 1$ region. In particular, the solution is still valid for $\epsilon_z \sim 1$, provided that $u \ll 1/\sqrt{\epsilon_z}$ (i.e., provided that we are close enough to the equatorial plane). The solution can also be continued in the $\epsilon_z > 1$ region by noting that in this case, according to Eq. (21) above, α would be purely immaginary, and the trigonometric functions would become hyperbolic functions. We are then led to search a solution in the form:

$$Y' = F'(u^2)\sinh(\alpha' x) \tag{23}$$

$$\hat{P'} = L'(u^2)\cosh(\alpha' x) + M'(u^2)\sinh^2(\alpha' x)$$
(24)

$$\hat{D'} = d'(u^2)\cosh(\alpha' x), \qquad (25)$$

where, as before:

$$F' = A' \exp\left(-\frac{u^2}{2}\right); \quad L' = l'F; \qquad (26)$$

$$d' = k'F;$$
 $M' = C'(u^2)F'^2.$ (27)

Repeating the above procedure we find:

$$\alpha' = \frac{1}{2}\sqrt{3(\epsilon_z - 1)}; \quad l' = \frac{2}{3}\alpha';$$
 (28)

$$k' = \frac{\alpha'}{3\epsilon_z}; \qquad C'(u^2) = -\alpha'^2 - \epsilon_z u^2.$$
(29)

The Oscillatory Structure. Once the form of the solution has been fixed, we can analyze the physical implications for the disk structure. In particular, it is immediate to recognize that for $\epsilon_z < 1$ the mass density distribution acquires the behavior

$$\frac{\epsilon}{\epsilon_0} = D(u^2) + \frac{1}{\beta}\hat{D}(u^2) =$$
$$= 1 - \epsilon_z u^2 + A(\alpha\epsilon_z)e^{-\frac{u^2}{2}}\cos(\alpha x) \ge 0. \quad (30)$$

We see that the perturbations to the mass density profile is an odd function of x and has an oscillating radial dependence. On the contrary, it is clear that the oscillating behaviour is not present in the $\epsilon_z > 1$ regime, since in that case the density and pressure are expressed in terms of hyperbolic functions. We will then concentrate in the following in the analysis of the $\epsilon_z < 1$ case.

Since $\epsilon_z u^2$ is much smaller than unity, it is easy to realize that the positive character of the mass density is ensured by the request $A < 1/(\alpha \epsilon_z)$. The total pressure term in the disk plasma is given by

$$\frac{p}{p_0} = e^{-\epsilon_z u^2} + 3\epsilon_z^2 A \left[\frac{2}{3}\alpha\cos(\alpha x) + A\left(\alpha^2 - \epsilon_z u^2\right)e^{-\frac{u^2}{2}}\sin^2(\alpha x)\right]e^{-\frac{u^2}{2}}.$$
 (31)

and the perturbation is an even function of x. The term in squared brackets can provide a negative contribution to the total pressure, but, in the limit of small enough $\epsilon_z \ll 1$, its weight is limited by the coefficient ϵ_z^2 and the expression above, for values of u in the range of some units over the equatorial plane, rewrites as

$$\frac{p}{p_0} \simeq 1 + 3\epsilon_z^2 A \left[\frac{1}{\sqrt{3}} \cos(\alpha x) + \frac{3}{2} A e^{-\frac{u^2}{2}} \sin^2(\alpha x) \right] e^{-\frac{u^2}{2}}$$
(32)

and the total plasma pressure is clearly positive when $A < 1/(\alpha \epsilon_z^2)$. It can be seen that, since $\epsilon_z < 1$, this is always ensured once the condition for the positiveness of the density $(A < 1/(\alpha \epsilon_z))$ is fulfilled. However, in the general case, the positive character of p/p_0 in the point $\alpha x = \pi$, requires that $2\alpha \epsilon_z^2 A \exp[(\epsilon_z - 1/2)u^2] \ll 1$, that constrains the possible values of A, with implications on the morphology of the ring profile.

However, the very important feature we get, is that, differently from the analysis in Refs. [2,3], here the mass density can have nodes even for small values of ϵ_z , i. e. for high β values of the plasma (see Fig. 1). This behavior is a consequence of dealing with a solution in which the quantity $\hat{\epsilon}$ is of order ϵ_z , instead of order ϵ_z^2 like in [2,3]. We show the radial density and pressure profiles in Figs. 1 and 2, for different values of the vertical coordinate u, and for values of the parameters A = 100 and $\epsilon_z = 10^{-2}$ [the value of α is fixed by the relation (21)]. It is clearly seen that there are "empty" regions, i.e. regions where $\epsilon/\epsilon_0 \ll 1$, indicating the presence of a ring-like structure of the disk. This a general feature of the mass distribution described by Eq. (30) when $A\alpha\epsilon_z \simeq 1$. The relevance of this result relies on the existence of a local ring profile in the disk plasma even if the magnetic pressure does not dominate the termostatic one. As a consequence, we can infer that the oscillatory structure of the disk is expectedly a very diffuse phenomenon in accreting astrophysical sources.

Comparison with previous works

In this section we compare our results with those obtained in previous works, mainly focusing on Ref. [3]. The main assumption underlying the analysis presented in Ref. [3] is that Y satisfies the eigenvalue equation (11). In our analysis, this equation is not satisfied given the positions that we have made concerning the form the functions Y and \hat{P} [see Eq. (12) above]. In fact, substituting $Y = F(u^2) \sin(\alpha x)$ in the above eigenvalue equation, we get:

$$\epsilon_z \frac{d^2 F}{du^2} - \alpha^2 F + \bar{D}F = 0. \tag{33}$$

On the other hand, we have that instead, in our analysis, the corresponding equation satisfied by F is Eq. (17), namely:

$$\epsilon_z \frac{d^2 F}{du^2} - \alpha^2 F + \bar{D}F - \frac{\alpha}{2}L = 0.$$
(34)

We recall that the function $L(u^2)$ appearing in the additional term is related to the cosine part of the pressure. It





Figure 1: Top: Radial density profile of the disk for u = 0.5, 1, 2 (black [solid], red [dashed] and blue [dotted] curves, respectively). Bottom: Normalized mass density as a function of the dimensionless radial and vertical coordinates x and u. In both panels, A = 100 and $\epsilon_z = 0.01$.

it is thus clear that, if $L(u^2) \neq 0$, the two approaches will bear different results. So the basic difference between the two analyses can be traced back to the presence, in the pressure function, of a term proportional to $\cos(\alpha x)$.

Numerical Analysis. – In the region of the parameters where ϵ_z is not much smaller than unity, the system (15)-(17) has to be integrated numerically. To this purpose, we change the vertical variable to $\xi = u^2/2$. Observing that $\frac{d^2F}{du^2} = 2\xi \frac{d^2F}{d\xi^2} + \frac{dF}{d\xi}$ we rewrite the configuration system in the form:

$$\frac{dL}{d\xi} + 2\epsilon_z d = 0 \ (35)$$

$$\frac{dM}{d\xi} + 2\left[-\alpha^2 F + \epsilon_z \left(2\xi \frac{d^2 F}{d\xi^2} + \frac{dF}{d\xi}\right)\right] \frac{dF}{d\xi} = 0. \ (36)$$

$$e^{-2\epsilon_z\xi} F - \alpha^2 F + \epsilon_z \left(2\xi \frac{d^2 F}{d\xi^2} + \frac{dF}{d\xi}\right) - \frac{\alpha}{2}L = 0 \ (37)$$

$$3\epsilon_z^2 dF + \alpha M + \alpha F \left[-\alpha^2 F + \epsilon_z \left(2\xi \frac{d^2 F}{d\xi^2} + \frac{dF}{d\xi} \right) \right] = 0. (38)$$

This system of four ordinary differential equations in the four unknown functions $F(\xi)$, $L(\xi)$, $M(\xi)$ and $d(\xi)$ can be integrated numerically by fairly standard methods. However, in order to perform the numerical integration, initial conditions should be given. These can be found by noting

Figure 2: Top: Normalized pressure as a function of the dimensionless radial coordinate x, for u = 0.5, 1, 2 (black [solid], red [dashed] and blue [dotted] curves, respectively). Bottom: Normalized pressure as a function of the dimensionless radial and vertical coordinates x and u. In both panels, A = 100 and $\epsilon_z = 0.01$.

that, when ξ is small enough, the analytic solution found in the previous section is an acceptable solution to the differential system even if ϵ_z is of order unity or larger. In fact, the parameter of the expansion that leads to the above analytical solution is $\epsilon_z u^2 = 2\epsilon_z \xi$, so that for every value of ϵ_z it is always possible to find a (however small) starting value ξ_0 for the numerical integration such that $2\epsilon_z \xi_0 \ll 1$.

Then, we give the initial conditions so that they match the analytic solution in ξ_0 :

$$F_{0} \equiv F(\xi_{0}) = Ae^{-\xi_{0}}; \quad L(\xi_{0}) = \frac{2}{3}\alpha F_{0};$$

$$d(\xi_{0}) = \frac{\alpha}{3\epsilon_{z}}F_{0}; \quad M(\xi_{0}) = (\alpha^{2} - 2\epsilon_{z}\xi_{0})F_{0}^{2};$$

$$\frac{dF}{d\xi}(\xi_{0}) = -F_{0} \qquad (39)$$

The fact that we are matching the analytical solution in ξ_0 means that we have to enforce the condition $\alpha = \frac{1}{2}\sqrt{3(1-\epsilon_z)}$ like we did before, in order to ensure the consistency of the differential system. Hence, initial conditions are completely fixed once A and ϵ_z (or equivalently α) are fixed. The results of our numerical integration are shown in Figs. 3-4.



Figure 3: Behaviour of the functions F(u), L(u), M(u) and d(u) for $D(u) = e^{\epsilon_z u^2}$, A = 1 and $\epsilon_z = 0.9$ (Red [solid] lines). Also shown is the corresponding analytical solution for $D(u) = 1 - \epsilon_z u^2$ (Black [dotted] lines).

Concluding Remarks. – In this paper we presented an exact separable solution of the radial and vertical equilibrium equations at the basis of the oscillating morphology emerging in the plasma configuration of an accretion disk. We reduced the original system of two partial differential equations into a set of four independent ordinary differential equations in the vertical coordinate, one for the flux surface function, one for the mass density perturbations and two related to different components of the pressure. This reduced system admits an analytical solution only in the limit of high values of the β parameter of the plasma, i. e. only when the thermostatic pressure of the disk is much larger than the magnetic pressure contribution. We also performed a numerical analysis of the system for small values of β .

The main results of our analysis can be summarized in the following three points.

i)-We have derived a solution showing how the radial gradient of the thermostatic pressure is relevant in establishing the equilibrium, even in the linear limit and for small values of ϵ_z . This feature makes our solution intrinsically different from the analyses developed in Refs. [2, 3], except for the discussion of the extreme non-linear regime in Section IX of Ref. [3]. In fact, in such a limit, our approach is reconcilied with that one, because the latter accounts on an equivalent level the role of the radial gradient of the pressure.

ii)-We have obtained the fundamental feature that the ring-like structure emerges as a strong structural feature of the plasma disk confined in magnetic field, since the radial oscillation of the mass density takes place even in the linear weakly magnetized limit $A \ll 1$ and $\epsilon_z \ll 1$. Indeed a suitable choice of these free parameters of the model is always possible in order to arrange for nodes in the mass density profile. This morphology is relevant because suggests that the ring profile can be expected to be a general character of the magnetized accreting compact objects we observe in the Universe, and it is due to the direct link existing between the radial pressure gradient and the mass density perturbations. This output of our



Figure 4: The same as Fig. 4, but for $\epsilon_z = 1.5$

separation algorithm indicates that more general configuration scenarios can be contained in the radial and vertical equilibrium equation with respect to the one investigated in [2,3], though they could exist in a non-separable regime.

iii)-We have found that the disk profile undergoes a transition when the parameter ϵ_z becomes greater than one, going from the oscillating structure to an hyperbolic behavior. This fact suggests the possible co-existence of two different disk components in the same global profile. In fact in the present local model we addressed ϵ_z as a constant because it refers to a generic value of the radial coordinate r_0 , but throught the disck it is clearly a function of the radial coordinate itself, taking values above and below unity in different radial regions.

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