



A DIRECT AND ELEMENTARY DERIVATION OF THE MULTIVARIATE FAÀ DI BRUNO FORMULA

C. BOITI^{1,*,\dagger} and R. MANFRIN²

¹Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli n. 30,
I-44121 Ferrara, Italy
e-mail: chiara.boiti@unife.it

²Dipartimento di Culture del Progetto, Università IUAV di Venezia, Dorsoduro n. 2196,
I-30123 Venezia, Italy
e-mail: manfrin@iuav.it

(Received June 9, 2025; revised January 19, 2026; accepted January 26, 2026)

Abstract. By applying Taylor expansion and the Leibniz rule for partial derivatives, we provide a straightforward deduction of the multivariate Faà di Bruno formula, yielding a concise and simple expression for the derivatives of the composite function $g \circ \mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}$, with $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g: \mathbb{R}^m \rightarrow \mathbb{R}$. Our formulation extends the original Faà di Bruno formula to the multidimensional case in a natural way.

1. Introduction and results

The well-known formula for the derivatives of composite functions, published by Faà di Bruno [4] in 1855 (see also [9] for a modern formulation and a simple proof) and dating back to [1], has been widely extended to the case of functions of several variables. In literature there are several versions and proofs of the multivariate Faà di Bruno formula for $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g: \mathbb{R}^m \rightarrow \mathbb{R}$. See, among others, [3, 5–8]. But, typically, the proofs are either excessively long or involve unnecessarily sophisticated tools.

Here we just apply the Taylor expansion for g and Leibniz rule for partial derivatives of the powers of $\mathbf{f}$ to give a straightforward proof of the multivariate Faà di Bruno formula. Furthermore, by these elementary methods, we obtain a simple formulation, (1.3) below, which is the natural extension to the multidimensional case of the original formula of Faà di Bruno.

* Corresponding author.

^{\dagger} The first author was partially supported by the Research Project GNAMPA-INdAM 2024: “Analisi di Gabor ed analisi microlocale: connessioni e applicazioni a equazioni a derivate parziali”.

Key words and phrases: Faà di Bruno’s formula, multivariate Taylor expansion, Leibniz rule.
Mathematics Subject Classification: 26B05.

Notation. We use boldface for vectors and multi-indices: we write $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ and, for $\mathbb{N} = \{0, 1, 2, \dots\}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m) \in \mathbb{N}^m$ with $|\boldsymbol{\alpha}| = \alpha_1 + \dots + \alpha_n$ and $|\boldsymbol{\nu}| = \nu_1 + \dots + \nu_m$; for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$, $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ means $\beta_i \leq \alpha_i$ for all $1 \leq i \leq n$; $\boldsymbol{\beta}_p, \boldsymbol{\omega}_q \in \mathbb{N}^n$ (for suitable indices $p, q \in \mathbb{N}$) will denote n -dimensional multi-indices that will be used in the calculations.

Moreover, $\mathbf{y}^\nu = \prod_{i=1}^m y_i^{\nu_i}$ and, for $\mathbf{f} = (f_1, \dots, f_m): \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\mathbf{f}^\nu = \prod_{i=1}^m f_i^{\nu_i}: \mathbb{R}^n \rightarrow \mathbb{R}$, with the convention that if h is a given function, then $h^0 \equiv 1$.

Also for partial derivatives we use standard notation: for $g: \mathbb{R}^m \rightarrow \mathbb{R}$ we set

$$\partial^\nu g = g^{(\nu)} = \frac{\partial^{|\nu|} g}{\partial y_1^{\nu_1} \dots \partial y_m^{\nu_m}}.$$

Similar notation will be used for any function $h: \mathbb{R}^n \rightarrow \mathbb{R}$. We shall apply the following Leibniz rule for the partial derivatives of a product of functions $h_1, \dots, h_k: \mathbb{R}^n \rightarrow \mathbb{R}$:

$$(1.1) \quad \partial^\alpha (h_1 \dots h_k) = \sum_{\boldsymbol{\omega}_1 + \dots + \boldsymbol{\omega}_k = \boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\omega}_1! \dots \boldsymbol{\omega}_k!} h_1^{(\boldsymbol{\omega}_1)} \dots h_k^{(\boldsymbol{\omega}_k)},$$

where $\boldsymbol{\alpha}! = \alpha_1! \dots \alpha_n!$, and $\boldsymbol{\omega}_1!, \dots, \boldsymbol{\omega}_k!$ are defined similarly. As usual, let $0! = 1$.

Finally, we shall always mean that the value of a given sum over a given set of indices is zero when that set is empty.

THEOREM 1.1. *Let $\mathbf{f} = (f_1, \dots, f_m): \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g: \mathbb{R}^m \rightarrow \mathbb{R}$, be smooth functions. Then, for $\boldsymbol{\alpha} \in \mathbb{N}^n$, $|\boldsymbol{\alpha}| \geq 1$, $\partial^\alpha (g \circ \mathbf{f})$ has the following equivalent expressions:*

$$(1.2) \quad \partial^\alpha (g \circ \mathbf{f}) = \boldsymbol{\alpha}! \sum_{1 \leq |\boldsymbol{\nu}| \leq |\boldsymbol{\alpha}|} \frac{g^{(\boldsymbol{\nu})}(\mathbf{f})}{\boldsymbol{\nu}!} \\ \times \sum_{\substack{\boldsymbol{\omega}_1 + \dots + \boldsymbol{\omega}_m = \boldsymbol{\alpha} \\ \nu_i \leq |\boldsymbol{\omega}_i| \leq |\boldsymbol{\alpha}| \nu_i}} \prod_{\substack{1 \leq i \leq m \\ \nu_i \geq 1}} \left(\sum_{\substack{\boldsymbol{\beta}_1 + \dots + \boldsymbol{\beta}_{\nu_i} = \boldsymbol{\omega}_i \\ |\boldsymbol{\beta}_j| \geq 1}} \prod_{j=1}^{\nu_i} \frac{f_i^{(\boldsymbol{\beta}_j)}}{\boldsymbol{\beta}_j!} \right),$$

where $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m) \in \mathbb{N}^m$ and $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_m, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{\nu_i} \in \mathbb{N}^n$;

$$(1.3) \quad \partial^\alpha(g \circ \mathbf{f}) = \alpha! \sum_{1 \leq |\nu| \leq |\alpha|} g^{(\nu)}(\mathbf{f}) \\ \times \sum_{\omega_1 + \dots + \omega_m = \alpha} \prod_{1 \leq i \leq m} \left(\sum_{\substack{\sum_{\beta \in J} k_\beta = \nu_i \\ \sum_{\beta \in J} k_\beta \beta = \omega_i}} \prod_{\beta \in J} \frac{1}{k_\beta!} \left(\frac{f_i^{(\beta)}}{\beta!} \right)^{k_\beta} \right),$$

where $J = J(\alpha) = \{\beta \in \mathbb{N}^n \mid |\beta| \geq 1, \beta \leq \alpha\}$, $k_\beta \in \mathbb{N}$ for all $\beta \in J$.

REMARK 1.2. Formula (1.3) is the natural extension to the multidimensional case of the original Faà di Bruno's formula. Note that in (1.3) it is not necessary to impose the conditions $\nu_i \leq |\omega_i| \leq |\alpha| \nu_i$ and $\nu_i \geq 1$ that we have in (1.2) for the summation over ω_i and the product over $1 \leq i \leq m$ respectively, because of the different expression in the parentheses of (1.3) with respect to that in (1.2). See the last part of the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. To begin with, we write the Taylor expansion of order $k \geq 1$ of g at a given point $\mathbf{b} \in \mathbb{R}^m$. For $\mathbf{y} \in \mathbb{R}^m$, we have (see [2], [10]):

$$(1.4) \quad g(\mathbf{y}) = g(\mathbf{b}) + \sum_{1 \leq |\nu| \leq k} \frac{g^{(\nu)}(\mathbf{b})}{\nu!} (\mathbf{y} - \mathbf{b})^\nu + R_k(\mathbf{y}, \mathbf{b}),$$

with the integral remainder

$$(1.5) \quad R_k(\mathbf{y}, \mathbf{b}) = (k+1) \int_0^1 \left[\sum_{|\nu|=k+1} \frac{g^{(\nu)}(\mathbf{b} + t(\mathbf{y} - \mathbf{b}))}{\nu!} (\mathbf{y} - \mathbf{b})^\nu \right] (1-t)^k dt.$$

Hence, setting $\mathbf{b} = \mathbf{f}(\mathbf{a})$, with $\mathbf{a}$ a given point of $\mathbb{R}^n$, we find

$$(1.6) \quad g(\mathbf{f}(\mathbf{x})) = g(\mathbf{f}(\mathbf{a})) + \sum_{1 \leq |\nu| \leq k} \frac{g^{(\nu)}(\mathbf{f}(\mathbf{a}))}{\nu!} (\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}))^\nu + R_k(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{a})).$$

We shall now obtain $\partial^\alpha(g \circ \mathbf{f})(\mathbf{a})$ by deriving the right-hand side of the expression (1.6) with respect to $\mathbf{x}$ and then setting $\mathbf{x} = \mathbf{a}$. Given $\nu = (\nu_1, \dots, \nu_m) \in \mathbb{N}^m$, such that $|\nu| \geq 1$, from Leibniz rule (1.1), we have

$$(1.7) \quad \partial^\alpha((\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}))^\nu) = \partial^\alpha \left(\prod_{i=1}^m (f_i(\mathbf{x}) - f_i(\mathbf{a}))^{\nu_i} \right) \\ = \sum_{\omega_1 + \dots + \omega_m = \alpha} \frac{\alpha!}{\omega_1! \dots \omega_m!} \left(\prod_{i=1}^m \partial^{\omega_i} ((f_i(\mathbf{x}) - f_i(\mathbf{a}))^{\nu_i}) \right)$$

$$= \sum_{\substack{\boldsymbol{\omega}_1 + \dots + \boldsymbol{\omega}_m = \boldsymbol{\alpha} \\ |\boldsymbol{\omega}_i| \leq |\boldsymbol{\alpha}| \nu_i}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\omega}_1! \dots \boldsymbol{\omega}_m!} \left(\prod_{\substack{1 \leq i \leq m \\ \nu_i \geq 1}} \partial^{\boldsymbol{\omega}_i} ((f_i(\mathbf{x}) - f_i(\mathbf{a}))^{\nu_i}) \right),$$

where in the last expression we took into account that $(f_i(\mathbf{x}) - f_i(\mathbf{a}))^{\nu_i} \equiv 1$ if $\nu_i = 0$, thus

$$(1.8) \quad \partial^{\boldsymbol{\omega}_i} ((f_i(\mathbf{x}) - f_i(\mathbf{a}))^{\nu_i}) = 0 \quad \text{if } \nu_i = 0 \text{ and } |\boldsymbol{\omega}_i| > 0;$$

since $\boldsymbol{\omega}_1 + \dots + \boldsymbol{\omega}_m = \boldsymbol{\alpha}$, the condition $|\boldsymbol{\omega}_i| \leq |\boldsymbol{\alpha}| \nu_i$, added in the last expression of (1.7), simply means that we take $|\boldsymbol{\omega}_i| = 0$ when $\nu_i = 0$.

Furthermore, for $\nu_i \geq 1$, Leibniz rule (1.1) for $k = \nu_i$ and $h_1 = \dots = h_{\nu_i} = f_i(\mathbf{x}) - f_i(\mathbf{a})$ gives that

$$(1.9) \quad \begin{aligned} & \partial^{\boldsymbol{\omega}_i} ((f_i(\mathbf{x}) - f_i(\mathbf{a}))^{\nu_i}) \\ &= \sum_{\boldsymbol{\beta}_1 + \dots + \boldsymbol{\beta}_{\nu_i} = \boldsymbol{\omega}_i} \frac{\boldsymbol{\omega}_i!}{\boldsymbol{\beta}_1! \dots \boldsymbol{\beta}_{\nu_i}!} \left(\prod_{j=1}^{\nu_i} \partial^{\boldsymbol{\beta}_j} (f_i(\mathbf{x}) - f_i(\mathbf{a})) \right) \\ &= \boldsymbol{\omega}_i! \sum_{\boldsymbol{\beta}_1 + \dots + \boldsymbol{\beta}_{\nu_i} = \boldsymbol{\omega}_i} \left(\prod_{j=1}^{\nu_i} \frac{1}{\boldsymbol{\beta}_j!} \partial^{\boldsymbol{\beta}_j} (f_i(\mathbf{x}) - f_i(\mathbf{a})) \right). \end{aligned}$$

For $\mathbf{x} = \mathbf{a}$ we have

$$(1.10) \quad \partial^{\boldsymbol{\beta}_j} (f_i(\mathbf{x}) - f_i(\mathbf{a})) \Big|_{\mathbf{x}=\mathbf{a}} = \begin{cases} 0 & \text{for } |\boldsymbol{\beta}_j| = 0, \\ f_i^{(\boldsymbol{\beta}_j)}(\mathbf{a}) & \text{for } |\boldsymbol{\beta}_j| \geq 1, \end{cases}$$

and also

$$(1.11) \quad \partial^{\boldsymbol{\alpha}} R_k(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{a})) \Big|_{\mathbf{x}=\mathbf{a}} = 0 \quad \text{for } |\boldsymbol{\alpha}| \leq k,$$

since, by (1.5), the remainder $R_k(\mathbf{y}, \mathbf{b})$ in (1.4) satisfies the condition

$$\partial^{\boldsymbol{\nu}} R_k(\mathbf{y}, \mathbf{b}) \Big|_{\mathbf{y}=\mathbf{b}} = 0 \quad \text{for } \boldsymbol{\nu} \in \mathbb{N}^m, |\boldsymbol{\nu}| \leq k.$$

Therefore, applying (1.7), (1.9), (1.10) and (1.11) to (1.6) for $\mathbf{x} = \mathbf{a}$ and $|\boldsymbol{\alpha}| = k \geq 1$, we get the expression (1.2) of $\partial^{\boldsymbol{\alpha}}(g \circ \mathbf{f})$:

$$(1.12) \quad \begin{aligned} & \partial^{\boldsymbol{\alpha}}(g \circ \mathbf{f})(\mathbf{a}) \\ &= \sum_{1 \leq |\boldsymbol{\nu}| \leq |\boldsymbol{\alpha}|} \frac{g^{(\boldsymbol{\nu})}(\mathbf{f}(\mathbf{a}))}{\boldsymbol{\nu}!} \partial^{\boldsymbol{\alpha}}((\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}))^{\boldsymbol{\nu}}) \Big|_{\mathbf{x}=\mathbf{a}} + \partial^{\boldsymbol{\alpha}} R_{|\boldsymbol{\alpha}|}(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{a})) \Big|_{\mathbf{x}=\mathbf{a}} \end{aligned}$$

$$= \alpha! \sum_{1 \leq |\nu| \leq |\alpha|} \frac{g^{(\nu)}(f(\mathbf{a}))}{\nu!} \sum_{\substack{\omega_1 + \dots + \omega_m = \alpha \\ \nu_i \leq |\omega_i| \leq |\alpha| \nu_i}} \prod_{\substack{1 \leq i \leq m \\ \nu_i \geq 1}} \left(\sum_{\substack{\beta_1 + \dots + \beta_{\nu_i} = \omega_i \\ |\beta_j| \geq 1}} \prod_{j=1}^{\nu_i} \frac{f_i^{(\beta_j)}(\mathbf{a})}{\beta_j!} \right),$$

where we have added the conditions $|\beta_j| \geq 1$ in the third sum (because of (1.10)) and, consequently, $\nu_i \leq |\omega_i|$ in the second sum by $\beta_1 + \dots + \beta_{\nu_i} = \omega_i$.

Finally, to prove that $\partial^\alpha(g \circ f)$ satisfies (1.3), we consider the terms inside the parentheses in the second line of (1.12). We claim that, for $i \in \{1, \dots, m\}$ such that $\nu_i \geq 1$,

(1.13)

$$\sum_{\substack{\beta_1 + \dots + \beta_{\nu_i} = \omega_i \\ |\beta_j| \geq 1}} \prod_{j=1}^{\nu_i} \frac{f_i^{(\beta_j)}(\mathbf{a})}{\beta_j!} = \sum_{\substack{\sum_{\beta \in J} k_\beta = \nu_i \\ \sum_{\beta \in J} k_\beta \beta = \omega_i}} \frac{\nu_i!}{\prod_{\beta \in J} k_\beta!} \prod_{\beta \in J} \left(\frac{f_i^{(\beta)}(\mathbf{a})}{\beta!} \right)^{k_\beta},$$

where $J = J(\alpha) = \{\beta \in \mathbb{N}^n \mid 1 \leq |\beta|, \beta \leq \alpha\}$ and $k_\beta \in \mathbb{N}$ for $\beta \in J$. Indeed, we have that

$$(1.14) \quad \left(\sum_{\beta \in J} \frac{f_i^{(\beta)}(\mathbf{a})}{\beta!} \right)^{\nu_i}$$

contains all addends of the sum inside the left-hand side of (1.13) and can be written, by the multinomial theorem, as

$$(1.15) \quad \left(\sum_{\beta \in J} \frac{f_i^{(\beta)}(\mathbf{a})}{\beta!} \right)^{\nu_i} = \sum_{\substack{\sum_{\beta \in J} k_\beta = \nu_i}} \frac{\nu_i!}{\prod_{\beta \in J} k_\beta!} \prod_{\beta \in J} \left(\frac{f_i^{(\beta)}(\mathbf{a})}{\beta!} \right)^{k_\beta}.$$

Therefore, in order to select in (1.14) only the addends that appear in the left-hand side of (1.13), it is necessary and sufficient to add the condition

$$(1.16) \quad \sum_{\beta \in J} k_\beta \beta = \beta_1 + \dots + \beta_{\nu_i} = \omega_i$$

in the right-hand side of (1.15). Then, substituting (1.13) into (1.12) we obtain (1.3), but with the conditions $\nu_i \leq |\omega_i| \leq |\alpha| \nu_i$ in the second sum and $\nu_i \geq 1$ in the first product.

Now, due to the fact that we use the convention that $h^0 \equiv 1$, it is clear that the right-hand side of (1.13) is equal to 1 when $\nu_i = 0$ (and hence $|\omega_i| = 0$, because of $|\omega_i| \leq |\alpha| \nu_i$). Thus, we can avoid the condition $\nu_i \geq 1$ in the first product of (1.3). Furthermore, when in the second sum of (1.3) the condition $\nu_i \leq |\omega_i| \leq |\alpha| \nu_i$ does not hold for some $i \in \{1, \dots, m\}$, the right-hand side of (1.13) is 0. In fact, in this case the conditions $\sum_{\beta \in J} k_\beta = \nu_i$ and

$\sum_{\beta \in J} k_\beta \beta = \omega_i$ cannot both be satisfied, so the sum is empty. This means that in the second sum of (1.3) we can avoid the condition $\nu_i \leq |\omega_i| \leq |\alpha| \nu_i$. $\square$

In the following example we see an application of Theorem 1.1 in the case $n = 1, m = 2$.

EXAMPLE 1.3. Compute the (ordinary) derivative $D^h g(f_1(x), f_2(x))$ for $x \in \mathbb{R}$, $h \in \mathbb{N}$, $h \geq 1$. From (1.3) we have that

$$\begin{aligned} D^h g(f_1, f_2) &= h! \sum_{1 \leq \nu_1 + \nu_2 \leq h} g^{(\nu_1, \nu_2)}(f_1, f_2) \\ &\times \sum_{r_1 + r_2 = h} \left(\sum_{\substack{k_1 + \dots + k_h = \nu_1 \\ 1 \cdot k_1 + 2 \cdot k_2 + \dots + h \cdot k_h = r_1}} \frac{1}{k_1! \dots k_h!} \left(\frac{Df_1}{1!} \right)^{k_1} \dots \left(\frac{D^h f_1}{h!} \right)^{k_h} \right) \\ &\times \left(\sum_{\substack{k_1 + \dots + k_h = \nu_2 \\ 1 \cdot k_1 + 2 \cdot k_2 + \dots + h \cdot k_h = r_2}} \frac{1}{k_1! \dots k_h!} \left(\frac{Df_2}{1!} \right)^{k_1} \dots \left(\frac{D^h f_2}{h!} \right)^{k_h} \right), \end{aligned}$$

since $J = J(h) = \{1, 2, \dots, h\}$.

In the particular case of $g(y_1, y_2) = \frac{1}{y_1 y_2}$, assuming $f_1(x), f_2(x) \neq 0$, we get

$$\begin{aligned} (1.17) \quad D^h \frac{1}{f_1 f_2} &= \frac{h!}{f_1 f_2} \\ &\times \sum_{\substack{1 \leq \nu_1 + \nu_2 \leq h \\ r_1 + r_2 = h}} \sum_{\substack{k_1 + \dots + k_h = \nu_1 \\ 1 \cdot k_1 + 2 \cdot k_2 + \dots + h \cdot k_h = r_1}} \frac{(-1)^{\nu_1} \nu_1!}{k_1! \dots k_h!} \left(\frac{Df_1}{1! f_1} \right)^{k_1} \dots \left(\frac{D^h f_1}{h! f_1} \right)^{k_h} \\ &\times \sum_{\substack{k_1 + \dots + k_h = \nu_2 \\ 1 \cdot k_1 + 2 \cdot k_2 + \dots + h \cdot k_h = r_2}} \frac{(-1)^{\nu_2} \nu_2!}{k_1! \dots k_h!} \left(\frac{Df_2}{1! f_2} \right)^{k_1} \dots \left(\frac{D^h f_2}{h! f_2} \right)^{k_h}, \end{aligned}$$

since

$$g^{(\nu)}(y_1, y_2) = (-1)^{\nu_1 + \nu_2} \frac{\nu_1!}{y_1^{\nu_1 + 1}} \cdot \frac{\nu_2!}{y_2^{\nu_2 + 1}}, \quad \nu = (\nu_1, \nu_2).$$

Note that formula (1.17) may be useful for estimating $|D^h \frac{1}{f_1 f_2}|$ since

$$\sum_{\substack{k_1 + \dots + k_h = \nu_i \\ 1 \cdot k_1 + 2 \cdot k_2 + \dots + h \cdot k_h = r_i}} \frac{\nu_i!}{k_1! \dots k_h!} \leq 2^{\nu_i}, \quad i = 1, 2.$$

Acknowledgement. We thank the referees for their remarks, which led us to clarify several steps in the exposition.

References

- [1] L. F. A. Arbogast, *Du calcul des dérivations*, Levrault (Strasbourg, 1800).
- [2] S. Campanato, *Lezioni di Analisi Matematica*, seconda parte, Libreria Scientifica G. Pellegrini (Pisa, 1977).
- [3] G. M. Constantine and T. H. Savits, A multivariate Faà di Bruno formula with applications, *Trans. Amer. Math. Soc.*, **348** (1996), 503–520.
- [4] F. Faà di Bruno, Sullo sviluppo delle funzioni, *Annali di Scienze Matematiche e Fisiche*, **6** (1855), 479–480.
- [5] L. E. Fraenkel, Formulae for high derivatives of composite functions, *Mat. Proc. Cambridge Philos. Soc.*, **83** (1978), 159–165.
- [6] H. Gzyl, Multidimensional extension of Faà di Bruno’s formula, *J. Math. Anal. Appl.*, **116** (1986), 450–455.
- [7] L. Hernández Encinas and J. Muñoz Masqué, A short proof of the generalized Faà di Bruno’s formula, *Appl. Math. Lett.*, **16** (2003), 975–979.
- [8] R. B. Leipnik and C. E. M. Pearce, The multivariate Faà di Bruno formula and multivariate Taylor expansions with explicit integral remainder term, *ANZIAM J.*, **48** (2007), 327–341.
- [9] K. Spindler, A short proof of the formula of Faà di Bruno, *Elem. Math.*, **60** (2005), 33–35.
- [10] M. Taylor, *Introduction to Analysis in Several Variables–Advanced Calculus*, Pure Appl. Undergrad. Texts, 46, Amer. Math. Soc. (Providence, RI, 2020).

Funding Open access funding provided by Università degli Studi di Ferrara within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.