Highlights

- A Two-phase strategy based on Langlie and D-optimality is proposed for a better fatigue limit estimation.
- The proposed method avoids the fixed step size in the conventional staircase method.
- A simulation-based investigation is performed to analyse the proposed two-phase method.
- The Two-phase is outperformed on estimated quality even with less information on input bounds.

Two-phase optimized experimental design for fatigue limit testing

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Abstract

This study proposes an innovative Two-phase method, based on the Langlie method and the D-optimality criterion, to overcome the intrinsic shortcomings of the staircase method used in estimating the fatigue limit distribution. This paper identifies the current challenges and provides an overview of existing solutions, setting the goal of developing an efficient data collection protocol. It further explains the application of D-optimality criterion and describes the Two-phase protocol, accompanied by a relevant example. The most significant advantage of this approach is its minimal requirement for pre-test information. A simulation-based study was executed to analyze the sensitivity of the input parameters and compare the effectiveness of the proposed method with the traditional staircase and Bayesian optimized method. The numerical simulations reveal that the proposed method offers improved estimation performance for the mean and standard deviation of the fatigue limit distribution, even with minimal pre-test information.

Keywords: Fatigue limit distribution, Staircase method, Langlie method, D-optimality criterion, Sensitivity testing

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Nomenclature

$\Phi(\cdot)$	Cumulative distribution function of the standard normal distribution
$\phi(\cdot)$	Probability density function of the standard normal distribution
$\mathrm{E}(\cdot)$	Expectation
$\mathrm{F}(\cdot)$	Distribution function
$\mathrm{I}(\cdot)$	Fisher information
$\mathrm{P}(\cdot)$	Probability
μ	Mean
σ	Standard deviation
d	Step size in staircase method
d^*	Normalised step size
fl	Fatigue limit (MPa)
i	Ordinal number of specimens
j	Number of specimens in Phase I
n	Number of specimens
s	stress amplitude (MPa)
s_L	Lower bound of stress amplitude (MPa)
s_U	Upper bound of stress amplitude (MPa)
y	Test results (failure or survival)

Subscript

f	specimen failure in the test
s	specimen survival in the test

Glossary

CDF	Cumulative Distribution Function
DM	Dixom-Mood
KDE	Kernel Density Estimation
MLE	Maximum Likelihood Estimation
PDF	Probability Distribution Function

1. Introduction

Multiaxial fatigue criteria such as those of Sines [1] or Crossland [2] are usually expressed in terms of the fatigue limits given for a fixed finite number of cycles. For structures subjected to random loads, Pitoiset et al. [3, 4] and Lambert et al. [5] have developed criteria that that require the correct characterisation of the probability distribution of the fatigue limits. The staircase methods [6, 7, 8] are known for efficiency and ease of application in obtaining these probability distributions.

Dixon and Mood [6] proposed the classical staircase method for sensitivity tests in 1948. This test procedure is recommended for fatigue limit estimates by standards [9] and research [10, 11, 12, 13, 14]. In the staircase procedure, fatigue tests are carried out sequentially on specimens subjected to constant amplitude stress cycles until a predetermined number of cycles N_L. The first specimen is tested at an initial stress amplitude selected arbitrarily. If the specimen "survival" (runout) until N_L cycles, the stress amplitude applied to the next specimen is increased by a step size d. Conversely, the specimen is marked as "failure" and the stress amplitude for the next specimen will be decreased by a step size d. The step size is usually selected to be constant during the entire experiment process. This procedure is repeated in sequence, with stress levels increasing and decreasing until the number of specimens is reached. Therefore, the staircase method reasonably estimates the mean fatigue limit [15].

There is no doubt that the staircase method remains the most widely used fatigue limit method in practice until now. In the literature, the staircase with traditional post-processing evaluation techniques still occupies majority of fatigue limit tests with different materials [16, 17, 18, 19, 20]. However, despite its benefits, the staircase method has significant drawbacks, particularly regarding the data extraction and its impact on fatigue characterization. The sequential nature of the staircase method, coupled with the predetermined selection of the initial value and test step size, inevitably results in a poorly distributed data sample derived from the fatigue test data [21].

Considering the remaining limitations of the staircase method, many studies in the literature have been devoted to improving the approach [22, 15, 23, 24, 25, 26]. These studies have focused on the minimizing the error of the standard deviation estimation value by creating the correct coefficient, improving data processing and optimizing the experimental design. A detailed literature review can be found in Section 2.2.

In the field of mathematical statistics, the tests specially designed to estimate distribution of a physical parameter, such as the fatigue limit, are called sensitivity tests, which aim to infer various critical values that cannot be measured in the field of engineering (e.g. in testing the sensitivity of explosives to shock [27]). The study aim is to draw inspiration from sensitivity testing methods, improve them and apply them in the context of fatigue limit estimate. At present, the sensitivity test methods [28, 29, 30] are mainly including Langlie method [31], Neyer-D method [27], Robbins-Monro method [32], Wu method [33]. The main characteristics of these methods are summarized in Tab. 1.

Robbins and Monro [34] proposed a sensitivity procedure but a large number of specimens are needed. After, Wu [33] and Joseph [32] proposed its adaptive procedure. Langlie [31] published a reliability method in 1962, which is easy to calculate the test levels. However, the disadvantage of the Langlie method is that the selection of the next test level does not take full advantage of the historical information which may result in a wasted sample size. Neyer [27] used D-optimality algorithm for optimizing the test levels in the sensitivity test. The Langlie and Neyer-D (LND) [37] and 3pod [38, 39] methods are recent works in this field, but they require a guessed standard deviation and their application to limit fatigue testing has never been studied.

We drew on previous research (reported in Tab. 1) to help improve the conventional staircase

Table 1: Summary of sensitivity test methods

Method	Advantages	Disadvantages
Robbins-Monro [34]	- good estimation on specific quantile point	- a large number of specimens are required
Probit [35]	- only one specimen	- no information on standard deviation
Langlie [31, 36]	- requires only guessed boundings - attempts to equal survivals and failures - easy to calculate the test levels	- suboptimal convergence - extrapolation outside search range not allowed - wider search range reduces efficiency - the standard deviation estimation is greatly affected by range
Neyer-D [27]	- use prior test to optimize the next test level	- require a pre-guessed scale parameter - affected by the type of distribution
Wu [33]	- a natural analog of the Robbins—Monro method - good estimation with small sample size	- require a design for initial conditions
Langlie-Neyer-D [37]	- combines the Langlie method and Neyer-D method	- require a pre-guessed scale parameter
3pod [38]	- good estimation on specific quantile point	- require a pre-guessed scale parameter

method [40, 22, 41, 42]. In this work, we propose a method based on the D-optimality criterion in order to obtain a better selection of stress levels. The Two-phase method implements the Langlie [31] and the Neyer-D procedures [27], to improve the fatigue limit test protocol. With the application of D-optimality criterion, the step size in the proposed strategy is not constant and is calculated after every single test. Moreover, it reduces the number of the inputs for the test and is insensitive to the predefined load bounds. The detailed problem statement and the proposed improvement are presented in Section 3. The acquisition of better estimates of the fatigue limit distribution may contribute to the development of more efficient multiaxial fatigue criteria or reliability estimation [5, 43].

This paper is structured as follows: a background of the problem in the staircase method and the existent possible solutions are summarized in Section 2. The proposed Two-phase test method, including detailed test protocol and an example, are presented in Section 3. A simulation procedure based on Monte-Carlo simulation is described in Section 4. In Section 5, a sensitivity analysis of the input parameters is performed, and the effectiveness of the proposed method is validated by comparing it to the conventional staircase and other latest methods. Finally, a brief conclusion and future work are given in Section 6.

2. Literature review

This section collects some of the most significant research about staircase method in the field of fatigue limit test. Our aim is to highlight the limitations of the conventional staircase method, to trace the development trajectory of research related to the staircase methods, and define our research objectives. In this section, some significant methods are introduced, that will be used for comparison analysis in numerical simulation studies.

2.1. The traditional staircase method and problem statement

Following staircase testing, an evaluation technique must be used to obtain the fatigue limit distribution parameters. The Dixon-Mood (DM) method [6] provides classical equations to calculate the mean and standard deviation. It assumes that the fatigue limit should follow the normal distribution. Only the less frequent events, failure or survival, are used to evaluate the distribution. The stress amplitude span is split by a step size d into several stress levels numbered by l, where $l = 0$ stands for the lowest stress level and l_{max} stands for the maximum stress level. Denoting $n_{c,l}$ as the number of the fewer frequency events (survival or failure) at the stress level l, two auxiliary values A and B can be calculated by Eq. 1:

$$
\begin{aligned}
A &= \sum_{l=0}^{l_{max}} l \times n_{c,l} \\
B &= \sum_{l=0}^{l_{max}} l^2 \times n_{c,l} \\
n_c &= \sum_{l=0}^{j_{max}} n_{c,l}
\end{aligned}
\tag{1}
$$

The auxiliary values are used to estimate the mean value μ_{DM} with Eq. 2 and the standard deviation σ_{DM} with Eq. 3.

$$
\mu_{DM} = s_0 + d\left(\frac{A}{n_c} \pm \frac{1}{2}\right) \tag{2}
$$

$$
\begin{aligned}
\sigma_{DM} &= 1.62 \times d\left(\frac{Bn_c - A^2}{n_c^2} + 0.029\right) \quad \text{if} \quad \frac{Bn_c - A^2}{n_c^2} \geqslant 0.3 \\
\sigma_{DM} &= 0.53 \times d \qquad\qquad\qquad\qquad\quad \text{if} \quad \frac{Bn_c - A^2}{n_c^2} < 0.3
\end{aligned}
\tag{3}
$$

In Eq. 2, s_0 is first stress level and the minus sign is used if the failed specimens are evaluated and otherwise, the plus sign is applied.

Following the staircase method procedure, all stress levels are controlled by a constant, i.e. the step size. From the literature [44, 45] and authors' previous study [42], the staircase method gives poor results for estimating the standard deviation because of the constant size of the steps, particularly for tests on small samples. On the one hand, it is more difficult to obtain an accurate measure of dispersion since results are clustered in the central region of the distribution, whereas only an insufficient number of results corresponding to low probabilities are available. On the other hand, since the step size is fixed in advance, the width of the staircase test data may not reflect the true standard deviation. Otherwise, a large number of specimens are required to accurately estimate the fatigue limit.

Under the assumptions of DM [6, 46], the step size must remains constant for each stress level and must be between 0.5σ and 2σ, σ corresponding to the standard deviation of the fatigue limit. Otherwise, a number of studies have begun to investigate the step size in the staircase method. The "5% criterion" [21] is a consensus and from engineering practice. When the fatigue limit determined by the conventional fatigue test method is known, the 5% of the fatigue strength can be taken as the increment. However, this assumption is conditional on the constant increment being less than twice the actual standard deviation of the tested population. Grove and Campean [47] assumed that it is more robust to the choice of the step size in the range $\sigma - 2\sigma$. Pollak et al. [26] carried out a characterisation simulation study for an increment range and found that using larger step sizes in the $1.6 - 1.75\sigma$ range can reduce the estimation bias.

Even after a large number of investigations, it is still hard to decide on the step size before the test, even though it has a major influence on the estimation of the standard deviation [48].

2.2. Advances in improving the staircase method

Several researchers have investigated with the aim to minimize the error of the standard deviation estimation value. In this paper, all these studies are classified as different topics including creating the correct coefficient, improving the data processing and optimizing the experimental design.

Early research proposed corrections to the DM method to reduce the effect of the step size, such as Svensson-Lorén (SL) [26], Pollak (PO) [48] and Braam-Zwaag (BZ) [49] corrections. Moreover, Pollak et al. [26] formulated a non-linear correction on the standard deviation of fatigue limit and involved bootstrapping sampling for a small number of specimens. By the study in Reference [40], the correction factor proposed by SL [26] gives a better improvement to the standard deviation estimation:

$$\sigma_{SL} = \frac{n}{n-3} \times \sigma_{DM} \tag{4}$$

where σ_{SL} is the standard deviation corrected by SL method, σ_{DM} is the standard deviation estimated by DM method in Eq. 3. The SL correction is used for comparison in the simulation-based investigation.

Over the last few decades, research has focused on improving the data processing method after testing. Zhang-Kececioglu method [50] obtained the distribution of the fatigue limit by calculating the coefficient of one–sided tolerance limit. Zhao and Yang [51] developed a general maximum likelihood approach (GMLA) to assess the fatigue limit by constructing physically paired local S-N relations for all failure or survival specimens from staircase tests. Wallin [21] applied a binomial probability function to estimate the standard deviation. Müller et al. [24] compared several evaluation techniques through Monte-Carlo simulations and demonstrated that the IABG (Industrieanlagen-Betriebsgesellschaft) method provides an estimator by omitting invalid test results and adding fictitious data. Çalişkan and Gürbüz [23] summarized different data processing methods to determine the fatigue limit of steels.

Recently, the evaluation method based on Kernel Density Estimation (KDE) was proposed by Shi et al. [22] to estimate the fatigue limit distribution from staircase tests without prior knowledge. If $\mathbf{S} = \{s_1, s_2, ..., s_n\}$ denote all the stress amplitudes in the staircase test, the Probability Density Function (PDF) of the fatigue limit distribution estimated by KDE is:

$$\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} \mathrm{K}\left(\frac{x - s_i}{h}\right). \tag{5}$$

where, $h > 0$ is a smoothing parameter called bandwidth, K is the non-negative Kernel function using the standard Gaussian Kernel function. Further information on the KDE method and its application can be found in [22, 41].

However, these researches focused only on evaluation techniques, and the fatigue scatter is still strongly dependent on the step size. The most recent studies focus on improving the test protocol in order to avoid fixed step size. Wallin [21] proposed a binomial probability-based method with a small step size, but the inaccurate estimation of the fatigue scatter remains. Roué et al. [15] developed a new experiment staircase procedure and reduced the uncertainty of standard deviation estimation by reloading unbroken specimens. This procedure extracts more information from the fatigue damage process, but a pre-estimated step size is still required.

Currently, the application of Bayes' theorem provides more reliable material fatigue limits without a fixed step sizes. Engler-Pinto et al. [52] proposed the Bayesian staircase strategy, then incorporated it into Life-Regression Models (S-N curve) [53]. Alcalá-Quintana and García-Pérez [54] carried out a simulation study to compare the fixed-step-size and Bayesian optimized methods, and he found that the standard deviation of the Bayesian estimates is more reliable than that of the conventional staircase in the same condition.

Recent research by Magazzeni et al. [55] has proposed protocols using Bayesian maximum entropy sampling, where all test levels are determined by prior information and the step size parameter is eliminated by Bayesian theory. This method is the latest and most advanced study for experimental fatigue limit test. The main idea is to incorporate a posterior information for

testing the next specimen. Bayesian maximum entropy sampling contributes to predict the expected result by maximizing the entropy gain. After testing i specimens, the stress level for the $(i+1)^{th}$ specimen can be selected by maximizing the expected gain of Shannon information from the posterior. The function U to be maximized with the stress amplitude s_{i+1} is defined as:

$$U(s_{i+1}) = H^s(s_{i+1}, \boldsymbol{S})p^s(\boldsymbol{S}|s_i) + H^f(s_{i+1}, \boldsymbol{S})p^f(\boldsymbol{S}|s_i) \tag{6}$$

where, U is the objective function that describes the entropy gain with the result y_{i+1}. $\boldsymbol{S}$ is a vector of all previous tests. The superscript "s" represents the "survival" and superscript "f" represents the "failure". Further information on Bayesian optimized method can be found in [40, 55].

2.3. Research objective

In summary, all studies aimed at obtaining a correct coefficient and improving the data processing, as well as some studies aimed at optimizing the experimental design have not succeeded in avoiding the fixed step size in the experimental test. Only the Bayesian optimized method has the idea of using a variable step size by optimization.

Therefore, improving the staircase experimental protocol is the current direction of research, among which the use of variable step sizes is considered an important advance. By learning from sensitive analysis methods (as described in Tab. 1), this study focuses mainly on the implementation of Langlie method and D-optimality criterion to improve the conventional staircase protocol, in which the stress step size is not constant and is calculated after every test. The sensitivity analysis methods are normally applied to determine the shock levels to explosives [33], the the ignition energy of dust cloud [36], and so on. To our knowledge, Langlie method and D-optimality has not been applied to the fatigue limit estimate.

3. Two-phase experimental protocol for fatigue limit test

To estimate the distribution of fatigue limits, a Two-phase method is proposed. The implementation stages are illustrated in the flowchart of Fig. 1. In this study, specimens are assumed to have fatigue limit values originating from a normal distribution determined by mean μ and standard deviation σ.

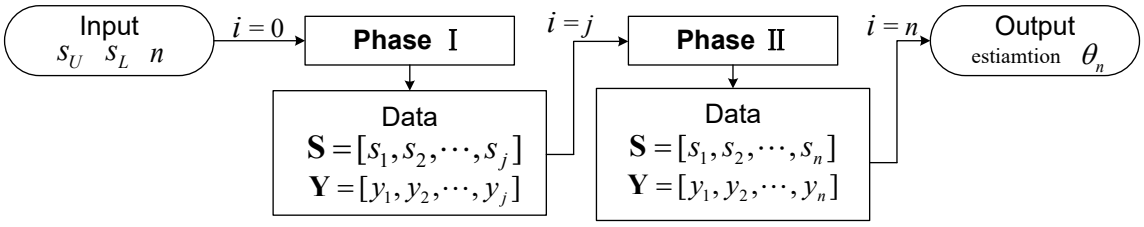

Figure 1: Implementation stages of the Two-phase method

As described in Fig. 1, this Two-phase method is divided into two phases using different procedures. For the start with, only three parameters need to be pre-defined as input: the upper bound s_U, the lower bound s_L and the total number of specimens n. Specimens are tested using the phase I procedure presented in Section 3.1. The stress level s_i is obtained on the specimen i, as well as response y_i (which gives a binary outcome of survival or failure). A stopping criterion was defined for the first phase and total j specimens were tested. After this, phase II is initiated to continue the experiment tests until all n specimens have been tested.

At the end, the stress levels and responses of all specimens are obtained and the fatigue limit distribution with parameter θ_n is then deduced, as reported in Section 3.2.

3.1. Phase I based on Langlie method

The Langlie method [31] is a sensitivity test method with a variable step size. The goal is to achieve an equal number of survival ($y = 0$) and failure ($y = 1$) outcomes, thereby equalizing the probability of experimental stress levels being either above or below the fatigue limit. Since the step size is adjusted during the experiment, the load value can quickly converge near the mean of the fatigue limit distribution. Consequently, the Langlie method is more advantageous for obtaining the sample information of the mean value [36].

The phase I follows Langlie test procedure as shown in Fig. 2.

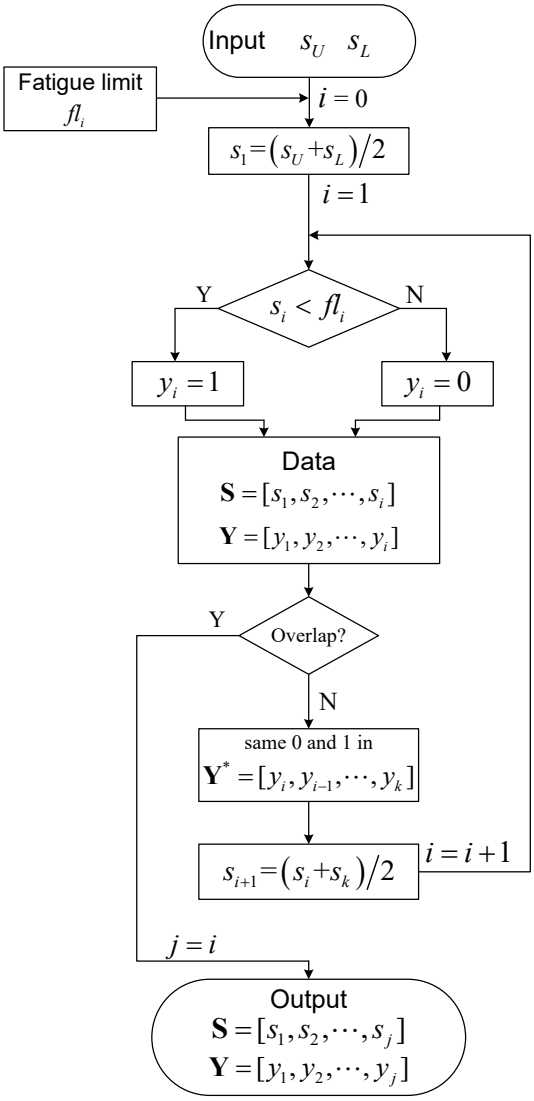

Figure 2: Flow chart of Phase I

s_U and s_L are the upper and lower bounds that defined before the test. The fatigue limit fl, indexed by i, comes from specimens in experimental test. The steps of phase I are as follows:

Step 1: Conduct the first test at an average level of upper and lower bounds:

$$s_1 = \frac{s_U + s_L}{2} \tag{7}$$

Step 2: After having completed i^{th} trials ($(i \geqslant 1)$), the stress levels $\mathbf{S} = \{s_1, s_2, ..., s_i\}$ and corresponding test results $\mathbf{Y} = \{y_1, y_2, ..., y_i\}$ are obtained. The rule for obtaining the $(i+1)^{th}$ stress level is to work backwards through the test sequence, starting at the i^{th} test, until a previous trial (defined as k^{th} test), for which there are as many "survivals" as "failures" between the k^{th} and the i^{th} trials. Hence, the selection of the next stress level is the average of s_i and s_k as:

$$s_{i+1} = \frac{s_i + s_k}{2} \tag{8}$$

If no such k exists, then x_k is assigned as s_U or s_L:

$$s_k = \begin{cases} s_L & \text{if} \quad y_i = 1 \\ s_U & \text{if} \quad y_i = 0 \end{cases} \tag{9}$$

Particularly, the second test at s_2 is determined by :

$$s_2 = \begin{cases} \dfrac{s_1 + s_L}{2} & \text{if} \quad y_1 = 1 \\ \dfrac{s_1 + s_U}{2} & \text{if} \quad y_1 = 0 \end{cases} \tag{10}$$

Step 3: Check the overlap of the test results $\mathbf{Y}$. If yes, then stop Phase I, otherwise, return to step 2.

The overlap check, used as the stopping criteria for Phase I, includes the following: (1) At least one survival result has a higher value than a failure result; (2) The average stress levels of failures are larger than that of survivals. The overlap is a sufficient and necessary condition for the maximum likelihood estimation for the test data [56, 57].

In summary, the objective of the Phase I design is to quickly identify a reasonable experimental range that contains the mean value of the fatigue limit and to move the stress levels to achieve an overlapping pattern. Assume that Phase I stops at the j^{th} trial, we have stress levels $\mathbf{S} = \{s_1, s_2, ..., s_j\}$ and results $\mathbf{Y} = \{y_1, y_2, ..., y_j\}$, which will be used as input for Phase II.

3.2. Phase II based on Neyer-D method

Phase II procedure is constructed based on the Neyer-D method. When applied to designing an experiment for fatigue limit, the aim is to select a stress level, denoted as s, that offers the most insightful data for estimating the parameters of the fatigue limit distribution. D-optimality is a criterion that seeks to maximize the determinant of the Fisher information matrix. Its objective is to minimize the volume of the confidence region for the parameters. This can be interpreted as an endeavor to maximize the experiment's informational yield about these parameters.

Phase II adheres to the Neyer-D test procedure, as depicted in Fig. 3. The results and tested levels derived from Phase I serve as the inputs. Initially, the mean and standard deviation of the set $\mathbf{S} = \{s_1, s_2, ..., s_j\}$ are estimated using Maximum Likelihood Estimation (MLE). These estimates are then utilized within the D-optimality framework to determine s_{j+1}. This sequence of steps is iteratively executed until n specimens have been tested.

The steps about MLE and D-optimality are described in the following subsections.

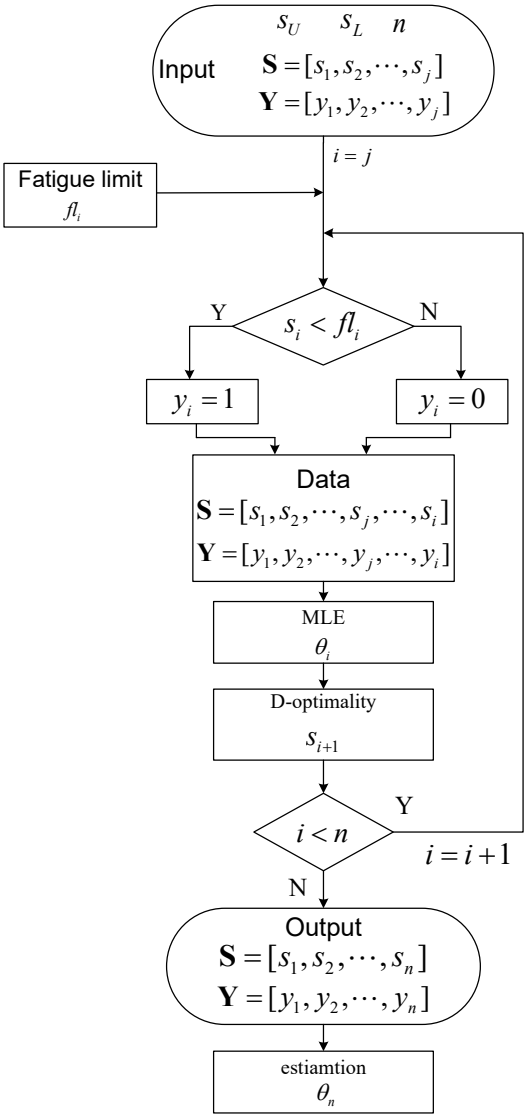

Figure 3: Flow chart of Phase II

3.2.1. Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) is a statistical method with a goal to find the parameter values that maximize the likelihood function, which is used in the fatigue limit characterization [15, 24].

Without loss of generality, the fatigue limit is assumed to follow a distribution with the following Cumulative Density Function (CDF) $\mathrm{F}(\cdot)$:

$$\mathrm{F}(z) = \mathrm{F}(\frac{x-\mu}{\sigma}) \tag{11}$$

where μ and σ are mean and standard deviation of the distribution.

For the fatigue limit test, the failed and survived specimens give information about the probability:

$$\begin{aligned} \mathrm{P}(x > z) &= 1 - F(z) \\ \mathrm{P}(x \leqslant z) &= F(z) \end{aligned} \tag{12}$$

Let $\theta = (\mu, \sigma)$, the maximum likelihood function L based on Eq. 11 is given by:

$$\mathrm{L}(\theta) = \prod_{i=0}^{n} \mathrm{F}(z_i)^{y_i} \left(1 - \mathrm{F}(z_i)\right)^{1-y_i} \tag{13}$$

where y_i is the result (failure=1, survival=0) of the i^{th} specimen.

The maximization of L results in the estimation of the distribution parameters. The logarithm of the minimized equation is strongly recommended in order to reduce some numerical computational problems. The new likelihood function is:

$$l(\theta) = \mathrm{lnL}(\theta) = \sum_{i=1}^{n} [y_i \mathrm{lnF}(z_i) + (1 - y_i)\mathrm{ln}(1 - \mathrm{F}(z_i))] \tag{14}$$

where "ln" holds for natural logarithm.

3.2.2. Fisher information

In this study, the Fisher information serves as a measure of the amount of information that an observable test samples can provide about the unknown fatigue limit distribution. The Fisher information matrix [27, 58] can be obtained by computing the expectation of the derivative of the logarithm of the likelihood function as Eq. 15:

$$\mathrm{I}(\theta) = \mathrm{E}\left(\frac{\partial l(\theta)}{\partial \theta_p} \frac{\partial l(\theta)}{\partial \theta_q}\right) \tag{15}$$

For the fatigue limit estimation under normal distribution, $\theta_1 = \mu$ and $\theta_2 = \sigma$. Hence, Eq. 15 could be rewritten as:

$$\mathrm{I}(\theta) = \begin{bmatrix} \mathrm{I}_{11} & \mathrm{I}_{12} \\ \mathrm{I}_{21} & \mathrm{I}_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{E}(\frac{\partial l}{\partial \mu}\frac{\partial l}{\partial \mu}) & \mathrm{E}(\frac{\partial l}{\partial \mu}\frac{\partial l}{\partial \sigma}) \\ \mathrm{E}(\frac{\partial l}{\partial \mu}\frac{\partial l}{\partial \sigma}) & \mathrm{E}(\frac{\partial l}{\partial \sigma}\frac{\partial l}{\partial \sigma}) \end{bmatrix} \tag{16}$$

Taking Eq. 14 into Eq. 16, the Fisher information can be expressed as Eq. 17. The detailed derivation is presented in Appendix.

$$\mathrm{I}(\theta) = \begin{bmatrix} \mathrm{I}_{11} & \mathrm{I}_{12} \\ \mathrm{I}_{21} & \mathrm{I}_{22} \end{bmatrix} = \sum_{i=1}^{n} \frac{[\mathrm{F}'(z_{i+1})]^2}{\mathrm{F}(z_{i+1})(1 - \mathrm{F}(z_{i+1}))\sigma^2} \begin{bmatrix} 1 & z_{i+1} \\ z_{i+1} & z_{i+1}^2 \end{bmatrix} \tag{17}$$

3.2.3. D-optimality

In the context of the fatigue limit test, after having completed i^{th} trials ($i \geqslant 1$), there are stress levels $\mathbf{S} = \{s_1, s_2, ..., s_i\}$ and corresponding test results $\mathbf{Y} = \{y_1, y_2, ..., y_i\}$. The estimated θ by MLE after i^{th} specimen is donated as $\hat{\theta}_i$. By Eq. 17, the information for the $(i+1)^{th}$ specimen can be deduced by the previous tests:

$$\mathrm{I}_{i+1}(\theta) = \mathrm{I}_i(\theta) + \frac{[\mathrm{F}'(z_{i+1})]^2}{\mathrm{F}(z_{i+1})(1 - \mathrm{F}(z_{i+1}))\sigma^2} \begin{bmatrix} 1 & z_{i+1} \\ z_{i+1} & z_{i+1}^2 \end{bmatrix} \tag{18}$$

According to Cramér-Rao inequality [59], the variance of a unbiased estimator $\hat{\theta}$ of θ is bounded by the inverse of Fisher information $\mathrm{I}(\theta)$. Therefore, the covariance matrix of any unbiased estimator $\hat{\theta}$ satisfies:

$$\mathrm{Cov}(\hat{\theta}) \geqslant \frac{1}{\mathrm{I}(\theta)} \tag{19}$$

That is, the Fisher information matrix measures the size of the confidence domain of the estimate parameters $\hat{\theta}$. The object function can be expressed as:

$$\max \quad |\mathrm{I}_{\mathrm{i}+1}(\hat{\theta}_{\mathrm{i}})| \tag{20}$$

With a normal distribution $\theta = (\mu, \sigma)$, the stress level s_{i+1} for $(i+1)^{th}$ specimen can be selected by maximizing the determinant of the Fisher information matrix:

$$g_{i+1}(x|\hat{\mu}_i, \hat{\sigma}_i) = |\mathrm{I}_{i+1}(\hat{\theta}_i)| = \left| \mathrm{I}_i(\hat{\mu}_i, \hat{\sigma}_i) + \frac{[\phi(z_{i+1})]^2}{\Phi(\hat{z}_{i+1})(1-\Phi(\hat{z}_{i+1}))\sigma_i^2} \begin{bmatrix} 1 & \hat{z}_{i+1} \\ \hat{z}_{i+1} & \hat{z}_{i+1}^2 \end{bmatrix} \right| \tag{21}$$

where z is defined as:

$$\begin{aligned} \hat{z}_i &= \frac{s_i - \hat{\mu}_i}{\hat{\sigma}_i} \\ \hat{z}_{i+1} &= \frac{x - \hat{\mu}_i}{\hat{\sigma}_i} \end{aligned} \tag{22}$$

In Eq. 21 and Eq. 22, ϕ and Φ are the probability density function (PDF) and cumulative density function (CDF) of the standard normal distribution. $\hat{\mu}_i$ and $\hat{\sigma}_i$ are the estimated mean and estimated standard deviation after i tests. x is the variable to be calculated, representing the stress level s_{i+1} for the next test.

In summary, D-optimality in the context of experimental design is a method to choose the best experiment to run based on the criterion of maximizing the determinant of the Fisher information matrix. In this study, the optimization function is calculated by "L-BFGS-B" algorithm [60], and the value corresponding to the maximum Fisher is the expected stress level for the next specimen.

3.3. Numerical example of the Two-phase test procedure

The simulation study of the proposed Two-phase method is conducted on the basis of Algorithm 1. As in the real experiment, three parameters must be determined in advance, including upper and lower bounds and the number of specimens. The fatigue limit value for each tested specimen, which are unknown in the experiments, are sampled from a normal distribution. Similar to the conventional staircase method, the ordinal number of the simulated specimens is marked as i. The stress level for the i^{th} specimens is generated successively concerning the procedures in Phase I. The fatigue limit of the i^{th} specimen is randomly extracted from the fatigue limit set in the first step. This value is then compared to the stress level of the i^{th} specimen. As in experimental tests, the specimen is considered as "survival" if the applied stress level is below the fatigue limit. Otherwise, the specimen is "failure". Then, the stress level for the $(i+1)^{th}$ specimen is selected and compared until there is an overlap in the stress levels.

After that, the current stress levels and results are imported as the initial conditions of Phase II. In this Phase, the stress level for the $(i+1)^{th}$ specimen is optimized by maximizing Fisher information and then compared to the simulated fatigue limit. The above steps are repeated until n specimens are tested.

Algorithm 1: Two-phase protocol

Data: input: Upper bound s_U, Lower bound s_L, the number of specimens: n
Prepare specimens and the corresponding fatigue limits $[fl_1, fl_2, ..., fl_n]$
`// Phase I`
The stress level for the first test: $s_1 = (s_U + s_L)/2$
foreach i **do**
  Compare s_i with fl_i
  Append $[s_1, s_2, ..., s_i]$ and $[y_1, y_2, ..., y_i]$
  if overlap **then**
    $j = i$
    Store $[s_1, s_2, ..., s_j]$ and $[y_1, y_2, ..., y_j]$
    Finish Phase I
  else
    Reverse $[y_1, y_2, ..., y_i]$
    if Find k that there are as many 0 as 1 in the range of $[y_i, y_{i-1}, ..., y_k]$ **then**
      Determine $s_{i+1} = (s_i + s_k)/2$
    else
      if $y_i = 1$ **then**
        $s_k = s_L$
      else
        $s_k = s_U$
      end
      Determine $s_{i+1} = (s_i + s_k)/2$
    end
  end
end
`// Phase II`
foreach i in $[j, n]$ **do**
  Obtain $\theta_i = (\mu_i, \sigma_i)$ by MLE
  Create Fisher information
  Create object function based on determinant of Fisher information matrix
  Maximize the object function $g(x)$
  Determine $s_{i+1} = x$
end
Result: All stress levels $\mathbf{S} = \{s_1, s_2, ..., s_n\}$ and results $\mathbf{Y} = \{y_1, y_2, ..., y_n\}$
Estimation θ_n

Following the procedures in Algorithm 1, an example of $n = 30$ samples is presented. The test assumes that the fatigue limit follows a normal distribution with true mean $\mu_0 = 400$ and true standard deviation $\sigma_0 = 10$. Two initial parameters for the Two-phase method are selected as $s_L = 200$ and $s_U = 600$ for simulation study. The up-and-down diagram is shown in Fig. 4.

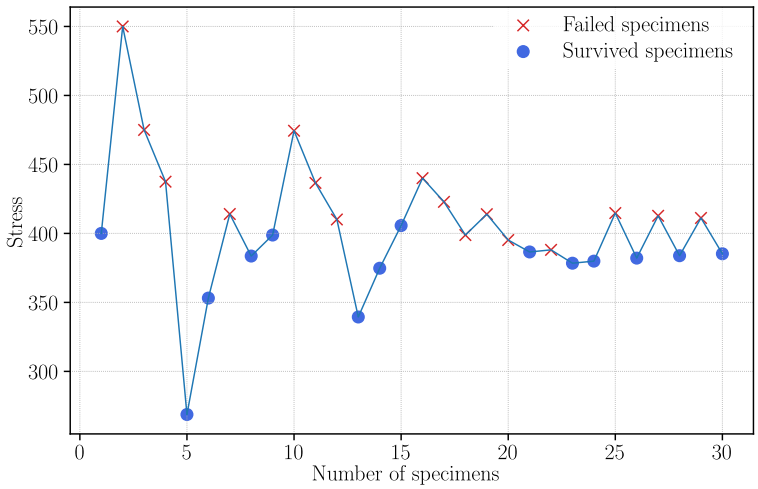

Figure 4: Up-and-down diagram of a simulated two-phase test

Similar to the conventional staircase procedure, the specimens are tested separately and sequentially. The first 18 specimens in Phase I perform a preliminary search of the mean fatigue limit, and the test data gradually converge toward the solution. After posterior convergence to a single estimated distribution, the D-optimality picks the stress levels for each sample using the previous Fisher information. Ultimately, the estimated mean and standard deviation values with MLE represent the fatigue limit distribution. The evolution of MLE estimation during the test is reflected in Fig. 5.

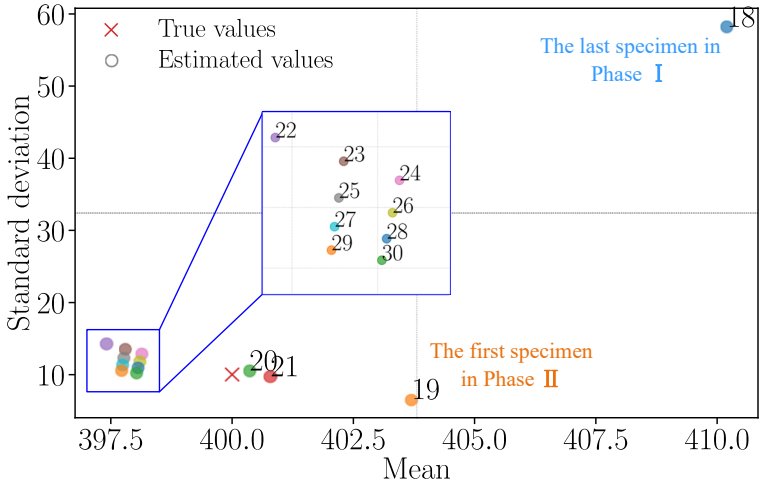

Figure 5: Evolution of estimated mean and standard deviation value (the ordinal number 19 to 30 of specimens in Phase II are marked)

As shown in Fig. 5, the last specimen (marked as 18) from Phase I deviates significantly from the true values, particularly in the estimation of the standard deviation. However, after applying the D-optimality approach, the estimated standard deviation (marked as 19 in Phase II) becomes substantially closer to the actual value. Thereafter, the estimates progressively converge towards the true values for the data points ranging from 22 to 30.

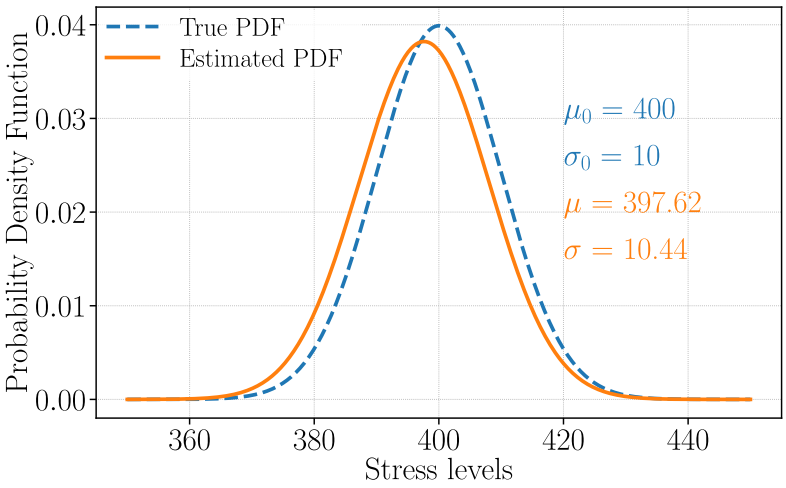

Figure 6: Estimated fatigue limit PDF

The final estimated PDF is compared with the true PDF in Fig. 6. From this figure, it is apparent that the estimated PDF closely matches the true PDF. For further comparison, numerical simulations are conducted in Section 4 to demonstrate the effectiveness of the Two-phase method.

4. Numerical validation of the proposed test protocol

In order to evaluate the effectiveness of the proposed Two-phase test method, a comparison study is conducted. The numerical simulation is configured as shown in Fig. 7.

Fatigue limit is a random variable due to variations in material properties, loading conditions, and other factors in experiment [61, 62]. Using a distribution to describe the fatigue limit allows for the quantification of these variations [63]. For the comparison, all the simulated fatigue limits are sampled by a normal distribution with the true mean value μ_0 and the true standard deviation σ_0. Here, the μ_0 and σ_0 are only used as benchmarks for comparison in numerical simulations.

In order to evaluate the effectiveness of Two-phase method in fatigue limit estimation, two other methods, the conventional staircase method, and Bayesian optimized method as described in Section 2.2, are constructed. It should be noted that all results obtained from the staircase simulation are post-processed by two different evaluation techniques, including Dixon-Mood (DM) method and Kernel Density Estimation (KDE) method as described in Section 2.1 and Section 2.2 respectively.

For each method, a total number of $m = 1000$ trials have been performed to investigate the quality of the estimates with respect to the number of specimens $n \in [10, 15, 20, 25, 30, 35, 40, 50, 70, 100]$. Although it's unrealistic that conducting real fatigue tests on 100 specimens due to the associated costs and time constraints, these numbers of specimens chosen for the numerical study is intended to make the example more comprehensive. Each method estimates the mean value μ^k and standard deviation σ^k for the k^{th} simulation test with $k = 1, \cdots, m$, and m is set as 5000 in this numerical study.

The quality of estimation is evaluated by normalising the estimated mean value μ_k and the estimated standard deviation s_k to the true mean value μ_0 and the true standard deviation σ_0 of the given probability distribution, respectively. The 5^{th}, 50^{th} (median), and 95^{th} percentiles of all μ results are used to evaluate the uncertainty of the compared method performance.

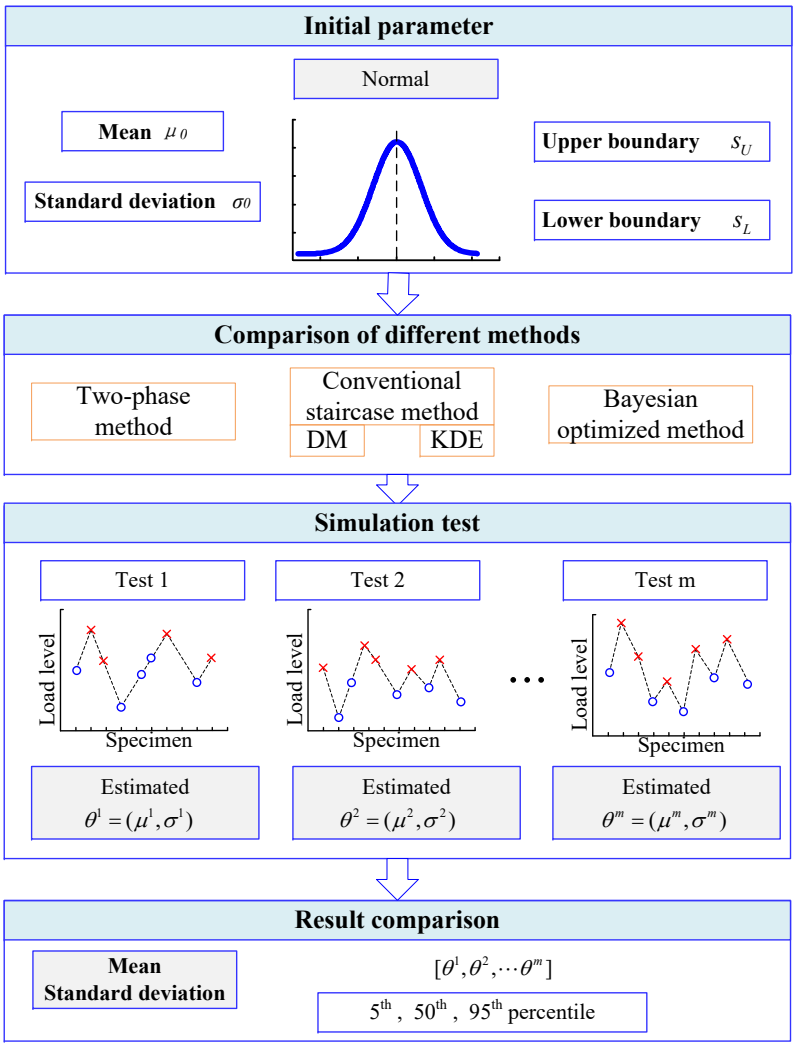

Figure 7: Simulation procedure [22]

5. Results and discussion

5.1. Sensitivity analysis of bounds in the Two-phase method

For this new proposed strategy, it is necessary to analyze the sensitivity to input parameters, including the lower and upper bounds. To simplify the comparison, a scale parameter r is defined as the ratio of the difference between the upper and lower bounds normalized by the true mean value:

$$r = \frac{s_U - s_L}{2\mu_0} \tag{23}$$

Table 2: Different input parameters

Lower bound s_L	Upper bound s_U	r
300	500	0.25
200	600	0.5
100	700	0.75
0	800	1

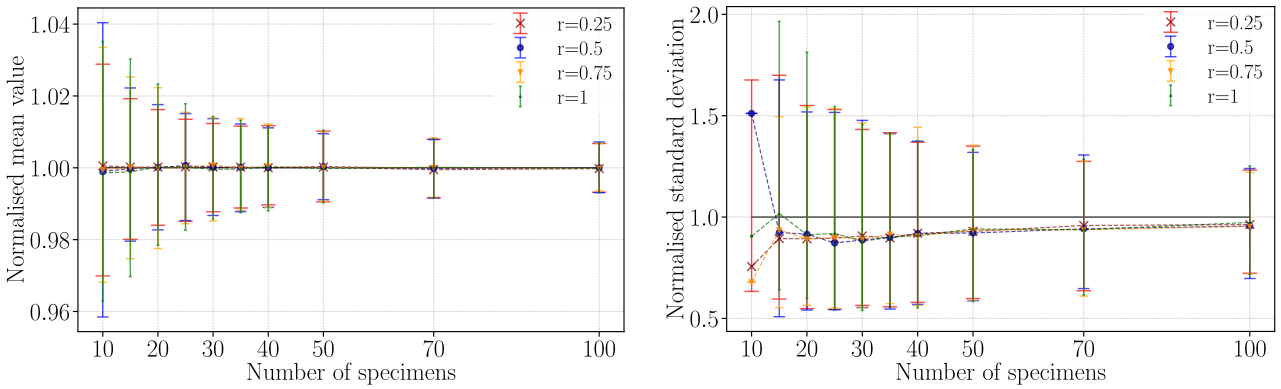

Figure 8: Normalised (a) mean and (b) standard deviation value from Two-phase method results for different range of lower and upper bounds

Fig. 8 shows the mean and standard deviation estimations for the different numbers of specimens. In this figure, $r = 0.25$, $r = 0.5$, $r = 0.75$ and $r = 1$ (as listed in Tab. 2) are illustrated by the red, blue, orange and green line, respectively. Each vertical bar represents the 5^{th} and 95^{th} percentiles and the central marker represent the 50^{th} percentile of a total of m trials. For a pair of mean and standard deviation, the estimated results of the Two-phase method become more accurate with more specimens. Indeed, we can observe from Fig. 8(a), the difference occurs only for several specimens less than 30. For the test with a number of specimens greater than 30, the input range has almost no influence on the estimation results. It is possible to choose between a small value (even much smaller than the fatigue limit) and the yield stress as the lower and upper bounds in a real experiment. The same findings can be observed in Fig. 8(b). The estimated standard deviation does not rely on the input range.

Significant deviations exist between the estimated and true values for the standard deviation estimation when the number of specimens is lower than 15. The horizontal dash is below the horizontal black line. This means the Two-phase method has the disadvantage of underestimating the standard deviation.

From Fig. 8, the estimate values, for both mean and standard deviation, have a trend that approaching a horizontal asympthote. Every additional specimen tested contributes further information about the fatigue limit distribution (thereby reducing the epistemic uncertainty). However, the rate of convergence towards the true estimates is not particularly rapid for a very large number of specimens. This observation supports the effectiveness of the Two-phase method, as it suggests that there is no significant need to test more than 40 specimens. Beyond this number of specimens, the amount of information gained does not increase substantially.

In summary, the Two-phase method exhibits low sensitivity to input parameters when a sufficient number of specimens ($n \geqslant 30$) are available for testing. In practical experiments, a range of 30-40 specimens proves to be sufficient.

5.2. Comparison with the staircase method

To make a general comparison, the normalized step size [26, 15] d^* is defined as the ratio of step size d and true standard deviation σ_0. Due to the step size only existing in the conventional staircase method, the $d^* = 0.5, 1.0, 1.5$ are selected for comparison. The DM and KDE methods are applied to estimate the fatigue limit distribution from the staircase test. The conventional staircase is simulated using the procedure described in Section 1.

To provide a proper comparison, the input for the Two-phase method is fixed to the worst situation as $r = 1$. Based on simulations, the estimated mean and standard deviation are shown

392 in Fig. 9, Fig. 10, and Fig. 11.

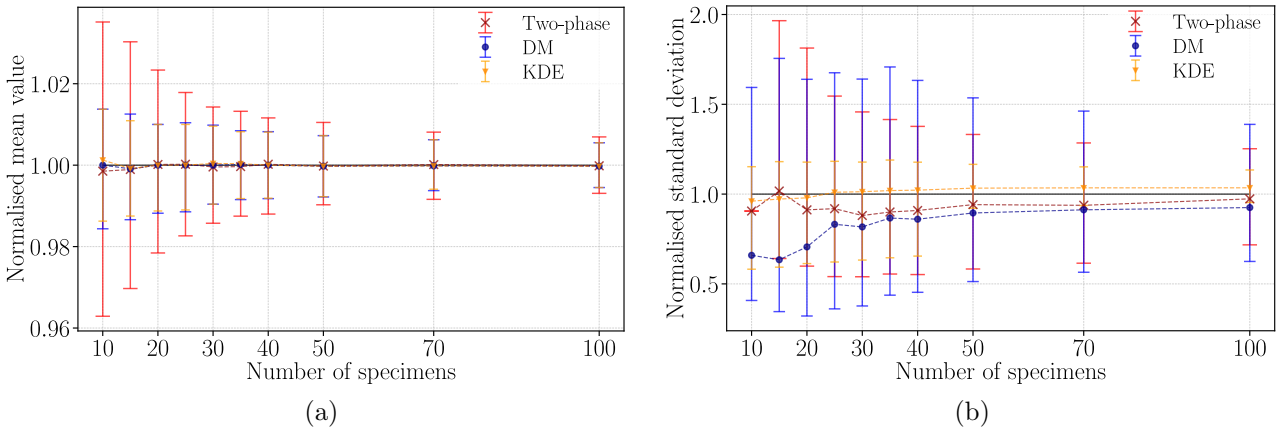

Figure 9: Normalised (a) mean value and (b) standard deviation estimated with two phase method and staircase method with $d^* = 0.5$

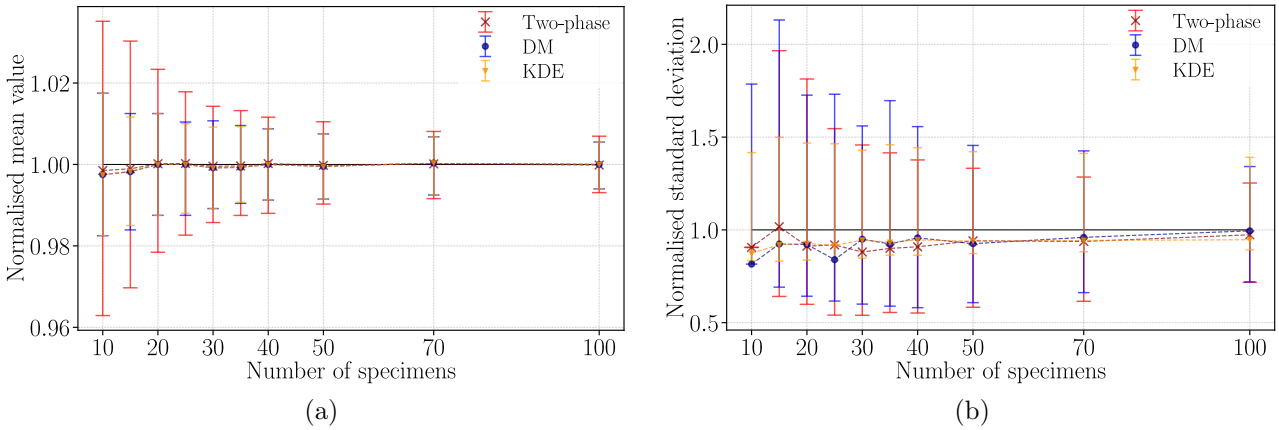

Figure 10: Normalised (a) mean value and (b) standard deviation estimated with two phase method and staircase method with $d^* = 1.0$

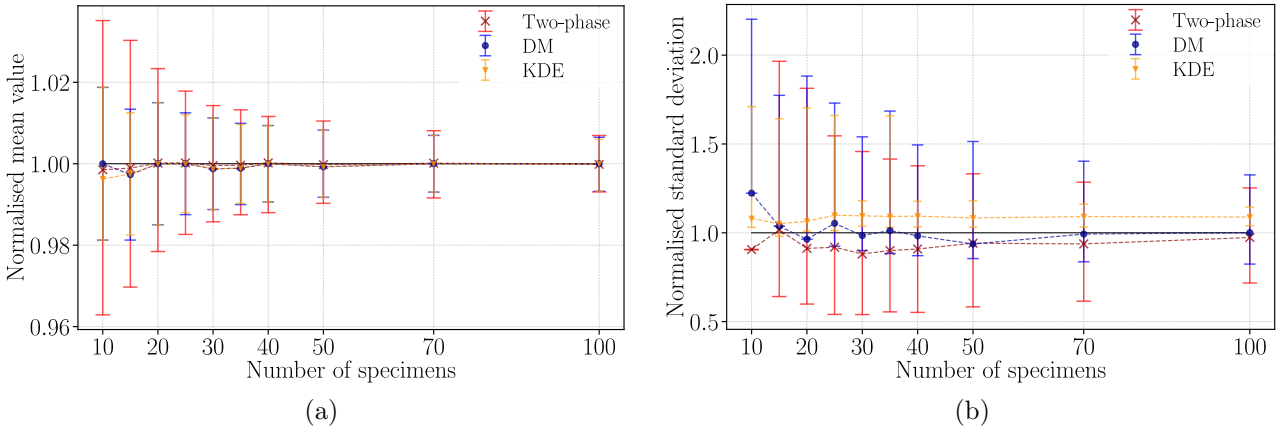

Figure 11: Normalised (a) mean value and (b) standard deviation estimated with two phase method and staircase method with $d^* = 1.5$

393 From Fig. 9(a), Fig. 10(a) and Fig. 11(a), the vertical red line is longer than others. That

means the Two-phase method has a larger uncertainty in estimating the mean value. This difference decreases as the sample size increases.

While from Fig. 9(b), Fig. 10(b) and Fig. 11(b), the advantage of using Two-phase method lies in the estimation of the standard deviation value. It can be observed that the results from DM and KDE will change with different step sizes. The staircase with the KDE method reduces the estimation uncertainty but only has a better estimation in the case of $d^* = 0.5$. When $d^* = 1.5$, DM method provides a better estimation. To sum up, the estimated standard deviation from Two-phase method always has a stable estimation even choosing a very imprecise input as $r = 1.0$.

5.3. Comparison with the Bayesian optimized method

To demonstrate the potential of the proposed method, a comparison between Two-phase method and the latest Bayesian optimized method is presented in this section.

These two methods have the same input parameters, including the lower bound, upper bound and number of specimens. By choosing $r = 0.25$, $r = 0.5$ and $r = 0.75$, the estimated results for the mean and standard deviation are shown in Fig. 12, Fig. 13, and Fig. 14, respectively.

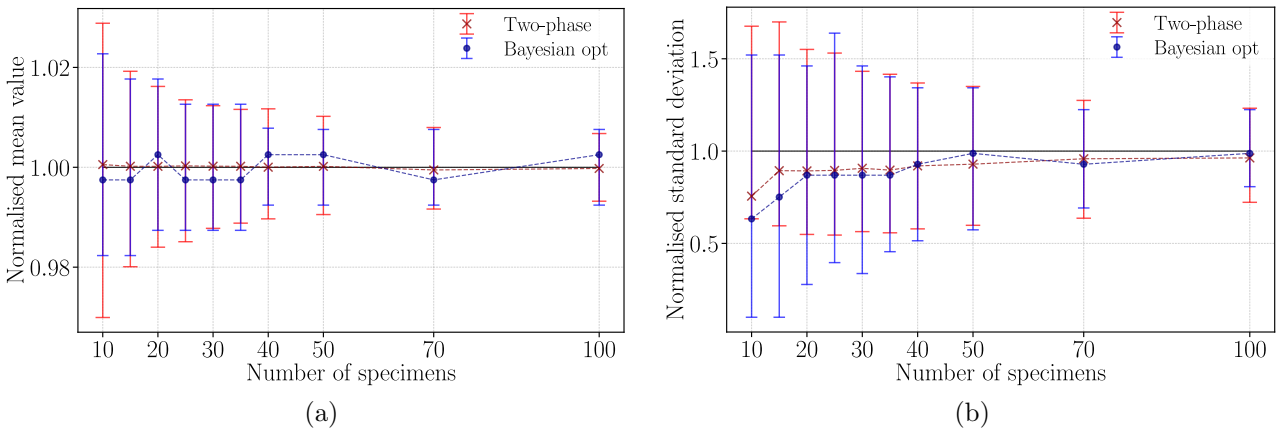

Figure 12: Normalised (a) mean value and (b) standard deviation estimated with two-phase method and Bayesian optimized method with $r = 0.25$

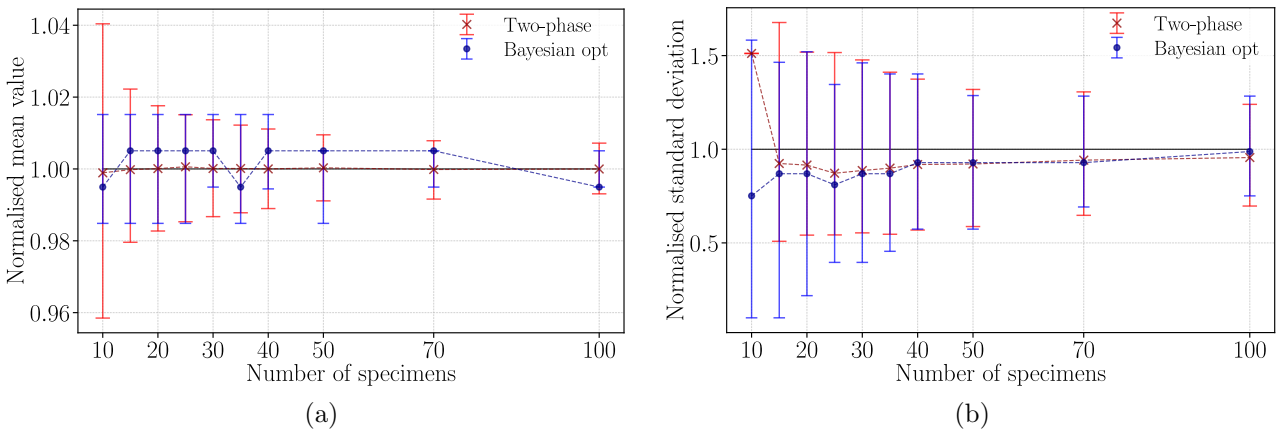

Figure 13: Normalised (a) mean value and (b) standard deviation estimated with two-phase method and Bayesian optimized method with $r = 0.5$

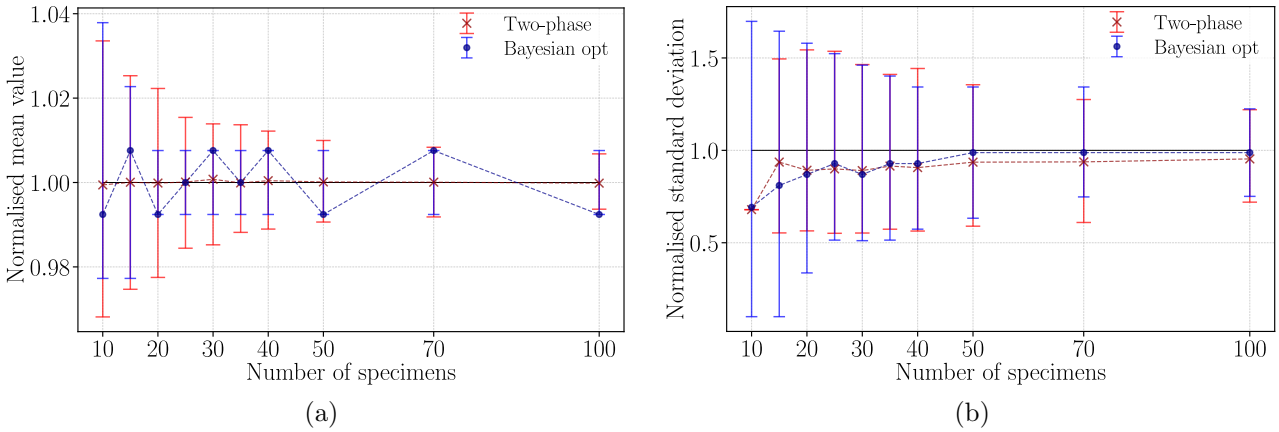

Figure 14: Normalised (a) mean value and (b) standard deviation estimated with two-phase method and Bayesian optimized method with $r = 0.75$

In the figures, the Two-phase method and Bayesian optimized method are illustrated by the red and blue lines, respectively. From Fig. 12(a), Fig. 13(a), and Fig. 14(a), the Two-phase method is more stable on the mean estimation, which is centered on the true value.

From Fig. 12(b), Fig. 13(b), and Fig. 14(b), the estimated standard deviation obtained from the Two-phase method has a smaller uncertainty than that from the Bayesian optimized method only when the number of specimens is lower than 20. While with the increase in the number of specimens, this difference becomes negligible. Moreover, both methods have an underestimated bias (less than 1) for the standard deviation, especially with fewer specimens.

Another notable difference between the Bayesian optimized method and the Two-phase method pertains to the methodologies employed to acquire the final distribution parameters(μ and σ). The Bayesian optimized method employs a predefined a cluster of possible distribution parameters and seeks to find the set of parameters that maximizes the posterior probability given the observed data, hence determining the most probable parameters within all the possible distribution parameters. On the other hand, the Two-phase method utilizes MLE to determine the set of parameters which maximizes the likelihood function, given the observed data. In the context of the Two-phase method, the MLE approach is applied after conducting the experiment, without a predefined all possible distribution parameters.

6. Conclusion

With the intent to overcome the step size limitation inherent in the staircase method, this study proposes an alternative Two-phase protocol for the fatigue limit tests. This protocol, based on the Langlie method and D-optimality, aims to enhance the conventional fatigue limit tests. The proposed approach uniquely introduces the concept of D-optimality into the fatigue limit testing, providing a detailed application of D-optimality in fatigue tests context.

The Two-phase protocol is elaborated upon with an example for practical understanding. A simulation-based study was subsequently carried out to evaluate the quality of the estimates produced by the two-phase method. The study thoroughly investigated the influence of the input bounds on the estimated quality. Numerical tests involved a comparative analysis against the conventional staircase method and the Bayesian optimized method.

Notably, the proposed Two-phase method has a good performance in estimating the fatigue limit mean and standard deviation. The protocol strength lies in its ability to utilize information obtained during testing, eliminating the need for estimated starting stress and fixed step size. A primary advantage is that minimal pre-defined information is required, with the protocol

relying only on lower and upper bounds. Therefore, the test results are not heavily dependent on prior knowledge and remain unaffected by the initial bounds.

The numerical comparison further revealed that the Two-phase method provides superior estimates of the fatigue limit standard deviation, even in situations of non-accurate input bounds. The method also shows good performance when compared with Bayesian optimized methods. Essentially, the Two-phase method yielded more accurate estimates of the mean and standard deviation, bringing the estimated distributions closer to the presumed normal distribution.

For practical experimental fatigue limit testing, a small value and yield stress could be chosen as the lower and upper bounds. To ensure the accuracy of the estimates, the number of specimens should exceed 30. Further investigations could explore the multi-distribution aspect of the fatigue limit, contributing to a broader understanding of the method's applicability and performance.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix: Fisher information matrix

The detailed derivation of Fisher information matrix is presented in this Appendix. A Gaussian random variable x with mean μ and standard deviation σ can be normalised as:

$$z = \frac{x - \mu}{\sigma} \tag{A.1}$$

Therefore, the partial derivative of z are easily obtained as:

$$\begin{aligned} \frac{\partial z}{\partial \mu} &= -\frac{1}{\sigma} \\ \frac{\partial z}{\partial \sigma} &= -\frac{x - \mu}{\sigma^2} \end{aligned} \tag{A.2}$$

As Eq. 14 in Section 3.2.1, the logarithmic maximum likelihood function is:

$$l(\theta) = \mathrm{lnL}(\theta) = \sum_{i=1}^{n} [y_i \mathrm{lnF}(z_i) + (1 - y_i)\ln(1 - \mathrm{F}(z_i))] \tag{A.3}$$

Therefore, the partial derivative of maximum likelihood function is obtained as:

$$\begin{aligned}\frac{\partial l}{\partial \mu} &= \sum_{i=1}^{n}\left(-\frac{y_i \mathrm{F}'(z_i)}{\mathrm{F}(z_i)\sigma}+\frac{(1-y_i)\mathrm{F}'(z_i)}{(1-\mathrm{F}(z_i))\sigma}\right)\\ &= \sum_{i=1}^{n}\left(-\frac{y_i \mathrm{F}'(z_i)(1-\mathrm{F}(z_i))}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}+\frac{(1-y_i)\mathrm{F}(z_i)\mathrm{F}'(z_i)}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\right)\\ &= \sum_{i=1}^{n}\frac{\mathrm{F}'(z_i)(\mathrm{F}(z_i)-y_i)}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\end{aligned} \tag{A.4}$$

$$\begin{aligned}\frac{\partial l}{\partial \sigma} &= (x-\mu)\sum_{i=1}^{n}\left(-\frac{y_i \mathrm{F}'(z_i)}{\mathrm{F}(z_i)\sigma^2}+\frac{(1-y_i)\mathrm{F}'(z_i)}{(1-\mathrm{F}(z_i))\sigma^2}\right)\\ &= (x-\mu)\sum_{i=1}^{n}\left(-\frac{y_i \mathrm{F}'(z_i)(1-\mathrm{F}(z_i))}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma^2}+\frac{(1-y_i)\mathrm{F}(z_i)\mathrm{F}'(z_i)}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma^2}\right)\\ &= \sum_{i=1}^{n}\frac{\mathrm{F}'(z_i)(\mathrm{F}(z_i)-y_i)z_i}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\end{aligned} \tag{A.5}$$

As the Eq. 15 in Section 3.2.2, the Fisher information matrix is:

$$\mathrm{I}(\theta)=\begin{bmatrix}\mathrm{I}_{11} & \mathrm{I}_{12}\\ \mathrm{I}_{21} & \mathrm{I}_{22}\end{bmatrix}=\begin{bmatrix}\mathrm{E}(\frac{\partial l}{\partial \mu}\frac{\partial l}{\partial \mu}) & \mathrm{E}(\frac{\partial l}{\partial \mu}\frac{\partial l}{\partial \sigma})\\ \mathrm{E}(\frac{\partial l}{\partial \mu}\frac{\partial l}{\partial \sigma}) & \mathrm{E}(\frac{\partial l}{\partial \sigma}\frac{\partial l}{\partial \sigma})\end{bmatrix} \tag{A.6}$$

Due to there are only two results, survival and failure, in the fatigue limit test, the results y obeys a Bernoulli distribution with parameter $\mathrm{F}(z)$. It could be found that the mean of y is $\mathrm{F}(z_i)$, and the standard deviation is $\mathrm{F}(z_i)(1-\mathrm{F}(z_i))$. For the tested specimens indexed by sequence $1, 2, \cdots, n$, the Fisher information includes:

$$\begin{aligned}\mathrm{I}_{11} &= \mathrm{E}\left(\frac{\partial l(\theta)}{\partial \mu}\frac{\partial l(\theta)}{\partial \mu}\right)\\ &= \mathrm{E}\left\{\sum_{i=1}^{n}\left(\frac{\mathrm{F}'(z_i)(\mathrm{F}(z_i)-y_i)}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\right)\sum_{j=1}^{n}\left(\frac{\mathrm{F}'(z_j)(\mathrm{F}(z_j)-y_j)}{\mathrm{F}(z_j)(1-\mathrm{F}(z_j))\sigma}\right)\right\}\\ &= \sum_{i=1}^{n}\left(\frac{\mathrm{F}'(z_i)}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\right)^2\frac{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))}{1}\\ &= \sum_{i=1}^{n}\frac{(\mathrm{F}'(z_i))^2}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma^2}\end{aligned} \tag{A.7}$$

$$\begin{aligned}\mathrm{I}_{12}=\mathrm{I}_{21} &= \mathrm{E}\left(\frac{\partial l(\theta)}{\partial \mu}\frac{\partial l(\theta)}{\partial \sigma}\right)\\ &= \mathrm{E}\left\{\sum_{i=1}^{n}\left(\frac{\mathrm{F}'(z_i)(\mathrm{F}(z_i)-y_i)}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\right)\sum_{j=1}^{n}\left(\frac{\mathrm{F}'(z_j)(\mathrm{F}(z_j)-y_j)z_j}{\mathrm{F}(z_j)(1-\mathrm{F}(z_j))\sigma}\right)\right\}\\ &= \sum_{i=1}^{n}\left(\frac{\mathrm{F}'(z_i)}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\right)^2\frac{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))z_i}{1}\\ &= \sum_{i=1}^{n}\frac{(\mathrm{F}'(z_i))^2 z_i}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma^2}\end{aligned} \tag{A.8}$$

$$
\begin{aligned}
\mathrm{I}_{22} &= \mathrm{E}\left(\frac{\partial l(\theta)}{\partial \sigma}\frac{\partial l(\theta)}{\partial \sigma}\right) \\
&= \mathrm{E}\left\{\sum_{i=1}^{n}\left(\frac{\mathrm{F}'(z_i)(\mathrm{F}(z_i)-y_i)z_j}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\right)\sum_{j=1}^{n}\left(\frac{\mathrm{F}'(z_j)(\mathrm{F}(z_j)-y_j)z_j}{\mathrm{F}(z_j)(1-\mathrm{F}(z_j))\sigma}\right)\right\} \\
&= \sum_{i=1}^{n}\left(\frac{\mathrm{F}'(z_i)z_i}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma}\right)^2\frac{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))z_i}{1} \\
&= \sum_{i=1}^{n}\frac{(\mathrm{F}'(z_i))^2 z_i^2}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma^2}
\end{aligned}
\tag{A.9}
$$

To sum up, the Fisher information matrix can be expressed as:

$$
\mathrm{I}(\theta) = \sum_{i=1}^{n}\frac{(\mathrm{F}'(z_i))^2}{\mathrm{F}(z_i)(1-\mathrm{F}(z_i))\sigma^2}\begin{bmatrix}1 & z_i \\ z_i & z_i^2\end{bmatrix} \tag{A.10}
$$

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