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On the unirationality of quadric bundles

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ABSTRACT

We prove that a general $(n-1)$-fold quadric bundle $\mathcal{Q}^{n-1} \to \mathbb{P}^1$, over a number field, with $(-K_{\mathcal{Q}^{n-1}})^n > 0$ and discriminant of odd degree $\delta_{\mathcal{Q}^{n-1}}$ is unirational, and that the same holds for quadric bundles over an arbitrary infinite field provided that $\mathcal{Q}^{n-1}$ has a point, is otherwise general and $n \leq 5$. As a consequence we get the unirationality of any smooth quadric surface bundle $\mathcal{Q}^2 \to \mathbb{P}^2$, over an algebraically closed field, with $\delta_{\mathcal{Q}^2} \leq 12$.

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1. Introduction

An n-dimensional variety X over a field k is rational if it is birational to $\mathbb{P}^n_k$, while X is unirational if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$. The Lüroth problem, asking whether every unirational variety was rational, dates back to the second half of the nineteenth century [32]. These two notions turned out to be equivalent for curves and complex surfaces. Only in the 1970s examples of unirational but non rational 3-folds, over an algebraically closed field of characteristic zero, were given by M. Artin and D. Mumford [1], V. Iskovskih and I. Manin [24], and C. Clemens and P. Griffiths [12].

The problem of determining whether a variety is rational or unirational is in general very hard and unirationality is very poorly understood. For instance, the rationality of certain smooth cubic 4-folds has been proved only recently [9], [40], and even today not many examples of unirational non rational varieties have been worked out. Also due to this difficulty, unirationality has gradually been replaced by the notion of rational connection [11], [27]. A variety X is rationally connected if two general points of X can be joined by a rational curve. We refer to [3] for a comprehensive survey on the subject.

Moreover, the set of rational points of a unirational variety, over an infinite field, is dense. This fact makes unirationality over a number field an interesting property not only in birational geometry but also in number theory.

When dealing with unirationality questions ad hoc constructions are often needed. In this paper we introduce instead a unifying strategy that can be applied to quadric bundles and more generally to fibrations $\pi : X \to Y$ over a field k when Y is unirational and the generic fiber of π is unirational over $k(Y)$.

The heart of our approach lies in the Enriques's unirationality criterion which states that X is unirational if and only if π has a unirational multisection. In order to produce such multisection we proceed as follows:

- either we construct unirational subvarieties of X mapping dominantly onto Y as intersections of special divisors on X; or
- we consider birational transformations $X \dashrightarrow X'$ where $X' \to Y'$ is a fibration whose general fiber F' is unirational and such that the strict transform of F' in X maps dominantly onto Y.

All the arguments presented in the paper rely on this general approach. More specifically, we apply the above strategy to quadric bundles X over $\mathbb{P}^1$ seen as hypersurfaces embedded in splitting projective bundles. In this case, taking advantage of the toric quotient construction of such splitting projective bundles, we can write down the equation of X in Cox coordinates. This way we are able to describe explicitly special unirational subvarieties of X and also the birational transformations that X inherits from the ambient projective bundle.

Once this is done we investigate quadric bundles over higher-dimensional projective spaces by studying their restrictions to a general line and applying to these our results for quadric bundles over $\mathbb{P}^1$.

Rationality of conic bundles has been extensively studied and we have a precise conjectural rationality statement, we refer to [39] for a comprehensive survey. Moreover, quadric bundles have recently received great attention especially concerning stable rationality [18], [45], [2], [41], [10], [20], [22], [42], [43], [35], [8], [33]. We recall that a variety X is stably rational if $X \times \mathbb{P}^m_k$ is rational for some $m \geq 0$. Hence, a rational variety is stably rational, and a stably rational variety is unirational. As an application of one of our main results together with [20], [22] we will derive new examples of unirational but non stably rational quadric bundles.

On the contrary unirationality is still widely open and not much is known. Classically, conjectures on unirationality of conic bundles take into account the degree of the discriminant and, mimicking what is known abut rationality, conic bundles with discriminant of large degree are not expected to be unirational [39, Section 14.2].

We will denote an h-fold quadric bundle over $\mathbb{P}^{n-h}$ by $\pi : \mathcal{Q}^h \rightarrow \mathbb{P}^{n-h}$. Its discriminant $D_{\mathcal{Q}^h} \subset \mathbb{P}^{n-h}$ is the divisor parametrizing the singular quadrics in the fibration $\pi : \mathcal{Q}^h \rightarrow \mathbb{P}^{n-h}$. We will denote by $\delta_{\mathcal{Q}^h}$ the degree of $D_{\mathcal{Q}^h}$.

By [26, Corollary 8] smooth conic bundles over $\mathbb{P}^1$ such that $(-K_{\mathcal{Q}^1})^2 > 0$ are unirational as soon as they have a point. The positivity condition $(-K_{\mathcal{Q}^1})^2 > 0$ translates into $\delta_{\mathcal{Q}^1} \leq 7$. We will investigate this problem for higher-dimensional quadric bundles.

Let $n \geq 2$. As a consequence of a result due to C. Tsen [46] and S. Lang [31, Corollary on page 378] all quadric bundles $\mathcal{Q}^{n-1} \rightarrow \mathbb{P}^1$ over an algebraically closed field are rational. Furthermore, J. Kollár proved that any smooth quadric bundle $\mathcal{Q}^{n-1} \rightarrow \mathbb{P}^1$ over a local field is unirational if and only if it has a point [28, Corollary 1.8]. We will prove results of this type over more general fields. As a sample over number fields we have the following:

Theorem 1.1. *Let $\pi : \mathcal{Q}^{n-1} \rightarrow \mathbb{P}^1$ be a general quadric bundle over a number field. If $(-K_{\mathcal{Q}^{n-1}})^n > 0$ and $\delta_{\mathcal{Q}^{n-1}}$ is odd then $\mathcal{Q}^{n-1}$ is unirational.*

We refer to 2.6 for the notion of generality we will adopt throughout the paper. Furthermore, in Corollary 4.16 we will extend Theorem 1.1 to an arbitrary infinite field provided that either $\delta_{\mathcal{Q}^{n-1}} \leq 3n+1$ or $n \leq 5$ and $\mathcal{Q}^{n-1}$ has a point.

Theorem 1.1 and Corollary 4.16, whose proofs rely on the strategy outlined in the first part of the introduction, are the core of the paper, and starting from them we will derive several unirationality results for quadric bundles over higher-dimensional projective spaces. For instance, by Corollary 4.11 for quadric surface bundles over the projective plane we have the following:

Theorem 1.2. *Let $\pi : \mathcal{Q}^2 \rightarrow \mathbb{P}^2$ be a smooth complex quadric surface bundle. If $\delta_{\mathcal{Q}^2} \leq 12$ then $\mathcal{Q}^2$ is unirational.*

Moreover, in Corollary 4.8 and Remark 4.9 we extend Theorem 1.2 to quadric bundles $\pi : \mathcal{Q}^h \to \mathbb{P}^{n-h}$ over more general fields and with $h, n-h \geq 2$.

Theorem 1.2, together with [8, Corollary 1.3], yields that a very general quadric bundle $\pi : \mathcal{Q}^2 \to \mathbb{P}^2$ with discriminant of degree $10 \leq \delta_{\mathcal{Q}^2} \leq 12$ is unirational but not stably rational.

For a special class of quadric bundles, namely divisors of bidegree $(d,2)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^{h+1}$, in Propositions 3.2, 4.13, Corollary 3.3 and Remark 3.4 we give more refined results. In particular, when k is the field of complex numbers as a consequence of our results and [20], [22] we have that a very general divisor of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^{h+1}$ for $h = 1,2$ is unirational but not stably rational.

The situation is very different when $\delta_{\mathcal{Q}^{n-1}}$ is even. For instance, the quadric bundles, over a real closed field, of a fixed multidegree and without points form a semialgebraic set containing an open ball in the Euclidean topology. However, in Corollary 4.6 we prove a unirationality result over a quadratic extension of the base field which in particular implies the potential density of the rational points.

Note that Theorem 1.1 and Corollary 4.6 imply [21, Conjecture 1.3] for quadric bundles over $\mathbb{P}^1$. We recall that [21, Conjecture 1.3] predicts that if X is a smooth projective variety with ample anti-canonical divisor over a number field k then for some finite extension k' of k the set $X(k')$ of k'-rational points of X is Zariski dense.

Finally, in Section 5 we will apply the techniques introduced in the previous sections to give unirationality results for quadric bundles over finite fields.

Conventions on the base field and terminology. All along the paper the base field k will be of characteristic different form two. In Sections 3, 4 k will be an infinite field while in Section 5 k will be a finite field.

Let X be a variety over k. When we say that X is rational or unirational, without specifying over which field, we will always mean that X is rational or unirational over k. Similarly, we will say that X has a point or contains a variety with certain properties meaning that X has a k-rational point or contains a variety defined over k with the required properties.

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2. Quadric bundles and the Enriques's unirationality criterion

In this section we introduce the notation and a version of the Enriques's unirationality criterion for quadric bundles.

Definition 2.1. Let W be a smooth $(n-h)$-dimensional variety, $\mathcal{E}$ a rank $h+2$ vector bundle over W, $\overline{\pi} : \mathbb{P}(\mathcal{E}) \to W$ the associated projective bundle with tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and $\mathcal{L}$ a line bundle over W. A quadratic form with values in $\mathcal{L}$ is a global section

$$\sigma \in H^0(W, \operatorname{Sym}^2 \mathcal{E}^\vee \otimes \mathcal{L}) \cong H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \overline{\pi}^*\mathcal{L}).$$

An n-dimensional quadric bundle over W is a variety of the form $\mathcal{Q}^h := \{\sigma = 0\} \subset \mathbb{P}(\mathcal{E})$ where $\sigma \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \overline{\pi}^*\mathcal{L})$ is a generically non degenerate quadratic form, endowed with the projection $\pi : \mathcal{Q}^h \rightarrow W$ induced by $\overline{\pi}$. A conic bundle is a quadric bundle with $h = 1$.

The discriminant of $\pi : \mathcal{Q}^h \rightarrow W$ is the divisor $D_{\mathcal{Q}^h} \subset W$ where σ does not have full rank. When $W = \mathbb{P}^{n-h}$ the discriminant $D_{\mathcal{Q}^h} \subset \mathbb{P}^{n-h}$ is a hypersurface and we will denote by $\delta_{\mathcal{Q}^h} := \deg(D_{\mathcal{Q}^h})$ its degree.

Remark 2.2. Hence, a general fiber of $\pi : \mathcal{Q}^h \rightarrow W$ is a quadric hypersurface in $\mathbb{P}^{h+1}$. Note that in Definition 2.1 the map $\pi : \mathcal{Q}^h \rightarrow W$ is not required to be flat. Often quadric bundles are defined as morphisms $\pi : \mathcal{Q}^h \rightarrow W$ whose fibers are isomorphic to quadric hypersurfaces of constant dimension and with $\mathcal{Q}^h$ smooth. Then π is necessarily flat and there exists a rank $h+2$ vector bundle $\mathcal{E} \rightarrow W$, a line bundle $\mathcal{L} \rightarrow W$, and a section $\sigma \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \overline{\pi}^*\mathcal{L})$ as above such that $\mathcal{Q}^h$ identifies with the zero locus of σ in $\mathbb{P}(\mathcal{E})$ and $\pi = \overline{\pi}_{|\mathcal{Q}^h}$ [39, Theorem 3.2], [4, Proposition 1.2].

Let $a_0, \dots, a_{h+1} \in \mathbb{Z}_{\geq 0}$, with $a_0 \geq a_1 \geq \dots \geq a_{h+1}$, be non negative integers, and consider the simplicial toric variety $\mathcal{T}_{a_0,\dots,a_{h+1}}$ with Cox ring

$$\operatorname{Cox}(\mathcal{T}_{a_0,\dots,a_{h+1}}) \cong k[x_0, \dots, x_{n-h}, y_0, \dots, y_{h+1}]$$

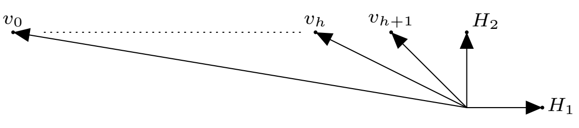
$\mathbb{Z}^2$-grading given, with respect to a fixed basis (H_1, H_2) of $\operatorname{Pic}(\mathcal{T}_{a_0,\dots,a_{h+1}})$, by the following matrix

$$\left(\begin{array}{cccccc} x_0 & \dots & x_{n-h} & y_0 & \dots & y_{h+1} \\ \hline 1 & \dots & 1 & -a_0 & \dots & -a_{h+1} \\ 0 & \dots & 0 & 1 & \dots & 1 \end{array}\right)$$

and irrelevant ideal $(x_0, \dots, x_{n-h}) \cap (y_0, \dots, y_{h+1})$. Then

$$\mathcal{T}_{a_0,\dots,a_{h+1}} \cong \mathbb{P}(\mathcal{E}_{a_0,\dots,a_{h+1}})$$

with $\mathcal{E}_{a_0,\dots,a_{h+1}} \cong \mathcal{O}_{\mathbb{P}^{n-h}}(a_0) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^{n-h}}(a_{h+1})$. The secondary fan of $\mathcal{T}_{a_0,\dots,a_{h+1}}$ is as follows



where $H_1 = (1, 0)$ corresponds to the sections $x_0, \dots, x_{n-h}$, $H_2 = (0, 1)$, and $v_i = (-a_i, 1)$ corresponds to the section y_i for $i = 0, \dots, h+1$.

Definition 2.3. A splitting h-fold quadric bundle $\pi : \mathcal{Q}^h \rightarrow \mathbb{P}^{n-h}$ is given by an equation of the following form

$$\mathcal{Q}^h := \left\{ \sum_{0\leq i\leq j\leq h+1} \sigma_{i,j}(x_0,\dots,x_{n-h})y_iy_j = 0 \right\} \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-h}}(a_0)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^{n-h}}(a_{h+1})) \tag{2.4}$$

where $\sigma_{i,j} \in k[x_0,\dots,x_{n-h}]_{d_{i,j}}$ is a homogeneous polynomial of degree $d_{i,j}$, and

$$d_{0,0} - 2a_0 = d_{0,1} - a_0 - a_1 = \cdots = d_{i,j} - a_i - a_j = \cdots = d_{h+1,h+1} - 2a_{h+1}. \tag{2.5}$$

The multidegree of a splitting quadric bundle is $(d_{0,0}, d_{0,1},\dots,d_{h+1,h+1}) \in \mathbb{Z}^{\binom{h+3}{2}}$.

Note that the discriminant of a splitting quadric bundle of multidegree $(d_{0,0}, d_{0,1},\dots, d_{h+1,h+1})$ has degree $\delta_{\mathcal{Q}^h} = d_{0,0}+\cdots+d_{i,i}+\cdots+d_{h+1,h+1}$. Note that $a_0 \geq a_1 \geq \cdots \geq a_{h+1}$ and (2.5) yield $d_{0,0} \geq d_{1,1} \geq \cdots \geq d_{h+1,h+1}$. Furthermore, the degrees $d_{i,i}$ of all the $\sigma_{i,i} \neq 0$ must have the same parity.

2.6. About the notion of generality

Let $k^{N(n-h,d_{i,j})}$, with $N(n-h, d_{i,j}) = \binom{d_{i,j}+n-h}{n-h}$, be the vector space of degree $d_{i,j}$ homogeneous polynomials in $n-h+1$ variables. Then splitting quadric bundles of multidegree $(d_{0,0}, d_{0,1},\dots,d_{h+1,h+1})$ over $\mathbb{P}^{n-h}$ correspond to the elements of

$$V^{n-h}_{d_{0,0},\dots,d_{h+1,h+1}} = k^{N(n-h,d_{0,0})} \oplus k^{N(n-h,d_{0,1})} \oplus \cdots \oplus k^{N(n-h,d_{h+1,h+1})}$$

up to multiplication by a non zero scalar. We will say that a splitting quadric bundle $\mathcal{Q}^h$ is general if it corresponds to a general element of $V^{n-h}_{d_{0,0},\dots,d_{h+1,h+1}}$. When referring to a general splitting quadric bundle satisfying certain properties we will mean that the quadric bundle is general among those satisfying the required properties.

Remark 2.7. By the Birkhoff–Grothendieck splitting theorem [19, Theorem 4.1] all quadric bundles $\mathcal{Q}^{n-1} \rightarrow \mathbb{P}^1$ are splitting.

2.7. From quadric bundles over $\mathbb{P}^{n-h}$ to quadric bundles over $\mathbb{P}^1$

Let $\pi : \mathcal{Q}^h \rightarrow \mathbb{P}^{n-h}$ be a quadric bundle $\mathcal{Q}^h \subset \mathbb{P}(\mathcal{E})$. Take a point $p \in \mathbb{P}^{n-h}$ and set $Q_p = \pi^{-1}(p)$. Let $\pi_p : \mathbb{P}^{n-h} \dashrightarrow \mathbb{P}^{n-h-1}$ be the projection from p, W the blow-up of $\mathbb{P}^{n-h}$ at p, $\widetilde{\mathcal{Q}}^h$ the blow-up of $\mathcal{Q}^h$ along Q_p, and $\widetilde{\pi}_p : W \rightarrow \mathbb{P}^{n-h-1}$ the morphism induced by π_p. By the universal property of the blow-up [17, Chapter II, Proposition 7.14] the morphism $\pi : \mathcal{Q}^h \rightarrow \mathbb{P}^{n-h}$ induces a morphism $\widetilde{\pi} : \widetilde{\mathcal{Q}}^h \rightarrow W$. Note that a general line L_p through p intersects $D_{\mathcal{Q}^h}$ in $\delta_{\mathcal{Q}^h}$ points counted with multiplicity. Set

$\mathcal{Q}^h_{L_p} = \pi^{-1}(L_p)$. Then $\mathcal{Q}^h_{L_p} \to L_p \cong \mathbb{P}^1$ is a quadric bundle $\mathcal{Q}^h_{L_p} \subset \mathbb{P}(\mathcal{E}_{|L_p})$ and hence by Remark 2.7 it is splitting. We sum up the situation in the following diagram

$$\begin{tikzcd} \widetilde{\mathcal{Q}}^h \arrow[r] \arrow[d, "\widetilde{\pi}"'] & \mathcal{Q}^h \arrow[d, "\pi"] & \\ W \arrow[r] \arrow[rr, bend right, "\widetilde{\pi}_p"'] & \mathbb{P}^{n-h} \arrow[r, dashed, "\pi_p"] & \mathbb{P}^{n-h-1}. \end{tikzcd}$$

The generic fiber $\widetilde{\mathcal{Q}}^h_\eta$ of $\widetilde{\pi}_p \circ \widetilde{\pi} : \widetilde{\mathcal{Q}}^h \to \mathbb{P}^{n-h-1}$ is a quadric bundle $\widetilde{\mathcal{Q}}^h_\eta \to \mathbb{P}^1_F$ over $F = k(t_1, \dots, t_{n-h-1})$. Note that $\delta_{\widetilde{\mathcal{Q}}^h_\eta} = \delta_{\mathcal{Q}^h}$.

Lemma 2.8. *If $\widetilde{\mathcal{Q}}^h_\eta$ is F-unirational then $\mathcal{Q}^h$ is k-unirational.*

Proof. If $\widetilde{\mathcal{Q}}^h_\eta$ is F-unirational then $\widetilde{\mathcal{Q}}^h$ is k-unirational, and since $\widetilde{\mathcal{Q}}^h$ is a blow-up of $\mathcal{Q}^h$ we conclude that $\mathcal{Q}^h$ is k-unirational as well. $\square$

The following fundamental results will be our main tools to investigate the birational geometry of quadric bundles over the projective space.

Remark 2.9. (Lang's theorem) Fix a real number $r \in \mathbb{R}_{\geq 0}$. A field k is C_r if and only if every homogeneous polynomial $f \in k[x_0, \dots, n_n]_d$ of degree $d > 0$ in $n+1$ variables with $n+1 > d^r$ has a non trivial zero in k^{n+1}.

If k is a C_r field, $f_1, \dots, f_s \in k[x_0, \dots, n_n]_d$ are homogeneous polynomials of the same degree and $n+1 > sd^r$ then $f_1, \dots, f_s$ have a non trivial common zero in k^{n+1} [38, Proposition 1.2.6]. Furthermore, if k is C_r then $k(t)$ is C_{r+1} [38, Theorem 1.2.7].

Let $\pi : \mathcal{Q}^h \to \mathbb{P}^{n-h}$ be a quadric bundle over a C_r field k with generic fiber $\mathcal{Q}^h_\eta$. If $h > 2^{r+n-h} - 2$ we have that $\mathcal{Q}^h_\eta$ has an F-point, where $F = k(t_1, \dots, t_{n-h})$. Projecting from such F-point we see that $\mathcal{Q}^h_\eta$ is F-rational and hence, arguing as in the proof of Lemma 2.8, we get that $\mathcal{Q}^h$ is rational.

Remark 2.10. (Chevalley–Warning's theorem) Let k be a finite field and $f_1, \dots, f_s$ homogeneous polynomials in $n+1$ variables of degree $d_1, \dots, d_s$ with coefficients in k. If $n+1 > d_1 + \dots + d_s$ then $f_1, \dots, f_s$ have a non trivial common zero in k^{n+1} [13], [47].

Furthermore, we will extensively make use of a unirationality criterion due to F. Enriques [25, Proposition 10.1.1].

Proposition 2.11. *Let $\pi : \mathcal{Q}^h \to W$ be a quadric bundle over a unirational variety W. Then $\mathcal{Q}^h$ is unirational if and only if there exists a unirational subvariety $Z \subset \mathcal{Q}^h$ such that $\pi_{|Z} : Z \to W$ is dominant.*

Proof. Assume that $\mathcal{Q}^h$ is unirational. Then there exists a dominant rational map $\psi : \mathbb{P}^n \dashrightarrow \mathcal{Q}^h$. If $H \subset \mathbb{P}^n$ is a general $(n-h)$-plane then $Z = \overline{\psi(H)} \subset \mathcal{Q}^h$ is unirational and transverse to the fibration $\pi : \mathcal{Q}^h \rightarrow W$ that is $\pi_{|Z} : Z \rightarrow W$ is dominant.

Now, assume that there exists a unirational subvariety $Z \subset \mathcal{Q}^h$ such that $\pi_{|Z} : Z \rightarrow W$ is dominant. Consider the fiber product

$$\begin{tikzcd} \mathcal{Q}^h \times_W Z \arrow[r] \arrow[d] & \mathcal{Q}^h \arrow[d, "\pi"] \\ Z \arrow[r, "\pi_{|Z}"] & W \end{tikzcd}$$

and note that $\mathcal{Q}^h \times_W Z \rightarrow Z$ is a quadric bundle admitting a rational section $Z \dashrightarrow \mathcal{Q}^h \times_W Z$. Such rational section yields a point of the quadric $\mathcal{Q} := \mathcal{Q}^h \times_W Z$ over the function field $k(Z)$, and by projecting from this point we get that $\mathcal{Q}$ is rational over $k(Z)$, and hence $\mathcal{Q}^h \times_W Z$ is birational to $Z \times \mathbb{P}^h$ over k. Since Z is unirational then $\mathcal{Q}^h \times_W Z$ is unirational, and since $\mathcal{Q}^h$ is dominated by $\mathcal{Q}^h \times_W Z$ we conclude that $\mathcal{Q}^h$ is unirational as well. $\square$

Remark 2.12. Consider a quadric bundle of the form (2.4) and assume that one of the $\sigma_{i,i}$, say $\sigma_{0,0}$, is identically zero. As noticed in [42, Definition 21] $\{y_1 = \cdots = y_{h+1} = 0\}$ yields a rational section of $\mathcal{Q}^h \rightarrow \mathbb{P}^{n-h}$ and hence $\mathcal{Q}^h$ is rational. Furthermore, if $d_{i,i} < 0$ then $\sigma_{i,i} = 0$ and hence we have, from the previous discussion, that $\mathcal{Q}^h$ is rational. So all through the paper we will assume that $d_{i,i} \geq 0$ and $\sigma_{i,i} \neq 0$ for all $i = 0, \dots, h+1$.

Next we derive some explicit formulas for the anti-canonical divisor of a quadric bundle.

Proposition 2.13. *Let $\pi : \mathcal{Q}^h \subset \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^{n-h}$ be a quadric bundle with discriminant of degree $\delta_{\mathcal{Q}^h}$. Then*

$$-K_{\mathcal{Q}^h} = \frac{(n-h+1)(h+2) - hc_1(\mathcal{E}) - \delta_{\mathcal{Q}^h}}{h+2}\overline{H}_1 + h\overline{H}_2$$

where $\overline{H}_1 = \pi^*\mathcal{O}_{\mathbb{P}^{n-h}}(1)_{|\mathcal{Q}^h}$ and $\overline{H}_2 = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)_{|\mathcal{Q}^h}$.

Proof. First note that $\mathcal{Q}^h \subset \mathbb{P}(\mathcal{E})$ is a divisor of class

$$\mathcal{Q}^h \sim \frac{\delta_{\mathcal{Q}^h} - 2c_1(\mathcal{E})}{h+2}H_1 + 2H_2 \tag{2.14}$$

where H_1 is the pull-back to $\mathbb{P}(\mathcal{E})$ of $\mathcal{O}_{\mathbb{P}^{n-h}}(1)$ and H_2 is the divisor class corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. To conclude it is enough to use the relative Euler's sequence on $\mathbb{P}(\mathcal{E})$ and the adjunction formula. $\square$

Proposition 2.15. *Let $\pi : \mathcal{Q}^h \subset \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-h}$ be a quadric bundle with discriminant of degree $\delta_{\mathcal{Q}^h}$. Denote by $c_i = c_i(\mathcal{E})$ the Chern classes of $\mathcal{E}$, and define recursively the polynomials*

$$g_i(c_1, \dots, c_i) = c_1 g_{i-1} - c_2 g_{i-2} + \dots + (-1)^{i-1} c_i g_0$$

setting $g_0 = 1$. Then

$$\begin{aligned}(-K_{\mathcal{Q}^h})^n = &\sum_{i=0}^{n-h} \binom{n}{n-h-i} \left(\frac{\delta_{\mathcal{Q}^h} - 2c_1}{h+2} g_{i-1} + 2g_i \right) \\ &\times \left(\frac{(n-h+1)(h+2) - hc_1 - \delta_{\mathcal{Q}^h}}{h+2} \right)^{n-h-i} h^{h+i}.\end{aligned}$$

Proof. Note that $H_1^i = 0$ for $i > n-h$ and $H_1^{n-h} H_2^{h+1} = 1$. Now, set $g_i(c_1, \dots, c_i) := H_1^{n-h-i} H_2^{h+i+1}$. Since

$$H_1^{n-h-i} H_2^{h+i+1} = c_1 H_1^{n-h-i+1} H_2^{h+i} - c_2 H_1^{n-h-i+2} H_2^{h+i-1} + \dots + (-1)^{i-1} c_i H_1^{n-h} H_2^{h+1}$$

we get that $g_i = c_1 g_{i-1} - c_2 g_{i-2} + \dots + (-1)^{i-1} c_i g_0$ and so the g_i can be computed recursively from $g_0 = H_1^{n-h} H_2^{h+1} = 1$. Therefore, plugging-in (2.14) we have

$$\overline{H}_1^{n-h-j} \overline{H}_2^{h+j} = H_1^{n-h-j} H_2^{h+j} \left(\frac{\delta_{\mathcal{Q}^h} - 2c_1(\mathcal{E})}{h+2} H_1 + 2H_2 \right) = \frac{\delta_{\mathcal{Q}^h} - 2c_1}{h+2} g_{j-1} + 2g_j$$

where we set $g_{-1} = 0$. Finally, Proposition 2.13 yields the claim. $\square$

Remark 2.16. Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a quadric bundle with discriminant of degree $\delta_{\mathcal{Q}^{n-1}}$. Then Proposition 2.15 yields

$$\begin{aligned}(-K_{\mathcal{Q}^{n-1}})^n &= \frac{2n(n-1)^{n-1}}{n+1}(2n+2-nc_1+c_1-\delta_{\mathcal{Q}^{n-1}}) + (n-1)^n \frac{\delta_{\mathcal{Q}^{n-1}} + 2nc_1}{n+1} \\ &= (n-1)^{n-1}(4n - \delta_{\mathcal{Q}^{n-1}}).\end{aligned}$$

For $n = 2$, that is when $\mathcal{Q}^1$ is a conic bundle, we get that $(-K_{\mathcal{Q}^1})^2 = 8 - \delta_{\mathcal{Q}^1}$ [30, Page 5].

3. Divisors in products of projective spaces

In this section we study the unirationality of divisors of bidegree $(d, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$. This will be crucial in the proof of Theorem 1.1.

Proposition 3.1. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a divisor of bidegree $(d, 2)$ with $1 < d < n$ defined by an equation of the form*

$$\mathcal{Q}^{n-1} = \left\{ \sum_{i=0}^{d} x_0^{d-i} x_1^i f_i = 0 \right\} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(y_0,\dots,y_n)}$$

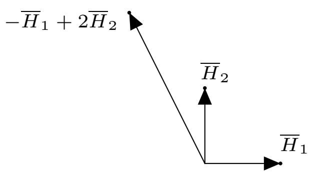
with $f_i \in k[y_0, \dots, y_n]_2$. Consider the matrix

$$M_{(z_0,\dots,z_{d-1})} = \begin{pmatrix} 0 & z_0 & f_0 \\ -z_0 & z_1 & f_1 \\ \vdots & \vdots & \vdots \\ -z_{d-2} & z_{d-1} & f_{d-1} \\ -z_{d-1} & 0 & f_d \end{pmatrix}$$

and let $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})} = \{\mathrm{rank}(M_{(\overline{z}_0,\dots,\overline{z}_{d-1})}) < 3\} \subset \mathbb{P}^n$ be the complete intersection of the $d-1$ quadrics defined by the 3×3 minors of $M_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ for a fixed $(\overline{z}_0, \dots, \overline{z}_{d-1}) \in k^d \setminus \{(0, \dots, 0)\}$.

If for some $(\overline{z}_0, \dots, \overline{z}_{d-1}) \in k^d \setminus \{(0, \dots, 0)\}$ the complete intersection $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})} \subset \mathbb{P}^n$ is unirational then $\mathcal{Q}^{n-1}$ is unirational.

Proof. By [34, Theorem 1.1 (ii)] if $\mathcal{Q}^{n-1}$ is normal and $\mathbb{Q}$-factorial then it is a Mori dream space and its Mori chamber decomposition is as follows



where the effective, movable and nef cones are given by

$$\mathrm{Eff}(\mathcal{Q}^{n-1}) = \mathrm{Mov}(\mathcal{Q}^{n-1}) = \left\langle \overline{H}_1, -\overline{H}_1 + 2\overline{H}_2 \right\rangle, \ \mathrm{Nef}(\mathcal{Q}^{n-1}) = \left\langle \overline{H}_1, \overline{H}_2 \right\rangle$$

and the chamber delimited by $\overline{H}_2$ and $-\overline{H}_1 + 2\overline{H}_2$ corresponds to a small transformation $\mathcal{Q}^{n-1}_+$ of $\mathcal{Q}^{n-1}$. The variety $\mathcal{Q}^{n-1}_+ \subset \mathbb{P}^{d-1}_{(z_0,\dots,z_{d-1})} \times \mathbb{P}^n_{(y_0,\dots,y_n)}$ is defined by

$$\mathcal{Q}^{n-2}_+ = \{\mathrm{rank}(M_{(z_0,\dots,z_{d-1})}) < 3\} \subset \mathbb{P}^{d-1}_{(z_0,\dots,z_{d-1})} \times \mathbb{P}^n_{(y_0,\dots,y_n)}.$$

Consider the rational map

$$\begin{array}{cccc} \rho: & \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(y_0,\dots,y_n)} & \dashrightarrow & \mathbb{P}^{d-1}_{(z_0,\dots,z_{d-1})} \\ & ([x_0 : x_1], [y_0 : \dots : y_n]) & \mapsto & [\rho_0 : \dots : \rho_{d-1}] \end{array}$$

where $\rho_i([x_0 : x_1], [y_0 : \dots : y_n]) = x_0^i x_1^{d-i-1} f_0 + x_0^{i+1} x_1^{d-i} f_1 + \dots + x_1^{d-1} f_i$ for $i = 0, \dots, d-1$. The small transformation $\psi : \mathcal{Q}^{n-1} \dashrightarrow \mathcal{Q}^{n-1}_+$ is given by the restriction to $\mathcal{Q}^{n-1}$ of the map

$$
\begin{array}{ccc}
\mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^n_{(y_0,\dots,y_n)} & \dashrightarrow & \mathbb{P}^{d-1}_{(z_0,\dots,z_{d-1})} \times \mathbb{P}^n_{(y_0,\dots,y_n)} \\
([x_0:x_1],[y_0:\dots:y_n]) & \mapsto & (\rho([x_0:x_1],[y_0:\dots:y_n]),[y_0:\dots:y_n]).
\end{array}
$$

We sum up the situation in the following diagram

$$
\begin{tikzcd}
 & \mathcal{Q}^{n-1} \arrow[r, dashed, "\psi"] \arrow[dl, "\pi"'] & \mathcal{Q}^{n-1}_{+} \arrow[dr, "\widetilde{\pi}"] & \\
\mathbb{P}^1_{(x_0,x_1)} & & & \mathbb{P}^2_{(z_0,\dots,z_{d-1})}
\end{tikzcd}
$$

where $\widetilde{\pi}:\mathcal{Q}^{n-1}_{+}\rightarrow\mathbb{P}^2_{(z_0,\dots,z_{d-1})}$ is the restriction of the first projection. Now, by hypothesis there exists a fiber $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}=\widetilde{\pi}^{-1}([\overline{z}_0:\dots:\overline{z}_{d-1}])\subset\mathbb{P}^n$ of $\widetilde{\pi}$ that is unirational. We have $\overline{z}_i\neq 0$ for some i, say $\overline{z}_0\neq 0$. So

$$
X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}=\{\overline{z}_0^2f_1+(\overline{z}_1^2-\overline{z}_0\overline{z}_i)f_0-\overline{z}_0\overline{z}_1f_i=0;\ \text{for } i=2,\dots,d\}\subset\mathbb{P}^n_{(y_0,\dots,y_n)}.
$$

The strict transform $\widetilde{X}_{(\overline{z}_0,\dots,\overline{z}_{d-1})}\subset\mathcal{Q}^{n+1}$ of $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ via ψ is cut out in $\mathbb{P}^1_{(x_0,x_1)}\times\mathbb{P}^n_{(y_0,\dots,y_n)}$ by the equation of $\mathcal{Q}^{n+1}$ together with the relations coming from $\operatorname{rank}(\widetilde{M}_{(\overline{z}_0,\dots,\overline{z}_{d-1})})<2$ where

$$
\widetilde{M}_{(\overline{z}_0,\dots,\overline{z}_{d-1})}=\begin{pmatrix} x_1^{d-1}f_0 & x_0x_1^{d-2}f_0+x_1^{d-1}f_1 & \dots & x_0^{d-1}f_0+x_0^{d-2}x_1f_1+\dots+x_1^{d-1}f_{d-1} \\ \overline{z}_0 & \overline{z}_1 & \dots & \overline{z}_{d-1}\end{pmatrix}.
$$

Then

$$
\widetilde{X}_{(\overline{z}_0,\dots,\overline{z}_{d-1})}\cap\{x_1\neq 0\}=\begin{cases}
\overline{z}_{d-1}x_1^{d-1}f_0-\overline{z}_0(x_0^{d-1}f_0+x_0^{d-2}x_1f_1+\dots+x_1^{d-1}f_{d-1})=0;\\
\vdots\\
\overline{z}_1x_1f_0-\overline{z}_0(x_0f_0+x_1f_1)=0;\\
x_0^df_0+\dots+x_1^df_d=0.
\end{cases}
$$

Therefore, $\widetilde{X}_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ is unirational and $\pi_{|\widetilde{X}_{(\overline{z}_0,\dots,\overline{z}_{d-1})}}:\widetilde{X}_{(\overline{z}_0,\dots,\overline{z}_{d-1})}\rightarrow\mathbb{P}^1_{(x_0,x_1)}$ is dominant. Hence, Proposition 2.11 yields that $\mathcal{Q}^{n-1}$ is unirational as well. □

Proposition 3.2. *Let $\pi:\mathcal{Q}^{n-1}\rightarrow\mathbb{P}^1$ be a divisor of bidegree $(3,2)$ in $\mathbb{P}^1\times\mathbb{P}^n$ with a point and otherwise general. If $n\geq 4$ then $\mathcal{Q}^{n-1}$ is unirational.*

Proof. We will adopt the notation of Proposition 3.1. Up to a change of coordinates we may assume that $p=([0:1],[0:\dots:0:1])\in\mathcal{Q}^{n-1}$, so that f_d must be of the form

$$
f_d=A(y_0,\dots,y_{n-1})+y_nL(y_0,\dots,y_{n-1})
$$

with $A\in k[y_0,\dots,y_{n-1}]_2$ and $L\in k[y_0,\dots,y_{n-1}]_1$. The f_i are otherwise general. Note that

$$\psi(p) = ([f_0(0,\dots,0,1) : f_1(0,\dots,0,1) : \dots : f_{d-1}(0,\dots,0,1)], [0 : \dots : 0 : 1])$$

and since the f_i are general the point $q = \psi(p) \in \mathcal{Q}^{n-1}_+$ is well-defined. Now, set

$$[\overline{z}_0 : \dots : \overline{z}_{d-1}] = \widetilde{\pi}(q) = [f_0(0,\dots,0,1) : f_1(0,\dots,0,1) : \dots : f_{d-1}(0,\dots,0,1)].$$

By Proposition 3.1 to conclude it is enough to prove that $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})} \subset \mathbb{P}^n$ is unirational. Since the f_i are general the variety $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})} \subset \mathbb{P}^n$ is a smooth complete intersection of two quadrics with a point $q \in X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$. If $n = 4$ then $X_{(\overline{z}_0,\dots,\overline{z}_2)}$ is a del Pezzo surface of degree four with a point and hence it is unirational [38, Summary 9.4.12]. If $\mathrm{char}(k) = 0$ the unirationality of a smooth complete intersection of two quadrics with a point for $n \geq 4$ follows from [14, Proposition 2.3].

If $n > 5$ and $\mathrm{char}\, k > 0$ we take general hyperplane sections of $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ through $q \in X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ until we get a surface S with a point. Since k is infinite [16, Corollary 3.4.14] yields that S is a smooth del Pezzo surface of degree four and hence it is unirational. The strict transform $\widetilde{S}$ of S via ψ is a unirational surface dominating $\mathbb{P}^1_{(x_0,x_1)}$. $\square$

Corollary 3.3. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a smooth divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$ over a C_r field k with $\mathrm{char}(k) = 0$. If $n > 2^r - 1$ and $n \geq 4$ then $\mathcal{Q}^{n-1}$ is unirational.*

Proof. Since $n > 2^r - 1$ and k is C_r all fibers of π have a point. The general fiber of π is a smooth quadric and hence the set of its rational points is dense. So, the set of rational points $\mathcal{Q}^{n-1}(k)$ of $\mathcal{Q}^{n-1}$ is also dense.

Then $\psi(\mathcal{Q}^{n-1}(k))$ is dense in $\mathcal{Q}^{n-1}_+$ and since the smooth locus of $\widetilde{\pi}$ is open there is a point $q \in \psi(\mathcal{Q}^{n-1}(k))$ such that the fiber $X_{(\overline{z}_0,\overline{z}_1,\overline{z}_2)}$ of $\widetilde{\pi}$ through q, where $[\overline{z}_0 : \overline{z}_1 : \overline{z}_2] = \widetilde{\pi}(q)$, is smooth at q.

Hence, [14, Proposition 2.3] yields that $X_{(\overline{z}_0,\overline{z}_1,\overline{z}_2)}$ is unirational and to conclude it is enough to apply Proposition 3.1. $\square$

Remark 3.4. A divisor of bidegree $(1,2)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^{h+1}$ is rational. Furthermore, if $\mathcal{Q}^{n-1} \subset \mathbb{P}^1 \times \mathbb{P}^n$ is a divisor of bidegree $(2,2)$ the variety $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ in Proposition 3.1 is a quadric hypersurface. So arguing as in the proof of Proposition 3.2 we get that $\mathcal{Q}^{n-1}$ is unirational provided it has a point.

Now, let $\mathcal{Q}^h \subset \mathbb{P}^{n-h} \times \mathbb{P}^{h+1}$ be a divisor of bidegree $(2,2)$ having a point $p \in \mathcal{Q}^h$, and $L \subset \mathbb{P}^{h+1}$ a general line through the image of p. The preimage $\pi_2^{-1}(L)$ of L via the second projection is a divisor of bidegree $(2,2)$ in $\mathbb{P}^{n-h} \times \mathbb{P}^1$ with a point. Hence, by the first part of the remark $\pi_2^{-1}(L)$ is unirational and since it dominates $\mathbb{P}^{n-h}$ Proposition 2.11 yields that $\mathcal{Q}^h$ is unirational.

4. Quadric bundles with positive volume

Let $\pi : \mathcal{Q}^{n-1} \subset \mathcal{T}_{a_0,\dots,a_n} \to \mathbb{P}^1$ be a quadric bundle of multidegree $(d_{0,0},\dots,d_{n,n})$ cut out by an equation as in (2.4) and set

$$\mathcal{Q}_j^{n-j-2} = \mathcal{Q}^{n-1} \cap \{y_0 = \cdots = y_j = 0\} \subset \mathcal{T}_{a_{j+1},\dots,a_n}$$

for $j = 0, \dots, n-2$. Then $\mathcal{Q}_j^{n-j-2} \to \mathbb{P}^1$ is a quadric bundle with discriminant of degree $\delta_{\mathcal{Q}_j^{n-j-2}} = d_{j+1,j+1} + \cdots + d_{n,n}$. Note that by Proposition 2.11 if $\mathcal{Q}_j^{n-j-2}$ is unirational for some $j = 0, \dots, n-2$ then $\mathcal{Q}^{n-1}$ is also unirational.

We will denote by $\sigma = \sigma(x_0, x_1, y_{n-2}, y_{n-1}, y_n)$ the polynomial defining $\mathcal{Q}_{n-3}^1$ in $\mathcal{T}_{a_{n-2},a_{n-1},a_n}$ and by $\rho = \rho(x_0, x_1)$ the discriminant polynomial of the conic bundle $\pi_{|\mathcal{Q}_{n-3}^1} : \mathcal{Q}_{n-3}^1 \to \mathbb{P}^1$.

We will assume that σ does not have a factor of positive degree depending just on x_0, x_1 in order to rule out the cases in which the conic bundle $\mathcal{Q}_{n-3}^1$ splits as the union of some fibers of the projective bundle in which it is embedded and a conic bundle with discriminant of smaller degree.

Lemma 4.1. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a quadric bundle. Assume that one of the following holds:*

(a) *$n \geq 3$ and either σ splits as the product of two polynomials both depending on y_{n-2}, y_{n-1}, y_n or σ does not have a factor of positive degree depending just on x_0, x_1 and either*
   (i) *$\delta_{\mathcal{Q}_{n-3}^1}\{0, 2, 4, 6\}$ and $\mathcal{Q}_{n-3}^1$ has a smooth point; or*
   (ii) *$\delta_{\mathcal{Q}_{n-3}^1} \in \{1, 3, 5, 7\}$;*

(b) *$n = 2$ and either σ splits as the product of two polynomials both depending on y_{n-2}, y_{n-1}, y_n or σ does not have a factor of positive degree depending just on x_0, x_1, the discriminant polynomial ρ is not identically zero and either*
   (i) *$\delta_{\mathcal{Q}_{n-3}^1}\{0, 2, 4, 6\}$ and $\mathcal{Q}_{n-3}^1$ has a smooth point; or*
   (ii) *$\delta_{\mathcal{Q}_{n-3}^1} \in \{1, 3, 5, 7\}$.*

Then $\mathcal{Q}^{n-1}$ is unirational.

Proof. Assume that σ splits as the product of two polynomials both depending on y_{n-2}, y_{n-1}, y_n. Then $\mathcal{Q}_{n-3}^1$ splits as a union of surfaces S_1, S_2 such that $S_1 \cap S_1$ is a rational section of $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$. Now, assume that σ does not split as the product of two polynomials both depending on y_{n-2}, y_{n-1}, y_n.

If $\delta_{\mathcal{Q}_{n-3}^1} = 0$ then $\mathcal{Q}_{n-3}^1 = \mathbb{P}^1 \times C$ where C is smooth conic. Consider the projection $\pi_2 : \mathcal{Q}_{n-3}^1 \to C$. If $p \in \mathcal{Q}_{n-3}^1$ is a point then $\mathbb{P}^1 \times \pi_2(p)$ is a section of $\mathcal{Q}_{n-3}^1 \to \mathbb{P}^1$ and hence $\mathcal{Q}_{n-3}^1$ is rational.

If $\delta_{\mathcal{Q}_{n-3}^1} = 1$ then one of the $\sigma_{i,i}$ must be zero and by Remark 2.12 $\mathcal{Q}^{n-1}$ is rational.

If $\delta_{\mathcal{Q}_{n-3}^1} = 2$ then $d_{n-2,n-2} = 2$, $d_{n-1,n-1} = d_{n,n} = 0$. If the point of $\mathcal{Q}_{n-3}^1$ lies on $\{y_{n-2} = 0\}$ then $\mathcal{Q}_{n-3}^1 \cap \{y_{n-2} = 0\}$ is rational. Assume that the point of $\mathcal{Q}_{n-3}^1$ does not lie on $\{y_{n-2} = 0\}$. Note that there is a blow-down morphism $\mathcal{Q}_{n-3}^1 \to Q \subset \mathbb{P}^3$ onto a quadric surface contracting $\mathcal{Q}_{n-3}^1 \cap \{y_{n-2} = 0\}$. If $\mathcal{Q}_{n-3}^1$ has a smooth point then Q also has a smooth point and hence it is rational.

If $\delta_{\mathcal{Q}^1_{n-3}} = 3$, keeping in mind Remark 2.12, we must have $d_{n-2,n-2} = d_{n-1,n-1} = d_{n,n} = 1$. Then $\mathcal{Q}^1_{n-3}$ is a surface of bidegree $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ and hence it is rational by Remark 3.4.

If $\delta_{\mathcal{Q}^1_{n-3}} = 4$ we have the following two possibilities:

$$\begin{pmatrix} d_{n-2,n-2} & d_{n-2,n-1} & d_{n-2,n} \\ d_{n-2,n-1} & d_{n-1,n-1} & d_{n-1,n} \\ d_{n-2,n} & d_{n-1,n} & d_{n,n} \end{pmatrix} = \left\{ \begin{pmatrix} 4 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}.$$

If the point of $\mathcal{Q}^1_{n-3}$ lies on $\{y_{n-2} = 0\}$ then $\mathcal{Q}^1_{n-3} \cap \{y_{n-2} = 0\}$ is rational. Otherwise we proceed as follows: consider the case

$$\begin{pmatrix} d_{n-2,n-2} & d_{n-2,n-1} & d_{n-2,n} \\ d_{n-2,n-1} & d_{n-1,n-1} & d_{n-1,n} \\ d_{n-2,n} & d_{n-1,n} & d_{n,n} \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}; \quad (a_{n-2}, a_{n-1}, a_n) = (2,0,0).$$

The divisor $\overline{H}_2$ induces the morphism

$$\begin{array}{ccc} \mathcal{Q}^1_{n-3} & \longrightarrow & S \subset \mathbb{P}^4_{(\xi_0,\dots,\xi_4)} \\ ([x_0 : x_1], [y_0 : y_1 : y_2]) & \mapsto & [x_0^2 y_0 : x_0 x_1 y_0 : x_1^2 y_0, y_1, y_2] \end{array}$$

contracting $\mathcal{Q}^1_{n-3} \cap \{y_{n-2} = 0\}$ where

$$S = \{\xi_1^2 - \xi_0\xi_2 = P(\xi_0, \dots, \xi_4) = 0\}$$

with $P \in k[\xi_0, \dots, \xi_4]_2$. Set $L_v = \{\xi_0 = \xi_1 = \xi_2 = 0\}$. The smooth point of $\mathcal{Q}^1_{n-3}$ yields a smooth point $q \in S$. Up to a change of variables we may assume that $q = [1 : 0 : 0 : 0 : 0]$. The projection of S from q is a cubic surface $S' \subset \mathbb{P}^3_{(z_0,\dots,z_3)}$ containing a line L given by the projection of L_v and singular at a 0-dimensional cycle of length two supported on L given by the projection of $L_v \cap \{P = 0\}$.

Assume that S' has a triple point $p \in S'$. Up to a change of variables we may assume that $p = [0 : 0 : 0 : 1]$. Then the equation of S' does not depend on z_3 and hence the equation of $\mathcal{Q}^1_{n-3} \subset \mathcal{T}_{2,0,0}$ does not depend on y_2. In particular, ρ is identically zero. If $n = 2$ this contradicts the hypotheses. Therefore S' does not have a triple point and hence [29, Theorem 1.2] yields that S' is unirational. If $n \geq 3$ the singular locus of $\mathcal{Q}^1_{n-3}$ yields a section of $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ and hence $\mathcal{Q}^{n-1}$ is rational.

Now, consider the case

$$\begin{pmatrix} d_{n-2,n-2} & d_{n-2,n-1} & d_{n-2,n} \\ d_{n-2,n-1} & d_{n-1,n-1} & d_{n-1,n} \\ d_{n-2,n} & d_{n-1,n} & d_{n,n} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}; \quad (a_{n-2}, a_{n-1}, a_n) = (1,1,0).$$

If the point of $\mathcal{Q}^1_{n-3}$ lies on $\{y_{n-2} = y_{n-1} = 0\}$ then $\sigma_{n,n} = 0$ and hence $\mathcal{Q}^1_{n-3}$ is rational. So we can assume that the point of $\mathcal{Q}^1_{n-3}$ lies either in the chart $\{y_{n-2} \neq 0\}$ or in the chart $\{y_{n-1} \neq 0\}$.

Since the roles of y_{n-2} and y_{n-1} are completely symmetric we can assume without loss of generality that the point is in the chart $\{y_{n-1} \neq 0\}$. Dehomogenizing the polynomial of $\mathcal{Q}^1_{n-3}$ with respect to x_1 and y_{n-1} and homogenizing the resulting polynomial by adding a new variable we get a quartic surface $Y \subset \mathbb{P}^3_{(x_0,y_{n-2},y_n,z)}$ that is birational to $\mathcal{Q}^1_{n-3}$ and has a point $p \in Y \setminus \{z = 0\}$.

Writing down the equation of Y we see that its singular locus consists of two double lines $L_1, L_2 \subset \{z = 0\}$ intersecting at $[0 : 0 : 1 : 0]$. The projections from these two lines show that Y admits two conic bundle structures: one is the conic bundle structure with started with and the other one is induced by projecting $\mathcal{Q}^1_{n-3}$ onto $\mathbb{P}^1_{(y_0,y_1)}$. A fiber of one of these conic bundle structure it then transverse to the other conic bundle structure.

If $\delta_{\mathcal{Q}^1_{n-3}} = 5$ then

$$\begin{pmatrix} d_{n-2,n-2} & d_{n-2,n-1} & d_{n-2,n} \\ d_{n-2,n-1} & d_{n-1,n-1} & d_{n-1,n} \\ d_{n-2,n} & d_{n-1,n} & d_{n,n} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}; \quad (a_{n-2}, a_{n-1}, a_n) = (1, 0, 0)$$

and arguing as in the first subcase of the case $\delta_{\mathcal{Q}^1_{n-3}} = 4$ one sees that there is a morphism $\mathcal{Q}^1_{n-3} \to S \subset \mathbb{P}^3$ where S is a cubic surface containing a line. If S has a triple point then we find a contradiction as in the case $\delta_{\mathcal{Q}^1_{n-3}} = 4$. If S does not have triple points then by [29, Theorem 1.2] it is unirational.

If $\delta_{\mathcal{Q}^1_{n-3}} = 7$ we have two possibilities:

$$\begin{pmatrix} d_{n-2,n-2} & d_{n-2,n-1} & d_{n-2,n} \\ d_{n-2,n-1} & d_{n-1,n-1} & d_{n-1,n} \\ d_{n-2,n} & d_{n-1,n} & d_{n,n} \end{pmatrix} = \left\{ \begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix} \right\}.$$

When

$$\begin{pmatrix} d_{n-2,n-2} & d_{n-2,n-1} & d_{n-2,n} \\ d_{n-2,n-1} & d_{n-1,n-1} & d_{n-1,n} \\ d_{n-2,n} & d_{n-1,n} & d_{n,n} \end{pmatrix} = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}; \quad (a_{n-2}, a_{n-1}, a_n) = (2, 0, 0)$$

note that $C = \mathcal{Q}^1_{n-3} \cap \{y_{n-2} = 0\}$ is a curve of bidegree $(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and hence it is rational.

Now, consider the second subcase. If the point $p = \{y_0 = y_1 = \sigma_{n,n} = 0\} \in \mathcal{Q}^1_{n-3}$ is smooth then the projection onto $\mathbb{P}^1_{(y_1,y_2)}$ yields an elliptic fibration. The elliptic involution yields on $\mathcal{Q}^1_{n-3}$ a birational automorphism called the Bertini's involution of $\mathcal{Q}^1_{n-3}$. Arguing as in [26, Section 4] one gets that such involution does not preserve the conic bundle structure and maps the fiber of the conic bundle through p to a rational multisection.

Assume that $p \in \mathcal{Q}^1_{n-3}$ is singular. The curve $C = \mathcal{Q}^1_{n-3} \cap \{y_{n-2} = 0\}$ lies on a Hirzebruch surface $\mathbb{F}_1$ and is mapped via the blow-down morphism $\mathbb{F}_1 \to \mathbb{P}^2$ onto a nodal cubic curve, and hence C is rational.

If $\delta_{\mathcal{Q}^1_{n-3}} = 6$ and $\mathcal{Q}^1_{n-3}$ has a smooth point, following [26, Section 8], we can consider the conic bundle with discriminant of degree seven constructed by blowing-up such point. Such conic bundle is unirational thanks to the previous analysis for the case $\delta_{\mathcal{Q}^1_{n-3}} = 7$.

Finally, to get the unirationality of $\mathcal{Q}^{n-1}$ it is enough to apply Proposition 2.11. □

Remark 4.2. The existence of a smooth point in Lemma 4.1 is necessary. For instance, consider the conic bundle

$$S = \{x_0^2y_0^2 + x_0^2y_1^2 + x_1^2y_2^2 = 0\} \subset \mathbb{P}^1_{(x_0,x_1)} \times \mathbb{P}^2_{(y_0,y_1,y_2)}$$

over $k = \mathbb{Q}$. Note that S is singular at $([0 : 1], [0 : 0 : 1])$ and along $\{x_0 = y_2 = 0\}$, and that S is not unirational. Indeed, if it were the set of rational points of S would be dense in S. However, setting $x_0 = y_2 = 1$ we get the conic fibration $\{y_0^2+y_1^2+x_1^2 = 0\} \subset \mathbb{A}^3_{(x_1,y_0,y_1)}$ and the conic $C_t = \{y_0^2 + y_1^2 + t^2 = 0\}$ does not have points for all $t \neq 0$.

Proposition 4.3. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a quadric bundle of multidegree $(d_{0,0}, \dots, d_{n,n})$. Assume that the polynomial σ defining the conic bundle*

$$\mathcal{Q}^1_{n-3} = \mathcal{Q}^{n-1} \cap \{y_0 = \cdots = y_{n-3} = 0\} \subset \mathcal{T}_{a_{n-2},\dots,a_n}$$

does not have a factor of positive degree depending just on x_0, x_1 and that if $n = 2$ then ρ is not identically zero. If

(i) *either $d_{n-2,n-2} + d_{n-1,n-1} + d_{n,n} \in \{1, 3, 5, 7\}$; or*
(ii) *$d_{n-2,n-2} + d_{n-1,n-1} + d_{n,n} \in \{0, 2, 4, 6\}$ and $\mathcal{Q}^1_{n-3}$ has a smooth point;*

then $\mathcal{Q}^{n-1}$ is unirational.

Proof. Since $\delta_{\mathcal{Q}^1_{n-3}} = d_{n-2,n-2} + d_{n-1,n-1} + d_{n,n} \leq 7$ the claim follows from Lemma 4.1. □

Lemma 4.4. *Let $\pi : \mathcal{Q}^2 \to \mathbb{P}^1$ be a smooth quadric surface bundle. Then σ does not have a factor of positive degree depending just on x_0, x_1.*

Proof. If σ has a factor of positive degree depending just on x_0, x_1 we may write

$$\mathcal{Q}^2 = \left\{ y_0 \sum_{i=0}^{3} \sigma_{0,i} y_i + \alpha(x_0, x_1) \sum_{1 \leq i \leq j \leq 3} \sigma_{i,j} y_i y_j = 0 \right\} \subset \mathcal{T}_{a_0,\dots,a_3}.$$

Hence, $\{y_0 = \alpha(x_0, x_1) = \sum_{i=1}^{3} \sigma_{0,i} y_i = \sum_{1 \le i \le j \le 3} \sigma_{i,j} y_i y_j = 0\} \subseteq \mathrm{Sing}(\mathcal{Q}^2)$ and so $\mathcal{Q}^2$ would be singular. $\square$

Theorem 4.5. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a quadric bundle. Assume that $(-K_{\mathcal{Q}^{n-1}})^n > 0$ and $\delta_{\mathcal{Q}^{n-1}}$ is odd. If either*

(i) *$n \le 5$, $\mathcal{Q}^{n-1}$ has a point and is otherwise general; or*
(ii) *$\delta_{\mathcal{Q}^{n-1}} \le 3n + 1$ and $\mathcal{Q}^{n-1}$ is general;*

then $\mathcal{Q}^{n-1}$ is unirational. Furthermore, if $n \le 3$ the above statements hold for any smooth quadric bundle.

Proof. By Proposition 2.15 we have that $(-K_{\mathcal{Q}^{n-1}})^n > 0$ if and only if $\delta_{\mathcal{Q}^{n-1}} \le 4n - 1$. If $n = 2$ then $\delta_{\mathcal{Q}^{n-1}} \le 7$ and we conclude by Proposition 4.3. Assume that $n \ge 3$.

If $d_{n-2,n-2}+d_{n-1,n-1}+d_{n,n} > 7$ then $d_{n-2,n-2} \ge 4$ unless $(d_{n-2,n-2}, d_{n-1,n-1}, d_{n,n}) = (3, 3, 3)$. If $d_{n-2,n-2} = 4$ then $d_{n-1,n-1} \ge 2$ and $d_{n,n} \ge 2$. So $\delta_{\mathcal{Q}^{n-1}} \ge 4(n-1)+2+2 = 4n > 4n-1$, a contradiction. If $d_{n-2,n-2} \ge 5$ then $d_{n-1,n-1} \ge 1$ and $d_{n,n} \ge 1$. So $\delta_{\mathcal{Q}^{n-1}} \ge 5(n-1)+1+1 = 5n-3 > 4n-1$ for $n > 2$, a contradiction.

Since $\mathcal{Q}^{n-1}$ is general we may assume that $\mathcal{Q}^1_{n-3}$ satisfies the hypotheses of Proposition 4.3. Moreover, since $\delta_{\mathcal{Q}^{n-1}}$ is odd all the $d_{i,i}$ are odd and hence $d_{n-2,n-2} + d_{n-1,n-1} + d_{n,n}$ is also odd. Then $d_{n-2,n-2} + d_{n-1,n-1} + d_{n,n} \in \{1, 3, 5, 7\}$ unless $(d_{n-2,n-2}, d_{n-1,n-1}, d_{n,n}) = (3, 3, 3)$ and by Proposition 4.3 $\mathcal{Q}^{n-1}$ is unirational.

Now, consider the case $(d_{n-2,n-2}, d_{n-1,n-1}, d_{n,n}) = (3, 3, 3)$. Since $n \le 5$ the quadric bundle $\mathcal{Q}^{n-1}$ is a divisor of bidegree $(3, 2)$ either in $\mathbb{P}^1 \times \mathbb{P}^4$ or in $\mathbb{P}^1 \times \mathbb{P}^5$ and since by hypothesis it has a point we conclude by Proposition 3.2.

If $\delta_{\mathcal{Q}^{n-1}} \le 3n+1$ we will show that the case $(d_{n-2,n-2}, d_{n-1,n-1}, d_{n,n}) = (3, 3, 3)$ can be ruled out and so we will not need the existence of a point anymore.

First, note that $d_{0,0} + \cdots + d_{n,n} \le 3n+1$ implies $d_{n-2,n-2} + d_{n-1,n-2} + d_{n,n} \le 7$. For $n = 2$ the claim is trivial. We proceed by induction on $n \ge 2$. If $d_{0,0} = 2$ then $d_{i,i} \le 2$ for all $i = 0, \dots, n$ and then $d_{n-2,n-2} + d_{n-1,n-2} + d_{n,n} \le 6$. Similarly $d_{0,0} = 1$ yields $d_{n-2,n-2} + d_{n-1,n-2} + d_{n,n} \le 3$, and $d_{0,0} = 0$ implies $d_{n-2,n-2} + d_{n-1,n-2} + d_{n,n} = 0$. So we may assume that $d_{0,0} \ge 3$.

Now, $d_{0,0}+\cdots+d_{n,n} \le 3n+1$ and $d_{0,0} \ge 3$ yield that $d_{1,1}+\cdots+d_{n,n} \le 3n+1-d_{0,0} \le 3n+1-3 = 3(n-1)+1$ and by induction we get that $d_{n-2,n-2} + d_{n-1,n-2} + d_{n,n} \le 7$. Since $\delta_{\mathcal{Q}^{n-1}}$ is odd to conclude it is enough to apply Proposition 4.3. Finally, if $n \le 3$ it is enough to apply Lemma 4.4 when $n = 3$ and [26, Corollary 8] for $n = 2$. $\square$

Corollary 4.6. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a general quadric bundle over an infinite field k. If $(-K_{\mathcal{Q}^{n-1}})^n > 0$ then there exists a quadratic extension k' of k such that $\mathcal{Q}^{n-1}$ is unirational over k'. Furthermore, if $n \le 3$ the above statement holds for any smooth quadric bundle.*

Proof. In (2.4) set $x_0 = y_0 = \cdots = y_{n-2} = 0$, $x_1 = 1$. Then we get a homogeneous polynomial $f(y_{n-1}, y_n)$ of degree two with coefficients in k. Set $\overline{f}(y_n) = f(1, y_n)$ and let k' be the splitting field of $\overline{f}$ over k. Then $\mathcal{Q}^{n-1}$ has a point over k'. Arguing as in the proof of Theorem 4.5 we have that $d_{n-2,n-2} + d_{n-1,n-1} + d_{n,n} \leq 7$ with the only exception $(d_{n-2,n-2}, d_{n-1,n-1}, d_{n,n}) = (3, 3, 3)$. Hence, to conclude it is enough to apply Propositions 3.2 and 4.3. $\square$

Definition 4.7. When $n-h \geq 2$ we will say that a quadric bundle $\pi : \mathcal{Q}^h \subset \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-h}$ satisfies condition † if $\mathcal{Q}^h_{|L}$ is general as a splitting quadric bundle, where $L \subset \mathbb{P}^{n-h}$ is a general line.

Corollary 4.8. *Let $\pi : \mathcal{Q}^h \to \mathbb{P}^{n-h}$ be a quadric bundle with discriminant of odd degree $\delta_{\mathcal{Q}^h}$ and satisfying condition †. If either*

(i) $\delta_{\mathcal{Q}^h} \leq 3h + 4$*; or*
(ii) $\delta_{\mathcal{Q}^h} \leq 4h + 3$, $h \leq 4$, *k is C_r and* $h + 2 > 2^{r+n-h-1}$*;*

then $\mathcal{Q}^h$ is unirational. Furthermore, if $h \leq 2$ the above statements hold for any smooth quadric bundle.

Proof. By Lemma 2.8 it is enough to prove that the quadric bundle $\widetilde{\mathcal{Q}}^h_\eta \to \mathbb{P}^1_F$ is unirational over $F = k(t_1, \ldots, t_{n-h-1})$. Recall that by the construction in Section 2.7 we have that $\delta_{\widetilde{\mathcal{Q}}^h_\eta} = \delta_{\mathcal{Q}^h}$. If $\delta_{\widetilde{\mathcal{Q}}^h_\eta} = \delta_{\mathcal{Q}^h} \leq 3h + 4$ we conclude by Theorem 4.5.

When $\delta_{\mathcal{Q}^h} \leq 4h + 3$ and $h \leq 4$ in order to apply Theorem 4.5 we need to produce a point on $\widetilde{\mathcal{Q}}^h_\eta$. Fix a general hyperplane $H \subset \mathbb{P}^{n-h}$. Keeping in mind the construction in Section 2.7 notice that $\widetilde{\mathcal{Q}}^h_\eta$ has a point if and only if $\mathcal{Q}^h_{|H} \to H$ has a rational section. Finally, since k is C_r and $h + 2 > 2^{r+n-h-1}$ Remark 2.9 yields that such a rational section exists. $\square$

Remark 4.9. By Corollary 4.8 we have that a quadric bundle $\mathcal{Q}^h \to \mathbb{P}^{n-h}$ with discriminant of odd degree $\delta_{\mathcal{Q}^h} \leq 4h + 3$ satisfying † is unirational in the following cases:

(i) k is algebraically closed and $n - h = 2$, $h \in \{1, 2\}$, or $n - h = 3$, $h \in \{3, 4\}$;
(ii) k is C_1 and $n - h = 1$, $1 \leq h \leq 4$, or $n - h = 2$, $h \in \{3, 4\}$;
(iii) k is C_2 and $n - h = 1$, $h \in \{3, 4\}$.

Corollary 4.10. *Assume that k is algebraically closed and let $\mathcal{Q}^1 \to \mathbb{P}^2$ be a smooth conic bundle. If the discriminant curve $D_{\mathcal{Q}_1}$ has a point $p \in D_{\mathcal{Q}_1}$ of multiplicity m_p and $\delta_{\mathcal{Q}^1} \leq m_p + 7$. Then $\mathcal{Q}^1$ is unirational.*

Proof. Consider the conic bundle $\widetilde{\mathcal{Q}}^1_\eta$ over $F = k(t)$ in Section 2.7 constructed by projecting from the point $p \in D_{\mathcal{Q}_1}$ of multiplicity m_p. Assume that $m_p \geq 2$. The conic bundle $\widetilde{\mathcal{Q}}^1_\eta$ has then a multiple fiber F_p defined over k with two A_1 singularities on it also defined over k.

By blowing-up these two singular points and then blowing-down the strict transform of F_p we get a conic bundle $\overline{\mathcal{Q}}^1_\eta$ with a new reducible fiber whose components are defined over k. So we may blow-down one of these components to get a conic bundle $\hat{\mathcal{Q}}^1_\eta$ with seven reducible fibers which is therefore unirational by [26, Corollary 8].

If $m_p = 1$ then $\widetilde{\mathcal{Q}}^1_\eta$ is already in the form $\overline{\mathcal{Q}}^1_\eta$ and to conclude we may proceed as in the last part of the proof for the case $m_p \geq 2$. Let $L \subset \mathbb{P}^2$ be a general line. By Remark 2.9 $\mathcal{Q}^1_L \rightarrow L$ has a section and hence $\widetilde{\mathcal{Q}}^1_\eta$ has a point. Therefore, by Proposition 4.3 $\widetilde{\mathcal{Q}}^1_\eta$ is unirational and to conclude it is enough to apply Lemma 2.8. $\square$

Corollary 4.11. *Assume that k is algebraically closed and let $\mathcal{Q}^2 \rightarrow \mathbb{P}^2$ be a smooth quadric surface bundle. If $\delta_{\mathcal{Q}^2} \leq 12$ then $\mathcal{Q}^2$ is unirational.*

Proof. Fix a point $p \in \mathbb{P}^2$ and consider the quadric bundle $\widetilde{\mathcal{Q}}^2_\eta$ in Section 2.7. Then $\widetilde{\mathcal{Q}}^2_\eta$ is smooth and $\delta_{\widetilde{\mathcal{Q}}^2_\eta} \leq 12$. Arguing as in the proof of Theorem 4.5 we get that the conic bundle $\widetilde{\mathcal{Q}}^1_{\eta,n-3} \subset \widetilde{\mathcal{Q}}^2_\eta$, as defined in the beginning of Section 4, has discriminant of degree $\delta_{\widetilde{\mathcal{Q}}^1_{\eta,n-3}} \leq 8$. Now, $\widetilde{\mathcal{Q}}^1_{\eta,n-3}$ spreads to a conic bundle $\mathcal{Q}^1 \rightarrow \mathbb{P}^2$ which is contained in $\mathcal{Q}^2$. We distinguish two cases: either $D_{\mathcal{Q}^1}$ contains p with multiplicity m_p and $\delta_{\mathcal{Q}^1} = m_p + 7$, or $D_{\mathcal{Q}^1}$ does not contain p and $\delta_{\mathcal{Q}^1} = 8$. In the first case arguing as in the proof of Corollary 4.10 we get that $\widetilde{\mathcal{Q}}^1_{\eta,n-3}$ is unirational. In the second case we project from a smooth point $q \in D_{\mathcal{Q}^1}$ and proceeding as in the proof of Corollary 4.10 we construct a conic bundle $\hat{\mathcal{Q}}^1_\eta$ with seven reducible fibers birational to $\widetilde{\mathcal{Q}}^1_{\eta,n-3}$. Finally, by Corollary 4.10 we get that $\widetilde{\mathcal{Q}}^1_{\eta,n-3}$ is unirational and Proposition 2.11 yields the unirationality of $\mathcal{Q}^2$. $\square$

Remark 4.12. Corollaries 4.10 and 4.11 hold for all smooth conic and quadric surface bundles, with no generality assumption, since their proofs rely on Theorem 4.5 which holds for all smooth quadric bundles as long as $n \leq 3$. The proof of Theorem 4.5 relies in turn on [26, Corollary 8], which holds for all smooth conic bundles, and on Lemma 4.4 which says that if $\mathcal{Q}^2 \rightarrow \mathbb{P}^1$ is smooth then the polynomial σ defining the conic bundle

$$\mathcal{Q}^1_0 = \mathcal{Q}^2 \cap \{y_0 = 0\} \subset \mathcal{T}_{a_1,a_2,a_3}$$

can not have a factor of positive degree depending just on x_0, x_1.

Proposition 4.13. *Let $\mathcal{Q}^h \subset \mathbb{P}^{n-h} \times \mathbb{P}^{h+1}$ be a divisor of bidegree $(3,2)$. Assume that k is C_r with $h+2 > 2^{r+n-h-1}$ and $h \geq 3$. If either $\mathcal{Q}^h$ is general or $\mathcal{Q}^h$ is smooth and* $\mathrm{char}(k) = 0$ *then $\mathcal{Q}^h$ is unirational.*

Proof. The quadric bundle $\widetilde{\mathcal{Q}}^h_\eta$ is a divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^{h+1}$ over $F = k(t_1,\dots,t_{n-h-1})$. Arguing as in the proof of Corollary 4.8 we produce a point of $\widetilde{\mathcal{Q}}^h_\eta$. So

Proposition 3.2 and Corollary 3.3 imply that $\widetilde{\mathcal{Q}}^h_\eta$ is unirational and Lemma 2.8 yields the unirationality of $\mathcal{Q}^h$. $\square$

Finally, we consider quadric bundles $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ over a number field.

Lemma 4.14. *Let $\pi : \mathcal{Q}^h \to \mathbb{P}^{n-h}$ be a smooth quadric bundle over a number field k with discriminant of odd degree. If $h \geq 3$ then $\mathcal{Q}^h$ has a point.*

Proof. For a place v of k we will denote by k_v the completion of k at v. Consider a general line $L \subset \mathbb{P}^{n-h}$ and the quadric bundle $\pi_{|L} : \mathcal{Q}^h_{|L} \to L$. Then $\delta_{\mathcal{Q}^h_{|L}}$ is odd and $\mathcal{Q}^h_{|L}$ has a point over the reals and hence $\mathcal{Q}^h_{|L}(k_v)$ is not empty for all real places of k. Therefore, also $\mathcal{Q}^h(k_v)$ is not empty for all real places of k.

Now, a general fiber $Q^h_q = \pi^{-1}(q) \subset \mathbb{P}^{h+1}$ is a quadric hypersurface of dimension $h \geq 3$ over k. By [44, Chapter IV, Theorem 6] Q^h_q has a point over all the p-adic places of k. We conclude that $\mathcal{Q}^h$ has a point over k_v for all places v, both real and p-adic, of k. Finally, [14, Proposition 3.9] yields that $\mathcal{Q}^h$ has a point over the base field k. $\square$

Theorem 4.15. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a general quadric bundle over a number field. If $(-K_{\mathcal{Q}^{n-1}})^n > 0$ and $\delta_{\mathcal{Q}^{n-1}}$ is odd then $\mathcal{Q}^{n-1}$ is unirational. Furthermore, if $n \leq 3$ the above statement holds for any smooth quadric bundle.*

Proof. Note that the only case in the proof of Theorem 4.5 for which the existence of a point is needed is $(d_{n-2,n-2}, d_{n-1,n-1}, d_{n,n}) = (3,3,3)$. In this case we have $d_{n-3,d-3} = d_{n-4,n-4} = 3$ otherwise $\delta_{\mathcal{Q}^{n-1}} > 4n-1$. So $\mathcal{Q}^3_{n-4}$ is a divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$.

By Lemma 4.14 a smooth divisor of bidegree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ has point, and hence the claim follows by Propositions 2.11, 3.2. $\square$

We sum up the main results of this section as follows:

Corollary 4.16. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a quadric bundle over an infinite field k. Assume that $(-K_{\mathcal{Q}^{n-1}})^n > 0$ and $\delta_{\mathcal{Q}^{n-1}}$ is odd. If either*

(i) *$n \leq 5$, $\mathcal{Q}^{n-1}$ has a point and is otherwise general; or*
(ii) *$\mathcal{Q}^{n-1}$ is general and $\delta_{\mathcal{Q}^{n-1}} \leq 3n+1$; or*
(iii) *$\mathcal{Q}^{n-1}$ is general and k is a number field;*

then $\mathcal{Q}^{n-1}$ is unirational.

Proof. It is enough to apply Theorems 4.5 and 4.15. $\square$

Corollary 4.17. *Let $\pi : \mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a general quadric bundle over a number field k. If $(-K_{\mathcal{Q}^{n-1}})^n > 0$ and $\delta_{\mathcal{Q}^{n-1}}$ is odd then there exists an $\epsilon > 0$ such that for any open subset $U \subset \mathcal{Q}^{n-1}$*

$$N(U,B) = \sharp\{p \in U(k) \mid ht(p) \leq B\} \geq c_{\mathcal{Q}^{n-1}} B^{\epsilon}$$

for $B \mapsto \infty$, where $c_{\mathcal{Q}^{n-1}}$ depends on $\mathcal{Q}^{n-1}$, and ht is the multiplicative height [5, Definition 1.5.4].

Proof. By Theorem 4.15 there is a dominant rational map $\theta : \mathbb{P}^n \dashrightarrow \mathcal{Q}^{n-1} \subset \mathbb{P}^N$ given by polynomials of a certain degree, say d. Then points of $\mathbb{P}^n$ of height at most $B^{\frac{1}{d}}$ are mapped to points of X of height at most B.

Let $V \subset \mathcal{Q}^{n-1}$ be the subset over which θ is finite. Then the number of points of height at most B of U grows at least as the number of points of height at most $B^{\frac{1}{d}}$ of $\theta^{-1}(U \cap V)$ which in turn grows at least as the number of points of height at most $B^{\frac{1}{d}}$ of $\mathbb{P}^n$ minus the number of points of height at most $B^{\frac{1}{d}}$ of a closed subset $Z \subset \mathbb{P}^n$.

Now, to conclude it is enough to observe that $N(\mathbb{P}^n, B^{\frac{1}{d}})$ grows as $c(n)B^{\frac{n+1}{d}}$ [36, Theorem 2.1] while $N(Z_i, B^{\frac{1}{d}})$ grows as $c'(\dim(Z_i), a_i, n)B^{\frac{1}{d}(\dim(Z_i)+\frac{1}{a_i}+\epsilon)}$ where Z_i is an irreducible component of Z of degree a_i and $0 < \epsilon \ll 1$ [37, Theorem B]. $\square$

Recently, the distribution of rational points on divisors in products of projective spaces has been much investigated, see for instance [6], [7].

5. Quadric bundles over finite fields

In this section $k = \mathbb{F}_q$ will be a finite field with q elements. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of c quadric hypersurfaces and $F_{c-1}(X) \subset \mathbb{G}(c-1,n)$ the variety parametrizing $(c-1)$-dimensional linear subspaces contained in X, where $\mathbb{G}(c-1,n)$ is the Grassmannian of $(c-1)$-dimensional linear subspaces of $\mathbb{P}^n$ in its Plücker embedding.

Remark 5.1. Thanks to the study of the geometry and of the canonical class of $F_{c-1}(X)$ in [15], and arguing as in the proof of [23, Theorem 2.1] we have that if $n \geq c(c+1)$ then $F_{c-1}(X)$ has a point and X is rational.

Lemma 5.2. *Let $X \subset \mathbb{P}^n$ be a complete intersection of two quadric hypersurfaces. Assume that $n \geq 4$. If either X is singular but not a cone or X is smooth and contains a line then X is rational. Furthermore, if X is smooth then X is unirational.*

Proof. Write $X = Q_1 \cap Q_2$ with $Q_i \subset \mathbb{P}^n$ quadric hypersurface. Since $n \geq 4$ Remark 2.10 yields that X has a point $p \in X$.

First assume X to be smooth and that there is a line L through p intersecting X in at least three points counted with multiplicity. Then L is contained in both Q_1 and Q_2. So $L \subset X$ and Remark 5.1 yields that X is rational. If all the lines through p intersect X in at most two points counted with multiplicity then the image of the birational projection of X from p is a cubic hypersurface $Y \subset \mathbb{P}^{n-1}$ with no triple points. Since $n \geq 4$ Remark 2.10 implies that Y has a point and hence by [29, Theorem 1] Y is unirational.

If X has a double point $p \in X$ the projection of X from p is a quadric Y which again by Remark 2.10 has a smooth point, and if X has a triple point $p \in X$ the projection of X from p is a hyperplane Y. In both cases Y is rational. $\square$

Proposition 5.3. *Let $\mathcal{Q}^{n-1} \subset \mathbb{P}^1 \times \mathbb{P}^n$ be a divisor of bidegree $(d, 2)$. If $d = 1$ then $\mathcal{Q}^{n-1}$ is rational. Furthermore, in the same notation of Proposition 3.1, if either $d = 2$ or*

- (i) *$d = 3$ and for some $[\overline{z}_0 : \overline{z}_1 : \overline{z}_2] \in \mathbb{P}^2$ the complete intersection $X_{(\overline{z}_0,\overline{z}_1,\overline{z}_2)}$ is not a cone; or*
- (ii) *$d \geq 4$, $n \geq d(d-1)$ and for some $[\overline{z}_0 : \cdots : \overline{z}_{d-1}] \in \mathbb{P}^{d-1}$ the complete intersection $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ is smooth;*

then $\mathcal{Q}^{n-1}$ is unirational.

Proof. The case $d \in \{1, 2\}$ follows from Remark 3.4. Assume that $d = 3$. Then $X_{(\overline{z}_0,\overline{z}_1,\overline{z}_2)}$ is a complete intersection of two quadrics and the claim follows from Proposition 3.1 and Lemma 5.2. If $d \geq 4$, $n \geq d(d-1)$ then $X_{(\overline{z}_0,\dots,\overline{z}_{d-1})}$ is a complete intersection of $d-1$ quadrics and the claim follows from Remark 5.1. $\square$

Proposition 5.4. *Let $\mathcal{Q}^{n-1} \to \mathbb{P}^1$ be a quadric bundle with $\delta_{\mathcal{Q}^{n-1}} \leq 4n-1$. In the notation of Section 4 assume that ρ does not vanish at all points of $\mathbb{P}^1$, σ does not have a factor of positive degree depending just on x_0, x_1 and $\mathcal{Q}^1_{n-3}$ has a smooth point; and if $d_{n-4} = d_{n-3} = d_{n-2} = d_{n-1} = d_n = 3$ assume in addition, in the notation of Proposition 3.1, that there exists $[\overline{z}_0 : \overline{z}_1 : \overline{z}_2] \in \mathbb{P}^2$ such that $X_{(\overline{z}_0,\overline{z}_1,\overline{z}_2)}$ is irreducible and not a cone. Then $\mathcal{Q}^{n-1}$ is unirational.*

Proof. For the first part it is enough to argue as in the proof of Theorem 4.5. Assume that $d_{n-2} = d_{n-1} = d_n = 3$. Then $n \geq 3$ and $d_{n-4} = d_{n-3} = d_{n-2} = d_{n-1} = d_n = 3$. So $\mathcal{Q}^3_{n-5} \subset \mathbb{P}^1 \times \mathbb{P}^4$ is a divisor of bidegree $(3, 2)$ and to conclude it is enough to apply Proposition 5.3. $\square$

Remark 5.5. Over a finite field $k = \mathbb{F}_q$ the conic bundle S in Remark 4.2 is unirational. Indeed, the fiber $\{y_0^2 + y_1^2 + y_2^2 = 0\}$ over $[x_0, x_1] = [1 : 1]$ is smooth and hence has $q + 1$ points which by the description of $\mathrm{Sing}(S)$ in Remark 4.2 are smooth points of S.

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