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A Nonlinear Dynamics for Risk Contagion: Analyzing the Risk-Free Equilibrium

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Abstract

In this paper we perform the stability analysis of the risk-free equilibrium point which characterizes a financial contagion dynamics. The model is formulated in the Susceptible-Infected-Recovered approach by employing an analogy between economic sectors and ecosystems. The dynamics is nonlinear and characterized by a time delay which represents a period of financial immunity got after risk infection. In addition, contagion phenomenon is modelled by employing a Holling Type II functional response taking into account an incubation time for risk infection. The analysis around the risk-free steady state is performed in terms of both local asymptotic stability and global asymptotic stability by classical approach. Our results highlight the crucial role of the incubation time in establishing whether risk crisis can be eliminated from the economic sector at the long run or it continues to exist in.

Mathematics Subject Classification: 91B05, 91B55, 34K04

Keywords: Risk contagion, Financial immunity, SIR model, Delay Differential equation, Stability analysis

1 Introduction

This work represents the second part of a complete analysis performed on the differential problem proposed in [2]. As an application, this model aims to describe the nonlinear dynamics related to the spreading of risk contagion throughout a given economic system or sector.

The topic of financial contagion is of great interest in the economic literature in different frameworks. Actually, a financial distress can be regarded like an infection occurring in different contexts, not only in the banking system or in financial markets, but also infections in terms of credit and debit relationships between companies. In this framework, the study developed in [2] relies on the analogy established between economic systems and ecosystems so that the Susceptible-Infected-Recovered (SIR) approach can be reinterpreted and applied to model risk contagion among economic players. As it is well-known, the SIR method has been introduced in the literature in order to model the spreading of a disease during an epidemic (see for instance [7], [8], [9]); anyway, it also reveals to be useful for modelling risk contagion (the reader is referred to [1], [4], [5], [6], [10]).

As already mentioned, here we focus on a compartmental approach where the susceptible players refer to low risk and are vulnerable to be infected by other economic agents with a high risk, therefore the possibility of becoming new high risk players is increasing. Infected agents have their risk at an extremely high level and they can infect other agents. Recovered players refer to economic agents with capability of risk control after infection, they can keep their risk at a low level and be not infectious. This recovered agents get a temporary immunity ability (but not lifelong) after curing; the immunity period τ represents the time lag of the model. We also assume that some economic

players indefinitely leave the economic sector at an elimination rate which depends on the risk level of compartments; then, we denote by γ_L and γ_H the death rates of agents with low risk and high risk, respectively, under the assumption that $0 < \gamma_L \leq \gamma_H < 1$. In addition, the phenomenon of contagion is modelled by a Holling Type II functional response with attack rate $a > 0$ and incubation time $h > 0$.
Therefore, as in [2], we deal with the following delay differential system

$$\frac{dS(t)}{dt} = b - \gamma_L S(t) - \frac{aS(t)}{1 + haS(t)} I(t) + \delta e^{-\gamma_L \tau} I(t - \tau), \tag{1}$$

$$\frac{dI(t)}{dt} = \frac{aS(t)}{1 + haS(t)} I(t) - (\gamma_H + \delta) I(t), \tag{2}$$

$$\frac{dR(t)}{dt} = \delta I(t) - \gamma_L R(t) - \delta e^{-\gamma_L \tau} I(t - \tau), \tag{3}$$

where $S(t)$, $I(t)$ and $R(t)$ are the densities of susceptible players, infected and recovered agents, respectively, and the parameters $b > 0$ and $0 < \delta < 1$ represent the birth rate of new players and the recovery rate from the high risk. We remark that (1) and (2) do not depend on $R(t)$, therefore equation (3) can be omitted without loss of generality. For this reason, we focus on system (1)-(2) equipped by suitable initial conditions

$$S(0) = S_0 > 0, \tag{4}$$

$$I(s) = I_0(s) \geq 0, \quad \text{for all } s \in [-\tau, 0], \quad \text{with } I_0(0) > 0, \tag{5}$$

where $I_0(\cdot)$ is a continuous function and represents the history of the infected class in the whole time lag interval $[-\tau, 0]$.
As we point out in [2], this problem admits a unique solution which takes on positive values. Moreover, model (1)-(2) has got the risk-free equilibrium

$$E_0^* = \left(\frac{b}{\gamma_L}, 0 \right),$$

and, for certain values of the parameters, another non-trivial steady state $E_\tau^* = (S_\tau^*, I_\tau^*)$ such that

$$S_\tau^* = \frac{\gamma_H + \delta}{a(1 - h(\gamma_H + \delta))}, \qquad I_\tau^* = \frac{\gamma_L(\gamma_H + \delta)}{(\gamma_H + \delta(1 - e^{-\gamma_L \tau}))\, a(1 - h(\gamma_H + \delta))} (\rho_0 - 1),$$

where

$$\rho_0 = \frac{ba(1 - h(\gamma_H + \delta))}{\gamma_L(\gamma_H + \delta)},$$

is the basic reproduction number of the risk infection. In the next Sections, we discuss both local and global asymptotic stability concerning the risk-free equilibrium point E_0^* at the long run.

2 Local stability of the risk-free equilibrium

We start our analysis by arguing about the local stability around the risk-free steady state. In this respect, we notice that the characteristic equation at the risk-free equilibrium is

$$(\lambda + \gamma_L)\left(\lambda - \frac{\gamma_L(\gamma_H + \delta)}{\gamma_L + hab}(\rho_0 - 1)\right) = 0,$$

which has real roots

$$\lambda_1 = -\gamma_L, \qquad \lambda_2 = \frac{\gamma_L(\gamma_H + \delta)}{\gamma_L + hab}(\rho_0 - 1).$$

Due to the fact that both roots are negative in the case when $\rho_0 < 1$, then the following result is proved.

Proposition 2.1 *If the following condition holds*

$$\rho_0 < 1, \tag{6}$$

then the equilibrium point $E_0^* = (b/\gamma_L, 0)$ *is locally asymptotically stable. Under the opposite assumption* $\rho_0 > 1$*, the risk-free steady state is unstable.*

3 Global stability of the risk-free equilibrium

With the aim of going deep inside and investigating the global stability conditions for the risk-free equilibrium, wed prove the following result.

Proposition 3.1 *Under the assumption that*

$$\rho_0 < 0, \tag{7}$$

the risk-free equilibrium $E_0^* = (b/\gamma_L, 0)$ *is globally asymptotically stable.*

The proof can be carried out by performing the following transformation

$$\widehat{S}(t) = S(t) - \frac{b}{\gamma_L}, \qquad \widehat{I}(t) = I(t);$$

which yields the following equivalent formulation of model (1)-(2):

$$\frac{d\widehat{S}(t)}{dt} = -\gamma_L \widehat{S}(t) - \frac{aS(t)}{1 + haS(t)}\widehat{I}(t) + \delta e^{-\gamma_L \tau}\,\widehat{I}(t-\tau), \tag{8}$$

$$\frac{d\widehat{I}(t)}{dt} = \left(\frac{aS(t)}{1 + haS(t)} - (\gamma_H + \delta)\right)\widehat{I}(t). \tag{9}$$

In [2] we prove that $S(t) > 0$ and $I(t) > 0$ for any time t, therefore the estimates $0 < \frac{aS(t)}{1+haS(t)} < \frac{1}{h}$ holds. As a consequence, there exists a constant

$$C = \frac{1}{h} - (\gamma_H + \delta) = \frac{1 - h(\gamma_H + \delta)}{h},$$

such that equation (9) yields

$$\frac{d\widehat{I}(t)}{dt} \leq C\,\widehat{I}(t).$$

By integration we have

$$\widehat{I}(t) \leq I_0(0)\,e^{Ct}, \tag{10}$$

for any time $t \geq 0$. We remark that $C < 0$, due to assumption (7). Under this argument, it follows that $\widehat{I}(t) \to 0$ in the case when $t \to +\infty$. Therefore, the original $I(t)$ has the same behaviour, i.e. $I(t) \to 0$ as $t \to +\infty$.
Concerning the behaviour of $\widehat{S}(t)$ at the long run, we plug (10) into (8) and obtain

$$\frac{d\widehat{S}(t)}{dt} \leq -\gamma_L \widehat{S}(t) + \delta e^{-\gamma_L \tau}\, I_0(0)\, e^{C(t-\tau)}.$$

The previous inequality is integrated so that:

- in the case when $\gamma_L + C \neq 0$, we obtain

$$\widehat{S}(t) \leq \widehat{S}(0)e^{-\gamma_L t} + \frac{\delta\, I_0(0)\, e^{-(\gamma_L + C)\tau}}{\gamma_L + C}\left(e^{Ct} - e^{-\gamma_L t}\right);$$

- in the opposite case when $\gamma_L = -C$, we have

$$\widehat{S}(t) \leq \widehat{S}(0)e^{-\gamma_L t} + \delta\, I_0(0) t e^{-\gamma_L t}.$$

In both cases $\widehat{S}(t)$ is bounded from above by the an exponentially decaying function. As a result, at $t \to +\infty$ we get $\widehat{S}(t) \to 0$ which is equivalent to have $S(t) \to \frac{b}{\gamma_L}$. Thus, we have proved that E_0^* is globally asymptotically stable.

4 Concluding remarks

The analysis of stability we have performed so far points out that the risk-free equilibrium E_0^* is globally asymptotically stable and attracts all risk trajectories in the case when incubation time h is large enough such that

$$h > \frac{1}{\gamma_H + \delta}. \tag{11}$$

Indeed, it is evident that the bound in (11) yields the basic reproduction number ρ_0 assume a negative value according to assumption (7) in Proposition 3.1. As expected, this finding suggests that the incubation time period is so long that the infected class empties because the rate of susceptible capture by contagion is lower than the rates at which high risk agents may leave the economic sector and are recovered. In this case, risk infection and financial crisis are dispatched to be overcome.
On the other hand, the length of the financial immunity does not affect the stability of the risk-free equilibrium point.

As a further development, the analysis of the problem is completed in [3] where the risk trajectory behaviour is studied at the long run in terms of local and global stability of the not risk-free equilibrium point E^*_τ.

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