DOTTORATO DI RICERCA IN
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Prof. **Lorenzo Brasco**

Perspectives in materials science: variational models, asymptotic analysis and homogenization

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Abstract

The central themes explored in my research include different areas within the field of Calculus of Variations, with a particular emphasis on applications to materials science and image analysis.
The first chapter collects the results that are a common base for other chapters, among which the main ones are related to the topics of Orlicz–Sobolev spaces, Supremal functionals, Young measures, BD spaces.
In the second chapter, we prove Γ-convergence results in the framework of generalized Orlicz spaces, extending theorem from [46, 83]. The proofs follow the general approach of these papers but developing improved techniques with weaker assumptions. Indeed, we can consider non-doubling functionals as well in our scenario. We cover both norm- and modular-type energies with much the same method. See [38] as a reference.
The third chapter presents the results from [39], which can be regarded as a continuation of the previous work. We aim to weaken the convexity assumptions on the density f. More precisely, we assume that the density satisfies a milder condition (know as $\mathrm{curl}_{(p>1)}$- Young quasiconvexity, introduced in [137]) than level convexity of $f(x, u, \cdot)$ adopted to prove the results in [83, 38]. On the other hand, we are also able to remove this assumption, under the restriction that $f(x, \cdot)$ satisfies a growth condition of type $|\cdot|^\gamma$, also from above, and assuming that $\varphi_n(x,t) := t^{p_n(x)}$, with $\sup_n \frac{p_n^+}{p_n^-} \leq \beta$, for a $\beta > 0$, where $p_n^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$, $p_n^- := \operatorname{ess\,inf}_{x\in\Omega} p_n(x)$.
The fourth chapter is dedicated to homogenization. It includes the introduction of two-scale Young measures, their characterization, and homogenization results, as developed in [40]. Our aim consists of studying the asymptotic behavior as $\varepsilon \searrow 0^+$ of $I_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to \mathbb{R}^+$ defined as

$$I_\varepsilon(u) := \iint_{\Omega\times\Omega} W\left(x, x', \frac{x}{\varepsilon}, \frac{x'}{\varepsilon}, u(x), u(x')\right) dx dx'.$$

This kind of non-local functional appears in the mathematical modeling of peridynamics [143], ferromagnetics [139], etc. Here, we follow an analogous approach in terms of suitable Young measures, considering L^p embedded in the space of Young measures equipped with the narrow topology.
The fifth chapter focuses on Orlicz spaces and the relaxation of variational functionals defined in such settings, referring to the results of [41]. We start from the study of integral functionals defined in variable exponent, both in the context of regularity theory and applications to electrorheological fluids and homogenization. The main contribution of [41] is to introduce the first model which features both quasi-convex anisotropy of [9, 88, 89] and the combination of linear and superlinear growth of [100, 101, 82]. We obtain an integral representation in the space

of functions with bounded variation of generalized growth of this functional in terms of a suitable point-wise *recession function.*
In the last chapter, we deal with image reconstruction problem. In order to counteract such problems, several higher-order models have been introduced. A particularly successful one is the second-order Total Generalized Variation, defined as

$$TGV^2_\alpha(u) := \sup\left\{\int_\Omega u\,\mathrm{div}^2\psi\,dx \;:\; \psi\in C^2_c(\Omega;\mathbb{R}^{n\times n}_{sym}), \|\psi\|_\infty\le\alpha_1, \|\mathrm{div}\,\psi\|_\infty\le\alpha_2\right\},$$

for every $u\in L^1_{loc}(\Omega)$, with $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^+\times\mathbb{R}^+$. Our contribution is twofold. First, we introduce a novel anisotropic functional setting of functions with bounded deformation, namely the space BD^φ of bounded deformation fields with generalized Orlicz growth. Second, we analyze a Musielak-Orlicz anisotropic Total Generalized Variation class of regularizers ($TGV^{\varphi,2}_\alpha$). After studying the basic properties of $TGV^{\varphi,2}_\alpha$, we identify a dual representation, and show how classical existence and stability for classical Total Generalized Variation carry over to our Musielak-Orlicz setting. See [42] as a reference.

Introduction

The aim of this thesis is to present the results of my research work, including published articles, preprints, and ongoing projects. These works have been developed in collaboration with various co-authors across different areas within the field of Calculus of Variations, with a particular emphasis on applications to materials science and image analysis. The central themes explored in my research include:

- A significant part of my research has focused on the study of generalized Orlicz–Sobolev spaces, developed in collaboration with P. Harjulehto and P. Hästö. These spaces can be seen as an extension of the classical Sobolev or variable exponent Lebesgue spaces. They were first introduced and studied by Z. W. Birnbaum and W. Orlicz in [43], with the aim of providing a more flexible functional framework. The main motivation behind their use is to construct an ideal setting for establishing Γ-convergence results for more general classes of functionals. These results are later complemented by more specific results obtained in variable exponent Lebesgue spaces—which are, in a sense, special cases of Musielak-Orlicz spaces (also known as Generalized Orlicz spaces). These later contributions were carried out with several collaborators, notably including M. Eleuteri.
- Supremal functionals, first introduced by G. Aronsson (see [13], [14], [17]), are functionals defined in terms of the essential supremum of a given function. These are particularly well-suited for modeling situations where the interest lies not in the average behavior of a system, but rather in the worst-case scenario. Moreover, their formal Euler–Lagrange equation is the so-called ∞-Laplacian, a degenerate and highly nonlinear operator that arises in the theory of absolutely minimizing Lipschitz extensions. For this reason, the study of approximations of supremal models with integral functionals via Γ-convergence becomes essential.
- Homogenization theory is a fundamental tool for deriving effective models in the presence of fine-scale oscillations. In this context, nonlocal functionals play a central role: unlike classical integral energies depending only on

pointwise or gradient information, they account for long-range interactions through kernels that couple pairs of points in the domain. Such energies naturally arise in models of peridynamics, phase transitions, image processing, and fracture mechanics, where nonlocality captures behaviors that cannot be described by purely local variational principles. Within the framework of Γ-convergence, homogenization theoryallows one to identify the effective energy, ensure stability of minimization problems, and understand how nonlocal microscopic interactions influence the limiting local or nonlocal models. In a dedicated chaper, an homogenization result —obtained in collaboration with E. Zappale and A. Torricelli— for nonlocal integral functionals is presented.

- Among the various topics addressed in this thesis, problems related to BD (functions of bounded deformation) spaces and image reconstruction are presented. The focus has been on extending classical results to a more general Orlicz setting, where the growth of the functional depends on a function φ. In this context—as part of a broader effort to lay the groundwork for future developments—certain statements required considering the specific case $\varphi(x,t) = t^{p(x)}$ corresponding to the variable exponent framework.

Although not central research themes of this work, several fundamental tools play a crucial role in establishing the results of this thesis. Among them:

- Γ-convergence is a fundamental tool in the Calculus of Variations, particularly for the optimal approximation of functionals. Its strength lies in its ability to preserve the convergence of minimizers and minimization problems under suitable compactness assumptions. In particular, by the fundamental theorem of Γ-convergence, if a sequence of functionals $\{F_n\}$ Γ-converges to a functional F and is equi-coercive, then not only do minimizers of F_n converge to minimizers of F, but the existence of minimizers is also inherited in the limit. This notion, first introduced by De Giorgi ([76]), and then developed by and refined in the works of authors such as, Dal Maso and Braides (see [49] and [73]), plays a central role throughout this thesis, both as a methodological foundation and as a unifying theme across various research directions.

- Young measures, originally introduced by L. C. Young in [150], are probability measures designed to capture oscillatory and concentration effects in sequences of bounded functions. They are particularly useful in describing weak limits of nonlinear functionals and in preserving fine information about the limiting behavior of variational problems. Beyond their extensive use in the proofs of various Γ-convergence results throughout this thesis, a dedicated chapter presents a characterization result as well as a specific homogenization result formulated in the topology induced by Young measures.

- Convex analysis also plays an essential supporting role throughout this thesis. Tools such as convexity and lower semicontinuity of functionals, properties of convex conjugate provide a robust framework for establishing compactness, stability, and variational inequalities. Although not a primary research topic in itself, convex analysis offers the foundational techniques needed to handle nonlocal energies, prove Γ-convergence results, and characterize effective limits of variational models.

The structure of this thesis is the following:

- **Chapter 1** collects the results that are a common base for almost all the following chapters;
- **Chapter 2** collects the results from [38], which include Γ-convergence results for supremal functionals in the Orlicz-Sobolev space setting under power-law type growth;
- **Chapter 3** presents the results from [39], which can be regarded also as a generalization of the previous work. Similar variational results are established under weaker convexity assumptions and, in some specific cases—such as those involving variable exponent spaces—even without convexity assumptions;
- **Chapter 4** is dedicated to non-locals functionals and homogenization. It includes the introduction of two-scale Young measures, their characterization as developed in [40];
- **Chapter 5** focuses on Orlicz spaces and on the relaxation of variational functionals defined in this framework, building upon the results of [41]. In particular, it introduces—for the first time in this context—a variational model involving functions of bounded variation in the Orlicz setting, thereby extending classical relaxation techniques to a broader and more flexible functional framework.
- **Chapter 6**, the final chapter, builds upon the previous one to study total variation within the Orlicz space framework, following the results of [42]. This chapter further introduces a Musielak–Orlicz version of the Total Generalized Variation (TGV), naturally leading to an associated image reconstruction analysis based on these generalized regularization tools.

Each chapter begins with an introduction aimed at contextualizing the results and presenting the relevant mathematical background.

Contents

Chapter 1

Preliminaries

In this chapter we collect the main results that will be used as a base for the theorems presented in future chapters. Most of these results will be used in all, or at least in most, of the chapters.

1.1 Notation

Throughout this work, $\Omega \subset \mathbb{R}^N$ is an open bounded set, $\mathcal{A}(\Omega)$ denotes the family of the open subsets of Ω, and for any $m \in \mathbb{N}, \mathcal{L}^m$ is the Lebsegue measure in $\mathbb{R}^m$. Also, we denote with $\mathcal{B}_k$ the family of all Borel sets of $\mathbb{R}^k$. $Q := (0,1)^N$ is the unit cube in $\mathbb{R}^N$. The symbols $\langle\cdot\rangle$ and $[\cdot]$ stand, respectively, for the fractional and integer part of a number, or a vector componentwise. The Dirac mass at a point $a \in \mathbb{R}^m$ is denoted by δ_a. For every Lebesgue measurable set $A \subset \mathbb{R}^m$, the symbol $⨍_A$ stands for the average $(\mathcal{L}^m(A))^{-1} \int_A$.
Let U be an open subset of $\mathbb{R}^m$. Then:

- For any linear space X, by X' we denote its dual space.
- $\mathcal{C}_c(U)$ is the space of the continuous functions $f : U \to \mathbb{R}$ with compact support.
- $\mathcal{C}^\infty_{per}(Q)$ is the space of Q-periodic functions on $\mathbb{R}^N$.
- $\mathcal{C}^\infty_c(U)$ is the space of the smooth functions $f : U \to \mathbb{R}$ with compact support.
- $\mathcal{C}_0(U)$ is the closure of $\mathcal{C}_c(U)$ with respect to the uniform convergence; it coincides with the space of the continuous functions $f : U \to \mathbb{R}$ such that, for every $\eta > 0$, there exists a compact set $K_\eta \subset U$ with $|f| < \eta$ on $U \setminus K_\eta$.
- $\mathcal{C}_b(U; \mathbb{R}^m)$ is the space of continuous and bounded functions from U to $\mathbb{R}^m$.

- $\mathcal{M}(U)$ is the space of real-valued Radon measures with finite total variation. We recall that, by the Riesz Representation Theorem, $\mathcal{M}(U)$ can be identified with the dual space of $\mathcal{C}_0(U)$ through the duality

$$\langle \mu, \phi \rangle = \int_U \phi \, d\mu, \;\; \mu \in \mathcal{M}(U), \;\; \phi \in \mathcal{C}_0(U).$$

- $\mathcal{P}(U)$ denotes the space of probability measures on U, i.e. the space of all $\mu \in \mathcal{M}(U)$ such that $\mu \geq 0$ and $\mu(U) = 1$.

- Instead, $\mathcal{M}(\Omega, \mathbb{R}^m)$ denotes the space of $\mathbb{R}^m$-valued Radon measures with finite total variation.

- $L^0(\Omega)$ is the set of measurable functions $f : \Omega \to \mathbb{R}$

- $L^1(\Omega, \mathcal{C}_0(U))$ is the space of maps $\phi : \Omega \to \mathcal{C}_0(U)$ such that:

  (i) ϕ is strongly measurable, i.e. there exists a sequence of simple functions $s_n : \Omega \to \mathcal{C}_0(U)$ such that $\|s_n(x) - \phi(x)\|_{\mathcal{C}_0(U)} \to 0$ for a.e. $x \in \Omega$;

  (ii) $x \mapsto \|\phi(x)\|_{\mathcal{C}_0(U)} \in L^1(\Omega)$.

- $L^\infty_w(\Omega, \mathcal{M}(U))$ is the space of maps $\nu : \Omega \to \mathcal{M}(U)$ such that:

  (i) ν is weak* measurable, i.e. $x \mapsto \langle \nu_x, \phi \rangle$ is measurable for every $\phi \in \mathcal{C}_0(U)$;

  (ii) $x \mapsto \|\nu_x\|_{\mathcal{M}(U)} \in L^\infty(\Omega)$. The space $L^\infty_w(\Omega, \mathcal{M}(U))$ can be identified with the dual space of $L^1(\Omega, \mathcal{C}_0(U))$ via the duality

$$\langle \mu, \phi \rangle = \int_\Omega \int_U \phi(x, \xi) d\mu_x(\xi) dx, \;\; \mu \in L^\infty_w(\Omega, \mathcal{M}(U)), \phi \in L^1(\Omega, \mathcal{C}_0(U)),$$

  where $\phi(x, \xi) := \phi(x)(\xi)$ for all $(x, \xi) \in \Omega \times U$.

- In case one of the previous sets is considered with respect a specific probability measure μ, the set will be indicated with μ as a subscript, for example as in $L^1_\mu(U, \mathbb{R}^N)$

- The space $L^p_{per}(Q)$ stands for the L^p-closure of all functions $f \in \mathcal{C}(\mathbb{R}^N)$ which are $Q-$periodic.

Furthermore, the notation $f \approx g$ means that there exist two constants $c_1, c_2 > 0$ such that $c_1 f \leq g \leq c_2 f$. Given an exponent $p \in (1, +\infty)$, we denote by p' its Hölder conjugate exponent, namely $p' := \frac{p}{p-1}$. If $p = 1$, then its Hölder conjugate exponent is $p' = +\infty$ and viceversa. We denote with $|\cdot|$ the Frobenius norm for

matrices, i.e. $|M|^2 := \mathrm{tr}(MM^{\mathrm{T}})$ where $M \in \mathbb{R}^{N\times N}$, and with : the scalar product associated with the Frobenius norm, i.e. $A : B := \mathrm{tr}(AB^{\mathrm{T}})$ for $A, B \in \mathbb{R}^{N\times N}$. With the same notation, namely $|\cdot|$, we also indicate the standard vector Euclidean norm, and with $\cdot$ the scalar product in $\mathbb{R}^N$.
We say that a function $\omega : [0,+\infty) \to [0,+\infty]$ is a *modulus of continuity* if it is increasing and such that $\omega(0) = \lim_{t\to 0^+} \omega(t) = 0$. Note that we do not require concavity and we allow extended real values
We denote by $\mathbb{R}^{N\times N}_{\mathrm{sym}}$ the space of real-valued $N \times N$ symmetric matrices. Given a set $O \subset X$, we denote by I_O the indicator function of the set O, namely

$$I_O(x) := \begin{cases} 0 & \text{if } x \in O, \\ +\infty & \text{if } x \notin O, \end{cases}$$

and with χ_O the characteristic function of the set O, namely

$$\chi_O(x) := \begin{cases} 1 & \text{if } x \in O, \\ 0 & \text{if } x \notin O. \end{cases}$$

Given a function $u : \Omega \to \mathbb{R}^m$, we denote by Du its distributional gradient. We define $BV(\Omega;\mathbb{R}^m)$ as the space of functions $u \in L^1(\Omega;\mathbb{R}^m)$ such that $Du \in \mathcal{M}(\Omega;\mathbb{R}^m)$. If $m = 1$, we simply write $BV(\Omega)$.
When $N = m$, for any function $u : \Omega \to \mathbb{R}^N$ we use the notation Eu for the distributional symmetric gradient of u, namely $Eu = \frac{1}{2}(Du + Du^{\mathrm{T}})$, where Du^{T} stands for the transpose of Du. We write

$$BD(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^N) \ : \ Eu \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})\}.$$

We will often indicate by c a generic constant, whose value may change from line to line.

1.2 Level convexity

Definition 1.2.1 (Lower semicontinuity)**.** A function $f : \mathbb{R}^N \to \mathbb{R}$ is said lower semicontinuous in $x_0 \in \mathbb{R}^N$, if

$$\liminf_{x\to x_0} f(x) \geq f(x_0).$$

We are in position to recall the convexity notions which are key in the supremal setting, referring to [137] for a general review.

Definition 1.2.2 (Level convexity). We say that $f : \mathbb{R}^N \to \mathbb{R}$ is *level convex* if the level-set $\{\xi \in \mathbb{R}^N : f(\xi) \leq t\}$ is convex for every $t \in \mathbb{R}$.

The following is a substitute for Jensen's inequality for level convex functions.

Lemma 1.2.3 (Theorem 1.2, [30]). *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a lower semicontinuous and level convex function and let μ be a probability measure in the open set $U \subset \mathbb{R}^N$. Then*

$$f\left(\int_U u\, d\mu\right) \leq \mu\text{-}ess\text{-}sup_U f \circ u$$

for every $u \in L^1_\mu(U, \mathbb{R}^N)$.

1.3 Young measures

In this subsection we briefly recall some results on the theory of Young measures (see e.g. [26], [37], [147, 148] and [87, 138]). Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $d, k \geq 1$, we denote by $C_c(\Omega; \mathbb{R}^d)$ the set of continuous functions with compact support in Ω, endowed with the supremum norm. The dual of the closure of $C_c(\Omega; \mathbb{R}^d)$ may be identified with the set of $\mathbb{R}^d$-valued Radon measures with finite mass $\mathcal{M}(\Omega; \mathbb{R}^d)$, through the duality

$$\langle \mu, \varphi \rangle := \int_\Omega \varphi(\xi) d\mu(\xi)\,, \qquad \mu \in \mathcal{M}(\Omega; \mathbb{R}^d)\,, \qquad \varphi \in C_c(\Omega; \mathbb{R}^d)\,.$$

We recall also the normal integrand and the weak*-measurability notion:

Definition 1.3.1 (Normal integrand). A function $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is called a *normal integrand* if

- f is $\mathcal{L}^N \times \mathcal{B}_d \times \mathcal{B}_k$-measurable;
- $f(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$.

Definition 1.3.2 (Weak*-measurability). A map $\mu : \Omega \mapsto \mathcal{M}(\Omega; \mathbb{R}^d)$ is said to be weak*-measurable if $x \mapsto \langle \mu_x, \varphi \rangle$ are measurable for all $\varphi \in C_c(\Omega; \mathbb{R}^d)$.

The next result is known as the Fundamental result on Young measures (in [122, Theorem 3.1] the reader can also find a proof). Young measures, indicated in the statement as μ_x, have the property to describe the oscillatory behavior of a sequence of bounded functions, in probabilistic terms. For this reason they will be exploited in the proof of several main results of this thesis.

Theorem 1.3.3 (Fundamental Theorem on Young Measures). *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and let $\{V_n\}_n$ be a sequence of measurable functions, $V_n : E \to \mathbb{R}^d$. Then there exists a subsequence $\{V_{n_k}\}_k$ and a weak* measurable map $\mu : E \to \mathcal{M}(\mathbb{R}^d)$ such that the following statements hold:*

(1) $\mu_x \geq 0$, $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} d\mu_x \leq 1$ *for a.e.* $x \in E$;

(2) $\forall \varphi \in C_0(\Omega, \mathbb{R}^d)$

$$\varphi(V_{n_k}) \overset{*}{\rightharpoonup} \bar{\varphi} \quad \textit{in } L^\infty(\mathbb{R}^d),$$

where

$$\bar{\varphi}(x) := \langle \mu_x, \varphi \rangle;$$

(3) *for every compact subset* $K \subset \mathbb{R}^d$, *if* $dist(V_{n_k} K) \to 0$ *in measure, then*

$$supp\, \mu_x \subset K \quad \textit{for a.e. } x \in E;$$

(4) $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.e.* $x \in E$ *if and only if*

$$\lim_{M \to \infty} \sup_k \mathcal{L}^N(\{|V_{n_k} \geq M|\}]) = 0; \tag{1.3.1}$$

(5) *if* $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.e.* $x \in E$ *then in* (3) *we may replace "if" with "if and only if";*

(6) *if* $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.e.* $x \in E$ *and* $A \subset E$ *is measurable and if* $\varphi \in C(\mathbb{R}^d)$ *is such that* $\varphi(V_{n_k})$ *is equi-integrable in* $L^1(A)$ *then*

$$\varphi(V_{n_k}) \rightharpoonup \bar{\varphi} \quad \textit{in } L^1(A).$$

Remark 1.3.4. If $p > 1$ and $\Omega \subseteq \mathbb{R}^N$ is a bounded open set and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$, then the sequence $\{Du_n\}_n$ is equi-integrable and generates a Young measure $\mu = \{\mu_x\}_{x\in\Omega}$ such that $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ for a.e. $x \in \Omega$ and

$$Du(x) = \int_{\mathbb{R}^{Nd}} \xi d\mu_x(\xi)\,.$$

Such a Young measure μ is usually called a $W^{1,p}$-*gradient Young measure*, see [127].

The previous theorem has been presented in order to provide a complete framework for this kind of object to the reader, but the main properties that will be needed in the proof can be summerized in the following Lemma.

Lemma 1.3.5. *Suppose that* Ω *is bounded and* $u_n \rightharpoonup u$ *in* $W^{1,q}(\Omega; \mathbb{R}^d)$, $q \in (1, \infty)$. *Then there exists a family of probability measures* $\{\mu_x\}_{x\in\Omega}$ *on* $\mathbb{R}^{Nd}$ *such that*

$$Du(x) = \int_{\mathbb{R}^{Nd}} \xi \, d\mu_x(\xi) \quad \textit{for a.e. } x \in \Omega$$

and, for any normal integrand $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}^+$,

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n(x), Du_n(x))\, dx \geq \int_\Omega \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi)\, d\mu_x(\xi)\, dx.$$

With the notation of the previous lemma and $v \in L^0(\Omega, \mathbb{R}^d)$, [83, Remark 2.11] states that

$$\lim_{p\to\infty}\left(\int_\Omega \int_{\mathbb{R}^{Nd}} f(x,v(x),\xi)^p \, d\mu_x(\xi)\, dx\right)^{\frac{1}{p}} = \text{ess-sup}_{x\in\Omega}\, \mu_x\text{-ess-sup}_{\xi\in\mathbb{R}^k} f(x,v(x),\xi).$$

Combined with the inequality μ_x-ess-sup$_{\xi\in\mathbb{R}^k} f(x,v(x),\xi) \geq f(x,v(x),Du(x))$ from Lemma 1.2.3, this implies the following:

Corollary 1.3.6. *With the assumptions and notation of the previous lemma, we have*

$$\lim_{p\to\infty}\left(\int_\Omega \int_{\mathbb{R}^{Nd}} f(x,v(x),\xi)^p \, d\mu_x(\xi)\, dx\right)^{\frac{1}{p}} \geq \textit{ess-sup}_{x\in\Omega} f(x,v(x),Du(x))$$

provided that $f(x,z,\cdot)$ is level convex for every $z \in \mathbb{R}^d$ and a.e. $x \in \Omega$.

1.4 Generalized Orlicz spaces

In this subsection, we provide background on generalized Orlicz spaces. The main references have been [66, 98], which are an ideal starting point for Orlicz spaces literature. The main goal for Orlicz-Spaces is to give a suitable setting in order to extend the notion of classic Lebesgue spaces, where instead of controlling a function with its p power, we will work on the composition $\varphi(f)$, where φ is a suitable function yet to define.
Firstly, we need to define the space (and so the properties) that given function has to satisfy. The next conditions, which measure the lower and upper growth-rate of the function φ, will play a fundamental role in defining the main properties we want to exploit.
In order to define the spaces we use in our work, we need to give different notions of norms and modulars that generate our spaces. Note that our terminology differs from the original one in [123], but follows the lines of [82, 104] in order to get a clearer connection between the modulars and their associated norms.

Definition 1.4.1. Let X be a real vector space and let $\|\cdot\| : X \to [0, +\infty]$ be a function on X. Consider the following conditions:

(N1) $\|f\| = 0$ if and only if $f = 0$,

(N2) $\|af\| = |a|\,\|f\|$ for all $f \in X$ and $a \in \mathbb{R}$,

(N3) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$, (N3') there exists $c > 0$ such that $\|f + g\| \leq c\,(\|f\| + \|g\|)$ for all $f, g \in X$.

We say that $\|\cdot\| : X \to [0, +\infty]$ is

- a quasi-seminorm if and only if it satisfies (N2) and (N3'),
- a seminorm if and only if it satisfies (N2) and (N3),
- a quasinorm if and only if it satisfies (N1), (N2) and (N3'),
- a norm if and only if it satisfies (N1), (N2) and (N3).

Note that conditions (N2) and (N3), equivalently (N2) and (N3'), already imply that if $f = 0_X$ then $\|f\| = 0$, namely condition (N1) only adds one of the two implications.

Definition 1.4.2. Let X be a real vector space. A function $\varrho : X \to [0, +\infty]$ is called a quasi-semimodular on X if and only if

(i) $\varrho(0_X) = 0$,

(ii) the function $\lambda \mapsto \varrho(\lambda x)$ is non-decreasing (i.e. $\varrho(\lambda_1 x) \leq \varrho(\lambda_2 x)$ if $\lambda_1 \leq \lambda_2$) on $[0, +\infty)$ for every $x \in X$,

(iii) $\varrho(-x) = \varrho(x)$ for every $x \in X$,

(iv) there exists $\beta \in (0, 1]$ such that for every $x, y \in X$ and every $\theta \in [0, 1]$

$$\varrho\left(\beta(\theta x + (1 - \theta)y)\right) \leq \theta\varrho(x) + (1 - \theta)\varrho(y).$$

If (iv) holds with $\beta = 1$, then ϱ is called semimodular. A quasi-semimodular is called a quasimodular when $\varrho(x) = 0$ if and only if $x = 0_X$. A modular is defined analogously from a semimodular.

Definition 1.4.3. Let X be a real vector space. If ϱ is a quasi-semimodular in X, then the modular space

$$X_\varrho := \{x \in X \ : \ \|x\|_\varrho < +\infty\}$$

is defined by the quasi-seminorm

$$\|x\|_\varrho := \inf\left\{\lambda > 0 \ : \ \varrho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

1.4.1 Generalized Φ-functions and Orlicz spaces

In this subsection we give some definitions regarding Φ-functions and we then define generalized Orlicz (or Musielak-Orlicz) spaces, following the lines of [98]. Since we are also going to make use of singular measures, the assumptions always need to get adjusted to hold for every point, as in [104].

Definition 1.4.4. A function $f : \Omega \to \mathbb{R}$ is called L-almost increasing if there exists $L \geq 1$ such that $f(s) \leq Lf(t)$ for every $s \leq t$. If this inequality holds for $L = 1$, the function is called non-decreasing. We define analogously L-almost decreasing and non-increasing functions.

Definition 1.4.5. We say that a function $\varphi : \Omega \times [0, +\infty) \to [0, +\infty]$ is a weak Φ-function, and we write $\varphi \in \Phi_w(\Omega)$, if the following conditions hold:

- the function $x \mapsto \varphi(x, |f(x)|)$ is measurable for every function $f \in L^0(\Omega)$,
- the function $t \mapsto \varphi(x, t)$ is non-decreasing for every $x \in \Omega$,
- $\varphi(x, 0) = \lim_{t \to 0^+} \varphi(x, t) = 0$ and $\lim_{t \to +\infty} \varphi(x, t) = +\infty$ for every $x \in \Omega$,
- the function $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing on $(0, +\infty)$ for every $x \in \Omega$ with a constant L independent of x.

If, moreover, $\varphi \in \Phi_w(\Omega)$ is convex and left-continuous with respect to t for every $x \in \Omega$, then φ is a convex Φ-function, and we write $\varphi \in \Phi_c(\Omega)$. If φ does not depend on x, then we omit the set and write $\varphi \in \Phi_w$ or $\varphi \in \Phi_c$.

Note that since we deal with conjugates of functions possibly with linear growth at infinity, it is needed to allow for extended real-valued Φ-functions. Now we are in the position of defining generalized Orlicz spaces.

Definition 1.4.6. Let $\varphi \in \Phi_w(\Omega)$ and define

$$\varrho_\varphi(f) := \int_\Omega \varphi(x, |f|)\, dx \qquad \text{for every } f \in L^0(\Omega).$$

Then, the set

$$L^\varphi(\Omega) := (L^0(\Omega))_{\varrho_\varphi} = \Big\{ f \in L^0(\Omega) \ : \ \varrho_\varphi(\lambda f) < +\infty \quad \text{for some } \lambda > 0 \Big\}$$

is called a *generalized Orlicz space* with quasinorm given by $\|f\|_\varphi := \|f\|_{\varrho_\varphi}$. We use the abbreviation $\|f\|_\varphi := \||f|\|_\varphi$ for vector-valued functions.

Thanks to [98, Lemma 3.2.2], it is known that $\|\cdot\|_\varphi$ is a quasinorm in $L^\varphi(\Omega)$ if $\varphi \in \Phi_w(\Omega)$, and a norm if $\varphi \in \Phi_c(\Omega)$.
Now we need to define some standard conditions on Φ-functions that grant them additional properties, which sometimes are inherited by their conjugate functions as well.

Definition 1.4.7. Let $\varphi : \Omega \times [0, +\infty) \to [0, +\infty]$ and let $p, q > 0$. Then we define the following conditions:

(A0) There exists $\beta \in (0, 1]$ such that $\varphi(x, \beta) \leq 1 \leq \varphi(x, \frac{1}{\beta})$ for every $x \in \Omega$.

(aInc)$_p$ There exists $L_p \geq 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L_p-almost increasing on $(0, +\infty)$ for every $x \in \Omega$.

(aDec)$_q$ There exists $L_q \geq 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L_q-almost decreasing on $(0, +\infty)$ for every $x \in \Omega$.

We say that (aInc), respectively (aDec), holds if (aInc)$_p$, respectively (aDec)$_q$, holds for some $p > 1$, respectively $q > 1$.

Remark 1.4.8. Note that, by Definition 1.4.5, every function $\varphi \in \Phi_w(\Omega)$ satisfies (aInc)$_1$.

Note that if $\varphi \in \Phi_w(\Omega)$, then $\varphi(\,\cdot\,, 1) \approx 1$ implies (A0). Moreover, if φ satisfies (aDec), then (A0) and $\varphi(\,\cdot\,, 1) \approx 1$ are equivalent. (aInc) and (aDec) are quantitative versions of the ∇_2 and Δ_2 conditions and measure lower and upper growth rates.

Example 1.4.9 shows that generalized Orlicz spaces cover also L^∞-spaces without the need for special cases. This is of special importance in this thesis as the main focus for the use of these spaces is the limit when the growth-rate tends to infinity.

In this example and in future analogous cases, we use the convention $\infty \cdot 0 = 0$.

Example 1.4.9. Define $\varphi_\infty \in \Phi_w(\Omega)$ by $\varphi_\infty(x, t) := \infty\chi_{(1,\infty)}(t)$. Then φ_∞ is a generalized weak Φ-function. From the definition of modular we see that

$$\rho_{\varphi_\infty}(f) = \begin{cases} 0, & \text{if } |f| \leq 1 \text{ a.e.} \\ \infty, & \text{otherwise.} \end{cases}$$

It follows from the Luxemburg norming procedure that $\|\cdot\|_{\varphi_\infty} = \|\cdot\|_\infty$ and so $L^{\varphi_\infty}(\Omega) = L^\infty(\Omega)$.

Other examples of generalized Orlicz spaces are (ordinary) Orlicz spaces where $\varphi(x,t)$ is independent of x, variable exponent spaces $L^{p(x)}$ where $\varphi(x,t) = t^{p(x)}$ [79], double phase spaces $\varphi(x,t) = t^p + a(x)t^q$ [29, 96], variable exponent double phase spaces $\varphi(x,t) = t^{p(x)} + a(x)t^{q(x)}$ [71] and many other variants (e.g. [22]).

The Orlicz–Sobolev space is defined based on $L^\varphi(\Omega)$ as usual:

Definition 1.4.10 (Orlicz–Sobolev space)**.** Let $\varphi \in \Phi_w(\Omega)$. The function $u \in L^\varphi(\Omega) \cap W^{1,1}_{loc}(\Omega)$ belongs to the *Orlicz–Sobolev space* $W^{1,\varphi}(\Omega)$, if the weak partial derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \ldots, N$, belong to $L^\varphi(\Omega)$. We define a modular and quasinorm on $W^{1,\varphi}(\Omega)$ by

$$\rho_{1,\varphi}(u) := \rho_\varphi(u) + \sum_{i=1}^N \rho_\varphi\left(\frac{\partial u}{\partial x_i}\right) \quad \text{and} \quad \|u\|_{1,\varphi} := \inf\left\{\lambda > 0 : \rho_{1,\varphi}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

1.5 Γ−convergence

This subsection introduces Γ-convergence and justifies its importance in the framework of the Calculus of Variations. The idea is to formalize a framework that allows one to approximate a given functional by a sequence of functionals, in such a way that the corresponding infima converge to the minimum of the original functional. In the first place, we recall the topological definition of Γ-convergence.

Definition 1.5.1 (Definition 4.1, [73])**.** Given a topological space X, the Γ-lower limit and the Γ-upper limit of the sequence $\{F_n\}_n$, $F_n : X \to \overline{\mathbb{R}}$, are functions from X to $\overline{\mathbb{R}}$ defined as

$$\begin{aligned} \Gamma - \liminf_{n\to\infty} F_n(x) &:= \sup_{U\in\mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y\in U} F_n(y) \\ \Gamma - \limsup_{n\to\infty} F_n(x) &:= \sup_{U\in\mathcal{N}(x)} \limsup_{n\to\infty} \inf_{y\in U} F_n(y), \end{aligned}$$

where $\mathcal{N}(x)$ is the family of all neighborhoods of the point x.
If there exists a function $F : X \to \overline{\mathbb{R}}$ such that $F = \Gamma - \liminf_{n\to\infty} F_n(x) = \Gamma - \limsup_{n\to\infty} F_n(x)$, then we say $F = \Gamma - \lim_{n\to\infty} F_n$ and that $\{F_n\}_n$ is Γ-converging to F or that F is the Γ-limit of $\{F_n\}_n$.

In the following, whenever we refer to the Γ-limit (provided that X is a metric space), we will use the sequential characterization, proved to be equivalent to Definition 1.5.1 as described in the next Proposition 1.5.2.

Proposition 1.5.2. *[Proposition 8.1, [73]] Let X be a metric space and let $F_n : X \to \mathbb{R} \cup \{\pm\infty\}$ for every $n \in \mathbb{N}$. Then $\{F_n\}_n$ Γ-converges to F with respect to the strong topology of X (and we write $\Gamma(X)\text{-}\lim_{n\to\infty} F_n = F$) if and only if*

(i) (Γ-lim inf inequality) *for every $x \in X$ and for every sequence $\{x_n\}_n$ converging to x, it is*
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n);$$

(ii) (Γ-lim sup inequality) *for every $x \in X$, there exists a sequence $\{x_n\}_n$ converging to x such that*
$$F(x) = \lim_{n\to\infty} F_n(x_n).$$

It follows from the previous definition that the $\Gamma\text{-}\lim_{n\to\infty} F_n$ is lower semicontinuous on X (see [73, Proposition 6.8]). This implies, as the next example shows, that a constant sequence of functionals may converge to a different Γ-limit.

Example 1.5.3. Let's consider the sequence of functionals $F_n = F : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, defined as
$$F_n(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0. \end{cases}$$
In this case F_n converges pointwise to F everywhere, but F_n does not Γ-converge to F: indeed, $F_n \xrightarrow{\Gamma} G$, where $G(x) = 0$, for every $x \in \mathbb{R}$.

An analogous definition can be given for Γ-convergence with respect to the weak* topology on X, since it is metrizable on compact sets. We denote it by $\Gamma(w^*\text{-}X)\text{-}\lim_{n\to\infty} F_n$. Hence we recall also that the function $F = \Gamma(w^*\text{-}X)\text{-}\lim_{n\to\infty} F_n$ is weakly* lower semicontinuous on X and when $F_n = \psi$ $\forall n \in \mathbb{N}$ then F coincides with the weakly* lower semicontinuous envelope of ψ, i.e.
$$F(x) = \sup\Big\{h(x) : \ \forall\, h : X \to \mathbb{R} \cup \{\pm\infty\} \ \ w^* \text{ l.s.c.}, \ h \leq \psi \text{ on } X\Big\}$$
(see Remark 4.5 in [73]).
The reasons for the interest in Γ-convergence rely on the statement of the next theorem, which provides a converging sequence for the minimum of the limit functional, under a Γ-convergence assumption (see [73, Chapter 7]).

Theorem 1.5.4 (Fundamental Theorem of Γ-convergence)**.** *Let $F_n : X \to (-\infty, +\infty]$ be a sequence of functionals such that*
$$\inf_X F_n(x) > -\infty, \forall n \in \mathbb{N}.$$
Let $x_n \in X, \forall n \in \mathbb{N}$ and let $\varepsilon_n \geq 0$ such that $\varepsilon_n \to 0$, as $n \to \infty$ and
$$F_n(x_n) \leq \inf_X F_n(x) + \varepsilon_n, \ \forall n \in \mathbb{N}.$$

If $x_n \to x$ *for some* $x \in X$ *and if* $F_n \xrightarrow{\Gamma} F$ *for some* $F : X \to [-\infty, +\infty]$ *then*

$$F(x) = \min_X F \text{ and } \lim_{n\to+\infty} F_n(x_n) = \min_X F.$$

For more details about Γ-convergence, the reader can refer to [73] and [49].

Chapter 2

Convergence of generalized Orlicz norms with lower growth rate tending to infinity

2.1 Introduction

In recent years, supremal functionals, i.e.

$$F(u) := \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), Du(x)), \tag{2.1.1}$$

with $\Omega \subset \mathbb{R}^N$ open, $u \in W^{1,\infty}(\Omega; \mathbb{R}^d)$, and $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ a suitable Borel function, have being object of growing interest in view of the many applications. Indeed many variational models are expressed by means of energies which are not of integral form since the relevant quantities entering in the modeling do not express a mean property. As a matter of fact, minimization of L^∞ functionals enters in the description of yield set in plasticity, in optimal design, in the description of the dielectric breakdown (see e.g. [48, 61, 95, 94]), in optical tomography ([109]), in connection with the infinite laplacian as in the seminal papers [13, 14, 15, 16, 17] and recently in [21, 68, 136]. Indeed functionals as in (2.1.1) can be seen as the counterpart of integral functionals, which give rise to the Euler-Lagrange (Euler-Aronsson in this setting) equations (or systems) of Δ_∞-type. On the other hand, for minimization problems involving (2.1.1), it is important to consider the pointwise behavior of the energy density also on very small sets as long as they appear in a very natural way in variational problems, where the relevant quantities do not express a mean property and the values of the energy densities on very small subsets of Ω cannot be neglected.
Motivated both by the above mentioned applications of power-law approximation, and by the fact that L^∞-norm of a measurable function is the limit, as $p \to +\infty$ of

its L^p-norms, many authors (see e.g. [11, 47, 48, 58, 63, 83, 77, 133, 134]) studied the variational convergence of the energies

$$I_p(u) := \left(\int_\Omega f(x, u, Du)^p \, dx \right)^{1/p} \tag{2.1.2}$$

as $p \to \infty$, obtaining a (Γ-)limit of supremal type described in terms of a suitable limiting density, not necessarily coinciding with the original one f. More precisely, the limiting density satisfies suitable convexity properties, for instance in the scalar case (i.e. d or $N = 1$) it is level convex in the last variable.
Still under level convexity assumptions, the energy I_p has been generalized to two complementary cases: Orlicz growth by Bocea and Mihăilescu [46] and variable exponent growth by Eleuteri and Prinari [83]. Instead, in the paper [38], we consider the same problem in the context of generalized Orlicz spaces, which covers both of these as special cases. Generalized Orlicz spaces (also known as Musielak–Orlicz spaces) and related PDE have been intensely studied recently, see, e.g., [66, 102, 105, 106, 108] and references therein. In fact, both the L^∞- and the non-standard growth energies are related to image processing [61, 64, 99].
The proofs follow the general approach of previous papers but developing improved techniques with weaker assumptions. The tools used in [46, 83] led to the unexpected assumption that ratios of the lower and upper growth-rates of the approximating functionals (denoted $\phi_n^\pm$ and $p_n^\pm$ in these papers) are bounded. Thanks to more precise estimates we improve in the paper, we manage to avoid this assumption. Indeed, we need no assumptions about the upper growth-rate and can consider non-doubling functionals as well. For instance, the energy $|Du|^{n+1/|x|}$ that was excluded from previous results. In addition, we manage to deal with open sets of finite measure, whereas previous results required regularity of the boundary. We cover both norm- and modular-type energies with much the same method (cf. Theorems 2.3.3). The general φ-growth highlights the exact properties that are needed for the results more clearly than the L^p-case, as a comparison of the assumptions in Theorems 2.3.2 and 2.3.3 shows, see also Example 2.3.4.
In Section 2.2, key tools for transferring results for constant exponents to the generalized Orlicz case are proved. Section 2.3 with the main convergence results concludes the chapter.

2.2 Asymptotically sharp embeddings

In this section, we present some auxiliary results, which are the main tools to prove the Γ-convergence result. These Orlicz-related propositions are a generalization of classical ones in different settings, such as variable-exponent Lebesgue spaces.

The first in order, the unit ball property, is a fundamental relation between the Orlicz quasinorm and the related modular.

Lemma 2.2.1 (Unit ball property, Lemma 3.2.3, [98])**.** *If $\varphi \in \Phi_w(\Omega)$, then*

$$\|f\|_\varphi < 1 \quad \Rightarrow \quad \rho_\varphi(f) \le 1 \quad \Rightarrow \quad \|f\|_\varphi \le 1.$$

A stronger result can be obtained adding left-continuity on the φ, but this version is enough for our purpouses.
The following proposition plays a significant role in reducing our main problem to the classic Lebesgue space $L^q(\Omega)$, where we can use the results about Young measures previously presented. The embedding is well-known (see [98, Lemma 3.7.7]) but we need a version with an asymptotically sharp constant as $p \to \infty$.

Proposition 2.2.2. *Let Ω have finite measure, $\varphi \in \Phi_w(\Omega)$ and $c, L \ge 1$. Assume that*

- $\frac{1}{c} \le \varphi(x, 1) \le c$ *for a.e.* $x \in \Omega$*;*
- φ *satisfies* $(aInc)_p$ *with constant* L *for some* $p \ge 1$*.*

Then $L^\varphi(\Omega) \hookrightarrow L^p(\Omega)$ *and*

$$\| \cdot \|_{L^p} \le (2L(|\Omega| + c))^{\frac{1}{p}} \| \cdot \|_\varphi. \tag{2.2.1}$$

Proof. The condition $(A0)$ follows from $\frac{1}{c} \le \varphi(x, 1) \le c$ and $(aInc)_p$, and so the embedding holds [98, Lemma 3.7.7]. By the unit ball property (Lemma 2.2.1), the norm inequality $\|u\|_{L^p} \le C\|u\|_\varphi$ follows once we prove that

$$\rho_\varphi\left(\frac{C\,|u|}{\|u\|_p}\right) = \int_\Omega \varphi\left(x, \frac{C\,|u|}{\|u\|_p}\right) dx > 1.$$

Let us derive a lower bound for φ. By $(aInc)_p$,

$$\varphi(x, t) \ge \frac{1}{L} t^p \varphi(x, 1) \ge \frac{1}{Lc} t^p$$

when $t \ge L^{1/p} \ge 1$. If we subtract $\frac{1}{c}$ from the right-hand side, we obtain a function which is negative when $t < L^{1/p}$ and therefore a suitable lower bound also when t is in this range. It follows that

$$\varphi(x, t) \ge \frac{1}{Lc} t^p - \frac{1}{c}$$

for every $t \ge 0$. Applying this in the modular, we see that

$$\int_\Omega \varphi\left(x, \frac{C\,|u|}{\|u\|_p}\right) dx \ge \int_\Omega \frac{1}{Lc}\left(\frac{C\,|u|}{\|u\|_p}\right)^p - \frac{1}{c}\, dx = \frac{C^p}{Lc} - \frac{|\Omega|}{c}.$$

Thus the desired norm-inequality holds if the right-hand side is greater than 1, which is equivalent to $C^p > L(|\Omega| + c)$. $\square$

The next proposition essentially takes care of the lim sup-part of the estimate in the main results. The almost increasing assumption is satisfied (with $L = 1$) for example when φ_n^{1/p_n} is convex.

Proposition 2.2.3. *Let Ω have finite measure, $\varphi_n \in \Phi_w(\Omega)$ for $n \in \mathbb{N}$ and $c, L \geq 1$. Assume for $n \in \mathbb{N}$ that*

- $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ *for a.e. $x \in \Omega$;*
- φ_n *satisfies $(aInc)_{p_n}$ with constant L for some $p_n \geq 1$.*

If $p_n \to \infty$, then $\lim_{n\to\infty} \|u\|_{\varphi_n} = \|u\|_\infty$ for all $u \in L^\infty(\Omega)$.

Proof. If $\|u\|_\infty = 0$, then $u = 0$ and there is nothing to prove. It suffices to consider the case $\|u\|_\infty = 1$, since the general case can be reduced to it by the normalization $\tilde{u} := \frac{u}{\|u\|_\infty}$.

By the definition of limit, we have to find for every $\varepsilon \in (0, 1)$ a number $n_0 \in \mathbb{N}$ such that

$$\frac{1}{1+\varepsilon} \leq \|u\|_{\varphi_n} \leq \frac{1}{1-\varepsilon}$$

for all $n \geq n_0$. By the unit ball property (Lemma 2.2.1) this inequality follows from

$$\rho_{\varphi_n}((1-\varepsilon)u) \leq 1 < \rho_{\varphi_n}((1+\varepsilon)u).$$

Fix $\varepsilon \in (0, 1)$. By $(aInc)_{p_n}$, $\varphi_n(x, 1-\varepsilon) \leq L(1-\varepsilon)^{p_n}\varphi_n(x, 1) \leq Lc(1-\varepsilon)^{p_n}$. Since $|u| \leq \|u\|_\infty = 1$ a.e. and φ_n is increasing, we conclude that

$$\rho_{\varphi_n}((1-\varepsilon)u) = \int_\Omega \varphi_n(x, (1-\varepsilon)|u|)\, dx \leq \int_\Omega \varphi_n(x, 1-\varepsilon)\, dx \leq Lc(1-\varepsilon)^{p_n}|\Omega| \overset{n\to\infty}{\longrightarrow} 0.$$

Thus we can find n_0 such that $\rho_{\varphi_n}((1-\varepsilon)u) < 1$ for all $n \geq n_0$.

To deal with the second inequality, we consider $A := \{(1+\varepsilon)|u| > 1 + \frac{\varepsilon}{2}\}$. Since $\|u\|_\infty = 1$, it follows that $|A| > 0$. By $(aInc)_{p_n}$,

$$\varphi_n(x, 1+\tfrac{\varepsilon}{2}) \geq \tfrac{1}{L}(1+\tfrac{\varepsilon}{2})^{p_n}\varphi_n(x, 1) \geq \tfrac{1}{Lc}(1+\tfrac{\varepsilon}{2})^{p_n}.$$

The inequality $1 < \rho_{\varphi_n}((1+\varepsilon)u)$ follows for large n from the estimate

$$\int_\Omega \varphi_n(x, (1+\varepsilon)|u|)\, dx \geq \int_A \varphi_n(x, 1+\tfrac{\varepsilon}{2})\, dx \geq \tfrac{1}{Lc}(1+\tfrac{\varepsilon}{2})^{p_n}|A| \overset{n\to\infty}{\longrightarrow} \infty. \qquad \square$$

The previous two results used the assumption $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$, which is stronger than $(A0)$. This is because $(A0)$ is not sufficient, as the next example demonstrates. Indeed, the sequence φ_n defined in the next example satisfies all the assumptions of Proposition 2.2.3, but does not satisfy $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$, although it does satisfy $(A0)$.

Example 2.2.4. Define $\varphi_n \in \Phi_w$ by $\varphi_n(t) := (a_n t)^n$, where $\beta \leq a_n \leq \frac{1}{\beta}$. Then φ_n satisfies $(A0)$ with constant β. If $a_n \to a \neq 1$, then $\|\cdot\|_{\varphi_n} \to \|\cdot\|_{\varphi_{a,\infty}} = a\|\cdot\|_\infty$, where $\varphi_{a,\infty}(t) := \infty\chi_{(1/a,\infty)}(t)$. Therefore, the conclusion of the previous proposition does not hold in this case. If $a > 1$, then the inequality $\|u\|_{\varphi_n} \leq c_n\|u\|_\infty$ of Proposition 2.2.2 similarly does not hold with any constant $c_n \to 1$.

The second example shows that the assumption that each φ_n satisfy $(aInc)_{p_n}$ with the same constant L is also needed for the conclusion.

Example 2.2.5. Define $\varphi \in \Phi_w$ by $\varphi(t) := \max\{0, 2t-1\} + \varphi_\infty(t)$. We consider the constant sequence $\varphi_n = \varphi$, whose Luxemburg quasinorms converge to $\|\cdot\|_\varphi$, not $\|\cdot\|_\infty$. Since $\varphi_n(1) = 1$, the first condition of Proposition 2.2.3 is satisfied. Let $\{p_n\}_n$ be a sequence converging to ∞. We show that φ_n satisfies $(aInc)_{p_n}$ with a constant $L_n \geq 1$. If $t \leq \frac{1}{2}$, then $\varphi(t) = 0$, and if $t > 1$, then $\varphi(t) = \infty$. Thus it is enough to show the second inequality in the following when $\frac{1}{2} \leq \lambda t \leq t \leq 1$:

$$\varphi_n(\lambda t) = 2\lambda t - 1 \leq (2t-1)\lambda \leq L_n\lambda^{p_n}(2t-1) = L_n\lambda^{p_n}\varphi_n(t).$$

This inequality holds when $L_n \geq \lambda^{1-p_n}$. Since $\lambda \geq \frac{1}{2}$, the choice $L_n := 2^{p_n-1}$ works for $(aInc)_{p_n}$.

Note that both examples are of Orlicz type as the dependence of φ on x is not used here.

2.3 Main theorems

In this chapter, the main results of the paper are presented and proved. In particular, two different Γ-convergence results are proved for a sequence of functionals defined in the Orlicz spaces, which rely respictevely on a norm-based and modular-based version.
Let $\varphi \in \Phi_w(\Omega)$ and $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$. We define $E_\varphi, F_\varphi : L^1(\Omega, \mathbb{R}^d) \to [0, \infty]$ by

$$E_\varphi(u) := \begin{cases} \rho_\varphi(f(\cdot, u, Du)) & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$F_\varphi(u) := \begin{cases} \|f(\cdot, u, Du)\|_\varphi & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

The abbreviations E_∞ and F_∞ refer to the case $\varphi = \varphi_\infty$ from Example 1.4.9, related L^∞.

Remark 2.3.1. It follows from Theorems 2.3.2 and 2.3.3 that $\{F_{\phi_n}\}_n$ and $\{E_{\phi_n}\}_n$ Γ-converge to F_∞ and E_∞ respectively, with respect to the L^1-weak topology.

Specifically, we prove the lim inf-property of Γ-convergence and show that we can use a constant recovery sequence.

Theorem 2.3.2. *Assume that* Ω *has finite measure. Let* $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ *be a normal integrand such that*

- $f(x, u, \cdot)$ *is level convex for a.e.* $x \in \Omega$ *and every* $u \in \mathbb{R}^d$*;*
- $f(x, u, \xi) \geq \alpha|\xi|^\gamma$ *for some* $\alpha, \gamma > 0$*, a.e.* $x \in \Omega$ *and every* $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd}$*.*

Let $\varphi_n \in \Phi_w(\Omega)$ *for* $n \in \mathbb{N}$ *and* $c, L \geq 1$ *and assume that*

- $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ *for a.e.* $x \in \Omega$*;*
- φ_n *satisfies* $(aInc)_{p_n}$ *with constant* L *for some* $p_n \geq 1$*.*

If $p_n \to \infty$ *as* $n \to \infty$*, then*

$$\limsup_{n\to\infty} F_{\varphi_n}(u) \leq F_\infty(u) \leq \liminf_{n\to\infty} F_{\varphi_n}(u_n)$$

for all $u, u_n \in L^1(\Omega, \mathbb{R}^d)$ *with* $u_n \rightharpoonup u$ *in* $L^1(\Omega, \mathbb{R}^d)$*.*

Proof. For simplicity we abbreviate $F_n := F_{\varphi_n}$, $n \in \mathbb{N}$. Let us first show that

$$\limsup_{n\to\infty} F_n(u) \leq F_\infty(u)$$

for $u \in L^1(\Omega, \mathbb{R}^d)$. If $F_\infty(u) = \infty$, this is clear. Otherwise, $F_\infty(u) < \infty$ so $f(\cdot, u, Du) \in L^\infty(\Omega)$ and Proposition 2.2.3 gives that

$$\lim_{n\to\infty} F_n(u) = \lim_{n\to\infty} \|f(\cdot, u, Du)\|_{\varphi_n} = \|f(\cdot, u, Du)\|_\infty = F_\infty(u).$$

We next deal with the lim inf-inequality. Let $\{u_n\}_n \subset L^1(\Omega, \mathbb{R}^d)$ converge weakly to $u \in L^1(\Omega, \mathbb{R}^d)$. Without loss of generality, we assume that

$$\liminf_{n\to\infty} F_n(u_n) = \lim_{n\to\infty} F_n(u_n) = M < \infty.$$

Then every subsequence of $\{u_n\}_n$ also has limit M. Recall that $p_n \to \infty$. Fix $q > \gamma$ and let $n_0 \in \mathbb{N}$ be such that

$$p_n \geq \frac{q}{\gamma} \quad \text{and} \quad F_n(u_n) \leq M + 1 \qquad \text{for all } n \geq n_0.$$

From $p_n \geq \frac{q}{\gamma}$ it follows that φ_n satisfies $(aInc)_{q/\gamma}$ with the same constant L as in the assumption. By Proposition 2.2.2,

$$\|f(\cdot, u_n, Du_n)\|_{q/\gamma} \leq \underbrace{\big(2L(|\Omega| + c)\big)^{\frac{\gamma}{q}}}_{=:C_q} \|f(\cdot, u_n, Du_n)\|_{\varphi_n} \leq C_1(M + 1). \tag{2.3.1}$$

for every $n \geq n_0$. We note that $C_q \searrow 1$ as $q \to \infty$. We use this inequality and the assumed lower bound on f to estimate the norm of the gradient:

$$\|Du_n\|_q^\gamma = \left\| |Du_n|^\gamma \right\|_{q/\gamma} \leq \tfrac{1}{\alpha} \|f(\cdot, u_n, Du_n)\|_{q/\gamma} \leq \tfrac{C_1}{\alpha}(M+1).$$

It follows that $\{\|Du_n\|_q\}_n$ is bounded. Then, up to a subsequence (depending on q), $\{Du_n\}_n$ weakly converges to a function w in $L^q(\Omega, \mathbb{R}^{Nd})$. Since $\{u_n\}_n$ weakly converges to u in $L^1(\Omega, \mathbb{R}^d)$, w is the distributional gradient of u.

Fix an open ball $B \subset \Omega$. We use a version of the Poincaré inequality where L^q-integrability is required only for the gradient. By the theorem in Section 1.5.2 [118, p. 35], we obtain that

$$\begin{aligned}\|u_n\|_{L^q(B,\mathbb{R}^d)} &\leq \|u_n - (u_n)_B\|_{L^q(B,\mathbb{R}^d)} + \|(u_n)_B\|_{L^q(B,\mathbb{R}^d)} \\ &\leq c(N,B)\|Du_n\|_{L^q(B,\mathbb{R}^d)} + |B|^{\frac{1}{q}-1}\|u_n\|_{L^1(B,\mathbb{R}^d)},\end{aligned}$$

Since $\{\|Du_n\|_q\}_n$ is bounded and $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$, the right hand side is uniformly bounded. Thus reflexivity in $L^q(B, \mathbb{R}^d)$ yields that $\{u_n\}_n$ has a weakly convergent subsequence. We have found a subsequence, denoted still by $\{u_n\}_n$, with $u_n \rightharpoonup u$ in $W^{1,q}(B, \mathbb{R}^d)$. Let $\{\mu_x\}_{x \in B}$ be the family of probability measures (the "Young measure") from Lemma 1.3.5 corresponding to this subsequence. We use the first inequality from (2.3.1) in B and Lemma 1.3.5 for the normal integrand $f^{q/\gamma}$ to conclude that

$$\begin{aligned}\liminf_{n\to\infty} F_n(u_n) &\geq \liminf_{n\to\infty} \frac{1}{C_q}\left(\int_B f(x, u_n(x), Du_n(x))^{\frac{q}{\gamma}}\, dx\right)^{\frac{\gamma}{q}} \\ &\geq \frac{1}{C_q}\left(\int_B \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi)^{\frac{q}{\gamma}}\, d\mu_x(\xi)\, dx\right)^{\frac{\gamma}{q}}.\end{aligned}$$

Take the limit inferior on the right-hand side as $q \to \infty$ and use $C_q \to 1$ as well as Corollary 1.3.6:

$$\begin{aligned}\liminf_{n\to\infty} F_n(u_n) &\geq \liminf_{q\to\infty} \left(\int_B \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi)^{\frac{q}{\gamma}}\, d\mu_x(\xi)\, dx\right)^{\frac{\gamma}{q}} \\ &\geq \text{ess-sup}_{x\in B} f(x, u(x), Du(x)) = \sup_{x \in B\setminus A} f(x, u(x), Du(x)),\end{aligned}$$

where $A \subset B$ is the exceptional set of measure zero related to the essential supremum. We can write $\Omega = \cup_{i\in\mathbb{N}} B_i$ as a countable union of balls B_i and then we obtain an estimate for the supremum in $\Omega \setminus \cup_{i\in\mathbb{N}} A_i \subset \cup_{i\in\mathbb{N}}(B_i \setminus A_i)$, where A_i is the exceptional set corresponding to B_i. Since also $\cup_{i\in\mathbb{N}} A_i$ has measure zero, this gives the essential supremum in Ω. Thus we have proved the lim inf-inequality and completed the proof. $\square$

We conclude with a similar result for the modular-based energy functional. We use the notation $\varphi^-(1) := \operatorname*{ess\,inf}_{x\in\Omega} \varphi(x,1)$ and $\varphi^+(1) := \text{ess-sup}_{x\in\Omega}\varphi(x,1)$.

Theorem 2.3.3. *Assume that Ω has finite measure. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ be a normal integrand such that*

- *$f(x,u,\cdot)$ is level convex for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^d$;*
- *$f(x,u,\xi) \geq \alpha|\xi|^\gamma$ for some $\alpha, \gamma > 0$, a.e. $x \in \Omega$ and every $(u,\xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd}$.*

Let $\varphi_n \in \Phi_w(\Omega)$ for $n \in \mathbb{N}$ and $L \geq 1$ and assume that

- *$\limsup_{n\to\infty} \varphi_n^+(1) = 0$ and $\liminf_{n\to\infty} \varphi_n^-(1)^{1/p_n} \geq 1$;*
- *φ_n satisfies $(aInc)_{p_n}$ with constant L for some $p_n \geq 1$.*

If $p_n \to \infty$ as $n \to \infty$, then

$$\limsup_{n\to\infty} E_n(u) \leq E_\infty(u) \leq \liminf_{n\to\infty} E_n(u_n)$$

for all $u, u_n \in L^1(\Omega,\mathbb{R}^d)$ with $u_n \rightharpoonup u$ in $L^1(\Omega,\mathbb{R}^d)$.

Proof. We again abbreviate $E_n := E_{\varphi_n}$, $n \in \mathbb{N}$. Let us first show that

$$\limsup_{n\to\infty} E_n(u) \leq E_\infty(u)$$

for any $u \in L^1(\Omega,\mathbb{R}^d)$. If $E_\infty(u) = \infty$, this is clear. Otherwise, $E_\infty(u) = 0$ so $\|f(\cdot,u,Du)\|_\infty \leq 1$. Thus we obtain that

$$\limsup_{n\to\infty} E_n(u) = \limsup_{n\to\infty} \int_\Omega \varphi_n(x, f(\cdot,u,Du))\,dx \leq \limsup_{n\to\infty} |\Omega|\, \varphi_n^+(1) = 0$$

and the inequality follows.
To deal with the lim inf-inequality, let $\{u_n\}_n \subset L^1(\Omega,\mathbb{R}^d)$ converge weakly to $u \in L^1(\Omega,\mathbb{R}^d)$. Without loss of generality, we assume that

$$\liminf_{n\to\infty} E_n(u_n) = \lim_{n\to\infty} E_n(u_n) = M < \infty.$$

Define $\hat\varphi_n \in \Phi_w(\Omega)$ by $\hat\varphi_n(x,t) := \frac{\varphi_n(x,t)}{\varphi_n(x,1)}$. Let $n_0 \in \mathbb{N}$ be such that $E_n(u_n) \leq M+1$ for all $n \geq n_0$, so that

$$\int_\Omega \hat\varphi_n(x, f(x,u_n,Du_n))\,dx \leq \frac{M+1}{\varphi_n^-(1)}.$$

Since $\hat{\varphi}_n$ satisfies $(aInc)_{p_n}$ with the same constant L, this yields by Corollary 3.2.10 of [98] that

$$\|f(\cdot, u_n, Du_n)\|_{\hat{\varphi}_n} \le \max\left\{\left(L\rho_{\hat{\varphi}_n}(f(\cdot, u_n, Du_n))\right)^{\frac{1}{p_n}}, 1\right\} \le \max\left\{\left(\frac{L(M+1)}{\varphi_n^-(1)}\right)^{\frac{1}{p_n}}, 1\right\},$$

and consequently

$$\limsup_{n\to\infty} \|f(\cdot, u_n, Du_n)\|_{\hat{\varphi}_n} \le \max\left\{\limsup_{n\to\infty} \varphi_n^-(1)^{-\frac{1}{p_n}}, 1\right\} = 1.$$

Since $\hat{\varphi}_n$ satisfies the requirements of Theorem 2.3.2 with the same L and $c = 1$, it follows that $F_\infty(u) \le 1$, i.e. $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d)$ and $\|f(\cdot, u, Du)\|_\infty \le 1$. Thus $E_\infty(u) \le \rho_\infty(1) = 0$, and hence the lim inf-inequality holds. □

The assumptions of Theorem 2.3.3 differ from Theorem 2.3.2. The next example shows that the condition $\frac{1}{c} \le \varphi_n(x, 1) \le c$ is not sufficient in the latter theorem.

Example 2.3.4. Let $p_n := n$ for all n. Then $\varphi_n(t) := t^n$ satisfies $(aInc)_n$ and $\frac{1}{c} \le \varphi_n(x, 1) \le c$ but not the condition $\limsup_{n\to\infty} \varphi_n^+(1) = 0$ of the previous theorem. Moreover, the claim of the theorem also does not hold: if f and u are such that $f(x, u, Du) = 1$ a.e., then $E_n(u) = |\Omega|$ for every $n \in \mathbb{N}$ but $E_\infty(u) = 0$ so that

$$\limsup_{n\to\infty} E_n(u) > E_\infty(u).$$

On the other hand, the energy $\varphi_n(t) := \frac{1}{n}t^n$ does satisfy the conditions the previous theorem.

The theorems proved above extend the results previously obtained in [46] and [83]. In contrast to these works, which focus respectively on Orlicz growth and variable exponent growth, within the present analysis we are able to consider the same problem in the context of generalized Orlicz spaces (or Musielak–Orlicz spaces) under more general assumptions. In particular, we manage to avoid the assumption that ratios of the lower and upper growth-rates of the approximating functionals (denoted $\phi_n^\pm$ and $p_n^\pm$ in previous papers) are bounded. More precisely, we need no assumptions about the upper growth-rate and can consider non-doubling functionals as well. In addition, we are able to deal with open sets of finite measure, whereas previous results required regularity of the boundary. As a consequence, the classical results of [46] and [83] are recovered as particular cases of our framework.

Chapter 3

Approximation of L^∞ functionals with generalized Orlicz norms

3.1 Introduction

In this chapter, as in the previous one, the main focus is Γ-convergence of a sequence of energies whose growth-rates tend to infinity. As mentioned in the last chapter, we recall here some motivation and earlier studies: Garroni, Nesi and Ponsiglione [95] studied dielectric breakdown using Γ-convergence of power-growth functions as the exponent tends to infinity. A more complicated functional of power-type was considered by Katzourakis [109] in a model for fluorescent optical tomography. Prinari and co-authors [63, 133, 134] considered an abstract version with the convergence of the energy

$$I_p(u) := \int_\Omega \frac{1}{p} f(x, u, Du)^p \, dx$$

as $p \to \infty$. A similar problem without dependence on the derivative was considered in [11, 47, 48] and related to applications in polycrystal plasticity. If the energy is of non-standard growth-type [116], then the Γ-limit can contain singular parts [119]. The energy I_p has been generalized to two complementary cases: Orlicz growth by Bocea and Mihăilescu [46] and variable exponent growth by Eleuteri and Prinari [83].

Instead, the main idea in [38], presented in the last chapter, consists of generalizing the approach of previous papers by means of improved techniques, with streamlined proofs and weaker assumptions. In particular, assumptions about the lower and upper growth-rates (denoted $\phi_n^\pm$ and $p_n^\pm$ in [46, 83]) of the approximating functionals are not needed, differently from [46] and the just-mentioned literature. In addition, in [38] the authors are able to deal with open sets of finite

measure, without requiring boundedness of the sets and regularity of their boundaries. The new setting spaces are known as Generalized Orlicz spaces (also known as Musielak–Orlicz spaces). These spaces and related PDE have been intensely studied recently, see, e.g., [66, 102, 105, 106, 108] and references therein. Both the L^∞- and the non-standard growth energy have been related to image processing [61, 64, 99].
In the present chapter, the first aim is to weaken the convexity assumptions on the density f. More specifically, given $\varphi \in \Phi_w(\Omega)$, (see Section 3.2 for more details on the class Φ_w) and a normal integrand $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$, we define, as in [38], $F_\varphi, E_\varphi, F_\infty, E_\infty : L^1(\Omega, \mathbb{R}^d) \to [0, \infty]$ by

$$F_\varphi(u) := \begin{cases} \|f(\cdot, u, Du)\|_\varphi & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases} \tag{3.1.1}$$

$$E_\varphi(u) := \begin{cases} \rho_\varphi(f(\cdot, u, Du)) & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases} \tag{3.1.2}$$

$$F_\infty(u) = \begin{cases} \|f(\cdot, u, Du)\|_\infty & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases} \tag{3.1.3}$$

and

$$E_\infty(u) = \begin{cases} 0, & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d) \text{ and } |f(\cdot, u, Du)| \leq 1 \text{ a.e.} \\ \infty, & \text{otherwise,} \end{cases} \tag{3.1.4}$$

where ρ_φ stands for the classic notion of *modular*, i.e. $\rho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|)\, dx$ (see [98] and Definition 2.3 for the precise meaning). More precisely, we assume that f satisfies the following assumptions:

(H1) $f(x, u, \cdot)$ is $\mathrm{curl}_{(p>1)}-Young$ quasiconvex for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}^d$, i.e.

$$\operatorname*{ess\,sup}_{y\in Q} f\left(x, u(x), \int_{\mathbb{R}^{Nd}} \xi d\mu_y(\xi)\right) \leq \operatorname*{ess\,sup}_{y\in Q}\left(\mu_y - \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^{Nd}} f(x, u(x), \xi)\right)$$

whenever $\mu \equiv \{\mu_y\}_{y\in Q}$ is a $W^{1,p}-$gradient Young measure for every $p \in (1, \infty)$, where Q is the unit cube of $\mathbb{R}^n$ centered in the origin with side length 1, (see Chapter 1 for details about Young measures and convexity notions);

(H2) there exist $\alpha, \gamma > 0$ such that

$$f(x, u, \xi) \geq \alpha|\xi|^\gamma \qquad \text{for a.e } x \in \Omega \text{ and for every } (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd}.$$

Assuming also that for every $n, \varphi_n \in \Phi_w(\Omega)$, satisfies $(aInc)_{p_n}$ with constant L for some $p_n \geq 1$, i.e.

(H3) there exists $L \geq 1$ such that $\varphi_n(x, \lambda t) \leq L\lambda^{p_n}\varphi_n(x,t)$ for a.e. $x \in \Omega$ and all $t \geq 0$, $\lambda \leq 1$,

and there exists $c > 0$ such that

(H4) $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ for a.e. $x \in \Omega$,

then, letting $F_n := F_{\varphi_n}, E_n := E_{\varphi_n} : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ be defined by (3.1.1) and (3.1.2), respectively (with $\varphi = \varphi_n$) we prove that (F_n) sequentially Γ-converge to F_∞ with respect to the L^1-weak topology, as $p_n \to +\infty$, so we can state the following result.

Theorem 3.1.1. *Assume that Ω has finite measure. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ be a normal integrand such that* (**H1**) *and* (**H2**) *hold. Let $\varphi_n \in \Phi_w(\Omega)$ for $n \in \mathbb{N}$ and $c, L \geq 1$ and assume that* **(H3)** *and* **(H4)** *hold. If $p_n \to \infty$ as $n \to \infty$, then*

$$\limsup_{n\to\infty} F_n(u) \leq F_\infty(u) \leq \liminf_{n\to\infty} F_n(u_n)$$

for all $u, u_n \in L^1(\Omega, \mathbb{R}^d)$ with $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$.

Reinforcing **(H4)** by

(H5) $\limsup\limits_{n\to\infty} \varphi_n^+(1) = 0$ and $\liminf\limits_{n\to\infty} \varphi_n^-(1)^{1/p_n} \geq 1$,

where $\varphi_n^+(t) := \operatorname{ess\,sup}\limits_{x\in\Omega} \varphi_n(x,t)$ and $\varphi_n^-(t) := \operatorname{ess\,inf}\limits_{x\in\Omega} \varphi_n(x,t)$, in the following theorems we show the (sequential) Γ-convergence of $\{E_n\}_n$ towards E_∞ with respect to the L^1-topology.

Theorem 3.1.2. *Let Ω, f, φ_n and L as in Theorem 3.1.1, satisfying* (**H1**) – (**H3**) *and* **(H5)** *Let E_n be the functional defined by* (3.1.2) *(with $\varphi = \varphi_n$). If $p_n \to \infty$ as $n \to \infty$, then*

$$\limsup_{n\to\infty} E_n(u) \leq E_\infty(u) \leq \liminf_{n\to\infty} E_n(u_n)$$

for all $u, u_n \in L^1(\Omega, \mathbb{R}^d)$ with $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$. where E_∞ is the functional in (3.1.4)*.*

At this point it is worth to observe that **(H1)** is a milder condition than level convexity (i.e. the convexity of the sublevel sets) of $f(x, u, \cdot)$ adopted to prove the results in [83, 38]. Indeed condition **(H1)** has been first introduced in [63, eq. (3.1)] to prove L^p-approximation for L^∞-type norm functionals, and later characterized in [137], see also [11, 12] for similar related notions.

Furthermore, if Ω is bounded with regular boundary, then Rellich theorem and (**H2**) guarantee that the weak L^1 convergence could be replaced by the strong one and the notion of Γ-convergence coincides with the one in Proposition 1.5.2.
On the other hand, in Theorems 3.4.2 and 3.4.4 we are able to remove assumption (**H1**), i.e. the $\mathrm{curl}_{(p>1)}$- Young quasiconvexity assumption on $f : \Omega \times \mathbb{R}^{Nd} \to \mathbb{R}$, in its last variable, under the restriction that $f(x, \cdot)$ satisfies a growth condition of type $|\cdot|^\gamma$, also from above, and assuming that $\varphi_n(x,t) := t^{p_n(x)}$, with $\sup_n \frac{p_n^+}{p_n^-} \leq \beta$, for a $\beta > 0$, where $p_n^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$, $p_n^- := \operatorname{ess\,inf}_{x\in\Omega} p_n(x)$.
The result obtained gives an extension of [133, Theorem 2.2], to the modular case allowing also to consider $p(\cdot)$-type Sobolev spaces and more general growth conditions. We would like to underline that, differently from [134, 77], we dealt with Carathéodory integrands instead of measurable ones, since the available relaxation results in $W^{1,p}$ and $W^{1,p(\cdot)}$ (crucial for the proofs) for integral functionals with only measurable integrands (see [59, 72] and [119], respectively) provide an integral representation in terms of a generic quasiconvex function, which in principle, in the case p constant, may differ from the one obtained in the case p variable. Hence no comparison among the two densities obtained relaxing F_φ either in $W^{1,p}$ ($\varphi(x,\xi) = |\xi|^p$) or in $W^{1,p(\cdot)}$ ($\varphi(x,\xi) = |\xi|^{p(\cdot)}$) would be possible. In contrast, in the arguments to obtain a supremal representation by means $L^{p(\cdot)}$-approximation, we rely in the case p constant, passing from $p(\cdot)$ to a suitable constant exponent p. On the other hand, in this setting, due to our techniques, authors had to require a certain regularity on Ω.
The chapter is organized as follows.
In Section 3.2 we recall some specific results needed for the main theorems (mainly concerning variable-exponent spaces and convexity notions).
In Section 3.3, we derive supremal functionals as limits of the generalized Orlicz norms under mild convexity assumptions.
We remove in Section 3.4 this latter assumption in the case where the Orlicz spaces reduce to Lebesgue spaces with variable exponents.

3.2 Preliminaries

For the purpose of this chapter, we consider the case when $\Omega \subset \mathbb{R}^N$ is an open set (where $N \geq 1$) and denote by $\mathcal{L}^N(U)$ the N-dimensional Lebesgue measure of any measurable set U. In the sequel we consider functions $u : \Omega \to \mathbb{R}^d$, with $d \geq 1$ and we denote by k any dimension different from Nd.

3.2.1 Variable exponents Lebesgue-Sobolev spaces

In this subection, we present the variable exponent, as a special case of generalized Orlicz, with $\varphi(x,t) = t^{p(x)}$ for some measurable function $p : \Omega \to [1, +\infty]$, called *variable exponent* on Ω. In this subsection can be found a collection of some specific results concerning variable exponent Lebesgue and Sobolev spaces. For more details we refer to the monograph [79].

Definition 3.2.1. For any (Lebesgue) measurable function $p : \Omega \to [1, +\infty]$ we define

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \qquad\qquad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

If $p^+ < +\infty$ we call p a *bounded variable exponent*, and we can define the variable exponent Lebesgue space as

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\},$$

which is a Banach space endowed with the *Luxemburg norm.*
By Corollary 3.3.4 in [79], if $0 < \mathcal{L}^N(\Omega) < +\infty$ and p and q are variable exponents such that $p \le q$ a.e. in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous. The embedding constant is less or equal to $2(\mathcal{L}^N(\Omega) + 1)$ and $2 \max\left\{ \mathcal{L}^N(\Omega)^{\left(\frac{1}{q}-\frac{1}{p}\right)^+}, \mathcal{L}^N(\Omega)^{\left(\frac{1}{q}-\frac{1}{p}\right)^-} \right\}$.
For any variable exponent p, we define p' by setting

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

with the convention that, if $p(x) = +\infty$ then $p'(x) = 1$. The function p' is called *the dual variable exponent of* p.
Consider the following result (for more details see Lemma 3.2.20 in [79]).

Theorem 3.2.2. *(Hölder's inequality) Let p, q, s be measurable exponents such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$$

a.e. in Ω. Then

$$\|fg\|_{s(\cdot)} \le \left(\left(\frac{s}{p}\right)^+ + \left(\frac{s}{q}\right)^+ \right) \|f\|_{p(\cdot)}\, \|g\|_{q(\cdot)}$$

for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, where in the case $s = p = q = \infty$, we use the convention $\frac{s}{p} = \frac{s}{q} = 1$.
In particular, in the case $s = 1$, we have

$$\left| \int_\Omega f\, g\, dx \right| \le \int_\Omega |f|\, |g|\, dx \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|f\|_{p(\cdot)}\, \|g\|_{p'(\cdot)}.$$

The *modular* in this setting specializes as,

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx.$$

In particular, when $p^+ < +\infty$, by Lemma 3.2.5 in [79], for every $u \in L^{p(\cdot)}(\Omega)$ it holds

$$\min\left\{\left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^-}}, \left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^+}}\right\} \le \|u\|_{p(\cdot)} \le \max\left\{\left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^-}}, \left(\rho_{p(\cdot)}(u)\right)^{\frac{1}{p^+}}\right\}. \quad (3.2.1)$$

In conclusion, for what concerns variable exponent Sobolev spaces, the reader can refer to[79, Definition 8.1.2], for more details.

Definition 3.2.3. Let $k, d \in \mathbb{N}$, $k \ge 0$, and let p be a measurable exponent. We define

$$W^{k,p(\cdot)}(\Omega, \mathbb{R}^d) := \Big\{u : \Omega \to \mathbb{R}^d : u, \partial_\alpha u \in L^{p(\cdot)}(\Omega, \mathbb{R}^d), \forall\alpha \text{ multi-index such that } |\alpha| \le k\Big\},$$

where

$$L^{p(\cdot)}(\Omega, \mathbb{R}^d) := \{u : \Omega \to \mathbb{R}^d : \ |u| \in L^{p(\cdot)}(\Omega)\}.$$

We define the semimodular on $W^{k,p(\cdot)}(\Omega)$ by

$$\rho_{W^{k,p(\cdot)}(\Omega)}(u) := \sum_{0\le|\alpha|\le k} \rho_{L^{p(\cdot)}(\Omega)}(|\partial_\alpha u|)$$

which induces a norm by

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \ \rho_{W^{k,p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Clearly $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$.
Variable exponent Sobolev spaces will play a major role in the following chapters (in particular Chapters 5 and 6), so more properties will be presented in dedicated sections.

3.2.2 Convexity notions for supremal functionals

In this section, we provide more information following the Section 1.2.
The level convexity is proved to be necessary and sufficient condition for the lower semicontinuity of supremal functionals in the scalar setting, [1, 30] and sufficient

for L^p and $L^{p(\cdot)}$- approximation, see [83, 134], and the references therein.
On the other hand such condition is not necessary for the lower semicontinuity of supremal functionals in the vectorial setting as firstly shown in [30]. Indeed, in this latter paper the authors characterized the lower semicontinuity of L^∞-functionals by the so-called strong-Morrey quasiconvexity, which we recall next and rename, as in [137], *BJW quasi-level-convexity.*

Definition 3.2.4. A function $f : \mathbb{R}^{Nd} \to \mathbb{R}$ is said to be strong Morrey quasiconvex (*BJW quasi-level-convex*)if

$$\forall\, \varepsilon > 0\ \forall\, \xi \in \mathbb{R}^{Nd}\ \forall\, K > 0\ \exists\, \delta = \delta(\varepsilon, K, \xi) > 0 :$$

$$\left.\begin{array}{l} \varphi \in W^{1,\infty}(Q;\mathbb{R}^d) \\ ||D\varphi||_{L^\infty(Q;\mathbb{R}^{Nd})} \leq K \\ \max_{x\in\partial Q} |\varphi(x)| \leq \delta \end{array}\right\} \Longrightarrow f(\xi) \leq \operatorname*{ess\,sup}_{x\in Q} f(\xi + D\varphi(x)) + \varepsilon.$$

On the other hand the relaxation in the supremal setting is currently open, as well as the understanding of the sufficiency of BJW -quasi level convexity (strong Morrey quasiconvexity) for L^p-approximation, hence several other notions have been introduced in the literature [11, 12, 63, 137].
In particular, we recall the notion of $\mathrm{curl}_{p\geq 1}$-Young quasiconvexity as in [137], which generalizes the concept of level-convexity (see Definition 1.2.2) and reduces to it in the scalar setting. The reader can refer to [63, 11] for previously introduced related notions and to [137] for a comparison.

Definition 3.2.5. Assume that f is lower semicontinuous and bounded from below. f is said to be $\mathrm{curl}_{(p>1)} - Young$ quasiconvex, if

$$\operatorname*{ess\,sup}_{x\in Q} f\left(\int_{\mathbb{R}^{Nd}} \xi d\nu_x(\xi)\right) \leq \operatorname*{ess\,sup}_{x\in Q}\left(\nu_x - \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^{Nd}} f(\xi)\right)$$

whenever $\nu \equiv \{\nu_x\}_{x\in Q}$ is a $W^{1,p}-$gradient Young measure for every $p \in (1, \infty)$, where Q is the unit cube of $\mathbb{R}^n$ centered in the origin with side length 1.

The following result, which provides characterizations of $\mathrm{curl}_{(p>1)}$- Young quasiconvexity has been proven in [137].

Proposition 3.2.6. *Let $f : \mathbb{R}^{Nd} \to \mathbb{R}$ be a lower semicontinuous function and bounded from below. Then the following conditions are equivalent.*

(*i*) *f is $\mathrm{curl}_{(p>1)} - Young$ quasiconvex;*

(ii) *f verifies*

$$f\left(\int_{\mathbb{R}^{Nd}} \xi d\nu_x(\xi)\right) \leq \nu_x - \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^{Nd}} f(\xi), \quad \textit{for a.e. } x \in Q$$

whenever $\nu \equiv \{\nu_x\}_{x\in Q}$ is a $W^{1,p}-$gradient Young measure for every $p \in (1,\infty)$ and Q is the unit cube of $\mathbb{R}^n$ centered in the origin with side length 1;

(iii) *f verifies*

$$f\left(\int_{\mathbb{R}^{Nd}} \xi d\nu_x(\xi)\right) \leq \nu - \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^{Nd}} f(\xi),$$

whenever ν is a homogeneous $W^{1,p}-$gradient Young measure for every $p \in (1,\infty)$.

Moreover, in the definition of $\mathrm{curl}_{(p>1)} - Young$ quasiconvexity the domain Q can be replaced by any open, bounded, connected set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary.

In the main results we prove that $\mathrm{curl}_{(p>1)} - Young$ quasiconvexity is sufficient for power law approximation also in the generalized Orlicz-Sobolev setting.

3.3 Lower growth approximation

In this section we study the approximation as the lower growth rate tends to ∞, via Γ-convergence, of supremal functionals in terms of generalized Orlicz-type quasinorms. In the following we consider a normal integrand $f : \Omega\times\mathbb{R}^d\times\mathbb{R}^{Nd} \to \mathbb{R}$, satisfying **(H1)**, **(H2)**.

3.3.1 Statement of the main results

We start by recalling from the introduction all theorems to easily compare the results obtained according to the different set of hypotheses and topologies considered.
Let $\varphi \in \Phi_w(\Omega)$, we recall the functionals $F_\varphi, E_\varphi : L^1(\Omega,\mathbb{R}^d) \to [0,\infty]$, introduced in (3.1.1) and (3.1.2), defined by

$$F_\varphi(u) := \begin{cases} \|f(\cdot,u,Du)\|_\varphi & \text{if } u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

and

$$E_\varphi(u) := \begin{cases} \rho_\varphi(f(\cdot,u,Du)) & \text{if } u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

The functionals F_∞ and E_∞ in (3.1.3) and (3.1.4) refer to the case $\varphi = \varphi_\infty$ from Example 1.4.9, related to L^∞, i.e.

$$F_\infty(u) = \begin{cases} \|f(\cdot, u, Du)\|_\infty & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

and

$$E_\infty(u) = \begin{cases} 0, & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d) \text{ and } |f(\cdot, u, Du)| \leq 1 \text{ a.e.,} \\ \infty, & \text{otherwise,} \end{cases}$$

Let $\{F_n\}_n$ and $\{E_n\}_n$ defined by (3.1.1) and (3.1.2), respectively, as

$$F_n := F_{\varphi_n} \text{ and } E_n := E_{\varphi_n}.$$

We prove in Theorems 3.1.1 and 3.1.2 that $\{F_n\}_n$ and $\{E_n\}_n$ (sequentially) Γ-converge to F_∞ and E_∞ respectively, with respect to the L^1-weak topology. Specifically, we prove the lim inf-property of Γ-convergence, i.e. (i) of Proposition 1.5.2 and show that we can use a constant recovery sequence for (ii) in Proposition 1.5.2.

Theorem 3.1.1. *Assume that Ω has finite measure. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \to \mathbb{R}$ be a normal integrand such that* (**H1**) *and* (**H2**) *hold.*
Let $\varphi_n \in \Phi_w(\Omega)$ for $n \in \mathbb{N}$ and $c, L \geq 1$ and assume that **(H3)** *and* **(H4)** *hold. If $p_n \to \infty$ as $n \to \infty$, then*

$$\limsup_{n\to\infty} F_n(u) \leq F_\infty(u) \leq \liminf_{n\to\infty} F_n(u_n)$$

for all $u, u_n \in L^1(\Omega, \mathbb{R}^d)$ with $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$.

Corollary 3.3.1. *Let $X \in \{L^\infty(\Omega, \mathbb{R}^d), C(\Omega, \mathbb{R}^d)\}$ be endowed with the norm $|| \cdot ||_\infty$. Under the same assumptions of Theorem 3.1.1, let $F_n, F_\infty : X \to [0, +\infty]$ be the functionals defined by* (3.1.1) *(with $\varphi = \varphi_n$) and* (3.1.3)*, respectively. Then the sequence $\{F_n\}_n$ Γ- converges, with respect to the L^∞-strong convergence, to the functional F_∞, as $n \to +\infty$.*

We conclude with similar results for the modular-based energy functional. We use the notation

$$\varphi^-(1) := \operatorname*{ess\,inf}_{x\in\Omega} \varphi(x, 1) \text{ and } \varphi^+(1) := \text{ess-sup}_{x\in\Omega}\varphi(x, 1).$$

Arguing as in the proof of [38, Theorem 4.4] and exploiting the previous theorem, the following corollary holds.

Theorem 3.1.2. *Let Ω, f, φ_n and L as in Theorem 3.1.1, satisfying* (**H1**) – (**H3**) *and* **(H5)** *Let E_n be the functional defined by* (3.1.2) *(with $\varphi = \varphi_n$). If $p_n \to \infty$ as $n \to \infty$, then*

$$\limsup_{n\to\infty} E_n(u) \le E_\infty(u) \le \liminf_{n\to\infty} E_n(u_n)$$

for all $u, u_n \in L^1(\Omega, \mathbb{R}^d)$ with $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$, where E_∞ is the functional in (3.1.4).

Corollary 3.3.2. *Let X be as in Corollary 3.3.1. Under the same assumptions of Theorem 3.1.2, let $E_n, E_\infty : X \to [0, +\infty]$ be the functionals defined by equations* (3.1.2) *(with $\varphi = \varphi_n$) and* (3.1.4)*, respectively. Then,* (E_n) *Γ- converges E_∞, as $n \to +\infty$, with respect to the L^∞- strong convergence.*

3.3.2 Proofs of Theorems

Proof of Theorem 3.1.1 The proof is very similar to the one of [38, Theorem 4.2], presented in the previous chapter. The main steps are written for the readers' convenience.
Let $F_n := F_{\varphi_n}$, $n \in \mathbb{N}$, be the functional in (3.1.1), with $\varphi = \varphi_n$. To show that

$$\limsup_{n\to\infty} F_n(u) \le F_\infty(u)$$

for $u \in L^1(\Omega, \mathbb{R}^d)$, we assume, without loss of generality, that $F_\infty(u) < +\infty$, hence, by Proposition 2.2.3, it follows that

$$\lim_{n\to\infty} F_n(u) = \lim_{n\to\infty} \|f(\cdot, u, Du)\|_{\varphi_n} = \|f(\cdot, u, Du)\|_\infty = F_\infty(u).$$

We next deal with the lim inf-inequality, following line by line [38, Theorem 4.2], hence we can assume without loss of generality that Ω is a ball B. Let $\{u_n\}_n \subset L^1(B, \mathbb{R}^d)$ converge weakly to $u \in L^1(B, \mathbb{R}^d)$. Without loss of generality, we assume that

$$\liminf_{n\to\infty} F_n(u_n) = \lim_{n\to\infty} F_n(u_n) = M < +\infty.$$

Then every subsequence of $\{u_n\}_n$ also has limit M in energy. We will prove that it will converge in any L^q for every $q > \gamma$. Recall that $p_n \to \infty$. Fix such a q and let $n_0 \in \mathbb{N}$ be such that

$$p_n \ge \frac{q}{\gamma} \quad \text{and} \quad F_n(u_n) \le M + 1 \qquad \text{for all } n \ge n_0.$$

From $p_n \ge \frac{q}{\gamma}$ it follows that φ_n satisfies $(aInc)_{\frac{q}{\gamma}}$ with the same constant L as in the assumption. By Proposition 2.2.2,

$$\|f(\cdot, u_n, Du_n)\|_{q/\gamma} \le \underbrace{\Big(2L(|\Omega| + c)\Big)^{\frac{\gamma}{q}}}_{=:C_q} \|f(\cdot, u_n, Du_n)\|_{\varphi_n} \le C_1(M + 1).$$

for every $n \geq n_0$, and $C_q \searrow 1$ as $q \to \infty$.
By this inequality and (**H2**), it follows:

$$\|Du_n\|_q^\gamma = \left\| |Du_n|^\gamma \right\|_{q/\gamma} \leq \tfrac{1}{\alpha}\|f(\cdot, u_n, Du_n)\|_{q/\gamma} \leq \tfrac{C_1}{\alpha}(M+1),$$

hence $\{\|Du_n\|_q\}_n$ is bounded in n for every $q \geq \gamma$. Then, up to a subsequence (possibly depending on q), $\{Du_n\}_n$ weakly converges to a function w in $L^q(B, \mathbb{R}^{Nd})$. Since $\{u_n\}_n$ weakly converges to u in $L^1(B, \mathbb{R}^d)$, w is the distributional gradient of u. By Poincaré inequality where L^q-integrability is required only for the gradient, cf. [118, Section 1.5.2, p. 35], we obtain that

$$\|u_n\|_{L^q(B,\mathbb{R}^d)} \leq c(N,B)\|Du_n\|_{L^q(B,\mathbb{R}^d)} + |B|^{\frac{1}{q}-1}\|u_n\|_{L^1(B,\mathbb{R}^d)},$$

Consequently $\{u_n\}_n$ has a weakly convergent subsequence in $W^{1,q}(B, \mathbb{R}^d)$. The metrizability of weak topology $W^{1,q}(B; \mathbb{R}^d)$ in compact sets allows us to apply Urysohn property, hence to conclude that the entire sequence $\{u_n\}_n$ is such that $u_n \rightharpoonup u$ in $W^{1,q}(B; \mathbb{R}^d)$. Let $\{\nu_x\}_{x \in B}$ be the family of $W^{1,q}$-gradient Young measures generated by (Du_n) for every $q > \min\{\gamma, 1\}$ (with barycenter $Du(x)$). Arguing as in [38, Theorem 4.2]

$$\begin{aligned} \liminf_{n\to\infty} F_n(u_n) &\geq \liminf_{q\to\infty} \left(\int_B \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi)^{\frac{q}{\gamma}}\, d\nu_x(\xi)\, dx \right)^{\frac{\gamma}{q}} \\ &\geq \text{ess-sup}_{x\in B} f(x, u(x), Du(x)), \end{aligned}$$

where in the last inequality we have exploited the fact that $f(x, u, \cdot)$ is $\mathrm{curl}_{(p>1)}$-Young quasiconvex for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^d$ and (ii) in Theorem 3.2.6, taking into account that

$$Du(x) = \int_{\mathbb{R}^{Nd}} \xi d\nu_x(\xi). \tag{3.3.1}$$

Thus we have completed the proof. □

Proof of Corollary 3.3.1 As in the proof of Theorem 3.1.1, the Γ-lim sup inequality follows the same arguments as in Theorem 3.1.1. In order to get the Γ-lim inf inequality, it is sufficient to note that if $(u_n) \subseteq X$ is a sequence L^∞-converging to u in X, then $\{u_n\}_n$ weakly L^q-converges to u for every $q \geq 1$. As in the proof of Theorem [83, Theorem 5.3], we get that the sequence $\{Du_n\}_n$ weakly converges to Du in $L^q(\Omega, \mathbb{R}^d)$ for every $q > 1$. In particular $\{u_n\}_n$ converges weakly to u in $W^{1,q}(\Omega, \mathbb{R}^{Nd})$ for every $q > N$. Then $\{Du_n\}_n$ generates a Young measure $\{\nu_x\}_{x\in\Omega}$ such that $\nu_x(\mathbb{R}^{Nd}) = 1$ and (3.3.1) holds, for a.e. $x \in \Omega$. Thus the Γ-lim inf inequality follows by applying Jensen's inequality in Definition 3.2.5 and (iii) in Theorem 3.2.6, in order to get (3.1.3). □

Proof of Theorem 3.1.2. The proofs follows the lines of the proof of [38, Theorem 4.4], by applying Theorem 3.1.1 instead of [38, Theorem 4.2]. □

Proof of Corollary 3.3.2. It is sufficient to argue as in the proof of Corollary 3.3.1 ,applying Theorem 3.1.2 to the sequence $\{E_n\}_n$, defined in X by equation (3.1.2) (with $\varphi = \varphi_n$), to get that it Γ-converges to $E_\infty : X \to (0, +\infty)$ with respect to the L^∞ strong convergence. □

As emphasized in [38], the assumptions of Theorem 3.1.2 differ from Theorem 3.1.1, since the condition $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ is not sufficient in the latter theorem. To this end we refer to Example 2.2.4.

3.4 $\mathbf{L}^{p(\cdot)}$- approximation of supremal functionals under no convexity assumptions

In this section the main aim is to remove assumption (**H1**), i.e. the $\mathrm{curl}_{p>1}$- Young quasiconvexity assumption on $f : \Omega \times \mathbb{R}^{Nd} \to \mathbb{R}$, in its last variable, under the restriction that f in(3.1.1) and (3.1.2) does not depend on the second variable, i.e. $f : \Omega \times \mathbb{R}^{Nd} \to [0, +\infty)$ and $f(x, \cdot)$ satisfies a growth condition of type $|\cdot|^\gamma$, also from above, and assuming that $\varphi_n(x, t) := t^{p_n(x)}$, i.e. the functionals appearing in (3.1.1) and (3.1.2) will be specialized in (3.4.2) and (3.4.14), respectively. Indeed, the result below provides an extension of [134, Theorem 2.2], to the case of modulars in the variable exponent Sobolev spaces and with more general growth conditions. On the other hand, in contrast with [134], we consider Carathéodory integrands instead of measurable ones (or normal ones as in the previous section). This is due to the exploited technical tools relying on integral representation of envelopes of integral functionals. Indeed, the relaxation and representation results available for integral functionals with only measurable integrands in $W^{1,p}$ and $W^{1,p(\cdot)}$ (see [72] and [119], respectively) provide a representation in terms of a generic quasiconvex function (see Definition 3.4.1 below), which in principle, in the case p constant, may differ from the one obtained in the case p variable, and, in principle, no comparison among the two integrands is possible. In contrast, in our arguments to obtain a supremal representation by means $L^{p(\cdot)}$-approximation, we rely in the case p constant, passing from $p(\cdot)$ to a suitable constant exponent p. We also observed that, differently from the results in Section 3.3, we assume Lipschitz regularity for $\partial\Omega$ and a control on the growth rates, see assumptions (3.4.3) and (3.4.4) below. Furthermore in Remarks 3.4.3 and 3.4.5 we discuss the removal of the Lipschitz assumptions and the fact that (3.4.4) is not necessary for Theorem

3.4.4.
Now we recall the notion of quasiconvexity (see e.g. [72]), that will be used in the sequel.

Definition 3.4.1. Let $g : \mathbb{R}^{Nd} \to \mathbb{R}$ be a Borel function and let $Q := (0;1)^N$. Then g is said quasiconvex (in the sense of Morrey) if

$$g(\xi) \leq \int_Q g(\xi + Du(x))\, dx$$

for every and for every $u \in W_0^{1,\infty}(Q;\mathbb{R}^d)$, $\xi \in \mathbb{R}^{Nd}$.

Theorem 3.4.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{Nd} \to [0,+\infty[$ be a Carathéodory function satisfying* (**H2**)*, such that*

$$\exists C > 0 : f(x,\xi) \leq C(|\xi|^\gamma + 1) \text{ for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^{Nd}. \tag{3.4.1}$$

Let $F_n : L^1(\Omega,\mathbb{R}^d) \to [0,+\infty]$ be the functional defined by

$$F_n(u) := \begin{cases} \|f(\cdot,Du)\|_{p_n(\cdot)} & \text{if } u \in W^{1,p_n(\cdot)}(\Omega,\mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4.2}$$

where $p_n : \Omega \to [1,+\infty)$ is a sequence of bounded variable exponents such that

$$\lim_{n\to+\infty} p_n^- = +\infty, \tag{3.4.3}$$

and

$$\exists \beta > 0 : \frac{p_n^+}{p_n^-} \leq \beta, \text{ for every } n \in \mathbb{N}. \tag{3.4.4}$$

Then $\{F_n\}_n$ sequentially $\Gamma(L^1(\Omega))$-converges with respect to the $L^1(\Omega;\mathbb{R}^d)$ weak topology, as $n \to \infty$ to the functional

$$F_\infty(u) = \begin{cases} \|\mathcal{Q}_\infty f(\cdot,Du)\|_\infty, & \text{if } u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^d), \\ \infty, & \text{otherwise,} \end{cases}$$

where

$$\mathcal{Q}_\infty f(x,\cdot) := \sup_{n\geq 1} (\mathcal{Q}f^n)^{1/n}(x,\cdot), \tag{3.4.5}$$

being $\mathcal{Q}f^n = \mathcal{Q}(f^n)$ the quasiconvex envelope (in the last variable) of f^n , i.e.

$$\mathcal{Q}(f^n)(x,\cdot) := \sup\{h : \mathbb{R}^{Nd} \to [0,+\infty] : h \text{ quasiconvex and } h(\xi) \leq f^n(x,\xi)\}. \tag{3.4.6}$$

Remark 3.4.3. (i) We underline that, in general $Q_\infty f$ is not level convex, but it is a strong Morrey quasiconvex (also called BJW-quasi-level convex) function, see [134, Section 5] for a proof. We stress that this property is a consequence of the lower semiconinuity of the Γ-limit functional (3.1.3) (see [30] and [137]).

(ii) We observe also that $\mathcal{Q}_\infty f$ is curl $-\infty$ quasiconvex (see [11, 12, 137]), i.e.

$$\begin{aligned}\mathcal{Q}_\infty f(x,\xi) &= \lim_{p\to+\infty} \inf\left\{\left(\int_Q (\mathcal{Q}_\infty f)^p(x,\xi+Du(y))\,dy\right)^{\frac{1}{p}} : u \in W^{1,\infty}_{\mathrm{per}}(Q;\mathbb{R}^N)\right\}\\ &= \lim_{p\to+\infty} (\mathcal{Q}((\mathcal{Q}_\infty f)^p)^{\frac{1}{p}}(x,\xi),\end{aligned}$$

in view of [137, Proposition 4.6, Theorem 4.8] (we also refer to [11] where the curl $-\infty$-quasiconvexity is deduced as a consequence of L^p-approximation).

(iii) For the sake of completeness (again referring to [137, Theorem 4.8]), we also remark that $Q_\infty f(x,\cdot)$ is curl-Young-quasiconvex, i.e. it satisfies the inequality in Definition 3.2.5 for every $W^{1,\infty}$-gradient Young measure.

(iv) We conclude this remark, recalling that we have provided a direct proof of the lower bound in Theorems 3.4.2 and 3.4.4, since, besides we rely on L^p-approximation results of [134], our growth conditions differ (i.e. they are milder) from those considered therein.

(v) The boundedness of Ω and the Lipschitz regularity of $\partial\Omega$ are used only in the proof of the lower bound inequality. They can be removed relying on the results of Section 4, assuming that the constant γ in **(H1)** and (3.3.1) is 1. Indeed the same growth is inherited by $\mathcal{Q}_\infty f$, that turns out to be $\mathrm{curl}_{(p>1)}$-Young quasiconvex (see [137, (7) of Theorem 4.8] for a proof) and such that $\mathcal{Q}_\infty f \le f$, hence it suffices to exploit the lower bound inequality of Theorem 3.4.2.

Proof. The proof will be obtained in two steps, first by proving the lower bound estimate and then the existence of a recovery sequence, in accordance with Proposition 1.5.2.

Concerning the Γ-lim inf inequality, let $\{u_n\}_n \subset L^1(\Omega;\mathbb{R}^d)$ converge weakly to $u \in L^1(\Omega;\mathbb{R}^d)$. Without loss of generality, we assume that

$$\liminf_{n\to\infty} F_n(u_n) = \lim_{n\to\infty} F_n(u_n) = M < \infty.$$

For this first step we rely on the results contained in [134, Theorem 2.2] for the case $p(x) \equiv p$ constant, indeed, a careful inspection of the proof of [134, Theorem 2.2], guarantees that the latter result holds with f satisfying (**H2**) and (3.4.1) instead of the linear growth condition therein. Furthermore, the Lipschitz regularity of $\partial\Omega$ and Sobolev and Rellich's theorems ensure the validity of the Γ-convergence result in [134, Theorem 2.2] also with respect to the sequential weak convergence in $L^1(\Omega;\mathbb{R}^d)$.

In detail, we observe that in view of the Lipschitz regularity of $\partial\Omega$, (**H2**) and (3.4.1), for any constant p sufficiently large, the functionals

$$I_p(u) := \begin{cases} \left(\int_\Omega f^p(x, Du(x))dx\right)^{1/p} & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise in } C(\overline{\Omega};\mathbb{R}^d), \end{cases}$$
$$G_p(u) := \begin{cases} \left(\int_\Omega f^p(x, Du(x))dx\right)^{1/p} & \text{if } u \in W^{1,p\gamma}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise in } C(\overline{\Omega};\mathbb{R}^d), \end{cases}$$
$$H_p(u) := \begin{cases} \left(\int_\Omega f^p(x, Du(x))dx\right)^{1/p} & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise in } C(\overline{\Omega};\mathbb{R}^d), \end{cases}$$

coincide. For the readers' convenience we refer to [83] where similar arguments have been used to pass from one convergence to another one in the proof of their Γ-convergences results.

At this point, for all $n \in \mathbb{N}$, let $\{u_n\}_n \subset W^{1,p_n^-}(\Omega;\mathbb{R}^d)$ and

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}_\infty f(\cdot, Du(\cdot)) &\leq \liminf_{n\to+\infty} \frac{1}{C_{p_n^-}} \left(\int_\Omega \mathcal{Q}(f^{p_n^-}(x, Du_n(x))dx\right)^{\frac{1}{p_n^-}} \\ &\leq \liminf_{n\to+\infty} \frac{1}{C_{p_n^-}} \left(\int_\Omega f^{p_n^-}(x, Du_n(x))dx\right)^{\frac{1}{p_n^-}} \\ &\leq \liminf_{n\to\infty} \|f(\cdot, Du_n)\|_{p_n(\cdot)}, \end{aligned} \tag{3.4.7}$$

where $C_{p_n^-}$ denotes the quantity $\Big(2L_n(|\Omega|+c)\Big)^{\frac{1}{p_n^-}}$ appearing in (2.2.1) which goes to 1 when $p_n^- \to \infty$. We remark that the first inequality is given by [134, Theorem 2.2], taking into account that the Γ-convergence result stated therein with respect to the uniform convergence, holds also with respect to the sequential weak L^1-convergence, while the second one is obtained by using $\mathcal{Q}(g) \leq g$. Finally the last inequality is deduced by Proposition 2.2.2 with the choices $\varphi_n(x,t) = t^{p_n(x)}$ and $p = p_n^-$: moreover we can consider $L_n = L = 1$ in (2.2.1) as long as $\varphi_n(x,\cdot)^{\frac{1}{p_n^-}}$ is convex, with $\varphi_n(x,t)$ satisfying $(aInc)_{q_n}$. Taking into account that the convexity assumption applied to $t_1 = t$ and $t_2 = 0$, gives

$$\begin{aligned}\varphi_n(x,\lambda t)^{\frac{1}{p_n^-}} &= \varphi_n(x,\lambda t_1 + (1-\lambda)t_2)^{\frac{1}{p_n^-}} \\ &\leq \lambda\varphi_n(x,t_1)^{\frac{1}{p_n^-}} + (1-\lambda)\varphi_n(x,t_2)^{\frac{1}{p_n^-}} = \lambda\varphi_n(x,t)^{\frac{1}{p_n^-}}, \qquad (3.4.8)\end{aligned}$$

which is equivalent to $(aInc)_{q_n}$ with $L = 1$ uniformly in $n \in \mathbb{N}$.
This implies the $\Gamma-$liminf inequality.

For what concerns the upper bound, we first observe that it suffices to consider the case $\|\mathcal{Q}_\infty(f(x,Du))\|_\infty = 1$, since the general case can be reduced to it by the normalization $\tilde{v} := \frac{v}{\|v\|_\infty}$ with $v := \mathcal{Q}_\infty(f(x,Du))$.
In order to estimate $\inf\{\limsup\limits_{n\to+\infty} \|f(x,Du_n(x))\|_{p_n(\cdot)} : u_n \rightharpoonup u \text{ in } L^1\}$, taking into account (3.2.1) we exploit the following inequality

$$\begin{aligned}&\|f(x,Du_n(x))\|_{p_n(\cdot)} \\ &\leq \left[\max\left\{\left(\int_\Omega f^{p_n(x)}(x,Du_n(x))dx\right)^{\frac{1}{p_n^-}}, \left(\int_\Omega f^{p_n(x)}(x,Du_n(x))dx\right)^{\frac{1}{p_n^+}}\right\}\right], \qquad (3.4.9)\end{aligned}$$

for every $n \in \mathbb{N}$.
At this point, we remark that, since we are dealing with an upper bound estimate for Γ-convergence we are entitled to pass from sequences $u_n \rightharpoonup u$ in $L^1(\Omega;\mathbb{R}^d)$ to sequences $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. Furthermore by [73, Proposition 6.11], and by the fact that the right-hand side of (3.4.9) can be written as $\Phi_n(\int_\Omega f^{p_n(x)}(x,Du_n(x))dx)$, with $\Phi_n : [0,+\infty) \to [0,+\infty)$, defined as

$$\Phi_n(H) = \begin{cases} H^{\frac{1}{p_n^+}} \text{ if } |H| \leq 1, \\ H^{\frac{1}{p_n^-}} \text{ if } |H| > 1 \end{cases}$$

continuous and increasing, so that Φ_n commutes with the operation of taking the lower semicontinuous envelope, in view of [73, eq. (6.3) in Proposition 6.16], we are allowed to deal with the relaxed functionals appearing in the right hand side of (3.4.9) which, in view of [119, Theorem 4.11] leads to

$$\inf\{\limsup_n \|f(x, Du_n(x))\|_{p_n(\cdot)} : u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d)\}$$

$$\leq \inf\left\{\limsup_n \left[\max\left\{\left(\int_\Omega f^{p_n(x)}(x, Du_n(x))dx\right)^{\frac{1}{p_n^-}}, \left(\int_\Omega f^{p_n(x)}(x, Du_n(x))dx\right)^{\frac{1}{p_n^+}}\right\}\right]\right.$$

$$\left. : u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d)\right\} \tag{3.4.10}$$

$$= \inf\left\{\limsup_n \left[\max\left\{\left(\int_\Omega \mathcal{Q}(f^{p_n(x)})(x, Du_n(x))dx\right)^{\frac{1}{p_n^-}},\right.\right.\right.$$

$$\left.\left.\left.\left(\int_\Omega \mathcal{Q}(f^{p_n(x)})(x, Du_n(x))dx\right)^{\frac{1}{p_n^+}}\right\}\right] : u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d)\right\}$$

where $\mathcal{Q}(f^{p_n(x)})(x, \cdot)$ is defined in (3.4.6).
On the other hand, [134, Remark 5.1] and (3.4.5) ensure that

$$\mathcal{Q}(f^{p_n(x)})(x, Du(x)) \leq (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x)),$$

hence, it suffices to consider for any sequence $\{u_n\}_n$ converging to u in $L^1(\Omega; \mathbb{R}^d)$,

$$\limsup_n \left[\max\left\{\left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du_n(x))dx\right)^{\frac{1}{p_n^-}},\right.\right.$$

$$\left.\left.\left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du_n(x))dx\right)^{\frac{1}{p_n^+}}\right\}\right].$$

Since $\|\mathcal{Q}_\infty(f(x, Du))\|_\infty = 1$, we have that

$$\int_\Omega (\mathcal{Q}_\infty f)^{p_n^+}(x, Du(x))dx \leq \int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx \leq \mathcal{L}^N(\Omega). \tag{3.4.11}$$

Following [83, (3.4), (3.5) and (3.6)], for every $n \in \mathbb{N}$, and, thanks to (3.4.11), we get that

$$1 = \lim_{n\to+\infty} (\mathcal{L}^N(\Omega))^{1/p_n^+} \geq \limsup_{n\to+\infty} \left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{1}{p_n^+}}$$

$$\geq \limsup_{n\to+\infty} \left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n^+}(x, Du(x))dx\right)^{\frac{1}{p_n^+}} = \|\mathcal{Q}_\infty f(x, Du(x))\|_\infty = 1.$$

Observing that we can replace lim sup by lim inf we can conclude that the lim sup is a limit and

$$\lim_{n\to+\infty} \left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{1}{p_n^+}} = 1. \tag{3.4.12}$$

By (3.4.4), we have that $1 \leq \beta_n := \frac{p_n^+}{p_n^-} \leq \beta$ and

$$\lim_{n\to+\infty}\left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{1}{p_n^-}} = \lim_{n\to+\infty}\left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{\beta_n}{p_n^+}}$$
$$= 1. \tag{3.4.13}$$

By (3.4.12) and (3.4.13), we can conclude that

$$\limsup_{n\to+\infty} \max\left\{\left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{1}{p_n^-}}, \left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{1}{p_n^+}}\right\}$$
$$= \lim_{n\to+\infty} \max\left\{\left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{1}{p_n^-}}, \left(\int_\Omega (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))dx\right)^{\frac{1}{p_n^+}}\right\}$$
$$= 1,$$

which, together with (3.4.10), concludes the proof of the upper bound and proves our statement. □

Theorem 3.4.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{Nd} \to [0, +\infty[$ be a Carathéodory function satisfying* (**H2**) *and* (3.4.1). *Let $E_n : L^1(\Omega, \mathbb{R}^d) \to [0, +\infty]$ be the functional defined by*

$$E_n(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{p_n(x)} f^{p_n(x)}(x, Du(x))dx & \text{if } u \in W^{1,p_n(\cdot)}(\Omega, \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4.14}$$

where $p_n : \Omega \to [1, +\infty)$ is a sequence of bounded variable exponents satisfying (3.4.3).
Then $\{E_n\}_n$ Γ converges with respect to the $L^1(\Omega; \mathbb{R}^d)$ weak convergence, as $n \to \infty$ to the functional

$$E_\infty(u) = \begin{cases} 0, & \text{if } u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^d), \text{ and } |\mathcal{Q}_\infty f(\cdot, Du)| \leq 1 \text{ a.e.} \\ \infty, & \text{otherwise,} \end{cases} \tag{3.4.15}$$

where $\mathcal{Q}_\infty f(x, \cdot)$ has been introduced in (3.4.5).

Proof. Concerning the $\Gamma-$liminf inequality, let $\{u_n\}_n \subset L^1(\Omega; \mathbb{R}^d)$ converge weakly to $u \in L^1(\Omega; \mathbb{R}^d)$. Without loss of generality, we assume that

$$\liminf_{n\to\infty} E_n(u_n) = \lim_{n\to\infty} E_n(u_n) = M < \infty.$$

Define

$$\varphi_n(x,t) := \frac{t^{p_n(x)}}{p_n(x)}$$

By the right hand side of the inequality contained in [98, Corollary 3.2.10], with φ which satisfy condition $(aInc)_{p_n^-}$ and with the constant a replaced by the constant L_n given by condition $(aInc)_{p_n^-}$, see (3.4.8), with $q_n = p_n^-$, we get

$$\|f(\cdot, Du_n)\|_{\varphi_n} \leq \max\left\{\left(\int_\Omega \frac{1}{p_n(x)} f^{p_n(x)}(x, Du_n(x))\, dx\right)^{1/p_n^-}, 1\right\}$$

and hence

$$\limsup_{n\to\infty} \|f(\cdot, Du_n)\|_{\varphi_n} \leq 1.$$

For all $n \in \mathbb{N}$, let $\{u_n\}_n \subset W^{1,p_n^-}(\Omega;\mathbb{R}^d)$.
By (3.4.7), we get

$$\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}_\infty f(\cdot, Du(\cdot)) \leq \liminf_{n\to\infty} \|f(\cdot, Du_n)\|_{\varphi_n} \leq 1.$$

In this way we are able to conclude that

$$\text{ess-sup}_{x\in\Omega} \mathcal{Q}_\infty f(x, Du(x)) \leq 1,$$

which proves the lower bound taking into account definition (3.4.15).
Now, we prove the Γ-$\limsup$ inequality.
Without loss of generality we can assume that $E_\infty(u) < \infty$, otherwise the thesis is trivial.
Arguing as in the proof of Theorem 3.4.2, by [73, Proposition 6.11], we can assume that the functionals E_n under consideration coincide with their relaxed ones, i.e. we can assume, in view of [119, Theorem 4.11], that

$$E_n(u) = \int_\Omega \frac{1}{p_n(x)} \mathcal{Q}\left(f^{p_n(x)}(x, Du(x))\right) dx$$

for every $u \in W^{1,p_n(\cdot)}(\Omega;\mathbb{R}^d)$. Since $\|\mathcal{Q}_\infty f(\cdot, Du)\|_\infty \leq 1$, recalling that for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{Nd}$, $\mathcal{Q}(f^p)^{\frac{1}{p}}(x,\xi)$ is increasing in p,
thanks to the definition of $\mathcal{Q}_\infty$ (3.4.5), we have

$$\limsup_{n\to\infty} \int_\Omega \frac{1}{p_n(x)} \mathcal{Q}\left(f^{p_n(x)}(x, Du(x))\right) dx \leq \limsup_{n\to\infty} \int_\Omega \frac{1}{p_n^-} (\mathcal{Q}_\infty f)^{p_n(x)}(x, Du(x))\, dx$$
$$= 0,$$

and thus the desired inequality follows and the proof is concluded. $\square$

Remark 3.4.5. Observe that in both Theorems 3.4.2 and 3.4.4, the proof of the Γ-$\liminf$ inequality holds for more general $\varphi_n(x,t)$, not necessarily of p_n-power type, provided that $(aInc)_{p_n}$ holds for the same constant L for all n, without imposing (3.4.1),(3.4.3) and (3.4.4).
In view of this last observation, in the case Ω has Lipschitz boundary, when $f : \Omega \times \mathbb{R}^{Nd} \to \mathbb{R}$, as a consequence of Theorems 3.4.2 and 3.4.4, in view of (iv) in Remark 3.4.3, and taking into account that the proofs of the upper-bound inequalities in Theorems 3.1.1, 3.1.2, Corollaries 3.3.1 and 3.3.2 (for which it is sufficient to consider a constant recovery sequence), we could replace in Theorems 3.1.1, 3.1.2, Corollaries 3.3.1 and 3.3.2 the $\mathrm{curl}_{(p>1)}$-Young quasiconvexity of $f(x,\cdot)$ by curl_∞-quasiconvexity. This latter assumption is in general weaker than the previous one, see [137].

The same observations regarding the Lipschitz continuity of $\partial\Omega$ contained in Remark (3.4.3)(v), apply to Theorem 3.4.4.
We also remark that (3.4.3) is not been exploited in the proof of this theorem, in contrast to Theorem 3.4.2, where it has been used in the proof of the upper bound inequality.

Chapter 4

Homogenization of non-local integral functionals via two-scale Young measures

4.1 Introduction

Homogenization has attracted much attention in last five decades, due to the many applications that mixtures and finely heterogeneous media have in applied science and technology, see for instance [117] and [69] among a wider pioneering literature. In fact both classical and more recent theories of ferro-magnetics and mechanics provide an accurate macroscopic description of phenomena occuring in materials which incorporate fine structures at the microscopic level. The reader can refer to [7, 23, 25, 31, 74, 77, 90, 91, 92] for applications to the theory of structured deformations, thin structures, non simple materials, evolution problems with nonstandard growth, elastomers, electromagnetic phenomena and constrained models. Here the idea is to adopt the homogenization techniques to formalize how the effects at micro-level emerge at the macro-level in the study of more general phenomena, such as peridynamics, where classical gradient are replaced by non classical one and, consequently non-local integral functions take over the role of standard integrals.
Indeed, adopting a different view point than [50], the aim of this chapter consists of studying the asymptotic behavior as $\varepsilon \searrow 0^+$ of $I_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to \mathbb{R}^+$ defined as

$$I_\varepsilon(u) := \iint_{\Omega\times\Omega} W\left(x, x', \frac{x}{\varepsilon}, \frac{x'}{\varepsilon}, u(x), u(x')\right) dx dx' \qquad (4.1.1)$$

where $\Omega \subset \mathbb{R}^N$, is an open subset, $u : \Omega \to \mathbb{R}^d$ is in some Lebesgue space L^p, $1 < p < \infty$ and the integrand $W : \Omega \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ has some measurability and continuity properties to be specified later and $W(\cdot,\cdot,\xi,\eta)$ is periodic in the first two variables, i.e. we will equivalently denote I_ε by

$$I_\varepsilon(u) = \iint_{\Omega\times\Omega} W\left(x, x,' \left\langle \frac{x}{\varepsilon}\right\rangle, \left\langle \frac{x'}{\varepsilon}\right\rangle, u(x), u(x')\right) dxdx',$$

where $\langle t\rangle$ denotes the vector in $(0,1)^N$ whose components are the fractional parts of $t \in \mathbb{R}^N$.

This kind of non-local functional appears in the mathematical modeling of peridynamics [143], ferromagnetics [139], etc. For instance, in the first setting, i.e. peridynamics: Ω represents the reference configuration, u is the deformation and I_ε is the associated energy.

In general I_ε is not lower semicontinuous hence the existence of minimizers is not guaranteed, and one should look for its relaxed version, for which we refer to [36], for instance, in order to detect the microstructure of the material in the context of nonlinear elasticity. Here we follow an analogous approach in terms of suitable Young measures, extending first L^p to the space of Young measures equipped with the narrow topology.

Thus, we can provide a full characterization of the Γ-limit. In fact, the Young measures are so far a crucial tool for obtaining a limiting formula. The reader can refer to [20, 87, 138], etc., for background on Young measures and to [24, 27, 28, 129] for the two-scales Young measures that are a key tool in this homogenization setting. As already emphasized in [36, 113, 131, 132], the relaxation and homogenization of (4.1.1) in L^p is a very delicate issue; in fact, it is not clear if the limiting energy (as $\varepsilon \to 0$) has still the same form.

In the subsequent analysis, the two-scale Young measures introduced by [81] and later developped by [149] , defined in Definition 4.2.1, will play a crucial role to compute, via Γ-convergence with respect to a suitable topology, the limit energy of (4.1.1), under the assumption on the energy density W of being an admissible integrand (as introduced in [45]) in the sense of Definition 4.2.9. We also stress that there is no loss of generality in assuming that W is *symmetric*, as observed in [130] and [36], i.e.

$$W(x, x_0, y, y_0, \xi, \xi_0) = W(x_0, x, y_0, y, \xi_0, \xi) \tag{4.1.2}$$

for a.e. $x, x_0 \in \Omega$, and all $y, y_0 \in Q, \xi, \xi_0 \in \mathbb{R}^d$.

Theorem 4.1.1. *Let $p > 1$ and assume $W : \Omega\times\Omega\times Q\times Q\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty]$ is a symmetric admissible integrand and there exists $a, \alpha \in L^1(\Omega\times\Omega)$ and $c > 0$ such that*

$$\alpha(x, x') + \frac{1}{c}|\xi|^p \le W(x, x', y, y', \xi, \xi') \le a(x, x') + c(|\xi|^p + |\xi'|^p) \tag{4.1.3}$$

for a.e. $x, x' \in \Omega$, *all* $y, y' \in \mathbb{R}^d$ *and* $\xi, \xi' \in \mathbb{R}^d$. *Let* $\varepsilon > 0$ *and let* $\{I_\varepsilon\}_\varepsilon$ *be the family of functionals in* (4.1.1). *Then,* $\{I_\varepsilon\}_\varepsilon$ Γ*-converges, with respect to the weak* $L^p(\Omega;\mathbb{R}^d)$ *convergence, to* $I_{hom} : L^p(\Omega,\mathbb{R}^d) \to [0,+\infty)$, *where* I_{hom} *is defined as*

$$I_{hom}(u) := \min_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x,x',y,y',\xi,\xi')d\nu_{(x,y)}(\xi)d\nu_{(x',y')}(\xi')dydy'dxdx'$$

where

$$\mathcal{M}_u := \Big\{\nu \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d)) : \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \text{ is a two-scale Young measure such that } \int_Q\int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi)dy = u(x)\Big\}. \quad (4.1.4)$$

Remark 4.1.2. We would like to underline that in the previous theorem we chose the image set $[0,+\infty]$ for semplicity's sake. Indeed, the result still holds if we assume W bounded from below and having values in $\mathbb{R}\cup\{\infty\}$.

In order to prove Theorem 4.1.1, we characterize two-scale Young measures by proving the following result. We use the strategy introduced in [110, 111] and later adopted in [24], (and in [93] for the Sobolev-Orlicz setting). See also [27] and [138] where some parts of our results have been proven.

Theorem 4.1.3. *Let* $Q := (0,1)^N$, Ω *be an open and bounded subset of* $\mathbb{R}^N$ *with Lipschitz boundary, and* $\nu \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$ *be such that* $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^d)$ *for a.e.* $(x,y) \in \Omega\times Q$. *Then, the family* $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ *is a two-scale Young measure if and only if the following conditions hold*

i) there exists $1 < p < +\infty$ *such that*

$$(x,y) \mapsto \int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega\times Q);$$

ii) there exists $u_1 \in L^p(\Omega, L^p_{per}(Q))$ *such that*

$$\int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi) = u_1(x,y),$$

for a.e. $(x,y) \in \Omega\times Q$;

iii) for every $f \in \mathcal{E}_p$

$$\int_Q\int_{\mathbb{R}^{d\times N}} f(y,\xi)d\nu_{(x,y)}(\xi)dy \geq f_{hom}(u(x)) \quad \text{for a.e. } x\in\Omega,$$

where

$$\mathcal{E}_p := \Big\{ f : \overline{Q} \times \mathbb{R}^d \to \mathbb{R} \textit{ continuous and such that exists} \tag{4.1.5}$$

$$\lim_{|\xi|\to+\infty} \frac{f(y,\xi)}{1+|\xi|^p} \textit{ uniformly with respect to } y \in \overline{Q} \Big\}, \tag{4.1.6}$$

$$u(x) := \int_Q u_1(x,y)dy,$$

and for every $\xi \in \mathbb{R}^d$, $f_{hom}(\xi) :=$

$$\lim_{T\to+\infty} \frac{1}{T^N} \inf \left\{ \int_{(0,T)^N} f(\langle x \rangle, \xi + v(x))dx : v \in L^p((0,T)^N, \mathbb{R}^d), \right.$$

$$\left. \int_{(0,T)^N} v(x)dx = 0 \right\}. \tag{4.1.7}$$

For a generalization of the above characterization when $p = 1$, in the constrained case (i.e., u satisfying a suitable PDE, not necessarily curl), we refer to [18], leaving to future investigation the study of non-local models as in (4.1.1) when W satisfies a linear growth condition (i.e. (4.1.3) for $p = 1$). To conclude this introduction, we point out Theorem 4.1.1 holds true in the multi-scale case (see [27, 129] for a discussion on the topic). Indeed, the techniques used in the proof (under appropriate symmetry conditions) can be similarly replicated in the case of multiple scales.
The chapter is organized as follows. Section 4.2 is devoted to notation and preliminaries. In section 4.3 some technical results are presented and, in particular, Theorem 4.1.3. Finally, in section 4.4 the homogenization result, i.e. Theorem 4.1.1 is proved and a comparison between our limiting formula with the relaxed energy obtained in [36] in the homogeneous isotropic setting is presented.

4.2 Preliminaries

4.2.1 Young measures

In this section, we provide further and more specific results concerning Young measures. For previous notions we refer to Section 1.3.
Two-scale Young measures (introduced in [81] and later developed by Pedregal in [128] and Valadier in [149]), are a key tool in our subsequent analysis. We recall the definition, as introduced by [129] and later developed by [27, 28, 24] and recently extended to the case $p = 1$ (i.e. generalized two-scale Young measures) by [18]

Definition 4.2.1 (Two-scale Young measures)**.** Let $\nu \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^d))$. The family $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is said to be a two-scale Young measure if $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $(x,y) \in \Omega \times Q$ and if for every sequence $\{\varepsilon_n\}_n \to 0$ there exists a bounded sequence $\{u_n\}_n$ in $L^p(\Omega, \mathbb{R}^d)$ such that for every $z \in L^1(\Omega)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^{N\times d})$,

$$\lim_{n\to\infty} \int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n(x)\right) dx = \int_\Omega \int_Q \int_{\mathbb{R}^d} z(x)\varphi(y,\xi)d\nu_{(x,y)}(\xi)dydx.$$

Namely, $\{(\langle\cdot/\varepsilon_n\rangle, u_n)\}_n$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x\in\Omega})$. Finally, we may call $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ the two-scale Young measure associated to the sequence $\{u_n\}_n$.

We refer to [27, Theorem 3.5] and [24, Subsection 2.3] to compare the two-scale Young measure associated with $\{u_n\}_n$ (according to the above definition), with the Young measure associated to $\left\{\left(\langle \frac{x}{\varepsilon_n}\rangle, u_n(x)\right)\right\}_n$, which, in principle, at a naive look, could seem a different element in $L^\infty_w(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$, where $\mathbb{T}^N$ is the torus in $\mathbb{R}^N$. In fact, for a better understanding of the above notion, it is important to recall the notion of two-scale convergence introduced by [124] and later developed by [3] and exploited by many authors.

Definition 4.2.2 (Two-scale convergence)**.** Let $\{u_h\}_h$ be a sequence in $L^1(\Omega, \mathbb{R}^d)$. The sequence $\{u_h\}_h$ is said to be two-scale convergent to a function $u = u(x,y) \in L^1(\Omega \times Q, \mathbb{R}^d)$ if, for every component u_h^i of u_h and u^i of u, $i = 1, \dots, d$.

$$\lim_{h\to\infty} \int_\Omega \varphi(x)\phi\left(\frac{x}{\varepsilon_h}\right) u_h^i(x)dx = \iint_{\Omega\times Q} \varphi(x)\phi(y)u^i(x,y)dxdy,$$

for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and any $\phi \in \mathcal{C}^\infty_{per}(Q)$. We simply write $u_h \rightsquigarrow u$.

We recall [129, Proposition 2.3].

Proposition 4.2.3. *Given a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ as in Definition 4.2.1, then the mapping $u_1 : \Omega \times Q \to \mathbb{R}^d$ defined by*

$$u_1(x,y) = \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda)$$

is the two-scale limit of the sequence $\{u_n\}_n$ generating $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$.

Now, we also recall the definition for the underlying deformation and we show that it does not depend on the generating sequence but only on the two-scale Young measures.

Remark 4.2.4. Consider $\{\varepsilon_n\}_n$, $\{u_n\}_n$ and $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ as in Definition 4.2.1. Since $\{u_n\}_n$ is bounded in L^p, then there exists a subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \rightsquigarrow u_1$ (see for instance [3, Theorem 0.1]). By [27, Proposition 3.4], $u_{n_k} \rightharpoonup u$ in L^p, with

$$u(x) = \int_Q u_1(x,y)dy.$$

It is important to notice that u is uniquely defined. Indeed, by Proposition 4.2.3 it holds

$$u_1(x,y) = \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda).$$

So, if we consider another subsequence $\{u_{n_j}\}_j$ two-scale converging to some function v_1, then since ν is the same for both $\{\varepsilon_{n_k}\}_k$ and $\{\varepsilon_{n_j}\}_j$, it holds

$$u_1(x,y) = \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda) = v_1(x,y).$$

In particular, it follows that

$$u(x) = \int_Q \int_{\mathbb{R}^d} \lambda d\nu_{(x,y)}(\lambda)dy,$$

and that u does not depend on the chosen subsequence $\{\varepsilon_{n_k}\}_k$. The function u is called the "underlying deformation" of $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$.

The proof of the following result is omitted, being easier than the one of [24, Lemma 3.4], since the generating sequences are only in L^p and not in $W^{1,p}$.

Lemma 4.2.5. *Let D be a Lebesgue measurable subset of Ω, and consider μ and ν in $L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$ such that $\{\mu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ and $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ are two-scale Young measures with the same underlying deformation $u \in L^p(\Omega,\mathbb{R}^d)$. Let*

$$\sigma_{(x,y)} := \begin{cases} \mu_{(x,y)} & \text{if } (x,y) \in D\times Q, \\ \nu_{(x,y)} & \text{if } (x,y) \in (\Omega\setminus D)\times Q. \end{cases}$$

Then $\sigma \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$ and $\{\sigma_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure with underlying deformation u.

Next we consider the homogeneous case.

Definition 4.2.6 (Homogeneous two-scale Young measure)**.** Given a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$ we say that it is homogeneous if the map $(x,y)\mapsto \nu_{(x,y)}$ is independent of x. In this case it can be identified with an element of $L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d))$ and we may write $\{\nu_y\}_{y\in Q} \equiv \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$.

This yields the definition of average for a two-scale Young measure and we can also prove that this is an homogeneous two-scale Young measure.

Definition 4.2.7 (Average)**.** Given a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$, its average is the family $\{\bar{\nu}_y\}_{y\in Q}$ defined by

$$\langle \bar{\nu}_y, \phi\rangle := \fint_\Omega \int_{\mathbb{R}^d} \phi(\xi) d\nu_{(x,y)}(\xi) dx, \quad \phi \in C_0(\mathbb{R}^d).$$

As observed in [24], if $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure, then it can be seen that $\overline{\mu} := \overline{\nu}_y \otimes dy$ is the average of $\{\mu_x\}_{x\in\Omega}$ with $\mu_x := \nu_{(x,y)} \otimes dy$ and $\mu \in L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^N\times\mathbb{R}^d))$. Thus, $\overline{\mu}$ is a homogeneous Young measure by [127, Theorem 7.1, pg. 117]. Following the same strategy as [24, Lemma 2.9], we prove that $\{\overline{\nu}_y\}_{y\in Q}$ is actually a homogeneous two-scale Young measure. We consider the case of constant underlying deformation $F \in \mathbb{R}^d$.

Lemma 4.2.8. *Let $\nu \in L^\infty_w(Q\times Q, \mathcal{M}(\mathbb{R}^d))$ be such that $\{\nu_{(x,y)}\}_{(x,y)\in Q\times Q}$ is a two-scale Young measure with underlying deformation F, with $F\in\mathbb{R}^d$. Then $\{\bar{\nu}_y\}_{y\in Q}$ is a homogeneous two-scale Young measure with the same underlying deformation.*

Proof. First we note that $\bar{\nu} \equiv \{\overline{\nu}_y\}_{y\in Q} \in L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d))$, indeed by Fubini's Theorem and by Definition 4.2.7 we have that $y \mapsto \bar{\nu}_y$ is weakly* measurable. Our aim is to show that for every sequence $\{\varepsilon_n\}_n$ there exists $\{v_n\}_n \in L^p(Q,\mathbb{R}^d)$ such that the sequence $\left\{\left(\langle\frac{\cdot}{\varepsilon_n}\rangle, v_n\right)\right\}_n$ generates the measure $\bar{\mu} = \bar{\nu}_y \otimes dy$ and $v_n \rightharpoonup F$ in L^p. Let $\{u_n\}_n \subset L^p(Q,\mathbb{R}^d)$ be the sequence generating the two-scale Young measure ν and assume that $u_n \rightharpoonup F$ in L^p. Let $\varepsilon_n \to 0$ and define $\rho_n := \varepsilon_n \left[\frac{1}{\sqrt{\varepsilon_n}}\right]$. By construction there exists $m_n \in \mathbb{N}$, $a_i^n \in \rho_n\mathbb{Z}\cap Q$, and a measurable set $E_n \subset Q$ such that $\mathcal{L}^N(E_n) \to 0$ and

$$Q = \bigcup_{i=1}^{m_n} (a_i^n + \rho_n Q) \cup E_n.$$

Let

$$v_n(x) := \begin{cases} u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right), & \text{if } x \in a_i^n + \rho_n Q \text{ and } i \in \{1,\dots,m_n\} \\ F \quad \text{otherwise} \end{cases}$$

By construction, $v_n \in L^p(Q,\mathbb{R}^d)$ and $v_n \rightharpoonup F$. Let $z \in \mathcal{C}_c(Q)$ and $\phi \in \mathcal{C}_0(\mathbb{R}^N\times\mathbb{R}^d)$, then

$$\begin{aligned}\int_Q z(x)\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right)dx &= \sum_{i=1}^{m_n}\int_{a_i^n+\rho_n Q} z(x)\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right)\right)dx\\ &+\int_{E_n} z(x)\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, F\right)\\ &=\sum_{i=1}^{m_n} z(a_i^n)\int_{a_i^n+\rho_n Q}\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right)\right)dx\\ &+\sum_{i=1}^{m_n}\int_{a_i^n+\rho_n Q}(z(x)-z(a_i^n))\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}\left(\frac{x-a_i^n}{\rho_n}\right)\right)dx\\ &+\int_{E_n} z(x)\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, F\right).\end{aligned}$$

By the uniform continuity of z and the fact that $\mathcal{L}^N(E_n)\to 0$ it follows that

$$\int_Q z(x)\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right)dx = \sum_{i=1}^{m_n}\rho_n^N z(a_i^n)\int_Q\phi\left(\left\langle\frac{\rho_n x+a_i^n}{\varepsilon_n}\right\rangle, u_{\rho_n/\varepsilon_n}(x)\right)dx + o(1),$$

i.e. that

$$\int_Q z(x)\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right)dx = \sum_{i=1}^{m_n}\rho_n^N z(a_i^n)\int_Q\phi\left(\left\langle\frac{x}{\varepsilon_n/\rho_n}\right\rangle, u_{\rho_n/\varepsilon_n}(x)\right)dx + o(1),$$

and so passing to the limit for $n\to\infty$ and by Definition 4.2.7 we get

$$\begin{aligned}\lim_{n\to\infty}\int_Q z(x)\phi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, v_n(x)\right)dx &= \int_Q z(x)dx\int_Q\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\nu_{(x,y)}(\xi)dydx\\ &=\int_Q z(x)dx\int_Q\int_{\mathbb{R}^d}\phi(y,\xi)d\bar\nu_y(\xi)dy\\ &=\langle\bar\nu,\phi\rangle\int_Q z(x)dx.\end{aligned}$$

By a density argument it is easy to see that the previous identity holds for every $z\in L^1(Q)$ and so that $\{\bar\nu_y\}_{y\in Q}$ is the homogeneous two scale Young measure generated by $\{v_n\}_n$. □

Now we introduce our class of functions, according to the notion introduced in [45], and later adopted by [24, 27, 149].

Definition 4.2.9 (Admissible integrand)**.** A function $f:\Omega\times\Omega\times Q\times Q\times\mathbb{R}^d\times\mathbb{R}^d\to[0,+\infty]$ is said to be an admissible integrand if, for any $\eta>0$, there exist compact sets $X^\eta\subset\Omega\times\Omega$, $Y^\eta\subset Q\times Q$, with $\mathcal{L}^{2N}((\Omega\times\Omega)\setminus X^\eta)<\eta$, $\mathcal{L}^{2N}((Q\times Q)\setminus Y^\eta)<\eta$ and such that $f_{|_{X^\eta\times Y^\eta\times\mathbb{R}^d\times\mathbb{R}^d}}$ is continuous.

Remark 4.2.10. i) We observe that by [27, Lemma 4.11], if f is an admissible integrand then for any fixed $\varepsilon > 0$ the functional

$$(x, x', \xi, \xi') \mapsto f(x, x', x/\varepsilon, x'/\varepsilon, \xi, \xi')$$

is $\mathcal{L}^{2N}(\Omega \times \Omega) \otimes \mathcal{B}(\mathbb{R}^{2d})$-measurable, where $\mathcal{B}(\mathbb{R}^{2d})$ is the σ-algebra of Borel subsets of $\mathbb{R}^{2d}$. In particular, the functional I_ε (defined in (4.1.1)) is well defined in $L^p(\Omega, \mathbb{R}^d)$.

ii) Moreover, we recall that by the Scorza-Dragoni Theorem it is possible to prove that a Carathéodory function is an admissible integrand.

The next result, stated for generic spaces and dimensions, is a two-scale analog of the [Fundamental Theorem of Young Measures [(iv), (v)], and it was proved in [28, Theorem 2.8]. We recall it here since it is a crucial tool to prove Theorem 4.1.1.

Theorem 4.2.11. *Let $k, m \in \mathbb{N}$, $R \subset \mathbb{R}^k$ be open and bounded, and S be the unit cube in $\mathbb{R}^k$. Let $\{\varepsilon_n\}_n$ be a vanishing positive sequence. Let $\{u_n\}_n \subset L^1(R, \mathbb{R}^m)$ be a bounded sequence. Then, there exist non relabeled subsequences and a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in R\times S}$ such that it holds*

(i) if $\mathcal{W} : R \times S \times \mathbb{R}^m \to [0, +\infty]$ is an admissible integrand then

$$\liminf_{n\to+\infty} \int_R \mathcal{W}\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n(x)\right) dx \geq \iint_{R\times S} \overline{\mathcal{W}}(x, y) dx dy, \tag{4.2.1}$$

where

$$\overline{\mathcal{W}}(x, y) := \int_{\mathbb{R}^M} \mathcal{W}(x, y, \lambda) d\nu_{(x,y)}(\lambda);$$

(ii) if $\mathcal{W} : R \times S \times \mathbb{R}^m \to \mathbb{R}$ is an admissible integrand and $\left\{\mathcal{W}\left(\cdot, \left\langle \frac{\cdot}{\varepsilon_n} \right\rangle, u_n(\cdot)\right)\right\}$ is equi-integrable for any $x \in R$, then $\mathcal{W}(x, y, \cdot)$ is $\nu_{(x,y)}$-integrable for a.e. $(x, y) \in R \times S$, the function $\overline{\mathcal{W}}$ is in $L^1(R \times S)$ and

$$\lim_{n\to+\infty} \int_R \mathcal{W}\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n(x)\right) dx = \iint_{R\times S} \overline{\mathcal{W}}(x, y) dx dy. \tag{4.2.2}$$

Remark 4.2.12. We would like to point out that the statement of Theorem 4.2.11 is slightly different compared to [28, Theorem 2.8]. Indeed, in [28, Theorem 2.8-i)] the integrand function $\mathcal{W}$ is assumed to be positive and finite. As far as we understand the proof still works for $\mathcal{W}$ having values in $[0, +\infty]$ as long as the set where $\mathcal{W} = \infty$ has null measure.

Remark 4.2.13. i) We remark that it is possible to look at Young measures from a different perspective than previous definition. In fact, if $\Omega \subset \mathbb{R}^N$ is an open bounded subset and $p > 1$, we can say that a Young measure in $\Omega \times \mathbb{R}^m$ is a positive measure μ in $\Omega \times \mathbb{R}^m$, such that its push-forward measure $\pi_\Omega \# \mu$, obtained through the projection π_Ω is the $\mathcal{L}^N$ measure on Ω, i.e. for all Borel subsets B of Ω

$$\pi_\Omega \# \mu(B) := \mu(B \times \mathbb{R}^m) = \mathcal{L}^N(B).$$

This set of measures is denoted by $\mathcal{Y}(\Omega; \mathbb{R}^m)$. By [20, Theorem 4.2.4], $\mathcal{Y}(\Omega; \mathbb{R}^m) \subset L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$, with the identification $\mu = \{\mu_x\}_{x\in\Omega} \otimes \mathcal{L}^N$.

ii) On the other hand, [20, Theorem 4.3.1] ensures that in $\mathcal{Y}(\Omega; \mathbb{R}^m)$, the weak* convergence in $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^m))$ of $\{\mu_n\}_n \subset \mathcal{Y}(\Omega; \mathbb{R}^m)$ towards $\mu \in \mathcal{Y}(\Omega; \mathbb{R}^m)$ is equivalent to the narrow convergence, where the latter is defined as follows

$$\mu_n \overset{nar}{\rightharpoonup} \mu \Longleftrightarrow \lim_n \iint_{\Omega\times\mathbb{R}^m} \psi(x,\lambda) d\mu_n(x,\lambda) = \iint_{\Omega\times\mathbb{R}^m} \psi(x,\lambda)\mu(x,\lambda),$$

for every $\psi \in C_b(\Omega; \mathbb{R}^m)$. For more details on the narrow topology we refer to [20, Section 4.3].

Equivalently, by the identification $\mu_n = \{(\mu_n)_x\}_{x\in\Omega}$ and $\mu = \{\mu_x\}_{x\in\Omega}$, we say that $\{\mu_n\}$ narrowly converges to μ if and only if for every $g \in L^1(\Omega)$ and $h \in \mathcal{C}_0(\mathbb{R}^m)$ it holds

$$\lim_{n\to\infty} \int_\Omega g(x) \int_{\mathbb{R}^m} h(y) d(\mu_n)_x(y) dx = \int_\Omega g(x) \int_{\mathbb{R}^m} h(y) d\mu_x(y) dx.$$

iii) Given $p \geq 1$, $\mathcal{Y}^p(\Omega; \mathbb{R}^m)$ is the subset of $\mathcal{Y}(\Omega; \mathbb{R}^m)$ such that

$$\iint_{\Omega\times\mathbb{R}^m} |y|^p d\mu(x,y) < +\infty. \tag{4.2.3}$$

As a consequence of Hölder's inequality, $\mathcal{Y}^p(\Omega; \mathbb{R}^m) \subset \mathcal{Y}^q(\Omega; \mathbb{R}^m)$ if $1 \leq q \leq p$.

We recall also that $\mathcal{Y}^p(\Omega; \mathbb{R}^m)$ is not closed in $\mathcal{Y}(\Omega; \mathbb{R}^m)$ under the narrow topology. Nevertheless, given a bounded sequence $\{v_n\}_n \subset L^p(\Omega; \mathbb{R}^m)$ there exists $\mu \in \mathcal{Y}(\Omega, \mathbb{R}^m)$ such that $\{v_n\}_n$ (up to a subsequence) generates μ; and $\mu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)$. Conversely, for any $\mu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)$, there exists a bounded sequence $\{v_n\}_n \subset L^p(\Omega; \mathbb{R}^m)$, generating μ and such that $\{|v_n|^p\}_n$ is equi-integrable. We refer to [20] and [36] for details.

Moreover, considering a bounded sequence $\{v_n\}_n \subset L^p(\Omega, \mathbb{R}^m)$ and $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^m)$, up to identifying each v_n with the Dirac mass $\delta_{v_n} \in \mathcal{Y}^p(\Omega, \mathbb{R}^m)$, by [20, Theorem 4.3.1] we have that $\{v_n\}_n$ generates ν (in the sense of Theorem 1.3.3) if and only if δ_{v_n} narrowly converges to ν.

iv) Following [149] and [27], we can extend the definition of Young measure of finite p moment to parametrized measures defined in Ω with values in S, with $S = \mathbb{T}^N \times \mathbb{R}^d$, equipped with the Borel σ-algebra, i.e. $\mathcal{Y}^p(\Omega; S)$. In this setting the test functions for the narrow convergence can be given by $C_c(\Omega; \mathbb{R}^d) \otimes C_c(S; \mathbb{R}^d)$, see [149, page 149].

These latter observations motivate the comparison of the homogenized functional I_{hom} of Theorem (4.1.1) with the functional $\overline{I}_{hom}$ in Corollary 4.4.3 computed in terms of suitable Young measures in $\mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$.

In fact, we also recall that the relaxation of non-local functionals as in (4.1.1), but not dependent on ε, has been obtained in [36] in term of narrow convergence in $\mathcal{Y}^p(\Omega; \mathbb{R}^d)$. The analogous stand point for the homogenization will be given in Corollary 4.4.3, with the use of the above mentioned subset of $\mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$.

4.2.2 Γ-convergence

In this subsection, we expand the Section 1.5. Since Proposition 1.5.2, we can write a Γ-limit as

$$F(x) = \inf\{\liminf F_k(x_k) : d(x_k, x) \to 0, \text{ as } k \to +\infty\}.$$

A fundamental property we want to underline is that the Γ-limit is lower semicontinuous with respect to the convergence induced by d, see [73, Proposition 6.8]. We also provide the definition for Γ-convergence for a family of functionals.

Definition 4.2.14. Let (X, d) be a metric space. For a positive parameter ε, we say that a family $\{F_\varepsilon\}_\varepsilon$ of functionals, with $F_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$, Γ-converges to $F : X \to \mathbb{R} \cup \{+\infty\}$, with respect to the metric d as $\varepsilon \to 0^+$, if for all vanishing sequences $\{\varepsilon_k\}_k$, $\{F_{\varepsilon_k}\}_k$ Γ-converges to F, when $k \to \infty$.

We will write, as for the case of sequences

$$F(x) : \inf\{\liminf F_\varepsilon(x_\varepsilon) : d(x_\varepsilon, x) \to 0, \text{ as } \varepsilon \to 0\}.$$

Finally, we state a theorem, whose proof is omitted being very similar to the one of [60, Theorem 1.1] (for related results see also [52]).

Theorem 4.2.15. *Let Ω be a bounded open set of $\mathbb{R}^N$, let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory integrand satisfying*

(H1) $f(\cdot, b)$ is Q-periodic, for all $b \in \mathbb{R}^d$;

(H2) there exist $p > 1$ and a positive constant C such that

$$\frac{1}{C}|b|^p - C \leq f(x,b) \leq C(1 + |b|^p),$$

for a.e. $x \in \Omega$ and for every $b \in \mathbb{R}^d$.

For every $\varepsilon > 0$, consider the family of functionals $F_\varepsilon : L^p(\mathbb{R}^d) \to (-\infty, +\infty]$, defined as

$$F_\varepsilon(u) := \int_\Omega f\left(\frac{x}{\varepsilon}, u(x)\right) dx.$$

Let $\mathcal{F}$ be the Γ-limit of $\{F_\varepsilon\}_\varepsilon$ with respect to the weak convergence in $L^p(\Omega;\mathbb{R}^d)$, i.e. $\mathcal{F}(u) := \inf\{\liminf F_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\}$, then

$$\mathcal{F}(u) = \int_\Omega f_{hom}(u(x))dx,$$

where f_{hom} is the function in (4.1.7).

4.3 Characterization

This section is mainly devoted to the proof of Theorem 4.1.3 and to the attainment of other related results which will be used in the sequel.

We start by recalling the space $\mathcal{E}_p$ defined by (4.1.5) and (4.1.6). Its properties have been presented in [24, Section 3]. We briefly recall the main ones. It is a Banach space under the norm

$$\|f\|_{\mathcal{E}_p} := \sup_{y\in\overline{Q},\xi\in\mathbb{R}^d} \frac{|f(y,\xi)|}{1+|\xi|^p}.$$

Moreover, $\mathcal{E}^p$ is isomorphic to the space $\mathcal{C}(\overline{Q} \times (\mathbb{R}^d \cup \{\infty\}))$.
Before proving Theorem 4.1.3 we start commenting on the formulas (4.1.7).
Under the assumptions of Theorem 4.1.3, if f belongs to the class (4.1.5), in view of standard relaxation results (see [73, Proposition 6.11], [87, Theorem 6.68 and Remark 6.69]),

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\} = \tag{4.3.1}$$
$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega (\mathrm{co}f)\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right) dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\},$$

where co$f$ stands for the convex envelope of f with respect to the second variable. Theorem 4.2.15 (see the proof of [60, Theorem 1.1.] within a slightly more general context), guarantees that

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right)dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\} = \int_\Omega f_{\text{hom}}(u(x))dx,$$

where f_{hom} is as in (4.1.7). On the other hand, (4.3.1)

$$\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f\left(\left\langle\frac{x}{\varepsilon}\right\rangle, u_\varepsilon(x)\right)dx : u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^d)\right\} = \int_\Omega (\text{co}f)_{\text{hom}}(u(x))dx,$$

where, for every $\xi \in \mathbb{R}^d$, $(\text{co}f)_{hom}(\xi) =$

$$\lim_{T\to+\infty}\frac{1}{T^N}\inf\left\{\int_{(0,T)^N}(\text{co}f)(\langle x\rangle, \xi + v(x))dx : v \in L^p((0,T)^N, \mathbb{R}^d),\right.$$
$$\left.\int_{(0,T)^N} v(x)dx = 0\right\}. \tag{4.3.2}$$

Finally arguing as in [51, Theorem 14.7] it results that

$$(\text{co}f)_{\text{hom}}(\xi) = \inf\left\{\int_{(0,1)^N}(\text{co}f)(\langle x\rangle, \xi + v(x))dx : v \in L^p((0,1)^N, \mathbb{R}^d),\right.$$
$$\left.\int_{(0,1)^N} v(x)dx = 0\right\}.$$

Hence, by easier arguments than those in [87, Proposition 6.24], we can conclude that $f_{\text{hom}} = (\text{co}f)_{\text{hom}}$.
The next result, whose proof is omitted, as in [24, Corollary 3.8], follows from the fact that conditions *i), ii)*, and *iii)* in Theorem 4.1.3 do not depend on the sequence $\{\varepsilon_n\}_n$.

Corollary 4.3.1. *Let $\{u_n\}_n$ be a bounded sequence in $L^p(\Omega, \mathbb{R}^d)$ and assume that there exists a vanishing sequence $\{\varepsilon_n\}_n$ such that $\{(\langle\frac{\cdot}{\varepsilon_n}\rangle, u_n)\}_n$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x\in\Omega}$. Then $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure.*

We start by addressing the proof of the necessity of Theorem 4.1.3.

Theorem 4.3.2. *Let Ω be a bounded and open subset of $\mathbb{R}^N$.*
Let $\nu \equiv \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \in L^\infty_w(\Omega\times Q, \mathcal{M}(\mathbb{R}^d))$ be a two-scale Young measure. Then there exist $u_1 \in L^p(\Omega; L^p_{per}(Q, \mathbb{R}^d))$ and $u \in L^p(\Omega, \mathbb{R}^d)$ such that

$$u_1(x,y) = \int_{\mathbb{R}^d}\xi d\nu_{(x,y)}(\xi), \tag{4.3.3}$$

$$u(x) = \int_Q u_1(x,y)dy \text{ for a.e. } x \in \Omega, \tag{4.3.4}$$

$$f_{hom}(u(x)) \leq \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_{(x,y)}(\xi) dy, \text{ for a.e. } x \in \Omega, \text{ for any } f \in \mathcal{E}_p, \tag{4.3.5}$$

where $f_{hom}(F)$ *is defined as in* (4.1.7) *and*

$$\int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q). \tag{4.3.6}$$

Proof. By definition of two-scale Young measure there exists a sequence $\{u_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ which generates $\{\nu_{x,y}\}_{(x,y)\in\Omega\times Q}$, whose two-scale limit is a function u_1 as in the statement, which, in particular, as observed in Proposition 4.2.3, satisfies (4.3.3). Also the weak limit of $\{u_n\}_n$ is a function $u \in L^p(\Omega; \mathbb{R}^d)$ such that $\int_Q u_1(x,y)dy = u(x)$.
Property (4.3.6) follows from [27, Theorem 3.6 (iv)] applied to the function $|\cdot|^p$, once we replace the original generating sequence $\{u_n\}_n$ by a p-equi-integrable one, which is always possible, in view of the Decomposition Lemma [87, Lemma 8.13]. The proof of (4.3.5) is based on the same argument as the one for proving [24, Lemma 3.1-(ii)], with the difference that we use an argument entirely similar to [60, Theorem 1.1] instead of [51, Theorem 14.5]. We write the proof for the readers' convenience.

Since $f \in \mathcal{E}_p$, then, as observed in [24, Section 3] there exists a constant c such that $|f(x,\xi)| \leq c(1+|\xi|^p)$ for every $(x,\xi) \in Q \times \mathbb{R}^d$. In order to apply Theorem 4.2.15, the function $f(x,\cdot)$ must also be p-coercive. Since this is not the case we introduce an auxiliary function. Fix $M > 0$ and consider $f_M(x,\xi) := \max\{-M, f(x,\xi)\}$, with $(x,\xi) \in Q \times \mathbb{R}^d$. Now we fix $\alpha > 0$ and we define $f_{M,\alpha}(x,\xi) := f_M(x,\xi) + \alpha|\xi|^p$ for every $(x,\xi) \in Q \times \mathbb{R}^d$. By construction

$$\alpha|\xi|^p - M \leq |f_{M,\alpha}(x,\xi)| \leq (c+\alpha)(1+|\xi|^p), \quad (x,\xi) \in \overline{Q} \times \mathbb{R}^d.$$

By Theorem 4.2.15, it follows that for every $A \in \mathcal{A}(\Omega)$,

$$\liminf_{n\to\infty} \int_A f_{M,\alpha}\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n\right) dx \geq \int_A (f_{M,\alpha})_{hom}(u) dx \geq \int_A f_{hom}(u) dx, \tag{4.3.7}$$

with f_{hom} defined as in (4.1.7), since $f_{M,\alpha} \geq f$. By construction we have that

$$\liminf_{n\to\infty} \int_A f_{M,\alpha}\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n\right) dx \leq \liminf_{n\to\infty} \int_A f_M\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n\right) dx + \alpha \sup_{n\in\mathbb{N}} \int_A |u_n|^p dx. \tag{4.3.8}$$

Combinig (4.3.7) and (4.3.8) and passing to the limit for $\alpha \to 0$ we get

$$\liminf_{n\to\infty} \int_A f_M \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) dx \geq \int_A f_{hom}(u) dx. \tag{4.3.9}$$

Now we consider the set

$$A_n^M := \left\{ x \in A : f\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) \leq -M \right\}$$

and we observe that by Chebyshev's inequality it holds

$$\mathcal{L}^N(A_n^M) \leq \frac{c}{M},$$

for some constant c independent from n and M. By definition of f_M we have that

$$\begin{aligned} \int_A f_M \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) dx &= -M\mathcal{L}^N(A_n^M) + \int_{A\setminus A_n^M} f\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) dx \\ &\leq \int_{A\setminus A_n^M} f\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) dx \end{aligned} \tag{4.3.10}$$

Since $\{u_n\}_n$ converges weakly in L^p then, in view of the Decomposition Lemma ([87, Lemma 8.13]), up to a subsequence, we can replace $\{u_n\}_n$ by a p-equi-integrable one, still denoted by $\{u_n\}_n$, and so, together with the p-growth condition of $f(x,\cdot)$, $\{f(\cdot,u_n)\}_n$ is also equi-itegrable and so

$$\int_{A_n^M} f\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) dx \to 0 \quad \text{as } M \to +\infty \tag{4.3.11}$$

uniformly with respect to n. Combining (4.3.9), (4.3.10), and (4.3.11) we get

$$\liminf_{n\to+\infty} \int_A f\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) dx \geq \int_A f_{hom}(u) dx.$$

Since $\{f(\cdot,u_n)\}_n$ is equi-integrable, by Theorem 1.3.3 (6) we get

$$\lim_{n\to\infty} \int_A f\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n \right) dx = \int_A \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_{(x,y)}(\xi) dx,$$

the proof follows by combining the last two inequalities and using a localization argument. □

We start addressing the proof of the sufficiency of Theorem 4.1.3, by introducing some preliminary notions and by proving some intermediate steps. For every $F \in \mathbb{R}^d$ let

$$M_F := \left\{ \nu \in L_w^\infty(Q, \mathcal{M}(\mathbb{R}^d)) : \{\nu_y\}_{y\in Q} \text{ is a homogeneous two-scale} \right. \tag{4.3.12}$$

$$\left. \text{Young measure and } \int_Q \int_{\mathbb{R}^d} \xi d\nu_y(\xi) dy = F \right\}.$$

Remark 4.3.3. Exactly as observed in [24, Remark 3.3], the set M_F is independent of Ω, i.e. if $\nu \in M_F$ and $\Omega' \subset \mathbb{R}^N$ is another domain, then for all vanishing sequences $\{\varepsilon_n\}_n$ there exists a sequence $\{v_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ such that $\{(\langle \cdot/\varepsilon_n \rangle, v_n)\}_n$ generates $\nu_y \otimes dy$. Indeed, let $r > 0$ such that $\Omega' \subset r\Omega$. In fact, given an arbitrary vanishing sequence $\{\varepsilon_n\}_n$, define $\delta_n := \varepsilon_n/r$. Then there exists a sequence $\{u_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ such that $\{(\langle \cdot/\delta_n \rangle, u_n)\}_n$ generates the homogeneous Young measure $\nu_y \otimes dy$. Define now $v_n(x) := u_n(x/r)$ so that $\{v_n\}_n \subset L^p(r\Omega, \mathbb{R}^d)$ and thus $\subset L^p(\Omega', \mathbb{R}^d)$. By changing variables it follows that the sequence $\{(\langle \cdot/\varepsilon_n \rangle, v_n)\}_n$ generates the homogeneous Young measure $\nu_y \otimes dy$ as well.

Lemma 4.3.4. *Let $F \in \mathbb{R}^d$, M_F as in* (4.3.12) *and $\mathcal{E}_p$ be as in* (4.1.5)*, then M_F is a convex and weak* closed subset of $(\mathcal{E}_p)'$.*

Proof. First we show that M_F is a subset of $(\mathcal{E}_p)'$. Fix $\nu \in \mathcal{M}_F$ and identify it with the homogeneous Young measure $\nu_y \otimes dy$. Let $\{v_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ be the sequence such that $\left\{\left(\langle \frac{\cdot}{\varepsilon_n} \rangle, v_n\right)\right\}_n$ generates ν.
Up to a subsequence, one can assume that $\{v_n\}_n$ is p-equi-integrable, see [87, Lemma 8.12]. Consequently, by [27, Theorem 3.6, iv)] it holds that

$$K := \iint_{Q \times \mathbb{R}^d} |\xi|^p d\nu_y(\xi) dy < +\infty.$$

By assumption, for a.e. $y \in Q$, the measure ν_y is a probability measure on Ω, so for every $f \in \mathcal{E}_p$ we have

$$\int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_y(\xi) dy \leq ||f||_{\mathcal{E}_p} \int_Q \int_{\mathbb{R}^d} (1 + |\xi|^p) d\nu_y(\xi) dy = (1+K)||f||_{\mathcal{E}_p},$$

hence $M_F \subset (\mathcal{E}_p)'$. Since $M_F \subset \overline{M_F}$, then in order to show that M_F is weakly* closed it is enough to show that its weak* closure $\overline{M_F} \subset M_F$. Since $\mathcal{E}_p$ is separable then $(\mathcal{E}_p)'$ is metrizable, hence for every $\nu \in \overline{M_F}$ there exists a sequence $\{\nu^n\}_n \subset M_F$ such that $\nu^n \overset{*}{\rightharpoonup} \nu$ in $(\mathcal{E}_p)'$. For every $n \in \mathbb{N}$, let $\{v_k^n\}_k \subset L^p(\Omega, \mathbb{R}^d)$ be the sequence generating ν^n. Since the map $(x, \xi) \mapsto \xi$ is an element of $\mathcal{E}_p$, then by the definition of weak* convergence it follows that

$$\lim_{n \to +\infty} \int_Q \int_\Omega \xi d\nu_y^n(\xi) dy = \int_Q \int_\Omega \xi d\nu_y(\xi) dy = F,$$

and thus ν has F as underlying deformation. We have now to show that ν is an homogeneous two-scale Young measure. For every $z \in L^1(Q)$ and $\phi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d)$ we have

$$\begin{aligned}
&\lim_{k \to +\infty} \lim_{n \to +\infty} \int_Q z(x) \int_\Omega \phi\left(\left\langle \frac{x}{\varepsilon_k} \right\rangle, v_k^n(x)\right) dx \\
&= \lim_{n \to +\infty} \int_Q \int_Q \int_{\mathbb{R}^d} z(x) \phi(y,\xi) d\nu_y^n(\xi) dy dx = \int_Q z(x) dx \int_Q \int_{\mathbb{R}^d} \phi(y,\xi) d\nu_y(\xi) dy
\end{aligned}$$

where in the second equality we used the fact that $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d) \subset \mathcal{E}_p$. By a diagonalization argument we can find a sequence $\{k_n\}_n$ such that once defined $u_n := v^n_{k_n}$, then it holds

$$\lim_{n\to+\infty} \int_Q z(x) \int_\Omega \phi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, u_n(x)\right) dx = \int_Q z(x)dx \int_Q \int_\Omega \phi(y,\xi)d\nu_y(\xi)dy.$$

It follows that ν is a homogeneous two-scale Young measure, which implies that M_F is weakly* closed. Now we have to prove that M_F is a convex set. Fix $\nu, \mu \in M_F$ and $t \in (0,1)$. Consider $D := (0,t) \times (0,1)^{N-1}$ and define

$$\sigma_{(x,y)} := \begin{cases} \nu_y & \text{if } (x,y) \in D \times Q, \\ \mu_y & \text{if } (x,y) \in (Q \setminus D) \times Q. \end{cases}$$

By Lemma 4.2.5, $\{\sigma_{(x,y)}\}_{(x,y)\in Q\times Q}$ is a two-scale Young measure, while by Lemma 4.2.8 we have that $\{\overline{\sigma}_y\}_{y\in Q}$ is a homogeneous two-scale Young measure, hence an element of M_F.

Finally, observe that for every $\phi \in L^1(Q, \mathcal{C}_0(\mathbb{R}^d))$ it holds

$$\begin{aligned} &\int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\overline{\sigma}_y(\xi)dy = \int_Q \int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\sigma_{(x,y)}(\xi)dydx \\ &= \int_D \int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\nu_y(\xi)dydx + \int_{Q\setminus D} \int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\mu_y(\xi)dydx \\ &= t\int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\nu_y(\xi)dy + (1-t)\int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\mu_y(\xi)dy. \end{aligned} \tag{4.3.13}$$

which means that $\overline{\sigma} = t\nu + (1-t)\mu$. In order to conclude that $\overline{\sigma} \in M_F$, we have to show that $\int_Q \int_{\mathbb{R}^d} \xi d\overline{\sigma}_y(\xi)dy = F$. In fact, this is the case: it suffices to apply (4.3.13) to $\phi(y,\xi) = \xi$, taking into account that $\int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\mu_y(\xi)dy = \int_Q \int_{\mathbb{R}^d} \phi(y,\xi)d\nu_y(\xi)dy = F$. □

Now we show that conditions i), ii) and iii) of Theorem 4.1.3 are sufficient to characterize two-scale Young measures. As in [24] we start from the homogeneous case. The non-homogeneous one will be obtained through a suitable approximation of two-scale Young measures by piecewise constant ones.

Lemma 4.3.5. *Let $F \in \mathbb{R}$ and $\nu \in L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d))$ be such that $\nu_y \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $y \in Q$. Assume that*

$$F = \int_Q \int_{\mathbb{R}^d} \xi d\nu_y(\xi)dy, \tag{4.3.14}$$

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_y(\xi)dy < +\infty, \tag{4.3.15}$$

and that

$$f_{hom}(F) \leq \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_y(\xi) dy, \tag{4.3.16}$$

for every $f \in \mathcal{E}_p$ where $f_{hom}(F)$ is defined in (4.1.7) *and $\mathcal{E}_p$ is the space introduced in* (4.1.5)*.*
Then $\{\nu_y\}_{y\in Q}$ is a homogeneous two-scale Young measure.

Proof. Let $F \in \mathbb{R}^d$ and $\nu \in L^\infty_w(Q, \mathcal{M}(\mathbb{R}^d)$ be such that $\nu_y \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $y \in Q$, and that (4.3.14), (4.3.15), and (4.3.16) hold. We will proceed by contradiction using the Hahn-Banach Separation Theorem. Assume that ν is not an element of M_F. By Lemma 4.3.4, M_F is a convex and weak* closed subset of $(\mathcal{E}_p)'$. Moreover, by (4.3.15) and the fact that $\{\nu_y\}_{y\in Q}$ is a family of probability measures, we get that $\nu \in (\mathcal{E}_p)'$ as well. Since $\nu \notin M_F$, according to Hahn-Banach Separation Theorem, we can separate ν from M_F i.e. there exists a linear weak* continuous map $L : (\mathcal{E}_p)' \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $\langle L, \nu\rangle_{(\mathcal{E}_p)'',(\mathcal{E}_p)'} < \alpha$ and $\langle L, \mu\rangle_{(\mathcal{E}_p)'',(\mathcal{E}_p)'} \geq \alpha$ for all $\mu \in M_F$. Let $f \in \mathcal{E}_p$ be such that

$$\alpha \leq \langle L, \mu\rangle_{(\mathcal{E})''_p,(\mathcal{E}_p)'} = \langle \mu, f\rangle_{(\mathcal{E}_p)',\mathcal{E}_p} = \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\mu_y(\xi) dy \tag{4.3.17}$$

for all $\mu \in M_F$ and

$$\alpha > \langle L, \nu\rangle_{(\mathcal{E})''_p,(\mathcal{E}_p)'} = \langle \nu, f\rangle_{(\mathcal{E})'_p,\mathcal{E}_p} = \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_y(\xi) dy \geq f_{hom}(F). \tag{4.3.18}$$

Let f_H be defined as

$$f_H(F) := \inf_{\mu\in M_F} \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\mu_y(\xi) dy, \quad F \in \mathbb{R}^d.$$

Then by (4.3.17), we have that $\alpha \leq f_H(F)$. We are going to show that

$$f_H(F) \leq f_{hom}(F), \tag{4.3.19}$$

which is in contradiction with (4.3.18) and proves the lemma.
To prove (4.3.19), let $T \in \mathbb{N}$ and $\phi \in L^p((0,T)^N; \mathbb{R}^d)$ such that $\int_{(0,T)^N} \phi(x)dx = 0$. Extend ϕ to $\mathbb{R}^N$ by $(0,T)^N$-periodicity and consider the sequence

$$\phi_n(x) := F + \phi\left(\frac{x}{\varepsilon_n}\right),$$

where $\{\varepsilon_n\}_n$ is an arbitrary vanishing sequence. Let $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d)$ and $z \in L^1(Q)$. Then, since $T \in \mathbb{N}$, the function $F \mapsto \varphi(\langle y\rangle, F + \phi(y))$ is $(0,T)^N$-periodic and according to the Riemann-Lebesgue Lemma, we get that

$$\begin{aligned}\lim_{n\to+\infty}\int_Q z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\phi_n(x)\right)dx &= \lim_{n\to+\infty}\int_Q z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,F+\phi\left(\frac{x}{\varepsilon_n}\right)\right)dx\\ &= \int_Q z(x)dx \fint_{(0,T)^N}\varphi(\langle y\rangle,F+\phi(y))dy.\end{aligned} \tag{4.3.20}$$

Observe that

$$\fint_{(0,T)^N}\varphi(\langle y\rangle,F+\phi(y))dy = \frac{1}{T^N}\sum_{a_i\in\mathbb{Z}^N\cap[0,T)^N}\int_{a_i+Q}\varphi(\langle y\rangle,F+\phi(y))dy \tag{4.3.21}$$
$$= \frac{1}{T^N}\sum_{a_i\in\mathbb{Z}^N\cap[0,T)^N}\int_Q\varphi(\langle a_i+y\rangle,F+\phi(a_i+y))dy$$
$$= \frac{1}{T^N}\sum_{a_i\in\mathbb{Z}^N\cap[0,T)^N}\int_Q\varphi(y,F+\phi(a_i+y))dy.$$

Thus, from (4.3.20) and (4.3.21), the pair $\{(\langle\cdot/\varepsilon_n\rangle;\phi_n)\}_n$ generates the homogeneous Young measure

$$\mu := \sum_{a_i\in\mathbb{Z}^N\cap[0,T)^N}\frac{1}{T^N}\delta_{F+\phi(a_i+y)}\otimes dy.$$

Then,

$$\int_Q\int_{\mathbb{R}^d}\xi d\mu_y(\xi)dy = F,$$

which implies that $\mu\in M_F$. In addition,

$$\fint_{(0,T)^N}f(\langle y\rangle,F+\phi(y))dy = \int_Q\int_{\mathbb{R}^d}f(y,\xi)d\mu_y(\xi)dy,$$

and then

$$\fint_{(0,T)^N}f(\langle y\rangle,F+\phi(y))dy \geq f_H(F).$$

As a consequence, taking the infimum over all $\phi\in L^p((0,T)^N,\mathbb{R}^d)$ such that $\int_{(0,T)^N}\phi(x)dx=0$ and the limit as $T\to+\infty$ we get that $f_H(F)\leq f_{hom}(F)$ which implies (4.3.19). □

The following result is an unconstrained version of [24, Proposition 3.6].

Lemma 4.3.6. *Let $\nu\in L^\infty_w(\Omega\times Q;\mathcal{M}(\mathbb{R}^d))$ be such that the family $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure. Then for a.e. $a\in\Omega$, $\{\nu_{(a,y)}\}_{y\in Q}$ is a homogeneous two-scale Young measure.*

Proof. Since $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure, it satisfies (4.3.3),(4.3.5) and (4.3.6) Since $u_1(x,\cdot)$ is Q-periodic for a.e. $x \in \Omega$, integrating (4.3.3) we get (4.3.4). Furthermore, (4.3.6) implies that

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)} dy < +\infty \text{ for a.e. } x \in \Omega. \tag{4.3.22}$$

Let $E \subset \Omega$ be a set of Lebesgue measure zero such that (4.3.4), (4.3.5) and (4.3.22) do not hold. Then for every $a \in \Omega \setminus E$,

$$\int_Q \int_{\mathbb{R}^d} \xi d\nu_{(a,y)}(\xi) dy = u(a),$$

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_{(a,y)}(\xi) dy < +\infty,$$

and

$$\int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_{(a,y)}(\xi) dy \geq f_{hom}(u(a))$$

for every $f \in \mathcal{E}_p$. As a consequence of Lemma 4.3.5, for every $a \in \Omega \setminus E$, the family $\{\nu_{(a,y)}\}_{y\in Q}$ is a homogeneous two-scale Young measure. □

Theorem 4.3.7. *Let Ω be a bounded and open subset of $\mathbb{R}^d$. Let $\nu \in L^\infty_w(\Omega \times Q, \mathcal{M}(\mathbb{R}^d))$ be such that $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^d)$ for a. e. $(x,y) \in \Omega \times Q$. Assume that i)-iii) of Theorem 4.1.3 hold, i.e.*

$$\int_{\mathbb{R}^d} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q), \tag{4.3.23}$$

there exist $u_1 \in L^p(\Omega; (L^p_{per}(Q;\mathbb{R}^d)))$ and $u \in L^p(\Omega;\mathbb{R}^d)$ such that

$$u_1(x,y) = \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi), \tag{4.3.24}$$

with $\int_Q u_1(x,y)dy = u(x)$. Assume also that

$$f_{hom}(u(x)) \leq \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_{(x,y)}(\xi) dy, \text{ for a.e. } x \in \Omega, \text{ for any } f \in \mathcal{E}_p, \tag{4.3.25}$$

where $f_{hom}(F)$ is the function in (4.1.7) Then $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure with underlying deformation u.

Proof. Let $u = 0$ a.e. on Ω and Let (φ, z) be in a countable dense subset of $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d) \times L^1(\Omega)$. Define

$$\bar{\varphi}(x) := \int_Q \int_{\mathbb{R}^d} \varphi(y,\xi) d\nu_{(x,y)}(\xi) dy.$$

Since the average with respect to y of $u_1(x,y)$ is $u(x)$ for a.e. $x \in \Omega$, integrating (4.3.24) with respect to $y \in Q$, it follows that

$$\int_Q \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi) dy = 0, \quad \text{for a.e. } x \in \Omega. \tag{4.3.26}$$

Let $E \subset \Omega$ be the null Lebesgue measure set such that (4.3.26), (4.3.25) or (4.3.23) do not hold. Then for every $a \in \Omega \setminus E$

$$\int_Q \int_{\mathbb{R}^d} \xi d\nu_{(a,y)}(\xi) dy = 0,$$

$$f_{hom}(0) \leq \int_Q \int_{\mathbb{R}^d} f(y,\xi) d\nu_{(a,y)}(\xi) dy$$

for every $f \in \mathcal{E}_p$, and

$$\int_Q \int_{\mathbb{R}^d} |\xi|^p d\nu_{(a,y)}(\xi) dy < +\infty.$$

Now, let $k \in \mathbb{N}$. According to [127, Lemma 7.9, pg. 129], there exist points $a_i^k \in \Omega \setminus E$ and positive numbers $\rho_i^k \leq 1/k$ such that $a_k^i + \rho_i^k \Omega$ are pairwise disjoint for each k,

$$\bar{\Omega} = \bigcup_{i \geq 1} (a_i^k + \rho_i^k \bar{\Omega}) \cup E_k, \ \text{ with } \mathcal{L}(E_k) = 0$$

and

$$\int_\Omega z(x) \bar{\varphi}(x) dx = \lim_{k \to +\infty} \sum_{i \geq 1} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx. \tag{4.3.27}$$

For each $k \in \mathbb{N}$, let $m_k \in \mathbb{N}$ be large enough so that

$$\left| \sum_{i=1}^{m_k} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx - \sum_{i \geq 1} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx \right| < \frac{1}{k}. \tag{4.3.28}$$

For fixed i and k, by choice of a_i^k and by Lemma 4.3.6, the family $\{\nu_{(a_i^k, y)}\}_{y \in Q}$ is a homogeneous two-scale Young measure. Hence, by Remark 4.3.3, for every vanishing sequence $\{\varepsilon_n\}_n$, there exist sequences $\{u_n^{i,k}\}_n \subset L^p(a_i^k + \rho_i^k \Omega, \mathbb{R}^d)$ such that

$$\lim_{n \to +\infty} \int_{a_i^k + \rho_i^k \Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n^{i,k}(x) \right) dx = \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx.$$

Summing up

$$\lim_{n\to+\infty}\sum_{i=1}^{m_k}\int_{a_i^k+\rho_i^k\Omega} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^{i,k}\right)dx = \sum_{i=1}^{m_k}\bar{\varphi}(a_i^k)\int_{a_i^k+\rho_i^k\Omega} z(x)dx.$$

Let us define

$$u_n^k(x) := \begin{cases} u_n^{i,k}(x) & \text{if } x \in a_i^k+\rho_i^k\Omega, \\ 0 & \text{otherwise} \end{cases}$$

and remark that $u_n^k \in L^p(\Omega,\mathbb{R}^d)$. Since the sets $a_i^k+\rho_i^k\Omega$ are pairwise disjoint for each k we have that

$$\int_\Omega z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^k(x)\right)dx = \sum_{i\geq 1}\int_{a_i^k+\rho_i^k\Omega} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^{i,k}(x)\right)dx$$

$$= \sum_{i=1}^{m_k}\int_{a_i^k+\rho_i^k\Omega} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^{i,k}(x)\right)dx + \int_{\Omega\cap\bigcup_{i>m_k}(a_i^k+\rho_i^k\Omega)} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}, u_n^k(x)\right\rangle\right)dx.$$

But as $z \in L^1(\Omega)$ and $\mathcal{L}^N(\Omega\cap\bigcup_{i>m_k}(a_i^k+\rho_i^k\Omega)) \to 0$, as $k\to+\infty$, it follows that

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\left|\int_{\Omega\cap\bigcup_{i>m_k}(a_i^k+\rho_i^k\Omega)} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^k(x)\right)dx\right| = 0. \tag{4.3.29}$$

Then, gathering ((4.3.27) - (4.3.29)) we obtain that

$$\begin{aligned}
&\lim_{k\to+\infty}\lim_{n\to+\infty}\int_\Omega z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^k(x)\right)dx \\
&= \lim_{k\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{m_k}\int_{a_i^k+\rho_i^k\Omega} z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n^k(x)\right)dx \\
&= \lim_{k\to+\infty}\sum_{i=1}^{m_k}\bar{\varphi}(a_i^k)\int_{a_i^k+\rho_i^k\Omega} z(x)dx \\
&= \lim_{k\to+\infty}\sum_{i\geq 1}\bar{\varphi}(a_i^k)\int_{a_i^k+\rho_i^k\Omega} z(x)dx \\
&= \int_\Omega z(x)\bar{\varphi}(x)dx.
\end{aligned}$$

A diagonalization argument implies the existence of a diverging sequence $\{k_n\}$, as $n\to\infty$, such that upon setting $u_n := u_n^{k_n}$, then

$$\lim_{n\to+\infty}\int_\Omega z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle, u_n(x)\right)dx = \int_\Omega z(x)\bar{\varphi}(x)dx$$

and $u_n \rightharpoonup 0$ in $L^p(\Omega, \mathbb{R}^d)$, which completes the proof whenever $u = 0$.
Consider now a general $u \in L^p(\Omega, \mathbb{R}^d)$ and ν satisfying properties (4.3.23)-(4.3.25). We define $\tilde{\nu} \in L^\infty_w(\Omega \times Q, \mathcal{M}(\mathbb{R}^d))$ by

$$\langle \tilde{\nu}, \psi \rangle := \int_\Omega \int_Q \int_{\mathbb{R}^d} \psi(x, y, \xi - u(x)) d\nu_{(x,y)}(\xi) dy dx, \tag{4.3.30}$$

for every $\psi \in L^1(\Omega \times Q, \mathcal{C}_0(\mathbb{R}^d))$. We can easily check that $\tilde{\nu}$ satisfies the analog properties of (4.3.23)-(4.3.25) with $u = 0$. Hence, applying the first step of the proof, for vanishing every sequence $\{\varepsilon_n\}_n$ we may find a sequence $\{\tilde{u}_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ such that $\{(\langle \cdot/\varepsilon_n \rangle, \tilde{u}_n)\}_n$ generates the Young measure $\{\tilde{\nu}_{(x,y)} \otimes dy\}_{x \in \Omega}$. Define $u_n := \tilde{u}_n + u$. It is easily seen that $\{(\langle \cdot/\varepsilon_n \rangle, u_n)\}_n$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$. Indeed, let $\psi \in L^1(\Omega, \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d))$ and define $\tilde{\psi}(x, y, \xi) := \psi(x, y, \xi + u(x))$ where $\tilde{\psi} \in L^1(\Omega, \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^d))$ as well. Then, by (4.3.30)

$$\begin{aligned} &\lim_{n \to +\infty} \int_\Omega \psi \left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n(x) \right) dx = \lim_{n \to +\infty} \int_\Omega \tilde{\psi} \left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \tilde{u}_n(x) \right) dx \\ &= \int_\Omega \int_Q \int_{\mathbb{R}^d} \tilde{\psi}(x, y, \xi) d\tilde{\nu}_{(x,y)}(\xi) dy dx = \int_\Omega \int_Q \int_{\mathbb{R}^d} \psi(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx \end{aligned}$$

which completes the proof. $\square$

4.4 Homogenization

This section is devoted to the proof of Theorem 4.1.1. To this end we start by showing that two-scale Young measures behave as measure products when generated by couples of sequences. This fact is a fundamental tool for achieving our result, and it represents an extension of the pioneering [128, Proposition 2.3].

Proposition 4.4.1. *Let $\Lambda = \{\Lambda_{(x,x,y,y')}\}_{(x,x',y,y') \in \Omega^2 \times Q^2} \subset L^\infty_w(\Omega^2 \times Q^2; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ be a family of probability measures supported on $\mathbb{R}^d \times \mathbb{R}^d$. Λ is the two-scale Young measure associated with a sequence $\{(u_n(x), u_n(x'))\}_n$, with $\{u_n\} \subset L^p(\Omega, \mathbb{R}^d)$ if and only if $\Lambda_{(x,x',y,y')} = \nu_{(x,y)} \otimes \nu_{(x',y')}$, where $\nu := \{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q} \subset L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^d))$ is the two-scale Young measure associated (in the sense of Definition 4.2.1) with the sequence $\{u_n\}_n$.*

Proof. Let $\{\varepsilon_n\}_n$ be any vanishing sequence and let Λ be the two-scale Young measure associated to the sequence $\{(u_n(x), u_n(x'))\}_n$ for some bounded sequence $\{u_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$. We test Λ against functions $\theta(x, x')\varphi(y, \xi, y', \xi')$ with $\theta \in L^1(\Omega \times \Omega)$ and $\varphi \in C_0(\mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^d)$, (in view of the density of linear combinations of such functions in $L^1(\Omega \times \Omega; C_0(\mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^d))$. In particular, we choose φ such that $\varphi(y, \xi, y', \xi') = \varphi_1(y, \xi)\varphi_2(y', \xi')$ and $\theta(x, x') = \theta_1(x)\theta_2(x')$.

$$
\begin{aligned}
&\iint_{\Omega\times\Omega} \theta_1(x)\theta_2(x') \left(\iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_1(y',\xi')\varphi_2(y',\xi') d\Lambda_{(x,x',y,y')}(\xi,\xi') dy dy' \right) dx dx' \\
&= \lim_{n\to+\infty} \iint_{\Omega\times\Omega} \theta(x,x')\varphi\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \left\langle \frac{x'}{\varepsilon_n} \right\rangle, u_n(x), u_n(x') \right) dx dx' \\
&= \lim_{n\to+\infty} \iint_{\Omega\times\Omega} \theta_1(x)\theta_2(x')\varphi_1\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n(x) \right) \varphi_2\left(\left\langle \frac{x'}{\varepsilon_n} \right\rangle, u_n(x') \right) dx dx' \\
&= \lim_{n\to+\infty} \left(\int_\Omega \theta_1(x)\varphi_1\left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, u_n(x) \right) dx \right) \left(\int_\Omega \theta_2(x')\varphi_2\left(\left\langle \frac{x'}{\varepsilon_n} \right\rangle, u_n(x') \right) dx' \right) \\
&= \int_\Omega \theta_1(x) \int_Q \int_{\mathbb{R}^d} \varphi(y,\xi) d\nu_{(x,y)}(\xi) dy dx \int_\Omega \theta_2(x') \int_Q \int_{\mathbb{R}^d} \varphi(y',\xi') d\nu_{(x',y')}(\xi') dy' dx' \\
&= \iint_{\Omega\times\Omega} \theta_1(x)\theta_2(x') \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_1(y,\xi)\varphi_2(y',\xi') d(\nu_{(x,y)}\otimes\nu_{(x',y')})(\xi,\xi') dy dy' dx dx'
\end{aligned}
$$

The fourth equality is due to the boundedness of the sequence $\{u_n\}$, and the fact that $\left\{ \left(\langle \frac{x}{\varepsilon_n} \rangle, u_n(x) \right) \right\}_n$ generates the two-scale Young measure ν. The thesis follows from the chain of identities. The reverse implication can be proved in a similar way. $\square$

Now, we prove Theorem 4.1.1. This result is a generalization of [24, Theorem 1.2]. We observe also that we will refer to the Γ-convergence as it has been introduced in Definition 4.2.14, since by the growth assumption (4.1.3), we can work in bounded subsets of $L^p(\Omega;\mathbb{R}^m)$, which are metrizable.

Proof of Theorem 4.1.1. We recall the family of functionals $I_\varepsilon : L^p(\Omega,\mathbb{R}^d) \to [0,+\infty)$ defined in (4.1.1) as

$$
I_\varepsilon(u) := \int_\Omega \int_\Omega W\left(x, x', \left\langle \frac{x}{\varepsilon} \right\rangle, \left\langle \frac{x'}{\varepsilon} \right\rangle, u(x), u(x') \right) dx dx'
$$

where $\Omega \subset \mathbb{R}^N$, $u \in L^p(\Omega;\mathbb{R}^d)$ and $W : \Omega\times\Omega\times Q\times Q\times\mathbb{R}^d\times\mathbb{R}^d \to [0,+\infty)$ is a symmetric admissible integrand, satisfying (4.1.3).
Let $u \in L^p(\Omega,\mathbb{R}^d)$ and let $\{\varepsilon_n\}_n$ be such that $\varepsilon_n \to 0$. We start by showing that

$$
\begin{aligned}
&\Gamma - \limsup_{n\to+\infty} I_{\varepsilon_n}(u) \\
&\leq \inf_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x,x',y,y',\xi,\xi') d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx'
\end{aligned}
\tag{4.4.1}
$$

where $\mathcal{M}_u$ is defined in (4.1.4). Let $\nu \in \mathcal{M}_u$, by Remark 4.2.4, Theorem 4.1.3 and Proposition 4.4.1, there exists a sequence $\{u_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$, generates the two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$, and clearly $\{(u_n(x), u_n(x'))\}_n$ is the sequence associated to the two-scale Young measure $\{\nu_{(x,y)} \otimes \nu_{(x',y')}\}_{(x,x',y,y')\in\Omega^2\times Q^2}$. Extract a subsequence $\{\varepsilon_{n_k}\}_k \subset \{\varepsilon_n\}_n$ such that

$$\limsup_{n\to\infty} I_{\varepsilon_n}(u_n) = \lim_{k\to\infty} I_{\varepsilon_{n_k}}(u_{n_k})$$

and that $\{|u_{n_k}|^p\}_k$ is equi-integrable, which is always possible by the Decomposition Lemma [87, Lemma 8.13]. In particular, due to the p-growth condition (4.1.3), it follows that the sequence $\{W(\cdot, \cdot', \langle \cdot/\varepsilon_{n_k}\rangle, \langle \cdot'/\varepsilon_{n_k}\rangle, u_{n_k}(\cdot), u_{n_k}(\cdot'))\}_k$ is equi-integrable in $\Omega \times \Omega$ as well, hence applying Theorem 4.2.11-*ii*) we get that

$$\begin{aligned} &\Gamma - \limsup_{n\to+\infty} I_{\varepsilon_n}(u) \leq \lim_{k\to+\infty} \iint_{\Omega\times\Omega} W\left(x, x', \left\langle \frac{x}{\varepsilon_{n_k}} \right\rangle, \left\langle \frac{x'}{\varepsilon_{n_k}} \right\rangle, u_{n_k}(x), u_{n_k}(x')\right) dx dx' \\ &= \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi')\, d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx'. \end{aligned} \tag{4.4.2}$$

Taking the infimum over all $\nu \in \mathcal{M}_u$ in the right hand side of (4.4.2), yields (4.4.1). Let us prove now that

$$\begin{aligned} &\Gamma - \liminf_{n\to+\infty} I_{\varepsilon_n}(u) \\ &\geq \inf_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx'. \end{aligned} \tag{4.4.3}$$

Let $\eta > 0$ and $\{u_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} I_{\varepsilon_n}(u) \leq \Gamma - \liminf_{n\to+\infty} I_{\varepsilon_n}(u) + \eta. \tag{4.4.4}$$

For a subsequence $\{n_k\}_k$, still thanks to Proposition 4.4.1, we can assume that there exists $\nu \in L^\infty_w(\Omega \times Q, \mathcal{M}(\mathbb{R}^d))$ such that $\{(u_n(x), u_n(x'))\}$ is the sequence associated to the two-scale Young measure $\{\nu_{(x,y)} \otimes \nu_{(x',y')}\}_{(x,y,x',y')\in(\Omega\times Q)^2}$ and

$$\lim_{k\to+\infty} I_{\varepsilon_{n_k}}(u_{n_k}) = \liminf_{n\to+\infty} I_{\varepsilon_n}(u_n). \tag{4.4.5}$$

We remark that $\{u_{n_k}\}_k$ is equi-integrable since it is bounded in $L^p(\Omega, \mathbb{R}^d)$ and $p > 1$. Thus by the Fundamental Theorem on Young Measures (Theorem 1.3.3) we get that for every $A \in \mathcal{A}(\Omega)$,

$$\int_A u(x) dx = \lim_{k\to+\infty} \int_A u_{n_k}(x) dx = \int_A \int_Q \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi) dy dx.$$

By the arbitrariness of the set A, it follows that

$$u(x) = \int_Q \int_{\mathbb{R}^d} \xi d\nu_{(x,y)}(\xi) dy \text{ a.e. in } \Omega. \tag{4.4.6}$$

As a consequence of Corollary 4.3.1, $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale Young measure and, by (4.4.6), we also have that $\nu \in \mathcal{M}_u$. Applying now Theorem 4.2.11-i) we get that

$$\begin{aligned}
&\lim_{k\to+\infty} \iint_{\Omega\times\Omega} W\left(x, x', \left\langle \frac{x}{\varepsilon_{n_k}} \right\rangle, \left\langle \frac{x'}{\varepsilon_{n_k}} \right\rangle, u_{n_k}(x), u_{n_k}(x')\right) dx dx' \\
&\geq \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') \, d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx' \\
&\geq \inf_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') \, d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx'.
\end{aligned}$$

Hence, by (4.4.4), (4.4.5) and the arbitrariness of η we get the desired result. Gathering (4.4.1) and (4.4.3) , we obtain that

$$\begin{aligned}
&\Gamma - \lim_{n\to+\infty} I_{\varepsilon_n}(u) \\
&= \inf_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') \, d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx'.
\end{aligned}$$

To conclude the proof of the upper bound, we have to prove that the minimum is attained. Consider a recovering sequence $\{\bar{u}_n\}_n \subset L^p(\Omega, \mathbb{R}^d)$ for $\Gamma - \lim_n I_{\varepsilon_n}(u)$. Arguing exactly as before we can assume that (a subsequence of) $\{\bar{u}_n\}_n$ generates a two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$, that $\nu \in \mathcal{M}_u$ and the couple $\{(u_n(x), u_n(x')\}_n$ generates $\{\nu_{(x,y)} \otimes \nu_{(x',y')}\}_{x,y,x',y')\in(\Omega\times Q)^2}$, and

$$\{W(\cdot, \cdot', \langle \cdot/\varepsilon_{n_k} \rangle, \langle \cdot'/\varepsilon_{n_k} \rangle, u_{n_k}(\cdot), u_{n_k}(\cdot'))\}_n$$

is equi-integrable. According to Theorem 4.2.11-ii), we get

$$\begin{aligned}
\Gamma - \lim_{n\to+\infty} I_{\varepsilon_n}(u) &= \lim_{n\to+\infty} \iint_{\Omega\times\Omega} W\left(x, x', \left\langle \frac{x}{\varepsilon_n} \right\rangle, \left\langle \frac{x'}{\varepsilon_n} \right\rangle, \bar{u}_n(x), \bar{u}_n(x')\right) dx dx' \\
&= \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi') \, d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx'.
\end{aligned}$$

which completes the proof. $\square$

Remark 4.4.2. i) We observe that if in the above theorem W does not depend on $y, y' \in Q$, the above Γ-convergence result reduces to a relaxation result with respect to the L^p-weak convergence, with I_{hom} therein replaced by

$$\min_{\nu\in\mathcal{M}_u} \iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', \xi, \xi') d\nu_{(x,y)}(\xi) d\nu_{(x',y')}(\xi') dy dy' dx dx',$$

with $\mathcal{M}_u$ in (4.1.4). I_{hom}, in turn, becomes

$$I_{hom}(u) = \min_{\mu \in \mathcal{M}'_u} \iint_{\Omega \times \Omega} \iint_{\mathbb{R}^d \times \mathbb{R}^d} W(x, x', \xi, \xi') d\mu_x(\xi) d\mu_{x'}(\xi') dx dx', \quad (4.4.7)$$

where

$$\mathcal{M}'_u := \Big\{ \mu \in L^\infty_w(\Omega, \mathcal{M}(\mathbb{R}^d)) : \{\mu_x\}_{x \in \Omega} \text{ is a Young measure such that } \int_{\mathbb{R}^d} \xi d\nu_x(\xi) = u(x) \Big\}.$$

The above equality is easily obtained in view of Definition 4.2.1, which ensures that any measure $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q} \in \mathcal{M}_u$ is such that $\nu_{(x,y)} \otimes dy = \mu_x \in \mathcal{M}'_u$.

ii) We observe that the same strategy adopted in the proof of Theorem 4.1.1, would lead directly to (4.4.7), without using two-scale Young measures but adopting the classical Young measures generated by sequence in $L^p(\Omega; \mathbb{R}^d)$. It suffices to replace Theorem 4.2.11 by Theorem 1.3.3 and the characterization of Young measures proved in [127, Theorem 7.7].

iii) Observe that equation 4.4.7 provides a statement analogous of [36, Theorem 6.1] replacing the narrow convergence by the L^p-weak convergence, under a slightly more general set of assumptions on W, cf. Remarks 4.1.2, 4.2.10 and 4.2.12.

We point out that analogous arguments as those in the proof of Theorem 4.1.1 allow us to obtain a Γ-convergence result in terms of a suitable narrow convergence (see Remark 4.2.13). In order to do so, a preliminary step is needed, i.e. a suitable extension of the functionals I_ε in (4.1.1) to $\mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$, where we recall that $\mathbb{T}^N$ represents the N-dimensional torus in $\mathbb{R}^N$.

Corollary 4.4.3. *Let p, Ω, W and $\{I_\varepsilon\}_\varepsilon$ be defined as in Theorem 4.1.1. Let $\mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$ be the space introduced in Remark 4.2.13 and let $\bar{I}_\varepsilon : \mathcal{Y}^p(\Omega, \mathbb{T}^N \times \mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}$*

$$\bar{I}_\varepsilon(\mu) := \begin{cases} I_\varepsilon(u) & \text{if } \mu = \{\delta_{(\langle \frac{x}{\varepsilon} \rangle, u(x))}\}_{x \in \Omega}, \\ +\infty & \text{otherwise} \end{cases} \quad (4.4.8)$$

Then, $\{\bar{I}_\varepsilon\}_\varepsilon$ Γ-converges, with respect to the narrow convergence, i.e. testing *with functions in* $L^1(\Omega; C_0(\mathbb{T}^N \times \mathbb{R}^d))$, *to* $\overline{I}_{hom} : \mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d) \to [0, +\infty]$, *where*

$$\overline{I}_{hom}(\mu) = \begin{cases} \displaystyle\iint_{\Omega\times\Omega} \iint_{(Q\times\mathbb{R}^d)^2} W(x, x', (y, \xi), (y', \xi'))d\mu_x(y, \xi)d\mu_{x'}(y', \xi')dxdx', \\ \qquad\qquad\qquad\qquad\qquad \textit{if } \{\mu_x\}_{x\in\Omega} = \{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q} \otimes dy, \\ +\infty \qquad\qquad\qquad\qquad \textit{otherwise,} \end{cases}$$

with ν *a two-scale Young measure.*

Before proving the result we observe that, when $\overline{I}_{\text{hom}}$ is finite, the right hand side coincides with

$$\iint_{\Omega\times\Omega} \iint_{Q\times Q} \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x, x', y, y', \xi, \xi')d\nu_{(x,y)}(\xi)d\nu_{(x',y')}(\xi')dydy'dxdx'.$$

Proof. The proof follows the same argument as [36, Theorem 6.1]. Let $\mu \in \mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$ and $\{\mu_n\}_n \subset \mathcal{Y}^p(\Omega; \mathbb{T}^N \times \mathbb{R}^d)$ be a sequence which converges narrowly to μ. With no loss of generality we can assume that $\liminf_{n\to\infty} \overline{I}_{\varepsilon_n}(\mu_{\varepsilon_n}) < +\infty$. Then, up to passing to a suitable subsequence (not relabelled) for which the liminf is a limit, $\mu_n = \delta_{(\langle \frac{x}{\varepsilon_n}\rangle, u_n(x))}$ for a suitable sequence $\{u_n\}_n \subset L^p(\Omega; \mathbb{R}^d)$.
By (4.1.3), $\{u_n\}_n$ is bounded in $L^p(\Omega; \mathbb{R}^d)$, hence (up to a subsequence) weakly convergent, then as observed in [27] $\left\{\left(\langle \frac{x}{\varepsilon_n}\rangle, u_n\right)\right\}_n$ is tight and in view of Theorem 4.1.3 the narrow limit μ is of the type $\nu_{(x,y)} \otimes dy$ for a suitable two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$. Hence the lower semicontinuity of $\iint_{\Omega\times\Omega} \iint_{(Q\times\mathbb{R}^d)^2} W(x, x', (y, \xi), (y', \xi'))d\mu_x(y, \xi)d\mu_{x'}(y', \xi')dxdx'$ with respect to the narrow convergence, observed in [36, Proposition 3.7] proves the lower bound.
For what concerns the upper bound, if μ is of the type $\nu_{(x,y)} \otimes dy$ for a suitable two-scale Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$, then Theorem 4.1.3 ensures the existence of a sequence $\{u_n\}_n \subset L^p(\Omega; \mathbb{R}^d)$, weakly convergent in $L^p(\Omega)$ and such that $\left\{\left(\langle \frac{x}{\varepsilon_n}\rangle, u_n\right)\right\}_n$ generates the two-scale Young measure $\nu_{(x,y)}$. The Decomposition Lemma ensures that the $\{u_n\}_n$ can be chosen p-equi-integrable. Clearly, by [27], the sequence is also narrowly convergent to $\mu_x = \nu_{(x,y)} \otimes dy$. Then (4.1.3) ensures that the integrand of $I_{\varepsilon_n}(u_n)$ is equi-integrable and, exactly as in the proof of Theorem 4.1.1 we can invoke Theorem 4.2.11(ii) which, in turn, guarantees the convergence towards

$$\iint_{\Omega\times\Omega} \iint_{(Q\times\mathbb{R}^d)^2} W(x, x', (y, \xi), (y', \xi'))d\mu_x(y, \xi)d\mu_{x'}(y', \xi')dxdx',$$

which concludes the proof in this case.
Finally if μ is not of the type $\nu_{(x,y)} \otimes dy$ for a suitable two-scale Young measure ν, then Theorem 4.1.3 says that $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ cannot be generated by any $\{\delta_{(\langle \frac{x}{\varepsilon_n}\rangle, u_n(x))}\}_n$, hence it cannot be the narrow limit of such a sequence. Consequently, the $\{\mu_n\}_n$ narrowly converging to μ is such that the energy $\overline{I}_{\varepsilon_n}(\mu_n) = +\infty$ for any $n \in \mathbb{N}$. This concludes the proof. $\square$

Chapter 5

Relaxation of Quasi-convex Functionals with Variable Exponent Growth

5.1 Introduction and main result

Variable exponent spaces have been studied since the 1980s [112, 142]. In the 1990s, motivated by the study of composite materials, Zhikov focused on variational integrals with non-standard growth [151, 152, 153, 154, 155]. Since then, integral functionals defined in variable exponent spaces have been actively studied, both in the context of regularity theory and applications to electrorheological fluids and homogenization, see [79] and the bibliography contained therein.

Existence theory of several models have been considered, related to optimal design and double phase problems, thin structures, dielectric materials, homogenization and fluids: we mention [32, 33, 39, 38, 70, 83, 84, 119, 145], among a wider literature, where lower semicontinuity, relaxation and variational convergence for integral functionals with densities satisfying a $p(\cdot)$-growth condition have been considered, under a variety of continuity assumptions on the exponent p. In these papers integral representations are obtained, more or less explicitly, according to the regularity hypotheses on p.

In the Sobolev space $W^{1,p(\cdot)}(\Omega;\mathbb{R}^m)$ in [70, 84, 145] the Γ-limit (see [73]) of such energy functionals is still an integral functional of the same type and growth, provided the exponent is log-Hölder continuous (Definition 5.2.3). This condition ensures that we can freeze the exponent on small balls and use blow-up methods introduced by Fonseca and Müller [88, 89]. Similarly, lack of log-Hölder continuity entails a singular term in the measure representation of functionals, see [154] and [32, 33].

An additional challenge arises in variational problems related to imaging problems where linear growth ($p = 1$) energies are useful. Chen, Levine and Rao [64] proposed a variable exponent generalization of the so-called ROF total variation model in classical BV-space. This was later extended to variable exponent [100, 101, 115] and double phase [99] BV-spaces, and recently to general Orlicz BV^{φ}-spaces [82]. However, the models which combine linear and super-linear growth have so far only been considered for convex energies while for linear growth energies there is a well-developed theory of quasi-convex energies, see [10, Section 5.5].

More recently, another point of view has been considered in [4, 141], taking into account not only bulk energies but a coupling of the latter ones with surface energies, thus leading to the study of variable exponent versions of the so-called Mumford–Shah-type functionals (see [10, Chapter 6]) with bulk energy of type

$$\int_{\Omega} f(\nabla u)^{p(x)} dx, \tag{5.1.1}$$

for superlinear growth, $\inf_{\Omega} p > 1$. In these papers' framework of special functions with bounded variation with variable exponent growth, quasiconvexity of the bulk integrand appears naturally as necessary and sufficient for a lower semicontinuous energy, as in the standard growth case.

When dealing with composite materials, we can describe failure phenomena, such as fracture, by zones of the domain with linear energy behavior in the relaxation process. Orlicz–Sobolev space bulk energies cannot captured this, which leads us to the extension of the above non-standard growth functional (5.1.1) to a free discontinuity setting, where singularities may appear in the form of jump discontinuities or Cantor measures.

The main contribution of this paper is to introduce the first model which features both quasi-convex anisotropy of [9, 88, 89] and the combination of linear and super-linear growth of [100, 101, 82]. We provide sufficient conditions for lower semicontinuity and relaxation of functionals of the form (5.1.1) when the exponent p may take the value 1. Once again, the log-Hölder continuity ensures the lack of Lavrentiev phenomenon and that the only concentration effect is the expected one from the BV-space described by the recession function. However, the subtlety of the problem is illustrated by the fact that a slightly stronger vanishing log-Hölder continuity is required in the linear growth set $\{p = 1\}$.

To state our main result let $A \subset \Omega \subset \mathbb{R}^N$ be a open. We consider the lower semicontinuous envelope of (5.1.1) with respect to the strong $L^{p(\cdot)}$ convergence, i.e. the functional

$$\mathcal{F}(u,A) := \inf\left\{ \liminf_{h\to\infty} \int_A f(\nabla u_h)^{p(x)}\,dx \,\middle|\, u_h \in W^{1,p(\cdot)}(A;\mathbb{R}^m), \right. \left. u_h \to u \text{ in } L^{p(\cdot)}(A;\mathbb{R}^m) \right\} \tag{5.1.2}$$

for $u \in L^{p(\cdot)}(\Omega;\mathbb{R}^m)$; we abbreviate $\mathcal{F}(u) := \mathcal{F}(u,\Omega)$. We obtain an integral representation in the space of functions with generalized growth bounded variation of this functional in terms of the point-wise *recession function* (see Remark 5.4.1)

$$f^\infty(\xi) := \limsup_{t\to+\infty} \frac{f(t\xi)}{t}.$$

Theorem 5.1.1. *Let Ω be a bounded open set with Lipschitz boundary, $p:\Omega\to[1,\infty)$ be* log-*Hölder continuous and assume that $f:\mathbb{R}^{m\times N}\to[0,\infty)$ satisfies the following assumptions.*

(H0) f is continuous and $f(0)=0$.

(H1) $f(\xi) \le \int_{(0,1)^n} f(\xi+\nabla u)\,dx$ for every $\xi\in\mathbb{R}^{N\times m}$ and $u\in W_0^{1,\infty}((0,1)^N;\mathbb{R}^m)$ (quasiconvexity).

(H2) There exists a constant $m>0$ such that $m\,|\xi| \le f(\xi)$ for every $\xi\in\mathbb{R}^{N\times m}$.

(H3) There exists a constant $M\ge 1$ such that $f(\xi)\le M(1+|\xi|)$ for every $\xi\in\mathbb{R}^{N\times m}$.

Then the relaxed functional has the integral representation

$$\mathcal{F}(u,\Omega) = \int_\Omega f(\nabla u)^{p(x)}\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u|$$

for all $u\in BV^{p(\cdot)}(\Omega;\mathbb{R}^m)$.

Remark 5.1.2. If A has Lipschitz boundary and if f satisfies $(H3)$, then

$$\mathcal{F}(u,A) = \inf\left\{ \liminf_{h\to\infty} \int_A f(\nabla u_h)^{p(x)}\,dx \,\middle|\, u_h\in W^{1,p(\cdot)}(A;\mathbb{R}^m), u_h\to u \text{ in } L^1(A;\mathbb{R}^m) \right\}$$

with convergence in L^1 instead of $L^{p(\cdot)}$, by [84, Proposition 2.8].

Our proof is based on approximation by linear growth functionals, for which we can use an improved version of the linear result due to Breit, Diening and Gmeinder [57], see Lemma 5.2.4. Based on this lemma we obtain in Proposition 5.4.2 a lower semi-continuity results which allows us to prove the inequality "$\geq$" in the main theorem. The proof of the opposite inequality uses the so-called blow-up method from [10, Section 5.5], which, as a first step, requires to prove that $\mathcal{F}(u,\cdot)$ in (5.1.2) is a Radon measure absolutely continuous with respect to the generalized total variation of Du, in the sense of [82, 100, 101]. In Proposition 5.4.4 we refine techniques from [101] to show how the strong log-Hölder continuity allows us to separate the effect of the exponent into a multiplicative constant which tends to 1 in the blow-up process. The remaining linear growth, quasi-convex part is then handled as in [10, Section 5.5]. Before the main result, we introduce background material in Section 5.2 and the appropriate vector-valued BV-type space with variable growth in Section 5.3.

5.2 Preliminaries

In this chapter, we consider a bounded open set Ω in $\mathbb{R}^N$, $N \geq 2$. We recall the following notation: for a set $E \subset \mathbb{R}^N$, χ_E is the *characteristic function* of E such that $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. We denote the *Hölder conjugate* exponent of $p \in [1,\infty]$ by $p' = \frac{p}{p-1}$. A generic constant denoted by $c > 0$ without subscript may vary between appearances. Let $f, g : E \to \mathbb{R}$. The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f(y) \leq Cg(y)$ all $y \in E$ and $f \approx g$ means that $f \lesssim g \lesssim f$.

5.2.1 Generalized Orlicz spaces

Here we extend the results from Section 1.4. The extension to the vector valued setting is the following. Let $\varphi \in \Phi_w(\Omega)$. A measurable function $u \equiv (u_1, \dots, u_m)\colon \Omega \to \mathbb{R}^m$ belongs to $L^\varphi(\Omega;\mathbb{R}^m)$ if $u_\alpha \in L^\varphi(\Omega)$ for every $\alpha = 1, \dots, m$. For $u\colon \Omega \to \mathbb{R}^m$ we write

$$\|u\|_\varphi := \big\||u|\big\|_\varphi < \infty,$$

and note that

$$L^\varphi(\Omega;\mathbb{R}^m) = \Big\{u : \Omega \to \mathbb{R}^m \text{ measurable } \Big|\, \rho_\varphi(|\lambda u|) < \infty \quad \text{for some } \lambda > 0\Big\}.$$

Next we define some standard conditions on the Φ-functions that imply additional properties of the spaces.

Definition 5.2.1. Let $\varphi : \Omega \times [0,\infty] \to [0,\infty]$ and let $p, q > 0$. Then we define the following conditions:

(A0) There exists $\beta \in (0,1]$ such that $\varphi(x,\beta) \le 1 \le \varphi(x,\frac{1}{\beta})$ for every $x \in \Omega$.

(A1) For every $K > 0$ there exists $\beta \in (0,1]$ such that, for every $x, y \in \Omega$,

$$\varphi(x,\beta t) \le \varphi(y,t) + 1 \quad \text{when} \quad \varphi(y,t) \in \left[0, \frac{K}{|x-y|^N}\right].$$

Note that this slight variation of (A1) is equivalent to that of [98] if (A0) and (aDec) hold (see [82, Section 3]). Assumption (A1) is an almost continuity condition whereas (aInc) and (aDec) are quantitative versions of the ∇_2 and Δ_2 conditions from Orlicz space theory and measure lower and upper growth rates. We denote by φ^* the conjugate function of φ, i.e.

$$\varphi^*(x,t) := \sup_{s \ge 0}\{st - \varphi(x,s)\},$$

for $x \in \Omega$ and $t \in [0,\infty)$. We define the associate space of the generalized Orlicz space $L^\varphi(\Omega)$, a variant of the dual function space which works better at the end-points $p = 1$ and $p = \infty$.

Definition 5.2.2. Let $\varphi \in \Phi_w(\Omega)$. The *associate space* $(L^\varphi(\Omega))'$ is the subset of measurable functions with finite norm

$$\|f\|_{(L^\varphi(\Omega))'} := \sup_{\|g\|_\phi \le 1} \int_\Omega fg\,dx.$$

For weak Φ-functions the associate space equals the generalized Orlicz space $L^{\varphi^*}(\Omega)$ [98, Theorem 3.4.6], up to equivalence of norms.

5.2.2 Variable exponent spaces

A measurable function $p : \Omega \to [1,\infty)$ is called a *variable exponent*. We denote for $A \subset \Omega$,

$$p_A^+ := \text{ess-sup}_{x\in A} p(x), \; p_A^- := \operatorname*{ess\,inf}_{x \in A} p(x), \; p^+ := p_\Omega^+, \; p^- := p_\Omega^-.$$

By the symbol Y we denote the set where p equals one,

$$Y := \{x \in \Omega \mid p(x) = 1\}.$$

Let $u : \Omega \to [-\infty,\infty]$. We recall that the variable exponent modular is defined as

$$\rho_{L^{p(\cdot)}(\Omega)}(u) := \rho_{p(\cdot)}(u) := \int_\Omega \tfrac{1}{p(x)}|u(x)|^{p(x)}\,dx.$$

This defines the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega;\mathbb{R}^m)$ as the special case of $L^\varphi(\Omega;\mathbb{R}^m)$ with $\varphi(x,t) = \frac{1}{p(x)}t^{p(x)}$. In this case, the conjugate Φ-function is related to the point-wise Hölder conjugate exponent:

$$\varphi^*(x,s) = \begin{cases} \frac{1}{p'(x)}s^{p'(x)} & \text{if } x \in \Omega \setminus Y \\ 0 & \text{if } x \in Y \text{ and } s \leq 1 \\ \infty & \text{if } x \in Y \text{ and } s > 1 \end{cases}$$

[82, Example 4.3]; we will abbreviate this function as $\frac{1}{p'(x)}s^{p'(x)}$ also in Y. We consider Definition 5.2.1 in this context. Now (A0) and $(\mathrm{aInc})_{p^-}$ always hold and $(\mathrm{aDec})_{p^+}$ holds if $p^+ < \infty$ [98, Lemma 7.1.1]. For (A1), we need the log-Hölder continuity of the exponent [98, Proposition 7.1.2].

Definition 5.2.3. A variable exponent p is log-*Hölder continuous* if there is a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{\log(e + \frac{1}{|x-y|})}$$

for all $x, y \in \Omega$. It is *strongly* log-*Hölder continuous* if additionally

$$p(x) - 1 \leq \frac{\omega(|x-y|)}{\log(e + \frac{1}{|x-y|})},$$

for some $\omega : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0}\omega(t) = 0$ and every $y \in Y$ and $x \in \Omega$.

A bounded exponent p is log-Hölder continuous in Ω if and only if there exists a constant $C > 0$ such that

$$\mathcal{L}^n(B)^{p_B^- - p_B^+} \leq C$$

for every ball $B \subset \Omega$ [79, Lemma 4.1.6]. Under the log-Hölder condition, smooth functions are dense in variable exponent Sobolev spaces [79, Theorem 9.1.8]. Strong log-Hölder continuity is a generalization of vanishing log-Hölder continuity [100], where the log-Hölder constant is controlled in only the set Y rather than in all of Ω.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient Du belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{1,p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)}$$

The norm of the latter term is the $L^{p(\cdot)}$-norm of the scalar-valued function $|Du|$, the Euclidean norm of Du, i.e. $\|Du\|_{L^{p(\cdot)}(\Omega)} := \big\||Du|\big\|_{L^{p(\cdot)}(\Omega)}$.

5.2.3 Functions of linear growth

We collect here some results about functions with linear growth that will be used in the proofs.
The following is a special case of Proposition 5.1, [57], with $\mathbb{A} = D$. The result was originally proved by Fonseca and Müller [89] with an additional assumption (H5), which we avoid by using the more general recent reference.

Lemma 5.2.4 (Weak lower semicontinuity). *Let Ω be a bounded open set with Lipschitz boundary. Assume that $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ is continuous, and $f(x,\cdot)$ satisfies $(H1)$, $(H2)$ and $(H3)$ uniformly in $x \in \overline{\Omega}$ and that*

$$|f(x,\xi) - f(y,\xi)| \leq \omega(|x-y|)(1+|\xi|)$$

for some modulus of continuity ω. Then $F^1_{qc} : BV(\Omega;\mathbb{R}^m) \to [0,\infty)$, defined as

$$F^1_{qc}(u) := \int_\Omega f(x,\nabla u)\,dx + \int_\Omega f_s^\infty\Big(x, \frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u|,$$

is sequentially weakly lower semicontinuous with respect to $BV(\Omega;\mathbb{R}^m)$-convergence, where f_s^∞ is the* strong recession function*:*

$$f_s^\infty(x,\xi) := \lim_{\substack{t\to\infty\\ \xi'\to\xi\\ x'\to x}} \frac{f(x',t\xi')}{t}.$$

Lemma 5.2.5 (Reshetnyak continuity, Theorem 2.38, [10]). *Let $\Omega \subset \mathbb{R}^n$ be open and μ, μ_h be $\mathbb{R}^m$-valued finite Radon measures in Ω. If $\mu_h \to \mu$ weakly* in Ω and $|\mu_h|(\Omega) \to |\mu|(\Omega)$, then*

$$\lim_{h\to\infty} \int_\Omega f\Big(x, \frac{\mu_h}{|\mu_h|}\Big)\,d|\mu_h| = \int_\Omega f\Big(x, \frac{\mu}{|\mu|}\Big)\,d|\mu|$$

provided $f : \Omega \times S^{m-1} \to \mathbb{R}$ is continuous and bounded, where $S^{m-1} \subset \mathbb{R}^m$ is the unit sphere.

5.3 BV-type spaces

Following [10], we say that a function $u \in L^1(\Omega)$ has a *bounded variation*, denoted $u \in BV(\Omega)$, if

$$V(u,\Omega) := \sup\left\{ \int_\Omega u \operatorname{div} w\,dx \,\middle|\, w \in C_c^1(\Omega;\mathbb{R}^m), |w| \leq 1 \right\} < \infty.$$

Such functions have distributional first derivatives Du which are Radon measures [10, Section 3.1] and $V(u,\Omega)$ equals the total variation $|Du|(\Omega)$ of the measure Du [10, Proposition 3.6].
In the sequel, we use the Lebesgue decomposition

$$Du = D^a u + D^s u,$$

where $D^a u$ is the absolutely continuous (with respect to the Lebesgue measure) part of the derivative and $D^s u$ is the singular part. The density of $D^a u$ is the vector valued function ∇u such that

$$\int_\Omega w \cdot dD^a u = \int_\Omega w \cdot \nabla u \, dx,$$

for all $w \in C_c^\infty(\Omega;\mathbb{R}^n)$. In the sequel we will identify $D^a u$ with ∇u.
We generalize to the vector-valued case a formulation for BV-type spaces introduced in [82]. When $m = 1$, these spaces reduce to $BV^\varphi(\Omega) := BV^\varphi(\Omega;\mathbb{R})$ considered in [82] and when $\varphi(t) = t$ they reduce to ordinary BV-spaces.

Definition 5.3.1. Let $m \in \mathbb{N}$, $\varphi \in \Phi_w(\Omega)$ and $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$. We define the "dual norm"

$$V_\varphi^m(u,\Omega) := V_\varphi^m(u) := \sup\left\{ \sum_{\alpha=1}^m \int_\Omega u_\alpha \mathrm{div} w_\alpha \, dx \,\middle|\, w \in [C_c^1(\Omega;\mathbb{R}^N)]^m, \|w\|_{\varphi^*} \le 1 \right\},$$

and the "dual modular"

$$\rho_{V,\varphi}^m(u) := \sup\left\{ \int_\Omega \left(\sum_{\alpha=1}^m u_\alpha \mathrm{div} w_\alpha - \varphi^*(x,|w|) \right) dx \,\middle|\, w \in [C_c^1(\Omega;\mathbb{R}^N)]^m \right\}.$$

We say that $u \in L^\varphi(\Omega;\mathbb{R}^m)$ belongs to $BV^\varphi(\Omega;\mathbb{R}^m)$ if

$$\|u\|_{BV^\varphi(\Omega;\mathbb{R}^m)} := \|u\|_{L^\varphi(\Omega;\mathbb{R}^m)} + V_\varphi^m(u,\Omega) < \infty.$$

If $\varphi(x,t) = \frac{1}{p(x)} t^{p(x)}$, then we replace φ by $p(\cdot)$ in the notation, for example $V_{p(\cdot)}^m(u,\Omega)$.

Lemma 5.3.2. *If $\varphi \in \Phi_w(\Omega)$, then $u \equiv (u_1,\dots,u_m) \in BV^\varphi(\Omega;\mathbb{R}^m)$ if and only if $u_i \in BV^\varphi(\Omega)$, for every $i \in \{1,\dots,m\}$.*

Proof. By [98, Theorem 2.5.10 and Corollary 3.2.5], we first observe regarding the Lebesgue norms that $\|u\|_\varphi \lesssim \sum_{i=1}^m \|u_i\|_\varphi$, while $\|u_i\|_\varphi \lesssim \|u\|_\varphi$, for every $i = 1,\dots,m$. Thus it remains to compare the variations.
Assume first that $u \equiv (u_1,\dots,u_m) \in BV^\varphi(\Omega;\mathbb{R}^m)$. Let $w_i \in C_c^1(\Omega;\mathbb{R}^n)$ with

$\|w_i\|_{\varphi*} \leq 1$. Define $w := (0, \dots, 0, w_i, 0, \dots, 0)$ and note that $w \in [C_c^1(\Omega; \mathbb{R}^N)]^m$ with $\|w\|_{\varphi^*} = \|w_i\|_{\varphi^*} \leq 1$. Then

$$\int_\Omega u_i \mathrm{div} w_i \, dx = \int_\Omega \sum_{k=1}^m u_k \mathrm{div} w_k \, dx \leq V_\varphi^m(u, \Omega).$$

Taking the supremum over such w_i, we find that

$$V_\varphi(u_i, \Omega) \leq V_\varphi^m(u, \Omega),$$

for every $i \in \{1, \dots, m\}$, which proves that $u_i \in BV^\varphi(\Omega)$.

Assume then that $u_i \in BV^\varphi(\Omega)$ for every $i \in \{1, \dots, m\}$ and choose $w \in [C_c^1(\Omega; \mathbb{R}^N)]^m$ with $\|w\|_{\varphi*} \leq 1$. Then each component satisfies $w_i \in C_c^1(\Omega; \mathbb{R}^N)$ and $\|w_i\|_{\varphi*} \leq 1$ and it follows from the definitions of V_φ that

$$\int_\Omega \sum_{i=1}^m u_i \mathrm{div} w_i \, dx \leq \sum_{i=1}^m V_\varphi(u_i).$$

Taking the supremum over such w, we find that

$$V_\varphi^m(u, \Omega) \leq \sum_{i=1}^m V_\varphi(u_i, \Omega),$$

which proves that $u \equiv (u_1, \dots, u_m) \in BV^\varphi(\Omega; \mathbb{R}^m)$. □

The following results are vector-valued versions of results from [82]. The proofs of Lemmas 5.3.3 and 5.3.5 are essentially identical and are thus omitted altogether (cf. [82, Lemmas 4.6 and 5.4]). For the others we show the changed parts.

Lemma 5.3.3. *If $\varphi \in \Phi_w(\Omega)$, then V_φ^m is a seminorm and $\|\cdot\|_{BV^\varphi(\Omega;\mathbb{R}^m)}$ is a quasinorm in $BV^\varphi(\Omega; \mathbb{R}^m)$. Moreover, if $\varphi \in \Phi_c(\Omega)$, then $\|\cdot\|_{BV^\varphi(\Omega;\mathbb{R}^m)}$ is a norm.*

The next result shows that $\rho_{V,\varphi}^m$ is a left-continuous semimodular in $L^1(\Omega; \mathbb{R}^m)$. The proof follows [82, Lemma 4.7].

Lemma 5.3.4. *If $\varphi \in \Phi_w(\Omega)$, then*

1. *$\rho_{V,\varphi}^m(0) = 0$;*
2. *the function $\lambda \mapsto \rho_{V,\varphi}^m(\lambda u)$ is increasing on $[0, \infty)$ for every $u \in L^1(\Omega; \mathbb{R}^m)$;*
3. *$\rho_{V,\varphi}^m(-u) = \rho_{V,\varphi}^m(u)$ for every $u \in L^1(\Omega; \mathbb{R}^m)$;*
4. *$\rho_{V,\varphi}^m(\theta u + (1-\theta)v) \leqslant \theta \rho_{V,\varphi}^m(u) + (1-\theta)\rho_{V,\varphi}^m(v)$ for every $u, v \in L^1(\Omega; \mathbb{R}^m)$ and every $\theta \in [0, 1]$;*

5. $\lim_{\lambda\to 1^-} \rho^m_{V,\varphi}(\lambda u) = \rho^m_{V,\varphi}(u)$ *for every* $u \in L^1(\Omega;\mathbb{R}^m)$.

Proof. The proofs of properties (1) and (3) coincide verbatim with [82, Lemma 4.7], so we omit them.

To show that $\lambda \mapsto \rho^m_{V,\varphi}(\lambda u)$ is increasing for every u we let $\lambda \in (0,1)$ and $w \in [C^1_c(\Omega;\mathbb{R}^N)]^m$. Since φ^* is increasing,

$$\int_\Omega \Big(\sum_{i=1}^m \lambda u_i \mathrm{div} w_i - \varphi^*(x,|w|) \Big)\, dx \le \int_\Omega \Big(\sum_{i=1}^m u_i \mathrm{div}(\lambda w_i) - \varphi^*(x,|\lambda w|) \Big)\, dx \le \rho^m_{V,\varphi}(u),$$

as $\lambda w \in [C^1_c(\Omega;\mathbb{R}^N)]^m$. Taking the supremum over w, we get $\rho^m_{V,\varphi}(\lambda u) \le \rho^m_{V,\varphi}(u)$.

Let us pecrove that $\rho^m_{V,\varphi}$ is convex. Let $u, v \in L^1(\Omega;\mathbb{R}^m)$, $\theta \in (0,1)$ and $w \in [C^1_c(\Omega;\mathbb{R}^N)]^m$. Then

$$\begin{aligned}
&\int_\Omega \Big(\sum_{i=1}^m (\theta u_i + (1-\theta) v_i) \mathrm{div} w_i - \varphi^*(x,|w|) \Big)\, dx \\
&\quad = \theta \int_\Omega \Big(\sum_{i=1}^m u_i \mathrm{div} w_i - \varphi^*(x,|w|) \Big)\, dx + (1-\theta) \int_\Omega \Big(\sum_{i=1}^m v_i \mathrm{div} w_i - \varphi^*(x,|w|) \Big)\, dx \\
&\quad \le \theta \rho^m_{V,\varphi}(u) + (1-\theta)\rho^m_{V,\varphi}(v).
\end{aligned}$$

The claim follows when we take the supremum over such w.

Finally, we show that $\rho^m_{V,\varphi}$ is left-continuous. Since $\lambda \mapsto \rho^m_{V,\varphi}(\lambda u)$ is increasing, $\rho^m_{V,\varphi}(\lambda u) \le \rho^m_{V,\varphi}(u)$ for $\lambda \in (0,1)$. We next consider the opposite inequality at the limit. Let first $\rho^m_{V,\varphi}(u) < \infty$ and fix $\varepsilon > 0$. By the definition of $\rho^m_{V,\varphi}$ there exists a test function $w \in [C^1_c(\Omega;\mathbb{R}^N)]^m$ with $\rho_{\varphi^*}(|w|) < \infty$ such that

$$\int_\Omega \Big(\sum_{i=1}^m u_i \mathrm{div} w_i \Big)\, dx \ge \rho^m_{V,\varphi}(u) - \varepsilon + \rho_{\varphi^*}(|w|).$$

Multiplying this inequality by $\lambda \in (0,1)$ and subtracting $\rho_{\varphi^*}(|w|)$, we obtain that

$$\rho^m_{V,\varphi}(\lambda u) \ge \lambda(\rho^m_{V,\varphi}(u) - \varepsilon) + (\lambda - 1)\rho_{\varphi^*}(|w|).$$

Hence

$$\lim_{\lambda \to 1^-} \rho^m_{V,\varphi}(\lambda u) \ge \rho^m_{V,\varphi}(u) - \varepsilon.$$

The claim follows from this as $\varepsilon \to 0^+$. The case $\rho^m_{V,\varphi}(u) = \infty$ is proved similarly, we only need to replace $\rho^m_{V,\varphi}(u) - \varepsilon$ by $\frac{1}{\varepsilon}$. □

Lemma 5.3.5 (BV-type approximation by smooth functions)**.** *Assume that* $\varphi \in \Phi_w(\Omega)$ *satisfies* (A0), (A1) *and* (aDec). *Then there exists* $c \ge 1$ *such that for every* $u \in L^\varphi(\Omega;\mathbb{R}^m)$ *we can find* $u_k \in C^\infty(\Omega;\mathbb{R}^m)$ *with*

$$u_k \to u \text{ in } L^\varphi(\Omega;\mathbb{R}^m) \quad \text{and} \quad V^m_\varphi(u) \le \lim_{k\to\infty} V^m_\varphi(u_k) \le c V^m_\varphi(u).$$

The following result is the vectorial counterpart of [82, Theorem 5.2]

Lemma 5.3.6. *Let* $\varphi \in \Phi_w(\Omega)$ *and* $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$.
Then $V^m_\varphi(u) \le \|\nabla u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^m))'}$.

(1) If $C^1_c(\Omega;\mathbb{R}^{m\times N})$ *is dense in* $L^{\varphi^*}(\Omega;\mathbb{R}^{m\times N})$, *then* $V^m_\varphi(u) = \|\nabla u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^m))'}$.

(2) If φ *satisfies* (A0), (A1) *and* (aDec), *then* $V^m_\varphi(u) \approx \|\nabla u\|_{L^\varphi(\Omega;\mathbb{R}^m)}$.

Proof. Since $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, it follows from the definition of $V^m_{p(\cdot)}$ and integration by parts that

$$V^m_{p(\cdot)}(u) = \sup\bigg\{\int_\Omega \sum_{i=1}^m \nabla u_i\cdot w_i\,dx\colon w\in [C^1_c(\Omega;\mathbb{R}^N)]^m, \|w\|_{\varphi^*}\le 1\bigg\}, \qquad (5.3.1)$$

The definition of the associate space norm, implies that

$$\int_\Omega \sum_{i=1}^m \nabla u_i\cdot w_i\,dx \le \int_\Omega |\nabla u|\,|w|\,dx \le \|\nabla u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^m))'}\|w\|_{L^{\varphi^*}(\Omega;\mathbb{R}^{m\times N})}.$$

Taking the supremum over $w\in [C^1_c(\Omega;\mathbb{R}^N)]^m$ with $\|w\|_{L^{\varphi^*}(\Omega)}\le 1$, we conclude that $V^m_{p(\cdot)}(u)\le \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}$.

To prove claim (1), we next show the opposite inequality $\|\nabla u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^m))'}\le V^m_{p(\cdot)}(u)$ when smooth functions are dense. Let $w\in L^{\varphi^*}(\Omega;\mathbb{R}^{m\times n})$ with $\|w\|_{\varphi^*}=1$ and let (w_j) be a sequence from $C^1_c(\Omega;\mathbb{R}^{m\times N})$ with $w_j\to w$ in $L^{\varphi^*}(\Omega;\mathbb{R}^{m\times N})$ and almost everywhere. Since also $w_j/\|w_j\|_{\varphi^*}\to w$ in $L^{\varphi^*}(\Omega;\mathbb{R}^{m\times N})$, we may assume that $\|w_j\|_{\varphi^*}=1$. By Fatou's Lemma,

$$\liminf_{j\to\infty}\int_\Omega \sum_{i=1}^m \nabla u_i\cdot (w_j)_i\,dx \ge \int_\Omega \sum_{i=1}^m \nabla u_i\cdot w_i\,dx,$$

so it follows from (6.3.17) that

$$V^m_{p(\cdot)}(u) \ge \sup\bigg\{\int_\Omega \sum_{i=1}^m \nabla u_i\cdot w_i\,dx \,\bigg|\, w\in L^{\varphi^*}(\Omega;\mathbb{R}^{m\times N}), \|w\|_{\varphi^*}\le 1\bigg\}.$$

For $h\in L^{\varphi^*}(\Omega)$ we set $w := \frac{\nabla u}{|\nabla u|}h$ if $|\nabla u|\ne 0$ and 0 otherwise in the inequality above. Thus

$$V^m_{p(\cdot)}(u) \ge \sup\bigg\{\int_\Omega |\nabla u|\,h\,dx \,\bigg|\, h\in L^{\varphi^*}(\Omega), \|h\|_{\varphi^*}\le 1\bigg\} = \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}.$$

Hence $V^m_{p(\cdot)}(u) = \|\nabla u\|_{(L^{\varphi^*}(\Omega))'}$ and the proof of (1) is complete.

Then we prove (2). Fix $h \in C^1_c(\Omega)$. Since $w_{\varepsilon,\delta} := (\frac{\nabla u}{|\nabla u|+\varepsilon}) * \eta_\delta$ is bounded and converges to $\frac{\nabla u}{|\nabla u|+\varepsilon}$ a.e. as $\delta \to 0$, it follows by dominated convergence (with majorant $|\nabla u|\,|h|$) that

$$\lim_{\varepsilon\to 0^+}\lim_{\delta\to 0^+}\int_\Omega \sum_{i=1}^m \nabla u_i \cdot (w_{\varepsilon,\delta} h)_i \, dx = \int_\Omega |\nabla u|\, h\, dx.$$

Since $w_{\varepsilon,\delta} h \in C^1_c(\Omega; \mathbb{R}^{m\times N})$, this and (6.3.17) imply that

$$V^m_{p(\cdot)}(u) \geq \sup\left\{ \int_\Omega |\nabla u|\, h\, dx \,\middle|\, h \in C^1_c(\Omega), \|h\|_{\varphi^*} \leq 1 \right\}.$$

The final part of the proof coincides verbatim with [82, Theorem 5.2], so we omit it. □

Note that a sufficient condition for the density of $C^1_c(\Omega; \mathbb{R}^{m\times N})$ in $L^{\varphi^*}(\Omega; \mathbb{R}^{m\times N})$ is that φ satisfies (A0) and (aDec) [98, Theorem 3.7.5].

In [100, 101], a BV-type space with variable exponent was defined based on the set Y where the exponent equals 1 using the modular which is denoted ρ_{old} below. The next result connects this definition to the one given above based on [82].

Theorem 5.3.7. *Let $p : \Omega \to [1, \infty)$ be lower semicontinuous. Then*

$$BV^{p(\cdot)}(\Omega; \mathbb{R}^m) = BV(\Omega; \mathbb{R}^m) \cap W^{1,p(\cdot)}(\Omega \setminus Y; \mathbb{R}^m)$$

and $V^m_{p(\cdot)}(u; \Omega) \approx \|u\|_{\rho_{\mathrm{old}}}$, where $\rho_{\mathrm{old}}(u) := |Du|(Y) + \rho_{L^{p(\cdot)}(\Omega\setminus Y; \mathbb{R}^{N\times m})}(\nabla u)$.

Proof. By Lemma 5.3.2, we can argue using components and so it suffices to consider the case $m = 1$. Furthermore, it suffices to prove that $\|\cdot\|_{\rho_{V,\varphi}} \approx \|u\|_{\rho_{\mathrm{old}}}$ since $V^m_\varphi(u; \Omega) \approx \|\cdot\|_{\rho_{V,\varphi}}$ by [82, Lemma 4.7] and the Lebesgue norms of the function are the same in both spaces.

Assume first that $u \in BV(\Omega) \cap W^{1,p(\cdot)}(\Omega\setminus Y)$ and $\rho_{\mathrm{old}}(u) < \infty$. If $\rho_{\varphi^*}(|w|) < \infty$, then $\varphi^*(x, |w|) = \infty \chi_{(1,\infty)}(|w|) = 0$ for almost all $x \in Y$, so that $|w| \leq 1$ a.e. on Y. Thus

$$\begin{aligned} \rho_{V,p(\cdot)}(u) &= \sup\left\{ \int_\Omega u \operatorname{div} w\, dx - \int_{\Omega\setminus Y} \frac{1}{p'(x)} |w|^{p'(x)}\, dx \,\middle|\, w \in C^1_c(\Omega, \mathbb{R}^N), \rho_{\varphi^*}(|w|) < \infty \right\} \\ &\leq \sup\left\{ \int_\Omega u \operatorname{div} w\, dx - \int_{\Omega\setminus Y} \frac{1}{p'(x)} |w|^{p'(x)}\, dx \,\middle|\, w \in C^1_c(\Omega, \mathbb{R}^N), |w| \leq 1 \text{ a.e. on } Y \right\} \end{aligned}$$

where p' is the Hölder conjugate of p. Since p is lower semicontinuous, $Y = \{p \leq 1\}$ is closed. Since further $u \in W^{1,p(\cdot)}(\Omega \setminus Y)$, $D^s u = 0$ in the open set $\Omega \setminus Y$. Then

$|w| \le 1$ and Young's inequality implies that

$$\begin{aligned}\int_\Omega u \operatorname{div} w \, dx &= \int_{\Omega\setminus Y} w \cdot \nabla u \, dx + \int_Y w \, dDu \\ &\le \int_{\Omega\setminus Y} \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{p'(x)} |w|^{p'(x)} \, dx + |Du|(Y).\end{aligned}$$

Using this in the previous estimate for the modular, we find that

$$\rho_{V,p(\cdot)}(u) \le \int_{\Omega\setminus Y} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + |Du|(Y).$$

We have shown that $\rho_{V,p(\cdot)}(u) \le \rho_{\mathrm{old}}(u)$. It follows by the definition of the Luxemburg norm that $\|\cdot\|_{\rho_{V,p(\cdot)}} \le \|\cdot\|_{\rho_{\mathrm{old}}}$.

Assume then conversely that $u \in BV^{p(\cdot)}(\Omega)$ so that $V_{p(\cdot)}(u;\Omega) < \infty$. Denote $F_i := \{p \le 1+\frac{1}{i}\}$. Then $F_i \searrow Y$ and $p^-_{\Omega\setminus F_i} \ge 1+\frac{1}{i}$. Since p is lower semicontinuous, F_i is closed in Ω. Thus $(p')^+_{\Omega\setminus F_i} < \infty$ so that $C^1_0(\Omega \setminus F_i)$ is dense in $L^{p'(\cdot)}(\Omega \setminus F_i)$ by [98, Theorem 3.7.15]. Thus Lemma 6.3.13(1) yields that

$$\|\nabla u\|_{(L^{\varphi^*}(\Omega\setminus F_i))'} = V_{p(\cdot)}(u;\Omega\setminus F_i) \le V_{p(\cdot)}(u;\Omega).$$

By [98, Theorem 3.4.6 and Proposition 2.4.5], $\|\nabla u\|_{L^{\varphi}(\Omega\setminus F_i)} \approx \|\nabla u\|_{L^{\varphi^*}(\Omega\setminus F_i))'}$ where the implicit constant c_1 does not depend on $p^-_{\Omega\setminus F}$. The previous inequality for $v := u/(cV_{p(\cdot)}(u;\Omega))$ and monotone convergence yield

$$\int_{\Omega\setminus Y} |\nabla v|^{p(x)} \, dx = \lim_{i\to\infty} \int_{\Omega\setminus F_i} |\nabla v|^{p(x)} \, dx \le \lim_{i\to\infty} \max\Big\{1, \|\nabla v\|^{p^+}_{L^{p(\cdot)}(\Omega\setminus F_i)}\Big\} \le 1.$$

Hence $\|\nabla v\|_{L^{p(\cdot)}(\Omega\setminus Y)} \le$ and so $\|\nabla u\|_{L^{p(\cdot)}(\Omega\setminus Y)} \le cV_{p(\cdot)}(u;\Omega)$.

By [82, Example 4.2] we have $BV^{p(\cdot)}(\Omega) \subset BV(\Omega)$, and $|Du|(\Omega) = V(u,\Omega) \le cV_{p(\cdot)}(u;\Omega)$ since φ satisfies $(A0)$. Thus $\|u\|_{\rho_{\mathrm{old}}} \lesssim V_{p(\cdot)}(u;\Omega)$, which concludes the proof. $\square$

5.4 Proof of main result

We recall the definition of the weak recession function

$$f^\infty(\xi) = \limsup_{t\to+\infty} \frac{f(t\xi)}{t}.$$

Remark 5.4.1. If $f : \mathbb{R}^{m\times n} \to [0,\infty)$ is convex, then the limit superior in f^∞ is a limit as $t \mapsto \frac{f(t\xi)-f(0)}{t}$ is increasing. If, additionally, f is Lipschitz, then

$$f_s^\infty(\xi) = \lim_{\substack{t\to\infty \\ \xi'\to\xi}} \frac{f(t\xi')}{t} = \lim_{\substack{t\to\infty \\ \xi'\to\xi}} \frac{f(t\xi)-f(t\xi')}{t} + f^\infty(\xi) = f^\infty(\xi).$$

In this case there is no need distinguish the weak and strong recession functions. As we recall below, our assumptions imply that $f_s^\infty = f^\infty$ for rank-one matrices, in particular for $f^\infty(\frac{dD^s u}{d|D^s u|})$.

However, the weak and strong recession functions are not the same without uniformly linear growth: if $x_i \in \Omega \setminus Y$ with $x_i \to x \in Y$, then

$$(f^{p(x)})_s^\infty(\xi) \geq \lim_{x_i\to x} \lim_{\substack{t\to\infty \\ \xi'\to\xi}} \frac{f(t\xi')^{p(x_i)}}{t} = +\infty > f^\infty(\xi) = (f^{p(x)})_w^\infty(\xi),$$

provided f satisfies (H1) and (H3). Thus we need to be careful when considering recession functions in the absence of uniformly linear growth.

Proposition 5.4.2. *Let Ω be a bounded open set with Lipschitz boundary and $p \in C(\overline{\Omega})$. If $f : \mathbb{R}^{m\times N} \to [0,\infty)$ satisfies $(H0)$, $(H1)$, $(H2)$ and $(H3)$, then the functional $F_{qc} : BV^{p(\cdot)}(\Omega;\mathbb{R}^m) \to [0,\infty)$, defined as*

$$F_{qc}(u) := \int_\Omega f(\nabla u)^{p(x)}\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u|,$$

is lower semicontinuous with respect to $L^{p(\cdot)}(\Omega;\mathbb{R}^m)$-convergence.

Proof. For $j \in \mathbb{N}$, $x \in \Omega$ and $t \in [0,\infty)$, define

$$\phi_j(x,t) := \begin{cases} t^{p(x)} & \text{if } t \leq j, \\ j^{p(x)} + p(x) j^{p(x)-1}(t-j), & \text{otherwise.} \end{cases}$$

Observe that ϕ_j is continuous, $\phi_j(x,t) \leq t^{p(x)}$, $\phi_j(x,t) \nearrow t^{p(x)}$ as $j \to \infty$, and

$$t - 1 \leq \phi_j(x,t) \leq j^{p^+} + p^+ j^{p^+-1}(t-j).$$

Let us write $\psi_j(x,\cdot) := \phi_j(x, f(\cdot))$ and note it is quasiconvex as the composition of a convex function and a quasiconvex function, by Jensen's inequality. By $(H2)$ and $(H3)$ we find that

$$\begin{aligned} m|\xi| - 1 \leq f(\xi) - 1 \leq \psi_j(x,\xi) &\leq j^{p^+} + p^+ j^{p^+-1} f(\xi) \\ &\leq j^{p^+} + M p^+ j^{p^+}(1+|\xi|) \leq M p^+ j^{p^+}(2+|\xi|). \end{aligned}$$

We consider the strong recession function (of $\psi_j + 1$)

$$\Psi_j(x,\xi) := (\psi_j + 1)_s^\infty(x,\xi) = \lim_{\substack{t\to\infty\\ \xi'\to\xi\\ x'\to x}} \frac{\psi_j(x', t\xi') + 1}{t},$$

whenever the limit exists. If $f(\xi) \le j$, then

$$|\psi_j(x,\xi) - \psi_j(y,\xi)| \le |f(\xi)^{p(x)} - f(\xi)^{p(y)}| \le j^{p^+}|1 - j^{|p(x)-p(y)|}|.$$

If $f(\xi) > j$, then by $(H3)$

$$\begin{aligned}|\psi_j(x,\xi) - \psi_j(y,\xi)| &\le |j^{p(x)} - j^{p(y)}| + (f(\xi) - j)\,|p(x)j^{p(x)-1} - p(y)j^{p(y)-1}| \\ &\le j^{p^+}|1 - j^{|p(x)-p(y)|}| + M(1+|\xi|)\,|p(x)j^{p(x)-1} - p(y)j^{p(y)-1}|.\end{aligned}$$

Since $p \in C(\overline{\Omega})$ we obtain that $|\psi_j(x,\xi) - \psi_j(y,\xi)| \le \omega_j(|x-y|)(1+|\xi|)$ for suitable moduli of continuity ω_j. Moreover, by assumption Ω is a bounded domain with Lipschitz boundary. We have thus verified all assumptions of Lemma 5.2.4, and so

$$u \mapsto \int_\Omega \psi_j(x,\nabla u) + 1\,dx + \int_\Omega \Psi_j\Big(x, \frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u|$$

is weakly* lower semicontinuity in $BV(\Omega;\mathbb{R}^m)$. We observe that the "+1" in the integral does not affect the claim (it was only used to satisfy the assumptions of the lemma). By [57, Lemma 6.1], $\xi \mapsto \psi_j(x,\xi)$ is Lipschitz continuous, and so

$$\left|\lim_{\substack{t\to\infty\\ \xi'\to\xi\\ x'\to x}} \frac{\psi_j(x',t\xi')+1}{t} - \lim_{\substack{t\to\infty\\ x'\to x}} \frac{\psi_j(x',t\xi)}{t}\right| \le \lim_{\substack{t\to\infty\\ \xi'\to\xi\\ x'\to x}} \frac{|\psi_j(x',t\xi) - \psi_j(x',t\xi')|}{t} = 0.$$

By the definition of ψ_j, the continuity of p and Remark 5.4.1, we conclude that

$$\Psi_j(x,\xi) = \lim_{\substack{t\to\infty\\ x'\to x}} \frac{\psi_j(x',t\xi)}{t} = \lim_{x'\to x} p(x')j^{p(x')-1} \lim_{t\to\infty} \frac{f(t\xi)}{t} = p(x)j^{p(x)-1}f^\infty(\xi),$$

for every rank-one matrix $\xi \in \mathbb{R}^{m\times N}$. Hence $\Psi_j(x,\xi) \nearrow (f^{p(x)})^\infty(\xi)$ as $j \to \infty$, for every rank-one matrix $\xi \in \mathbb{R}^{m\times N}$, in particular, by Alberti's theorem [2], for $\xi = \frac{dD^s u}{d|D^s u|}$. Applying the lower semicontinuity result, we obtain

$$\begin{aligned}&\int_\Omega \psi_j(x,\nabla u)\,dx + \int_\Omega \Psi_j\Big(x, \frac{dD^s u}{d|D^s u|}\Big)d|D^s u| \\ &\qquad \le \liminf_{h\to\infty}\left[\int_\Omega \psi_j(x,\nabla u_h)\,dx + \int_\Omega \Psi_j\Big(x, \frac{dD^s u_h}{d|D^s u_h|}\Big)\,d|D^s u_h|\right] \\ &\qquad \le \liminf_{h\to\infty}\left[\int_\Omega f(\nabla u_h)^{p(x)}\,dx + \int_\Omega (f^{p(x)})^\infty\Big(\frac{dD^s u_h}{d|D^s u_h|}\Big)\,d|D^s u_h|\right],\end{aligned}$$

whenever $u_h \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^m)$, where we used $\psi_j(x,\xi) \le f(\xi)^{p(x)}$ and $\Psi_j(x,\xi) \le (f^{p(x)})^\infty$ in the second step. Now the right hand side is independent of j, and hence we obtain by monotone convergence as $j \to \infty$, that

$$\begin{aligned} F_{qc}(u) &= \int_\Omega f(\nabla u)^{p(x)}\,dx + \int_\Omega (f^{p(x)})^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u| \\ &\le \liminf_{h\to\infty}\left(\int_\Omega f(\nabla u_h)^{p(x)}\,dx + \int_\Omega (f^{p(x)})^\infty\Big(\frac{dD^s u_h}{d|D^s u_h|}\Big)\,d|D^s u_h|\right) \end{aligned}$$

whenever $u_h \overset{*}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^m)$.

Suppose next that $u_h \to u$ in $L^{p(\cdot)}(\Omega;\mathbb{R}^m)$ for $u, u_h \in BV^{p(\cdot)}(\Omega;\mathbb{R}^m)$. We may assume that

$$\liminf_{h\to\infty} F_{qc}(u_h) < \infty$$

and choose a subsequence with $\lim_{k\to\infty} F_{qc}(u_{h_k}) = \liminf_{h\to\infty} F_{qc}(u_h)$. We denote the subsequence by u_h, again. By $(H2)$, taking into account that Ω is bounded

$$\begin{aligned} \lim_h |Du_h|(\Omega) &= \lim_h \left(\int_\Omega |\nabla u_h|\,dx + |D^s u_h|(\Omega)\right) \\ &\le \lim_h \left(\int_\Omega 1 + |\nabla u_h|^{p(x)}dx + |D^s u_h|(\Omega)\right) \\ &\lesssim \lim_h \left(\int_\Omega f(\nabla u_h)^{p(x)}dx + \int_\Omega f^\infty\Big(\frac{dD^s u_h}{d|D^s u_h|}\Big)\,d|D^s u_h|\right) < \infty. \end{aligned}$$

By compactness, u_h converges to u weakly* in BV, up to a subsequence again denoted u_h (see [10, Proposition 1.59].) Applying the previous result to this subsequence, we obtain that

$$\begin{aligned} F_{qc}(u) &\le \liminf_{h\to\infty}\left(\int_\Omega f(\nabla u_h)^{p(x)}\,dx + \int_\Omega (f^{p(x)})^\infty\Big(\frac{dD^s u_h}{d|D^s u_h|}\Big)\,d|D^s u_h|\right) \\ &= \liminf_{h\to\infty} F_{qc}(u_h) \end{aligned}$$

since $(f^{p(x)})^\infty = f^\infty$ in Y, which includes the support of $D^s u_h$ by Theorem 5.3.7. □

Remark 5.4.3. Suppose that $u, u_h \in W^{1,p(\cdot)}(\Omega;\mathbb{R}^m)$ and $u_h \to u$ in $L^{p(\cdot)}(\Omega;\mathbb{R}^m)$. The previous proposition implies that

$$\int_\Omega f(\nabla u)^{p(x)}\,dx \le \liminf_h \int_\Omega f(\nabla u_h)^{p(x)}\,dx.$$

We remark that the continuity of f is not actually needed in the next proposition, it suffices to assume that it is Borel.

Proposition 5.4.4. *Let p be* log-*Hölder continuous and $f : \mathbb{R}^{N\times m} \to [0,\infty)$ satisfy $(H0)$, $(H2)$ and $(H3)$. Then*

$$\mathcal{F}(u,A) \leq C\Big(|A| + V^m_{p(\cdot)}(u;A) + V^m_{p(\cdot)}(u;A)^{p^+}\Big) \tag{5.4.1}$$

for every $u \in BV^{p(\cdot)}(\Omega;\mathbb{R}^m)$ and open $A \subset \Omega$.

Moreover, for fixed $u \in BV^{p(\cdot)}(\Omega;\mathbb{R}^m)$, the set function $A \mapsto \mathcal{F}(u,A)$ is the restriction to open subsets of Ω of a finite Radon measure in Ω.

Proof. Since p is log-Hölder continuous, we can choose functions $u_h \in W^{1,p(\cdot)}(A;\mathbb{R}^m) \cap C^\infty(A;\mathbb{R}^m)$ with $u_h \to u$ in $L^{p(\cdot)}(A;\mathbb{R}^m)$ and

$$\|\nabla u_h\|_{L^{p(\cdot)}(A;\mathbb{R}^m)} \approx \lim V^m_{p(\cdot)}(u_h;A) \leq cV^m_{p(\cdot)}(u;A)$$

by Lemmas 5.3.5 and 6.3.13(2). Assumption $(H3)$ and [98, Lemma 3.2.9] imply that

$$\begin{aligned}\mathcal{F}(u,A) &\leq \liminf_h \int_A f(\nabla u_h)^{p(x)}\,dx \leq (2M)^{p^+} \liminf_h \int_A 1 + |\nabla u_h|^{p(x)}\,dx \\ &\leq \liminf_h \Big(\mathcal{L}^n(A) + \max\Big\{\|\nabla u_h\|_{L^{p(\cdot)}(A;\mathbb{R}^m)}, \|\nabla u_h\|^{p^+}_{L^{p(\cdot)}(A;\mathbb{R}^m)}\Big\}\Big) \\ &\leq C(p^+)\Big(\mathcal{L}^n(A) + V^m_{p(\cdot)}(u;A) + V^m_{p(\cdot)}(u;A)^{p^+}\Big).\end{aligned}$$

This is the desired inequality.

It remains to show that $\mathcal{F}(u,\cdot)$ is the restriction of a Radon measure for every $u \in BV^{p(\cdot)}(\Omega;\mathbb{R}^m)$. We will establish this by checking the De Giorgi–Letta criteria: every increasing bounded set function $\mu : \mathcal{A} \to [0,\infty)$ defined on the open subsets $\mathcal{A}$ of Ω such that

(i) $A, A' \in \mathcal{A} \;\Rightarrow\; \mu(A \cup A') \leq \mu(A) + \mu(A')$,

(ii) $A, A' \in \mathcal{A}, A \cap A' = \emptyset \;\Rightarrow\; \mu(A \cup A') \geq \mu(A) + \mu(A')$,

(iii) $\mu(A) = \sup\{\mu(A') \mid A' \in \mathcal{A}, A' \subset\subset A\}$,

can be extended to Borel sets $B \subset \Omega$ by $\mu(B) = \inf\{\mu(A) \mid B \subset A \in \mathcal{A}\}$ as a positive Radon measure [10, Theorem 1.53].

If $u \in BV^{p(\cdot)}(\Omega;\mathbb{R}^m)$, then the previously shown inequality ensures that $\mathcal{F}(u,A)$ is finite; since $f \geq 0$, it is increasing and non-negative. The superadditivity condition (ii) is a consequence of the definition of $\mathcal{F}$ since $u_h \in W^{1,p(\cdot)}(A \cup A';\mathbb{R}^m)$ if and only if $u_h|_A \in W^{1,p(\cdot)}(A;\mathbb{R}^m)$ and $u_h|_{A'} \in W^{1,p(\cdot)}(A';\mathbb{R}^m)$ for disjoint and open A and A'.

Let $A_2 \subset\subset A_1 \subset A$. We will prove that

$$\mathcal{F}(u,A) \leq \mathcal{F}(u,A_1) + \mathcal{F}(u,A \setminus \overline{A}_2), \tag{5.4.2}$$

but first we show how this inequality implies (i) and (iii). By (5.4.1) and Theorem 5.3.7, we obtain

$$\mathcal{F}(u,A) \le C\Big(|A| + V^m_{p(\cdot)}(u;A) + V^m_{p(\cdot)}(u;A)^{p^+}\Big) \le C\Big(|A| + \|u\|_{\rho_{\mathrm{old}},A} + \|u\|^{p^+}_{\rho_{\mathrm{old}},A}\Big).$$

Since $p^+ < \infty$, the modular $\rho_{\mathrm{old}}(u)$ over A is finite, and hence form the definition of ρ_{old} we obtain that $\|u\|_{\rho_{\mathrm{old}},F} \to 0$ as $F \to \emptyset$. Since A is open, $A \setminus \bar{A}_2 \to \emptyset$ as $A_2 \nearrow A$. Then (iii) follows by $\mathcal{F}(u,A) \le \mathcal{F}(u,A_1) + \mathcal{F}(u, A \setminus \overline{A}_2)$ as $A_2 \nearrow A$.

To show (i), we choose $A_1 \subset\subset A$ in (5.4.2) with A replaced by $A \cup A'$ and use monotonicity:

$$\mathcal{F}(u, A\cup A') \le \mathcal{F}(u,A_1) + \mathcal{F}(u,(A\cup A')\setminus \overline{A}_2) \le \mathcal{F}(u,A) + \mathcal{F}(u,(A\cup A')\setminus \overline{A}_2).$$

Then (i) follows by (iii) as $A_2 \nearrow A$.

It remains to prove (5.4.2). By the definition of $\mathcal{F}$ and a diagonal argument we choose sequences $\{u_{1,h}\} \subset W^{1,p(\cdot)}(A_1;\mathbb{R}^m)$ and $\{u_{2,h}\} \subset W^{1,p(\cdot)}(A \setminus \overline{A}_2;\mathbb{R}^m)$ that converge to u in $L^1(A_1;\mathbb{R}^m)$ and $L^1(A\setminus\overline{A}_2;\mathbb{R}^m)$, respectively, with

$$\lim_h \int_{A_1} f(\nabla u_{1,h})^{p(x)}\,dx = \mathcal{F}(u,A_1) \quad \text{and} \quad \lim_h \int_{A\setminus\overline{A}_2} f(\nabla u_{2,h})^{p(x)}\,dx = \mathcal{F}(u, A\setminus\overline{A}_2). \tag{5.4.3}$$

For $t \in (0,1)$, let $A_t := \{x \in A_1 \mid \operatorname{dist}(x,\partial A_1) > t\}$. Let $\eta \in (0, \operatorname{dist}(\partial A_1, \partial A_2))$ to be chosen later. Consider a smooth cut-off function $\zeta_\delta \in C_0^\infty(A_{\eta-\delta};[0,1])$ such that $\zeta_\delta = 1$ on A_η and $\|\nabla\zeta_\delta\|_{L^\infty(A_{\eta-\delta})} \le \frac{C}{\delta}$. The sequence $v_{h,\delta} := \zeta_\delta u_{1,h} + (1-\zeta_\delta)u_{2,h}$ converges to u in $L^{p(\cdot)}(A;\mathbb{R}^m)$ as $h \to \infty$. Since $\nabla\zeta_\delta \neq 0$ only in $A_{\eta-\delta}\setminus A_\eta$, we obtain

$$\begin{aligned}\int_A f(\nabla v_{h,\delta})^{p(x)}\,dx \le \int_{A_\eta} f(\nabla u_{1,h})^{p(x)}\,dx + \int_{A\setminus A_{\eta-\delta}} f(\nabla u_{2,h})^{p(x)}\,dx \\ + \int_{A_{\eta-\delta}\setminus A_\eta} f(\nabla v_{h,\delta})^{p(x)}\,dx.\end{aligned}$$

In the first two integral on the right the set can be increased to A_1 and $A \setminus \overline{A_2}$ since $f \ge 0$. For the last term we use $(H3)$ and $\nabla v_{h,\delta} = \zeta_\delta \nabla u_{1,h} + 1 - \zeta_\delta)\nabla u_{2,h} + (u_{1,h} - u_{2,h})\nabla\zeta_\delta$ to obtain

$$\begin{aligned} f(\nabla v_{h,\delta})^{p(x)} &\le M^{p(x)}(1 + |\nabla v_{h,\delta}|)^{p(x)} \\ &\le C(p^+)\Big(1 + |\nabla u_{1,h}|^{p(x)} + |\nabla u_{2,h}|^{p(x)} + (\tfrac{1}{\delta}|u_{1,h} - u_{2,h}|)^{p(x)}\Big)\end{aligned}$$

Thus

$$\begin{aligned}\int_A f(\nabla v_{h,\delta})^{p(x)}\,dx \le \int_{A_1} f(\nabla u_{1,h})^{p(x)}\,dx + \int_{A\setminus A_2} f(\nabla u_{2,h})^{p(x)}\,dx \\ + C(p^+)\nu_h(A_{\eta-\delta}\setminus A_\eta) + \int_{A_1\setminus A_2} (\tfrac{1}{\delta}|u_{1,h} - u_{2,h}|)^{p(x)}\,dx.\end{aligned}$$

where, for $E \subset \mathbb{R}^N$,

$$\nu_h(E) := \int_{E\cap(A_1\setminus A_2)} 1 + |\nabla u_{1,h}|^{p(x)} + |\nabla u_{2,h}|^{p(x)}\,dx.$$

Passing to the limit as $h \to \infty$ and observing that $u_{1,h} - u_{2,h} \to u - u = 0$ in $L^{p(\cdot)}(A_1 \setminus A_2; \mathbb{R}^m)$, we obtain that

$$\mathcal{F}(u, A) \leq \mathcal{F}(u, A_1) + \mathcal{F}(u, A \setminus \overline{A}_2) + C(p^+) \limsup_{h\to 0} \nu_h(A_{\eta-\delta} \setminus A_\eta).$$

We complete the proof of (5.4.2) by showing that the last term tends to zero when $\delta \to 0$.

By the properties of Lebesgue integral ν_h is a Radon measure. By (5.4.3) and $(H2)$, $\sup_h \nu_h(K) < \infty$ for every compact $K \subset \mathbb{R}^N$. By the weak compactness of measures [86, Theorem 1.41], there exists a subsequence, still denoted by (ν_h), and a Radon measure ν such that

$$\limsup_{h\to\infty} \nu_h(K) \leq \nu(K)$$

for all compact sets $K \subset \mathbb{R}^N$, so that in particular

$$\limsup_{h\to 0} \nu_h(A_{\eta-\delta} \setminus A_\eta) \leq \nu(\overline{A_{\eta-\delta} \setminus A_\eta}).$$

Let (δ_i) be a positive sequence converging to zero. Since $\overline{A_{\eta-\delta_i} \setminus A_\eta}$ is a decreasing sequence of sets, we obtain

$$\limsup_{i\to\infty} \nu(\overline{A_{\eta-\delta_i} \setminus A_\eta}) = \nu\Big(\bigcap_{i=1}^{\infty} \overline{A_{\eta-\delta_i} \setminus A_\eta}\Big) = \nu(\partial A_\eta).$$

Since $\nu(\cup_{\eta>0}\partial A_\eta \cap A_1) \leq \nu(A_1) < \infty$, we can fix η such that $\nu(\partial A_\eta) = 0$. $\square$

Lemma 5.4.5. *Let Ω be a bounded open set with Lipschitz boundary, let p be strongly* log-*Hölder continuous, and let f satisfy* $(H0)$, $(H1)$, $(H2)$ *and* $(H3)$. *Then*

$$\mathcal{F}(u, \Omega) \leq \int_\Omega f(\nabla u)^{p(x)}\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u|$$

for all $u \in BV^{p(\cdot)}(\Omega; \mathbb{R}^m)$.

Proof. We adapt the strategy of [10, Proposition 5.49]. Recall that $Y := \{p = 1\}$ and denote $E^\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, E) \leq \delta\}$ for $E \subset \mathbb{R}^n$ and $\delta > 0$.

Fix $u \in BV^{p(\cdot)}(\Omega;\mathbb{R}^m)$. Let $0<\delta$ and consider the standard mollifiers η_δ, and denote $u_\delta := u * \eta_\delta$, so that $\nabla u_\delta := \nabla u * \eta_\delta + D^s u * \eta_\delta$. Let $A \subset\subset \Omega$ be open and observe that $p(x)>1$ in $A \setminus Y^\delta$ and thus also $\nabla u_\delta = \nabla u * \eta_\delta$. Hence

$$\int_A f(\nabla u_\delta)^{p(x)}\,dx \le \int_{A\setminus Y^\delta} f(\nabla u * \eta_\delta)^{p(x)}\,dx + \int_{Y^\delta \cap A} f(\nabla u_\delta)^{p(x)}\,dx.$$

We estimate the two integrals on the right-hand side in the next two paragraphs.

Since f satisfies $(H1)$ and $(H3)$ and $\bar{q}\ge 1$,

$$\Big|f(\xi)^q - f(\eta)^q\Big| \le C(\bar{q},M)(1+|\xi|^{q-1}+|\eta|^{q-1})|\xi-\eta|, \tag{5.4.4}$$

for every $q\in[1,\bar{q}]$ and $\xi,\eta\in\mathbb{R}^{N\times m}$ [10, Lemma 5.42]. Using this with $q=p(x)$, we find by Hölder's inequality that

$$\begin{aligned}
&\int_{A\setminus Y} \Big|f(\nabla u)^{p(x)} - f(\nabla u * \eta_\delta)^{p(x)}\Big|\,dx \\
&\quad\lesssim \int_{A\setminus Y} (1+|\nabla u|^{p(x)-1} + |\nabla u * \eta_\delta|^{p(x)-1})\,|\nabla u - \nabla u * \eta_\delta|\,dx \\
&\quad\lesssim \Big(1+\|\nabla u\|_{p(\cdot)} + \|\nabla u * \eta_\delta\|_{p(\cdot)}\Big)\|\nabla u - \nabla u * \eta_\delta\|_{p(\cdot)}.
\end{aligned}$$

Since $\nabla u * \eta_\delta \to \nabla u$ in $L^{p(\cdot)}(A\setminus Y;\mathbb{R}^m)$ [79, Theorem 4.6.4], the right-hand side above converges to zero as $\delta\to 0$. Thus

$$\limsup_{\delta\to 0}\int_{A\setminus Y^\delta} f(\nabla u*\eta_\delta)^{p(x)}\,dx \le \limsup_{\delta\to 0}\int_{A\setminus Y} f(\nabla u*\eta_\delta)^{p(x)}\,dx = \int_{A\setminus Y} f(\nabla u)^{p(x)}\,dx.$$

We then consider the second integral, over $Y^\delta\cap A$. There exists $c>0$ such that $|\nabla u_\delta| \le \|u\|_1 \sup|\nabla \eta_h| \le c\delta^{-1}$. Hence, by $(H3)$,

$$f(\nabla u_\delta)^{p(x)} = f(\nabla u_\delta)^{p(x)-1} f(\nabla u_\delta) \le (c\delta^{-1})^{p(x)-1} f(\nabla u_\delta)$$

for all small δ. Denote

$$\omega(\delta) := \sup_{x\in Y^\delta}(p(x)-1)\log\frac{1}{\operatorname{dist}(x,Y)}.$$

By the strong log-Hölder assumption, $\omega(0^+)=0$ and so

$$(c\delta^{-1})^{p(x)-1} \le e^{2(p(x)-1)\log 1/\delta} \le e^{2\omega(\delta)} \to 1$$

when $\delta \le \frac{1}{c}$. Hence

$$\int_{Y^\delta\cap A} f(\nabla u_\delta)^{p(x)}\,dx \le e^{2\omega(\delta)} \int_{Y^\delta\cap A} f(\nabla u_\delta)\,dx.$$

Now estimating the integral on the right hand side as in the proof of [10, Theorem 5.49], i.e. exploiting (5.4.4) with $q = 1$ and [10, Theorem 2.2(b)], we have

$$\begin{aligned}
&\limsup_{\delta\to 0} \int_{Y^\delta\cap A} f(\nabla u_\delta)\,dx \\
&\quad\le \limsup_{\delta\to 0} \Big[\int_{Y^\delta\cap A} f(\nabla u * \eta_\delta)\,dx + M\,|D^s u * \eta_\delta|(Y^\delta \cap A)\Big] \\
&\quad\le \limsup_{\delta\to 0} \Big[\int_{(Y^\delta\cap A)^\delta} f(\nabla u) + c\,|\nabla u * \eta_\delta - \nabla u|\,dx + M\,|D^s u|(Y^\delta \cap A)_\delta\Big] \\
&\quad= \int_{Y\cap A} f(\nabla u)\,dx + M\,|D^s u|(\overline{A}).
\end{aligned}$$

Since $p = 1$ in Y, the integrand on the right can be replaced by $f(\nabla u)^{p(x)}$.

Combining the estimates over $A\backslash Y^\delta$ and $Y^\delta \cap A$ and assuming that $|D^s u|(\partial A) = 0$ we obtain that

$$\mathcal{F}(u, A) \le \int_A f(\nabla u)^{p(x)}\,dx + C\,|D^s u|(A). \tag{5.4.5}$$

From the proof of Proposition 5.4.4 we know that $\mathcal{F}(u, A) = \sup_{B\subset A \text{ open}} \mathcal{F}(u, B)$. Since the sets B can be chosen with $|D^s u|(\partial B) = 0$ (as in the proof of the proposition), (5.4.5) holds for all open $A \subset \Omega$. By the proposition, $\mathcal{F}(u, \cdot)$ extends to a Radon measure and it follows from its Lebesgue decomposition $\mathcal{F} = \mathcal{F}^a + \mathcal{F}^s$ that

$$\mathcal{F}^a(u, B) \le \int_B f(\nabla u)^{p(x)}\,dx \quad \text{and} \quad \mathcal{F}^s(u, B) \le C\,|D^s u|(A)$$

for every Borel set B, since $D^s u$ and the Lebesgue measure are mutually singular.

We still need a better estimate for $\mathcal{F}^s$; we use a similar strategy but estimate $\int_{Y^\delta\cap A} f(\nabla u_\delta)\,dx$ in a different way. Define $g(\xi) := \sup_{t>0} \frac{f(t\xi)-f(0)}{t} = \sup_{t>0} \frac{f(t\xi)}{t}$. Since f is Lipschitz [10, Lemma 5.42], it follows that g is, as well. By $(H2)$ and $(H3)$, $m\,|\xi| \le g(\xi) \le M\,|\xi|$. We note that $f(\xi) \le g(\xi) = g(\frac{\xi}{|\xi|})|\xi|$ and use Lemma 5.2.5 (Reshetnyak continuity) for the continuous and bounded function g to conclude that

$$\begin{aligned}
\liminf_{\delta\to 0} \int_{Y^\delta\cap A} f(\nabla u_\delta)\,dx &\le \limsup_{\delta\to 0} \int_{Y^\delta\cap A} g\Big(\frac{\nabla u_\delta}{|\nabla u_\delta|}\Big)\,|\nabla u_\delta|\,dx \\
&= \int_{Y^{\delta_1}\cap A} g\Big(\frac{dDu}{d|Du|}\Big)\,d|Du|
\end{aligned}$$

for any $\delta_1 > 0$. We next use that $d|Du| = |\nabla u|\,dx + d|D^s u|$ and $\frac{dDu}{d|Du|} = \frac{dD^s u}{d|D^s u|}$ in the support of $D^s u$ which is a subset of Y. Thus

$$\int_{Y^\delta\cap A} g\Big(\frac{dDu}{d|Du|}\Big)\,d|Du| = \int_{Y^\delta\cap A} g\Big(\frac{dDu}{d|Du|}\Big)\,|\nabla u|\,dx + \int_{Y\cap A} g\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u|$$

Since f is quasiconvex, it and also g is convex along rank-one directions [10, pp. 299–300]; thus $g(\xi) = \lim_{t\to\infty} \frac{f(t\xi)}{t} = f^\infty(\xi)$ when ξ is a rank-one matrix. By Alberti's theorem, $\frac{dD^s u}{d|D^s u|}$ is a rank-one matrix [2]. Furthermore, $g(\xi) \le M$ when $|\xi| \le 1$. Hence we can continue our previous estimate by

$$\int_{Y^\delta\cap A} g\Big(\frac{dDu}{d|Du|}\Big)\, d|Du| \le M \int_{Y^\delta\cap A} |\nabla u|\, dx + \int_{Y\cap A} f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u|.$$

Combining the estimates from the previous paragraph with the earlier estimate over $A \setminus Y^{\delta_1}$ and assuming that $|D^s u|(\partial A) = 0$ we obtain as $\delta_1 \to 0$ that

$$\mathcal{F}(u, A) \le C \int_A f(\nabla u)^{p(x)} + 1\, dx + \int_{Y\cap A} f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u|.$$

In the same way as with (5.4.5), we conclude from this that

$$\mathcal{F}^s(u, B) \le \int_B f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u|. \qquad \square$$

We are now ready to prove the main result.

Proof of Theorem 5.1.1. It follows from Lemma 5.4.5 that

$$\mathcal{F}(u) \le \int_\Omega f(\nabla u)^{p(x)}\, dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u|$$

for all $u \in BV^{p(\cdot)}(\Omega; \mathbb{R}^m)$.

For the opposite inequality let $u \in BV^{p(\cdot)}(\Omega; \mathbb{R}^m)$ and $u_h \in W^{1,p(\cdot)}(\Omega; \mathbb{R}^m)$ with $u_h \to u$ in $L^{p(\cdot)}(\Omega; \mathbb{R}^m)$. By Proposition 5.4.2 and $u_h \in W^{1,p(\cdot)}(\Omega; \mathbb{R}^m)$,

$$F_{qc}(u) \le \liminf_{h\to\infty} F_{qc}(u_h) = \liminf_{h\to\infty} \int_\Omega f(\nabla u_h)^{p(x)}\, dx.$$

Taking the infimum over such sequences yields $F_{qc}(u) \le \mathcal{F}(u)$, as required. $\square$

The next result shows that the test-functions from $\mathcal{F}$ can alternatively be taken to be smooth. Note that uniform domains, including bounded domains with Lipschitz boundary, are extension domains [79, Theorem 8.5.12].

Proposition 5.4.6. *Let $\Omega \subset \mathbb{R}^n$ be a $W^{1,p(\cdot)}$-extension domain, p be* log-*Hölder continuous and $f : \mathbb{R}^{n\times m} \to [0, \infty)$ satisfy (H0), (H2) and (H3). Then*

$$\mathcal{F}(u) = \inf \Big\{ \liminf_{h\to\infty} \int_\Omega f(\nabla u_h)^{p(x)}\, dx \,\Big|\, u_h \in C^\infty(\overline{\Omega}; \mathbb{R}^m), u_h \to u \text{ in } L^1(\Omega; \mathbb{R}^m) \Big\},$$

for every $u \in BV^{p(\cdot)}(\Omega; \mathbb{R}^m)$.

Proof. Denote by $\mathcal{F}_{C^\infty}$ the right-hand side of the claimed equation. The inequality $\mathcal{F}(u) \leq \mathcal{F}_{C^\infty}(u)$ follows from $C^\infty(\overline{\Omega};\mathbb{R}^m) \subset W^{1,p(\cdot)}(\Omega;\mathbb{R}^m)$.

For the opposite inequality we follow the ideas from [88, Proposition 2.6(ii)]. Let $u_j \in W^{1,p(\cdot)}(\Omega;\mathbb{R}^m)$ with $u_j \to u \in L^{p(\cdot)}(\Omega;\mathbb{R}^m)$ and

$$\mathcal{F}(u) = \lim_{j\to\infty} \int_\Omega f(\nabla u_j)^{p(x)}\,dx.$$

By [79, Theorem 9.1.7], $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$, hence, arguing componentwise one can deduce that $C^\infty(\overline{\Omega};\mathbb{R}^m)$ is dense in $W^{1,p(\cdot)}(\Omega;\mathbb{R}^m)$. By the density we find for every u_j a sequence $(u_j^k) \subset C_0^\infty(\overline{\Omega};\mathbb{R}^m)$ with $u_j^k \to u_j$ in $W^{1,p(\cdot)}(\Omega;\mathbb{R}^m)$ and $\nabla u_j^k \to \nabla u_j$ almost everywhere. By the continuity of f and p, we find that $f(\nabla u_j^k(x))^{p(x)} \to f(\nabla u_j(x))^{p(x)}$ almost everywhere. Assumption $(H3)$ and $p^+ < \infty$ yield that $f(\xi)^{p(x)} \leq (M(1+|\xi|))^{p(x)} \leq (2M)^{p^+}(1+|\xi|^{p(x)})$ and hence $(2M)^{p^+}(1+|\nabla u_j^k|^{p(\cdot)}) - f(\nabla u_j^k)^{p(\cdot)} \geq 0$. Fatou's lemma yields that

$$\int_\Omega (2M)^{p^+}(1+|\nabla u_j|^{p(x)}) - f(\nabla u_j)^{p(x)}\,dx \\ \leq \liminf_{k\to\infty} \int_\Omega (2M)^{p^+}(1+|\nabla u_j^k|^{p(x)}) - f(\nabla u_j^k)^{p(x)}\,dx.$$

Since Ω is bounded and $u_j^k \to u_j$ in $W^{1,p(\cdot)}(\Omega,\mathbb{R}^m)$, the first terms on cancel and we find that

$$\int_\Omega f(\nabla u_j)^{p(x)}\,dx \geq \limsup_{k\to\infty} \int_\Omega f(\nabla u_j^k)^{p(x)}\,dx.$$

For every j we choose k_j such that $\|u_j - u_j^{k_j}\|_1 \leq \frac{1}{j}$ and

$$\int_\Omega f(\nabla u_j^{k_j})^{p(x)}\,dx < \int_\Omega f(\nabla u_j)^{p(x)}\,dx + \frac{1}{j}.$$

Then $(u_j^{k_j})$ converges to u in $L^1(\Omega,\mathbb{R}^m)$ so it is a valid test sequence for $\mathcal{F}_{C^\infty}$. Hence

$$\mathcal{F}_{C^\infty}(u) \leq \liminf_{j\to\infty} \int_\Omega f(\nabla u_j^{k_j})^{p(x)}\,dx \leq \liminf_{j\to\infty} \left(\int_\Omega f(\nabla u_j)^{p(x)} + \frac{1}{j} \right) \leq \mathcal{F}(u). \quad \square$$

Chapter 6

Functions BD of Musielak-Orlicz-type and anisotropic TGV for image-denoising problems

6.1 Introduction

Ever since the early 90s, the variational approach has been a cornerstone of the mathematical image reconstruction. Generally speaking, in its simpler formulation, this methodology can be summarized as follows: for a given bounded domain $\Omega \subset \mathbb{R}^2$, grayscale images are identified with functions $u : \Omega \to \mathbb{R}$. Starting from a datum f, possibly corrupted by noise, a clean image is obtained through the minimization of an energy functional of the form

$$\mathcal{I}(u) := \alpha \|Ku - f\|_H + \beta \, \mathcal{R}(u),$$

for $\alpha, \beta > 0$. The structure of $\mathcal{I}$ relies on the competition between the *fidelity term* $\|Ku - f\|_H$, for a given normed space H and a forward operator K, intended to keep track of the original datum f, and a *regularizing term* $\mathcal{R}(u)$, whose task is to filter out the noise and return a clean image. The strength of both effects is quantified by the two *tuning parameters* α and β.
In this chapter we propose a novel regularizer based on a Musielak-Orlicz generalization of classical Total Generalized Variation functionals, and provide a first study of the associated space of functions with bounded deformation with generalized Orlicz growth.
Depending on specific applications, various types of variational regularizers have been proposed in the literature. A classical example is the ROF model, where $\mathcal{R}(u)$

is the total variation of u, cf. [140]. A known drawback of such functional is the so-called staircasing effect, namely the tendency of producing artificially piecewise-affine artifacts in the image denoising, see [62, 107]. In order to counteract such problems, several higher-order models have been introduced. A particularly successful one is the second-order Total Generalized Variation, defined as

$$TGV_\alpha^2(u) := \sup \left\{ \int_\Omega u \operatorname{div}^2 \psi \, dx \ : \ \psi \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{sym}), \|\psi\|_\infty \le \alpha_1, \|\operatorname{div}\psi\|_\infty \le \alpha_2 \right\},$$

for every $u \in L^1_{loc}(\Omega)$, with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. We refer to the seminal paper [53] for a general introduction and higher-order generalizations, and to [54] for an in-depth study of the second-order formulation.
In the last few years, due to an increasing interest in accurate and adaptive image reconstruction, an emerging trend has been considering anisotropic models, allowing to tailor the denoising effects to specific features of different parts of the images. First steps in this direction have already been taken, e.g., in [44, 62, 144]. In particular, in [64] the authors have proposed an anisotropic ROF-type functional with $\mathcal{R}$ of the form $\mathcal{R}(u) = \int_\Omega \phi(x, |Du|)\, dx$ with

$$\phi(x,t) := \begin{cases} \frac{1}{p(x)}\, t^{p(x)} & \text{if } t \in [0,1], \\ t - 1 + \frac{1}{p(x)} & \text{if } t > 1, \end{cases}$$

and with the variable exponent $p : \Omega \to [c, 2]$, with $c > 1$.
In parallel, the study of variational problems in variable, Orlicz, and generalized Orlicz spaces has boomed in the past ten years. Among the vast literature, we single out the contributions [66, 79, 98, 100, 101, 135] and the references therein. The extension of the model in [64] to incorporate the case $c = 1$, as well as an Orlicz generalization for regularizers $\mathcal{R}(u) \simeq \int_\Omega \varphi(x, |Du|)$ for Φ-functions φ possibly having linear growth, is the subject of [82], where a thorough mathematical foundation of the space of anisotropic functions with bounded variation with generalized Orlicz growth has also been provided. A further step in this direction has been taken in [97], where the associated Euler-Lagrange equations and gradient flows have also been analyzed. The effect of an anisotropic norm in the Chan-Vese (or anisotropic Mumford-Shah) is the subject of [121]. Anisotropic models, at least in the ROF setting, have indeed proven to allow for more accurate computations, as well as for a staircasing reduction and a sharper texture recognition. We refer to [65, 85, 114] for an overview and further references.
Our contribution is twofold. First, we introduce a novel anisotropic functional setting of functions with bounded deformation, namely the space BD^φ of bounded deformation fields with generalized Orlicz growth. After analyzing its basic properties, we establish a link to a modular representation and provide a further study

in the special case of variable Lebesgue integrability.
Second, we analyze a Musielak-Orlicz anisotropic Total Generalized Variation class of regularizers. Let Ω be a bounded, connected and open set in $\mathbb{R}^N$, with $N \geq 2$. Inspired by [53], for a fixed generalized weak $\Phi-$function $\varphi \in \Phi_w(\Omega)$, we define our anisotropic Total Generalized Variation of order 2 with weight $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ as

$$TGV_\alpha^{\varphi,2}(u) := \sup\Bigg\{ \int_\Omega u \operatorname{div}^2\psi\, dx : \\ \psi \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{\text{sym}}), \|\psi\|_{\varphi^*} \leq \alpha_1, \|\operatorname{div}\psi\|_{\varphi^*} \leq \alpha_2 \Bigg\}, \tag{6.1.1}$$

for every $u \in L^1_{\text{loc}}(\Omega)$, where $\|\cdot\|_{\varphi^*}$ is the norm associated to the Fenchel conjugate function of φ, cf. Definition 6.2.4, and where the divergence of order 2 of $\xi \in \mathbb{R}^{n\times n}_{\text{sym}}$ is defined as

$$\operatorname{div}^2\xi := \sum_{i=1}^n \frac{\partial^2 \xi_{ii}}{\partial x_i^2} + 2\sum_{i<j} \frac{\partial^2 \xi_{ij}}{\partial x_i \partial x_j},$$

(see also [53] for higher-order notions of divergence). Note that, in principle, $TGV^{\varphi,2}(u)$ might take infinite values.

A prototypical example of admissible functions φ is given by $\varphi(t) := \frac{1}{p(x)} t^{p(x)}$ (see also [82, Example 4.3]), with $p : \Omega \to [1, +\infty]$, thus possibly describing linear, superlinear, or even infinite growth in different portions of the domain.
After studying the basic properties of $TGV_\alpha^{\varphi,2}$, we identify a dual representation, and show how classical existence and stability for classical Total Generalized Variation carry over to our Musielak-Orlicz setting.
The chapter is organized as follows. In Section 6.2 we collect some more definitions and technical results on Φ-functions and generalized Orlicz spaces. Section 6.3 contains the definition and main properties of the novel space of bounded deformation fields with generalized Orlicz growth BD^φ. In particular, in Subsection 6.3.1 we establish a connection between BD^φ and an associated weighted-modular representation, while in Subsection 6.3.2 we further establish a decomposition of the modular function in terms of the modular of the absolutely continuous part of the symmetric gradient and a singular part weighted by the recession function, cf. Theorem 6.3.14. Eventually, in Subsection 6.3.3 we specify our setting to the variable exponent case. The focus of Section 6.4 is on our anisotropic Total Generalized Variation. In Subsection 6.4.1 we establish its main properties, and in Subsection 6.4.2 we identify its dual reformulation in terms of a notion of anisotropic measure variation. Finally, in Subsection 6.4.3 we show stability with respect to the data and well-posedness of the minimization problem. In Appendices A.1 and B.1 we give the proofs of some auxiliary results we need for the proof of Theorem 6.3.14.

6.2 Preliminary results

In this subsection we give some definitions and results regarding Φ-functions, extending Section 1.4.

Definition 6.2.1. Let $\varphi : \Omega \times [0,+\infty) \to [0,+\infty]$. Then we define the following conditions:

(A1) For every $K > 0$ there exists $\beta \in (0,1]$ such that for every $x, y \in \Omega$

$$\varphi(x,\beta t) \leq \varphi(y,t) + 1 \quad \text{when } \varphi(y,t) \in \left[0, \tfrac{K}{|x-y|^N}\right],$$

(VA1) For every $K > 0$ there exists a modulus of continuity ω such that for every $x, y \in \Omega$

$$\varphi\left(x, \frac{t}{1+\omega(|x-y|)}\right) \leq \varphi(y,t) + \omega(|x-y|) \quad \text{when } \varphi(y,t) \in \left[0, \frac{K}{|x-y|^N}\right],$$

Assumption (A1) is an almost continuity condition and we define the condition (A1) as in [82]. We point out that it slightly differs from [98, 103], where it is assumed that

$$\varphi(x,\beta t) \leq \varphi(y,t) \quad \text{when } \varphi(y,t) \in \left[1, \tfrac{1}{|B|}\right]$$

and x and y belong to the ball B. Lastly, the "vanishing (A1)" condition (VA1) is a continuity condition for φ, introduced in [105] to prove maximal regularity of minimizers.
In order to state the last property which we need for our analysis, we first need to define the recession function of φ.

Definition 6.2.2. Let $\varphi \in \Phi_w(\Omega)$. We call recession function of φ the function $\varphi^\infty : \Omega \to [0,+\infty]$ defined by

$$\varphi^\infty(x) := \limsup_{t\to+\infty} \frac{\varphi(x,t)}{t} \qquad \text{for all } x \in \Omega.$$

Note that, since the continuity of φ is not necessarily inherited by φ^∞, this makes the non-autonomous case not trivial. For our analysis, we also need the following weaker version of (VA1) where at least one of the points has to belong to the set $\{\varphi^\infty < +\infty\}$.

Definition 6.2.3. Let $\varphi \in \Phi_w(\Omega)$. We say that φ satisfies restricted (VA1) if and only if it satisfies (A1) and for every $K > 0$ there exists ω modulus of continuity such that

$$\varphi\left(x, \frac{t}{1+\omega(|x-y|)}\right) \leq \varphi(y,t) + \omega(|x-y|) \quad \text{when } \varphi(y,t) \in \left[0, \frac{K}{|x-y|^N}\right],$$

for every $x, y \in \Omega$ such that $\varphi^\infty(x) < +\infty$ or $\varphi^\infty(y) < +\infty$.

We now need to define the associate space of the generalized Orlicz space $L^\varphi(\Omega)$. In general, the associate space is a variant of the dual function space which works better at the end-points $p = 1$ and $p = \infty$.

Definition 6.2.4. Let $\varphi \in \Phi_w(\Omega)$. We define the associate space $(L^\varphi)'(\Omega) = \{u \in L^0(\Omega) : \|u\|_{(L^\varphi)'} < +\infty\}$ with

$$\|u\|_{(L^\varphi)'} := \sup_{\|v\|_\varphi \le 1} \int_\Omega uv\, dx.$$

Thanks to [98, Theorem 3.4.6], it is well know that for weak Φ-functions the associate space corresponds to the generalized Orlicz space $L^{\varphi^*}(\Omega)$, where

$$\varphi^*(x,t) := \sup_{s\ge 0}[st - \varphi(x,s)]$$

is the (Fenchel) conjugate function of $\varphi \in \Phi_w(\Omega)$. Moreover, it can be shown that (see [82, Theorem 3.4.6]) for all $u \in L^0(\Omega)$, $\|u\|_{L^\varphi}$ is equivalent to $\sup_{\|v\|_{L^{\varphi^*}} \le 1} \int_\Omega uv\, dx$.

Let us state some well-known properties of the conjugate function.

Lemma 6.2.5. *Let $\varphi \in \Phi_w(\Omega)$. Then,*

(i) $\varphi^* \in \Phi_c(\Omega)$, *meaning that φ^* is always convex and left-continuous,*

(ii) If $p, q \in (1, +\infty)$, then φ satisfies (aInc$_p$) or (aDec$_q$) if and only if φ^* satisfies respectively (aDec$_{p'}$) or (aInc$_{q'}$),

(iii) If $\varphi \in \Phi_c(\Omega)$, then

$$\varphi^*\left(x, \tfrac{\varphi(x,t)}{t}\right) \le \varphi(x,t) \quad \text{for every } x \in \Omega \text{ and } t \in \mathbb{R}^+,$$

and $\varphi^{**} = \varphi$,

(iv) If φ satisfies (A0) or (A1), then so does φ^*.

$\square$

Conditions (i), (ii) and (iv) have been proven in [98], respectively in Lemma 2.4.1, Proposition 2.4.9 and Lemmas 3.7.6 and 4.1.7, while condition (iii) has been shown in [79, Corollary 2.6.3], see also [98, Proposition 2.4.5].

Lastly, we report the following result regarding some immersions of generalized Orlicz spaces into Lebesgue spaces.

Lemma 6.2.6. *Let $\Omega \subset \mathbb{R}^N$ be such that $|\Omega| < +\infty$ and let $\varphi \in \Phi_w(\Omega)$. If φ satisfies (A0), then*

$$L^\infty(\Omega) \hookrightarrow L^\varphi(\Omega) \hookrightarrow L^1(\Omega).$$

Moreover, if φ satisfies (aInc)$_p$, then $L^\varphi(\Omega) \hookrightarrow L^p(\Omega)$. $\square$

Note that the second continuous inclusion comes from the fact that every weak Φ-function satisfies $(\mathrm{aInc})_1$, see Remark 1.4.8. This Lemma was shown in [98, Corollary 3.7.9, Corollary 3.7.10]. We point out that condition (A0) is essential both for establishing the above embeddings and to ensure that $L^\varphi(\Omega)$ is a Banach space, cf. [98, Theorem 3.7.13]. In particular, of (A0) is violated, the first inclusion in Lemma 6.2.6 might fail to hold. We refer to [98, Example 3.7.11] for a counterexample.

6.3 The space of bounded deformations fields with generalized Orlicz growth

In this section we recall and extend reslts from previous chapter. In [82, Definition 4.1], given a function $\varphi \in \Phi_w(\Omega)$, an anisotropic total variation (there called *dual norm*) is introduced exploiting the conjugate function of φ, namely for any $u \in L^1(\Omega)$ we have

$$V_\varphi(u) := \sup\left\{\int_\Omega u \operatorname{div} w\, dx \ : \ w \in C_c^1(\Omega;\mathbb{R}^N), \|w\|_{\varphi^*} \le 1\right\}. \tag{6.3.1}$$

Moreover, the space of functions of bounded variation with generalized Orlicz growth is defined as

$$BV^\varphi(\Omega) := \{u \in L^\varphi(\Omega) \ : \ \|u\|_{BV^\varphi(\Omega)} < +\infty\}, \tag{6.3.2}$$

where

$$\|u\|_{BV^\varphi(\Omega)} := \|u\|_\varphi + V_\varphi(u). \tag{6.3.3}$$

Equivalently, given $\varphi \in \Phi_w(\Omega)$ in [41] a vectorial anisotropic total variation is introduced, namely for any $m \in \mathbb{N}^+$ and $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ we have

$$V_\varphi^m(u) := \sup\left\{\sum_{i=1}^m \int_\Omega u_i \operatorname{div} w_i\, dx \ : \ w \in \left[C_c^1(\Omega;\mathbb{R}^N)\right]^m, \|w\|_{\varphi^*} \le 1\right\}. \tag{6.3.4}$$

One can equivalently write

$$V_\varphi^m(u) = \sup\left\{\int_\Omega u \cdot \operatorname{div} w\, dx \ : \ w \in C_c^1(\Omega;\mathbb{R}^{m\times N}), \|w\|_{\varphi^*} \le 1\right\},$$

where the divergence $\operatorname{div} w$ is a vector in $\mathbb{R}^m$, namely taken a matrix-valued field ξ, it is computed row-wise, i.e.

$$(\operatorname{div} \xi)_i := \operatorname{div} \xi_i = \sum_{j=1}^n \frac{\partial \xi_{ij}}{\partial x_j}, \tag{6.3.5}$$

for any $i \in [1, m] \cap \mathbb{N}$ and any $\xi : \Omega \to \mathbb{R}^{m\times N}$. Moreover, we say that u belongs to $BV^{\varphi}(\Omega;\mathbb{R}^m)$ if

$$\|u\|_{BV^{\varphi}(\Omega;\mathbb{R}^m)} := \|u\|_{\varphi} + V_{\varphi}^{m}(u) < +\infty.$$

In the case $m = 1$, (6.3.4) reduces to (6.3.1) and we denote $BV^{\varphi}(\Omega;\mathbb{R})$ simply by $BV^{\varphi}(\Omega)$.

Remark 6.3.1. Clearly, when $\varphi(x,t) = |t|$, $BV^{\varphi}(\Omega;\mathbb{R}^m) = BV(\Omega;\mathbb{R}^m)$. A priori, without assuming any other property on φ except for it to be in the space $\Phi_w(\Omega)$, the spaces $BV(\Omega)$ and $BV^{\varphi}(\Omega)$ are not related to each other. Considering $\Omega \subset \mathbb{R}^N$ bounded, one gets the continuous inclusion $BV^{\varphi}(\Omega) \hookrightarrow BV(\Omega)$ assuming (A0) on the function φ, see Lemma 6.2.6. The fact that φ satisfies (aInc)$_1$ is already included in the definition of weak Φ-function, see Remark 1.4.8.

Remark 6.3.2. In [6] and [120], in the simple anisotropic case, (i.e. $\varphi(x,t)$ with linear behavior in t) the dual function φ^0 is used instead of the conjugate function φ^* in order to define a weighted total variation (they call it *generalized total variation*). But under the assumptions considered in [6] and [120], the two definitions coincide. Indeed, let us assume that $\varphi : \Omega \times [0,+\infty) \to [0,+\infty)$ is a continuous function which is convex in the second variable, satisfies (A0) and such that

$$\varphi(x,t) = t\,\varphi(x,1) \quad \text{for any } x \in \Omega \text{ and } t \in [0,+\infty).$$

Assume that there exist $0 < \lambda \le \Lambda < +\infty$ such that

$$\lambda t \le \varphi(x,t) \le \Lambda t \qquad \text{for any } x \in \Omega \text{ and for any } t \in [0,+\infty).$$

Moreover, let us consider $\psi : \Omega \times \mathbb{R}^N \to [0,+\infty)$ defined as $\psi(x,\xi) := \varphi(x,|\xi|)$ for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Note that ψ is continuous, convex in the second variable, and such that

$$\psi(x,t\xi) = \varphi(x,|t\xi|) = |t|\varphi(x,|\xi|) = |t|\psi(x,\xi)$$

and

$$\lambda|\xi| \le \psi(x,\xi) \le \Lambda|\xi|,$$

for every $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and every $x \in \Omega$.
In [6], considered $u \in BV(\Omega)$ and under these assumptions on ψ, the anisotropic total variation (there called *generalized total variation of u with respect to ψ in Ω*) has been characterized, see in particular [6, Proposition 3.2], as

$$\overline{V_{\psi}}(u) := \sup\left\{\int_{\Omega} u\,\mathrm{div}w\,dx \;:\; w \in C_c^1(\Omega;\mathbb{R}^N), \psi^0(x,w(x)) \le 1 \text{ for all } x \in \Omega\right\}, \tag{6.3.6}$$

where ψ^0 is the dual function of ψ, namely

$$\psi^0(x,\xi^*) := \sup\{(\xi^*,\xi) \ : \ \xi \in \mathbb{R}^n, \psi(x,\xi) \leq 1\}.$$

Note that in [6] they use the notation $C^1_0(\Omega;\mathbb{R}^N)$ to indicate the space that we denote with $C^1_c(\Omega;\mathbb{R}^N)$. Then, given a function $u \in BV(\Omega)$, the anisotropic total variation of u with respect to φ in Ω introduced in [82] as (6.3.1), coincides with the notion given by [6], namely (6.3.6).
Indeed, by [6, (5.10)] the right-hand side of (6.3.6) coincides with

$$\sup\left\{\int_\Omega u \operatorname{div} w \, dx \ : \ w \in C^1_c(\Omega;\mathbb{R}^N), \int_\Omega \psi^*(x, w(x))\, dx < +\infty\right\}. \tag{6.3.7}$$

Considering the even extension of $\varphi(x,\cdot)$, for any $x \in \Omega$, and since $\varphi(x,\cdot)$ is convex, in view of our definition of ψ and Lemma B.1.1, which guarantees that

$$\psi^*(x,\cdot) = \varphi^*(x,|\cdot|) \qquad \text{for a.e. } x \in \Omega, \tag{6.3.8}$$

we have that (6.3.7) is equal to

$$\sup\left\{\int_\Omega u \operatorname{div} w \, dx \ : \ w \in C^1_c(\Omega;\mathbb{R}^N), \int_\Omega \varphi^*(x, |w(x)|)\, dx < +\infty\right\}. \tag{6.3.9}$$

Taking into account [6, (2.18)], namely

$$\psi^*(x,\xi^*) = \begin{cases} 0 & \text{if } \psi^0(x,\xi^*) \leq 1, \\ +\infty & \text{if } \psi^0(x,\xi^*) > 1, \end{cases}$$

for every $x \in \Omega$ and every $\xi^* \in \mathbb{R}^N$, we have that (6.3.9) can be equivalently written as

$$\sup\left\{\int_\Omega u \operatorname{div} w \, dx \ : \ w \in C^1_c(\Omega;\mathbb{R}^N), \int_\Omega \varphi^*(x, |w(x)|)\, dx \leq 1\right\}, \tag{6.3.10}$$

where (6.3.8) has been exploited again. Now, using the definition

$$\|w\|_{\varphi^*} := \inf\left\{\lambda > 0 \ : \ \int_\Omega \varphi^*\left(x, \left|\frac{w(x)}{\lambda}\right|\right) dx < 1\right\},$$

we get that (6.3.10) is equal to (6.3.1).

In order to treat the minimization problem using $TGV^{\varphi,2}_\alpha$ as a regularizer, see (6.1.1), we need to introduce a variant of the generalized variation associated to φ defined in (6.3.1). For every $v \in \mathcal{M}(\Omega;\mathbb{R}^m)$ with $m \in \mathbb{N}^+$, we define

$$\widetilde{V}^m_\varphi(v) := \sup\left\{\int_\Omega \psi \cdot dv(x) \ : \ \psi \in C^1_c(\Omega;\mathbb{R}^m), \|\psi\|_{\varphi^*} \leq 1\right\}. \tag{6.3.11}$$

If we are working on a set $A \subsetneq \Omega$, then we will denote it by $\widetilde{V}^m_\varphi(v;A)$.

Remark 6.3.3. Note that for any $u \in BV(\Omega)$, in order to know that at least that $Du \in \mathcal{M}(\Omega;\mathbb{R}^n)$, an integration by parts yields

$$\widetilde{V}^N_\varphi(Du) = V_\varphi(u),$$

where the second term is defined as in (6.3.1), namely the anisotropic total variation introduced in [82].

In the following, we will either use the above notation for $m = N$ or $m = N \times N$. Note that in the latter case the scalar product coincides with the one underlying the Frobenius norm in $\mathbb{R}^{N\times N}$, namely reading for any $v \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$ as

$$\widetilde{V}^{N\times N}_\varphi(v) = \sup\left\{\int_\Omega \psi : dv(x) \ : \ \psi \in C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}), \|\psi\|_{\varphi^*} \le 1\right\}.$$

Considering test functions in $C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$ or $C^1_c(\Omega;\mathbb{R}^{N\times N})$ is equivalent when $v \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$, since the scalar product only takes into account the symmetric part of the test function. In general the notation $\widetilde{V}^{N\times N}_\varphi$ could be used also for measures in $\mathcal{M}(\Omega;\mathbb{R}^{N\times N})$ without any symmetry property, but that is not the case in this work. We avoid specifying the symmetry in order to lighten the notation.

Remark 6.3.4. Considering an open set $\Omega \subset \mathbb{R}^N$, if v has the special structure $v = Du - w$, with $u, w \in \mathcal{M}(\Omega;\mathbb{R}^N)$, an integration by parts entails

$$\widetilde{V}^n_\varphi(Du - w) = \sup\left\{\int_\Omega u \operatorname{div}\psi\, dx + \int_\Omega \psi \cdot dw(x) \ : \ \psi \in C^1_c(\Omega;\mathbb{R}^N), \|\psi\|_{\varphi^*} \le 1\right\},$$

see also [75]. Note that it is possible to rewrite it like this by [10, Exercise 3.2], indeed having $u \in \mathcal{M}(\Omega;\mathbb{R}^n)$ and $Du = v + w \in \mathcal{M}(\Omega;\mathbb{R}^n)$ implies that $u \in BV(\Omega)$.
In the same way, if Ω is also bounded and C^1, by [146, Chapter II, Theorem 2.3], having $Eu \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$ implies $u \in BD(\Omega)$. Namely, in the special case $v = Eu \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$, (6.3.11) rewrites as

$$\widetilde{V}^{N\times N}_\varphi(Eu) = \sup\left\{\int_\Omega E^*\psi \cdot u\, dx \ : \ \psi \in C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}), \|\psi\|_{\varphi^*} \le 1\right\}, \quad (6.3.12)$$

where $E^*\psi = -\operatorname{div}\psi$, and div denotes the matrix divergence of order 1, computed row-wise (i.e. with respect to columns), cf. (6.3.5).

Remark 6.3.5. We point out that, here and in the following, the set Ω is only required to have a C^1 boundary in order to apply [146, Chapter II, Theorem 2.3]. The extension of the above result to the case of Lipschitz boundary will be the subject of the forthcoming paper [78].

Thanks to the definition of generalized variation associated to φ in (6.3.11), we can now define the space of bounded deformation fields with generalized Orlicz growth, namely

$$BD^{\varphi}(\Omega) := \left\{ w \in L^{\varphi}(\Omega; \mathbb{R}^N) \ : \ \|w\|_{BD^{\varphi}(\Omega)} < +\infty \right\}, \tag{6.3.13}$$

where

$$\|w\|_{BD^{\varphi}(\Omega)} := \|w\|_{\varphi} + \widetilde{V}_{\varphi}^{N\times N}(Ew).$$

Lemma 6.3.6. *Let Ω be a bounded and open set and let $\varphi \in \Phi_w(\Omega)$. Then $\widetilde{V}_{\varphi}^{N\times N}$ is a seminorm.*

Proof. We need to check that $\widetilde{V}_{\varphi}^{N\times N}$ satisfies conditions (N2) and (N3) in Definition 1.4.1. The homogeneity property (N2) is obvious, in fact $\widetilde{V}_{\varphi}^{N\times N}(av) = |a|\widetilde{V}_{\varphi}^{N\times N}(v)$ for every $v \in \mathcal{M}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ and any $a \in \mathbb{R}$. In order to check (N3), namely the triangle inequality, let us consider $v, w \in \mathcal{M}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$. We have

$$\int_{\Omega} \psi : d(v+w) = \int_{\Omega} \psi : dv + \int_{\Omega} \psi : dw \leq \widetilde{V}_{\varphi}^{N\times N}(v) + \widetilde{V}_{\varphi}^{N\times N}(w)$$

for any $\psi \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ such that $\|\psi\|_{\varphi^*} \leq 1$. By taking the supremum over such ψ we have the thesis. $\square$

This result entails that if $\varphi \in \Phi_w(\Omega)$, then $\|\cdot\|_{BD^{\varphi}(\Omega)}$ is a quasinorm. In addition, if φ verifies (A0), or if $\varphi \in \Phi_c(\Omega)$, then $\|\cdot\|_{BD^{\varphi}(\Omega)}$ is a norm, see [98, Lemma 3.2.2, Theorem 3.7.13].

Remark 6.3.7. As for $BV(\Omega)$ and $BV^{\varphi}(\Omega)$, see Remark 6.3.1, without assuming any other property on φ except for it to be in the space $\Phi_w(\Omega)$, the spaces $BD(\Omega)$ and $BD^{\varphi}(\Omega)$ are not related to each other. Considering a set Ω bounded, one gets the continuous immersion $BD^{\varphi}(\Omega) \hookrightarrow BD(\Omega)$ assuming (A0) on the function φ, see Lemma 6.2.6.

Remark 6.3.8. Since a priori we define the space starting from a function $\varphi \in \Phi_w(\Omega)$, one may ask whether the space defined starting from φ^{**}, always a convex Φ-function, see Lemma 6.2.5(i), namely

$$BD^{\varphi^{**}}(\Omega) := \left\{ w \in L^{\varphi^{**}}(\Omega; \mathbb{R}^n) \ : \ \|w\|_{BD^{\varphi^{**}}(\Omega)} < +\infty \right\},$$

with

$$\|w\|_{BD^{\varphi^{**}}(\Omega)} := \|w\|_{\varphi^{**}} + \widetilde{V}_{\varphi^{**}}^{N\times N}(Ew),$$

differs from the space $BD^{\varphi}(\Omega)$. Thanks to [98, Proposition 2.4.5, Proposition 3.2.4] it is possible to prove that $BD^{\varphi}(\Omega) = BD^{\varphi^{**}}(\Omega)$, indeed the norms $\|\cdot\|_{\varphi}$ and $\|\cdot\|_{\varphi^{**}}$ are equivalent and the variations coincide, implying that the norms

$\|\cdot\|_{BD^\varphi(\Omega)}$ and $\|\cdot\|_{BD^{\varphi^{**}}(\Omega)}$ are equivalent as well. Note that some results needed for this consideration, e.g. [98, Lemma 2.4.3] or [98, Proposition 2.4.5], are only stated without the x-dependence, but they hold true even in the generalized Orlicz setting, while others, e.g. [98, Theorem 2.5.10] or [98, Proposition 3.2.4], are already stated with the x-dependence.

With the next example we want to show that, at least considering some Φ-functions for which a Korn-type inequality does not hold, the space $BV^\varphi(\Omega)$ is a proper subset of $BD^\varphi(\Omega)$. The example follows the lines of [8, Example 7.7], exploiting a construction developed in [126], see also [146, Chapter I, Subsection 1.3].

Example 6.3.9. We first remark that (6.3.4), $BV^\varphi(\Omega;\mathbb{R}^n) \subset BD^\varphi(\Omega)$. We show below that, at least in some situations, the inclusion is strict.
Indeed, without loss of generality we can consider an open set $\Omega \supset\supset B$, with B a small ball centered at 0, and a function such that $\varphi(x,t) = |t|$ for a.e. $x \in B$. Then, the same field u provided by [8, Example 7.7], which has compact support in B and is in $BD(\Omega) \setminus BV(\Omega;\mathbb{R}^N)$ provides the desired example of a field in $BD^\varphi(\Omega) \setminus BV^\varphi(\Omega;\mathbb{R}^N)$.
Note that many Φ-functions, e.g. many out of the ones listed in [98, Example 2.5.3], fulfill the assumption of coinciding with $|t|$ on a ball B, meaning that this example holds true in many contexts.

The validity of Korn-type inequalities in the context of Orlicz spaces has been studied in different contexts, for example [56, Theorem 1.1] where it was shown that the ∇_2-condition and the Δ_2-condition are sharp conditions for Korn-type inequalities in Orlicz spaces (for the definition of these two conditions see [135, Subsection 2.3] or [55, Equations (2.6), (2.7)]). In [67, Theorems 3.1, 3.3] the validity of Korn-type inequalities for Young functions was shown, while in [55, Theorems 3.1, 3.3] the equivalence of these Korn-type inequalities to two balance conditions, similar to the ∇_2- and Δ_2-conditions, has been proven. Note that, under the assumptions considered in [67] and [55], their definition of Young functions and our definition of convex Φ-functions, when independent of x, are equivalent. In the context of generalized Orlicz spaces (or Musielak-Orlicz spaces) instead, there are results in the case of variable exponents Lebesgue spaces when the exponent is bounded away from 1, see [80, Section 5] and [79, Theorems 14.3.21, 14.3.23], but not much else is known.

6.3.1 Weighted modular and associated norms

Analogously to [82, Definition 4.1], we define the anisotropic modular, or *dual modular*, associated to the anisotropic total variation $\widetilde{V}_\varphi^{N\times N}$, defined as in (6.3.11).

In particular, we define for any $v \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$

$$\varrho_{\widetilde{V}_\varphi^{N\times N}}(v) := \sup\left\{\int_\Omega \psi : dv(x) - \int_\Omega \varphi^*(x,|\psi|)\,dx \ : \ \psi \in C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})\right\}. \tag{6.3.14}$$

Lemma 6.3.10. *Let Ω be a bounded and open set and let $\varphi \in \Phi_w(\Omega)$. Then $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ is a semimodular.*

Proof. We need to check that $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ satisfies the conditions of Definition 1.4.2. Conditions (i) and (iii) are obvious, respectively thanks to the properties of φ^* and that if ψ is an admissible test function, then also $-\psi$ is. Condition (iv) is also easy to prove using the linearity of the first integral in the definition of $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ and the fact that the supremum of a sum is smaller than the sum of the suprema.
We are only left to prove condition (ii), namely that the function $\lambda \mapsto \varrho_{\widetilde{V}_\varphi^{n\times n}}(\lambda v)$ on $[0,+\infty)$ is non-decreasing for any $v \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$. Indeed, if $\lambda = 0$, then $\varrho_{\widetilde{V}_\varphi^{N\times N}}(\lambda v) = 0$, but we know already that for any other $\mu > \lambda = 0$ then $\varrho_{\widetilde{V}_\varphi^{N\times N}}(\mu v) \geq 0$, since we can always choose $\psi = 0$ as a test function. Let us instead fix $0 < \lambda < \mu < +\infty$. We point out that since both ψ and $-\psi$ are possible test functions, we can also only consider without loss of generality functions in the space

$$\overline{T}^+ := \left\{\psi \in C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}) \ : \ \int_\Omega \psi : dv(x) \geq 0\right\},$$

since the second integral in the definition of the modular is always zero or negative. Then we have

$$\begin{aligned}
\varrho_{\widetilde{V}_\varphi^{N\times N}}(\lambda v) &= \sup\left\{\int_\Omega \lambda\psi : dv(x) - \int_\Omega \varphi^*(x,|\psi|)\,dx \ : \ \psi \in C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})\right\}\\
&= \sup\left\{\lambda\int_\Omega \psi : dv(x) - \int_\Omega \varphi^*(x,|\psi|)\,dx \ : \ \psi \in \overline{T}^+\right\}\\
&\leq \sup\left\{\mu\int_\Omega \psi : dv(x) - \int_\Omega \varphi^*(x,|\psi|)\,dx \ : \ \psi \in \overline{T}^+\right\}\\
&= \sup\left\{\int_\Omega \mu\psi : dv(x) - \int_\Omega \varphi^*(x,|\psi|)\,dx \ : \ \psi \in C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})\right\}\\
&= \varrho_{\widetilde{V}_\varphi^{N\times N}}(\mu v),
\end{aligned}$$

for any $v \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$, proving condition (ii).
Since condition (iv) holds with the constant $\beta = 1$, then $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ is a semimodular. $\square$

Note that, analogously to the anisotropic total variation, see Remark 6.3.4, considering a bounded, C^1 and open set $\Omega \subset \mathbb{R}^N$, in the special case $v = Eu \in \mathcal{M}(\Omega; \mathbb{R}^{N\times N}_{\rm sym})$ this rewrites as

$$\varrho_{\widetilde{V}_\varphi^{N\times N}}(Eu) = \sup \left\{ \int_\Omega [u \cdot E^*\psi - \varphi^*(x, |\psi|)]\, dx \;:\; \psi \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\rm sym}) \right\}, \quad (6.3.15)$$

where $E^*\psi = -\operatorname{div}\psi$.

Remark 6.3.11. On the one hand, note that testing (6.3.15) with $\psi = 0$, we get that the supremum in $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ is always non-negative. On the other hand, if $\varrho_{\varphi^*}(|\psi|) = +\infty$, then the integral in (6.3.15) is equal to $-\infty$, since the first term is finite. Here we used again [146, Chapter II, Theorem 2.3] yielding that $Eu \in \mathcal{M}(\Omega; \mathbb{R}^{N\times N}_{\rm sym})$ implies $u \in BD(\Omega)$.

This means that test functions ψ with $\varrho_{\varphi^*}(|\psi|) = +\infty$ can be omitted and $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ can be equivalently rewritten as

$$\varrho_{\widetilde{V}_\varphi^{N\times N}}(Eu) := \sup \left\{ \int_\Omega [u \cdot E^*\psi - \varphi^*(x, |\psi|)]\, dx : \psi \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\rm sym}),\, \varrho_{\varphi^*}(|\psi|) < +\infty \right\}.$$

The analogous reasoning can be repeated for the general expression of $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ in (6.3.14).

In Lemma 6.3.10 we showed that $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ is a semimodular, implying that, thanks to the Luxemburg method, we obtain a seminorm $\|\cdot\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}}$. The following result relates the anisotropic variation we introduced in (6.3.11) with the seminorm associated to the modular defined in (6.3.14), similarly to [82, Lemma 4.7].

Lemma 6.3.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, C^1 and open set. Let $\varphi \in \Phi_w(\Omega)$ and let $u \in BD^\varphi(\Omega)$. Then*

$$\|Eu\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}} \leq \widetilde{V}_\varphi^{N\times N}(Eu) \leq 2\, \|Eu\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}}.$$

Proof. Since $u \in BD^\varphi(\Omega)$, the case $\widetilde{V}_\varphi^{N\times N}(Eu) = +\infty$ is excluded a priori. Moreover, if $\widetilde{V}_\varphi^{N\times N}(Eu) = 0$, then $\varrho_{\widetilde{V}_\varphi^{N\times N}}(\frac{Eu}{\lambda}) = 0$ for every $\lambda > 0$, yielding $\|Eu\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}} = 0$. Lastly, since the claim is homogeneous, the case $\widetilde{V}_\varphi^{N\times N}(Eu) \in (0, +\infty)$ can always be reduced to $\widetilde{V}_\varphi^{N\times N}(Eu) = 1$.
By the definition of $\widetilde{V}_\varphi^{N\times N}$, see (6.3.12), for any $w \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\rm sym})$ such that $\|w\|_{\varphi^*} \leq 1$ it follows that

$$\int_\Omega u \cdot E^* w\, dx \leq \widetilde{V}_\varphi^{N\times N}(Eu)\, \|w\|_{\varphi^*} = \|w\|_{\varphi^*} \leq 1 + \varrho_{\varphi^*}(|w|),$$

where the last inequality comes from a general property of the Luxemburg norms, see [79, Corollary 2.1.15]. Thus, by the definition of $\varrho_{\widetilde{V}_\varphi^{N\times N}}$ given in (6.3.14),

$$\varrho_{\widetilde{V}_\varphi^{N\times N}}(Eu) \le \sup_{w\in C_c^1(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})} [1 + \varrho_{\varphi^*}(|w|) - \varrho_{\varphi^*}(|w|)] = 1,$$

implying $\|Eu\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}} \le 1$. This concludes the proof of the first inequality.
We next establish the opposite inequality $2\,\|Eu\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}} \ge 1$, which is equivalent to

$$\varrho_{\widetilde{V}_\varphi^{n\times n}}\left(\frac{2Eu}{\lambda}\right) > 1 \qquad \text{for every } \lambda < 1. \tag{6.3.16}$$

Indeed, on the one hand if $\|v\|_{\varrho_{\widetilde{V}_\varphi^{n\times n}}} \ge 1$, by Definition 1.4.3 we have that

$$\inf\left\{k > 0 : \varrho_{\widetilde{V}_\varphi^{n\times n}}\left(\frac{v}{k}\right) \le 1\right\} \ge 1,$$

entailing that for any $\lambda < 1$ it holds $\|v\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}} > \lambda$. By [98, Lemma 3.2.3] then this implies that $\varrho_{\widetilde{V}_\varphi^{N\times N}}(\frac{v}{\lambda}) > 1$ for any $\lambda < 1$. On the other hand, having $\varrho_{\widetilde{V}_\varphi^{N\times N}}(\frac{v}{\lambda}) > 1$ for any $\lambda < 1$ entails that, by Definition 1.4.3,

$$\|v\|_{\varrho_{\widetilde{V}_\varphi^{N\times N}}} = \inf\left\{\lambda > 0 \ : \ \varrho_{\widetilde{V}_\varphi^{N\times N}}\left(\frac{v}{\lambda}\right) \le 1\right\} \ge 1$$

directly from the definition of inf. Now, by [98, Lemma 3.2.3] and thanks to the fact that $\varphi^* \in \Phi_c(\Omega)$, see Lemma 6.2.5(iii), $\varrho_{\varphi^*}(|w|) \le 1$ if and only if $\|w\|_{\varphi^*} \le 1$, and we can immediately conclude that

$$\begin{aligned}\varrho_{\widetilde{V}_\varphi^{N\times N}}\left(\frac{2Eu}{\lambda}\right) &\ge \sup\left\{\int_\Omega \left[\frac{2u}{\lambda}\cdot E^*w - \varphi^*(x,|w|)\right] dx : w\in C_c^1(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}), \|w\|_{\varphi^*} \le 1\right\}\\ &\ge \frac{2}{\lambda}\,\widetilde{V}_\varphi^{N\times N}(Eu) - 1 > 1.\end{aligned}$$

This proves (6.3.16) and yields the thesis. □

The following result, instead, relates the seminorm $\widetilde{V}_\varphi^{N\times N}$, see Lemma 6.3.6, with the norm of the symmetrized gradient, in particular considering the norm associated to the associate space $(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'$. Obviously, in the associate space norm we test with functions in $L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N})$, while in $\widetilde{V}_\varphi^{N\times N}$ the test functions are smooth. This means that some approximation is needed, but a priori we are not able to use density in $L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N})$ since φ^* is in general not doubling.
In order to state the result, we need the classical decomposition of the symmetrized gradient, see for instance [35],

$$Eu = \mathcal{E}u\,\mathcal{L}^N + E^s u.$$

Moreover, let us remark the definition of the space $LD_{\mathrm{loc}}(\Omega)$, namely

$$LD_{\mathrm{loc}}(\Omega) := \{u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^N) \ : \ Eu \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})\}.$$

The result, analogous to [82, Theorem 5.2], reads as follows.

Lemma 6.3.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected and open set. Let $\varphi \in \Phi_w(\Omega)$ and let $u \in LD_{\mathrm{loc}}(\Omega)$. Then*

$$\widetilde{V}^{N\times N}_{\varphi}(Eu) \leq \|\mathcal{E}u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'}.$$

Moreover, if additionally $C^1_c(\Omega; \mathbb{R}^{N\times N})$ is dense in $L^{\varphi^*}(\Omega; \mathbb{R}^{N\times N})$, then

$$\widetilde{V}^{N\times N}_{\varphi}(Eu) = \|\mathcal{E}u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'}.$$

Proof. Since $u \in LD_{\mathrm{loc}}(\Omega)$, it follows from the definition of $\widetilde{V}^{N\times N}_{\varphi}$ that

$$\widetilde{V}^{N\times N}_{\varphi}(Eu) = \sup\left\{\int_\Omega \mathcal{E}u : w\, dx \ : \ w \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}), \|w\|_{\varphi^*} \leq 1\right\}. \qquad (6.3.17)$$

The definition of the associate space norm (extended in a standard way to the vector valued setting), see Definition 6.2.4, implies that, chosen $w \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ such that $\|w\|_{\varphi^*} \leq 1$,

$$\int_\Omega \mathcal{E}u : w\, dx \leq \int_\Omega |\mathcal{E}u||w|\, dx \leq \|\mathcal{E}u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'}\|w\|_{\varphi^*}.$$

Taking the supremum over those w, we conclude that

$$\widetilde{V}^{N\times N}_{\varphi}(\mathcal{E}u) \leq \|\mathcal{E}u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'}.$$

Under the density assumption, namely $C^1_c(\Omega; \mathbb{R}^{N\times N})$ being dense in $L^{\varphi^*}(\Omega; \mathbb{R}^{N\times N})$, we next show the opposite inequality, namely $\|\mathcal{E}u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'} \leq \widetilde{V}^{N\times N}_{\varphi}(\mathcal{E}u)$. Indeed, let $w \in L^{\varphi^*}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ with $\|w\|_{\varphi^*} = 1$ and let $(w_j)_j \subset C^1_c(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ be a sequence such that $w_j \to w$ in $L^{\varphi^*}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ and pointwise almost everywhere. Since we know that also $w_j/\|w_j\|_{\varphi^*} \to w$ in $L^{\varphi^*}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$, we may assume that $\|w_j\|_{\varphi^*} = 1$ for every $j \in \mathbb{N}$. By Fatou's Lemma, we have

$$\liminf_{j\to\infty} \int_\Omega \mathcal{E}u : w_j\, dx \geq \int_\Omega \mathcal{E}u : w\, dx.$$

So, from (6.3.17) follows that

$$\widetilde{V}^{N\times N}_{\varphi}(\mathcal{E}u) \geq \sup\left\{\int_\Omega \mathcal{E}u : w\, dx \ : \ w \in L^{\varphi^*}(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}), \|w\|_{\varphi^*} \leq 1\right\}.$$

Let $h \in L^{\varphi^*}(\Omega)$. We can set

$$w := \begin{cases} \dfrac{\mathcal{E}u}{|\mathcal{E}u|}\, h & \text{if } |\mathcal{E}u| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$\widetilde{V}_{\varphi}^{n\times n}(\mathcal{E}u) \geq \sup\left\{ \int_{\Omega} |\mathcal{E}u|\, h\, dx \ :\ h \in L^{\varphi^*}(\Omega), \|h\|_{\varphi^*} \leq 1 \right\} = \|\mathcal{E}u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'}.$$

Hence $\widetilde{V}_{\varphi}^{N\times N}(\mathcal{E}u) = \|\mathcal{E}u\|_{(L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N}))'}$. This concludes the proof. □

In the second part of Lemma 6.3.13 we assume that $C^1_c(\Omega;\mathbb{R}^{N\times N})$ is dense in $L^{\varphi^*}(\Omega;\mathbb{R}^{N\times N})$. If φ^* satisfies (A0) and (aDec), then this holds automatically by [98, Theorem 3.7.15] (note that this theorem is stated for scalar functions, but the vectorial case can be deduced working components-wise thanks to the equivalence of euclidean norms in spaces of finite dimension). We know already that, by Lemma 6.2.5(ii), (iv), φ^* satisfies these conditions if and only if φ satisfies (A0) and (aInc), meaning that it is not clear whether this density is there or not in the subsets where the function φ is linear.

6.3.2 Explicit expression of the anisotropic modular

In this subsection we derive an explicit formula for the dual modular $\varrho_{\widetilde{V}_{\varphi}^{N\times N}}$ in terms of the modular ϱ_{φ} of the absolutely continuous part of the symmetric gradient and its singular part with weight given by the recession function, see Definition 6.2.2. In particular, analogously to [82, Theorem 6.4], the main result of the section reads as follows.

Theorem 6.3.14. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected and open set. Let $\varphi \in \Phi_c(\Omega) \cap C(\Omega \times [0,+\infty))$ satisfy (A0), (aDec) and restricted (VA1). If $u \in BD(\Omega)$, then*

$$\varrho_{\widetilde{V}_{\varphi}^{N\times N}}(Eu) = \varrho_{\varphi}(|\mathcal{E}u|) + \int_{\Omega} \varphi^{\infty}\, d|E^s u|.$$

□

The following result is an immediate consequence of Theorem 6.3.14, when the Φ-function φ does not depend on x, i.e. when it is a classical Orlicz Φ-function.

Corollary 6.3.15. *Let $\varphi \in \Phi_c$ be independent of x and satisfy (aDec). If $u \in BD(\Omega)$, then*

$$\varrho_{\widetilde{V}_{\varphi}^{N\times N}}(Eu) = \varrho_{\varphi}(|\mathcal{E}u|) + \varphi^{\infty}|E^s u|$$

in any Borel subset of Ω and so

(1) $BD^{\varphi}(\Omega) = BD(\Omega)$ *if* $\varphi^{\infty} < \infty$,

(2) $BD^{\varphi}(\Omega) = LD^{\varphi}(\Omega)$ *if* $\varphi^{\infty} = \infty$.

□

We observe that in (2) we cannot say that $BD^{\varphi}(\Omega) = W^{1,\varphi}(\Omega)$, see e.g. [34, Definition 3], since the latter equality would follow from a Korn's inequality. But, as proven in [56, Theorem 1.1], the latter, for classical convex Φ-functions holds if and only if φ satisfies both ∇_2 and Δ_2 conditions. Now, recall first that $\varphi \in \Phi_c$ implies that $\varphi = \varphi^{**}$ and that the ∇_2 condition for φ is equivalent, by [79, Definition 2.4.10], to φ^* satisfying Δ_2. Furthermore, [79, Corollary 2.4.11] states that $\varphi \in \Phi_w$ satisfies ∇_2 if and only if it satisfies $(aInc)$. By assumption φ satisfies $(aDec)$ but not $(aInc)$, thus φ satisfies Δ_2 but not ∇_2. As a particular case, one can consider $\varphi(t) = t\log(1+t)$, see [79, Remark 1.3].

In order to prove Theorem 6.3.14 we need the following lemma, proven in [82, Lemma 6.1], which shows the importance of the recession function φ^{∞}, see Definition 6.2.2.

Lemma 6.3.16. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded, connected and open set. Let* $\varphi \in \Phi_c(\Omega)$ *satisfy restricted (VA1). If* $\psi \in C(\Omega)$ *with* $\varrho_{\varphi^*}(\psi) < +\infty$*, then* $|\psi| \le \varphi^{\infty}$.

In [100, Section 3 and Example A.1] it was shown that log-Hölder continuity is not sufficient when working in $BV^{p(\cdot)}$ and here, similarly, the (A1) condition (corresponding to log-Hölder continuity in the general case of convex Φ-functions) is not sufficient in the previous result, as in the following ones, in view of [82, Example 4.3]. That is why we make use of the restricted (VA1) condition, which corresponds to strong log-Hölder continuity in the case of the variable exponent function, see [82, Proposition 3.2].
In the next two results we consider the singular and absolutely continuous parts of the symmetric gradient separately. Then we combine them to handle the whole function in Theorem 6.3.14. Since the proofs are similar to the ones of [82, Propositions 6.2, 6.3], they can be found in our setting in Appendix A.1.

Proposition 6.3.17. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded, connected and open set. Let* $\varphi \in \Phi_c(\Omega)$ *satisfy (A0), (aDec) and restricted (VA1). If* $u \in BD(\Omega)$*, then*

$$\sup_{\psi \in T^{\varphi}} \int_{\Omega} \psi : dE^s u = \int_{\Omega} \varphi^{\infty} \, d|E^s u|,$$

where

$$T^{\varphi} := \left\{ \psi \in C_c^1(\Omega; \mathbb{R}^{N\times N}_{\text{sym}}) \;:\; \varrho_{\varphi^*}(|\psi|) < +\infty \right\}. \tag{6.3.18}$$

□

Proposition 6.3.18. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected and open set. Let $\varphi \in \Phi_c(\Omega) \cap C(\Omega \times [0, +\infty))$ satisfy (A0) and (aDec). If $u \in BD(\Omega)$, then*

$$\sup_{\psi \in T^\varphi} \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)]\, dx = \varrho_\varphi(|\mathcal{E}u|).$$

□

Note that removing the assumption that φ is continuous in both variables is not trivial.
Having the two previous propositions at hands, we can now prove Theorem 6.3.14, exploiting the ideas used in [82, Theorem 6.4].

Proof of Theorem 6.3.14. Since $Eu = \mathcal{E}u + E^s u$ and since $u \in BD(\Omega)$, an integration by parts implies that

$$\int_\Omega u \cdot E^* \psi \, dx = \int_\Omega \psi : dEu = \int_\Omega \mathcal{E}u : \psi \, dx + \int_\Omega \psi : dE^s u$$

for any $\psi \in T^\varphi$. Hence, the claim follows from Propositions 6.3.17 and 6.3.18 once we prove that

$$\begin{aligned} &\sup_{\psi \in T^\varphi} \left\{ \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)]\, dx + \int_\Omega \psi : dE^s u \right\} \\ &= \sup_{\psi \in T^\varphi} \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)]\, dx + \sup_{\psi \in T^\varphi} \int_\Omega \psi : dE^s u. \end{aligned}$$

The first inequality, namely "$\leq$", is obvious, so we only need to prove the opposite one.
Let us first assume that $\varrho_\varphi(|\mathcal{E}u|) + \int_\Omega \varphi^\infty \, d|E^s u| < +\infty$ and let us fix $\varepsilon > 0$. We can always choose $\psi_1, \psi_2 \in T^\varphi$ such that

$$\sup_{\psi \in T^\varphi} \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)]\, dx \leq \int_\Omega [\mathcal{E}u : \psi_1 - \varphi^*(x, |\psi_1|)]\, dx + \varepsilon < +\infty \quad (6.3.19)$$

and

$$\sup_{\psi \in T^\varphi} \int_\Omega \psi : dE^s u \leq \int_\Omega \psi_2 : dE^s u + \varepsilon < +\infty. \quad (6.3.20)$$

Since $u \in BD(\Omega)$ and $\psi_1, \psi_2 \in T^\varphi$, we have $|\mathcal{E}u|\,|\psi_i| \in L^1(\Omega)$ and $\varrho_{\varphi^*}(|\psi_i|) < +\infty$ for $i \in \{1, 2\}$. Thus, by the absolute continuity of the integral, we can find $\delta > 0$ such that

$$\left| \int_{\Omega \setminus \Omega_1} [\mathcal{E}u : \psi_i - \varphi^*(x, |\psi_i|)]\, dx \right| \leq \int_{\Omega \setminus \Omega_1} [|\mathcal{E}u|\,|\psi_i| + \varphi^*(x, |\psi_i|)]\, dx \leq \varepsilon \quad (6.3.21)$$

for $i \in \{1,2\}$ and any $\Omega_1 \subset \Omega$ with $|\Omega \setminus \Omega_1| < \delta$ and

$$\left| \int_{\Omega\setminus\Omega_2} \psi_2 : dE^s u \right| \le \int_{\Omega\setminus\Omega_2} \varphi^\infty \, d|E^s u| \le \varepsilon$$

for any $\Omega_2 \subset \Omega$ with $|E^s u|(\Omega \setminus \Omega_2) < \delta$.
Since $\operatorname{supp} E^s u$ is a set with zero Lebesgue measure, we can find a finite collection of open rectangles $Q_i \subset \Omega$ such that $|E^s u|(\bigcup_{i\in\mathbb{N}} Q_i) > |E^s u|(\Omega) - \delta$ and $|\bigcup_{i\in\mathbb{N}} 2Q_i| < \delta$. Let us now choose $\theta \in C^1_c(\Omega)$ such that $0 \le \theta \le 1$, $\theta = 1$ in $\Omega_2 := \bigcup_{i\in\mathbb{N}} Q_i$ and $\theta = 0$ in $\Omega_1 := \Omega \setminus \bigcup_{i\in\mathbb{N}} 2Q_i$, and let us fix $\psi_\varepsilon := \theta\,\psi_2 + (1-\theta)\psi_1 \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\text{sym}})$. Since ψ_ε is a pointwise convex combination, we get

$$\varphi^*(\cdot, |\psi_\varepsilon|) \le \varphi^*(\cdot, \max\{|\psi_2|, |\psi_1|\}) \le \varphi^*(\cdot, |\psi_2|) + \varphi^*(\cdot, |\psi_1|).$$

This yields that $\varrho_{\varphi^*}(|\psi_\varepsilon|) \le \varrho_{\varphi^*}(|\psi_2|) + \varrho_{\varphi^*}(|\psi_1|) < +\infty$, yielding $\psi_\varepsilon \in T^\varphi$. We then get that

$$\begin{aligned}
\varrho_{\widetilde{V}^{N\times N}_\varphi}(Eu) &= \sup_{\psi\in T^\varphi} \left\{ \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)] \, dx + \int_\Omega \psi : dE^s u \right\} \\
&\ge \int_\Omega [\mathcal{E}u : \psi_\varepsilon - \varphi^*(x, |\psi_\varepsilon|)] \, dx + \int_\Omega \psi_\varepsilon : dE^s u \\
&\ge \int_{\Omega_1} [\mathcal{E}u : \psi_1 - \varphi^*(x, |\psi_1|)] \, dx + \int_{\Omega_2} \psi_2 : dE^s u - c_\theta \\
&\ge \int_\Omega [\mathcal{E}u : \psi_1 - \varphi^*(x, |\psi_1|)] \, dx + \int_\Omega \psi_2 : dE^s u - 5\varepsilon,
\end{aligned}$$

where, thanks to (6.3.21),

$$c_\theta := \int_{\Omega\setminus\Omega_1} [|\mathcal{E}u| \, |\psi_\varepsilon| + \varphi^*(x, |\psi_\varepsilon|)] \, dx + \int_{\Omega\setminus\Omega_2} |\psi_\varepsilon| \, d|E^s u| \le 3\varepsilon$$

by the absolute integrability assumptions and by Lemma 6.3.16. By the choices of ψ_1 and ψ_2, namely (6.3.19) and (6.3.20), we then get

$$\varrho_{\widetilde{V}^{N\times N}_\varphi}(Eu) \ge \sup_{\psi\in T^\varphi} \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)] \, dx + \sup_{\psi\in T^\varphi} \int_\Omega \psi : dE^s u - 7\varepsilon.$$

The lower bound follows by letting $\varepsilon \to 0^+$. This concludes the proof in the case $\varrho_\varphi(|\mathcal{E}u|) + \int_\Omega \varphi^\infty \, d|E^s u| < +\infty$.
If, instead, $\varrho_\varphi(|\mathcal{E}u|) = +\infty$ and $\int_\Omega \varphi^\infty \, d|E^s u| < +\infty$, we simply estimate

$$\begin{aligned}
&\sup_{\psi\in T^\varphi} \left\{ \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)] \, dx + \int_\Omega \psi : dE^s u \right\} \\
&\ge \sup_{\psi\in T^\varphi} \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)] \, dx - \sup_{\psi\in T^\varphi} \int_\Omega \psi : dE^s u \\
&= \varrho_\varphi(|\mathcal{E}u|) - \int_\Omega \varphi^\infty \, d|E^s u| = +\infty.
\end{aligned}$$

We are only left to consider the case $\int_\Omega \varphi^\infty \, d|E^s u| = +\infty$. By the proof of Proposition 6.3.17, see Appendix A.1, there exists $\psi_\varepsilon \in C^1_c(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ such that

$$\int_\Omega \psi_\varepsilon : dE^s u > \frac{1}{\varepsilon}$$

and $|\psi_\varepsilon| \le \frac{\varphi(\cdot,k)}{k}$ for some $k = k_\varepsilon$. For any function $\theta : \Omega \to [0,1]$, we have that

$$\mathcal{E}u : (\theta\psi_\varepsilon) - \varphi^*(\cdot, |\theta\psi_\varepsilon|) \ge \mathcal{E}u : (\theta\psi_\varepsilon) - \varphi^*\left(\cdot, \frac{\varphi(\cdot,k)}{k}\right) \ge -\left(\frac{\varphi^+(k)}{k} |\mathcal{E}u| + \varphi^+(k)\right),$$

see Lemma 6.2.5(iii). Since the function on the right-hand side is integrable, we can choose $\delta_k > 0$ such that its integral over any measurable A with $|A| < \delta_k$ is at least -1. Furthermore, since $\operatorname{supp} E^s u$ has measure zero, we can choose $\theta \in C^\infty_c(\Omega)$ as before to have support with Lebesgue measure at most δ_k and such that satisfies

$$\int_\Omega (\theta\psi_\varepsilon) : dE^s u > \frac{1}{2} \int_\Omega \psi_\varepsilon : dE^s u > \frac{1}{2\varepsilon}.$$

Then,

$$\begin{aligned}
&\sup_{\psi \in T^\varphi} \left\{ \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x, |\psi|)] \, dx + \int_\Omega \psi : dE^s u \right\} \\
&\ge \int_\Omega (\theta\psi_\varepsilon) : dE^s u + \int_\Omega [\mathcal{E}u : (\theta\psi_\varepsilon) - \varphi^*(x, |\theta\psi_\varepsilon|)] \, dx \ge \frac{1}{2\varepsilon} - 1.
\end{aligned}$$

Letting $\varepsilon \to +\infty$, the claim follows. This concludes the proof of the theorem. □

6.3.3 The variable exponent case

In this section we focus on the special Φ-function

$$\varphi(x,t) := \frac{1}{p(x)} t^{p(x)}, \tag{6.3.22}$$

with $p : \Omega \to [1, +\infty)$ a measurable function. Let us recall that, choosing φ as in (6.3.22) and considering the set

$$Y := \{x \in \Omega \ : \ p(x) = 1\}, \tag{6.3.23}$$

we have that the conjugate φ^* is of the form

$$\varphi^*(x,t) = \begin{cases} \frac{1}{q(x)} t^{q(x)} & \text{if } x \in \Omega \setminus Y, \\ \infty \chi_{(1,\infty)}(t) & \text{if } x \in Y, \end{cases} \tag{6.3.24}$$

see [82, Example 4.3], where q is the function defined pointwise in $\Omega \setminus Y$ by $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ and with the convention that $\infty \cdot 0 = 0$. From now, given a set $A \subset \Omega$, we write

$$p_A^+ := \operatorname*{ess-sup}_{x \in A} p(x) \qquad \text{and} \qquad p_A^- := \operatorname*{ess-inf}_{x \in A} p(x),$$

respectively $q_A^\pm$ for the conjugate exponent function q. In the following, we will assume $p_\Omega^+ < +\infty$.

Remark 6.3.19. The explicit definition of the function φ in (6.3.22) entails some additional properties other than the explicit form of its conjugate function given by (6.3.24). Indeed, it is immediate to see that $\varphi \in \Phi_c(\Omega)$, see Definition 1.4.5 and [98, Section 7.1]. Moreover, arguing as in [98, Lemma 7.1.1], it is easy to see that φ satisfies (A0), $(\mathrm{aInc})_{p_\Omega^-}$ and $(\mathrm{aDec})_{p_\Omega^+}$, see Definition 6.2.1.

The first result we prove in this subsection is the equivalent of Theorem 6.3.14 in this specific setting. In particular, we show that the same conclusion of Theorem 6.3.14 holds under less restrictive hypotheses on our function φ. Before doing so, we need to recall the definition of the modular associated to the variable exponent space when $p_\Omega^+ < +\infty$, given in an equivalent way to the one in [100, Subsection 2.2], namely

$$\varrho_{L^{p(\cdot)}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})}(u) := \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)}\, dx$$

for any $u \in \mathcal{M}(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$. Now, the theorem reads as follows.

Theorem 6.3.20. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected and open set. Let φ be defined as in* (6.3.22) *and let $p : \Omega \to [1, +\infty)$ be a lower semicontinuos function with $p_\Omega^+ < +\infty$ and such that Y is closed, see* (6.3.23)*. If $u \in BD(\Omega)$, then*

$$\varrho_{\widetilde{V}_\varphi^{N\times N}}(Eu) = \varrho_{\mathrm{old}}(Eu) := |Eu|(Y) + \varrho_{L^{p(\cdot)}(\Omega\setminus Y;\mathbb{R}^{N\times N}_{\mathrm{sym}})}(\mathcal{E}u). \tag{6.3.25}$$

Proof. First, let us observe that the claim of Lemma 6.3.16 is automatically satisfied in the case of φ given by (6.3.22) with no continuity required on p, but only under the assumption that p_Ω^+ is finite. Indeed, on the one hand, in view of (6.3.24) and Definition 6.2.2 (which is a limit in this specific case), it suffices to prove the inequality in the set Y given by (6.3.23), since in $\Omega \setminus Y$ the recession function is $+\infty$. On the other hand though, whenever $\psi \in C(\Omega)$ with $\varrho_{\varphi^*}(|\psi|) < +\infty$, then by (6.3.24) we have that $\varphi^*(x, |\psi|) = 0$ for almost every $x \in Y$, implying that $|\psi| \le 1$ almost everywhere in Y. This means that $|\psi(x)| \le 1 = \varphi^\infty(x)$ in Y, which proves our claim.

As a second step, let us remark that choosing φ as in (6.3.22), a careful inspection of the proof of Proposition 6.3.18 guarantees that the result is still true if we remove the assumptions therein and, again, we only require that $p_\Omega^+ < +\infty$. In fact, by Remark 6.3.19, we know that (A0) and (aDec) are satisfied and this implies that we can apply [98, Theorem 3.7.15], meaning the required continuity on φ can be bypassed exploiting the density of $C_c^\infty(\Omega)$ in $L^\varphi(\Omega)$. Hence, we can conclude that

$$\sup_{\psi\in T^\varphi}\int_\Omega [\mathcal{E}u:\psi-\varphi^*(x,|\psi|)]\,dx=\frac{1}{p(x)}\int_\Omega |\mathcal{E}u|^{p(x)}\,dx$$

for any $u\in BD(\Omega)$, where T^φ is defined as in (6.3.18).
We are left to prove that Proposition 6.3.17 holds true under these hypotheses and choosing φ as in (6.3.22), namely that

$$\sup_{\psi\in T^\varphi}\int_\Omega \psi : dE^s u=\int_\Omega \varphi^\infty\, d|E^s u|.$$

Note that the inequality "$\leq$" is immediately verified thanks to the fact that Lemma 6.3.16 holds under our assumptions. So it suffices to prove that

$$\int_\Omega \varphi^\infty d|E^s u|\leq \sup_{\psi\in T^\varphi}\int_\Omega \psi : dE^s u.$$

Thanks to the lower semicontinuity assumption on p on Ω, we obtain that (A.1.1) and (A.1.2) hold. In particular, the second one holds for any $\beta\in[0,1]$, due to the non decreasing monotonicity of $(\cdot)^{p(x)}$. Observe that h_k is lower semicontinuous by definition (it is a lower semicontinuous envelope, see [73, Chapter 1]) and increasing in k. Consequently the same arguments as in the proof of Proposition 6.3.17 lead to the desired conclusion. Moreover, let us remark that for every $u\in BD(\Omega)$ it holds

$$\sup_{\psi\in T^\varphi}\int_\Omega \psi : dE^s u=\int_Y d|E^s u|=|E^s u|(Y),$$

since $\varphi^\infty(x)=1$ in Y as observed before, which indeed says that $E^s u$ is concentrated on Y.
Finally, following the same steps of the proof of Theorem 6.3.14, since under our hypotheses the equivalent formulations of Lemma 6.3.16, Proposition 6.3.17 and Proposition 6.3.18 hold, we get the thesis. □

Remark 6.3.21. As a corollary of Theorem 6.3.20, under the same hypotheses we get the coincidence of the spaces BD^φ and $BD\cap LD^{p(\cdot)}$, with

$$LD^{p(\cdot)}(U):=\{u\in L^{p(\cdot)}(U;\mathbb{R}^n)\ :\ Eu\in L^{p(\cdot)}(U;\mathbb{R}^{N\times N}_{\mathrm{sym}})\},$$

when φ is as in (6.3.22). Alternatively, this latter equality can be proven directly arguing as in [41, Theorem 3.8].

6.4 The imaging problem

In this section we finally deal with the novel regularizer based on a Musielak-Orlicz generalization of classical Total Generalized Variation functionals that we defined in (6.1.1), namely the anisotropic Total Generalized Variation of order 2, with weight $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, given by

$$TGV_\alpha^{\varphi,2}(u) := \sup\left\{ \int_\Omega u \,\mathrm{div}^2\psi \, dx \ : \right. \\ \left. \psi \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}), \|\psi\|_{\varphi^*} \le \alpha_1, \|\mathrm{div}\,\psi\|_{\varphi^*} \le \alpha_2 \right\},$$

where we consider $u \in L^1_{\mathrm{loc}}(\Omega)$.
In Subsection 6.4.1 we prove some properties of this functional, while in Subsection 6.4.2 we give a dual characterization of this functional exploiting the space BD^φ we defined in Section 6.3. Finally, in Subsection 6.4.3 we show how the anisotropic Total Generalized Variation inherits classical existence and stability properties.

6.4.1 Basic properties of $TGV_\alpha^{\varphi,2}(u)$

Following the lines of [53], we define the space of functions of bounded variation of order 2 with Orlicz growth and weight α as

$$BGV_\alpha^{\varphi,2}(\Omega) := \{u \in L^\varphi(\Omega) \ : \ TGV_\alpha^{\varphi,2}(u) < +\infty\}, \tag{6.4.1}$$

endowed with the norm

$$\|u\|_{BGV_\alpha^{\varphi,2}} := \|u\|_\varphi + TGV_\alpha^{\varphi,2}(u),$$

where the functional $TGV_\alpha^{\varphi,2}$ is defined as in (6.1.1).

Remark 6.4.1. Note that it would also be possible to define the functionals $TGV_\alpha^{\varphi,k}$ and the spaces $BGV_\alpha^{\varphi,k}(\Omega)$ for general $k \in \mathbb{N}^+$ following the lines of [53]. Moreover, for $k = 1$ and $\alpha \in \mathbb{R}^+$, and chosen $u \in L^1_{\mathrm{loc}}(\Omega)$, we would have

$$TGV_\alpha^{\varphi,1}(u) = \sup\left\{ \int_\Omega u \,\mathrm{div}\psi \, dx \ : \ \psi \in C_c^1(\Omega; \mathbb{R}^N), \|\psi\|_{\varphi^*} \le \alpha \right\} = \alpha V_\varphi(u),$$

where the right-hand side is defined as in (6.3.1). Hence, we can indeed speak of a generalization of an anisotropic total variation. Contrary to Remark 6.3.3, which was true for $u \in BV(\Omega)$ in order to know at least that $Du \in \mathcal{M}(\Omega; \mathbb{R}^N)$, here it is enough to take $u \in L^1_{\mathrm{loc}}(\Omega)$.

In the following result we prove some basic properties of the anisotropic Total Generalized Variation.

Proposition 6.4.2. *Let Ω be a bounded, connected and open set in $\mathbb{R}^N$, with $N \geq 2$. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0) and let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then, the following statements hold:*

(i) $TGV_\alpha^{\varphi,2}$ is a seminorm on the normed space $BGV_\alpha^{\varphi,2}(\Omega)$.

(ii) For $u \in L^1_{\mathrm{loc}}(\Omega)$, there holds $TGV_\alpha^{\varphi,2}(u) = 0$ if and only if u is a polynomial of degree 0 or 1.

(iii) For eevery $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, the seminorms $TGV_\alpha^{\varphi,2}$ and $TGV_{\tilde{\alpha}}^{\varphi,2}$ are equivalent.

(iv) $TGV_\alpha^{\varphi,2}$ is rotationally invariant, namely for any matrix $R \in SO(N)$ and $u \in BGV_\alpha^{\varphi,2}(\Omega)$ we can define $\tilde{u}(x) := u(Rx)$ and we have that $\tilde{u} \in BGV_\alpha^{\varphi,2}(R^{\mathrm{T}}\Omega)$ and $TGV_\alpha^{\varphi,2}(\tilde{u}) = TGV_\alpha^{\varphi,2}(u)$.

(v) For $r > 0$ and $u \in BGV_\alpha^{\varphi,2}(\Omega)$, define $\bar{u}(x) := u(rx)$. Then $\bar{u} \in BGV_\alpha^{\varphi,2}(r^{-1}\Omega)$ and

$$TGV_\alpha^{\varphi,2}(\bar{u}) = r^{-N}\, TGV_{\bar{\alpha}}^{\varphi,2}(u) \qquad \text{with } \bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) := (\alpha_1 r^2, \alpha_2 r).$$

Proof. (i). Thanks to the definition of the functional $TGV_\alpha^{\varphi,2}$ given in (6.1.1), it is obvious to see that the two conditions (N2) and (N3) in Definition 1.4.1 hold, implying that $TGV_\alpha^{\varphi,2}$ is a seminorm. Note that $BGV_\alpha^{\varphi,2}(\Omega)$ is a normed space because $\varphi \in \Phi_w(\Omega)$ and satisfies (A0), so $\|\cdot\|_\varphi$ is a norm, see [98, Theorem 3.7.13].
(ii). Let us denote the set of the test functions for the functional $TGV_\alpha^{\varphi,2}$ by

$$K_\alpha^{\varphi,2}(\Omega) := \{\psi \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}) \ : \ \|\psi\|_{\varphi^*} \leq \alpha_1, \|\operatorname{div}\psi\|_{\varphi^*} \leq \alpha_2\}.$$

Let us first assume that u is a polynomial of degree 0 or 1, implying that $\nabla^2 u = 0$. Then, since the test function ψ is in $K_\alpha^{\varphi,2}(\Omega)$ and thanks to two integration by parts, it is obvious that $TGV_\alpha^{\varphi,2}(u) = 0$.
Suppose now that $TGV_\alpha^{\varphi,2}(u) = 0$ for some $u \in L^1_{\mathrm{loc}}(\Omega)$. For any $\psi \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$, one can find a $\lambda > 0$ such that

$$\|\lambda\psi\|_\infty \leq \alpha_1 \quad \text{ and } \quad \|\operatorname{div}(\lambda\psi)\|_\infty \leq \alpha_2.$$

Since φ satisfies (A0), meaning that φ^* satisfies (A0) as well, see Lemma 6.2.5(iv), and since Ω is bounded, thanks to Lemma 6.2.6 we have that $\lambda\psi \in K_\alpha^{\varphi,2}(\Omega)$. Testing in (6.1.1) with $\lambda\psi$, we then get

$$\int_\Omega u \operatorname{div}^2\psi \, dx = 0 \qquad \text{for any } \psi \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}),$$

implying that $\nabla^2 u = 0$ in the weak sense. Since Ω is connected, this implies that u has to be a polynomial of degree less than 2.
(iii). The equivalence of the seminorms $TGV_{\alpha}^{\varphi,2}$ and $TGV_{\tilde{\alpha}}^{\varphi,2}$ can be proven noticing that, chosen

$$c := \frac{\max\{\alpha_1, \alpha_2\}}{\min\{\tilde{\alpha}_1, \tilde{\alpha}_2\}},$$

we have that $c\, K_{\alpha}^{\varphi,2}(\Omega) \subset K_{\tilde{\alpha}}^{\varphi,2}(\Omega)$, meaning that $c\, TGV_{\alpha}^{\varphi,2}(u) \leq TGV_{\tilde{\alpha}}^{\varphi,2}(u)$ for any $u \in L^1_{\mathrm{loc}}(\Omega)$. Interchanging the roles of α and $\tilde{\alpha}$ gives us the desired equivalence.
(iv). Thanks to [53, (A.4)], we know that given a matrix $R \in SO(N)$, then

$$\psi \in C_c^2(R^{\mathrm{T}}\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}) \qquad \Longleftrightarrow \qquad \widetilde{\psi} := (\psi \circ R^{\mathrm{T}})R^{\mathrm{T}} \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}),$$

with $\operatorname{div}^l \widetilde{\psi} = ((\operatorname{div}^l \psi) \circ R^{\mathrm{T}})R^{\mathrm{T}}$, with $l = 0, 1$ where $\operatorname{div}^0 = id$. Hence, for $l = 0, 1$,

$$\|\operatorname{div}^l \widetilde{\psi}\|_{\varphi^*} = \|((\operatorname{div}^l \psi) \circ R^{\mathrm{T}})R^{\mathrm{T}}\|_{\varphi^*} = \|(\operatorname{div}^l \psi) \circ R^{\mathrm{T}}\|_{\varphi^*} = \|\operatorname{div}^l \psi\|_{\varphi^*},$$

implying that $\psi \in K_{\alpha}^{\varphi,2}(R^{\mathrm{T}}\Omega)$ if and only if $\widetilde{\psi} \in K_{\alpha}^{\varphi,2}(\Omega)$. Lastly, for any $\psi \in K_{\alpha}^{\varphi,2}(R^{\mathrm{T}}\Omega)$ we have

$$\int_{R^{\mathrm{T}}\Omega} u(Rx) \operatorname{div}^2\psi(x)\, dx = \int_{\Omega} u(x) \operatorname{div}^2\psi(R^{\mathrm{T}}x)\, dx = \int_{\Omega} u(x) \operatorname{div}^2\widetilde{\psi}(x)\, dx,$$

yielding $TGV_{\alpha}^{\varphi,2}(u \circ R) = TGV_{\alpha}^{\varphi,2}(u)$.
(v). Note that

$$\operatorname{div}(v \circ r^{-1}\mathbb{I}_n) = r^{-1}(\operatorname{div} v) \circ r^{-1}\mathbb{I}_n,$$

implying that, chosen $\psi \in C_c^l(r^{-1}\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ and $\bar{\psi} := r^2\, \psi \circ r^{-1}\mathbb{I}_n$, we have

$$\|\operatorname{div}^l \bar{\psi}\|_{\varphi^*} = r^{2-l}\|\operatorname{div}^l \psi\|_{\varphi^*},$$

for $l = 0, 1$. As a consequence, $\psi \in K_{\alpha}^{\varphi,2}(r^{-1}\Omega)$ if and only if $\bar{\psi} \in K_{\bar{\alpha}}^{\varphi,2}(\Omega)$ with $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2) := (\alpha_1 r^2, \alpha_2 r)$. Lastly, this yields

$$\int_{r^{-1}\Omega} u(rx) \operatorname{div}^2\psi(x)\, dx = r^{-N} \int_{\Omega} u(x) \operatorname{div}^2\psi(r^{-1}x)\, dx = \int_{\Omega} u(x) \operatorname{div}^2\bar{\psi}(x)\, dx,$$

implying $TGV_{\alpha}^{\varphi,2}(u \circ r\, \mathbb{I}_N) = r^{-N}\, TGV_{\bar{\alpha}}^{\varphi,2}(u)$. □

Note that it is possible to prove that $TGV_{\alpha}^{\varphi,2}$ is weakly lower semicontinuous with respect to the weak L^p convergence, as well as to the weak* convergence in L^∞.

Lemma 6.4.3. *Let Ω be a bounded set in $\mathbb{R}^N$ and let $\varphi \in \Phi_w(\Omega)$. Then, $TGV_\alpha^{\varphi,2}$ is weakly lower semicontinuous in $L^p(\Omega)$ for every $1 \le p < +\infty$, as well as weak* lower semicontinuous in $L^\infty(\Omega)$ for any $\alpha \in \mathbb{R}^+ \times \mathbb{R}^+$.*

Proof. We only prove the lemma for $p \in [1, +\infty)$, as the case $p = +\infty$ is analogous. Chosen $\psi \in C_c^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$ with $\|\psi\|_{\varphi^*} \le \alpha_1$ and $\|\operatorname{div}\psi\|_{\varphi^*} \le \alpha_2$ as in (6.1.1), then $\operatorname{div}^2\psi \in L^\infty(\Omega)$. Then, chosen $(u_k)_k \subset L^p(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $L^p(\Omega)$ we get

$$\int_\Omega u \operatorname{div}^2\psi\, dx = \liminf_{k\to+\infty} \int_\Omega u_k \operatorname{div}^2\psi\, dx \le \liminf_{k\to+\infty} TGV_\alpha^{\varphi,2}(u_k).$$

The thesis follows by taking the supremum over all such ψ. $\square$

The next proposition is the equivalent of [53, Proposition 3.5] in our setting, proving that $BGV_\alpha^{\varphi,2}(\Omega)$ is a Banach space.

Proposition 6.4.4. *Let Ω be a bounded, connected and open set in $\mathbb{R}^N$, with $N \ge 2$. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0). Then, for any weight $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ the space $BGV_\alpha^{\varphi,2}(\Omega)$ is a Banach space if endowed with the norm $\|\cdot\|_{BGV_\alpha^{\varphi,2}}$.*

Proof. Thanks to Proposition 6.4.2(iii), we know that all the seminorms $TGV_\alpha^{\varphi,2}$, respectively norms $\|\cdot\|_{BGV_\alpha^{\varphi,2}}$, are equivalent for any $\alpha \in \mathbb{R}^+ \times \mathbb{R}^+$, so we only need to prove that $BGV_\alpha^{\varphi,2}(\Omega)$ is complete for some $\alpha \in \mathbb{R}^+ \times \mathbb{R}^+$.
Note that, since Ω is bounded and φ satisfies (A0), then by Lemma 6.2.6 we have $L^\varphi(\Omega) \hookrightarrow L^1(\Omega)$. This, together with Lemma 6.4.3, implies that $TGV_\alpha^{\varphi,2}$ is a lower semicontinuous functional with respect to $L^\varphi(\Omega)$ for any α and φ chosen as in the hypotheses.
Now let $\{u_n\}_n$ be a Cauchy sequence in $BGV_\alpha^{\varphi,2}(\Omega)$. This yields that $\{u_n\}_n$ is a Cauchy sequence also in $L^\varphi(\Omega)$, implying the existence of a limit $u \in L^\varphi(\Omega)$ such that

$$TGV_\alpha^{\varphi,2}(u) \le \liminf_{n\to+\infty} TGV_\alpha^{\varphi,2}(u_n),$$

thanks to the lower semicontinuity of the functional. As a consequence, $u \in BGV_\alpha^{\varphi,2}(\Omega)$ and the only thing left to be shown is that u is the limit in the corresponding norm as well. In order to do so, let us choose $\varepsilon > 0$ and $n \in \mathbb{N}$ such that for any $\bar{n} \ge n$ it holds that $TGV_\alpha^{\varphi,2}(u_n - u_{\bar{n}}) \le \varepsilon$. Thanks again to the lower semicontinuity of the functional (with respect to the L^1 weak convergence), letting $\bar{n} \to +\infty$ we then get

$$TGV_\alpha^{\varphi,2}(u_n - u) \le \liminf_{\bar{n}\to+\infty} TGV_\alpha^{\varphi,2}(u_n - u_{\bar{n}}) \le \varepsilon,$$

implying that $u_n \to u$ in $BGV_\alpha^{\varphi,2}(\Omega)$. $\square$

6.4.2 Dual formulation

The first step in order to treat minimization problems in image denoising with this new regularizer is to prove an equivalent result to [54, Theorem 3.1], namely proving that (6.1.1) can be rewritten equivalently as a minimization problem which can be interpreted as an optimal balancing between the first and second derivative of u in terms of "sparse" penalization, see [54, Remark 3.2]. In [54] this was done via the Radon norm (i.e. the norm in $\mathcal{M}$ associated to the Radon measure Du), while here we need to use the variant of the generalized variation associated to φ defined in (6.3.11). Given the anisotropic nature of such regularizers, we expect them to better capture fine texture details in the images. The result, providing a dual characterization of $TGV_\alpha^{\varphi,2}$, reads as follows.

Theorem 6.4.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected, C^1 and open set. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0) and let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then for any $u \in L^1(\Omega)$ we have*

$$TGV_\alpha^{\varphi,2}(u) = \min_{w \in BD(\Omega)} \left\{ \alpha_2 \widetilde{V}_\varphi^N(Du - w) + \alpha_1 \widetilde{V}_\varphi^{N\times N}(Ew) \right\}, \tag{6.4.2}$$

where $TGV_\alpha^{\varphi,2}$ is defined as in (6.1.1) *and where we have adopted the convention that the above right-hand side is infinite if $\widetilde{V}_\varphi^{N\times N}(Ew) = +\infty$. If, additionally, $u \in BV^\varphi(\Omega)$, defined as in* (6.3.2)*, then*

$$TGV_\alpha^{\varphi,2}(u) = \min_{w \in BD^\varphi(\Omega)} \left\{ \alpha_2 \widetilde{V}_\varphi^N(Du - w) + \alpha_1 \widetilde{V}_\varphi^{N\times N}(Ew) \right\}, \tag{6.4.3}$$

where the space $BD^\varphi(\Omega)$ is defined as in (6.3.13)*.*

Proof. Let us define $X_2 := C_0^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$, $X_1 := C_0^1(\Omega; \mathbb{R}^N)$ and $\Lambda := \mathrm{div} \in \mathcal{L}(X_2, X_1)$. We stress that

$$X_2 = \overline{C_c^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})}^{\|\cdot\|_{C^2}} \qquad \text{and} \qquad X_1 = \overline{C_c^1(\Omega; \mathbb{R}^N)}^{\|\cdot\|_{C^1}}. \tag{6.4.4}$$

Chosen $u \in L^1(\Omega)$, we define

$$F_1(M) := I_{\{\|\cdot\|_{\varphi^*} \le \alpha_2\}}(M), \qquad \text{for any } M \in X_2$$

and

$$F_2^u(v) := I_{\{\|\cdot\|_{\varphi^*} \le \alpha_1\}}(v) - \int_\Omega u \, \mathrm{div} v \, dx, \qquad \text{for any } v \in X_1.$$

Since φ satisfies (A0), then by Lemma 6.2.5(iv) also φ^* satisfies (A0) and, by Lemma 6.2.6, we have that F_1 and F_2^u are continuous in the C^2- and C^1-topologies, respectively. Then, thanks to a density argument and to (6.4.4), we have that

$$TGV_\alpha^{\varphi,2}(u) = -\inf_{M \in X_2} \left[F_1(M) + F_2^u(\Lambda M) \right].$$

Let us remark that we can write the space X_1 equivalently as

$$X_1 = \bigcup_{\lambda \geq 0} \lambda \left(\mathrm{dom} F_2^u - \Lambda \, \mathrm{dom} F_1 \right). \tag{6.4.5}$$

Indeed, if $z \in \mathrm{dom} F_2^u$ then, trivially, $z \in X_1$ implies $z \in \bigcup_{\lambda \geq 0} \lambda \left(\mathrm{dom} F_2^u - \Lambda \, \mathrm{dom} F_1 \right)$ and for some $\lambda \geq 0$ it is always verified that $\frac{z}{\lambda} \in \mathrm{dom} F_2^u$. This implies that $X_1 \subset \bigcup_{\lambda \geq 0} \lambda \left(\mathrm{dom} F_2^u - \Lambda \, \mathrm{dom} F_1 \right)$. On the other hand, the converse is true since X_1 is a closed vector space (and $\mathrm{dom} F_2^u$ and $\mathrm{dom} F_1$ are subsets of X_1).
Now, since X_1 is a closed set, it means that also the right-hand side of (6.4.5) is closed, which allows us to use [19, Corollary 2.3], a Fenchel-Rockafellar duality result, getting

$$TGV_\alpha^{\varphi,2}(u) = - \inf_{M \in X_2} \left[F_1(M) + F_2^u(\Lambda M) \right] = \min_{w \in X_1^*} \left[(F_2^u)^*(w) + F_1^*(-\Lambda^* w) \right].$$

Now, exploiting the definition of Fenchel conjugate, we get

$$(F_2^u)^*(w) = \alpha_2 \sup_{v \in X_1} \left\{ \langle w, v \rangle_Y + \int_\Omega u \, \mathrm{div} v \, dx \ : \ \|v\|_{\varphi^*} \leq 1 \right\},$$

for every $w \in \mathcal{D}'(\Omega; \mathbb{R}^n)$, and

$$F_1^*(-\Lambda^* w) = \alpha_1 \sup_{M \in X_2} \left\{ \langle -\Lambda^* w, M \rangle_X \ : \ \|M\|_{\varphi^*} \leq 1 \right\},$$

for every $w \in \mathcal{D}'(\Omega; \mathbb{R}^n)$ with $-\Lambda^* w \in \mathcal{D}'(\Omega; \mathbb{R}^{N \times N})$, where $\langle \cdot, \cdot \rangle_{X_1}$ and $\langle \cdot, \cdot \rangle_{X_2}$ denote, respectively, the duality products between X_1 and X_1^* and between X_2 and X_2^*. Now, we observe that if $F_1^*(-\Lambda^* w) = +\infty$, then by Lemma 6.2.6 we find $\| - \Lambda^* w\|_{\mathcal{M}} = +\infty$. Moreover, since $-\Lambda^* = E$, we infer that for every $w \in \mathcal{D}'(\Omega; \mathbb{R}^n)$ with $-\Lambda^* w \in \mathcal{D}'(\Omega; \mathbb{R}^{N \times N})$ we have

$$F_1^*(-\Lambda^* w) = \begin{cases} \alpha_1 \widetilde{V}_\varphi^{N \times N}(Ew) & \text{if } Ew \in \mathcal{M}(\Omega; \mathbb{R}^{N \times N}) \\ +\infty & \text{if } Ew \in \mathcal{D}'(\Omega; \mathbb{R}^{N \times N}) \setminus \mathcal{M}(\Omega; \mathbb{R}^{N \times N}). \end{cases}$$

Analogously, by observing that, since $u \in L^1(\Omega)$,

$$(F_2^u)^*(w) = \alpha_2 \sup_{v \in X_1} \left\{ \langle Du - w, v \rangle_{X_1} \ : \ \|v\|_{\varphi^*} \leq 1 \right\},$$

for every $w \in \mathcal{D}'(\Omega; \mathbb{R}^N)$, the same argument as for F_1^* yields

$$(F_2^u)^*(w) = \begin{cases} \alpha_2 \widetilde{V}_\varphi^{N}(Du - w) & \text{if } Du - w \in \mathcal{M}(\Omega; \mathbb{R}^N) \\ +\infty & \text{if } Du - w \in \mathcal{D}'(\Omega; \mathbb{R}^N) \setminus \mathcal{M}(\Omega; \mathbb{R}^N), \end{cases}$$

again for every $w \in \mathcal{D}'(\Omega;\mathbb{R}^N)$. Thus, to summarize, by classical properties of BD and BV, see [146, Chapter II, Theorem 2.3] and [10], we have that the following characterization holds:

$$TGV_\alpha^{\varphi,2}(u) = \begin{cases} \min\limits_{w\in BD(\Omega)} \left\{\alpha_2 \widetilde{V}_\varphi^N(Du-w) + \alpha_1 \widetilde{V}_\varphi^{N\times N}(Ew)\right\} & \text{if } u \in BV(\Omega;\mathbb{R}^N), \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^N). \end{cases} \tag{6.4.6}$$

To conclude the proof we observe that if $u \in BV^\varphi(\Omega)$, then by [98, Theorem 3.4.6] the above infimum problem can be considered on the subspace of $BD(\Omega)$ where additionally $\|w\|_{\varphi^{**}} < +\infty$, namely $BD^\varphi(\Omega)$, see Remark 6.3.8. □

Note that in the proof we exploited the condition (A0) on the function φ in order to have the inclusions $BV^\varphi(\Omega) \hookrightarrow BV(\Omega)$ and $BD^\varphi(\Omega) \hookrightarrow BD(\Omega)$, see Remarks 6.3.1 and 6.3.7.
The following result reduces the functional-analytic setting to the space $BV^\varphi(\Omega)$, see (6.3.2), by establishing equivalence of the two spaces $BGV_\alpha^{\varphi,2}$ and BV^φ.

Corollary 6.4.6. *Under the same assumptions of Theorem 6.4.5, we have that* $TGV_\alpha^{\varphi,2}(u) < +\infty$ *if and only if* $u \in BV^\varphi(\Omega)$.

Proof. By Remark 6.3.3, choosing $w = 0$ as a test function in the characterization of $TGV_\alpha^{\varphi,2}(u)$, namely (6.4.2), we directly infer

$$TGV_\alpha^{\varphi,2}(u) \le \alpha_2 \widetilde{V}_\varphi^N(Du) = \alpha_2 V_\varphi(u),$$

(see (6.3.1)), which, in turn, yields

$$\|u\|_\varphi + TGV_\alpha^{\varphi,2}(u) \le \max\{1,\alpha_2\}\|u\|_{BV^\varphi(\Omega)}.$$

Conversely, since $\widetilde{V}_\varphi^N$ satisfies the triangle inequality, letting w_u be the minimum in (6.4.2), we deduce

$$\alpha_2 V_\varphi(u) = \alpha_2 \widetilde{V}_\varphi^N(Du) \le \alpha_2 \widetilde{V}_\varphi^N(Du - w_u) + \alpha_2 \widetilde{V}_\varphi^N(w_u) \le TGV_\alpha^{\varphi,2}(u) + \alpha_2 \widetilde{V}_\varphi^N(w_u).$$

□

Remark 6.4.7. Thanks to Proposition 6.4.2, in particular thesis (i), and to Corollary 6.4.6, we have that $TGV_\alpha^{\varphi,2}$ is a seminorm also on the space $BV^\varphi(\Omega)$.

Remark 6.4.8. Differently from the classical TGV-setting, it is not clear whether it is possible to show that there exists a constant $C > 0$ such that

$$\frac{1}{C}\|u\|_{BV^\varphi(\Omega)} \le \|u\|_\varphi + TGV_\alpha^{\varphi,2}(u)$$

or not. Roughly speaking, this is due to the absence of a Musielak-Orlicz equivalent of Sobolev-Korn inequality. We refer the interested reader to the discussion and the references in Section 6.3.

We also stress that, as it follows from the analysis in Proposition 6.4.9,

$$\frac{1}{C}(\|u\|_\varphi + |Du|(\Omega)) \le \|u\|_\varphi + TGV_\alpha^{\varphi,2}(u)$$

for every $u \in BV^\varphi(\Omega)$.

Thanks to what we have proven in Subsections 6.3.2 and 6.3.3, we expect also the variation $\widetilde{V}_\varphi^m$ and, as a consequence, the total generalized variation $TGV_\alpha^{\varphi,2}$ to have a decomposition depending on the absolutely continuous part and the singular part of the gradient of the functions $u \in BV^\varphi(\Omega)$. We leave this open question for future works.

6.4.3 Existence and stability of solutions to the minimization problem

In this section we show how our notion of total generalized variation inherits classical existence and stability properties. Throughout this section we will always tacitly assume that $TGV_\alpha^{\varphi,2}$ is extended to $+\infty$ outside of its domain. We first prove that a Poincaré-Wirtinger inequality also holds in our setting, entailing the coercivity needed to prove the existence of solutions to minimum problems in image denoising.
To this aim, from now on we denote by $\mathcal{P}^1(\Omega)$ the space of affine functions, as well as by $\Pi : L^1(\Omega) \to \mathcal{P}^1(\Omega)$ a linear projection.

Proposition 6.4.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected, open set with C^1 boundary. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0) and let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. Let $1 < p \le \frac{N}{N-1}$. Then, there exists a constant $c > 0$ such that*

$$\|u\|_{L^p(\Omega)} \le c\, TGV_\alpha^{\varphi,2}(u),$$

for every $u \in \ker \Pi \cap L^p(\Omega)$.

Proof. Because φ satisfies (A0), by Remark 6.3.1 we have $BV^\varphi(\Omega) \hookrightarrow BV(\Omega)$ and so

$$|Du - w|(\Omega) \le \widetilde{V}_\varphi^N(Du - w).$$

Analogously, thanks to Remark 6.3.7 we get $BD^\varphi(\Omega) \hookrightarrow BD(\Omega)$, entailing

$$|Ew|(\Omega) \le \widetilde{V}_\varphi^{N\times N}(Ew),$$

for every $w \in BD(\Omega)$. In view of [54, Theorem 3.1, Proposition 4.1], we infer the existence of a constant $c > 0$ such that

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\le c\, TGV_\alpha^2(u) \le c\,[\alpha_2|Du - w|(\Omega) + \alpha_1|Ew|(\Omega)]\\ &\le c\left[\alpha_2 \widetilde{V}_\varphi^N(Du - w) + \alpha_1 \widetilde{V}_\varphi^{N\times N}(Ew)\right],\end{aligned}$$

for every $w \in BD(\Omega)$. The thesis follows then by Theorem 6.4.5. □

In view of the Lemma 6.4.3, we deduce existence of minimizers whenever $TGV_\alpha^{\varphi,2}$ is augmented by suitable fidelity terms. In what follows, given two Hilbert spaces X and H, we denote by $\mathcal{L}(X; H)$ the set of linear and continuous operators between X and H.

Theorem 6.4.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with C^1 boundary. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0) and let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. Let $1 < p \le \frac{N}{N-1}$. Let further H be an Hilbert space, and let $K \in \mathcal{L}(L^p(\Omega); H)$ be injective on $\mathcal{P}^1(\Omega)$. Then, for every $f \in H$, the minimum problem*

$$\min_{u \in L^p(\Omega)} \frac{1}{2}\|Ku - f\|_H + TGV_\alpha^{\varphi,2}(u) \tag{6.4.7}$$

has a solution.

Proof. Arguing exactly as in the proof of [54, Theorem 4.2], in view of Propositions 6.4.2 and 6.4.9, we find that any minimizing sequence for (6.4.7) is uniformly bounded in $L^p(\Omega)$ for every $1 \le p \le \frac{N}{N-1}$. The thesis follows then by Lemma 6.4.3. □

We conclude this section by showing stability of minimizers of (6.4.7) with respect to the fidelity datum f.

Theorem 6.4.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with C^1 boundary. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0) and let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$. Let $1 < p \le \frac{N}{N-1}$. Let H be an Hilbert space and let $K \in \mathcal{L}(L^p(\Omega); H)$ be injective on $\mathcal{P}^1(\Omega)$. Let $\{f_k\}_k \subset H$ be such that $f_k \to f$ strongly in H. Then, setting*

$$\mathcal{F}_k(u) := \frac{1}{2}\|Ku - f_k\|_H + TGV_\alpha^{\varphi,2}(u), \qquad \mathcal{F}_0(u) := \frac{1}{2}\|Ku - f\|_H + TGV_\alpha^{\varphi,2}(u),$$

for every $u \in L^p(\Omega)$, we have that $\{\mathcal{F}_k\}_k$ Γ-converges to $\mathcal{F}_0$ with respect to the weak L^p-topology.

Proof. The limsup inequality is immediately verified by considering for each $u \in L^p(\Omega)$ the constant recovery sequence $\bar{u}_k = u$ for every $k \in \mathbb{N}$. To show the liminf inequality, let $u \in L^p(\Omega)$ and let $\{u_k\}_k \subset L^p(\Omega)$ be such that $u_k \rightharpoonup u$ weakly in $L^p(\Omega)$. The fact that

$$\mathcal{F}_0(u) \leq \liminf_{k \to +\infty} \mathcal{F}_k(u_k)$$

is a direct consequence of Lemma 6.4.3. □

As a consequence of Theorem 6.4.11 we infer convergence of the associated minimizers.

Corollary 6.4.12. *Under the same hypotheses of Theorem 6.4.11, we have*

(i) $\min \mathcal{F}_k \to \min \mathcal{F}_0$.

(ii) Letting $u_k \in \operatorname{argmin} \mathcal{F}_k$ *for every* $k \in \mathbb{N}$, *we have that* $\{u_k\}_k$ *is precompact in* $L^{\frac{N}{N-1}}(\Omega)$, *and that every limiting point* u *satisfies* $u \in \operatorname{argmin} \mathcal{F}_0$.

Proof. The result follows from the Fundamental Theorem of Γ-convergence, cf. [49], by combining Theorem 6.4.11 and Proposition 6.4.9. □

Appendix A

A.1 Proofs of the results in Subsection 6.3.2

In Proposition 6.3.17 and Proposition 6.3.18 we consider respectively the singular and absolutely continuous parts of the symmetric gradient separately and we then combine them to handle the whole function in Theorem 6.3.14.
In order to prove Proposition 6.3.17, we need the following result, analogous to [82, Lemma 3.5], which holds a characterization of the singular part of the symmetric gradient of our function.

Lemma A.1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected and open set. If $u \in BD(\Omega)$, then*

$$|E^s u|(\Omega) = \sup \left\{ \int_\Omega \psi : dE^s u \ : \ \psi \in C_c^1(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}),\ |\psi| \leq 1 \right\}.$$

Proof. By the definitions of total variation of a measure and of weak derivative, together with the fact that $Eu = \mathcal{E}u + E^s u$, we see that

$$\begin{aligned}
|Eu|(\Omega) &= \sup_{\psi \in C_c^1(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}), |\psi|\leq 1} \left(\int_\Omega \psi : d\mathcal{E}u + \int_\Omega \psi : dE^s u \right) \\
&\leq \sup_{\psi \in L^\infty(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}), |\psi|\leq 1} \int_\Omega \psi : d\mathcal{E}u + \sup_{\psi \in L^\infty(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}), |\psi|\leq 1} \int_\Omega \psi : dE^s u \\
&= |\mathcal{E}u|(\Omega) + |E^s u|(\Omega).
\end{aligned}$$

Since $|\mathcal{E}u|(\Omega) + |E^s u|(\Omega) = |Eu|(\Omega)$ as $\mathcal{E}$ and E^s are mutually singular, each inequality has to be an equality, and so the claim follows. □

We can now give the proof of Proposition 6.3.17, following the lines of [82, Proposition 6.2].

Proof of Proposition 6.3.17. By the definition of the total variation of a measure, Lemma 6.3.16 and the definition of T^φ, see (6.3.18), we have

$$\sup_{\psi\in T^\varphi}\int_\Omega \psi : dE^s u \le \sup_{\psi\in T^\varphi}\int_\Omega |\psi|\, d|E^s u| \le \int_\Omega \varphi'_\infty\, d|E^s u|.$$

In order to tackle the opposite inequality, let us define $h_k:\Omega\to[0,+\infty]$ by

$$h_k(x) := \lim_{r\to 0^+}\inf_{y\in B(x,r)}\frac{\varphi(y,k)}{k} \qquad \text{for any } x\in\Omega.$$

Notice that h_k is lower semicontinuous with

$$h_k \le \tfrac{\varphi(\cdot,k)}{k} \le \varphi^\infty. \tag{A.1.1}$$

From the first inequality it follows that $\varphi^*(\cdot,h_k)\le\varphi(\cdot,k)$, implying that $\varrho_{\varphi^*}(h_k)\le \varrho_\varphi(k)<+\infty$ since φ satisfies (A0) and (aDec), and Ω is bounded.
Let us now remark that $h_k\to\varphi^\infty$ as $k\to+\infty$. Indeed, if $\varphi^\infty(x)=+\infty$, then we can use (A1) in all sufficiently small balls, since $\varphi^+(k)<+\infty$, to conclude that

$$h_k(x) = \lim_{r\to 0^+}\inf_{y\in B(x,r)}\frac{\varphi(y,k)}{k} \ge \frac{\varphi(x,\beta k)-1}{k} \to \beta\varphi^\infty(x) = +\infty \tag{A.1.2}$$

as $k\to+\infty$. If instead $\varphi^\infty(x)<+\infty$, then we use the same inequality but now with $\beta:=\frac{1}{1+\omega(r)}$, where ω is the modulus of continuity coming from the restricted (VA1) condition, and we obtain the desired convergence as $\beta\to 1^-$.
Note that h_k is increasing in k since φ is convex. Then, by monotone convergence,

$$\int_\Omega \varphi^\infty\, d|E^s u| = \lim_{k\to\infty}\int_\Omega h_k\, d|E^s u|.$$

Let us fix $\varepsilon>0$ and assume that $\int_\Omega \varphi^\infty\, d|E^s u|<+\infty$. Then, we can find $h=h_k$ and $K>0$ such that

$$\begin{aligned}\int_\Omega \varphi^\infty\, d|E^s u| &\le \int_\Omega h\, d|E^s u| + \varepsilon \le \sum_{j=1}^{K^2}\int_\Omega \frac{1}{K}\chi_{\{h>\frac{j}{K}\}}\, d|E^s u| + 2\varepsilon \\ &= \frac{1}{K}\sum_{j=1}^{K^2}|E^s u|\left(\left\{h>\frac{j}{K}\right\}\right) + 2\varepsilon.\end{aligned}$$

Since h is lower semicontinuous, then $\{h>\frac{j}{K}\}$ is open, and hence by Lemma A.1.1 we can choose $\psi_j\in C^1_c(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})$ with $|\psi_j|\le 1$ such that

$$\int_\Omega \varphi^\infty\, d|E^s u| \le \frac{1}{K}\sum_{j=1}^{K^2}\int_{\{h>\frac{j}{K}\}}\psi_j : dE^s u + 3\varepsilon = \int_\Omega\left(\sum_{j=1}^{K^2}\frac{1}{K}\psi_j\right) : dE^s u + 3\varepsilon.$$

Defining $\psi_\varepsilon := \sum_{j=1}^{K^2} \frac{1}{K}\psi_j$, we can notice that $\psi_\varepsilon \in C_c^1(\Omega;\mathbb{R}^{N\times N}_{\text{sym}})$ and

$$|\psi_\varepsilon| \le \frac{1}{K}\sum_{j=1}^{K^2} |\psi_j| \le \frac{1}{K}\sum_{j=1}^{K^2} \chi_{\{h>\frac{j}{K}\}} \le h,$$

giving $\varrho_{\varphi^*}(|\psi_\varepsilon|) < +\infty$. Therefore $\psi_\varepsilon \in T^\varphi$ and

$$\int_\Omega \varphi^\infty \, d|E^s u| \le \int_\Omega \psi_\varepsilon : dE^s u + 3\varepsilon \le \sup_{\psi\in T^\varphi} \int_\Omega \psi : dE^s u + 3\varepsilon.$$

The upper bound then follows letting $\varepsilon \to 0^+$. If instead $\int_\Omega \varphi^\infty \, d|E^s u| = +\infty$, then a similar argument gives $\frac{1}{3\varepsilon} \le \sup_{\psi\in T^\varphi} \int_\Omega \psi : dE^s u$ and the claim again follows. □

Lastly, we now give the proof of Proposition 6.3.18, following the lines of [82, Proposition 6.3].

Proof of Proposition 6.3.18. The upper bound follows directly from Young's inequality, indeed

$$\sup_{\psi\in T^\varphi} \int_\Omega [\mathcal{E}u : \psi - \varphi^*(x,|\psi|)]\, dx \le \int_\Omega \varphi(x,|\mathcal{E}u|)\, dx = \varrho_\varphi(|\mathcal{E}u|).$$

In order to prove the lower bound let us choose $g_i \in C(\Omega;\mathbb{R}^{N\times N}_{\text{sym}}) \cap L^\varphi(\Omega;\mathbb{R}^{N\times N}_{\text{sym}})$ such that $g_i \to \mathcal{E}u$ pointwise a.e and in $L^1(\Omega;\mathbb{R}^{N\times N}_{\text{sym}})$. Then Fatou's Lemma and L^1-convergence yield

$$\int_\Omega \varphi(x,|\mathcal{E}u|)\, dx \le \liminf_{i\to+\infty} \int_\Omega \varphi(x,|g_i|)\, dx \quad \text{and} \quad \lim_{i\to+\infty} \int_\Omega g_i : \psi\, dx = \int_\Omega \mathcal{E}u : \psi\, dx$$

for any $\psi \in T^\varphi$. Thus, it suffices to show that

$$\int_\Omega \varphi(x,|g|)\, dx \le \sup_{\psi\in T^\varphi} \int_\Omega [g : \psi - \varphi^*(x,|\psi|)]\, dx$$

for any $g \in C(\Omega;\mathbb{R}^{N\times N}_{\text{sym}}) \cap L^\varphi(\Omega;\mathbb{R}^{N\times N}_{\text{sym}})$. Moreover, replacing ψ by $\frac{g}{\varepsilon+|g|}|\psi|$ and letting $\varepsilon \to 0^+$, we see that $g : \frac{g}{\varepsilon+|g|}|\psi| \to |g|\,|w|$ pointwise. Thus, by monotone convergence, the vector-values of g and ψ are irrelevant and we only need to show that

$$\int_\Omega \varphi(x,|g|)\, dx \le \sup_{\psi\in C_c^1(\Omega)} \int_\Omega [|g\psi| - \varphi^*(x,|\psi|)]\, dx \tag{A.1.3}$$

for any $g \in C(\Omega) \cap L^\varphi(\Omega)$. We can also exclude test functions such that $\varrho_{\varphi^*}(\psi) = +\infty$, see Remark 6.3.11.

Now let us denote with φ' the left-derivative of φ with respect to the second variable. Then φ' is left-continuous and $\varphi(x,s) \geq \varphi(x,s_0) + \varphi'(x,s_0)(s-s_0)$ by convexity. For any $\varepsilon > 0$, let us define

$$\psi_\varepsilon(x,t) := \int_{-\infty}^{+\infty} \varphi(x, \max\{\tau, 0\})\, \zeta_\varepsilon(t-\tau)\, d\tau = (\varphi *_t \zeta_\varepsilon)(x,t), \tag{A.1.4}$$

where ζ_ε is a mollifier in $\mathbb{R}$ with support in $[0,\varepsilon]$ and where $*_t$ denotes the convolution only in the second variable. Since φ and φ' are increasing in the second variable and left-continuous, we have that $\psi_\varepsilon \nearrow \varphi$ and $\psi'_\varepsilon \nearrow \varphi'$ as $\varepsilon \to 0^+$. Moreover, $\psi'_\varepsilon = \varphi *_t \zeta'_\varepsilon$ is continuous in both x, since φ is, and t, as a convolution with a smooth function.
Let $v_i \in C_c(\Omega)$ with $0 \leq v_i \leq 1$ and $v_i \nearrow 1$ as $i \to +\infty$. By uniform continuity in $\operatorname{supp} v_i$, we can choose $\delta = \delta_{\varepsilon,i} > 0$ such that, for any $\varepsilon > 0$ and $i \in \mathbb{N}$,

$$\psi'_\varepsilon(x, |g(x)|\, v_i(x)) - \varepsilon \leq \psi'_\varepsilon(y, |g(y)|\, v_i(y))$$

for all $x \in B(y,\delta)$ and $y \in \Omega$. Then

$$\psi_{\varepsilon,i} := \max\{\psi'_\varepsilon(\,\cdot\,, |g|\, v_i) - \varepsilon, 0\} *_x \eta_\delta \leq \psi'_\varepsilon(\cdot, |g|) \leq \varphi'(\cdot, |g|),$$

where η_δ is a mollifier in $\mathbb{R}$ with support in $[0,\delta]$ and where $*_x$ denotes the convolution only in the first variable. We have that $\psi_{\varepsilon,i} \to \varphi'(\cdot, |g|)$, so we conclude by Fatou's Lemma that

$$\int_\Omega |g|\, \varphi'(x,|g|)\, dx \leq \liminf_{i\to+\infty,\, \varepsilon\to 0} \int_\Omega |g\, \psi_{\varepsilon,i}|\, dx.$$

Since φ satisfies (A0) and (aDec), we see that

$$\varphi^*(x, |\psi_{\varepsilon,i}|) \leq \varphi^*(x, \varphi'(x,|g|)) \leq |g|\, \varphi'(x,|g|) \leq c\, \varphi(x,|g|),$$

where $c > 0$ is a constant. Since $g \in L^\varphi(\Omega)$ and φ satisfies (aDec), then $\varrho_\varphi(g) < +\infty$. Thus, dominated convergence with upper bound $c\,\varphi(\cdot, g)$ yields

$$\int_\Omega \varphi^*(x, \varphi'(x,|g|))\, dx = \lim_{i\to+\infty,\, \varepsilon\to 0} \int_\Omega \varphi^*(x, |\psi_{\varepsilon,i}|)\, dx.$$

Since ψ_ε is a valid test function and $\varphi'(\cdot,|g|) < +\infty$ a.e. in Ω, the last equality, together with "Young's equality", see [125, Lemma 1.7.3(i)], implies that

$$\begin{aligned}\sup_{\psi\in C^1_c(\Omega),\, \varrho_{\varphi^*}(\psi)<+\infty} \int_\Omega [|g\psi| - \varphi^*(x,|\psi|)]\, dx &\geq \liminf_{i\to+\infty,\, \varepsilon\to 0} \int_\Omega [|g|\,|\psi_{\varepsilon,i}| - \varphi^*(x,|\psi_{\varepsilon,i}|)]\, dx \\ &\geq \int_\Omega [|g|\,\varphi'(x,|g|) - \varphi^*(x,\varphi'(x,|g|))]\, dx = \int_\Omega \varphi(x,|g|)\, dx,\end{aligned}$$

proving (A.1.3) and completing the proof of the lower bound. □

Remark A.1.2. The natural idea would be to choose $\psi := \varphi'(x, |g|)$ in the supremum in (A.1.3), instead of the family of test functions that we chose in (A.1.4), where φ' is the left-derivative of φ with respect to the second variable. However, this function is not in general smooth and we cannot use regular approximation by smooth functions since φ^* is not doubling.

Appendix B

B.1 A Convex Analysis lemma

The following result is presented for readers convenience and it is due to [5], see Lemma 2.33 therein. It has been used in Section 6.3.

Lemma B.1.1. *Let X be a Banach space with norm $\|\cdot\|_X$, let $\phi : \mathbb{R} \to \mathbb{R}$ be an even and convex function, and define $f : X \to \mathbb{R}$ as $f(x) := \phi(\|x\|_X)$. Then*

$$f^*(x^*) = \phi^*(\|x^*\|_{X^*})$$

where X^* denotes the dual of X with norm $\|\cdot\|_{X^*}$.

Proof. By definition

$$f^*(x^*) = \sup_{x\in X}[\langle x^*, x\rangle - f(x)] = \sup_{x\in X}[\langle x^*, x\rangle - \phi(\|x\|_X)],$$

which can be equivalently written as

$$\begin{aligned} f^*(x^*) &= \sup_{t\geq 0} \sup_{\|x\|_X = t} [\langle x^*, x\rangle - \phi(t)] \\ &= \sup_{t\geq 0} \left\{ \left[\sup_{\|x\|_X = t} \langle x^*, x\rangle \right] - \phi(t) \right\} = \sup_{t\geq 0}[t\|x^*\|_{X^*} - \phi(t)]. \end{aligned} \tag{B.1.1}$$

On the other hand, by definition

$$\phi^*(\|x^*\|_{X^*}) = \sup_{t\in\mathbb{R}}[t\|x^*\|_{X^*} - \phi(t)], \tag{B.1.2}$$

hence it suffices to prove that (B.1.1) coincides with (B.1.2), i.e. that it suffices to take the supremum on nonnegative real numbers. Using the fact that ϕ is even, for any $t > 0$ we have

$$-t\|x^*\|_{X^*} - \phi(-t) = -t\|x^*\|_{X^*} - \phi(t) \leq t\|x^*\|_{X^*} - \phi(t),$$

hence

$$\sup_{t\in\mathbb{R}}[t\|x^*\|_{X^*} - \phi(t)] = \sup_{t\geq 0}[t\|x^*\|_{X^*} - \phi(t)],$$

which concludes the proof. □

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